Hamiltonian structure for a differential system from a modified Laguerre weight via the geometry of the modified third Painlevé equation

Anton Dzhamay[*], Galina Filipuk[†], Adam Ligęza[‡], Alexander Stokes[§]

March 17, 2021

Abstract

Recurrence coefficients of semi-classical orthogonal polynomials are often related to the solutions of special nonlinear second-order differential equations known as the Painlevé equations. Each Painlevé equation can be written in a standard form as a non-autonomous Hamiltonian system, so it is natural to ask whether differential systems satisfied by the recurrence coefficients also possess Hamiltonian structures. We consider recurrence coefficients for a modified Laguerre weight which satisfy a differential system known to be related to the modified third Painlevé equation and identify a Hamiltonian structure for it by constructing its space of initial conditions. We also discuss a transformation from this system to the modified third Painlevé equation which simultaneously identifies a discrete system for the recurrence coefficients with a discrete Painlevé equation.

MSC classification: 34M55

Key words: Painlevé equations, orthogonal polynomials, Hamiltonian systems, symplectic transformations

1 Introduction

Recurrence coefficients of semi-classical orthogonal polynomials are often related to the solutions of the Painlevé equations, either discrete or differential [11]. In this paper we revisit the modified Laguerre weight

$$w(x) = w(x,s) = x^{\alpha} e^{-x} e^{-s/x}, \;\; x \in (0,\infty), \;\; \alpha,\, s > 0,$$

which was studied extensively in [1]. It was shown that for polynomials $P_n(x)$ orthogonal with respect to this weight the recurrence coefficients α_n, β_n in the identity

$$xP_n(x) = P_{n+1}(x) + \alpha_n(s)P_n(x) + \beta_n(s)P_{n-1}(x), \quad n \in \mathbb{Z}_{\geq 1}$$

are related to new variables $c_n = c_n(s)$ and $b_n = b_n(s)$ by $\alpha_n = 2n+1+\alpha+c_n$, $\beta_n = n(n+\alpha) + b_n + \sum_{j=0}^{n-1} c_j$ which in turn satisfy the following systems of discrete and differential equations (see equations (2.16), (2.17) and (3.10), (3.11) in [1]):

$$\begin{aligned} b_{n+1} + b_n &= s - (2n+1+\alpha+c_n)c_n, \\ (b_n^2 - sb_n)(c_n + c_{n-1}) &= (ns - (2n+\alpha)b_n)c_n c_{n-1}, \end{aligned} \qquad n \in \mathbb{Z}_{\geq 1}, \tag{1}$$

[*]School of Mathematical Sciences, The University of Northern Colorado, Greeley, CO 80526, USA. Email: Anton.Dzhamay@unco.edu

[†]Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097, Warsaw, Poland. Email: filipuk@mimuw.edu.pl

[‡]Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097, Warsaw, Poland. Email: a.ligeza@mimuw.edu.pl

[§]Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom. Email: alexander.stokes.14@ucl.ac.uk

$$\begin{aligned} s\frac{dc_n}{ds} &= 2b_n + (2n+1+\alpha+c_n)c_n - s, \\ s\frac{db_n}{ds} &= \frac{2}{c_n}(b_n^2 - sb_n) + (2n+\alpha+1)b_n - ns, \end{aligned} \qquad n \in \mathbb{Z}_{\geq 0}. \tag{2}$$

In this paper we consider the identification of Hamiltonian structures for differential systems for recurrence coefficients of orthogonal polynomials in the context of their relations to Painlevé equations. As shown in [1], the system (2) gives a second-order differential equation which can be reduced to the modified third Painlevé equation, which has a well-known Hamiltonian form. Given this relation, it is natural to ask whether the system (2) admits a Hamiltonian description. The main task is to determine the symplectic form in terms of which the Hamiltonian structure should be defined, and we show how this may be done by constructing a space of initial conditions for the system via resolution of singularities as in the foundational work of Okamoto [6]. This approach relies on the fact that the system is transformable to a Painlevé equation, but does not rely on the transformation being known. This means that our method is applicable to cases where the system is suspected to be equivalent to one of the Painlevé equations but the relation is not known explicitly, which is often the case for systems coming from semi-classical orthogonal polynomials.

1.1 Background

There is a vast body of literature on the Painlevé equations, and among their solutions are the Painlevé transcendents, which have wide applications and a place in the modern library of special functions [1]. They also have an underlying geometric structure, see [4, 9] and references within. As shown in [1], the system (2) gives a second-order differential equation

$$c_n'' = \frac{(c_n')^2}{c_n} - \frac{c_n'}{s} + (2n+1+\alpha)\frac{c_n^2}{s^2} + \frac{c_n^3}{s^2} + \frac{\alpha}{s} - \frac{1}{c_n}, \tag{3}$$

which is a form of the modified third Painlevé equation $\mathrm{P}_{\mathrm{III}'}$:

$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{\alpha_3 q^2}{4t^2} + \frac{\beta_3}{4t} + \frac{\gamma_3 q^3}{4t^2} + \frac{\delta_3}{4q}. \tag{4}$$

This is achieved through the change of variables

$$c_n(s) = -q(t), \quad s = t, \tag{5}$$

with parameters in equations (3) and (4) related according to

$$\alpha_3 = -4(2n+1+\alpha),\ \beta_3 = -4\alpha,\ \gamma_3 = -\delta_3 = 4. \tag{6}$$

Each differential Painlevé equation can be written as a non-autonomous Hamiltonian system with polynomial Hamiltonian [7, 8], which in the case of $\mathrm{P}_{\mathrm{III}'}$ (4) with $\gamma_3 = -\delta_3 = 4$ is given by

$$\frac{dq}{dt} = \frac{\partial H}{\partial p},\ \ \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{7}$$

where the Hamiltonian [4, Sect. 8.5.17] is given by

$$H = \frac{1}{t}(p(p-1)q^2 + (a_1+a_2)qp + tp - a_2 q), \tag{8}$$

in which the parameters a_1, a_2 are related to those in (4) by

$$\alpha_3 = -4(a_1 - a_2),\ \ \beta_3 = -4(a_1+a_2-1),\ \ \gamma_3 = -\delta_3 = 4. \tag{9}$$

The standard form of a related discrete Painlevé equation [4, Sect. 8.1.20] arising from Bäcklund transformation symmetries of (4) is

$$\begin{aligned} q_n + q_{n+1} &= -\frac{a_2(n)}{p} - \frac{a_1(n)}{p-1}, \\ p_n + p_{n+1} &= 1 - \frac{t}{q_{n+1}^2} - \frac{a_1(n)+a_2(n)+1}{q_{n+1}}, \end{aligned} \tag{10}$$

[1] NIST Digital Library of Mathematical Functions, dlmf.nist.gov

for $n \in \mathbb{Z}$. This is constructed from a translation element of the extended affine Weyl group of Bäcklund transformations, which relates a solution $(q,p) = (q_n, p_n)$ of the system (7) with parameters $a_1 = a_1(n)$ and $a_2 = a_2(n)$ to a solution $(q,p) = (q_{n+1}, p_{n+1})$ of the same system but with parameters $a_1 = a_1(n+1)$ and $a_2 = a_2(n+1)$, subject to the following evolution with n:

$$a_1(n+1) = a_1(n) + 1, \;\; a_2(n+1) = a_2(n) + 1. \tag{11}$$

Another change of variables for (2) presented in [1] is

$$c_n(s) = s/q(s), \tag{12}$$

which also gives the modified third Painlevé equation but with parameters

$$\alpha_3 = -4\alpha, \; \beta_3 = -4(2n+1+\alpha), \; \gamma_3 = -\delta_3 = 4. \tag{13}$$

Moreover, in [2] it was shown that system (1) is equivalent to a pair of difference equations for c_n of second- and third-order respectively, and it was shown how to obtain c_{n-1} and c_{n+1} from the Bäcklund transformations of the third Painlevé equation.

2 Main results

Firstly, we have the following structure of the differential equations (2) from the modified Laguerre weight as a non-autonomous Hamiltonian system:

Theorem 1. *System (2) can be written in the Hamiltonian form*

$$\frac{1}{c_n^2}\frac{dc_n}{ds} = \frac{\partial K}{\partial b_n}, \qquad \frac{1}{c_n^2}\frac{db_n}{ds} = -\frac{\partial K}{\partial c_n}, \tag{14}$$

with the Hamiltonian

$$K = \frac{b_n(c_n^2 + (1+\alpha+2n)c_n + b_n - s)}{sc_n^2} - \frac{n}{c_n} + C(s), \tag{15}$$

where $C(s)$ is an arbitrary function of s.

This result can be verified by direct calculation, but our aim in what follows is to demonstrate how this Hamiltonian structure can be detected from the space of initial conditions for system (2). Secondly, we note that the relation (12) gives a transformation which simultaneously identifies the differential and discrete systems (1) and (2) with both the Hamiltonian form (7) of $\mathrm{P_{III'}}$ and the discrete Painlevé equation (10):

Theorem 2. *The birational transformation $(q,p) \mapsto (c_n, b_n)$ given by*

$$c_n = \frac{s}{q}, \qquad b_n = -\frac{s(s+q+2nq+\alpha q - q^2)}{q^2} - sp, \tag{16}$$

identifies the differential systems (2) and (7) with $t = s$, with parameters given by

$$a_1 = a_1(n) = 1+n+\alpha, \;\; a_2 = a_2(n) = n+1. \tag{17}$$

Under this transformation with $(q,p) = (q_n, p_n)$, the discrete system (1) is reduced to (10) with the same parameters as in (17)*.*

We remark that the other transformation (5) from [1] does not identify the discrete systems, which is essentially because the parameter correspondence (6) does not match the parameter evolution (11) for the discrete Painlevé equation. Another transformation between the discrete systems (1) and (10) is obtained in [5] with parameters $\alpha_3 = 4\alpha$, $\beta_3 = 4(1-2n-\alpha)$, $\gamma_3 = -\delta_3 = 4$, which corresponds to $a_1(n) = n$ and $a_2(n) = \alpha + n$. This was obtained through the method of [3] using techniques from the geometric theory of discrete Painlevé equations [9] and we remark that identifications with the standard form of the discrete Painlevé equation (10) are naturally non-unique and this transformation is conjugate to ours by a symmetry of the discrete Painlevé equation.

2.1 Finding the symplectic structure from the space of initial conditions

The main task in identifying the Hamiltonian structure in Theorem 1 is to determine the symplectic form with respect to which the Hamiltonian form of the system should be sought. For the system (2), we seek a symplectic form $G(c_n, b_n, s)dc_n \wedge db_n$ with respect to which the system is Hamiltonian, i.e. of the form

$$G(c_n, b_n, s)\frac{dc_n}{ds} = \frac{\partial K}{\partial b_n}, \qquad G(c_n, b_n, s)\frac{db_n}{ds} = -\frac{\partial K}{\partial c_n}, \tag{18}$$

for some Hamiltonian $K(c_n, b_n, s)$, also to be determined. If it is expected that there exists a birational transformation from the given system to one of the Painlevé equations, then the given system can be expected to possess a space of initial conditions in the sense of Okamoto [6], from which the appropriate symplectic form can be deduced.

We can perform the following procedure to find the symplectic form coming from the space of initial conditions. On first glance the phase space of the system is the trivial bundle over the independent variable space $\mathbb{C}\setminus\{0\}$ with fibre $\mathbb{C}^2$, but we compactify the fibres to $\mathbb{CP}^1 \times \mathbb{CP}^1$ to allow for poles of solutions. We do this by letting $x = c_n, y = b_n$ and introducing $X = 1/x, Y = 1/y$ so $\mathbb{CP}^1 \times \mathbb{CP}^1$ is covered by the four charts $(x, y), (x, Y), (X, y), (X, Y)$. Next we resolve all points of indeterminancy of the system (2), namely the points in charts at which the rational functions giving x', y' (or the equivalent in other charts) are indeterminate in the sense that their numerator and denominator simultaneously vanish. One introduces a suitable birational transformation and new coordinate charts and continues this process until all singularities are resolved in all charts. This amounts to lifting the system under blowups of certain points in the fibres; more information and explicit examples of this procedure can be found in [4, Sections 2.6.1, 2.6.2]. When introducing new coordinate charts, we use the convention from [3]: after blowing up a point p_i with coordinates $x = a$ and $y = b$ we introduce a pair of coordinate charts (u_i, v_i) and (U_i, V_i) according to $x = a + u_i = a + U_iV_i$ and $y = b + u_iv_i = b + V_i$, so the exceptional line replacing the point is given by $u_i = 0$ or $V_i = 0$.

With this notation, the points of indeterminancy of the system (2) form three cascades of infinitely near points. The first cascade is $p_1 : (x, y) = (0, 0)$ and $p_2 : (u_1, v_1) = (0, -n)$. The second cascade is $p_3 : (x, y) = (0, s)$ and $p_4 : (u_3, v_3) = (0, -n - \alpha)$. The third cascade is $p_5 : (X, Y) = (0, 0)$, $p_6 : (u_5, v_5) = (0, 0)$, $p_7 : (u_6, v_6) = (0, -1)$ and $p_8 : (u_7, v_7) = (0, 1+\alpha+2n)$. The sequence of points for the differential system (2) is the same as for the discrete system (1) as in [5] and we arrive at the same surfaces forming the spaces of initial conditions for these two systems, which are of type $D_6^{(1)}$ in Sakai's classification [9].

From the geometric theory of Painlevé equations, since we have resolved the singularities of the system through eight blowups we know that there should be a unique biquadratic curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ passing through these eight points, which will give the pole divisor of the symplectic form defining the Hamiltonian structure. This is given in coordinates by $x^2Y^2 = 0$ and the symplectic form can be chosen up to a constant multiple to be

$$\frac{dx \wedge dy}{x^2} = -dX \wedge dy = -\frac{dx \wedge dY}{x^2Y^2} = \frac{dX \wedge dY}{Y^2}.$$

This means we can take $G = 1/c_n^2$, which leads to the following partial differential equations for the Hamiltonian function K in (18):

$$\begin{aligned}\frac{\partial K}{\partial b_n} &= \frac{s}{c_n^2}\left(2b_n + (2n + 1 + \alpha + c_n)c_n - s\right), \\ \frac{\partial K}{\partial c_n} &= -\frac{s}{c_n^3}\left(2(b_n^2 - sb_n) + (2n + \alpha + 1)b_nc_n - nsc_n\right).\end{aligned}$$

These can be solved by elementary methods to arrive at the Hamiltonian K in Theorem 1.

2.2 Symplectic transformations and gauge normalisation of the space of initial conditions

Given a system related by a birational transformation to the standard form of one of the Painlevé equations, one can always recover its Hamiltonian structure from the space of initial conditions with some additional considerations. Like in the example above, this does not rely on knowing the transformation explicitly but merely that it exists.

We begin with a slight generalisation of a well-known fact about symplectic transformations used to define global non-autonomous Hamiltonian structures for the Painlevé equations on Okamoto's spaces of initial conditions [10]. Consider a birational transformation of complex variables

$$(q, p, t) \mapsto (x(q, p, t), y(q, p, t), t)\,, \tag{19}$$

where the $\mathbb{C}^3$-coordinate neighbourhoods are equipped with 2-forms

$$\eta = G(x, y, t) dx \wedge dy, \qquad \omega = F(q, p, t) dq \wedge dp, \tag{20}$$

in which F and G are rational in (x, y) and (q, p) respectively and locally analytic in t. Suppose that the transformation (19) is symplectic in the sense that

$$G(x, y, t)\delta x \wedge \delta y = F(q, p, t)\delta q \wedge \delta p, \tag{21}$$

where δ indicates the exterior derivative on the $\mathbb{C}^2$-fibre over t, so t is treated as a constant in the calculation. The following is proved by direct calculation.

Lemma 3. *Given a symplectic transformation* (19) *as above, if there exist functions* $H(q, p, t)$*,* $K(x, y, t)$ *such that*

$$G(x, y, t)dx \wedge dy + dK \wedge dt = F(q, p, t)dq \wedge dp + dH \wedge dt, \tag{22}$$

then the system of differential equations

$$F\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad F\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{23}$$

is transformed under (19) *to*

$$G\frac{dx}{dt} = \frac{\partial K}{\partial y}, \qquad G\frac{dy}{dt} = -\frac{\partial K}{\partial x}. \tag{24}$$

If we have a birational transformation relating a given system in (x, y, t)-coordinates to the standard Hamiltonian form of one of the Painlevé equations, then the symplectic form with respect to which the system should be Hamiltonian should be the pullback of the one from the standard form, so we can define the function G via

$$\delta q \wedge \delta p = G(x, y, t)\delta x \wedge \delta y. \tag{25}$$

If we have obtained a space of initial conditions for the given system through eight blowups of $\mathbb{CP}^1 \times \mathbb{CP}^1$, then this symplectic form is determined up to a constant multiple by the unique biquadratic curve in (x, y)-coordinates passing through the eight points, as in the example above. If more than eight blowups are required to regularise the system, then some curves will need to be blown down in order to arrive at a generalised Halphen surface [9].

With the symplectic form in (x, y)-coordinates in hand, we also have the function H from the standard form, so it remains to find $K(x, y, t)$ satisfying the relation (22). This is not possible for all birational transformations, but sufficient conditions are provided by the following.

Lemma 4. *Given a birational transformation* $(q, p, t) \mapsto (x(q, p, t), y(q, p, t), t)$*, define* $G(x, y, t)$ *according to* (25)*. Then there exists a function* $K(x, y, t)$ *such that*

$$G(x, y, t)dx \wedge dy + dK \wedge dt = dq \wedge dp + dH \wedge dt, \tag{26}$$

if either of the following conditions hold:

$$(1) \quad \frac{\partial x}{\partial t} = \frac{\partial y}{\partial t} = 0, \qquad (2) \quad \frac{\partial G}{\partial t} = 0,$$

Proof. For case (1), where the transformation is not t-dependent, the Hamiltonian K is determined up to functions of t as the result of applying the transformation to H, but in case (2) there will be a correction between the Hamiltonians. To obtain the second condition, note that the equality of 2-forms (26) gives two partial differential equations for $K(x, y, t)$, the compatibility of which can be confirmed by using condition (25) as well as its derivatives with respect to q, p and t. □

We remark that if neither of these conditions hold, we cannot find a Hamiltonian form for the original system in the variables (x, y). However, if we make use of the gauge freedom in the construction of the surfaces forming the space of initial conditions we can normalise the symplectic form to be independent of t. We apply Möbius transformations $x \mapsto \tilde{x}$, $y \mapsto \tilde{y}$, to each of the $\mathbb{CP}^1$-factors such that the function $\widetilde{G}(\tilde{x}, \tilde{y}, t)$ defined by $\delta q \wedge \delta p = \widetilde{G}(\tilde{x}, \tilde{y}, t) \delta\tilde{x} \wedge \delta\tilde{y}$ satisfies the second condition from Lemma 4. This is always possible for the surfaces providing spaces of initial conditions for the Painlevé equations and corresponds to choosing coordinates such that the biquadratic curve giving the anticanonical divisor does not move with t, see [9, 4].

In conclusion, we have demonstrated how Hamiltonian structures can be obtained for differential systems transformable to the Painlevé equations. This is achieved through the construction of a space of initial conditions, which will always lead to a Hamiltonian form for the system either in the original variables or after applying appropriate blowdowns in the case when more than eight blowups were required to resolve the indeterminacies of the system as well as gauge transformations to normalise the symplectic form coming from the anticanonical divisor.

Acknowledgements

AS was partially supported by a University College London Graduate Research Scholarship and Overseas Research Scholarship.

References

[1] Y. Chen, A. Its, *Painlevé III and a singular linear statistics in Hermitian random matrix ensembles, I*, J. Approx. Th. 162 (2010), 270–297.

[2] Y. Chen, G. Filipuk, M. N. Rebocho, *Nonlinear difference equations for a modified Laguerre weight: Laguerre-Freud equations and asymptotics*, Jaen J. Approx. 11 (1-2) (2019), 47–65.

[3] A. Dzhamay, G. Filipuk, A. Stokes, *Recurrence coefficients for discrete orthogonal polynomials with hypergeometric weight and discrete Painlevé equations*, J. Phys. A 53 (2020), no. 49, 495201–495230.

[4] K. Kajiwara, M. Noumi, Y. Yamada, *Geometric aspects of Painlevé equations*, J. Phys. A 50 (2017), no. 7, 073001, 164 pp.

[5] X. Li, A. Dzhamay, D. Zhang, *The MacDonald hierarchy and discrete Painlevé equations* (working title), in preparation.

[6] K. Okamoto, *Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé*, (French) [On foliations associated with second-order Painlevé equations with fixed critical points], Japan. J. Math. (N.S.) 5 (1979), no. 1, 1–79.

[7] K. Okamoto, *Polynomial Hamiltonians associated with Painlevé equations. I*, Proc. Japan Acad. Ser. A Math. Sci. 56 (1980), no. 6, 264–268.

[8] K. Okamoto, *Polynomial Hamiltonians associated with Painlevé equations. II. Differential equations satisfied by polynomial Hamiltonians*, Proc. Japan Acad. Ser. A Math. Sci. 56 (1980), no. 8, 367–371.

[9] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. 220 (2001), no. 1, 165–229.

[10] T. Shioda and K. Takano, *On some Hamiltonian structures of Painlevé systems, I*, Funkcial. Ekval. 40 (1997), no. 2, 271–291.

[11] W. Van Assche, *Orthogonal Polynomials and the Painlevé Equations*, Australian Mathematical Society Lecture Series, 27, Cambridge University Press, Cambridge, 2018.