Families of complete non-compact Spin(7) holonomy manifolds

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Abstract

We consider complete non-compact Spin(7)-manifolds which are either asymptotically locally conical (ALC) or asymptotically conical (AC). The thesis consists of two parts. In the first part we develop the deformation theory of AC Spin(7)-manifolds. We show that the moduli space of torsion-free AC Spin(7)-structures on a given 8-manifold M asymptotic to a fixed Spin(7)-cone is an orbifold for generic decay rates in the non-L^2 regime. Furthermore, we derive a formula for the dimension of the moduli space, which has contributions from the topology of M and from solutions of a first order PDE system on the link of the asymptotic cone.

In the second part we prove existence results of cohomogeneity one Spin(7) holonomy metrics for which a generic orbit is isomorphic to the Aloff–Wallach space $N(1,-1) \cong \mathrm{SU}(3)/\mathrm{U}(1)$. The unique non-trivial rank 3 vector bundle over the 5-sphere and the universal quotient bundle of $\mathbb{C}P^2$ each carry a 1-parameter family (up to scale) of such metrics. We show that these families share a common behaviour: a generic member of these families belongs to one of two open intervals, of which one consists of ALC Spin(7) holonomy metrics and the other one of incomplete metrics. These two intervals are separated by a distinguished parameter which gives rise to an AC Spin(7) holonomy metric. Another interesting phenomenon occurs at the other endpoint of the open interval of ALC metrics, where the family collapses to the Bryant–Salamon AC G_2 holonomy metric on $\Lambda^2_-\mathbb{C}P^2$. Notable is the existence of the two AC spaces. These are the first examples of smooth AC Spin(7) holonomy manifolds known to exist since Bryant–Salamon's original example on $\mathbf{S}_+(S^4)$ in 1989. Furthermore, they provide a Spin(7) analogue of the well-known conifold transition in the setting of Calabi–Yau 3-folds.

Impact Statement

This thesis addresses the study of non-compact Riemannian manifolds with holonomy group Spin(7), which is one of the exceptional holonomy groups on Berger's list. The geometric significance of Spin(7) holonomy metrics lies in the fact that they are Ricci flat. We prove the existence of new examples and explain their deformations. These new non-compact examples potentially can be used as building blocks in the construction of new compact examples.

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Contents

Chapter 1

Introduction

The theme of this thesis are complete non-compact Spin(7)-manifolds. Spin(7) and G_2 are the two exceptional holonomy groups in Berger's classification of holonomy groups of Riemannian manifolds. Their significance lies in the fact that exceptional holonomy metrics are Ricci-flat, but not Kähler. The first example of a metric with holonomy Spin(7) was given by Bryant in 1987 [Bry87]. Two years later Bryant–Salamon found the first complete example [BS89]. An important geometric aspect of complete non-compact Spin(7)-manifolds is the asymptotic behaviour. In this thesis we encounter two asymptotic types: *asymptotically conical* (AC) manifolds and *asymptotically locally conical* (ALC) manifolds. The complete example given by Bryant–Salamon, which lives on the bundle $\mathbf{S}_+(S^4)$ of positive spinors on the 4-sphere, is an AC manifold: at infinity the geometry converges to a cone with holonomy contained in Spin(7). In this sense it is a Spin(7)-analogue of the famous Eguchi–Hanson hyperkähler metric on T^*S^2. ALC manifolds at infinity locally look like the product of a cone and a circle of fixed size. These are analogues of *asymptotically locally flat* (ALF) hyperkähler 4-manifolds such as the well-known Taub-NUT metric on $\mathbb{C}^2$.

Deformation Theory of AC Spin(7)-manifolds

Thematically this thesis consists of two parts. In the first part we develop the deformation theory of AC Spin(7)-manifolds. An important feature of the asymptotic geometry of AC Spin(7)-manifolds is the rate of convergence to the asymptotic cone. In the setting of AC manifolds, we are interested in polynomial decay, i.e. where the Spin(7)-structure and the associated metric decay to the Spin(7)-cone like r^ν, $\nu < 0$, where r is the radial function of the cone. ν is called the *rate*. Tensors which decay with rate $\nu < -4$ are square-integrable. The main result of this part of the thesis is that the moduli space is an orbifold in the non-L^2 regime ($-4 < \nu < 0$), and we derive a formula for its dimension.

Theorem A. *Let (M, ψ) be an AC Spin(7)-manifold of rate ν. For generic $\nu \in (-4, 0)$ the moduli space $\mathcal{M}_\nu$ of torsion-free AC Spin(7)-structures on M asymptotic to the same Spin(7)-cone at the same rate modulo an appropriate notion of equivalence is an orbifold. The dimension of $\mathcal{M}_\nu$ is determined by topological data of M, and solutions to systems of differential equations on the link of the asymptotic cone.*

For a more precise formulation see Theorem 3.2.27. The deformation theory of compact Spin(7)-manifolds has been developed by Joyce [Joy00]. The moduli space is always smooth and infinitesimal deformations can be expressed in terms of harmonic forms. Several standard techniques in the compact setting do not carry over to the non-compact setting. For example, we frequently are in situations where integration by parts is not available. Furthermore, the mapping properties of differential operators behave rather differently in the non-compact setting as compared to the compact setting. Nordström [Nor08] developed the deformation theory of exponentially asymptotically cylindrical (EAC) Spin(7)-manifolds using analysis on non-compact manifolds. In contrast to the AC setting, in the EAC setting all decay rates lie in the L^2-regime because of the exponential decay, and the dimension of the moduli space only depends on topological data. Our work is most closely related to the deformation theory of G_2-conifolds developed by Karigiannis–Lotay [KL20]. In our set-up we consider the moduli space $\mathcal{M}_\nu$ of AC manifolds of a particular rate ν. To prove smoothness of an orbifold chart, our strategy is to compute infinitesimal deformations and then use the inverse function theorem adapted to appropriate Banach spaces. The computation of the infinitesimal deformations is carried out in several steps. Analogously to the use of Hodge theory in the compact setting yet more intricate, deformations which lie in L^2 can be related to the topology of the manifold M. As we vary the rate and enter the non-L^2 regime, we use the analysis on weighted Sobolev and Hölder spaces developed by Lockhart–McOwen [LM85]. Outside a discrete subset of so-called *critical* rates the relevant differential operators are Fredholm and the space of deformations remains constant. As we cross a critical rate the added deformations can be related to particular solutions of differential equations on the asymptotic cone. In the range of rates considered by us we can formulate these equations purely on the compact link of the cone. If $\nu < -4$, the above program cannot be carried out: the operator describing the infinitesimal deformations may not be surjective and hence the inverse function theorem cannot be applied. Therefore, deformations of AC Spin(7)-metrics with $\nu < -4$ may be obstructed. This resembles the G_2-setting [KL20].

In our moduli problem we only consider torsion-free Spin(7)-structures which are asymptotic to a fixed Spin(7)-cone. Denoting the link of the cone by Σ, the space of torsion-free, conical Spin(7)-structures on $(0, \infty) \times \Sigma$ corresponds to the space of nearly parallel G_2-structures on Σ. We do not expect this space to have a "nice" structure in general, e.g.

that of a manifold. Alexandrov–Semmelmann [AS12] showed that the homogeneous, nearly parallel G_2-structure on the Aloff–Wallach space $N(1,1)$ has an 8-dimensional space of infinitesimal deformations, but by a recent result of Dwivedi–Singhal [DS20] these are all obstructed. This picture is similar to the case of G_2-cones, where Foscolo [Fos16] showed that deformations of nearly Kähler manifolds in general are obstructed. This aspect is another difference to the EAC case [Nor08]. The link of a G_2-cylinder is a compact Calabi–Yau manifold, and the link of a Spin(7)-cylinder is a compact G_2-manifold. Deformations of these are well understood, which allows a more inclusive set-up for the moduli space.

That the moduli space is in general an orbifold rather than a manifold is owed to the fact that the stabiliser of a torsion-free AC Spin(7)-structure in the group of diffeomorphisms decaying to the identity can be non-trivial. While we can exclude any continuous such symmetries, the stabiliser can still be a non-trivial finite group. If this group does not act trivially on a slice for the diffeomorphism action, we only obtain an orbifold rather than a manifold chart. We present one criterion to check if the stabiliser acts trivially on the orbifold chart. The tangent space of the orbifold chart at a torsion-free AC Spin(7)-structure ψ of rate ν is related to closed 4-forms which are anti-self-dual with respect to ψ and decay with rate ν. If the projection of these forms to the fourth cohomology group of M is injective, we can conclude that each element in the orbifold chart represents a different point in the moduli space, i.e. that the quotient by the stabiliser is trivial.

There are several similar articles concerned with the deformation theory of calibrated submanifolds: Marshall [Mar02] and Pacini [Pac04] studied deformations of AC Special Lagrangian submanifolds in $\mathbb{C}^m$ and Lotay studied deformations of AC coassociative and associative submanifolds of G_2-manifolds, cf. [Lot09] and [Lot11], respectively.

As an application of Theorem A we show that the Bryant–Salamon Spin(7) holonomy metric on $\mathbf{S}_+(S^4)$ is locally rigid, modulo scaling, as a torsion-free AC Spin(7)-structure on $\mathbf{S}_+(S^4)$ asymptotic to the same Spin(7)-cone. We solve the differential equations on the link by following Alexandrov–Semmelmann [AS12], who compute infinitesimal Einstein deformations of normal homogeneous nearly parallel G_2-manifolds with standard invariant metrics. Under these constraints the differential equations can be solved with purely representation theoretic methods. This example strongly relies on the condition that the homogeneous metric on the link is normal and standard. In the second part of the thesis we prove the existence of new AC Spin(7)-manifolds asymptotic to more sophisticated Spin(7)-cones, which does not allow us to carry out similar computations.

An important subclass of Spin(7)-manifolds are Calabi–Yau 4-folds. For a given AC Calabi–Yau 4-fold we can use Theorem A to obtain deformations as a Spin(7)-structure. Given that a large number of AC Calabi–Yau 4-folds are known while only very few AC Spin(7) holonomy metrics are known to exist, this leads to the interesting question: can an AC Calabi–Yau metric on a manifold of real dimension 8 be deformed to an AC metric with holonomy Spin(7)? It is known that infinitesimal deformations of Calabi–Yau structures on compact manifolds are unobstructed (see [Got04, Tia87, Tod89]). By adjusting for example the approach in [Got04] to the analytic framework of AC manifolds, it is reasonable to believe that infinitesimal deformations of AC Calabi–Yau structures are unobstructed in an interesting range of decay rates. Therefore, if each infinitesimal Spin(7)-deformation is induced by an infinitesimal SU(4)-deformation, this would be strong evidence to provide a negative answer to the above question. A priori infinitesimal Spin(7)-deformations of AC Calabi–Yau 4-folds are rather complicated. We consider the above question for a simpler subclass: here we can solve the algebraic system characterising the leading order term of an SU(4)-deformation which induces at leading order term a given infinitesimal Spin(7)-deformation of rate $\nu \in (-4, 0)$. However, we have not been able to show that this algebraic deformation also solves the differential equation characterising infinitesimal deformations of torsion-free SU(4)-structures.

We do not consider the analogous question for AC hyperkähler manifolds of real dimension 8 because it is believed that apart from $T^*\mathbb{C}P^2$ there are no other resolutions of 8-dimensional hyperkähler cones [Fu06, Section 2.3]. Furthermore, Namikawa [Nam08] has shown that a hyperkähler cone has a smoothing if and only if it has a resolution. Thus we expect that $T^*\mathbb{C}P^2$ is the only 8-manifold supporting an AC hyperkähler metric. Dancer–Swann [DS97] have shown that apart from the Calabi metric on $T^*\mathbb{C}P^n$ there are no other cohomogeneity one hyperkähler metrics in real dimension greater than four.

Existence of cohomogeneity Spin(7)-manifolds

In the second part of the thesis we prove the existence of new examples of complete Spin(7) holonomy manifolds with AC and ALC asymptotics. After Bryant–Salamon's pioneering work in the late 1980s, in the 1990s most work in the area concentrated on compact manifolds [Joy00]. The search for metrics with exceptional holonomy has a different flavour in the non-compact setting compared to the compact setting. A Bochner type argument shows that every Killing field on a compact Ricci-flat manifold is parallel. Therefore, compact irreducible manifolds with exceptional holonomy do not admit any continuous symmetries. All known constructions in the compact setting rely on perturbative techniques starting from some degenerate limit. In contrast, symmetry reduction methods are a powerful approach in the non-compact setting. As all homogeneous Ricci-flat metrics are flat, the strongest reduction are symmetries with cohomogeneity one, i.e. where generic orbits have codimension one. The condition that a Spin(7)-structure is torsion-free reduces from a non-linear PDE system to a non-linear ODE system. The examples of an incomplete and complete Spin(7) holonomy metric by Bryant and Bryant–Salamon, respectively, both have a cohomogeneity one symmetry.

The condition that the holonomy of a Riemannian cone is contained in Spin(7) is equivalent to the condition that the metric on the link is induced by a nearly parallel G_2-structure. In the Bryant–Salamon example the link of the asymptotic cone is the "squashed" 7-sphere. In the early 2000s the study of cohomogeneity one Spin(7)-manifolds gained fresh impetus by the work of Cvetič–Gibbons–Lü–Pope. In [CGLP02b] they look for further examples with generic orbit S^7. On each of $\mathbb{R}^8$ and $\mathbf{S}_+(S^4)$ they find an explicit ALC Spin(7) holonomy metric, which they call the $\mathbb{A}_8$ and $\mathbb{B}_8$ metric, respectively. These were the first examples of ALC Spin(7) holonomy metrics. In [CGLP02a] they consider a more general ansatz. Based on numerics, they suggest that the $\mathbb{B}_8$ metric is part of a family of Spin(7) holonomy metrics in a neighbourhood of $S^4 \subset \mathbf{S}_+(S^4)$, which depends up to scale on one parameter q and exhibits a behaviour as sketched in diagram 1.1.

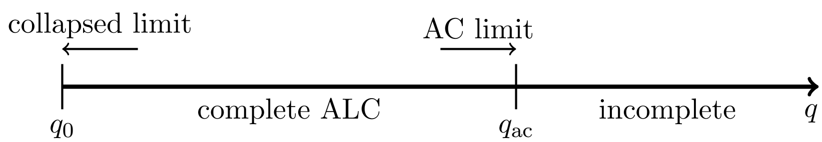


Figure 1.1: Typical behaviour of 1-parameter family of local cohomogeneity one torsion-free Spin(7)-structures.

In the interior of an interval (q_0, q_{ac}) the torsion-free Spin(7)-structures are complete and ALC. At the explicit value q_{ac} the asymptotic geometry transitions from ALC to AC, the so-called *AC limit*. This AC Spin(7)-manifold is the classical example by Bryant–Salamon. At the other endpoint of the interval the asymptotic circle of the ALC manifolds shrinks and disappears in the limit. The family converges in the Gromov–Hausdorff topology to the Bryant–Salamon AC G_2 holonomy metric on $\Lambda^2_-(S^4)$. This is called the *collapsed limit*. They predict a similar 1-parameter family on $K_{\mathbb{C}P^3}$, which they call the $\mathbb{C}_8$ family. Here the AC manifold, which appears at the AC limit, is asymptotic to the cone over a finite quotient of S^7 equipped with the round metric. It has first been discovered by Calabi [Cal79] and its holonomy is SU(4) rather than Spin(7). The existence of the $\mathbb{B}_8$ and $\mathbb{C}_8$ families has been established by Bazaikin (cf. [Baz07, Baz08]).

Cohomogeneity one cones with holonomy Spin(7), or equivalently homogeneous nearly parallel G_2-manifolds with one Killing spinor, have been classified in [FKMS97]. Apart from the "squashed" S^7 and the isotropy irreducible space SO(5)/SO(3), all other examples live on the Aloff–Wallach spaces. If (k, l) is a pair of integers which are not both zero, U(1) can be embedded via $e^{i\theta} \mapsto \mathrm{diag}(e^{ik\theta}, e^{il\theta}, e^{-i(k+l)\theta})$ into the maximal torus of diagonal matrices in SU(3). Denote this subgroup by $\mathrm{U}(1)_{k,l}$. The Aloff–Wallach space $N(k, l)$ is the quotient $\mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$. Each Aloff–Wallach space carries a homogeneous nearly parallel G_2-structure.

Bazaikin [Baz08] considered cohomogeneity one Spin(7)-structures with generic orbit

$N(1,1)$ or one of the related Aloff–Wallach spaces $N(1,-2)$ and $N(-2,1)$. He again finds 1-parameter families which behave as sketched in Figure 1.1. If the generic orbit is $N(1,1)$, the resulting space is the orbifold $T^*\mathbb{C}P^2/\mathbb{Z}_2$. The AC limit is obtained by replacing S^4 with the non-spin manifold $\mathbb{C}P^2$ in Bryant–Salamon's construction of an AC Spin(7) holonomy metric on the bundle of positive spinors on S^4. If the generic orbit is $N(1,-2)$, the underlying space is the canonical bundle of the flag manifold $F_3 = \mathrm{SU}(3)/\mathrm{U}(1)^2$. Similarly to the $\mathbb{C}_8$ family, the AC limit has been constructed earlier by Calabi and has holonomy SU(4).

In this thesis we consider cohomogeneity one Spin(7) holonomy metrics with generic orbit isomorphic to $N(1,-1)$. Our motivation stems from the collapsed limit sketched in Figure 1.1. In [BS89] Bryant–Salamon constructed an AC G_2 holonomy metric on $\Lambda^2_-\mathbb{C}P^2$ which is asymptotic to the cone over the homogeneous nearly Kähler structure on the flag manifold $F_3 = \mathrm{SU}(3)/\mathrm{U}(1)^2$. Because $N(1,-1)$ is a circle bundle over F_3 and $(0,\infty)\times N(1,-1)$ can topologically be completed by adding a S^5, which is a circle bundle over $\mathbb{C}P^2$, it is plausible that there exist torsion-free ALC Spin(7)-structures which collapse to the Bryant–Salamon metric on $\Lambda^2_-\mathbb{C}P^2$. Very close to the collapsed limit, this has recently been proved independently by Foscolo [Fos19] with PDE methods.

We now turn to the formulation of our two main results of this part of the thesis.

Theorem B. *Denote the adjoint bundle of the principal* $\mathrm{SU}(2)$*-bundle* $\mathrm{SU}(3)\to\mathrm{SU}(3)/\mathrm{SU}(2)\cong S^5$ *by* M_{S^5}. $\mathrm{SU}(3)\times\mathrm{U}(1)$ *acts on* M_{S^5} *with cohomogeneity one, and the generic orbit is the Aloff–Wallach space* $N(1,-1)$.

There exists a 1-parameter family (up to scale) Ψ_μ, $\mu\in(0,\infty)$, *of* $\mathrm{SU}(3)\times\mathrm{U}(1)$*-invariant Spin(7) holonomy metrics in a neighbourhood of* S^5 *in* M_{S^5} *and a distinguished parameter* $\mu_{\mathrm{ac}}>0$ *such that*

- Ψ_μ *is complete on* M_{S^5} *and asymptotically locally conical (ALC) if* $\mu\in(0,\mu_{\mathrm{ac}})$,
- Ψ_μ *is complete on* M_{S^5} *and asymptotically conical (AC) if* $\mu=\mu_{\mathrm{ac}}$,
- Ψ_μ *is incomplete if* $\mu\in(\mu_{\mathrm{ac}},\infty)$.

$\Psi_{\mu_{\mathrm{ac}}}$ *is asymptotic to the Spin(7)-cone over the unique* $\mathrm{SU}(3)\times\mathrm{U}(1)$*-invariant nearly parallel* G_2*-structure on* $N(1,-1)$.

Theorem C. *Denote the universal quotient bundle of* $\mathbb{C}P^2$ *by* $M_{\mathbb{C}P^2}$*: the fibre of* $M_{\mathbb{C}P^2}$ *at* $l\in\mathbb{C}P^2$, *which corresponds to a 1-dimensional linear subspace of* $\mathbb{C}^3$, *is the quotient* $\mathbb{C}^3/l$. $\mathrm{SU}(3)\times\mathrm{U}(1)$ *acts on* $M_{\mathbb{C}P^2}$ *with cohomogeneity one, and the generic orbit is the Aloff–Wallach space* $N(1,0)$, *which is* $\mathrm{SU}(3)$*-equivariantly diffeomorphic to* $N(1,-1)$.

There exists a 1-parameter family (up to scale) Υ_τ, $\tau\in\mathbb{R}$, *of* $\mathrm{SU}(3)\times\mathrm{U}(1)$*-invariant Spin(7) holonomy metrics in a neighbourhood of* $\mathbb{C}P^2$ *in* $M_{\mathbb{C}P^2}$ *and a distinguished parameter* $\tau_{\mathrm{ac}}\in\mathbb{R}$ *such that*

- *Υ_τ is complete on $M_{\mathbb{C}P^2}$ and asymptotically locally conical (ALC) if $\tau \in (-\infty, \tau_{\mathrm{ac}})$,*
- *Υ_τ is complete on $M_{\mathbb{C}P^2}$ and asymptotically conical (AC) if $\tau = \tau_{\mathrm{ac}}$,*
- *Υ_τ is incomplete if $\tau \in (\tau_{\mathrm{ac}}, \infty)$.*

$\Upsilon_{\tau_{\mathrm{ac}}}$ is asymptotic to the Spin(7)-cone over the unique $\mathrm{SU}(3) \times \mathrm{U}(1)$-invariant nearly parallel G_2-structure on $N(1, -1)$. For all τ the zero section $\mathbb{C}P^2 \subset M_{\mathbb{C}P^2}$ is a Cayley submanifold with respect to Υ_τ and the cohomology class of Υ_τ is non-trivial.

Again the 1-parameter families Ψ_μ and Υ_τ exhibit a behaviour as illustrated in Figure 1.1. Ψ_{ac} and Υ_{ac} are the first complete AC Spin(7) holonomy metrics proven to exist since Bryant–Salamon's original example on $\mathbf{S}_+S^4$. Theorems B and C were previously conjectured by Cvetič–Gibbons–Lü–Pope [CGLP02a] and Gukov–Sparks–Tong [GST03]. The 1-parameter families Ψ_μ and Υ_τ have first been constructed by Reidegeld [Rei10] in a neighbourhood of the exceptional orbits S^5 and $\mathbb{C}P^2$, respectively. Whether they give rise to complete metrics was left open. Similar to the $\mathbb{B}_8$-family, for a specific parameter $\tau^* < \tau_{\mathrm{ac}}$ the ALC Spin(7) holonomy metric Υ_{τ^*} has previously been described explicitly by Cvetič–Gibbons–Lü–Pope [CGLP02a], Gukov–Sparks [GS02] and Kanno–Yasui [KY02]. To the knowledge of the author, there is no explicit expression known of Ψ_μ for any μ.

There is a qualitative difference in the collapsed limit of the two families Ψ_μ and Υ_τ. Because S^5 is a circle bundle over $\mathbb{C}P^2$, M_{S^5} globally has the structure of a circle bundle over $\Lambda^2_-\mathbb{C}P^2$. As $\mu \to 0$, the collapse occurs with bounded curvature similar to the well-known collapse of Berger's sphere. In contrast, the manifold $M_{\mathbb{C}P^2}$ has the structure of a circle bundle only outside the zero section. The fixed locus of the circle action is precisely the zero section. Therefore, the curvature blows up on $\mathbb{C}P^2$ as $\tau \to -\infty$. Because Foscolo's analytic method [Fos19] only applies for collapse with bounded curvature, the result about the existence of a continuous family of complete ALC Spin(7) holonomy metrics on $M_{\mathbb{C}P^2}$ is new.

The AC Spin(7) holonomy manifolds $(M_{S^5}, \Psi_{\mu_{\mathrm{ac}}})$ and $(M_{\mathbb{C}P^2}, \Upsilon_{\tau_{\mathrm{ac}}})$ resemble the well-known conifold transition of Calabi–Yau 3-folds: the *smoothing* T^*S^3 of the conifold $\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 |\ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}$ and its *small resolution* $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$ are topologically different spaces which carry cohomogeneity one AC Calabi–Yau metrics asymptotic to the same Calabi–Yau cone, the conifold. In analogy to the Calabi–Yau setting, the Spin(7) conifold transition has been conjectured by Gukov–Sparks–Tong [GST03]. The collapsed limit also has a lower dimensional analogue. The $\mathbb{D}_7$ family of cohomogeneity one G_2 holonomy metrics on $\mathbf{S}(S^3) = S^3 \times \mathbb{R}^4$ collapses with bounded curvature to the AC Calabi–Yau metric on the small resolution of the conifold, while the curvature blows up on the zero section S^3 as the $\mathbb{B}_7$ family of cohomogeneity one G_2 holonomy metrics on $\mathbf{S}(S^3) = S^3 \times \mathbb{R}^4$ collapses to the AC Calabi–Yau metric on the smoothing of the conifold.

Again Gukov–Sparks–Tong conjecture that the families Ψ_μ and Υ_τ provide an analogue of this phenomenon.

Reidegeld [Rei08, Lemma 5.4.2] showed that a non-compact cohomogeneity one space with principal orbit isomorphic to $N(1,-1)$ can also be topologically completed by adding as a singular orbit either the flag manifold F_3 or $\mathrm{SU}(3)/\mathrm{SO}(3)$, the space of linear special Lagrangian subspaces of $\mathbb{C}^3$. However, there can be no torsion-free Spin(7)-structures on the resulting spaces which are left invariant by all of $\mathrm{SU}(3)\times\mathrm{U}(1)$ [Rei08, pp. 177-178, pp. 192-pp. 197].

The AC limits have another interpretation. In the 1-parameter families described above a natural scaling of the Ricci-flat manifolds has been fixed. In geometric terms this scaling can be understood as fixing the size of the singular orbit. Then as the parameter increases the asymptotic circle length of the complete ALC solutions increases until it approaches infinity at the AC limit. If, however, we rescale to keep the asymptotic circle length fixed, the size of the singular orbit shrinks as the parameter increases and approaches zero at the limit. This suggests that the limit should be a *conically singular* (CS) ALC space. Indeed, we prove the existance of an $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant CS ALC Spin(7)-metric with principal orbit $N(1,-1)$.

Theorem D. *There exists a 1-parameter family $\Psi^{\mathrm{cs}}_\lambda$, $\lambda\in\mathbb{R}$, of $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant Spin(7) holonomy metrics on $(0,\varepsilon(\lambda))_t\times N(1,-1)$, with $\varepsilon(\lambda)>0$ for every $\lambda\in\mathbb{R}$, which are conically singular (CS) as $t\to 0$ asymptotic to the Spin(7)-cone over the unique $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant nearly parallel G_2-structure on $N(1,-1)$. The 1-parameter family has the following properties:*

- *If $\lambda<0$, then $\Psi^{\mathrm{cs}}_\lambda$ extends to $(0,\infty)_t\times N(1,-1)$, is forward complete and asymptotically locally conical (ALC) as $t\to\infty$.*
- *If $\lambda=0$, then $\Psi^{\mathrm{cs}}_\lambda$ is the Spin(7)-cone over the unique $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant nearly parallel G_2-structure on $N(1,-1)$.*
- *If $\lambda>0$, then $\Psi^{\mathrm{cs}}_\lambda$ does not extend to a forward complete metric.*

For a fixed sign of λ the Spin(7)-structures $\Psi^{\mathrm{cs}}_\lambda$ are related by scaling.

The proof of Theorem D is significantly easier than the proof of Theorems B and C because the phase transition between ALC metrics and incomplete metrics happens at the Spin(7)-cone itself rather than at a new AC Spin(7)-manifold. Previously, other examples of CS Spin(7) metrics were given by the $\mathbb{A}_8$ metric on $\mathbb{R}^8$ and variations of its construction by replacing the S^7 with an arbitrary 3-Sasakian 7-manifold.

Chapter 2

Preliminaries

2.1 Spin(7)-geometry

We give a brief review of Spin(7)-geometry. For more details we refer to [Sal89], [Joy00] and [Nor08]. We first discuss the linear algebraic picture. The spin representation of Spin(7) has a real form which can be identified with $\mathbb{R}^8$. Under this action Spin(7) can be characterised as the stabiliser in $\mathrm{GL}(8, \mathbb{R})$ of the 4-form

$$\begin{aligned}\psi_0 =& dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} \\ & - dx_{2358} - dx_{2367} - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678},\end{aligned}$$

where $(x_1, \ldots, x_8)$ are coordinates on $\mathbb{R}^8$. The action of Spin(7) on $\mathbb{R}^8$ induces an action on the exterior algebra. We get the following decomposition into irreducible components:

$$\Lambda^2(\mathbb{R}^8)^* = \Lambda^2_7 \oplus \Lambda^2_{21}, \quad \Lambda^3(\mathbb{R}^8)^* = \Lambda^3_8 \oplus \Lambda^3_{48}, \quad \Lambda^4(\mathbb{R}^8)^* = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}.$$

Here Λ^k_q denotes a q-dimensional irreducible subspace. For higher degree forms we get an analogous decomposition by applying the Hodge star operator. Under the identification $\Lambda^2(\mathbb{R}^8)^* = \mathfrak{so}(8, \mathbb{R})$ the component Λ^2_{21} corresponds to the Lie algebra of Spin(7). $\mathrm{GL}(8, \mathbb{R})$ acts on $\Lambda^4(\mathbb{R}^8)^*$ by pulling back ψ_0. The derivative at the identity gives a map $\mathfrak{gl}(8, \mathbb{R}) \to \Lambda^4(\mathbb{R}^8)^*$. Under the decomposition

$$\mathfrak{gl}(8, \mathbb{R}) = \Lambda^2(\mathbb{R}^8)^* \oplus S^2(\mathbb{R}^8)^* = \Lambda^2_7 \oplus \Lambda^2_{21} \oplus \mathbb{R}\,\mathrm{Id} \oplus S^2_0(\mathbb{R}^8)^*,$$

where $S^2_0(\mathbb{R}^8)^*$ denote the trace-less symmetric bilinear forms on $\mathbb{R}^8$, the kernel corresponds to $\mathrm{Lie}(\mathrm{Spin}(7)) = \Lambda^2_{21}$, and Λ^4_1, Λ^4_7 and Λ^4_{35} are the images of $\mathbb{R}$, Λ^2_7 and $S^2_0(\mathbb{R}^8)^*$, respectively. In particular, the orbit of ψ_0 under the action of $\mathrm{GL}(8, \mathbb{R})$ has co-dimension 27 and its tangent space at ψ_0 is given by

$$T_{\psi_0}(\mathrm{GL}(8,\mathbb{R})\cdot\psi_0) = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}. \tag{2.1.1}$$

Λ^4_{27} can be identified with the normal directions at ψ_0.

We have the identity $*^2 = \mathrm{Id}$ for the Hodge star operator acting on 4-forms. The induced decomposition in spaces of self-dual and anti-self-dual 4-forms is given by

$$\Lambda^4_+ = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}, \quad \Lambda^4_- = \Lambda^4_{35}.$$

A further space which will be important to us is Λ^3_8, which has the description

$$\Lambda^3_8 = \{X \lrcorner \psi_0 \mid X \in \mathbb{R}^8\}. \tag{2.1.2}$$

The above discussion implies that as Spin(7) representations we have

$$\Lambda^3_8 \cong \mathbb{R}^8, \quad \Lambda^4_7 \cong \Lambda^2_7.$$

Because Spin(7) is simply connected, the inclusion Spin(7) $\subset$ SO(8) factors through Spin(8). If we denote the real positive and negative spin representations of Spin(8) by σ_8^+ and σ_8^-, respectively, as representations of Spin(7) there are isomorphisms

$$\sigma_8^+ \cong \Lambda^4_1 \oplus \Lambda^4_7 \cong \mathbb{R} \oplus \Lambda^2_7, \quad \sigma_8^- \cong \Lambda^3_8 \cong \mathbb{R}^8. \tag{2.1.3}$$

Now we turn to the global differential geometric picture. Let M be an oriented 8-manifold. We say that a 4-form ψ is *admissible* if at each point $p \in M$ there is an orientation-preserving isomorphism between T_pM and $\mathbb{R}^8$ which identifies $\psi|_p$ with ψ_0. We refer to ψ as a *Spin(7)-structure*. ψ reduces the structure group of the frame bundle of M to Spin(7) by considering the subbundle $\{u\colon \mathbb{R}^8 \xrightarrow{\sim} T_pM \mid u^*\psi_p = \psi_0\}$. Because Spin(7) is a subgroup of SO(8), ψ induces in a purely algebraic way a Riemannian metric g. The condition that the holonomy group of the induced metric is contained in Spin(7) is equivalent to

$$d\psi = 0. \tag{2.1.4}$$

In this case we say that ψ is *torsion-free* and (M,ψ) a *Spin(7)-manifold*. A key point is that g is Ricci-flat if ψ is torsion-free.

The above decomposition of the exterior algebra into irreducible components gives a global decomposition of the corresponding vector bundles. By abuse of notation, we will denote these subbundles by the same symbols as in the linear picture. This decomposition is preserved by the Hodge Laplacian if the Spin(7)-structure is torsion-free. We denote the space of all smooth admissible 4-forms on M by $\mathcal{A}(M)$. Just as the orbit of ψ_0 is a non-linear subspace of $\Lambda^4(\mathbb{R}^8)^*$, the space $\mathcal{A}(M)$ is a non-linear subspace of $\Omega^4(M)$. Therefore, the condition (2.1.4) is non-linear. (2.1.1) gives

$$T_\psi \mathcal{A}(M) = \Gamma(\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}). \tag{2.1.5}$$

An 8-manifold equipped with a Spin(7)-structure is spin because of the inclusion Spin(7) $\subset$ Spin(8). If we denote the real positive and negative spin bundle by $\mathbf{S}_+$ and $\mathbf{S}_-$, respectively, by (2.1.3) there are isomorphisms of vector bundles

$$\mathbf{S}_+ \cong \Lambda^4_1 \oplus \Lambda^4_7, \quad \mathbf{S}_- \cong \Lambda^3_8 \cong TM \cong T^*M. \tag{2.1.6}$$

In particular, if the Spin(7)-structure is torsion-free, the Dirac Laplacian can be identified with the Hodge Laplacian on the respective bundles. The identifications 2.1.6 can be chosen such that the positive and negative Dirac operator correspond to

$$\not{D}_+ : \Gamma(\Lambda^4_{1\oplus 7}) \to \Gamma(\Lambda^3_8), \quad \gamma \mapsto \pi_8(d^*\gamma), \tag{2.1.7a}$$

$$\not{D}_- : \Gamma(\Lambda^3_8) \to \Gamma(\Lambda^4_{1\oplus 7}), \quad \gamma \mapsto \pi_{1\oplus 7}(d\gamma). \tag{2.1.7b}$$

To describe the moduli space of torsion-free Spin(7)-structures on a manifold, we need to describe in a systematic way other admissible 4-forms close to a reference Spin(7)-structure ψ. As this can be done fibre-wise, we first return to the local picture in $\mathbb{R}^8$. The decomposition (2.1.1) in tangent and normal directions implies that the derivative of the map

$$(\mathrm{GL}(8,\mathbb{R}) \cdot \psi_0) \times \Lambda^4_{27} \to \Lambda^4, \quad (\psi, \zeta) \mapsto \psi + \zeta,$$

at $(\psi_0, 0)$ is an isomorphism. By the inverse function theorem every 4-form sufficiently close to ψ_0 can be written in a unique way as the sum of an element in $(\mathrm{GL}(8,\mathbb{R}) \cdot \psi_0)$ and Λ^4_{27}. In particular, if $\varepsilon > 0$ is chosen sufficiently small, we can apply this decomposition to $\psi + \eta$, where η is an element of $B_\varepsilon(\Lambda^4_{35}; 0)$, the ball of radius ε centred at 0 in Λ^4_{35}. We can write this decomposition as

$$\psi_0 + \eta = \Pi(\eta) + \Theta(\eta), \tag{2.1.8}$$

with unique smooth maps

$$\Pi : B_\varepsilon(\Lambda^4_{35}; 0) \to (\mathrm{GL}(8,\mathbb{R}) \cdot \psi_0), \quad \Theta : B_\varepsilon(\Lambda^4_{35}; 0) \to \Lambda^4_{27}. \tag{2.1.9}$$

By the uniqueness we have $\Pi(0) = \psi_0$ and $\Theta(0) = 0$. Differentiating the path in $(\mathrm{GL}(8,\mathbb{R})\cdot\psi_0)$ given by $\Pi(t\eta)$ gives

$$\frac{d}{dt}\bigg|_{t=0} \Pi(t\eta) = \eta - \frac{d}{dt}\bigg|_{t=0} \Theta(t\eta)$$

Taking the type decomposition with respect to ψ_0, with (2.1.1) we see that the derivatives of Π and Θ at 0 are given by

$$D\Pi|_0 = \mathrm{Id}, \quad D\Theta|_0 = 0.$$

$A \in \mathrm{Spin}(7)$ preserves the size and type of η and thus we have

$$A^*\Pi(\eta) + A^*\Theta(\eta) = \psi_0 + A^*\eta = \Pi(A^*\eta) + \Theta(A^*\eta).$$

By the uniqueness of the decomposition we see that the maps Π and Θ are Spin(7)-equivariant.

To sum up, the maps (2.1.9) defined by the decomposition (2.1.8) have the properties:

(i) $\Pi(0) = \psi_0$ and $\Theta(0) = 0$,

(ii) $D\Pi|_0 = \mathrm{Id}$ and $D\Theta|_0 = 0$,

(iii) Π and θ are Spin(7)-equivariant.

Back to the global picture, on a Spin(7)-manifold (M, ψ) we can piece the fibre-wise maps together to define such maps in a ε-neighbourhood of the zero section in Λ^4_{35}. The fibre-wise norms are taken with respect to the inner product induced by ψ.

Next we introduce other special geometric structures and discuss how Spin(7)-structures relate to them.

Definition 2.1.10. An $\mathrm{SU}(n)$-structure on a Riemannian manifold (M, g) of real dimension $2n$ with complex structure J and corresponding Kähler form $\omega(X,Y) = g(JX,Y)$ carries a complex volume form $\theta \in \Gamma(\Lambda^{n,0}M)$ such that the pair (ω, θ) satisfies the constraint equations (see [Got04, Definition 4.2.1])

$$\omega \wedge \theta = 0, \quad \frac{1}{n!}\omega^n = c_n\, \theta \wedge \bar{\theta}, \tag{2.1.11}$$

for some constant c_n depending on n. The $\mathrm{SU}(n)$-structure (ω, θ) is torsion-free if and only if

$$d\omega = 0, \quad d\theta = 0. \tag{2.1.12}$$

In this case the holonomy group reduces to $\mathrm{SU}(n)$ and (M, J, ω, θ) is called a *Calabi–Yau* manifold.

Definition 2.1.13. G_2 is the subgroup of $\mathrm{GL}(7,\mathbb{R})$ which preserves the 3-form

$$\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}.$$

Here $(x_1,\dots,x_7)$ are coordinates on $\mathbb{R}^7$ and we denote $dx_i \wedge dx_j \cdots \wedge dx_l$ by $dx_{ij\dots l}$. Now let M be an oriented 7-manifold. We say that a 3-form φ is *positive* if at each point $p \in M$ there exists an orientation-preserving linear isomorphism between T_pM and $\mathbb{R}^7$ such that $\varphi|_p$ is identified with φ_0. We refer to φ as a *G_2-structure.* φ reduces the structure group of the frame bundle of M to G_2 by considering the subbundle $\{u\colon \mathbb{R}^7 \xrightarrow{\sim} T_pM \mid u^*\varphi_p = \varphi_0\}$. G_2 is a subgroup of $\mathrm{SO}(7)$. In particular, φ induces in a purely algebraic way a Riemannian metric g. The Hodge star operator induced by g gives the Hodge dual 4-form $*\varphi$. The condition that the holonomy group of g is contained in G_2 is equivalent to $d\varphi = 0$ and $d * \varphi = 0$. We call such a G_2-structure *torsion-free.* This is a non-linear condition as the Hodge star operator depends in a non-linear way on φ.

Remark 2.1.14. The relation between the above geometric structures and Spin(7) is given by the inclusions

$$\mathrm{SU}(3) \subset \mathrm{G}_2 \subset \mathrm{Spin}(7),$$
$$\mathrm{SU}(4) \subset \mathrm{Spin}(7).$$

The first set of inclusions can be seen as follows. If (M,θ,ω,h) is a 6-dimensional manifold equipped with an SU(3)-structure, then we obtain a G_2-structure on $M \times \mathbb{R}$ by

$$\varphi = dt \wedge \omega + \mathrm{Re}\ \theta, \quad *\varphi = \frac{1}{2}\omega^2 - dt \wedge \mathrm{Im}\ \theta, \quad g = dt^2 + h. \tag{2.1.15}$$

If $(M,\varphi,*\varphi,h)$ is a 7-dimensional manifold equipped with a G_2-structure, then we obtain a Spin(7)-structure on $M \times \mathbb{R}$ by

$$\psi = dt \wedge \varphi + *\varphi, \quad g = dt^2 + h. \tag{2.1.16}$$

An SU(4)-structure (ω,θ) on an 8-dimensional manifold M induces the Spin(7)-structure

$$\psi = \frac{1}{2}\omega^2 + \mathrm{Re}\,\theta. \tag{2.1.17}$$

2.2 Asymptotic types of non-compact Spin(7)-manifolds

Let Σ be a 7-manifold equipped with a complete Riemannian metric g_Σ which is induced by the G_2-structure $\varphi_\Sigma \in \Omega^3_+(\Sigma)$. Then the conical metric

$$g_C = dr^2 + r^2 g_\Sigma$$

on $C(\Sigma) = (0,\infty) \times \Sigma$ is induced by the Spin(7)-structure

$$\psi_C = r^3 dr \wedge \varphi_\Sigma + r^4 *_\Sigma \varphi_\Sigma.$$

$(C(\Sigma), \psi_C)$ is said to be a Spin(7)-cone if ψ_C is torsion-free. The exterior derivative is given by

$$d\psi_C = -r^3 dr \wedge d\varphi_\Sigma + 4r^3 dr \wedge *_\Sigma \varphi_\Sigma + r^4 d *_\Sigma \varphi_\Sigma.$$

Hence, the condition $d\psi_C = 0$ is equivalent to

$$d\varphi_\Sigma = 4 *_\Sigma \varphi_\Sigma. \tag{2.2.1}$$

This means that the G_2-structure on Σ is *nearly parallel*. Nearly parallel G_2-manifolds are Einstein manifolds with positive scalar curvature. In particular, Σ must be compact. If the link is the 7-sphere with the round metric, then the cone is the euclidean $\mathbb{R}^8$ with the standard Spin(7)-structure. Apart from its quotients, this is the only Spin(7)-cone with trivial holonomy. All other Spin(7)-cones need to have holonomy group Sp(2), SU(4) or Spin(7). If the holonomy equals Sp(2), the link is a *3-Sasakian* manifold, and if it equals SU(4), then the link must be a 7-dimensional *Sasaki-Einstein* manifold. If the cone has full holonomy Spin(7), we say the nearly parallel G_2-structure is *proper*.

We are interested in Spin(7)-manifolds which are asymptotic to a Spin(7)-cone with a polynomial decay rate. By the Cheeger–Gromoll splitting theorem irreducible non-compact Spin(7)-manifolds can have only one end. Therefore, we assume that Σ is connected.

Definition 2.2.2. Let $C := (C(\Sigma), \psi_C, g_C)$ be the Spin(7)-cone over the nearly parallel G_2-manifold $(\Sigma, \varphi_\Sigma, g_\Sigma)$. A Spin(7)-manifold (M, ψ, g) is an *asymptotically conical* (AC) Spin(7)-manifold asymptotic to C with rate $\nu \in (-\infty, 0)$ if there exist a compact subset $K \subset M$, $R > 0$ and a diffeomorphism

$$F\colon (R,\infty) \times \Sigma \subset C(\Sigma) \to M - K$$

such that we have the decay

$$|\nabla_C^j(F^*\psi - \psi_C)|_{g_C} = \mathcal{O}(r^{\nu-j}) \quad \text{for all } j \in \mathbb{N}_0.$$

In particular, this implies

$$|\nabla_C^j(F^*g - g_C)|_{g_C} = \mathcal{O}(r^{\nu-j}) \quad \text{for all } j \in \mathbb{N}_0.$$

Definition 2.2.3. For a fixed F as above we fix a radial function ρ and a cut-off function χ for the remainder of this paper.

- On the compact piece K set $\rho \equiv 1$, on $F((R+1,\infty)\times\Sigma)$ set $\rho \equiv r$, and in the intermediate region interpolate smoothly in an increasing fashion. In particular, $\rho \geq 1$ everywhere.
- $\chi \colon C(\Sigma) \to [0,1]$ is a cut-off function supported on $(R,\infty)\times\Sigma$ and $\chi \equiv 1$ on $(R+1,\infty)\times\Sigma$.

This will allow us to introduce weighted function spaces. Furthermore, if γ is a differential form on $C(\Sigma)$ we can transplant it to M via $(F^{-1})^*(\chi\gamma)$. By abuse of notation we will suppress F in the rest of the paper and just write $\chi\gamma$ for the corresponding form on M.

Remark 2.2.4. Suppose that the AC Spin(7)-manifold (M,ψ,g) is asymptotic to the Euclidean Spin(7)-structure ψ_0 on $\mathbb{R}^8$. Fix an arbitrary point $p \in M$. Denote the distance from p by $r = \operatorname{dist}_g(\cdot,p)$ and the volume of a ball of radius r in 8-dimensional Euclidean space by $v(r)$. The asymptotic behaviour of the metric implies that the function

$$r \mapsto \frac{\operatorname{Vol} B(p,r)}{v(r)}$$

converges to 1 as $r \to \infty$. However, by the Bishop–Gromov volume comparison theorem this function is non-increasing and converges to 1 as $r \to 0$. This shows that every ball of radius r in M has the same volume as a corresponding ball in Euclidean space. This implies that (M,ψ,g) is isometric to $(\mathbb{R}^8,\psi_0,g_0)$. Therefore, we do not need to consider AC Spin(7)-manifolds asymptotic to Euclidean space, and exclude this case from all of our statements.

Definition 2.2.5. Let $C := (C(\Sigma),\psi_C,g_C)$ be the Spin(7)-cone over the nearly parallel G_2-manifold $(\Sigma,\varphi_\Sigma,g_\Sigma)$. A Spin(7)-manifold (M,ψ,g) has an isolated conical singularity asymptotic to C with rate $\nu \in (0,\infty)$ if there exists an open subset $U \subset M$ and a diffeomorphism

$$F : (0,\varepsilon)\times\Sigma \subset C(\Sigma) \to U$$

for some $\varepsilon > 0$ such that

$$|\nabla^j_C(F^*\psi - \psi_C)|_{g_C} = \mathcal{O}(r^{\nu-j}) \quad \text{for all } j \in \mathbb{N}_0 \text{ as } r \to 0.$$

In particular this implies

$$|\nabla^j_C(F^*g - g_C)|_{g_C} = \mathcal{O}(r^{\nu-j}) \quad \text{for all } j \in \mathbb{N}_0 \text{ as } r \to 0.$$

We say that (M, ψ, g) is a *conically singular* (CS) Spin(7)-manifold.

Analogous to Spin(7)-cones in dimension 8, we have G_2-cones in dimension 7. Circle bundles over G_2-cones provide another asymptotic model for non-compact Spin(7)-manifolds. To explain this in detail, suppose (Σ, h) is 6-dimensional Riemannian manifold equipped with an SU(3)-structure (Ω, ω) which induces h. Then the cone over Σ carries the G_2-structure

$$\varphi_C = r^2 dr \wedge \omega + r^3 \mathrm{Re}\ \Omega, \quad *\varphi_C = \frac{1}{2} r^4 \omega^2 - r^3 dt \wedge \mathrm{Im}\ \Omega,$$

which induces the cone metric g_C. $(C(\Sigma), \varphi_C, g_C)$ is said to be a *G_2-cone* if the G_2-structure is torsion-free. The exterior derivatives are

$$\begin{aligned} d\varphi_C &= -r^2 dr \wedge d\omega + 3r^2 dr \wedge \mathrm{Re}\ \Omega + r^3 d\,\mathrm{Re}\ \Omega, \\ d * \varphi_C &= 2r^3 dr \wedge \omega^2 + r^4 d\omega \wedge \omega + r^3 dr \wedge d\,\mathrm{Im}\ \Omega. \end{aligned}$$

Hence the condition $d\varphi_C = d * \varphi_C = 0$ is equivalent to

$$d\omega = 3\mathrm{Re}\ \Omega, \quad d\,\mathrm{Im}\ \Omega = -2\omega^2. \tag{2.2.6}$$

This means precisely that the SU(3)-structure on Σ is *nearly Kähler*. Because nearly Kähler manifolds are Einstein manifolds with positive scalar curvature, Σ has to be compact if it is complete.

Definition 2.2.7. Let $(C(\Sigma), \varphi_C, g_C)$ be a G_2-cone over the nearly Kähler manifold $(\Sigma, \Omega, \omega, h)$, ℓ a positive constant and $p : P \to C(\Sigma)$ a U(1)-principal bundle with a connection $\theta \in \Omega^1(P)$ which gives rise to a Spin(7)-structure on P via $\psi_P = \ell\theta \wedge \varphi_C + *\varphi_C$ with associated metric $g_P = g_C + \ell^2\theta^2$. A Spin(7)-manifold (M, ψ, g) is said to be an *asymptotically locally conical* (ALC) Spin(7)-manifold asymptotic to (P, ψ_P, g_P) with rate $\nu \in (-\infty, 0)$ and asymptotic circle length ℓ if there exists a compact subset $K \subset M$ and (possibly for a double cover of $M - K$) a diffeomorphism

$$F : p^{-1}((R, \infty) \times \Sigma) \subset P \to M - K$$

for some $R > 0$ such that

$$|\nabla_P^j (F^*\psi - \psi_P)|_{g_P} = \mathcal{O}(r^{\nu-j}) \quad \text{for all } j \in \mathbb{N}_0 \text{ as } r \to \infty.$$

This implies

$$|\nabla_P^j (F^*g - g_P)|_{g_P} = \mathcal{O}(r^{\nu-j}) \quad \text{for all } j \in \mathbb{N}_0 \text{ as } r \to \infty.$$

2.3 Analysis on AC Spin(7)-manifolds

Deformations of Spin(7)-structures have first been studied by Joyce on compact manifolds (see [Joy00, Section 10.7]). It relies on analysis and Hodge theory on compact manifolds. To study deformations of other geometries it is essential to have an analytic framework adapted to the situation. E.g. Nordström studied deformations of G_2- and Spin(7)-structures on EAC (exponentially asymptotically cylindrical) manifolds [Nor08]. We will need analysis on conifolds. In this section we collect the necessary analytical background. It is our aim to make this as concise as possible. For references for the statements in this section and a more detailed account of weighted analysis on conical and cylindrical spaces and its applications to geometry we refer the reader to [KL20,Nor08,Mar02,Pac13] and [FHN17, Appendix B]. The underlying theory was outlined by Lockhart–McOwen [LM85].

In the following V and W will be a subbundle of $\Lambda^\bullet T^*M$. Via the identification (2.1.6), $\mathbf{S}_\pm(M)$, the positive and negative spinor bundle on M, also fit into the discussion of this section. The metric g on M and its Levi-Civita connection induce a metric and a metric connection on V and W. By V_C, W_C we denote the corresponding vector bundles on the cone.

We set

$$\mathcal{C}^\infty_\lambda(V) = \{\gamma \in \mathcal{C}^\infty(V) \,|\, |\nabla^j \gamma| = \mathcal{O}(r^{\lambda-j}) \text{ for all } j \geq 0\}.$$

Next we define appropriate Banach spaces of sections of such bundles.

Definition 2.3.1. Let $p \geq 1$, $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$. For any $\gamma \in \mathcal{C}^\infty_0(V)$ the quantity

$$\|\gamma\|_{L^p_{k,\lambda}} = \left(\sum_{j=0}^k \int_M |\rho^{-\lambda+j} \nabla^j \gamma|^p \rho^{-8} \mathrm{vol}_g \right)^{\frac{1}{p}}$$

is well defined and a norm. Here ρ is the radial function from Definition 2.2.3. We define the weighted Sobolev space $L^p_{k,\lambda}(V)$ to be the completion of $\mathcal{C}^\infty_0(V)$ with respect to this norm. $L^2_{k,\lambda}(V)$ is a Hilbert space with the inner product

$$\langle \gamma, \xi \rangle_{L^2_{k,\lambda}} = \sum_{j=0}^{k} \int_M \langle \rho^{-\lambda+j} \nabla^j \gamma, \rho^{-\lambda+j} \nabla^j \xi \rangle \rho^{-8} \mathrm{vol}_g.$$

Remark 2.3.2. (i) Note that $L^2_{0,-4}(V) = L^2(V)$. We refer to rates $\nu < -4$ as the L^2-*setting* and to rates $\nu > -4$ as the *non-L^2 setting*.

(ii) From the definition it follows that $\rho^\mu L^2_{0,\lambda}(V) = L^2_{0,\lambda+\mu}(V)$. In particular, we have $L^2_{0,\lambda}(V) = \rho^{-4-\lambda} L^2$.

(iii) Set $\Omega^k_{l,\lambda} := L^2_{l,\lambda}(\Lambda^k)$, $\Omega^{\mathrm{even}}_{l,\lambda} := L^2_{l,\lambda}(\Lambda^{\mathrm{even}})$ and $\Omega^{\mathrm{odd}}_{l,\lambda} := L^2_{l,\lambda}(\Lambda^{\mathrm{odd}})$.

Definition 2.3.3. Let $p \geq 1$, $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$. For any $\gamma \in \mathcal{C}^\infty_0(V)$ the quantity

$$\|\gamma\|_{\mathcal{C}^{k,\alpha}_\lambda} = \sum_{j=0}^{k} \|\rho^{-\lambda+j} \nabla^j \gamma\|_{\mathcal{C}^0} + [\rho^{-\lambda+k} \nabla^k \gamma]_\alpha$$

is well defined and a norm. Here $[\,\cdot\,]_\alpha$ is the Hölder seminorm. We define the weighted Hölder space $\mathcal{C}^{k,\alpha}_\lambda(V)$ to be the closure of $\mathcal{C}^\infty_0(V)$ with respect to this norm.

Theorem 2.3.4. *[Mar02, Theorem 4.17]*

(i) If $l \geq m + \alpha + 4$, then there is a continuous embedding $L^2_{l,\lambda}(V) \hookrightarrow \mathcal{C}^{m,\alpha}_\lambda(V)$.

(ii) If $\lambda < \lambda'$ and $l > 0$, there is a compact embedding $L^2_{l,\lambda} \hookrightarrow L^2_{0,\lambda'}$.

In the following denote by $*_M$ the Hodge star operator on M induced by g, and by $*_C$ the Hodge star operator on the asymptotic cone $C(\Sigma)$ induced by the conical metric g_C. d^*_M and d^*_C denote the co-differential on M and $C(\Sigma)$, respectively. In our calculations it is useful to know that comparing the Hodge star operator on M and the asymptotic cone gives an additional decay of ν, the AC rate.

Lemma 2.3.5. *Suppose $\gamma \in \Omega^k_{l,\lambda}$. Then*

- $(*_M - *_C)\gamma \in \Omega^{8-k}_{l,\lambda+\nu}$.
- $(d + d^*_M)\gamma - (d + d^*_C)\gamma \in \Omega^{k-1}_{l-1,\lambda+\nu-1} \oplus \Omega^{k+1}_{l-1,\lambda+\nu-1}$.

Suppose $\gamma \in \mathcal{C}^\infty_\lambda(\Lambda^k)$. Then

- $(*_M - *_C)\gamma \in \mathcal{C}^\infty_{\lambda+\nu}(\Lambda^{8-k})$.
- $(d + d^*_M)\gamma - (d + d^*_C)\gamma \in \mathcal{C}^\infty_{\lambda+\nu-1}(\Lambda^{k-1} \oplus \Lambda^{k+1})$.

Proposition 2.3.6. *Suppose $\eta \in L^2_{0,\lambda}(V)$ and $\omega \in L^2_{0,\mu}(V)$. If $\lambda + \mu \leq -8$, then the L^2-pairing*

$$\langle \eta, \omega \rangle_{L^2} = \int_M \langle \eta, \omega \rangle \mathrm{Vol}$$

is finite and satisfies the inequality

$$\langle \eta, \omega \rangle_{L^2} \leq \|\eta\|_{L^2_{0,\lambda}} \|\omega\|_{L^2_{0,\mu}}$$

Proof. Using the Cauchy-Schwarz inequality both for the pointwise inner product and the L^2-version, $\lambda + \mu \leq -8$ and $\rho \geq 1$, we get

$$\begin{aligned}
\langle \eta, \omega \rangle_{L^2} &= \int_M \langle \eta, \omega \rangle \mathrm{Vol} \\
&\leq \int_M |\eta||\omega| \mathrm{Vol} \\
&= \int_M (|\rho^{-\lambda}\eta|\rho^{-4})(|\rho^{-\mu}\omega|\rho^{-4})\rho^{8+\lambda+\eta} \mathrm{Vol} \\
&\leq \int_M (|\rho^{-\lambda}\eta|\rho^{-4})(|\rho^{-\mu}\omega|\rho^{-4}) \mathrm{Vol} \\
&\leq \left(\int_M |\rho^{-\lambda}\eta|^2 \rho^{-8} \mathrm{Vol} \right)^{1/2} \left(\int_M |\rho^{-\mu}\omega|^2 \rho^{-8} \mathrm{Vol} \right)^{1/2} \\
&= \|\eta\|_{L^2_{0,\lambda}} \|\omega\|_{L^2_{0,\mu}} < \infty.
\end{aligned}$$

□

Proposition 2.3.7. *We have* $(L^2_{0,\lambda}(V))^* \cong L^2_{0,-8-\lambda}(V)$.

Proof. Proposition 2.3.6 gives a pairing

$$\langle \cdot, \cdot \rangle_{L^2} \colon L^2_{0,\lambda}(V) \times L^2_{0,-8-\lambda}(V) \to \mathbb{R}.$$

This defines a continuous linear map

$$L^2_{0,-8-\lambda}(V) \to (L^2_{0,\lambda}(V))^*, \quad \omega \mapsto \langle \cdot, \omega \rangle_{L^2}.$$

Under the Hilbert space isomorphism $(L^2_{0,\lambda}(V))^* \cong L^2_{0,\lambda}(V)$ this corresponds to the map

$$L^2_{0,-8-\lambda}(V) \to L^2_{0,\lambda}(V), \quad \omega \mapsto \rho^{8+2\lambda}\omega.$$

This clearly is an isomorphism. □

In the remainder of the section let

$$P : \Gamma(V) \to \Gamma(W)$$

be one of the elliptic differential operators

$$d + d^* : \Gamma(\Lambda^\bullet) \to \Gamma(\Lambda^\bullet), \tag{2.3.8}$$
$$\Delta : \Gamma(\Lambda^k) \to \Gamma(\Lambda^k), \tag{2.3.9}$$
$$\not{D}_+ : \Gamma(\mathbf{S}_+) \to \Gamma(\mathbf{S}_-), \tag{2.3.10}$$
$$\not{D}_- : \Gamma(\mathbf{S}_-) \to \Gamma(\mathbf{S}_+), \tag{2.3.11}$$

Denote by k the order of P and by P_C the corresponding differential operator

$$P_C : \Gamma(V_C) \to \Gamma(W_C)$$

on the cone. P is asymptotic to P_C in an appropriate sense and thus is an example of an *asymptotically conical operator* [Mar02, section 4.3.2]. In this case P extends to a bounded linear map on all weighted Sobolev and Hölder spaces [Mar02, Proposition 4.20]. By $P_{l+k,\lambda}$ we denote the induced operator

$$P_{l+k,\lambda} : L^2_{l+k,\lambda}(V) \to L^2_{l,\lambda-k}(W) \tag{2.3.12}$$

on weighted Sobolev spaces.

Denote by P^* the formal adjoint of P. (2.3.8) and (2.3.9) are formally self-adjoint and $\not{D}_+^* = \not{D}_-$. The next Lemma shows that integration by parts is available for sections of weighted Sobolev spaces if the decay rates are fast enough.

Lemma 2.3.13. *Let $\eta \in L^2_{k,\lambda}(V)$ and $\omega \in L^2_{k,\mu}(V)$. If $\lambda + \mu \leq -8 + k$, then the quantities $\langle P\eta, \omega\rangle_{L^2}$ and $\langle \eta, P^*\omega\rangle_{L^2}$ are finite and equal, i.e.*

$$\langle P\eta, \omega\rangle_{L^2} = \langle \eta, P^*\omega\rangle_{L^2}.$$

Proof. We first show that the two quantities are finite. If $\lambda + \mu \leq -8 + k$, by Proposition 2.3.6 and the continuity of $P_{k,\lambda}$ we have

$$\langle P\eta, \omega\rangle_{L^2} \leq \|P\eta\|_{L^2_{0,\lambda-k}} \|\omega\|_{L^2_{0,\mu}} \leq C\|\eta\|_{L^2_{k,\lambda}} \|\omega\|_{L^2_{k,\mu}} < \infty.$$

Finiteness of $\langle \eta, P^*\omega\rangle$ follows analogously. Hence by the dominated convergence theorem it is enough to prove the statement for compactly supported forms, for which it is true by the definition of the formal adjoint. □

Remark 2.3.14. Lemma 2.3.13 says that if we consider the operators

$$\begin{aligned} P_{k,\lambda} : L^2_{k,\lambda}(V) &\longrightarrow L^2_{0,\lambda-k}(W), \\ L^2_{0,-8-\lambda}(V) &\longleftarrow L^2_{k,-8-\lambda+k}(W) : P^*_{k,-8-\lambda+k}, \end{aligned}$$

then with respect to the pairings

$$L^2_{0,\lambda}(V) \times L^2_{0,-8-\lambda}(V) \to \mathbb{R}, \quad L^2_{0,\lambda-k}(W) \times L^2_{0,-8-\lambda+k}(W) \to \mathbb{R},$$

the operator $P^*_{k,-8-\lambda+k}$ is the "adjoint" of $P_{k,\lambda}$. Using the Fourier transform, one can define Sobolev spaces $L^2_{s,\mu}$ for any $s \in \mathbb{R}$. Then similarly to Proposition 2.3.7 we have $(L^2_{k,\lambda}(V))^* = L^2_{-k,-8-\lambda}(V)$ and $(L^2_{0,\lambda-k}(W))^* = L^2_{0,-8-\lambda+k}(W)$. Therefore, $P^*_{k,-8-\lambda+k}$ is the restriction of the full adjoint $(P_{k,\lambda})^* = P^*_{0,-8-\lambda+k}$ to the more regular subspace $L^2_{k,-8-\lambda+k}(W)$. In all cases in this paper sections in the kernel and cokernel are smooth by elliptic regularity. Therefore, we do not have to use the full adjoint.

In the examples (2.3.8)-(2.3.11) that we consider, both P and the asymptotic model P_C are elliptic. Thus P is an example of an *uniformly elliptic* operator [Mar02, Section 4.3.2]. This control at infinity allows to develop a Fredholm and regularity theory similar to the setting of compact manifolds.

Theorem 2.3.15. *[Mar02, Theorem 4.21] Suppose $\gamma \in L^1_{\mathrm{loc}}(V)$ is a weak solution of $P\gamma = \zeta$ with $\zeta \in L^1_{\mathrm{loc}}(W)$. If ζ lies in $L^2_{l,\lambda-k}(W)$ (respectively $\mathcal{C}^{l,\alpha}_{\lambda-k}(W)$), then γ lies in $L^2_{l+k,\lambda}(V)$ (respectively $\mathcal{C}^{l+k,\alpha}_{\lambda}(V)$), and is a strong solution. Furthermore, there exists a positive constant $C > 0$ such that γ satisfies the estimate*

$$\|\gamma\|_{L^2_{l+k,\lambda}} \leq C \left(\|P\gamma\|_{L^2_{l,\lambda-k}} + \|\gamma\|_{L^2_{0,\lambda}} \right), \tag{2.3.16}$$

respectively the estimate

$$\|\gamma\|_{\mathcal{C}^{l+k,\alpha}_{\lambda}} \leq C \left(\|P\gamma\|_{\mathcal{C}^{l,\alpha}_{\lambda-k}} + \|\gamma\|_{\mathcal{C}^{0,\alpha}_{\lambda}} \right). \tag{2.3.17}$$

Remark 2.3.18. Using Theorems 2.3.15 and 2.3.4 it follows immediately that any closed and coclosed form on (M, ψ) is smooth. Therefore we can denote the kernel of

$$P_{l+k,\lambda} : L^2_{l+k,\lambda}(V) \to L^2_{l,\lambda-k}(W)$$

by $\ker P_\lambda$.

We will also have to deal with less regular linear differential operators $L : \Gamma(V) \to \Gamma(W)$ for which the coefficients only lie in some Hölder space $\mathcal{C}^{l,\alpha}$, and the convergence to the asymptotic model is with respect to the $\mathcal{C}^{l,\alpha}_{\lambda}$-norm. The above elliptic regularity result generalises to this setting.

Theorem 2.3.19. *[Nor08, Theorem 4.2.22] Let $L : \Gamma(V) \to \Gamma(W)$ be a linear elliptic differential operator of rank k with $\mathcal{C}^{l,\alpha}$-regular coefficients which is $\mathcal{C}^{l,\alpha}_\lambda$-asymptotic to the conical, elliptic differential operator L_C. If $u \in \mathcal{C}^{k,\alpha}_{\lambda+k}$ and $Lu \in \mathcal{C}^{l,\alpha}_\lambda$, then $u \in \mathcal{C}^{k+l,\alpha}_{\lambda+k}$ and*

$$\|u\|_{\mathcal{C}^{l+k,\alpha}_{\lambda+k}} \leq C\left(\|Lu\|_{\mathcal{C}^{l,\alpha}_\lambda} + \|u\|_{\mathcal{C}^{0,\alpha}_{\lambda+k}}\right).$$

The key in understanding the mapping properties of $P_{l+k,\lambda}$ is to study P_C acting on homogeneous sections.

Definition 2.3.20. $\gamma \in \Omega^k(C(\Sigma))$ is *homogeneous* of rate λ if there exist $\alpha \in \Omega^{k-1}(C(\Sigma))$ and $\beta \in \Omega^k(C(\Sigma))$ such that

$$\gamma = r^\lambda(r^{k-1}dr \wedge \alpha + r^k\beta).$$

$\gamma \in \Omega^\bullet(C(\Sigma))$ is said to be homogeneous of rate λ if each degree component is homogeneous of rate λ. In both cases this implies that $|\gamma|_{g_C}$ is a homogeneous function in r of rate λ.

Definition 2.3.21. $\lambda \in \mathbb{R}$ is a *critical rate* for P if there exists a non-zero homogeneous section γ of rate λ such that $P_C\gamma = 0$. Denote by

$$\mathcal{D}(P) = \{\lambda \in \mathbb{R} \,|\, \exists \gamma \in \mathcal{C}^\infty(V_C) \text{ non-zero and homogeneous of rate } \lambda \text{ such that } P_C\gamma = 0\}$$

the set of all critical rates. Set

$$\mathcal{K}_P(\lambda) = \{\gamma = \sum_{j=0}^m (\log r)^j \gamma_j \,|\, \text{each } \gamma_j \text{ is homogeneous of order } \lambda \text{ and } P_C\gamma = 0\}.$$

$\mathcal{K}_{\Lambda^k}(\lambda)$, $\mathcal{K}_{\text{even}}(\lambda)$, $\mathcal{K}_{\text{odd}}(\lambda)$ and $\mathcal{K}_{\text{ASD}}(\lambda)$ are defined by choosing P to be $d + d^*$ acting on k-forms, even degree forms, odd degree forms and anti-self dual 4-forms, respectively.

Remark 2.3.22. (i) In the general theory one also has to consider complex critical rates. However, all operator considered by us are formally self-adjoint, or restrictions thereof. In this case all critical rates need to be real.

(ii) An important property of the set $\mathcal{D}(P)$ is that it is discrete.

(iii) Suppose $\gamma = \sum_{j=0}^m (\log r)^j \gamma_j$, where each γ_j is homogeneous of order λ and $\gamma_m \neq 0$. As a polynomial in $\log r$, the leading order term of $P_C\gamma$ is $P_C\gamma_m$. Hence $\mathcal{K}_P(\lambda) = \{0\}$ if λ is not a critical rate.

Contrary to the compact setting, the elliptic estimate (2.3.16) cannot be used to prove that $P_{l+k,\lambda}$ is a Fredholm operator because $L^2_{l+k,\lambda}$ does not embed compactly into $L^2_{0,\lambda}$. However, if λ is a non-critical rate, the estimate can be strengthened to

Proposition 2.3.23. *[FHN17, Proposition B.9] If $\lambda' > \lambda$ and the interval $[\lambda, \lambda']$ does not contain a critical rate for P, then for any $\gamma \in L^2_{l+r,\lambda}(V)$ we have*

$$\|\gamma\|_{L^2_{l+k,\lambda}} \leq C\left(\|P\gamma\|_{L^2_{l,\lambda-k}} + \|\gamma\|_{L^2_{0,\lambda'}}\right). \tag{2.3.24}$$

The same statement holds for Hölder spaces, but we mainly use Sobolev spaces. In the improved estimate (2.3.24) the "error term" $\|\gamma\|_{L^2_{0,\lambda'}}$ is the norm of a space into which $L^2_{l+k,\lambda}$ embeds compactly by Theorem 2.3.4 (ii). This allows us to prove

Corollary 2.3.25. *Assume that λ is a non-critical rate for P. Then there exists a positive constant C such that for all $\gamma \in L^2_{l+k,\lambda}(V)$ which are orthogonal in $L^2_{l+k,\lambda}(V)$ to $\ker(P_{l+k,\lambda})$ we have*

$$\|\gamma\|_{L^2_{l+k,\lambda}} \leq C\|P\gamma\|_{L^2_{l,\lambda-k}}.$$

From Theorem 2.3.4 (ii), Proposition 2.3.23 and Corollary 2.3.25 we finally can clarify the mapping properties of $P_{l+k,\lambda}$.

Theorem 2.3.26. *[KL20, Theorems 4.11 and 4.13] The operator*

$$P_{l+k,\lambda} : L^2_{l+k,\lambda}(V) \to L^2_{l,\lambda-k}(W)$$

is Fredholm if and only if λ is not a critical rate. In this case we have:

(i) We can identify a complement of $P_{l+k,\lambda}(L^2_{l+k,\lambda}(V))$ in $L^2_{l,\lambda-k}(W)$ with $\ker P^*_{-8-\lambda+k}$, i.e. there exists a finite dimensional subspace U of $L^2_{l,\lambda-k}(W)$ such that

$$L^2_{l,\lambda-k}(W) = P_{l+k,\lambda}(L^2_{l+k,\lambda}(V)) \oplus U$$

and $U \cong \ker P^*_{-8-\lambda+k}$.

(ii) If $\lambda \geq -4+k$, then U is a subspace of $L^2_{l,\lambda-k}(W)$ and we can set $U = \ker P^*_{-8-\lambda+k}$.

For us it will be important to understand how $\ker P_\lambda$ changes as λ varies. The key to understanding how the kernel changes at a critical rate is the following

Theorem 2.3.27. *[KL20, Proposition 4.21] Let $\lambda_2 < \lambda_1$ be two non-critical rates of P and λ_0 the only critical rate in the interval (λ_2, λ_1). If $\gamma \in L^2_{l,\lambda_1}(V)$ and $P\gamma \in L^2_{l-k,\lambda_2-k}(W)$ (i.e. "$P\gamma$ decays faster than expected"), then there exist $\eta \in \mathcal{K}(\lambda_0)_{P_C}$ and $\tilde{\gamma} \in L^2_{l,\lambda_0+\nu}(V)$ which depend both linearly on γ, such that outside of a compact subset*

$$\gamma - \eta - \tilde{\gamma} \in L^2_{l,\lambda_2}(V).$$

A consequence of Theorem 2.3.27 we obtain the following two theorems.

Theorem 2.3.28. *[KL20, Theorem 4.20] For non-critical rates $\lambda_2 < \lambda_1$ of P the index change is given by*

$$\mathrm{ind}P_{\lambda_1} - \mathrm{ind}P_{\lambda_2} = \sum_{\lambda \in (\lambda_2, \lambda_1) \cap \mathcal{D}(P)} \dim \mathcal{K}_P(\lambda).$$

Theorem 2.3.29. *If the interval $[\lambda_1, \lambda_2]$ does not contain any critical rate, then* $\ker P_{\lambda_1} = \ker P_{\lambda_2}$.

2.4 Differential forms on AC Spin(7)-manifolds

2.4.1 Computations on the asymptotic Spin(7)-cone

In this paper it is important to have a good understanding of the Laplace-operator and $d + d^*$ acting on forms in weighted Sobolev spaces. In light of Theorem 2.3.26 we need to understand the critical rates of the corresponding operators acting on homogeneous sections on the cone. We first collect some explicit formulas. Let $\gamma = r^\lambda (r^{k-1} dr \wedge \alpha + r^k \beta)$ be a homogeneous k-form on $C(\Sigma)$, where $\alpha \in \Omega^{k-1}(\Sigma)$ and $\beta \in \Omega^k(\Sigma)$. The Hodge-star operator on the cone is given by

$$\begin{aligned} *_C(dr \wedge \alpha) &= r^{9-2k} *_\Sigma \alpha, \\ *_C \beta &= (-1)^k r^{7-2k} dr \wedge *_\Sigma \beta. \end{aligned}$$

Using this we get

$$d\gamma = r^{\lambda-1}\Big(r^k dr \wedge ((\lambda+k)\beta - d_\Sigma \alpha) + r^{k+1} d_\Sigma \beta\Big), \tag{2.4.1a}$$

$$*_C \gamma = r^{\lambda+8-k} *_\Sigma \alpha + (-1)^k r^{\lambda+7-k} dr \wedge *_\Sigma \beta, \tag{2.4.1b}$$

$$d_C^* \gamma = r^{\lambda+k-2}(-(\lambda+8-k)\alpha + d_\Sigma^* \beta) + r^{\lambda+k-3} dr \wedge (-d_\Sigma^* \alpha), \tag{2.4.1c}$$

$$\begin{aligned} \Delta_C \gamma = {} & r^{\lambda+k-3} dr \wedge \Big(\Delta_\Sigma \alpha - (\lambda+k-2)(\lambda-k+8)\alpha - 2 d_\Sigma^* \beta\Big) \\ & + r^{\lambda+k-2}\Big(\Delta_\Sigma \beta - (\lambda+k)(\lambda-k+6)\beta - 2 d_\Sigma \alpha\Big) \end{aligned} \tag{2.4.1d}$$

All of these formulas purely depend on the dimension of the cone. We make no use of the fact that our cone is a Spin(7)-cone.

Remark 2.4.2. For later use we give a brief characterisation of closed, homogeneous anti-self-dual 4-forms on the cone. Suppose

$$\gamma = r^{\lambda+3} dr \wedge \alpha + r^{\lambda+4} \beta$$

is a homogeneous 4-form of rate λ for $\alpha \in \Omega^3(\Sigma)$ and $\beta \in \Omega^4(\Sigma)$. Then γ is anti-self-dual if and only if $\beta = - *_\Sigma \alpha$, i.e.

$$\gamma = r^{\lambda+3} dr \wedge \alpha + r^{\lambda+4} (- *_\Sigma \alpha).$$

If $\lambda \neq -4$, γ is closed if and only if $d_\Sigma \alpha = -(\lambda+4) *_\Sigma \alpha$. Then we have in particular

$$\gamma = d\left(\frac{1}{\lambda+4} r^{\lambda+4} \alpha \right).$$

If $\lambda = -4$, then γ is closed if and only if α is harmonic on Σ.

Later on we need to know $\mathcal{K}_{\text{even}}(-4)$ and $\mathcal{K}_{\text{odd}}(-3)$. The calculation is analogous to [Kar09, Proposition 2.21] in the G_2-setting.

Lemma 2.4.3. *Let $\eta = \sum_{k=0}^4 \eta_{2k}$ be a closed and coclosed even degree form on $C(\Sigma)$ homogeneous of rate -4, i.e.*

$$\eta_{2k} = r^{2k-5} dr \wedge \alpha_{2k-1} + r^{2k-4} \beta_{2k},$$

where $\alpha_{2k-1} \in \Omega^{2k-1}(\Sigma)$ and $\beta_{2k} \in \Omega^{2k}(\Sigma)$. Then all components except α_3 and β_4 vanish, i.e.

$$\eta = r^{-1} dr \wedge \alpha_3 + \beta_4,$$

and α_3 and β_4 are both harmonic on Σ. In particular, η is of pure degree 4.

Proof. We have

$$d^* \eta_{2k} = r^{2k-7} dr \wedge (-d^* \alpha_{2k-1}) + r^{2k-6} (-(4-2k)\alpha_{2k-1} + d^* \beta_{2k})$$

and

$$d\eta_{2k-2} = r^{2k-7} dr \wedge (-d\alpha_{2k-3} + (2k-6)\beta_{2k-2}) + r^{2k-6} d\beta_{2k-2}.$$

Because of $d^* \eta_{2k} + d\eta_{2k-2} = 0$ we get

$$\begin{aligned} -d\alpha_{2k-3} + (2k-6)\beta_{2k-2} - d^* \alpha_{2k-1} &= 0, \\ d\beta_{2k-2} - (4-2k)\alpha_{2k-1} + d^* \beta_{2k} &= 0. \end{aligned}$$

This gives

$$(k-4)\beta_k = d\alpha_{k-1} + d^*\alpha_{k+1}, \quad k = 0,2,4,6,8, \tag{2.4.4}$$

$$(3-k)\alpha_k = d\beta_{k-1} + d^*\beta_{k+1}, \quad k = 1,3,5,7. \tag{2.4.5}$$

Applying d, d^* to (2.4.4) and (2.4.5) gives

$$\Delta\alpha_k = (k-5)d\beta_{k-1} + (k-3)d^*\beta_{k+1}, \quad k = 1,3,5,7, \tag{2.4.6}$$

$$\Delta\beta_k = (4-k)d\alpha_{k-1} + (2-k)d^*\alpha_{k+1}, \quad k = 0,2,4,6,8. \tag{2.4.7}$$

Combining (2.4.4), (2.4.5), (2.4.6) and (2.4.7) gives

$$\Delta\alpha_k = -(k-3)^2\alpha_k - 2d\beta_{k-1}, \quad k = 1,3,5,7, \tag{2.4.8}$$

$$\Delta\beta_k = (2-k)(k-4)\beta_k + 2d\alpha_{k-1}, \quad k = 0,2,4,6,8. \tag{2.4.9}$$

Now equations (2.4.8), (2.4.9) give successively: $\Delta\beta_0 = -8\beta_0$ and hence $\beta_0 = 0$. $\Delta\alpha_1 = -4\alpha_1$ and therefore $\alpha_1 = 0$. Then $\beta_2, \alpha_3, \beta_4$ are harmonic. For $k = 5$ we get $\Delta\alpha_5 = -4\alpha_5$ and α_5 vanishes. Then $\beta_6 = 0$ because $\Delta\beta_6 = -8\beta_6$. $\Delta\alpha_7 = -16\alpha_7$ gives $\alpha_7 = 0$ and finally $\Delta\beta_8 = -24\beta_8$ gives $\beta_8 = 0$. It is left to prove that β_2 vanishes. This immediately follows from (2.4.4). □

Lemma 2.4.10. *Let $\eta = \sum_{k=0}^{3} \eta_{2k+1}$ be a closed and coclosed odd degree form on $C(\Sigma)$ homogeneous of rate -3, i.e.*

$$\eta_{2k+1} = r^{2k-3} dr \wedge \alpha_{2k} + r^{2k-2}\beta_{2k+1},$$

where $\alpha_{2k} \in \Omega^{2k}(\Sigma)$ and $\beta_{2k+1} \in \Omega^{2k+1}(\Sigma)$. Then all components except α_4 and β_3 vanish, i.e.

$$\eta = \eta_3 + \eta_5 = \beta_3 + r dr \wedge \alpha_4,$$

and α_4 and β_3 are both harmonic on Σ. In particular, η_3 and η_5 are individually closed and co-closed.

Proof. We have

$$d^*\eta_{2k+1} = r^{2k-5} dr \wedge (-d^*\alpha_{2k}) + r^{2k-4}(-(5-(2k+1))\alpha_{2k} + d^*\beta_{2k+1})$$

and

$$d\eta_{2k-1} = r^{2k-5} dr \wedge (-d\alpha_{2k-2} + (2k-4)\beta_{2k-1}) + r^{2k-4} d\beta_{2k-1}.$$

Because of $d^*\eta_{2k+1} + d\eta_{2k-1} = 0$ we get

$$\begin{aligned} -d\alpha_{2k-2} + (2k-4)\beta_{2k-1} - d^*\alpha_{2k} &= 0, \\ -(5-(2k+1))\alpha_{2k} + d^*\beta_{2k+1} + d\beta_{2k-1} &= 0. \end{aligned}$$

This gives

$$(k-3)\beta_k = d\alpha_{k-1} + d^*\alpha_{k+1}, \quad k = 1,3,5,7, \tag{2.4.11}$$

$$(4-k)\alpha_k = d\beta_{k-1} + d^*\beta_{k+1}, \quad k = 0,2,4,6. \tag{2.4.12}$$

Applying d and d^* to (2.4.11) and (2.4.12) gives

$$\Delta\alpha_k = (k-4)d\beta_{k-1} + (k-2)d^*\beta_{k+1}, \quad k = 0,2,4,6, \tag{2.4.13}$$

$$\Delta\beta_k = (5-k)d\alpha_{k-1} + (3-k)d^*\alpha_{k+1}, \quad k = 1,3,5,7. \tag{2.4.14}$$

Combining (2.4.11), (2.4.12), (2.4.13) and (2.4.14) gives

$$\Delta\alpha_k = -(k-2)(k-4)\alpha_k - 2d\beta_{k-1}, \quad k = 0,2,4,6, \tag{2.4.15}$$

$$\Delta\beta_k = -(k-3)^2\beta_k + 2d\alpha_{k-1}, \quad k = 1,3,5,7. \tag{2.4.16}$$

For $k = 0$ we get $\Delta\alpha_0 = -8\alpha_0$ and therefore $\alpha_0 = 0$. For $k = 1$ we get $\Delta\beta_1 = -4\beta_1$ and consequentially β_1 vanishes. By (2.4.15) we first see that α_2 is harmonic. This in turn by (2.4.16) means that β_3 is harmonic. Again by (2.4.15) α_4 is harmonic. $k = 5$ gives $\Delta\beta_5 = -4\beta_5$ and hence $\beta_5 = 0$. Then $\Delta\alpha_6 = -8\alpha_6$ which gives $\alpha_6 = 0$. Finally β_7 vanishes because $\Delta\beta_7 = -16\beta_7$. We still have to show that $\alpha_2 = 0$. This now follows from (2.4.12). □

Lemma 2.4.17. *[FHN17, Proposition A.7] Let $\gamma = \sum_{j=0}^{m}(\log r)^j\gamma$, where all γ_j are differential forms on C homogeneous of order λ. If $(d+d^*)\gamma = 0$, then $m = 0$.*

As a consequence of Lemmas 2.4.3, 2.4.10 and 2.4.17 we get

Corollary 2.4.18. *We have*

$$\begin{aligned} \mathcal{K}_{\text{even}}(-4) &= r^{-1}dr \wedge H^3(\Sigma,\mathbb{R}) + H^4(\Sigma,\mathbb{R}), \\ \mathcal{K}_{\text{odd}}(-3) &= H^3(\Sigma,\mathbb{R}) + rdr \wedge H^4(\Sigma,\mathbb{R}), \\ \mathcal{K}_{\Lambda^4}(-4) &= r^{-1}dr \wedge H^3(\Sigma,\mathbb{R}) + H^4(\Sigma,\mathbb{R}), \\ \mathcal{K}_{\Lambda^3}(-3) &= H^3(\Sigma,\mathbb{R}), \\ \mathcal{K}_{\Lambda^5}(-3) &= rdr \wedge H^4(\Sigma,\mathbb{R}). \end{aligned}$$

Lemma 2.4.19. *Let γ be a harmonic 1-form on the Spin(7)-cone $C(\Sigma)$ homogeneous of rate $\lambda \in (-6, 1)$. We have:*

- *If $\lambda \in (-6, 0]$, then γ must vanish.*
- *If $\lambda \in (0, 1)$, then $\gamma = d(\frac{1}{\lambda+1} r^{\lambda+1}\alpha) = d^*(-\frac{1}{\lambda+7} r^{\lambda+2}\beta)$, where $\alpha \in \Omega^0(\Sigma)$ and $\beta \in \Omega^1(\Sigma)$ are eigenforms of Δ_Σ with eigenvalue $(\lambda+1)(\lambda+7)$. In particular, γ is closed and co-closed, and therefore an element of $\mathcal{K}_{\Lambda^1}(\lambda)$.*

Proof. Following earlier work by Cheeger [Che78], Foscolo–Haskins–Nordström [FHN17, Theorem A.2] have classified harmonic homogeneous forms of arbitrary pure degree on arbitrary cones. They find four types (i), (ii), (iii) and (iv). The proof of this Lemma is an application of their classification and the Lichnerowicz–Obata Theorem [Oba62], which says that the smallest positive eigenvalue of the scalar Laplace-operator is at least $\frac{\mathrm{Scal}}{6} = 7$ if we scale the metric on the link such that the scalar curvature equals 42.

Let $\alpha \in \Omega^0(\Sigma)$, $\beta \in \Omega^1(\Sigma)$, and $\gamma = r^\lambda(dr \wedge \alpha + r\beta)$ be a harmonic, homogeneous 1-form of rate λ. If γ is non-zero and of type (i), (ii) or (iii), then α must be a non-zero eigenfunction of the Laplace-operator with eigenvalue $(\lambda-1)(\lambda+7)$, $(\lambda+1)(\lambda+7)$ and $(\lambda-1)(\lambda+5)$, respectively. The first expression is always negative if $\lambda < 1$, the second expression is at most 7 if $\lambda \in (-6, 0]$, and the third expression is less than 7 if $\lambda \in (-6, 1)$. By the Lichnerowicz–Obata Theorem γ must be of type (ii) and $\lambda \in (0, 1)$. In the latter case $d\alpha = (\lambda+1)\beta$, $d^*\beta = (\lambda+7)\alpha$, and γ is of the desired form.

If γ is non-zero of type (iv), then $\alpha = 0$ and β is a co-closed, non-zero solution of $\Delta_\Sigma\beta = (\lambda+1)(\lambda+5)\beta$. By an application of the Bochner formula it follows that the smallest eigenvalue of the Laplacian on Σ on co-closed 1-forms is 12, see [CT94, Lemma 2.27] and [HS17, Lemma B.2]. But $(\lambda+1)(\lambda+5) < 12$ if $\lambda < 1$. Therefore, γ must vanish. □

Lemma 2.4.20. *If $\lambda \in [0, 1)$, then $\mathcal{K}_{\not{D}_-}(\lambda) \cong \mathcal{K}_{\Lambda^1}(\lambda)$.*

Proof. Under the identifications 2.1.6 up to constants we can write the negative Dirac operator as

$$\not{D}_- : \Gamma(\Lambda^1) \to \Gamma(\Lambda^0 \oplus \Lambda^2_7), \quad \gamma \to (d^*\gamma, \pi_7(d\gamma)).$$

The inclusion $\mathcal{K}_{\Lambda^1}(\lambda) \subset \mathcal{K}_{\not{D}_-}(\lambda)$ follows from the above formula for $\not{D}_-$ and the reverse inclusion follows from $\Delta = \not{D}^2$ and Lemma 2.4.19. □

Lemma 2.4.21. *[KL20, Proposition 3.3] Suppose that*

$$Z = r^{\lambda+1} f \partial_r + r^\lambda X$$

is a Killing vector field on the Spin(7)-cone $(C(\Sigma), \psi_C)$, where f is a function on Σ and X is a vector field on Σ. If $\lambda < 0$, then Z must vanish.

2.4.2 Harmonic spinors and closed and co-closed forms

By

$$\mathcal{H}^k_\lambda = \{\gamma \in \Omega^k_\lambda |\ d\gamma = 0 \text{ and } d^*\gamma = 0\}$$

we denote the space of all closed and co-closed k-forms on M decaying with rate λ. $\mathcal{H}^{\text{even}}_\lambda$ and $\mathcal{H}^{\text{odd}}_\lambda$ are defined analogously. Furthermore, we set

$$(\mathcal{H}^k_q)_\lambda = \{\gamma \in \mathcal{C}^\infty_\lambda(\Lambda^k_q) |\ d\gamma = 0 \text{ and } d^*\gamma = 0\}$$

if $\Lambda^k_q \subset \Lambda^k$ is a q-dimensional irreducible subrepresentation of Λ^k.

On compact manifolds any harmonic form is closed and co-closed. In general, this is not true in the non-compact setting because integration by parts is not always available. However, by Lemma 2.3.13 we can use integration by parts if the rate of decay is fast enough.

Lemma 2.4.22. *Suppose $\lambda \leq -3$.*
(i) If $\omega \in L^2_{2,\lambda}(\Lambda^k)$ is harmonic, then ω is closed and co-closed.
(ii) $\ker(\not{D}_+)_\lambda \cong (\mathcal{H}^4_1)_\lambda \oplus (\mathcal{H}^4_7)_\lambda$ and $\ker(\not{D}_-)_\lambda \cong (\mathcal{H}^3_8)_\lambda \cong \mathcal{H}^1_\lambda$.

Proof. **(i):** If $\lambda \leq -3$ Lemma 2.3.13 allows the following integration by parts:

$$0 = \langle \Delta\omega, \omega\rangle_{L^2} = \langle dd^*\omega, \omega\rangle_{L^2} + \langle \omega, d^*d\omega\rangle_{L^2} = \|d^*\omega\|^2_{L^2} + \|d\omega\|^2_{L^2}.$$

(ii): By applying Lemma 2.3.13 with $\not{D}$ as in the proof of (i) we get $\ker(\not{D}^2)_\lambda = \ker(\not{D})_\lambda$. Formula (2.1.6) and (i) give

$$\begin{aligned}\ker(\not{D}_+)_\lambda &= \ker(\not{D}_-\not{D}_+)_\lambda \cong \ker(\Delta|_{\Lambda^4_1})_\lambda \oplus \ker(\Delta|_{\Lambda^4_7})_\lambda = (\mathcal{H}^4_1)_\lambda \oplus (\mathcal{H}^4_7)_\lambda,\\ \ker(\not{D}_-)_\lambda &= \ker(\not{D}_+\not{D}_-)_\lambda \cong \ker(\Delta|_{\Lambda^3_8})_\lambda = (\mathcal{H}^3_8)_\lambda \cong \mathcal{H}^1_\lambda.\end{aligned}$$

□

Lemma 2.4.23. *Let $\omega \in L^2_{2,\lambda}(\Lambda^\bullet)$ be closed and coclosed. If $\lambda \leq -3$, then the individual degree components of ω are closed and coclosed.*

Proof. Denote by ω_k the degree k component of ω. The fact that $(d + d^*)\omega = 0$ gives

$d\omega_k = -d^*\omega_{k+2}$. The condition $\lambda \leq -3$ allows the following integration by parts:

$$\|d\omega_k\|^2_{L^2} = \langle d\omega_k, d\omega_k\rangle_{L^2} = -\langle d\omega_k, d^*\omega_{k+2}\rangle_{L^2} = -\langle \omega, d^*d^*\omega_{k+2}\rangle_{L^2} = 0.$$

□

Lemma 2.4.24. *If* $\lambda \leq -3$, *then* $\ker(\not{D})_\lambda = \{0\}$.

Proof. We use the Lichnerowicz formula

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}(g).$$

Because g is Ricci-flat, the scalar curvature vanishes and the Dirac Laplacian coincides with the rough Laplacian. If $s \in L^2_{k,\lambda}(\mathbf{S})$ for $\lambda \leq -3$, we can apply Lemma 2.3.13 to obtain

$$\langle \not{D}^2 s, s\rangle_{L^2} = \langle \nabla^*\nabla s, s\rangle_{L^2} = \|\nabla s\|_{L^2}.$$

Therefore, s is parallel if $s \in \ker(\not{D})_\lambda$. In particular, its point-wise norm is constant on M. Because the $L^2_{k,\lambda}$-norm of s is finite, s must vanish. □

The spin bundle identification (2.1.6), Lemma 2.4.22 (ii) and Lemma 2.4.24 imply

Corollary 2.4.25. $(\mathcal{H}^4_1)_\lambda$, $(\mathcal{H}^4_7)_\lambda$, $(\mathcal{H}^3_8)_\lambda$ *and* $\mathcal{H}^1_\lambda$ *are zero if* $\lambda \leq -3$.

For 1-forms this statement can be improved:

Lemma 2.4.26. *A harmonic 1-form* $\gamma \in \mathcal{C}^\infty_\lambda(T^*M)$ *vanishes if* $\lambda \leq 0$.

Proof. The statement is true for $\lambda \leq -3$ by Corollary 2.4.25. By Theorem 2.3.29 the kernel of the Laplace operator acting on 1-forms can only change at critical rates. However, by Lemma 2.4.19 there are no critical rates in the interval $[-6, 0]$. □

Lemma 2.4.27. *The negative Dirac operator*

$$(\not{D}_-)_{l+1,\lambda+1} \colon L^2_{l+1,\lambda+1}(\Lambda^3_8) \to L^2_{l,\lambda}(\Lambda^4_1 \oplus \Lambda^4_7).$$

is injective if $\lambda \leq -1$ *and surjective if* $\lambda \geq -5$.

Proof. Under the identification (2.1.6) the statement about injectivity follows from Lemma 2.4.26. The adjoint of $(\not{D}_-)_{l+1,\lambda+1}$ is the positive Dirac operator

$$(\not{D}_+)_{m+1,-8-\lambda} \colon L^2_{m+1,-8-\lambda}(\Lambda^4_1 \oplus \Lambda^4_7) \to L^2_{m,-9-\lambda}(\Lambda^3_8).$$

Then $\mathrm{Coker}(\not{D}_-)_{\lambda+1} = \ker(\not{D}_+)_{-8-\lambda}$ and with Lemma 2.4.24 the cokernel is zero if $\lambda \geq -5$. □

We will need the following analogues of the Hodge decomposition theorem on compact manifolds:

Proposition 2.4.28. *[KL20, Proposition 4.33] Suppose $\lambda+1$ is a non-critical rate for $d+d^*$. Let $0 \leq k \leq 8$. If $\lambda > -4$, we have*

$$\Omega^k_{l,\lambda} = d(\Omega^{k-1}_{l+1,\lambda+1}) + d^*(\Omega^{k+1}_{l+1,\lambda+1}) \oplus \mathcal{H}^k_{-8-\lambda}.$$

Proposition 2.4.29. *[KL20, Proposition 4.31, Corollary 4.32] Suppose $\lambda+1$ is a non-critical rate for $d+d^*$. Let $0 \leq k \leq 8$. If $\lambda < -4$, we have an L^2-orthogonal decomposition*

$$\Omega^k_{l,\lambda} = d(\Omega^{k-1}_{l+1,\lambda+1}) \oplus d^*(\Omega^{k+1}_{l+1,\lambda+1}) \oplus \mathcal{H}^k_{\lambda} \oplus W^k_{l,\lambda},$$

where $W^k_{l,\lambda}$ is isomorphic to $\mathcal{H}^k_{-8-\lambda}/\mathcal{H}^k_{\lambda}$.

In the next definition we define the main differential operator involved in studying the moduli space of AC Spin(7)-manifolds.

Definition 2.4.30. We denote the exterior derivative restricted to sections of $\Lambda^4_{35} = \Lambda^4_{\mathrm{ASD}}$ by

$$d_{\mathrm{ASD}} \colon \Omega^4_{35}(M) \to d\Omega^4(M), \quad \gamma \mapsto d\gamma.$$

By $(d_{\mathrm{ASD}})_{l,\nu}$ we denote its continuous extension

$$(d_{\mathrm{ASD}})_{l,\nu} \colon L^2_{l,\nu}(\Lambda^4_{35}) \to d(\Omega^4_{l,\nu}).$$

Lemma 2.4.31. *Suppose that $\nu+1$ is a non-critical rate of $d+d^*$. If $\nu > -4$, the operator $(d_{\mathrm{ASD}})_{l,\nu}$ is surjective.*

Proof. We can prove this by using the Hodge decomposition from Proposition 2.4.28. Let $\alpha \in d(\Omega^4_{l,\nu})$ be exact. Then we can write $\alpha = d\eta$ for some co-exact $\eta \in d^*(\Omega^5_{l+1,\nu+1})$. But then $\alpha = d(\eta - *\eta)$ is the exterior derivative of an anti-self-dual form and hence $(d_{\mathrm{ASD}})_{l,\nu}$ is surjective. □

Remark 2.4.32. The main reason why we have to restrict to rates $\nu > -4$ in Theorem A is that in the L^2-setting we cannot even expect that

$$d \colon L^2_{l,\nu}(\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}) \to d(\Omega^4_{l,\nu})$$

is surjective. The reason is that in the Hodge decomposition from Proposition 2.4.29 forms in the space $W^4_{l,\nu}$ are not necessarily closed. If we denote by $(W_c)^4_{l,\nu}$ the subspace of closed forms in $W^4_{l,\nu}$ and by $(W_\perp)^4_{l,\nu}$ its L^2-orthogonal complement in $W^4_{l,\nu}$, then every form in $d(\Omega^4_{l,\nu})$ can be written as

$$d(\eta + \omega)$$

for unique $\eta \in d^*(\Omega^5_{l+1,\nu+1})$ and $\omega \in (W_\perp)^4_{l,\nu}$. If $\omega \neq 0$, then we cannot use the same trick as in the proof of Lemma 2.4.31.

Lemma 2.4.33. *Suppose ν is a non-critical rate of the operator $d + d^*$. Then $d\Omega^4_{l,\nu}(M)$ is a closed subspace of $\Omega^5_{l-1,\nu-1}$, and therefore a Banach space.*

Proof. If $\nu \leq -4$, this is true because all components in the L^2-orthogonal decomposition from Proposition 2.4.29 are closed.

Next we consider the case $\nu > -4$. Let $\{d\gamma_j\}_{j\in\mathbb{N}}$ be a Cauchy sequence in $d\Omega^4_{l,\nu}(M)$. By Lemma 2.4.31 we can assume that $\gamma_j \in L^2_{l,\nu}(\Lambda^4_{35})$ for all $j \in \mathbb{N}$. Because γ_j is anti-self dual we have $d^*\gamma_j = *d\gamma_j$ and, therefore, $\{(d+d^*)\gamma_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\Omega^\bullet_{l-1,\nu-1}$. By Proposition 2.3.25 there exists $\gamma \in \Omega^4_{l,\nu}$ such that $(d + d^*)\gamma = \lim_{j\to\infty}(d + d^*)\gamma_j$. This finishes the proof. □

2.5 Cohomology of AC Spin(7)-manifolds

Suppose (M, ψ) is an AC Spin(7)-manifold. The compactly supported cohomology groups $H^k_{\text{cs}}(M,\mathbb{R})$ of M are the cohomology groups associated to the chain complex of compactly supported forms on M. Any representative of a class in $H^k_{\text{cs}}(M,\mathbb{R})$ is a closed k-form and therefore induces a class in $H^k(M,\mathbb{R})$. This is well-defined at the cohomology level and induces a map

$$\mathcal{I}^k : H^k_{\text{cs}}(M,\mathbb{R}) \to H^k(M,\mathbb{R}).$$

It follows straight from Definition 2.2.2 that if $r > R$, there is an embedding $\iota_r : \Sigma \to M$ given by $\iota_r = F(r, \cdot)$. This induces a restriction map $\iota_r^* : H^k(M,\mathbb{R}) \to H^k(\Sigma,\mathbb{R})$. Because the embeddings are homotopic for different values of r, the map ι_r^* does not depend on r. Henceforth we will denote it by

$$\Upsilon^k : H^k(M,\mathbb{R}) \to H^k(\Sigma,\mathbb{R}). \tag{2.5.1}$$

The maps $\mathcal{I}^k$ and Υ^k are part of a long exact sequence given by

$$\cdots \to H^k_{\text{cs}}(M,\mathbb{R}) \xrightarrow{\mathcal{I}^k} H^k(M,\mathbb{R}) \xrightarrow{\Upsilon^k} H^k(\Sigma,\mathbb{R}) \xrightarrow{\partial^k} H^{k+1}_{\text{cs}}(M,\mathbb{R}) \to \cdots \tag{2.5.2}$$

The boundary map ∂^k can be described as follows. If $[\alpha] \in H^k(\Sigma, \mathbb{R})$ set $\partial^k[\alpha] := [d(\chi\alpha)]$, where χ is the cut-off function from Definition 2.2.3. This is well-defined. Note that even though the form $d(\chi\alpha)$ is exact, the map ∂^k is non-trivial because $d(\chi\alpha)$ in general cannot be written as the exterior derivative of a compactly supported form.

For us it is important to have a good description of preimages of cohomology classes in $\mathrm{im}\Upsilon^k$. We say that a k-form γ on M is *translation invariant* if there exists $R' \geq R$ such that for all $r \geq R'$ we have

$$\iota_r^*(\gamma) = \iota_{R'}^*(\gamma),$$
$$\iota_r^*(\partial_r \lrcorner \gamma) = \iota_{R'}^*(\partial_r \lrcorner \gamma).$$

Equivalently there exist $R' \geq R$, $\alpha \in \Omega^{k-1}(\Sigma)$ and $\beta \in \Omega^k(\Sigma)$ such that on $F((R', \infty) \times \Sigma)$ we have $\gamma = dr \wedge \alpha + \beta$. If in addition $\alpha = 0$, we say that γ is a *lift*.

Lemma 2.5.3. *[Mar02, Corollary 5.9] Let $[\beta] \in \mathrm{im}\Upsilon^k$ where β is any representative. Then a preimage of $[\beta]$ under Υ^k can be represented by a lift, i.e. there exists $\zeta \in \Omega^k_{\mathrm{cs}}(M)$ such that $\xi = \chi\beta + \zeta$ is closed and $\Upsilon^k[\xi] = [\beta]$.*

Lemma 2.5.4. $\mathrm{im}\Upsilon^3$ *and* $\mathrm{im}\Upsilon^4$ *annihilate each other under the Poincaré pairing, i.e. with respect to harmonic representatives we have*

$$*\mathrm{im}\Upsilon^4 \perp_{L^2} \mathrm{im}\Upsilon^3 \quad \textit{and} \quad * \mathrm{im}\Upsilon^3 \perp_{L^2} \mathrm{im}\Upsilon^4.$$

Proof. Let α be a harmonic representative of a class $[\alpha] \in \mathrm{im}\Upsilon^3$ and β a harmonic representative of a class $[\beta] \in \mathrm{im}\Upsilon^4$. Then by Lemma 2.5.3 there exist a closed 3-form γ, a closed 4-form η and compactly supported forms γ_- and η_- such that $\gamma = \chi\alpha + \gamma_-$ and $\eta = \chi\beta + \eta_-$. Stokes' theorem gives

$$0 = \int_M d(\gamma \wedge \eta) = \lim_{r\to\infty} \int_{\{r\}\times\Sigma} (\gamma|_{\{r\}\times\Sigma}) \wedge (\eta|_{\{r\}\times\Sigma}) = \lim_{r\to\infty} r^7 \int_\Sigma \alpha \wedge \beta$$
$$+ \lim_{r\to\infty} \int_{\{r\}\times\Sigma} \Big(\alpha \wedge (\eta_-|_{\{r\}\times\Sigma}) + (\gamma_-|_{\{r\}\times\Sigma}) \wedge \beta + (\gamma_-|_{\{r\}\times\Sigma}) \wedge (\eta_-|_{\{r\}\times\Sigma})\Big)$$

Because the integrand in the second limit is compactly supported this limit is zero. Therefore, we get

$$\langle *\alpha, \beta\rangle_{L^2} = \langle \alpha, *\beta\rangle_{L^2} = \int_\Sigma \alpha \wedge \beta = 0.$$

□

The reason that topology is relevant for us is that we need to understand closed and co-closed forms decaying with the L^2-rate -4. The following Proposition relates them to the cohomology groups of M. The result is due to Lockhart [Loc87]. We will use a version adapted to the AC setting [Lot05, Theorem 6.5.2].

Proposition 2.5.5. *We have*

$$\mathcal{H}^k_{L^2} = \mathcal{H}^k_{-4} \cong \begin{cases} H^k(M,\mathbb{R}) & k > 4, \\ \mathcal{I}^4(H^4_{\mathrm{cs}}(M,\mathbb{R})) & k = 4, \\ H^k_{\mathrm{cs}}(M,\mathbb{R}) & k < 4. \end{cases}$$

Because we study deformations of Spin(7)-structures our main interest are 4-forms. By Proposition 2.5.5 harmonic 4-forms which lie in L^2 can be understood to be purely topological. More specifically we have

$$\mathcal{H}^4_{L^2} \cong \mathcal{I}^4(H^4_{\mathrm{cs}}(M,\mathbb{R})) \subset H^4(M,\mathbb{R}).$$

Splitting up into self dual and anti-self-dual 4-forms induces the decomposition

$$\mathcal{I}^4(H^4_{\mathrm{cs}}(M,\mathbb{R})) = (\mathcal{H}^4_+)_{L^2} \oplus (\mathcal{H}^4_-)_{L^2}. \tag{2.5.6}$$

This splitting can also be understood in a topological way. Let $[\xi], [\eta] \in \mathcal{I}^4(H^4_{\mathrm{cs}}(M,\mathbb{R}))$, where ξ and η are compactly supported representatives of the corresponding cohomology classes. Then

$$\int_M \xi \wedge \eta$$

is finite and defines a symmetric bilinear form on $\mathcal{I}^4(H^4_{\mathrm{cs}}(M,\mathbb{R}))$. To see that this is well-defined suppose that ξ' is another compactly supported representative of the cohomology class of ξ. Writing $[\xi]_{\mathrm{cs}}, [\xi']_{\mathrm{cs}}$ for the corresponding classes in $H^4_{\mathrm{cs}}(M,\mathbb{R})$, we get

$$\mathcal{I}^4([\xi]_{\mathrm{cs}} - [\xi']_{\mathrm{cs}}) = 0.$$

By the exactness of (2.5.2) and the description of the boundary map ∂^3, there exist $[\alpha] \in H^3(\Sigma)$ and $\gamma \in \Omega^3_{\mathrm{cs}}(M)$ such that

$$\xi' = \xi + d(\chi\alpha + \gamma).$$

Setting $\eta' = \eta + d\phi$, we have

$$\int_M d(\chi\alpha + \gamma) \wedge \eta' = \int_M d((\chi\alpha + \gamma) \wedge (\eta + d\phi)) = \lim_{r\to\infty} \int_M \alpha \wedge d(\phi|_{\{r\}\times\Sigma}) = 0.$$

This bilinear form is non-degenerate because the pairing of $H^4_{\mathrm{cs}}(M,\mathbb{R})$ and $H^4(M,\mathbb{R})$ is non-degenerate. $(\mathcal{H}^4_+)_{L^2}$ is a positive definite subspace of $\mathcal{I}^4(H^4_{\mathrm{cs}}(M,\mathbb{R}))$ with respect to this bilinear form and $(\mathcal{H}^4_-)_{L^2}$ is a negative definite subspace.

The last topological ingredient we need is that sufficiently fast decaying forms are exact on the end.

Lemma 2.5.7. *[Kar09, Lemma 2.12] Let γ be a smooth k-form on the cone $C(\Sigma)$. If*

$$|\nabla^j_C \gamma|_{g_C} = \mathcal{O}(r^{\lambda - j}) \text{ as } r \to \infty \text{ for all } j \in \mathbb{N}, \text{ for some } \lambda < -k,$$

then there exists a smooth $k-1$ form ξ on $(R,\infty)\times\Sigma$ such that $d\xi = \gamma$.

Chapter 3

Deformation theory of asymptotically conical Spin(7)-manifolds

3.1 The Moduli Space is an orbifold

In this section we consider the moduli space of torsion-free AC Spin(7)-structures of rate ν on the manifold M. As explained in the introduction we do not want to consider deformations of the Spin(7)-cone or, equivalently, of the link Σ because deformations of nearly parallel G_2-manifolds are not very well understood. Therefore, we fix an asymptotic Spin(7)-cone $C := (C(\Sigma), \varphi_C)$. The diffeomorphism group of M acts on the set of AC Spin(7)-structures asymptotic to C at a fixed rate. Indeed, if Φ is a diffeomorphism of M and ψ is asymptotic to ψ_C at rate ν with respect to some identification F of the cone and M outside a compact subset as in Definition 2.2.2, then $\Phi^*\psi$ is an AC Spin(7)-structure on M asymptotic to C at rate ν with respect to $F' := \Phi^{-1} \circ F$. Because $\Phi^*\psi$ does not decay to ψ_C at rate ν with respect to F unless Φ decays sufficiently fast to an automorphism of C, we can break the diffeomorphism invariance by fixing F and taking the quotient by suitably decaying diffeomorphisms. For the sake of simplicity, in this thesis we only quotient by diffeomorphisms which decay to the identity. Those decaying to some automorphism of C can in principle be divided out later. The above procedure also normalises a scale. For $\lambda > 0$ the rescaled Spin(7)-structure $\lambda^4\psi$ decays to ψ_C only after composing F with the diffeomorphism $(r, x) \mapsto (\lambda r, x)$ of the cone. Our results do not depend on the choice of F.

For the description of the moduli space it is convenient to choose a reference point. Let (M, ψ, g) be an AC Spin(7)-manifold at rate $\nu < 0$ with respect to F. Let $\mathcal{A}_\nu$ be the space of admissible 4-forms on M which decay with the same rate as ψ in the chosen gauge, i.e.

$$\mathcal{A}_\nu := \{\tilde{\psi} \in \mathcal{A}(M) |\ \tilde{\psi} - \psi \in \mathcal{C}^\infty_\nu(\Lambda^4 T^* M)\} \subset \psi + \mathcal{C}^\infty_\nu(\Lambda^4 T^* M). \tag{3.1.1}$$

The space of all (up to the choice of F) torsion-free AC Spin(7)-structures on M asymptotic to C at rate ν is denoted by

$$\mathcal{X}_\nu := \{\tilde{\psi} \in \mathcal{A}_\nu \,|\, d\tilde{\psi} = 0\}.$$

Denote by $\mathcal{D}_\lambda$ the group of diffeomorphisms generated by vector fields in $\mathcal{C}^\infty_\lambda(TM)$. The group $\mathcal{D}_{\nu+1}$ acts on $\mathcal{A}_\nu$ and $\mathcal{X}_\nu$ by pull-back. Then $\mathcal{M}_\nu := \mathcal{X}_\nu / \mathcal{D}_{\nu+1}$ is the moduli space of torsion-free AC Spin(7)-structures on M with decay rate ν asymptotic to C. We want to use the implicit function theorem for smooth maps between Banach spaces to prove that $\mathcal{M}_\nu$ is an orbifold for particular rates ν. Therefore, we equip $\mathcal{A}_\nu$ and $\mathcal{D}_{\nu+1}$ with the $L^2_{l,\nu}(\Lambda^4 T^* M)$ and $L^2_{l+1,\nu+1}(TM)$ topologies rather than the Frechet space topology of smooth forms and vector fields. We choose some $l \geq 6$, so that by Theorem 2.3.4 (i) we have an embedding $L^2_{l,\nu} \hookrightarrow \mathcal{C}^{1,\alpha}_\nu$. The action of $\mathcal{D}_{\nu+1}$ on $\mathcal{A}_\nu$ is continuous, $\mathcal{X}_\nu$ carries the subspace topology of $\mathcal{A}_\nu$, and $\mathcal{M}_\nu$ the quotient topology, with respect to which we want to prove the smooth manifold structure. As auxiliary objects we introduce $\mathcal{A}_{l,\nu}$ and $\mathcal{D}_{l+1,\nu+1}$, the completions of $\mathcal{A}_\nu$ and $\mathcal{D}_{\nu+1}$, respectively.

3.1.1 The space of torsion-free Spin(7)-structures and the stabiliser

Before we treat the moduli space $\mathcal{M}_\nu$, we first study the space of torsion-free Spin(7)-structures $\mathcal{X}_{l,\nu}$, which are $L^2_{l,\nu}$-regular, and the stabiliser

$$\mathcal{I}_\psi := \{\Phi \in \mathcal{D}_{l+1,\nu+1} \,|\, \Phi^* \psi = \psi\}$$

of ψ in $\mathcal{D}_{l+1,\nu+1}$. Because isometries of smooth Riemannian metrics are smooth by a result of Myers–Steenrod [MS39], $\mathcal{I}_\psi$ can alternatively be defined as the stabiliser of ψ in $\mathcal{D}_{\nu+1}$. Using the implicit function theorem, we show that $\mathcal{X}_{l,\nu}$ is a smooth manifold under a suitable assumption on the rate ν.

Proposition 3.1.2. *Suppose $\nu > -4$, and that ν and $\nu + 1$ are non-critical rates of the operator $d + d^*$. Then $\mathcal{X}_{l,\nu}$ is a smooth manifold and the tangent space $T_\psi \mathcal{X}_{l,\nu}$ is given by the kernel of the linear map*

$$d\colon T_\psi \mathcal{A}_{l,\nu} = L^2_{l,\nu}(\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}) \to d\Omega^4_{l,\nu}. \tag{3.1.3}$$

Proof. $\mathcal{X}_{l,\nu}$ is the zero level set of the exterior derivative

$$d\colon \mathcal{A}_{l,\nu} \to d\Omega^4_{l,\nu}. \tag{3.1.4}$$

By Lemma 2.4.33 this is a smooth map between Banach manifolds because ν is not a critical rate for $d+d^*$. By (2.1.5) we know that $T_\psi \mathcal{A}_{l,\nu} = L^2_{l,\nu}(\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35})$. The linearisation of (3.1.4) at ψ is the map (3.1.3). Under our assumptions this map is surjective by Lemma 2.4.31. The statement follows from the implicit function theorem for smooth maps between Banach spaces. □

Remark 3.1.5. By Remark 2.4.32 in the L^2-setting the linearisation of (3.1.4) cannot be expected to be surjective if $(W_\perp)^4_{l,\nu}$ is non-trivial. Therefore, the space $\mathcal{X}_{l,\nu}$ is in general not smooth for L^2-rates $\nu < -4$.

Next we show that $\mathcal{I}_\psi$ is finite. To prove this, it is enough to show that ψ cannot have any continuous symmetries in $\mathcal{D}_{\nu+1}$, i.e. that Killing vector fields of rate $\nu+1$ vanish. This implies that the stabiliser $\mathcal{I}_\psi$ is discrete, and thus finite because it is also compact.

Proposition 3.1.6. *Suppose $\nu < 0$. Let $\xi \in L^2_{l,\nu+1}(TM)$ be a Killing vector field for g, i.e. $\mathcal{L}_\xi g = 0$. Then ξ vanishes. Furthermore, the exterior derivative is injective on $L^2_{l,\nu}(\Lambda^3_8)$.*

Proof. Because (M, ψ, g) is Ricci-flat, any Killing vector field is harmonic. If $\nu \leq -1$, then $\xi^\flat$ is a harmonic 1-form of non-positive decay rate and hence has to vanish by Lemma 2.4.26. If $\xi \lrcorner \psi \in L^2_{l,\nu}(\Lambda^3_8)$ is closed, then ξ is a Killing vector field. The statement for $\nu < -1$ follows.

It is left to consider the case $\nu \in (-1, 0)$. By Theorem 2.3.27 there exists a critical rate $\lambda + 1 < \nu + 1$ for the Laplace operator such that

$$\xi = \chi Z + \mathcal{O}(r^{\lambda+1-\varepsilon}), \tag{3.1.7}$$

where

$$Z = r^{\lambda+1} f \partial_r + r^\lambda X.$$

Here f is a function on Σ and X a vector field on Σ, and Z is g_C-dual to a harmonic 1-form on $C(\Sigma)$ homogeneous of order $\lambda + 1$.

Our goal is to show that Z is a Killing vector field for g_C. Then by Lemma 2.4.21 Z must vanish. Repeating the argument for critical rates $\lambda + 1 \in (0, 1)$ shows that ξ is a Killing vector field for g with non-positive decay rate, and therefore vanishes as in the case

$\nu < -1$. For general vector fields X, Y, V on the cone we have

$$\mathcal{L}_V g(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V),$$
$$\mathcal{L}_V g_C(X,Y) = g_C(\nabla^C_X V, Y) + g_C(X, \nabla^C_Y V),$$

and therefore

$$\begin{aligned}(\mathcal{L}_V g - \mathcal{L}_V g_C)(X,Y) = &(g - g_C)(\nabla_X V, Y) + (g - g_C)(X, \nabla_Y V) \\ &+ g_C((\nabla_X - \nabla^C_X)V, Y) + g_C(X, (\nabla_Y - \nabla^C_Y)V)\end{aligned}$$

We have $|g - g_C| = \mathcal{O}(r^\nu)$, $|\nabla - \nabla^C| = \mathcal{O}(r^{\nu-1})$ as g is AC with rate ν, and $|\xi| = \mathcal{O}(r^{\lambda+1})$, $|\nabla\xi| = \mathcal{O}(r^\lambda)$ by (3.1.7) and elliptic regularity. Therefore, we get

$$\mathcal{L}_\xi g_C = \mathcal{L}_\xi g_C - \mathcal{L}_\xi g = \mathcal{O}(r^{\nu+\lambda}) = \mathcal{O}(r^{\lambda-\varepsilon})$$

for some $\varepsilon > 0$. On the other hand we have

$$\mathcal{L}_\xi g_C = \mathcal{L}_Z g_C + \mathcal{O}(r^{\lambda-\varepsilon})$$

and $\mathcal{L}_Z g_C$ is homogeneous of rate λ. Therefore, $\mathcal{L}_Z g_C = 0$. □

3.1.2 Slice construction for moduli space

Now we are ready to study the moduli space $\mathcal{M}_\nu = \mathcal{X}_\nu / \mathcal{D}_{\nu+1}$. We want to break the action of $\mathcal{D}_{\nu+1}$ on $\mathcal{A}_\nu$ and in each orbit close to ψ choose in a smooth fashion a representative which is unique up the the action of the stabiliser $\mathcal{I}_\psi$. Ebin [Ebi70] showed how to find a slice for the diffeomorphism action on the space of Riemannian metrics. Using Proposition 3.1.2 and simplifications of Ebin's approach in our setting, which were explained by Nordström [Nor08], we find that a good slice $\mathcal{S}_\psi$ around ψ needs to satisfy three properties:

Theorem 3.1.8. *[Nor08, Section 3.1.3] Let K be a closed subspace which is a complement of $T_\psi(\mathcal{D}_{l+1,\nu+1} \cdot \psi)$ in $T_\psi \mathcal{A}_{l,\nu}$. Let $\mathcal{S}_\psi$ be a smooth submanifold of $\mathcal{A}_{l,\nu}$ which contains ψ and satisfies*

(S.1) $T_\psi \mathcal{S}_\psi = K$,

(S.2) $\mathcal{S}_\psi$ is $\mathcal{I}_\psi$-invariant,

(S.3) all $\tilde{\psi} \in \mathcal{R}_\psi := \mathcal{S}_\psi \cap \mathcal{X}_{l,\nu}$ are smooth 4-forms on M.

Then we have:

(i) $\mathcal{R}_\psi$ is a smooth manifold,

(ii) the map

$$\mathcal{R}_\psi \subset \mathcal{X}_\nu \to \mathcal{M}_\nu = \mathcal{X}_\nu/\mathcal{D}_{\nu+1}, \quad \tilde{\psi} \mapsto \tilde{\psi}\mathcal{D}_{\nu+1}$$

is open,

(iii) the induced map $\mathcal{R}_\psi/\mathcal{I}_\psi \to \mathcal{M}_\nu$ is a homeomorphism onto its image,

(iv) the map

$$\mathcal{D}_{l+1,\nu+1} \times \mathcal{R}_\psi \to \mathcal{X}_{l,\nu} \tag{3.1.9}$$

is a smooth submersion onto a neighbourhood of ψ.

Once we find $\mathcal{S}_\psi$, we can use Theorem 3.1.8 (i)-(iv) to conclude that under the assumptions on ν in Proposition 3.1.2 $\mathcal{M}_\nu$ is an orbifold: around ψ we can use $\mathcal{R}_\psi/\mathcal{I}_\psi$ as a chart, and the transition to another such chart $\mathcal{R}_{\tilde{\psi}}/\tilde{\psi}$ centred at $\tilde{\psi} \in \mathcal{X}_\nu$ can be described via a section of the submersion (3.1.9). If $\mathcal{I}_\psi$ is trivial or acts trivially on $\mathcal{R}_\psi$, we can strengthen our conclusion and find that $\mathcal{M}_\nu$ is a smooth in a neighbourhood of ψ. One particular way to check if the action of $\mathcal{I}_\psi$ on $\mathcal{R}_\psi$ is trivial is to look at the projection $\mathcal{R}_\psi \to H^4(M, \mathbb{R})$ to the cohomology group. This is well-defined because elements in $\mathcal{R}_\psi$ are smooth, closed 4-forms. Because elements in $\mathcal{I}_\psi$ are isotopic to the identity, the projection is $\mathcal{I}_\psi$-invariant. If $\mathcal{R}_\psi \to H^4(M, \mathbb{R})$ is an embedding, all forms in $\mathcal{R}_\psi$ therefore represent different points in the moduli space and we can use $\mathcal{R}_\psi$ as a smooth chart.

To wrap up our discussion of the moduli space $\mathcal{M}_\nu$, we are left to find a good slice $\mathcal{S}_\psi$ as in Theorem 3.1.8. To do so, we first determine a complement of $T_\psi(\mathcal{D}_{l+1,\nu+1} \cdot \psi)$ in $T_\psi \mathcal{A}_{l,\nu} = L^2_{l,\nu}(\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35})$. To compute $T_\psi(\mathcal{D}_{l+1,\nu+1} \cdot \psi)$, let F_t be the 1-parameter subgroup of diffeomorphisms generated by some $X \in \mathcal{C}^\infty_{\nu+1}(TM)$. Then

$$\left.\frac{d}{dt}\right|_{t=0} F_t^* \psi = \mathcal{L}_X \psi = X \lrcorner d\psi + d(X \lrcorner \psi) = d(X \lrcorner \psi).$$

By (2.1.2) we get $T_\psi(\mathcal{D}_{\nu+1} \cdot \psi) = d(\mathcal{C}^\infty_{\nu+1}(\Lambda^3_8))$. Analogously $T_\psi(\mathcal{D}_{l+1,\nu+1} \cdot \psi) = d(L^2_{l+1,\nu+1}(\Lambda^3_8))$. In particular, $T_\psi(\mathcal{D}_{l+1,\nu+1} \cdot \psi)$ is closely related to the image of the negative Dirac operator (2.1.7b), which we will exploit to determine a complement. In the following let $K_{l,\nu}$ be a complement of $d \ker(\not{D}_-)_{\nu+1}$ in $L^2_{l,\nu}(\Lambda^4_{35})$. By Lemma 2.4.27 we have

$$K_{l,\nu} \cong \begin{cases} L^2_{l,\nu}(\Lambda^4_{35}) & \text{if } \nu \in (-4, -1], \\ L^2_{l,\nu}(\Lambda^4_{35})/d \ker(\not{D}_-)_{\nu+1} & \text{if } \nu \in (-1, 0). \end{cases} \tag{3.1.10}$$

Proposition 3.1.11. *We have the decomposition*

$$L^2_{l,\nu}(\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}) = T_\psi(\mathcal{D}_{l+1,\nu+1} \cdot \psi) \oplus K_{l,\nu}$$

Proof. Let $\alpha \in L^2_{l,\nu}(\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35})$. If $\nu \in (-4, 0)$, the negative Dirac-operator

$$(\not{D}_-)_{l+1,\nu+1} \colon L^2_{l+1,\nu+1}(\Lambda^3_8) \to L^2_{l,\nu}(\Lambda^4_1 \oplus \Lambda^4_7)$$

is surjective by Corollary 2.4.27. Therefore, we can write

$$\pi_{1+7}\alpha = \not{D}_-(X \lrcorner \psi) = \pi_{1+7} d(X \lrcorner \psi)$$

for some $X \lrcorner \psi \in L^2_{l+1,\nu+1}(\Lambda^3_8)$. Then

$$\alpha - d(X \lrcorner \psi) \in L^2_{l,\nu}(\Lambda^4_{35}).$$

This proves

$$L^2_{l,\nu}(\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}) = T_\psi(\mathcal{D}_{l+1,\nu+1} \cdot \psi) + L^2_{l,\nu}(\Lambda^4_{35})$$

for $\nu \in (-4, 0)$. Assume that $d(X \lrcorner \psi) \in T_\psi(\mathcal{D}_{l+1,\nu+1} \cdot \psi) \cap L^2_{l,\nu}(\Lambda^4_{35})$ for some $X \lrcorner \psi \in L^2_{l+1,\nu+1}(\Lambda^3_8)$. Then $\not{D}_-(X \lrcorner \psi) = \pi_{1+7} d(X \lrcorner \psi) = 0$. This proves the statement. □

By Proposition 3.1.11 we want to determine a slice $\mathcal{S}_\psi$ around ψ which satisfies the properties (S.1)-(S.3) from Theorem 3.1.8 and at ψ has the tangent space $K_{l,\nu}$. A candidate for such a slice is the graph of the map Θ, which we have defined in (2.1.9), over a neighbourhood of ψ in the affine space $\psi + K_{l,\nu}$. We set

$$\mathcal{S}_\psi := \{\Pi(\eta) = \psi + \eta - \Theta(\eta) \mid \eta \in U \subset K_{l,\nu}\},$$

where U is a sufficiently small neighbourhood of the origin in $K_{l,\nu}$. Properties (S.1) and (S.2) follow from the properties (i)-(iii) of the maps Π and Θ. Next we proof property (S.3):

Proposition 3.1.12. *All elements in a neighbourhood of ψ in $\mathcal{R}_\psi$ are smooth.*

Proof. By the definition of $\mathcal{S}_\psi$ each $\hat{\psi} \in \mathcal{R}_\psi$ sufficiently close to ψ can be written as

$$\Pi(\xi) = \psi + \xi - \Theta(\xi)$$

for some $\xi \in L^2_{l,\nu}(\Lambda^4_{35})$. Using the Hodge star operator with respect to ψ, the condition

$d\Pi(\xi) = 0$ leads to the equation

$$(d + d^*)\xi - d\Theta(\xi) - *d\Theta(\xi) = 0,$$

which can be re-written as

$$(d + d^*)\xi + Q(\xi, \nabla\xi) = R(\xi),$$

where $Q(x, y)$ and $R(x)$ are smooth maps which depend on ψ and its derivatives, and $Q(x, y)$ is linear in y and $Q(0, y) = 0$ for all y. Because the $\mathcal{C}^0_\nu$-norm of ξ is controlled by its $L^2_{l,\nu}$-norm via the Sobolev embedding $L^2_{l,\nu} \hookrightarrow \mathcal{C}^{1,\alpha}_\nu$, the linear operator $L = d+d^*+Q(\xi, \nabla\cdot)$ is $\mathcal{C}^0$-close to $d+d^*$ if the slice $\mathcal{S}_\psi$ is chosen sufficiently small and therefore elliptic. Again by Sobolev embedding we can assume the induction hypothesis $\xi \in \mathcal{C}^{q,\alpha}_\nu$ for $q \geq 1$. Theorem 2.3.19 implies $\xi \in \mathcal{C}^{q+1,\alpha}_\nu$. By induction we see that ξ is smooth. We conclude that $\Pi(\xi)$ is smooth because Π and ξ are smooth. □

We have now proved that $\mathcal{M}_\nu$ is an orbifold under suitable assumptions on the rate ν. To compute its dimension, we determine the tangent space of the pre-moduli space $\mathcal{R}_\psi$ at ψ.

Lemma 3.1.13. *The tangent space of the pre-moduli space $\mathcal{R}_\psi$ at ψ is given by*

$$T_\psi\mathcal{R}_\psi \cong \begin{cases} (\mathcal{H}^4_{35})_\nu & \text{if} \nu \in (-4, -1], \\ (\mathcal{H}^4_{35})_\nu / d\ker(\not{D}_-)_{\nu+1} & \text{if} \nu \in (-1, 0). \end{cases}$$

Proof. By construction we have

$$T_\psi\mathcal{R}_\psi = T_\psi\mathcal{X}_{l,\nu} \cap T_\psi\mathcal{S}_\psi.$$

Therefore, by Lemma 3.1.2 $T_\psi\mathcal{R}_\psi$ is the kernel of the linear map

$$d\colon T_\psi\mathcal{S}_{l,\nu} \to d\Omega^4_{l,\nu}.$$

The statement follows with formula (3.1.10). □

Remark 3.1.14. As explained in the beginning of this section the gauge fixing normalises the scale of the AC Spin(7)-structures. However, scaling is still seen by the moduli space $\mathcal{M}_\nu$. In the following we explain that an AC Spin(7)-manifold with decay rate ν always induces a canonical Spin(7)-deformation via scaling. Therefore, if there is a torsion-free AC Spin(7)-structure on M which decays to the cone C precisely at rate $\nu \in (-4, 0)$, by

Lemma 3.1.13 the dimension of the space $(\mathcal{H}^4_{35})_\lambda$ must increase as λ crosses ν. This gives a criterion to exclude the existence of torsion-free AC Spin(7)-structures at certain rates.

The vector field $V = r\partial_r$ generates the flow

$$\Phi_\lambda(r,x) = (e^\lambda r, x)$$

on the cone $C(\Sigma)$. The action of Φ_λ scales the conical Spin(7)-structure:

$$\Phi^*_{\log\lambda}\psi_C = \lambda^4\psi_C.$$

We can transplant the vector field to M by setting $\hat{V} = \chi V$. Denote its flow by $\hat{\Phi}_\lambda$. As noted above the rescaled AC Spin(7)-structures $\lambda^4\psi$ are not in $\mathcal{A}_\lambda$ and therefore do not contribute to $\mathcal{M}_\nu$. However, up to asymptotic decay we can reverse scaling by λ^4 by the action of $\hat{\Phi}_{1/\log\lambda}$. Set

$$\psi_\lambda := \lambda^4\hat{\Phi}^*_{(1/\log\lambda)}\psi.$$

To see that ψ_λ decays to ψ with rate ν write

$$\psi(r,x) = dr \wedge \varphi(r,x) + *\varphi(r,x).$$

Then we have

$$\psi_\lambda(r,x) = \lambda^3 dr \wedge \varphi(r/\lambda, x) + \lambda^4 * \varphi(r/\lambda, x).$$

With respect to the norm given by g_C we have

$$\begin{aligned}
&|\psi_\lambda(r,x) - \psi_C(r,x)|^2 = |\psi_\lambda(r,x) - \lambda^4\psi_C(r/\lambda,x)|^2\\
=&\lambda^3|\varphi(r/\lambda,x) - (r/\lambda)^3\varphi_\Sigma(x)|^2 + \lambda^4| *_\varphi \varphi(r/\lambda,x) - (r/\lambda)^4 *_\Sigma \varphi_\Sigma(x)|^2\\
=&\mathcal{O}(r^\nu)
\end{aligned}$$

because ψ decays to ψ_C with rate ν.

The family ψ_λ induces the infinitesimal deformation

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \psi_\lambda = 4\psi - \mathcal{L}_{\hat{V}}\psi.$$

The results in this section prove

Proposition 3.1.15. *Suppose $\nu \in (-4,0)$ and that ν and $\nu+1$ are non-critical rates of the operator $d+d^*$. Then the moduli space $\mathcal{M}_\nu$ is an orbifold and the orbifold chart $\mathcal{R}_\psi$*

at ψ has the tangent space

$$T_\psi \mathcal{R}_\psi \cong \begin{cases} (\mathcal{H}^4_{35})_\nu & \text{if} \nu \in (-4,-1], \\ (\mathcal{H}^4_{35})_\nu / d\ker(\not{D}_-)_{\nu+1} & \text{if} \nu \in (-1,0). \end{cases} \tag{3.1.16}$$

Furthermore, if the stabiliser $\mathcal{I}_\psi$ of ψ is trivial or acts trivially on the orbifold chart $\mathcal{R}_\psi$, the moduli space $\mathcal{M}_\nu$ is a smooth in a neighbourhood of ψ. In particular, this is true (after possibly shrinking $\mathcal{R}_\psi$) if the projection $T_\psi\mathcal{R}_\psi \to H^4(M,\mathbb{R})$ is injective.

3.2 Computation of infinitesimal deformations

The aim of this section is to give a more precise description of the infinitesimal deformations (3.1.16). With the terminology introduced in Definition 2.3.21 and section 2.5 we will show

Proposition 3.2.1. *Let (M,ψ,g) be an AC Spin(7)-manifold. If $\nu > -4$, we have*

$$(\mathcal{H}^4_{35})_\nu \cong (\mathcal{H}^4_-)_{L^2} \oplus \mathrm{im}\Upsilon^4 \oplus \bigoplus_{\lambda \in \mathcal{D}(d_{\mathrm{ASD}})\cap(-4,\nu)} \mathcal{K}_{\mathrm{ASD}}(\lambda).$$

Proposition 3.2.1 will follow from Proposition 3.2.7 and Corollary 3.2.9. We need to study how $(\mathcal{H}^4_{35})_\nu$ changes as ν passes a critical rate. Because $(\mathcal{H}^4_{35})_\nu = \ker(d_{\mathrm{ASD}})_{l,\nu}$ and $(d_{\mathrm{ASD}})_{l,\nu}$ is surjective for generic rates $\nu > -4$ by Lemma 2.4.31, this corresponds to the change in ind $(d_{\mathrm{ASD}})_{l,\nu}$ as ν crosses a critical rate. In the introduction we have explained the index change at critical rates for uniformly elliptic operators. However, d_{ASD} is not elliptic. Therefore, we need to adapt Theorem 2.3.28 to our non-elliptic setting. To simplify the presentation we will first explain how it can be adjusted to the non-elliptic operator $d + d^*|_{\Omega^k}$. The main ingredient in the proof of Theorem 2.3.28 is Theorem 2.3.27. We will first adapt this to our situation. Compare with [KL20, Lemma 4.28].

Proposition 3.2.2. *Let (M,ψ) be an AC Spin(7)-manifold of rate ν. Let λ_0 be a critical rate for $d+d^*$ and let $\beta_2 < \beta_1$ be two non-critical rates for $d+d^*$ such that λ_0 is the unique critical rate for the operator $d+d^*$ in the interval $[\beta_2,\beta_1]$ and $\lambda_0 + \nu < \beta_2$.*

If $\gamma \in \Omega^k_{l+1,\beta_1}$ with $(d+d^*_M)\gamma \in \Omega^\bullet_{l,\beta_2-1}$, then there exist unique $\eta \in \mathcal{K}_{\Lambda^k}(\lambda_0)$ and $\tilde{\gamma} \in \Omega^k_{l+1,\beta_2}$ with

$$\gamma = \chi\eta + \tilde{\gamma}. \tag{3.2.3}$$

Moreover, η and $\tilde{\gamma}$ depend linearly on γ. Here χ is the cut-off function from Definition 2.2.3.

Proof. We cannot apply Theorem 2.3.27 to $d+d^*_M|_{\Omega^k}$, but we can embed $\Omega^k \subset \Omega^\bullet$ and then apply Theorem 2.3.27 to $d + d^*_M : \Omega^\bullet \to \Omega^\bullet$. More specifically, there exist $\omega \in \mathcal{K}_{d+d^*}(\lambda_0)$

and $\tilde{\gamma} \in \Omega^{\bullet}_{l+1,\beta_2}$ such that

$$\gamma = \omega + \tilde{\gamma}$$

on the end. The price we have to pay for using Theorem 2.3.27 is that a priori ω can be a mixed degree form. Therefore, we need to show that all except the degree k part of ω vanish. Because γ is a k-form, for $l \neq k$ the l-form component of ω has to decay with rate β_2 to cancel with the degree l component of $\tilde{\gamma}$. However, each non-zero degree component of ω is homogeneous of rate $\lambda_0 > \beta_2$. Therefore, ω is a pure degree k-form. Finally, it is straightforward that $\tilde{\gamma}$ is purely of degree k as well. □

Proposition 3.2.4. *Let (M, ψ) be an AC Spin(7)-manifold of rate ν. Let λ_0 be a critical rate for d_{ASD} and let $\beta_2 < \beta_1$ be two non-critical rates for $d + d^*$ such that λ_0 is the unique critical rate for the operator $d + d^*$ in the interval $[\beta_2, \beta_1]$ and $\lambda_0 + \nu < \beta_2$.*

If $\gamma \in L^2_{l+1,\beta_1}(\Lambda^4_{35})$ with $d\gamma \in \Omega^5_{l,\beta_2-1}$, then there exist unique $\omega \in \mathcal{K}_{\mathrm{ASD}}(\lambda_0)$ and $\hat{\gamma} \in L^2_{l+1,\beta_2}(\Lambda^4)$ such that

$$\gamma = \chi\omega + \hat{\gamma}.$$

This decomposition depends linearly on γ.

Proof. Because γ is anti-self-dual, we have $d^*_M\gamma = - *_M d *_M \gamma = *_M d\gamma$. Because the Hodge-star is an isometry and $d\gamma \in \Omega^5_{l,\beta_2-1}$, we know that $d^*_M\gamma \in \Omega^3_{l,\beta_2-1}$. By embedding $\Lambda^4_{35} \subset \Lambda^4$ we can use Proposition 3.2.2 to get $\omega \in \mathcal{K}_{\Lambda^4}(\lambda_0)$ and $\hat{\gamma} \in \Omega^4_{l+1,\beta_2}$ such that

$$\gamma = \chi\omega + \hat{\gamma}.$$

Projecting on the self-dual part gives

$$0 = \chi(\omega + *_C\omega) + \chi(*_M - *_C)\omega + \hat{\gamma} + *_M\hat{\gamma}. \tag{3.2.5}$$

By Lemma 2.3.5 the middle term in (3.2.5) decays like $\lambda_0 + \nu$. Because $\lambda_0 + \nu < \beta_2$, all terms on the right-hand side of (3.2.5) except $(\omega + *_C\omega)$ decay with rate $\beta_2 < \lambda_0$ while $(\omega + *_C\omega)$ decays with rate λ_0. Therefore, $(\omega + *_C\omega)$ has to vanish, i.e. ω is of type 35 with respect to the Spin(7)-structure on the cone, and in particular $\omega \in \mathcal{K}_{\mathrm{ASD}}(\lambda_0)$.

□

Proposition 3.2.6. *Let λ_0 be a critical rate for the operator $d + d^*_M|_{\Omega^k}$, and choose $\varepsilon > 0$ small enough such that λ_0 is the unique critical rate for the operator $d + d^*_M|_{\Omega^k}$ in the* *interval* $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ *and* $\lambda_0 + \nu < \lambda_0 - \varepsilon$. *Then there exists an injective linear map*

$$T^k_{\lambda_0} \colon \mathcal{H}^k_{\lambda_0+\varepsilon}/\mathcal{H}^k_{\lambda_0-\varepsilon} \to \mathcal{K}_{\Lambda^k}(\lambda_0).$$

In the same setting for the operator d_{ASD} *there exists an injective linear map*

$$T^{\mathrm{ASD}}_{\lambda_0} \colon (\mathcal{H}^4_{35})_{\lambda_0+\varepsilon}/(\mathcal{H}^4_{35})_{\lambda_0-\varepsilon} \to \mathcal{K}_{\mathrm{ASD}}(\lambda_0).$$

Proof. Let $\gamma \in \mathcal{H}^k_{\lambda_0+\varepsilon}$. Because $(d+d^*_M)\gamma = 0$, we are in the situation of Proposition 3.2.2. Hence there is a unique $\eta \in \mathcal{K}_{\Lambda^k}(\lambda_0)$ such that

$$\gamma = \chi\eta + \mathcal{O}(r^{\lambda_0-\varepsilon}).$$

We set $T^k_{\lambda_0}(\gamma) = \eta$. It is clear that $\eta = 0$ if $\gamma \in \mathcal{H}^k_{\lambda_0-\varepsilon}$. Hence $T^k_{\lambda_0}$ is well-defined. Because η depends linearly on γ, the map is linear. If $T^k_{\lambda_0}(\gamma) = 0$, then $\gamma = \mathcal{O}(r^{\lambda_0-\varepsilon})$. Therefore, $T^k_{\lambda_0}$ is injective. The statement for d_{ASD} follows analogously by using Proposition 3.2.4. □

Proposition 3.2.7. *If* $\lambda_0 > -4$, *the map* $T^{\mathrm{ASD}}_{\lambda_0}$ *is an isomorphism.*

Proof. We need to prove the surjectivity of $T^{\mathrm{ASD}}_{\lambda_0}$. Suppose $\eta \in \mathcal{K}_{\mathrm{ASD}}(\lambda_0)$. As $\lambda_0 \neq -4$, η is exact on the cone by Remark 2.4.2, i.e. there exists a 3-form ξ on the cone such that $\eta = d\xi$. But then $d(\chi\eta) = d(\chi\eta - d(\chi\xi)) \in d\Omega^4_{\mathrm{cs}}$. $(d_{\mathrm{ASD}})_{l,\lambda-\epsilon}$ is surjective by Lemma 2.4.31 as ε can be chosen small enough such that $\lambda - \varepsilon > -4$. Therefore, there exists $\hat{\gamma} \in L^2_{l,\lambda-\varepsilon}(\Lambda^4_{35})$ such that $\gamma = \chi\eta + \hat{\gamma}$ is closed. □

3.2.1 The exceptional rate -4

To finish the proof of Proposition 3.2.1 it remains to compute the index change for the operator $(d_{\mathrm{ASD}})_{l,\lambda}$ at the exceptional critical rate $\lambda = -4$. We are going to prove a more general statement by considering the operator $d + d^*_M$ restricted to 4-forms at the critical rate $\lambda = -4$. The main result of this section is

Proposition 3.2.8. *The map* T^4_{-4} *takes values in* $r^{-1}dr \wedge (\mathrm{im}\Upsilon^3)^\perp + \mathrm{im}\Upsilon^4$, *and*

$$T^4_{-4} \colon \mathcal{H}^4_{-4+\varepsilon}/\mathcal{H}^4_{-4-\varepsilon} \to r^{-1}dr \wedge (\mathrm{im}\Upsilon^3)^\perp + \mathrm{im}\Upsilon^4$$

is an isomorphism.

The corresponding statement for the operator d_{ASD} is a simple consequence of Proposition 3.2.8.

Corollary 3.2.9. *The map T^{ASD}_{-4} takes values in the space*

$$\{r^{-1}dr \wedge (- *_\Sigma \beta) + \beta \mid \beta \in \mathrm{im}\Upsilon^4\} \cong \mathrm{im}\Upsilon^4,$$

and

$$T^{\mathrm{ASD}}_{-4} \colon (\mathcal{H}^4_{35})_{-4+\varepsilon}/(\mathcal{H}^4_{35})_{-4-\varepsilon} \to \mathrm{im}\Upsilon^4$$

is an isomorphism.

Proof. Because

$$(\mathcal{H}^4_{35})_{-4+\varepsilon}/(\mathcal{H}^4_{35})_{-4-\varepsilon} \subset \mathcal{H}^4_{-4+\varepsilon}/\mathcal{H}^4_{-4-\varepsilon},$$

the map T^{ASD}_{-4} is the map T^4_{-4} restricted to anti-self-dual forms. The statement follows from Theorem 3.2.8 and the description of $\mathcal{K}_{\mathrm{ASD}}(-4)$ in Remark 2.4.2. □

In the remainder of this subsection we will prove Proposition 3.2.8 in several steps.

Lemma 3.2.10. *The map T^4_{-4} takes values in $r^{-1}dr \wedge (\mathrm{im}\Upsilon^3)^\perp + \mathrm{im}\Upsilon^4$, i.e. there is an injective linear map*

$$T^4_{-4} \colon \mathcal{H}^4_{-4+\varepsilon}/\mathcal{H}^4_{-4-\varepsilon} \to r^{-1}dr \wedge (\mathrm{im}\Upsilon^3)^\perp + \mathrm{im}\Upsilon^4.$$

Proof. Let $\gamma \in \mathcal{H}^4_{-4+\varepsilon}$. Then by Proposition 3.2.6 and Corollary 2.4.18 there exist harmonic forms $\alpha \in \Omega^3(\Sigma)$ and $\beta \in \Omega^4(\Sigma)$ such that

$$\gamma = \chi(r^{-1}dr \wedge \alpha + \beta) + \mathcal{O}(r^{-4-\varepsilon})$$

and $T^4_{-4}(\gamma) = r^{-1}dr \wedge \alpha + \beta$. By Lemma 2.5.7 the part of γ which decays like $-4-\varepsilon$ is exact on the end. Therefore, $\Upsilon^4([\gamma]) = [\beta]$ and $[\beta] \in \mathrm{im}\Upsilon^4$. The same argument for $*\gamma$ gives $[*\alpha] \in \mathrm{im}\Upsilon^4$ and hence $[\alpha] \in (\mathrm{im}\Upsilon^3)^\perp$ by Lemma 2.5.4. □

The main difficulty in proving Proposition 3.2.8 is to show surjectivity. The next Lemma is a first step towards this goal. However, because of the structure of $\mathcal{K}_{\Lambda^4}(-4)$ more work will be needed later on.

Lemma 3.2.11. *(i) Let $\beta \in \mathrm{im}\Upsilon^4$. Then there exist $\alpha \in (\mathrm{im}\Upsilon^3)^\perp$ and $\gamma \in \mathcal{H}^4_{-4+\varepsilon}$ such that $T^4_{-4}(\gamma) = r^{-1}dr \wedge \alpha + \beta$.*

(ii) Let $\alpha \in (\mathrm{im}\Upsilon^3)^\perp$. Then there exist $\beta \in \mathrm{im}\Upsilon^4$ and $\gamma \in \mathcal{H}^4_{-4+\varepsilon}$ such that $T^4_{-4}(\gamma) = r^{-1}dr \wedge \alpha + \beta$.

Proof. (i) By Lemma 2.5.3 there exists $\zeta \in \Omega^4_{\mathrm{cs}}(M)$ such that $\xi = \chi\beta + \zeta$ is closed. Because $\xi \in \Omega^4_{l,-4+\varepsilon}$ we can apply the Hodge decomposition from Proposition 2.4.28 to deduce that there is a $\gamma \in \mathcal{H}^4_{-4+\varepsilon}$ cohomologous to ξ. In particular $\Upsilon^4([\gamma]) = \Upsilon^4([\xi]) = \beta$. By Lemma 3.2.10 there is some $\alpha \in \mathrm{im}(\Upsilon^3)^\perp$ with $\gamma = \chi(r^{-1}dr \wedge \alpha + \beta) + \mathcal{O}(r^{-4-\varepsilon})$ and $T^4_{-4}(\gamma) = r^{-1}dr \wedge \alpha + \beta$.
(ii) By Lemma 2.5.4 $*\alpha \in \mathrm{im}\Upsilon^4$. The statement follows from (i). □

Remark 3.2.12. Note that in Lemma 3.2.11 (i) we cannot choose α. It just says that there exists some. This in particular means that we cannot yet prove that $T^4_{-4}\colon \mathcal{H}^4_{-4+\varepsilon}/\mathcal{H}^4_{-4-\varepsilon} \to r^{-1}dr \wedge (\mathrm{im}\Upsilon^3)^\perp + \mathrm{im}\Upsilon^4$ is surjective, and therefore an isomorphism. The reason is that in the proof of Lemma 3.2.11 we pass from a closed form $\xi = \chi\beta + \mathcal{O}(r^{-4-\varepsilon})$ to a cohomologous closed and co-closed form $\gamma = \xi + d\zeta$ for some $\zeta \in \Omega^3_{l+1,-3+\varepsilon}$. Note that $d(\chi \log r\alpha) = \chi(r^{-1}dr \wedge \alpha) + \mathcal{O}(r^{-4-\varepsilon})$ and $\chi \log r\alpha \in \Omega^3_{l+1,-3+\varepsilon}$. Hence by transitioning from ξ to γ a priori there could be introduced a 3-form α such that $\gamma = \chi(r^{-1}dr \wedge \alpha + \beta) + \mathcal{O}(r^{-4-\varepsilon})$. It will take significantly more effort to rule this out. This difficulty is unique to change in harmonic middle dimensional degree forms at the L^2-rate of even dimensional AC manifolds. On odd dimensional cones such as in the G_2-setting this difficulty does not appear.

Our idea to overcome this difficulty is to interpret $\mathcal{H}^4_{-4+\varepsilon}/\mathcal{H}^4_{-4-\varepsilon}$ as the kernel change of the elliptic operator

$$d + d^* : \Omega^{\mathrm{even}}_{l,-4\pm\varepsilon} \to \Omega^{\mathrm{odd}}_{l-1,-5\pm\varepsilon}.$$

This will allow us to instead compute the change in kernel at the critical rate $\lambda = -3$ of the adjoint operator

$$d + d^* : \Omega^{\mathrm{odd}}_{l,-3\mp\varepsilon} \to \Omega^{\mathrm{even}}_{l-1,-4\mp\varepsilon},$$

which is easier. In the following we make this idea precise.

Lemma 3.2.13. *We have*

$$\mathcal{H}^4_{-4+\varepsilon}/\mathcal{H}^4_{-4-\varepsilon} \cong \mathcal{H}^{\mathrm{even}}_{-4+\varepsilon}/\mathcal{H}^{\mathrm{even}}_{-4-\varepsilon}.$$

Proof. By Lemma 2.4.23 every degree component of a form in $\mathcal{H}^{\mathrm{even}}_{-4+\varepsilon}$ is closed and co-closed and therefore

$$\mathcal{H}^{\mathrm{even}}_{-4+\varepsilon}/\mathcal{H}^{\mathrm{even}}_{-4-\varepsilon} \cong \bigoplus_{k=0}^{4} \mathcal{H}^{2k}_{-4+\varepsilon}/\mathcal{H}^{2k}_{-4-\varepsilon}.$$

By Corollary 2.4.18 we have $\mathcal{K}_{\Lambda^k}(-4) = 0$ for $k = 0, 2, 6, 8$. Therefore by Proposition 3.2.6 $\mathcal{H}^k_{-4+\varepsilon}/\mathcal{H}^k_{-4-\varepsilon} = 0$ for $k = 0, 2, 6, 8$. The statement follows. □

If we consider forms of degrees 3 and 5 at rate $\lambda = -3$, we don't have the problem described in Remark 3.2.12.

Lemma 3.2.14. *(i) The map T^3_{-3} takes values in* $\mathrm{im}\Upsilon^3$ *and*

$$T^3_{-3} \colon \mathcal{H}^3_{-3+\varepsilon}/\mathcal{H}^3_{-3-\varepsilon} \to \mathrm{im}\Upsilon^3$$

is an isomorphism.

(ii) The map T^5_{-3} takes values in $rdr \wedge (\mathrm{im}\Upsilon^4)^\perp$ and

$$T^5_{-3} \colon \mathcal{H}^5_{-3+\varepsilon}/\mathcal{H}^5_{-3-\varepsilon} \to rdr \wedge (\mathrm{im}\Upsilon^4)^\perp$$

is an isomorphism.

Proof. (i) By Corollary 2.4.18 we have $\mathcal{K}_{\Lambda^3}(-3) = H^3(\Sigma, \mathbb{R})$. The proof that the image of T^3_{-3} is contained in $\mathrm{im}\Upsilon^3$ is analogous to the proof of Lemma 3.2.10. The proof that it is surjective onto $\mathrm{im}\Upsilon^3$ is analogous to the proof of Lemma 3.2.11.
(ii) The statement follows from (i) by applying the Hodge-$*$ operator. $\square$

We have

$$\mathcal{H}^3_{-3+\varepsilon}/\mathcal{H}^3_{-3-\varepsilon} \oplus \mathcal{H}^5_{-3+\varepsilon}/\mathcal{H}^5_{-3-\varepsilon} \subset \mathcal{H}^{\mathrm{odd}}_{-3+\varepsilon}/\mathcal{H}^{\mathrm{odd}}_{-3-\varepsilon}.$$

To conclude equality we would need that for any element in $\mathcal{H}^{\mathrm{odd}}_{-3+\varepsilon}$ all individual degree components are closed and co-closed. In general this statement starts to fail at rate -3 (see Lemma 2.4.23). However, because each individual degree component of the leading order term in $\mathcal{K}_{\mathrm{odd}}(-3)$ is closed and co-closed, we can improve Lemma 2.4.23 to rate $-3+\varepsilon$ for odd degree forms.

Lemma 3.2.15. *For all elements in $\mathcal{H}^{\mathrm{odd}}_{-3+\varepsilon}$ each individual degree component is closed and co-closed.*

Proof. Let $\gamma = \gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 \in \mathcal{H}^{\mathrm{odd}}_{-3+\varepsilon}$. By Theorem 2.3.27 there exist $\eta \in \mathcal{K}_{\mathrm{odd}}(-3)$ and $\hat{\gamma} \in \Omega^{\mathrm{odd}}_{l,-3-\varepsilon}$ such that

$$\gamma = \chi\eta + \hat{\gamma}.$$

Because each degree component of η is closed and co-closed on the cone by Lemma 2.4.10, by Lemma 2.3.5 $d\gamma_k \in \Omega^{k+1}_{l-1,4-\varepsilon}$ and $d^*_M\gamma_k \in \Omega^{k-1}_{l-1,-4-\varepsilon}$ for $k = 1,3,5,7$. Therefore, we can apply Proposition 2.3.13 to integrate by parts:

$$\|d\gamma_k\|^2_{L^2} + \|d^*\gamma_k\|^2_{L^2} = \langle d\gamma_k, d\gamma_k\rangle_{L^2} + \langle d^*_M\gamma_k, d^*_M\gamma_k\rangle_{L^2} = \langle \Delta\gamma_k, \gamma_k\rangle_{L^2} = 0.$$

□

Corollary 3.2.16. *We have*

$$\mathcal{H}^{\mathrm{odd}}_{-3+\varepsilon}/\mathcal{H}^{\mathrm{odd}}_{-3-\varepsilon} = \mathcal{H}^{3}_{-3+\varepsilon}/\mathcal{H}^{3}_{-3-\varepsilon} \oplus \mathcal{H}^{5}_{-3+\varepsilon}/\mathcal{H}^{5}_{-3-\varepsilon} = \mathrm{im}\Upsilon^3 + r dr \wedge (\mathrm{im}\Upsilon^4)^{\perp}.$$

Proof. This follows immediately from Lemmas 3.2.14 and 3.2.15. □

Proof of Theorem 3.2.8. By Lemma 3.2.13 we can compute the change in kernel of the operator $(d + d^*_M)^{\mathrm{even}}_{\lambda}$ at rate $\lambda = -4$ instead of the operator $d + d^*_M$ restricted to 4-forms. The kernel change of $(d+d^*)^{\mathrm{even}}_{\lambda}$ corresponds to the cokernel change of the operator $(d+d^*_M)^{\mathrm{odd}}_{\lambda}$ at rate $\lambda = -3$. Using Theorem 2.3.28 and Corollary 3.2.16 we have a complete understanding of that. In formulas:

$$\begin{aligned}\dim \mathcal{H}^4_{-4+\varepsilon} - \dim \mathcal{H}^4_{-4-\varepsilon} &= \dim \mathcal{H}^{\mathrm{even}}_{-4+\varepsilon} - \dim \mathcal{H}^{\mathrm{even}}_{-4-\varepsilon} \\ &= \dim\ker(d + d^*_M)^{\mathrm{even}}_{-4+\varepsilon} - \dim\ker(d + d^*_M)^{\mathrm{even}}_{-4-\varepsilon} \\ &= \dim \mathrm{Coker}(d + d^*_M)^{\mathrm{odd}}_{-3-\varepsilon} - \dim \mathrm{Coker}(d + d^*_M)^{\mathrm{odd}}_{-3+\varepsilon} \\ &= (\mathrm{ind}(d + d^*_M)^{\mathrm{odd}}_{-3+\varepsilon} - \mathrm{ind}(d + d^*_M)^{\mathrm{odd}}_{-3-\varepsilon}) - (\dim\ker(d + d^*_M)^{\mathrm{odd}}_{-3+\varepsilon} - \dim\ker(d + d^*_M)^{\mathrm{odd}}_{-3-\varepsilon}) \\ &= (\dim H^3(\Sigma, \mathbb{R}) + \dim H^4(\Sigma, \mathbb{R})) - (\dim \mathrm{im}\Upsilon^3 + \dim(\Upsilon^4)^{\perp}) \\ &= \dim(\mathrm{im}\Upsilon^3)^{\perp} + \dim \mathrm{im}\Upsilon^4.\end{aligned}$$

Therefore the injection

$$T^4_{-4} \colon \mathcal{H}^4_{-4+\varepsilon}/\mathcal{H}^4_{-4-\varepsilon} \to r^{-1} dr \wedge (\mathrm{im}\Upsilon^3)^{\perp} + \mathrm{im}\Upsilon^4.$$

from Lemma 3.2.10 is surjective. □

In the following we describe an alternative proof of the surjectivity in Proposition 3.2.8. Let β be the harmonic representative of any class in $\mathrm{im}\Upsilon^4$. To prove Theorem 3.2.8 it is enough to find $\xi \in \mathcal{C}^{\infty}_{-4-\varepsilon}(\Lambda^4)$ such that

$$(d + d^*)(\chi\beta) = (d + d^*)\xi. \tag{3.2.17}$$

Then $\gamma = \chi\beta - \xi \in \mathcal{H}^4_{-4+\varepsilon}$ and $\Upsilon^4[\gamma] = [\beta]$. By using linearity and the Hodge-$*$ operator we then can solve equation (3.2.17) if we replace β by any element in $r^{-1}dr \wedge (\mathrm{im}\Upsilon^3)^{\perp} + \mathrm{im}\Upsilon^4$.

The main ingredient in the alternative proof is Lemma 2.5.4. It allows to show that the obstructions to solve (3.2.17) vanish. Let m be an arbitrary integer at least 1. Even though $\chi\beta$ lies just in $\Omega^4_{m,-4+\varepsilon}$, we get an improved decay rate for $(d + d^*)(\chi\beta)$. Because β is closed and co-closed on the cone by Lemma 2.3.5 we have $(d + d^*)(\chi\beta) \in \Omega^{\mathrm{odd}}_{m-1,-5-\varepsilon}$

if we choose ε small enough such that $2\varepsilon < -\nu$. To sum up we consider the operator

$$d + d^* \colon \Omega^{\text{even}}_{m,-4-\varepsilon} \to \Omega^{\text{odd}}_{m-1,-5-\varepsilon}$$

and want to determine if $(d + d^*)(\chi\beta)$ is in the image. The adjoint operator is given by

$$d + d^* \colon \Omega^{\text{odd}}_{l,-3+\varepsilon} \to \Omega^{\text{even}}_{l-1,-4+\varepsilon}.$$

By the Fredholm alternative we need to show

$$\langle (d + d^*)(\chi\beta), \sigma\rangle_{L^2} = 0$$

for any $\sigma \in \mathcal{H}^{\text{odd}}_{-3+\varepsilon}$. By Proposition 3.2.6 and Corollary 2.4.18 we can write $\sigma = \sigma_+ + \sigma_-$, where

$$\sigma_+ = \chi(\zeta + r dr \wedge \eta)$$

with $\zeta \in \operatorname{im}\Upsilon^3$, $\eta \in (\operatorname{im}\Upsilon^4)^\perp$, and $\sigma_- \in \Omega^{\text{odd}}_{l,-3-\varepsilon}$. Standard integration by parts from Lemma 2.3.13 gives

$$\begin{aligned}
\langle (d + d^*)(\chi\beta), \sigma\rangle &= \langle (d + d^*)(\chi\beta), \sigma_+\rangle + \langle (d + d^*)(\chi\beta), \sigma_-\rangle \\
&= \langle (d + d^*)(\chi\beta), \sigma_+\rangle + \langle \chi\beta, (d + d^*)\sigma_-\rangle \\
&= \langle (d + d^*)(\chi\beta), \sigma_+\rangle - \langle \chi\beta, (d + d^*)\sigma_+\rangle.
\end{aligned}$$

Therefore it is enough to prove the integration by parts

$$\langle (d + d^*)(\chi\beta), \sigma_+\rangle = \langle \chi\beta, (d + d^*)\sigma_+\rangle.$$

Note that this does not merely follow from Lemma 2.3.13 because the smallest possible λ such that $\chi\beta \in L^2_{k,\lambda}(\Lambda^{\text{even}})$ is strictly greater than -4 and the smallest possible μ such that $\sigma_+ \in L^2_{k,\mu}(\Lambda^{\text{odd}})$ is strictly greater than -3, and hence the sum is strictly greater than -7. Therefore, we are in a situation in which Lemma 2.3.13 fails. The idea is to use Lemma 2.5.4 to adapt the proof of integration by parts to this situation. First note that we can write $*_M(\chi r dr \wedge \eta) = \chi *_\Sigma \eta + \omega$ with some $\omega \in \mathcal{C}^\infty_{-3+\nu}(\Lambda^3)$. We have

$$\begin{aligned}
&\langle d(\chi\beta), \sigma_+\rangle = \langle d(\chi\beta), \chi r dr \wedge \eta\rangle = \int_M (d(\chi\beta)) \wedge *(\chi r dr \wedge \eta)\\
&= \int_M d\big((\chi\beta) \wedge *(\chi r dr \wedge \eta)\big) - \int_M (\chi\beta) \wedge d * (\chi r dr \wedge \eta)\\
&= \int_M d\big((\chi\beta) \wedge (\chi * \eta + \omega)\big) + \int_M (\chi\beta) \wedge * d^*(\chi r dr \wedge \eta)\\
&= \lim_{r\to\infty} \int_{\{r\}\times\Sigma} \beta \wedge *\eta + \lim_{r\to\infty} \int_{\{r\}\times\Sigma} \beta \wedge \omega|_{\{r\}\times\Sigma} + \langle \chi\beta, d^*\sigma_+\rangle\\
&= \lim_{r\to\infty} r^7 \langle \beta, \eta\rangle_{L^2(\Sigma)} + \lim_{r\to\infty} \int_{\{r\}\times\Sigma} \beta \wedge \omega|_{\{r\}\times\Sigma} + \langle \chi\beta, d^*\sigma_+\rangle\\
&= \langle \chi\beta, d^*\sigma_+\rangle.
\end{aligned}$$

In the second to last line the first limit vanishes because $\beta \perp_{L^2} \eta$ and the second limit vanishes because the integrand decays with rate $-4-3+\nu < -7$.

$\langle d^*(\chi\beta), \sigma_+\rangle = \langle \chi\beta, d\sigma_+\rangle$ follows similarly.

3.2.2 Summary

We now summarise our results and give a precise formulation of Theorem A. In Proposition 3.1.15 we have seen that for generic rates in the non-L^2 regime the moduli space $\mathcal{M}_\nu$ is an orbifold and that infinitesimal deformations of a torsion-free AC Spin(7)-structure ψ are related to closed anti-self-dual 4-forms on (M, ψ). So far we have proven Proposition 3.2.1 in this section, showing that the jump of $(\mathcal{H}^4_{35})_\nu$ at a critical rate λ is given by $\mathcal{K}_{\mathrm{ASD}}(\lambda)$. Before we formulate our main theorem, we relate forms in $\mathcal{K}_{\mathrm{ASD}}(\lambda)$ to solutions of a differential equation purely on the link (Σ, φ_Σ) of the asymptotic cone.

Proposition 3.2.18. *Define*

$$\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda) := \{\zeta \in \Omega^3_{27}(\Sigma) |\ d\zeta = -(\lambda+4) * \zeta\}.$$

Then we have

$$\mathcal{K}_{\mathrm{ASD}}(\lambda) \cong \begin{cases} \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda) & \text{if} \lambda \in (-4, -1],\\ \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda) \oplus \mathcal{K}_{\Lambda^1}(\lambda+1) & \text{if} \lambda \in (-1, 0). \end{cases}$$

Proof. By Remark 2.4.2 and Lemma 2.4.17 any $\eta \in \mathcal{K}_{\mathrm{ASD}}(\lambda)$ is of the form

$$\eta = r^\lambda (r^3 dr \wedge \alpha + r^4(- * \alpha))$$

with

$$d\alpha = -(\lambda + 4) * \alpha. \tag{3.2.19}$$

The G_2-structure φ_Σ induces the decomposition $\Omega^3(\Sigma) = \Omega^3_1(\Sigma) \oplus \Omega^3_7(\Sigma) \oplus \Omega^3_{27}(\Sigma)$ of 3-forms which we can use to write

$$\alpha = \alpha_1 + \alpha_7 + \alpha_{27} = f\varphi_\Sigma + X \lrcorner * \varphi_\Sigma + \zeta, \tag{3.2.20}$$

where f is a function and X a vector field on Σ, and $\zeta \in \Omega^3_{27}(\Sigma)$. Our goal is to show that if $\lambda \in (-4, 1)$ then f and X have to vanish. The main idea to prove this is to interpret the condition $d\eta = 0$ on the cone as an equation involving the Laplace operator and use the fact that the Laplace operator preserves the type decomposition on the cone with respect to the Spin(7)-structure. More specifically, because $\lambda \neq -4$ we can write $\eta = d\beta$, where $\beta = \frac{1}{\lambda+4} r^{\lambda+4}\alpha$. The fact that η is a closed anti-self-dual 4-form on $C(\Sigma)$ homogeneous of rate λ implies that β is a harmonic 3-form on $C(\Sigma)$ homogeneous of rate $\lambda + 1$. Indeed by (2.4.1) β is co-closed because α is co-closed, and hence

$$\Delta\beta = dd^*\beta + d^*d\beta = d^*\eta = 0.$$

Next we relate the type decomposition (3.2.20) of α with respect to the G_2-structure on the link to the type decomposition of $\beta = \beta_8 + \beta_{48}$ with respect to the Spin(7)-structure on the cone. Because the decomposition is linear in α, we compute the contributions of $\alpha_1, \alpha_7, \alpha_{27}$ separately. We have

$$\frac{r^{\lambda+4}}{\lambda+4}\alpha_1 = \frac{r^{\lambda+4}}{\lambda+4} f\varphi_\Sigma = \frac{r^{\lambda+1}}{\lambda+4}(f\partial_r) \lrcorner \psi_C, \tag{3.2.21}$$

and α_1 only contributes to β_8.

Write $\pi_8(r^{\lambda+4}\alpha_7) = Y \lrcorner \psi$ for some vector field on $C(\Sigma)$ to be determined. By the computation 3.2.21 Y does not contain a ∂_r component. Therefore, we get

$$\begin{aligned}\pi_8(r^{\lambda+4}\alpha_7) &= -r^3 dr \wedge (Y \lrcorner \varphi_\Sigma) + r^4 Y \lrcorner * \varphi_\Sigma,\\ \pi_{48}(r^{\lambda+4}\alpha_7) &= r^{\lambda+4} X \lrcorner \psi_C - \pi_8(r^{\lambda+4}\alpha_7) = dr \wedge (r^3 Y \lrcorner \varphi_\Sigma) + (r^{\lambda+4}X - r^4 Y) \lrcorner * \varphi_\Sigma.\end{aligned}$$

The characterising equation $\pi_{48}(r^{\lambda+4}\alpha_7) \wedge \psi = 0$ for forms of type 48 leads to the equation

$$\begin{aligned}dr \wedge *\varphi_\Sigma \wedge (r^7 Y \lrcorner \varphi_\Sigma) + dr \wedge \varphi_\Sigma \wedge ((r^{\lambda+7}X - r^7 Y) \lrcorner * \varphi_\Sigma)\\ + * \varphi_\Sigma \wedge ((r^{\lambda+8}X - r^8 Y) \lrcorner * \varphi_\Sigma) = 0,\end{aligned}$$

which is equivalent to the system

$$*\varphi_\Sigma \wedge ((r^{\lambda+8}X - r^8Y)\lrcorner * \varphi_\Sigma) = 0,$$
$$*\varphi_\Sigma \wedge (r^7Y\lrcorner\varphi_\Sigma) + \varphi_\Sigma \wedge ((r^{\lambda+7}X - r^7Y)\lrcorner * \varphi_\Sigma) = 0.$$

By [Kar05, Lemma 2.2.3] the first equation is always true while the second equation is equivalent to

$$3 * Y^\flat - 4 * (r^\lambda X - Y)^\flat = 0,$$

which gives $Y = \frac{4}{7}r^\lambda X$.

Because $\zeta \wedge \varphi = 0$ and $\zeta \wedge *\varphi = 0$, we immediately get $\zeta \wedge \psi = 0$. Hence, α_{27} only contributes to β_{48}.

Adding the individual contributions gives $\beta_8 = Z\lrcorner\psi$ with

$$Z = \frac{r^\lambda}{\lambda+4}\left(rf\partial_r + \frac{4}{7}X\right).$$

Because the Laplace operator preserves type decompositions, we get $\Delta\beta_8 = 0$ and that $Z^\flat = \frac{r^{\lambda+1}}{\lambda+4}(fdr + \frac{4}{7}rX^\flat)$ is a homogeneous harmonic 1-form of rate $\lambda+1$. Here $Z^\flat$ is dual to Z on the cone and $X^\flat$ is dual to X on Σ. If $\lambda \leq -1$, then $Z^\flat$ must vanish by Lemma 2.4.19, and the statement follows in this case.

We are left to finish the proof for $\lambda \in (-1, 0)$. In this case $Z^\flat$ is closed and co-closed by Lemma 2.4.19. If $\alpha' = f\varphi + X\lrcorner*\varphi + \zeta'$ is another solution of (3.2.19), then $\zeta' - \zeta \in \mathcal{E}(\Sigma, \lambda)$. This proves

$$\dim(\mathcal{K}_{\mathrm{ASD}}(\lambda)/\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)) \leq \dim \mathcal{K}_{\Lambda^1}(\lambda+1). \tag{3.2.22}$$

By Lemma 2.4.20 $\beta_8 = Z^\flat\lrcorner\psi$ is in the kernel of $\not{D}_-$. Therefore, $d\beta_8$ lies in $\mathcal{K}_{\mathrm{ASD}}(\lambda)$. The equation $d\beta_8 = \mathcal{L}_Z\psi_C$ and Lemma 2.4.21 show that $d\beta_8$ is non-zero if Z is non-zero. This proves the reverse inequality of (3.2.22). □

It follows straight from the definition that all forms $\zeta \in \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ are co-closed eigenforms of the Hodge-Laplacian satisfying

$$\Delta\zeta = (\lambda+4)^2\zeta. \tag{3.2.23}$$

Similar to the Spin(7)-setting where trace-less symmetric 2-tensors can be identified with 4-forms of type 35, in the G_2-setting trace-less symmetric 2-tensors can be identified with 3-forms of type 27. We will denote this isomorphism by $\mathbf{i}\colon S^2_0(T^*\Sigma) \to \Lambda^3_{27}$. Even though the Laplace-operator does not preserve the type decompositions on (Σ, φ_Σ) because φ_Σ is not torsion-free, we can still relate the eigenforms $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ of the Hodge-Laplacian to eigenforms of the Lichnerowicz-Laplacian with a shift of the eigenvalue.

Proposition 3.2.24. *Suppose $\zeta \in \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$. Setting $u := \mathbf{i}^{-1}(\zeta)$ we have $\Delta_L u = (\lambda^2 + 6\lambda + 12)u$.*

Proof. The statement follows from [AS12, Proposition 6.1]: Alexandrov–Semmelmann derive the formula

$$\mathbf{i}\Delta\mathbf{i}^{-1}(\zeta) = \Delta\zeta - \frac{\tau_0}{2} * (d\zeta)_{\Lambda^4_7} + \frac{\tau_0}{2} * (d\zeta)_{\Lambda^4_{27}} + \frac{\tau_0^2}{4}\zeta,$$

where the constant τ_0 is given by the defining equation $d\varphi = \tau_0 * \varphi$ for the nearly parallel G_2-structure on Σ. Here one needs to take into account that in our chosen orientation both $*$ and τ_0 differ from those in [AS12] by a sign. Therefore, the formula carries over to our setting without changes. By (2.2.1) in our scale we have $\tau_0 = 4$. Because $d\zeta = -(\lambda + 4) * \zeta$ we have $(d\zeta)_{\Lambda^4_7} = 0$ and $(d\zeta)_{\Lambda^4_{27}} = d\zeta$. With (i) we get

$$\mathbf{i}\Delta\mathbf{i}^{-1}(\zeta) = (\lambda + 4)^2\zeta - 2(\lambda + 4)\zeta + 4\zeta = (\lambda^2 + 6\lambda + 12)\zeta.$$

□

On the Einstein manifold $(\Sigma, \varphi_\Sigma, g_\Sigma)$ with scalar curvature 42 the Lichnerowicz Laplacian on symmetric 2-tensors is given by

$$\Delta_L = \nabla^*\nabla - 2\mathring{R} + 12\,\mathrm{Id}.$$

The curvature operator $\mathring{R}$ acts on a symmetric 2-tensor h by

$$(\mathring{R}h)(X, Y) = \sum_{i,j} R(e_i, X, Y, e_j)h(e_i, e_j), \tag{3.2.25}$$

where $\{e_i\}$ is a local orthonormal frame. The operator

$$\Delta_E := \nabla^*\nabla - 2\mathring{R}$$

is called the *Einstein Laplacian* and (Σ, g_Σ) is linearly stable as a Einstein manifold if Δ_E is a non-negative operator. This implies

Corollary 3.2.26. *Suppose that the link (Σ, g_Σ) of the Spin(7)-cone $C := (C(\Sigma), \psi_C)$ is linearly stable as an Einstein manifold. Then $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda) = \{0\}$ for every $\lambda \in (-4, 0)$. In particular, every AC Spin(7)-manifold asymptotic to C has rate at most -4.*

Proof. The assumptions imply that the smallest positive eigenvalue of Δ_L is at least 12. But $\lambda^2+6\lambda+12$ takes values in $(3,12)$ for $\lambda \in (-4,0)$. The conclusion follows from Remark 3.1.14. $\square$

We can now summarise our results as

Theorem 3.2.27. *Let $C := (C(\Sigma), \psi_C)$ be a Spin(7)-cone, which is not isometric to Euclidean space, and (M,ψ) an AC Spin(7)-manifold asymptotic to C with rate ν. Suppose $\nu \in (-4,0)$ and that $\nu+1$ and ν are non-critical rates for the Laplace-operator on C.*

Then the moduli space $\mathcal{M}_\nu = \mathcal{X}_\nu/\mathcal{D}_{\nu+1}$ of all torsion-free AC Spin(7)-structures on M asymptotic to C with rate ν is an orbifold of dimension

$$\dim \mathcal{M}_\nu = \dim(\mathcal{H}^4_-)_{L^2} + \dim \operatorname{im} \Upsilon^4 + \sum_{\lambda \in \mathcal{D}(d+d^*)\cap(-4,\nu)} \dim \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda),$$

where

$$\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda) := \{\zeta \in \Omega^3_{27}(\Sigma) |\ d\zeta = -(\lambda+4) * \zeta\},$$

and Υ^4 is the restriction map (2.5.1)*.*

Furthermore, if the stabiliser of ψ in $\mathcal{D}_{\nu+1}$ is trivial or acts trivially in a neighbourhood of ψ in $\mathcal{X}_\nu$, the moduli space $\mathcal{M}_\nu$ is smooth in a neighbourhood of $\psi\mathcal{D}_{\nu+1}$. In particular, this is the case if the with respect to the type decomposition given by ψ the projection map

$$(\mathcal{H}^4_{35})_\nu \to H^4(M,\mathbb{R})$$

is injective.

Proof. The statement for $\nu \in (-4,-1]$ follows from Propositions 3.1.15, 3.2.1 and 3.2.18.

It is left to extend the statement to rates $\nu \in (-1,0)$ if (Σ, φ_Σ) is not the round 7-sphere. By Proposition 3.1.6 the exterior derivative is injective on $\ker(\not{D}_-)_{\nu+1}$. With Lemma 2.4.27, Theorem 2.3.28 and Lemma 2.4.20 we get

$$\begin{aligned}\dim d \ker(\not{D}_-)_{\nu+1} = \dim \ker(\not{D}_-)_{\nu+1} &= \sum_{\lambda \in \mathcal{D}(\not{D}_-)\cap(0,\nu+1)} \ker(\not{D}_-)_{\lambda+\varepsilon} - \ker(\not{D}_-)_{\lambda-\varepsilon} \\ = \sum_{\lambda \in \mathcal{D}(\not{D}_-)\cap(0,\nu+1)} \operatorname{ind}(\not{D}_-)_{\lambda+\varepsilon} - \operatorname{ind}(\not{D}_-)_{\lambda-\varepsilon} &= \sum_{\lambda \in \mathcal{D}(d+d^*)\cap(-1,\nu)} \dim \mathcal{K}_{\Lambda^1}(\lambda+1).\end{aligned}$$

With Proposition 3.2.18 we get

$$\begin{aligned}
&\dim \mathcal{M}_\nu = \dim(\mathcal{H}^4_{35})_\nu - \dim d\ker(\not{D}_-)_{\nu+1} \\
&= \dim(\mathcal{H}^4_-)_{L^2} + \dim \operatorname{im}\Upsilon^4 \\
&+ \sum_{\lambda\in\mathcal{D}(d+d^*)\cap(-4,-1]} \dim \mathcal{K}_{\mathrm{ASD}}(\lambda) + \sum_{\lambda\in\mathcal{D}(d+d^*)\cap(-1,\nu)} \dim \mathcal{K}_{\mathrm{ASD}}(\lambda) - \dim \mathcal{K}_{\Lambda^1}(\lambda+1) \\
&= \dim(\mathcal{H}^4_-)_{L^2} + \dim \operatorname{im}\Upsilon^4 + \sum_{\lambda\in\mathcal{D}(d+d^*)\cap(-4,\nu)} \dim \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda).
\end{aligned}$$

□

In the compact case the projection of the moduli space to $H^4(M)$ is an immersion [Joy00, Theorem 10.7.1]. In the AC case we can only prove this under the restriction that there are no critical rates in the interval $(-4,\nu)$, and in particular all Spin(7)-structures in $\mathcal{M}_\nu$ decay with rate -4.

Lemma 3.2.28. *Let $\nu \in (-4,-0)$ and suppose there is no critical rate in the interval $(-4,\nu]$. Then $\mathcal{M}_\nu$ is a smooth manifold and the map*

$$\begin{aligned}
\pi\colon \mathcal{M}_\nu &\to H^4(M), \\
\tilde{\psi}\mathcal{D}_{\nu+1} &\mapsto [\tilde{\psi}]
\end{aligned}$$

is an immersion.

Proof. By the assumption, equation (3.1.16) and Proposition 3.2.7 we have $(\mathcal{H}^4_{35})_\nu = (\mathcal{H}^4_{35})_{-4+\varepsilon}$ for some arbitrarily small $\varepsilon > 0$. We need to show that the projection

$$(\mathcal{H}^4_{35})_{-4+\varepsilon} \to H^4(M).$$

is injective. Assume that $\gamma \in (\mathcal{H}^4_{35})_{-4+\varepsilon}$ is exact. By Corollary 3.2.9 there exists $\beta \in \operatorname{im}\Upsilon^4$ and $\gamma_- \in \mathcal{C}^\infty_{-4-\varepsilon}(\Lambda^4 T^*M)$ such that

$$\gamma = \chi(r^{-1}dr \wedge (-*_\Sigma \beta) + \beta) + \gamma_-,$$

and $\Upsilon^4([\gamma]) = [\beta]$. By assumption $[\gamma] = 0$ and thus $\beta = 0$ and $\gamma = \gamma_- \in \mathcal{H}^4_{L^2}$. By Proposition 2.5.5 $[\gamma] = 0$ implies $\gamma = 0$. Therefore, the linearisation is injective and π is an immersion. □

3.3 Example: the Bryant–Salamon metric

The Bryant–Salamon metric on $\mathbf{S}_+(S^4)$ is a cohomogeneity one AC Spin(7) holonomy metric asymptotic with rate $-10/3$ to the cone over the "squashed" 7-sphere. In this section we will compute the contributions $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ to the moduli space following Alexandrov–Semmelmann [AS12] by using the fact that the squashed 7-sphere can be understood as a standard homogeneous space. We will briefly describe their method. Let G/H be a reductive 7-dimensional homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Denote by $\bar{\nabla}$ the canonical homogeneous connection on G/H. We say that G/H is a *naturally* reductive homogeneous space if the torsion tensor $T_{\bar{\nabla}}$ of $\bar{\nabla}$ is an alternating tensor, i.e. a 3-form. We are interested in the situation where G/H is equipped with a G-invariant nearly parallel G_2-structure φ_Σ such that $\varphi_\Sigma = \frac{2}{3} T_{\bar{\nabla}}$ (see [AS12, Lemma 7.1]). This allows Alexandrov–Semmelmann to relate the Laplacian

$$\bar{\Delta} = \bar{\nabla}^* \bar{\nabla} + q(\bar{R})$$

to the Laplacian with respect to the Levi-Civita connection. If $\zeta \in \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$, then similarly as in Proposition 3.2.24 in the eigenproblem (3.2.23) we get a shift of the eigenvalue with [AS12, Proposition 5.3]:

$$\bar{\Delta}\zeta = \Delta\zeta + \frac{2}{3} * d\zeta = \underbrace{\left((\lambda + 4)^2 - \frac{2}{3}(\lambda + 4)\right)}_{=:\mu} \zeta. \tag{3.3.1}$$

To compute the action of $\bar{\Delta}$ we need to make another restriction: we require that the Einstein metric induced by φ_Σ is *standard*. This means that it is induced by a negative multiple $-c^2 B$ of the Killing form B of $\mathfrak{g}$. The point of standard homogeneous spaces is that their curvature tensor with respect to the canonical homogeneous connection satisfies the same formula as the curvature tensor of symmetric spaces with respect to the Levi-Civita connection. Therefore, eigenproblems for $\bar{\Delta}$ can be solved with methods from representation theory. For a representation $\rho\colon H \to \mathrm{GL}(E)$ of H denote the associated vector bundle by $E_\rho = G \times_\rho E$. The left action of G on E_ρ induces a G-action $\ell\colon G \to \mathrm{GL}(\Gamma(E_\rho))$ on the space of sections of E_ρ. Then by [MS10, Lemma 5.2] the action of $\bar{\Delta}$ is given by

$$\bar{\Delta} = -\frac{1}{c^2} \mathrm{Cas}^G_\ell.$$

The Casimir operator acts on a G-representation $\gamma\colon G \to \mathrm{GL}(V)$ as

$$\mathrm{Cas}^G_\gamma = \sum_i (\gamma_* X_i)^2,$$

where γ_* denotes the induced action of the Lie algebra and X_i is an orthonormal basis of $\mathfrak{g}$ with respect to $-B$. The Peter–Weyl Theorem and the Frobenius reciprocity give an isomorphism

$$L^2(E_\rho) = \overline{\bigoplus_\gamma} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, E), \tag{3.3.2}$$

where γ runs over all isomorphism classes of irreducible representations V_γ of G. A section of E_ρ is the same as an H-invariant map $G \to E$. Under this identification an element $v \otimes A \in V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, E)$ gives rise to the section $g \mapsto A(\gamma(g^{-1})v)$. On each component $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, E)$ Cas^G_ℓ then acts as $\mathrm{Cas}^G_{V_\gamma}$. Therefore, the eigenspace of $\bar{\Delta}$ for the eigenvalue μ is isomorphic to the sum of the spaces $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, E)$ for which

$$\mathrm{Cas}^G_{V_\gamma} = -c^2\mu. \tag{3.3.3}$$

We can now compute $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ in two steps. Set $E = \Lambda^3_{27}\mathfrak{m}$ and suppose that in the orientation chosen by Alexandrov–Semmelmann $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ is characterised by solutions of the equation $\bar{d}\zeta + \bar{\lambda} * \zeta = 0$, where $\bar{d} = \mathrm{Alt} \circ \bar{\nabla}$ and $\bar{\lambda}$ is a constant related to λ, and each $\zeta \in \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ satisfies $\bar{\Delta}\zeta = \mu\zeta$. First, using (3.3.3) we determine all isomorphism classes of irreducible representations V_γ of G such that $\mathrm{Cas}^G_{V_\gamma} = -c^2\mu$. Secondly, having narrowed down the list of possible $V_\gamma \subset \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$, we need to solve the equation $\bar{d}\zeta + \bar{\lambda} * \zeta = 0$. All ζ in a subspace isomorphic to V_γ solving this equation is equivalent to the existence of $A \in \mathrm{Hom}_H(V_\gamma, E)$ such that (see [AS12, Equation (7.42)])

$$\sum_{1 \le i_1 < \cdots < i_4 \le 7} \sum_{j=1}^4 (-1)^j A(e_{i_j} \cdot v)(e_{i_1}, \ldots, \hat{e}_{i_j}, \ldots, e_{i_4}) e^{i_1 \ldots i_4} + \bar{\lambda} * A(v) = 0 \tag{3.3.4}$$

for all $v \in V_\gamma$, where $e_1, \ldots, e_7$ is a basis of $\mathfrak{m}$. With respect to the identification (3.3.2) V_γ is then embedded into $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ via

$$V_\gamma \to V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, E) \subset L^2(E_\rho), \quad v \mapsto v \otimes A.$$

Let us now apply this theory to compute the spaces $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ for the Bryant–Salamon metric. The "squashed" nearly parallel G_2-structure on S^7 is not naturally reductive if we write $S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$. However, it is both naturally reductive and standard if we write

$S^7 = \frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)_u\times\mathrm{Sp}(1)_d}$ [AS12, Example 8.2], where

$$\mathrm{Sp}(1)_u = \left\{ \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, 1 \right) : a \in \mathrm{Sp}(1) \right\}, \quad \mathrm{Sp}(1)_d = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, a \right) : a \in \mathrm{Sp}(1) \right\}.$$

This description leads to a nearly parallel G_2-structure satisfying $d\varphi = \tau_0 * \varphi$ with scalar curvature $\frac{21}{8}\tau_0^2$, where $\tau_0 = \frac{12}{\sqrt{5}}$. This means that we have to rescaled our original choice of φ by κ^3 and our original metric g_Σ by κ^2, where κ is given by the equation $\tau_0 = \frac{4}{\kappa}$. The eigenproblem (3.3.1) is replaced by

$$\bar{\Delta}\zeta = \kappa^{-2}\mu\zeta.$$

By [AS12, Lemma 7.1] c and τ_0 are related by $c^2 = \frac{6}{5\tau_0^2}$. In the light of equation (3.3.3) we need to determine those irreducible representations V_γ of $G = \mathrm{Sp}(2) \times \mathrm{Sp}(1)$ for which

$$\mathrm{Cas}_\gamma^G = -c^2\kappa^{-2}\mu = -\frac{6}{5\tau_0^2}\frac{\tau_0^2}{16}\mu = -\frac{3}{40}\mu. \tag{3.3.5}$$

Irreducible representations of $\mathrm{Sp}(2)$ are indexed by their highest weight $\gamma = (k_1, k_2), k_1 \geq k_2 \geq 0$, and irreducible representations of $\mathrm{Sp}(1)$ are indexed by their highest weight $\gamma = l, l \geq 0$. The Casimir operator is explicitly given by (see [AS12, p. 737])

$$\mathrm{Cas}_{V(k_1,k_2)\otimes V(l)}^{\mathrm{Sp}(2)\times\mathrm{Sp}(1)} = -\frac{1}{12}(4k_1 + k_1^2 + 2k_2 + k_2^2) - \frac{1}{8}(2l + l^2). \tag{3.3.6}$$

The eigenvalue μ in equation (3.3.1) takes values in $(-\frac{1}{9}, \frac{40}{3})$ for $\lambda \in (-4, 0)$. Therefore, by (3.3.5) we need to determine all $V(k_1, k_2, l) := V(k_1, k_2) \otimes V(l)$ such that

$$\mathrm{Cas}_{V(k_1,k_2)\otimes V(l)}^{\mathrm{Sp}(2)\times\mathrm{Sp}(1)} \in (-1, 0].$$

Using formula (3.3.6) we find that there are four possibilities: $(k_1, k_2, l) = (1, 1, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, $(0, 0, 0)$ with the Casimir operator equal to $-\frac{2}{3}, -\frac{5}{12}, -\frac{3}{8}, 0$, respectively. This will lead to the eigenvalues $\kappa^{-2}\mu = \frac{576}{25}, \frac{72}{5}, \frac{324}{25}, 0$, respectively.

Next we need to determine the corresponding Hom-spaces. If we denote the standard representations of $\mathrm{Sp}(1)_u$ and $\mathrm{Sp}(1)_d$ by U and D, then all irreducible representations of $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$ can be written via the symmetric powers as S^kUS^lW (omitting the tensor product sign and complexification sign). Then

$$\Lambda^3_{27}\mathfrak{m}^* \cong S^2US^2D \oplus US^3D \oplus UD \oplus S^4D \oplus \mathbb{C},$$
$$V(1,1,0) \cong \Lambda^2_0(\mathbb{C}^4)^* \cong UD \oplus \mathbb{C}, \quad V(1,0,0) \cong \mathbb{C}^4 \cong U \oplus D,$$
$$V(0,0,1) \cong D, \quad V(0,0,0) \cong \mathbb{C}.$$

$V(1,0,0)$ and $V(0,0,1)$ do not have common subrepresentations with $\Lambda^3_{27}\mathfrak{m}$, and therefore do not lead to any solutions.

$V(0,0,0)$ and $\Lambda^3_{27}\mathfrak{m}^*$ have the trivial representation $\mathbb{C}$ as a common component. We have $\mu = 0$, $\lambda = -10/3$, and $\zeta \in \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ satisfy $d\zeta = -\frac{2}{3} * \zeta$. Then we have $\bar{d}\zeta = d\zeta + \frac{2}{3} * \zeta = 0$ by [AS12, Lemma 5.2]. Therefore, we want to solve equation 3.3.4 with $\bar{\lambda} = 0$. This is trivially satisfied for the trivial representation. We get $\mathcal{E}(\Sigma, \varphi_\Sigma, -10/3) \cong \mathbb{R}$. The Bryant–Salamon metric has decay rate $-10/3$ and this is exactly the deformation given by rescaling as in Remark 3.1.14.

We are left to consider $V(1,1,0)$. Then $\mu = \frac{64}{5}$ and $\lambda = -\frac{-55+\sqrt{2905}}{15}$. Thus, at the scale of scalar curvature 42 and in our chosen orientation we want to solve $d\zeta = -\frac{5+\sqrt{2905}}{15} * \zeta$. Again by [AS12, Lemma 5.2] this is equivalent to $\bar{d}\zeta = d\zeta + \frac{2}{3} * \zeta = \frac{5-\sqrt{2905}}{15} * \zeta$, and at the scale with scalar curvature $\frac{21}{8}\tau_0^2$ and in the orientation chosen in [AS12] we want to solve

$$\big(\bar{d} + \tfrac{\sqrt{5}-\sqrt{581}}{5} *\big)\zeta = 0. \tag{3.3.7}$$

The common $\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d$-subrepresentation of $\Lambda^3_{27}\mathfrak{m}$ and $V(1,1,0)$ are UD and $\mathbb{C}$. Alexandrov–Semmelmann show that $\mathrm{Hom}_H(UD, \Lambda^3_{27}\mathfrak{m})$ is 1-dimensional. Furthermore, they show that UD can be identified with a submodule of $\mathfrak{sp}(2)$ and that a generator A of $\mathrm{Hom}_H(UD, \Lambda^3_{27}\mathfrak{m})$ satisfies with repsect to an orthonormal frame $e_1, \ldots, e_7$ of $\mathfrak{m}$

$$\begin{aligned}
A(e_1 \cdot e_4) &= -\tfrac{2}{\sqrt{5}}\big(3e^{467} + e^{137} + e^{126} + e^{234}\big),\\
A(e_2 \cdot e_4) &= -\tfrac{2}{\sqrt{5}}\big(-3e^{457} + e^{237} - e^{125} - e^{134}\big),\\
A(e_3 \cdot e_4) &= -\tfrac{2}{\sqrt{5}}\big(3e^{456} - e^{236} - e^{135} + e^{124}\big),\\
A(e_4 \cdot e_4) &= 0,\\
A(e_4) &= -3e^{567} - e^{235} + e^{136} - e^{127}.
\end{aligned}$$

Therefore, if in (3.3.4) we set $\bar{\lambda} = \frac{\sqrt{5}-\sqrt{581}}{5}$ and $v = e_4$, the coefficient of e^{1234} on the left-hand side is $3\frac{2}{\sqrt{5}} - 3\bar{\lambda} = 3\frac{\sqrt{5}+\sqrt{581}}{5} \neq 0$. Therefore, the common subrepresentation UD does not lead to any solutions. For the trivial representation formula (3.3.4) simplifies to

$$\bar{\lambda} * A(v) = 0$$

for all $v \in \mathbb{C}$. This implies $A = 0$. Again we get no solution. Therefore, $V(1,1,0)$ does not lead to any infinitesimal Spin(7)-deformations.

The computations in this section allow us to determine the dimension of the moduli space with Theorem 3.2.27.

Corollary 3.3.8. *The Bryant–Salamon AC Spin(7) holonomy metric on $\mathbf{S}_+(S^4)$ is locally rigid, modulo scaling, as a torsion-free AC Spin(7)-structure on $\mathbf{S}_+(S^4)$ asymptotic to the cone over the "squashed" 7-sphere, up to any rate $\nu < 0$.*

Proof. So far we have shown that the spaces $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$, $\lambda \in (-4, 0)$, vanish unless $\lambda = -10/3$. $\mathcal{E}(\Sigma, \varphi_\Sigma, -10/3)$ is 1-dimensional and the corresponding deformation is the scaling described in Remark 3.1.14. By Proposition 2.5.5 and the long exact sequence (2.5.2) we have

$$\mathcal{H}^4_{L^2} \cong \mathcal{I}^4(H^4_{\mathrm{cs}}(\mathbf{S}_+(S^4), \mathbb{R})) \cong H^4(\mathbf{S}_+(S^4), \mathbb{R}) \cong H^4(S^4, \mathbb{R}) \cong \mathbb{R}.$$

Cvetič–Lü–Pope [CLP01, Section 5.2] have constructed a square-integrable 4-form on $\mathbf{S}_+(S^4)$, which is harmonic with respect to the Bryant–Salamon metric and has the same duality as the Spin(7) 4-form (which is anti-self-dual in their convention and self-dual in our convention). Therefore, $(\mathcal{H}^4_-)_{L^2} = \{0\}$. Finally we have $\operatorname{im}\Upsilon^4 \subset H^4(S^7, \mathbb{R}) = \{0\}$. By Theorem 3.2.27 the moduli space $\mathcal{M}_\nu$ is 1-dimensional for any $\nu \in (-10/3, 0)$. □

3.4 AC Calabi–Yau 4-folds

In this section we apply our results to asymptotically conical Calabi–Yau manifolds of real dimension 8. These carry a torsion-free SU(4)-structure, and by the inclusion 2.1.17 form a subclass of AC Spin(7)-manifolds. Suppose that the Spin(7)-cone $C := (C(\Sigma), \psi_C, g_C)$ is a Calabi–Yau cone, i.e. there are dilation invariant $J_C, \omega_C = g_C(J_C\cdot, \cdot), \theta_C$ which satisfy the algebraic constraint equations (2.1.11) and the condition for torsion-freeness (2.1.12), and thus induce a torsion-free conical Spin(7)-structure ψ_C via the formula

$$\psi_C = \frac{1}{2}\omega_C^2 + \operatorname{Re}\theta_C.$$

(M, ω, θ) is an AC Calabi–Yau 4-fold asymptotic to $(C(\Sigma), \omega_C, \theta_C)$ with decay rate $\nu < 0$ if ω and θ are closed and satisfy decay conditions as in Definition 2.2.2. Then the Spin(7)-structure $\psi = \frac{1}{2}\omega^2 + \operatorname{Re}\theta$ is AC asymptotic to ψ_C with rate ν.

The conical SU(4)-structure induces the extra structure of a Sasaki–Einstein manifold on the link of the cone, the nearly parallel G_2-manifold $(\Sigma, \varphi_\Sigma, g_\Sigma)$. In the following we briefly review Sasaki–Einstein manifolds. Good references for this section are [BG00] and

[Spa11]. Because the cone is Kähler, Σ carries a Sasaki structure, the odd dimensional cousin of Kähler structures, which is compatible with g_Σ. The 1-form

$$\eta = J_C(r^{-1}dr)$$

induces a contact structure on Σ. The restriction of the vector field

$$\xi = J_C(r\partial_r)$$

to $\{1\} \times \Sigma$ is a unit length Killing vector field for g_Σ. Furthermore, ξ is metric-dual to η, and thus a Reeb vector field for the contact structure. ξ spans a line bundle $V \subset T\Sigma$ with orthogonal complement H, i.e.

$$T\Sigma = V \oplus H, \tag{3.4.1}$$

under which the metric splits as

$$g_\Sigma = \eta \otimes \eta + g_H.$$

By definition J_C preserves $\operatorname{span}\{\partial_r, \xi\}$, and therefore induces an almost complex structure Φ on H which is explicitly given by

$$\Phi(X) = \nabla_X \xi.$$

Φ preserves g_H and induces a Kähler form ω_H on H. Finally, θ_C induces a complex volume form θ_H on H. The triple $(g_\Sigma, \Phi, \theta_H)$ reduces the structure group of Σ to SU(3). The SU(3)-structure on Σ is related to the G_2-structure on Σ by

$$\begin{aligned} \varphi_\Sigma &= \eta \wedge \omega_H + \operatorname{Re}\theta_H, \\ *\varphi_\Sigma &= \frac{1}{2}\omega_H^2 - \eta \wedge \operatorname{Im}\theta_H, \end{aligned}$$

and to the conical SU(4)-structure by

$$\begin{aligned} g_C &= dr^2 + r^2\eta^2 + r^2 g_H, \\ \omega_C &= r dr \wedge \eta + r^2\omega_H, \\ \theta_C &= r^3(dr + ir\eta) \wedge \theta_H. \end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
\operatorname{Re}\theta_C &= r^3(dr\wedge \operatorname{Re}\theta_H - r\eta\wedge \operatorname{Im}\theta_H),\\
\operatorname{Im}\theta_C &= r^3(dr\wedge \operatorname{Im}\theta_H + r\eta\wedge \operatorname{Re}\theta_H),\\
\psi_C &= \frac{1}{2}\omega_C^2 + \operatorname{Re}\theta_C\\
&= r^3 dr\wedge\eta\wedge\omega_H + r^3 dr\wedge \operatorname{Re}\theta_H + r^4(\tfrac{1}{2}\omega_H^2 - \eta\wedge \operatorname{Im}\theta_H).
\end{aligned}
$$

Under the reduction of the structure group of Σ to SU(3), the decomposition of tensors given by the G_2-structure φ_Σ splits even further. We will discuss those relevant to us. The bundle of trace-less symmetric 2-tensors splits as

$$
S_0^2(T^*\Sigma) \cong \operatorname{span}\{\tfrac{7}{6}\eta\otimes\eta - \tfrac{1}{6}g_\Sigma\} \oplus (V^*\otimes H^*) \oplus S_0^2(H^*). \tag{3.4.2}
$$

The bundle of 3-forms splits as

$$
\Lambda^3 T^*\Sigma = \Lambda^3_{1\oplus 1}H \oplus \Lambda^3_6 H \oplus \Lambda^3_{12}H \oplus (V^*\otimes \Lambda^2_1 H) \oplus (V^*\otimes\Lambda^2_6) \oplus (V^*\otimes \Lambda^2_8 H)
$$

in irreducible components with a similar decomposition for 4-forms given by the Hodge star operator. Here

$$
\begin{aligned}
\Lambda^3_{1\oplus 1}H &= \operatorname{span}\{\operatorname{Re}\theta_H, \operatorname{Im}\theta_H\},\\
\Lambda^3_6 H &= \omega_H\wedge\Lambda^1 H^*,\\
\Lambda^3_{12}H &= \{\rho\in\Lambda^3 H^* \,|\, \rho \text{ is primitive of type } (2,1)+(1,2)\},\\
\Lambda^2_1 H &= \operatorname{span}\{\omega_H\},\\
\Lambda^2_6 H &= \{X\lrcorner \operatorname{Re}\theta_H \,|\, X\in H\},\\
\Lambda^2_8 H &= \{\sigma|\, \sigma \text{ is primitive of type } (1,1)\}.
\end{aligned}
$$

For more details on the decomposition of real differential forms with respect to an SU(3)-structure we refer to [Fos16, Section 2.2] and [MNS08, Section 2].

This splitting allows us to give a refined description of the space $\mathcal{E}(\Sigma,\varphi_\Sigma,\lambda)$ on Sasaki–Einstein 7-manifolds. Using the description of elements in $\mathcal{E}(\Sigma,\varphi_\Sigma,\lambda)$ as trace-less symmetric 2-tensors which solve the eigenvalue problem from Proposition 3.2.24 for the Lichnerowicz Laplacian, we can exclude certain tensors on which the Einstein Laplacian is non-negative by an explicit computation of the curvature operator (3.2.25).

Lemma 3.4.3. *The curvature operator* $\overset{\circ}{R}$ *acts on the bundle*

$$SV := \operatorname{span}\{\tfrac{7}{6}\eta\otimes\eta - \tfrac{1}{6}g_\Sigma\} \oplus (V^*\otimes H^*)$$

as minus the identity. In particular, under the identification of Λ^3_{27} *and* $S^2_0(T^*\Sigma)$ *from Proposition 3.2.24* $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ *does not contain pure sections of* SV *if* $\lambda \in (-4, 0)$.

Proof. The main ingredient in the proof is the fact that the curvature tensor of Sasaki manifolds simplifies, which is one of the common features of Kähler and Sasakian geometry. For vector fields $X, Y \in \Gamma(T\Sigma)$ we have (see [BG00, Proposition 2.1.2 (ii)])

$$R(\xi, X)Y = g_\Sigma(X,Y)\xi - g_\Sigma(\xi, Y)X. \tag{3.4.4}$$

With this formula we can compute the action of $\overset{\circ}{R}$ on the individual components separately. Set $e_0 = \xi$ and choose a local orthonormal frame $e_1, \ldots, e_6$ of H. Then $e_0, \ldots, e_6$ is a local orthonormal frame of g_Σ, and we have

$$\begin{aligned}
\overset{\circ}{R}(\eta\otimes\eta)(X,Y) &= \sum_{i,j=0}^{6} R(e_i, X, Y, e_j)(\eta\otimes\eta)(e_i, e_j)\\
&= g_\Sigma(R(\xi, X)Y, \xi)\\
&= g_\Sigma(g_\Sigma(X,Y)\xi - g_\Sigma(\xi,Y)X, \xi)\\
&= g_\Sigma(X,Y) - g_\Sigma(\xi, X)g_\Sigma(\xi, Y)\\
&= (g_\Sigma - \eta\otimes\eta)(X,Y).\\
\overset{\circ}{R}g_\Sigma(X,Y) &= \sum_{i,j=0}^{6} R(e_i, X, Y, e_j)g_\Sigma(e_i, e_j)\\
&= \sum_{i=0}^{6} R(e_i, X, Y, e_i) = Ric(X,Y) = 6g_\Sigma(X,Y).
\end{aligned}$$

Combining the above computations gives

$$\overset{\circ}{R}\left(\tfrac{7}{6}\eta\otimes\eta - \tfrac{1}{6}g_\Sigma\right) = \tfrac{7}{6}(g_\Sigma - \eta\otimes\eta) - g_\Sigma = -\tfrac{7}{6}\eta\otimes\eta + \tfrac{1}{6}g_\Sigma.$$

On $V^* \otimes H^*$ a similar computation gives

$$\begin{aligned}
\mathring{R}h(X,Y) &= \sum_{i,j=0}^{6} R(e_i, X, Y, e_j)h(e_i, e_j) \\
&= \sum_{i=1}^{6} R(e_i, X, Y, \xi)h(e_i, \xi) + R(\xi, X, Y, e_i)h(\xi, e_i) \\
&= \sum_{i=1}^{6} (g_\Sigma(R(\xi, X)Y, e_i) + g_\Sigma(R(\xi, Y)X, e_i))h(\xi, e_i) \\
&= \sum_{i=1}^{6} g_\Sigma(2g_\Sigma(X,Y)\xi - g_\Sigma(\xi, X)Y - g_\Sigma(\xi, Y)X, e_i)h(\xi, e_i) \\
&= \sum_{i=1}^{6} -(g_\Sigma(\xi, X)g_\Sigma(Y, e_i) + g_\Sigma(\xi, Y)g_\Sigma(X, e_i))h(\xi, e_i).
\end{aligned}$$

Setting $(X,Y) = (\xi,\xi)$ or (e_k, e_l) for $k, l = 1, \ldots, 6$ gives zero. Setting $(X,Y) = (\xi, e_k)$ for $k = 1, \ldots, 6$ gives

$$\mathring{R}h(\xi, e_k) = -h(\xi, e_k).$$

This proves the first statement, which implies that the Einstein-Laplacian Δ_Σ is a non-negative operator on sections of SV. The rest follows as in the proof of Corollary 3.2.26. □

Because the nearly parallel G_2-structure φ_Σ is not parallel, the Levi-Civita connection does not preserve the decomposition

$$S^2_0(T^*\Sigma) \cong SV \oplus S^2_0(H^*). \tag{3.4.5}$$

Therefore, Proposition 3.4.3 only allows us to exclude pure sections of SV in $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$, but not mixed ones. On each nearly parallel G_2-manifold there exists a unique metric connection $\bar{\nabla}$ with totally skew-symmetric torsion and holonomy contained in G_2, and Alexandrov–Semmelmann [AS12, Proposition 5.3] explain that an eigenvalue problem for the Lichnerowicz Laplacian Δ_L corresponds to an eigenvalue problem for the Lichnerowicz Laplacian $\bar{\Delta}_L$ with respect to $\bar{\nabla}$ with a shifted eigenvalue. However, in the case of Sasaki–Einstein manifolds the holonomy of $\bar{\nabla}$ does not reduce further to $\mathrm{SU}(3)$ [Fri07, Proposition 3.1]. Therefore, $\bar{\Delta}_L$ does not preserve the $\mathrm{SU}(3)$-decomposition (3.4.5) either.

For simplicity in the following we restrict attention to elements in $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ which correspond to pure sections of $S^2_0(H^*)$. Under the isomorphism $\Lambda^3_{27} \cong S^2_0(T^*\Sigma)$ we have

$$S^2_0(H^*) \cong (V^* \otimes \Lambda^2_8 H) \oplus \Lambda^3_{12} H.$$

We will use the splitting $d = d_f + d_H$ of the exterior derivative on horizontal differential forms, where

$$d_f = \eta \wedge \mathcal{L}_\xi \colon \mathcal{C}^\infty(\Lambda^k H) \to \mathcal{C}^\infty(V^* \otimes \Lambda^k H)$$

is the exterior derivative on the fibres and

$$d_H \colon \mathcal{C}^\infty(\Lambda^k H) \to \mathcal{C}^\infty(\Lambda^{k+1} H)$$

is the horizontal exterior derivative.

Lemma 3.4.6. *Let $\sigma \in \mathcal{C}^\infty(\Lambda^2_8 H)$, $\rho \in \mathcal{C}^\infty(\Lambda^3_{12} H)$. Then*

$$\zeta = \eta \wedge \sigma + \rho \tag{3.4.7}$$

is an element of $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ if and only if (σ, ρ) solves the system

$$d_H \rho = (\lambda + 2)\, \omega_H \wedge \sigma, \tag{3.4.8a}$$

$$-d_H \sigma + \mathcal{L}_\xi \rho = (\lambda + 4) *_H \rho. \tag{3.4.8b}$$

ζ induces the infinitesimal Spin(7)-deformation

$$r^{\lambda+3} dr \wedge (\eta \wedge \sigma + \rho) + r^{\lambda+4}(\omega_H \wedge \sigma + \eta \wedge *_H \rho) \in \mathcal{K}_{\mathrm{ASD}}(\lambda). \tag{3.4.9}$$

Proof. To describe the characterising differential equation $d\zeta = -(\lambda+4)*\zeta$ we first compute the Hodge star of ζ. We have $*_H \sigma = -\omega_H \wedge \sigma$, see [MNS08, Equation (19)]. Hence we get

$$*_\Sigma(\eta \wedge \sigma + \rho) = *_H \sigma + *_H \rho \wedge \eta = -\omega_H \wedge \sigma - \eta \wedge *_H \rho.$$

Using $\frac{1}{2} d\eta = \omega_H$ (see [BG00, Proposition 2.1.3 (iv)]) we have

$$d(\eta \wedge \sigma + \rho) = 2\omega_H \wedge \sigma - \eta \wedge d\sigma + d\rho.$$

Therefore, the equation $d(\eta \wedge \sigma + \rho) = -(\lambda + 4) *_\Sigma (\eta \wedge \sigma + \rho)$ splits into the system (3.4.8). □

In the following Proposition we pursue the question whether integrable infinitesimal deformations in $\mathcal{E}(\Sigma, \varphi_\Sigma, \lambda), \lambda \in (-4, 0)$, of the form (3.4.7) of the AC Calabi–Yau 4-fold as a Spin(7)-structure at the leading order term come from deformations of the underlying SU(4)-structure.

Proposition 3.4.10. *Suppose that $\zeta = \eta \wedge \sigma + \rho \in \mathcal{E}(\Sigma, \varphi_\Sigma, \lambda)$ with $\sigma \in \mathcal{C}^\infty(\Lambda^2_8 H)$ and*

$\rho \in \mathcal{C}^\infty(\Lambda^3_{12}H)$ at leading order induces the infinitesimal Spin(7)-deformation $\dot{\psi}$ of the AC Calabi–Yau structure (ω, θ). Then at leading order the Spin(7)-deformation is induced by an infinitesimal SU(4)*-deformation $(\dot{\omega}, \dot{\theta})$ with*

$$\dot{\omega} = r^{\lambda+2}\sigma + \mathcal{O}(r^{\lambda-\varepsilon}), \quad \operatorname{Re}\dot{\theta} = r^{\lambda+3}dr \wedge \rho + r^{\lambda+4}\eta \wedge *_H\rho + \mathcal{O}(r^{\lambda-\varepsilon}).$$

At leading order $(\dot{\omega}, \dot{\theta})$ is an infinitesimal deformation of torsion-free SU(4)*-structures if and only if (σ, ρ) solves the following system:*

$$d_H\rho = 0, \tag{3.4.11a}$$

$$\mathcal{L}_\xi\rho = (\lambda+4) *_H \rho, \tag{3.4.11b}$$

$$(\lambda+2)\sigma = 0, \tag{3.4.11c}$$

$$d_\Sigma\sigma = 0. \tag{3.4.11d}$$

In particular, σ must vanish if $\lambda \neq -2$.

Proof. Taking the derivatives of equations (2.1.11) and (2.1.17) with respect to a 1-parameter family of SU(4)-structures, we see that an infinitesimal deformation $(\dot{\omega}, \dot{\theta})$ of an AC SU(4)-structure satisfies the constraint equations

$$\dot{\omega} \wedge \theta + \omega \wedge \dot{\theta} = 0, \tag{3.4.12a}$$

$$\tfrac{1}{3!}\dot{\omega} \wedge \omega^3 = c_4(\dot{\theta} \wedge \bar{\theta} + \theta \wedge \dot{\bar{\theta}}) \tag{3.4.12b}$$

and induces the infinitesimal Spin(7)-deformation

$$\dot{\omega} \wedge \omega + \operatorname{Re}\dot{\theta}. \tag{3.4.13}$$

Writing $\dot{\omega} = \omega' + \mathcal{O}(r^{\lambda-\varepsilon})$, $\dot{\theta} = \theta' + \mathcal{O}(r^{\lambda-\varepsilon})$ and $\dot{\psi} = \psi' + \mathcal{O}(r^{\lambda-\varepsilon})$, at leading order the equations (3.4.12) and (3.4.13) are equivalent to

$$\omega' \wedge \theta_C + \omega_C \wedge \theta' = 0, \tag{3.4.14a}$$

$$\tfrac{1}{3!}\omega' \wedge \omega_C^3 = 2c_4\operatorname{Re}(\theta' \wedge \bar{\theta}_C) = 2c_4(\operatorname{Re}\theta' \wedge \operatorname{Re}\theta_C + \operatorname{Im}\theta' \wedge \operatorname{Im}\theta_C), \tag{3.4.14b}$$

$$\omega' \wedge \omega_C + \operatorname{Re}\theta' = \psi'. \tag{3.4.14c}$$

Writing

$$
\begin{aligned}
\omega' &= r^{\lambda+1} dr \wedge (\alpha_\omega \eta + \beta_\omega) + r^{\lambda+2}(\eta \wedge \gamma_\omega + \phi_\omega), \\
\operatorname{Re}\theta' &= r^{\lambda+3} dr \wedge (\eta \wedge \alpha_+ + \beta_+) + r^{\lambda+4}(\eta \wedge \gamma_+ + \phi_+), \\
\operatorname{Im}\theta' &= r^{\lambda+3} dr \wedge (\eta \wedge \alpha_- + \beta_-) + r^{\lambda+4}(\eta \wedge \gamma_- + \phi_-), \\
\psi' &= r^{\lambda+3} dr \wedge (\eta \wedge \alpha + \beta) + r^{\lambda+4}(\eta \wedge \gamma + \phi)
\end{aligned}
$$

where $\alpha_\omega \in \Omega^0(H), \beta_\omega, \gamma_\omega \in \Omega^1(H), \phi_\omega, \alpha_\pm, \alpha \in \Omega^2(H), \beta_\pm, \gamma_\pm, \beta, \gamma \in \Omega^3(H)$ and $\phi_\pm, \phi \in \Omega^4(H)$, we can reduce the system (3.4.14) to a system for horizontal differential forms.

We list some intermediate steps in the calculation: We have

$$
\begin{aligned}
\omega' \wedge \operatorname{Re}\theta_C =& r^{\lambda+5} dr \wedge (\eta \wedge (\beta_\omega \wedge \operatorname{Im}\theta_H + \gamma_\omega \wedge \operatorname{Re}\theta_H) + \phi_\omega \wedge \operatorname{Re}\theta_H) \\
&+ r^{\lambda+6} \eta \wedge (-\phi_\omega \wedge \operatorname{Im}\theta_H), \\
\omega_C \wedge \operatorname{Re}\theta' =& r^{\lambda+5} dr \wedge (\eta \wedge (\omega_H \wedge \alpha_+ + \phi_+) + \omega_H \wedge \beta_+) \\
&+ r^{\lambda+6}(\eta \wedge (\omega_H \wedge \gamma_+) + \omega_H \wedge \phi_+, \\
\omega' \wedge \operatorname{Im}\theta_C =& r^{\lambda+5} dr \wedge (\eta \wedge (-\beta_\omega \wedge \operatorname{Re}\theta_H + \gamma_\omega \wedge \operatorname{Im}\theta_H) + \phi_\omega \wedge \operatorname{im}\theta_H) \\
&+ r^{\lambda+6} \eta \wedge (\phi_\omega \wedge \operatorname{Re}\theta_H), \\
\omega_C \wedge \operatorname{Im}\theta' =& r^{\lambda+5} dr \wedge (\eta \wedge (\phi_- + \omega_H \wedge \alpha_-) + \omega_H \wedge \beta_-) \\
&+ r^{\lambda+6}(\eta \wedge \omega_H \wedge \gamma_- + \omega_H \wedge \phi_-) \\
\omega' \wedge \omega_C =& r^{\lambda+3} dr \wedge (\eta \wedge (\alpha_\omega \omega_H + \phi_\omega) + \omega_H \wedge \beta_\omega) \\
&+ r^{\lambda+4}(\eta \wedge \omega_H \wedge \gamma_\omega + \omega_H \wedge \phi_\omega), \\
\omega' \wedge \omega_C^3 =& r^{\lambda+7} dr \wedge \eta \wedge (\alpha_\omega \omega_H^3 + 3\,\omega_H^2 \wedge \phi_\omega), \\
\operatorname{Re}\theta' \wedge \operatorname{Re}\theta_C =& r^{\lambda+7} dr \wedge \eta \wedge (\beta_+ \wedge \operatorname{Im}\theta_H + \gamma_+ \wedge \operatorname{Re}\theta_H), \\
\operatorname{Im}\theta' \wedge \operatorname{Im}\theta_C =& r^{\lambda+7} dr \wedge \eta \wedge (-\beta_- \wedge \operatorname{Re}\theta_H + \gamma_- \wedge \operatorname{Im}\theta_H).
\end{aligned}
$$

The real part of (3.4.14a) and equation (3.4.14c) are equivalent to the system

$$\begin{aligned}\beta_\omega \wedge \operatorname{Im}\theta_H + \gamma_\omega \wedge \operatorname{Re}\theta_H + \phi_+ + \omega_H \wedge \alpha_+ &= 0,\\ \phi_\omega \wedge \operatorname{Re}\theta_H + \omega_H \wedge \beta_+ &= 0,\\ -\phi_\omega \wedge \operatorname{Im}\theta_H + \omega_H \wedge \gamma_+ &= 0,\\ \omega_H \wedge \phi_+ &= 0,\\ \alpha_\omega \omega_H + \phi_\omega + \alpha_+ &= \alpha,\\ \omega_H \wedge \beta_\omega + \beta_+ &= \beta,\\ \omega_H \wedge \gamma_\omega + \gamma_+ &= \gamma,\\ \omega_H \wedge \phi_\omega + \phi_+ &= \phi,\end{aligned}$$

which determines ω' and $\operatorname{Re}\theta'$. To solve for ψ' from (3.4.9) we need to specialise to $\alpha = \sigma$, $\beta = \rho$, $\gamma = *_H\rho$ and $\phi = \sigma \wedge \omega_H$. We find that the general solution is

$$\alpha_\omega = 0, \quad \beta_\omega = Z, \quad \gamma_\omega = JZ, \quad \phi_\omega = Z \lrcorner \operatorname{Re}\theta_H + \sigma,$$
$$\alpha_+ = -Z \lrcorner \operatorname{Re}\theta_H, \quad \beta_+ = -Z \wedge \omega_H + \rho, \quad \gamma_+ = -JZ \wedge \omega_H + *_H\rho, \quad \phi_+ = JZ \wedge \operatorname{Re}\theta_H,$$

where Z is a horizontal vector field. We will restrict our attention to $Z = 0$, i.e.

$$\omega' = r^{\lambda+2}\sigma, \quad \operatorname{Re}\theta' = r^{\lambda+3} dr \wedge \rho + r^{\lambda+4}\eta \wedge *_H\rho.$$

We will use that for $\sigma \in \Gamma(\Lambda^2_8 H)$ and $\rho \in \Gamma(\Lambda^3_{12})$ we have $\sigma \wedge \omega_H^2 = 0$, $\sigma \wedge \theta_H = 0$, $\rho \wedge \omega_H = 0$ and $\rho \wedge \theta_H = 0$. Furthermore, we also have $*_H\rho \in \Gamma(\Lambda^3_{12}H)$.

The imaginary part of (3.4.14a) now simplifies to

$$\phi_- + \omega_H \wedge \alpha_- = 0, \quad \omega_H \wedge \beta_- = 0, \quad \omega_H \wedge \gamma_- = 0, \quad \omega_H \wedge \phi_- = 0.$$

Unlike in the setting of SU(3)-structures, for SU(4)-structures $\operatorname{Re}\theta$ does not determine $\operatorname{Im}\theta$. For our purposes it is enough to note that some solution $\operatorname{Im}\theta'$ exists, which is clearly the case, e.g. if we choose all forms to be primitive. Equation (3.4.14c) is solved automatically, reducing to 0 on both sides. The derivation of the differential equations follows as in the proof of Lemma 3.4.6.

□

While we were able to solve the algebraic equations for an infinitesimal SU(4)-deformation at leading order term, we cannot say if this comes from a deformation of torsion-free SU(4)-structures, i.e. that the systems (3.4.8) and (3.4.11) are equivalent. They are clearly equivalent if $\sigma = 0$, but in general we cannot exclude solutions of (3.4.8) with $\sigma \neq 0$.

Chapter 4

Existence of cohomogeneity one Spin(7) holonomy metrics

4.1 Introduction

In this part of the thesis our aim is to prove Theorems B and C, which state the existence of complete cohomogeneity one AC and ALC Spin(7) holonomy metrics with principal orbit the Aloff–Wallach space $N(1,-1)$, and Theorem D, which states the existence of conically singular ALC Spin(7) holonomy metrics. In the introduction to this chapter we first review the theory of cohomogeneity one Spin(7)-manifolds, then we give a brief description of the difficulties in the proof of Theorems B and C, and outline our strategy to overcome them. The proof of Theorem D follows along similar lines but is easier. We conclude this introductory section with a plan for the remainder of the thesis.

4.1.1 Cohomogeneity one Spin(7)-manifolds

References for our brief introduction to cohomogeinity one manifolds are [Mos57, Rei08, Rei10]. Let G be a compact Lie group acting continuously on the connected manifold M. We say this action is of *cohomogeneity one* if there exists an orbit with codimension 1. In this case the quotient M/G has to be diffeomorphic to either S^1, $[0,1]$, $\mathbb{R}$ or $[0,\infty)$. In the first two cases M is compact. However, by a Bochner-type argument compact irreducible Ricci-flat manifolds cannot have any continuous symmetries. In the third case M has two ends. However, by the Cheeger–Gromoll splitting theorem complete irreducible Ricci-flat manifolds can have only one end. Therefore, in the context of complete cohomogeneity one manifolds with holonomy Spin(7) only the last case is interesting and from now on we only consider $M/G=[0,\infty)$. Denote by $q: M \to M/G$ the quotient map. Isotropy groups of orbits which q does not map to the end point of the half-open interval $[0,\infty)$ are conjugate to one another and there exists $H \subset G$ such that $q^{-1}(0,\infty)$ is G-equivariantly diffeomorphic to $(0,\infty) \times G/H$. These orbits are called *principal orbits*. The orbit $q^{-1}(0)$ is called the *singular orbit*. Denote its isotropy group by K, i.e. $q^{-1}(0) = G/K$. This allows us to write

$$M = (G/K) \cup (0,\infty) \times (G/H). \tag{4.1.1}$$

We can say more about the structure of M. Note that $G \to G/K$ is a principal K-bundle. We can choose $H \subset K$ such that K/H is diffeomorphic to a sphere. In fact there exists a representation V of K such that M has the structure of the total space of the associated vector bundle $G \times_K V \to G/K$ over the singular orbit, the principal orbits $\{t\} \times G/H$ are sphere bundles over G/K which foliate the vector bundle outside the zero section and the spherical fibres of the fibrations $\{t\} \times G/H \to G/K$ are isomorphic to K/H.

We say that a Spin(7)-manifold (M,ψ) is a *cohomogeneity one Spin(7)-manifold* if there exists a cohomogeneity one action by some compact Lie group G on M such that ψ is G-invariant. Then G also preserves the induced metric. The Spin(7)-structure ψ induces on each principal orbit $\{t\} \times G/H$ a G-invariant G_2-structure (φ_t, h_t) and on $q^{-1}(0,\infty)$ the Spin(7)-structure can be recovered as

$$\psi = dt \wedge \varphi_t + *\varphi_t, \tag{4.1.2}$$

$$g = dt^2 + h_t. \tag{4.1.3}$$

Here the Hodge star depends on φ_t. The condition $d\psi = 0$ for ψ to be torsion-free then is equivalent to the system

$$d_{G/H} *\varphi_t = 0, \tag{4.1.4a}$$

$$\frac{\partial}{\partial t} *\varphi_t = d_{G/H}\varphi_t. \tag{4.1.4b}$$

Here $d_{G/H}$ denotes the exterior derivative on G/H. The first equation is a static condition, i.e. it does not involve a derivative with respect to the parameter t. Therefore, we can interpret a torsion-free Spin(7)-structure on the dense subset $M - q^{-1}(0)$ as a solution of the evolution equation

$$\frac{\partial}{\partial t} *\varphi_t = d_{G/H}\varphi_t \tag{4.1.5}$$

in the space of co-closed, G-invariant G_2-structures on the homogeneous space G/H. Note that that this space is finite dimensional.

How can we approach the problem of constructing a complete G-invariant torsion-free

Spin(7)-structure on M? Fixing a co-closed G-invariant G_2-structure $\hat{\varphi}$ on a principal orbit $\{t_0\} \times G/H$ leads to a well-defined initial value problem. By the Picard–Lindelöf theorem there exists a torsion-free Spin(7)-structure on $(t_0-\varepsilon, t_0+\varepsilon)\times G/H$ of the form (4.1.2) with $\varphi_0 = \hat{\varphi}$ for some $\varepsilon > 0$. To investigate whether this Spin(7)-structure can be extended to a complete torsion-free Spin(7)-structure, two questions have to be addressed. First, does it extend backward and close smoothly on the singular orbit? Secondly, does it extend forward over the non-compact end? In general neither question is easy to answer. In the context of non-compact cohomogeneity one Einstein metrics Eschenburg–Wang [EW00] take a different approach. They instead consider a singular initial value problem on the singular orbit. Smooth solutions give rise to smooth Einstein metrics in a neighbourhood of the singular orbit. To investigate completeness it remains to check whether the solution extends over the non-compact end. This has become the standard approach in the construction of cohomogeneity one structures, e.g. [FHN18]. Also see the more recent treatment by Verdiani–Ziller [VZ18] on the problem of extending cohomogeneity one metrics smoothly over the singular orbit. In the realm of special holonomy a simplifying assumption made by Eschenburg–Wang is often not satisfied and their approach has to be adjusted accordingly. In particular, Reidegeld [Rei08, Rei10] studied this singular initial value problem in the context of Spin(7)-structures.

4.1.2 Strategy to prove Theorems B and C

The main difficulty in proving Theorems B and C is to establish the existence of the AC spaces. The behaviour of the remaining family members, which lead to ALC and incomplete metrics, can be deduced by a comparison argument. Compared to previous work, we face additional difficulties. In Bazaikin's work on the $\mathbb{B}_8$ family, the $\mathbb{C}_8$ family, and on cohomogeneity one Spin(7)-manifolds with generic orbit isomorphic to $N(1,1)$, the AC limit was known beforehand. Moreover, in these examples the AC spaces enjoy additional symmetry as compared to other family members and are given by an explicit expression. Foscolo–Haskins–Nordström [FHN18] consider problems in the context of cohomogeneity one G_2-manifolds in which the AC limits are not known beforehand. They solve this problem by "shooting from infinity": an AC end, i.e. a torsion-free AC G_2-structure defined outside a compact subset, corresponds in their case to a trajectory in a plane. Backwards degeneration of AC ends occurs if a trajectory hits one of two particular curves in the plane. The corresponding solution is a complete AC space—and, in particular, closes smoothly on the singular orbit—if and only if the trajectory degenerates at the intersection point of those two curves. There is a 1-parameter family of AC ends, and they show that degeneration must occur at either curve. By continuity there exists a complete AC solution. However, in our setting the space of AC ends is more complicated. Therefore, this strategy does not seem helpful in our situation.

We overcome this problem by following a reverse strategy: we first show the existence of ALC solutions and incomplete metrics and then deduce the existence of the AC metric. More specifically, we show that the corresponding sets of parameters which give rise to complete ALC solutions and incomplete metrics each are open and non-empty. In addition, we manage to show that the complement of these two sets are precisely the parameters which give rise to complete AC solutions. In particular, because the set of all parameters is connected, we deduce the existence of an AC solution. This idea, that the set of parameters giving rise to ALC solutions is open, results from geometric intuition: as we vary the parameter of an ALC solution slightly, we expect qualitatively the same asymptotic ALC geometry, but with an corresponding variation of the asymptotic circle length.

In the following we give a brief overview of how we carry out the program outlined above. Our main emphasis is the choice of "good coordinates" on the state space of the dynamical system. Outside the singular orbit a torsion-free G-invariant Spin(7)-structure can be interpreted as a trajectory in the space $\mathcal{S}$ of co-closed G-invariant G_2-structures on the principal orbit G/H given as a solution of the evolution equation (4.1.5). In our case $\mathcal{S}$ is 4-dimensional and we choose coordinates (a, b, c, f) which allow us to conveniently read off the asymptotic behaviour of complete solutions. Furthermore, the right-hand side of the ODE system is a homogeneous expression with respect to this set of coordinates. This allows us to consider the ODE system in projective space, thereby eliminating one dimension. After this projectivization, singular orbits and asymptotic models are given by fixed points of the dynamical system. Complete torsion-free Spin(7)-structures correspond to trajectories connecting these fixed points. In particular, we obtain a complete list of possible asymptotic geometries. This approach is inspired by Atiyah–Hitchin's work [AH88] on gravitational instantons. However, a further coordinate change on projective space is needed to gain control over all three remaining functions. In this new set of coordinates (X, Y, Z) the right-hand side of the ODE system is given by a purely polynomial expression. There are no intrinsic singularities and a solution can become singular only by shooting off to infinity in finite time. Initially the solution is contained in a compact cube, which it can exit only at the hypersurface $Y = 0$. It turns out that (in)completeness and asymptotic behaviour can be read off from the single function Y. This has a geometric interpretation. Any $\mathrm{SU}(3) \times \mathrm{U}(1)$-invariant metric on $\mathbb{C}P^2$ is determined by specifying two numbers b and c. The function $Y = b^2/c^2$ measures the ratio of these two numbers. $N(1, -1)$ fibres over $\mathbb{C}P^2$ and it is enough to follow the evolution of the induced metric on the base.

4.1.3 Structure of this chapter

In Section 4.2 we describe the action of $\mathrm{SU}(3)\times\mathrm{U}(1)$ on M_{S^5} and $M_{\mathbb{C}P^2}$ and derive the ODE system characterising $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant torsion-free Spin(7)-structures with principal orbit $N(1,-1)$. In particular, we find a set of coordinates on the projectivization of the space of homogeneous G_2-structures on $N(1,-1)$ which greatly simplifies the analysis of the ODE system.

In Section 4.3 we review Reidegeld's work [Rei10] on the existence of cohomogeneity one Spin(7)-metrics in a neighbourhood of a singular orbit. We state the existence of a 1-parameter family Ψ_μ, $\mu \in (0,\infty)$, of solutions in a neighbourhood of $S^5 \subset M_{S^5}$ and a 1-parameter family Υ_τ, $\tau \in \mathbb{R}$, of solutions in a neighbourhood of $\mathbb{C}P^2 \subset M_{\mathbb{C}P^2}$.

In Section 4.4 we construct a 1-parameter family $\Psi^{\mathrm{cs}}_\lambda, \lambda \in \mathbb{R}$, of local cohomogeneity one torsion-free Spin(7)-structures with an isolated conical singularity and a 2-parameter family $\Psi^{\mathrm{ac}}_{\alpha,\beta}$, $\alpha,\beta \in \mathbb{R}$, of AC ends. In both cases the link of the asymptotic cone is $N(1,-1)$ equipped with the unique $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant nearly parallel G_2-structure. As in [FHN18] we use an existence result for singular initial value problems due to Picard [Pic28].

Section 4.5 is the heart of this chapter. Here we carry out the qualitative analysis of the ODE system that we have outlined above. The main result of this section is that the sets of parameters giving rise to complete ALC solutions and incomplete solutions are open.

To finish the proof of the existence of complete AC solutions, we need to prove the existence of parameters which give rise to complete ALC solutions and incomplete metrics. In section 4.6 we prove the existence of ALC solutions closing smoothly on S^5. For the singular orbit $\mathbb{C}P^2$ we recall an explicit solution which was earlier derived by physicists. The existence of incomplete solutions for large values of μ and τ is established in Section 4.7 by a rescaling argument.

In Section 4.8 we put together the results from the previous sections and prove Theorems B, C and D.

4.2 Spin(7)-metrics with Principal Orbit $N(1,-1)$

4.2.1 The Aloff–Wallach space $N(1,-1)$

For every pair (k,l) of integers which are not both zero, $\mathrm{U}(1)$ can be embedded in the maximal torus of diagonal matrices in $\mathrm{SU}(3)$ as

$$e^{i\theta} \mapsto \begin{pmatrix} e^{ik\theta} & 0 & 0 \\ 0 & e^{il\theta} & 0 \\ 0 & 0 & e^{-i(k+l)\theta} \end{pmatrix}. \tag{4.2.1}$$

We also denote this subgroup of SU(3) by $\mathrm{U}(1)_{k,l}$. The Aloff–Wallach space $N(k,l)$ is the homogeneous space $\mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$. We work with the following basis of $\mathfrak{su}(3)$:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},$$

$$E_7 = \begin{pmatrix} -i/2 & 0 & 0 \\ 0 & -i/2 & 0 \\ 0 & 0 & i \end{pmatrix}, \qquad E_8 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We denote the dual basis of $E_1, \dots, E_8$ by $e_1, \dots, e_8$. The structure constants are

$[\cdot,\cdot]$	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
E_1	0	$2E_8$	$-E_5$	$-E_6$	E_3	E_4	0	$-2E_2$
E_2	$-2E_8$	0	E_6	$-E_5$	E_4	$-E_3$	0	$2E_1$
E_3	E_5	$-E_6$	0	$E_8 - 2E_7$	$-E_1$	E_2	$\frac{3}{2}E_4$	$-E_4$
E_4	E_6	E_5	$-E_8 + 2E_7$	0	$-E_2$	$-E_1$	$-\frac{3}{2}E_3$	E_3
E_5	$-E_3$	$-E_4$	E_1	E_2	0	$-2E_7 - E_8$	$\frac{3}{2}E_6$	E_6
E_6	$-E_4$	E_3	$-E_2$	E_1	$2E_7 + E_8$	0	$-\frac{3}{2}E_5$	$-E_5$
E_7	0	0	$-\frac{3}{2}E_4$	$\frac{3}{2}E_3$	$-\frac{3}{2}E_6$	$\frac{3}{2}E_5$	0	0
E_8	$2E_2$	$-2E_1$	E_4	$-E_3$	$-E_6$	E_5	0	0

Remark 4.2.2. We now discuss various relations between the Aloff–Wallach spaces $N(k,l)$ for different pairs of integers. First, the subgroups $\mathrm{U}(1)_{k,l}$ and $\mathrm{U}(1)_{ak,al}$ coincide and hence we can assume without loss of generality that the pair (k,l) is coprime. Secondly, complex conjugation on SU(3) generates a group of outer automorphisms isomorphic to $\mathbb{Z}_2$ and maps $N(k,l)$ to $N(-k,-l)$. Finally, homogeneous spaces G/H_1 and G/H_2 are G-equivariantly diffeomorphic if the isotropy groups H_1 and H_2 are conjugate in G. The Weyl group of SU(3) is isomorphic to the symmetric group S_3 and conjugation by its elements permutes the triple $(k,l,-k-l)$ in formula (4.2.1) accordingly. Therefore, it interchanges the subgroups $\mathrm{U}(1)_{k,l}, \mathrm{U}(1)_{l,-k-l}, \mathrm{U}(1)_{k,-k-l}$, etc., and partitions the set of Aloff–Wallach spaces into equivalence classes.

We have $\mathfrak{u}(1)_{1,-1} = \mathrm{span}\{E_8\}$ and the adjoint action of $\mathrm{U}(1)_{1,-1}$ maps the complement $\mathfrak{m} = \mathrm{span}\{E_1,\ldots,E_7\}$ into itself. Hence $T_{[\mathrm{Id}]}N(1,-1)$ can be identified with $\mathfrak{m}$ and an SU(3)-invariant tensor field on $N(1,-1)$ corresponds to a tensor on $\mathfrak{m}$ which is left invariant by the adjoint action of $\mathrm{U}(1)_{1,-1}$. With respect to the basis $E_1,\ldots,E_7$ the infinitesimal generator of the adjoint action is given by

$$\mathrm{ad}(E_8) = \begin{pmatrix} 0 & -2 & & & & & \\ 2 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & 0 & 1 & \\ & & & & -1 & 0 & \\ & & & & & & 0 \end{pmatrix}.$$

Hence $\mathfrak{m}$ splits into the four irreducible U(1)-modules

$$U_1 = \mathrm{span}\{E_1,E_2\}, \quad U_2 = \mathrm{span}\{E_3,E_4\}, \quad U_3 = \mathrm{span}\{E_5,E_6\}, \quad U_4 = \mathrm{span}\{E_7\}.$$

If we denote the irreducible representation of U(1) of weight m by $\mathbb{C}_m$ we get

$$\mathfrak{m} = U_1 \oplus U_2 \oplus U_3 \oplus U_4 = \mathbb{C}_2 \oplus \mathbb{C}_1 \oplus \mathbb{C}_{-1} \oplus \mathbb{R}, \tag{4.2.3}$$

Hence $\mathfrak{m}$ has two isotypical components. The equivalence classes of $N(1,-1)$ and $N(1,1)$ are the only equivalence classes with this property and therefore are called the *exceptional* Aloff–Wallach spaces. The other Aloff–Wallach spaces are called *generic*.

We fix the SU(3)-invariant metric q on $N(1,-1)$ for which $E_1,\ldots,E_8$ is an orthonormal basis as a background metric which allows us to consider any other SU(3)-invariant metric g

on $N(1,-1)$ as an SU(3)-invariant symmetric section of the endomorphism bundle. Because the submodules U_2 and U_3 are isomorphic, not every $\mathrm{U}(1)_{1,-1}$-invariant endomorphism of $\mathfrak{m}$ is diagonal. We can identify U_2 and U_3 with $\mathbb{C}$ by identifying xE_3+yE_4 and xE_5+yE_6 with $x+iy$, respectively. Over the real numbers the space of U(1)-equivariant endomorphisms $\mathbb{C}_1 \to \mathbb{C}_{-1}$ is generated by $z \mapsto \bar{z}$ and $z \mapsto i\bar{z}$. With the above identifications this corresponds to $e_3 \otimes E_5 - e_4 \otimes E_6$ and $e_3 \otimes E_6 + e_4 \otimes E_5$, respectively. Hence any invariant symmetric endomorphism on $\mathfrak{m}$ with respect to the basis $E_1, \ldots, E_7$ is of the form

$$\begin{aligned} &a^2 \operatorname{Id}_{U_1} + b^2 \operatorname{Id}_{U_2} + c^2 \operatorname{Id}_{U_3} + f^2 \operatorname{Id}_{U_4} \\ &+v(e_3 \otimes E_5 - e_4 \otimes E_6) + w(e_3 \otimes E_6 + e_4 \otimes E_5) \\ &+v(e_5 \otimes E_3 - e_6 \otimes E_4) + w(e_5 \otimes E_4 + e_6 \otimes E_3). \end{aligned} \tag{4.2.4}$$

In particular, the space of SU(3)-invariant metrics on $N(1,-1)$ is 6-dimensional.

Remark 4.2.5. In addition to the left multiplication of G on G/H, there is another action given by conjugation with elements of the normaliser $\mathrm{N}_G(H) = \{g \in G|\ gHg^{-1} = H\}$. In our case $\mathrm{N}_{\mathrm{SU}(3)}(\mathrm{U}(1)_{1,-1})$ is the maximal torus of diagonal matrices in SU(3) isomorphic to $\mathrm{U}(1)^2$. We are particularly interested in the subgroup of the normaliser given by the embedding

$$e^{i\theta} \mapsto \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{i2\theta} \end{pmatrix}, \tag{4.2.6}$$

which is generated by $2E_7$. The action of E_7 leaves the diagonal part of the endomorphism (4.2.4) invariant but we have

$$\begin{aligned} \operatorname{ad}(2E_7)(e_3 \otimes E_5 - e_4 \otimes E_6) &= -6(e_3 \otimes E_6 + e_4 \otimes E_5), \\ \operatorname{ad}(2E_7)(e_3 \otimes E_6 + e_4 \otimes E_5) &= 6(e_3 \otimes E_5 - e_4 \otimes E_6). \end{aligned}$$

This has several consequences. First, Reidegeld [Rei08, p. 154] concludes that in the non-diagonal case it suffices to consider 5 instead of 6 parameters. Secondly, any SU(3)-invariant metric on $N(1,-1)$ with this additional U(1)-symmetry is diagonal, i.e. of the form

$$a^2(e_1^2 + e_2^2) + b^2(e_3^2 + e_4^2) + c^2(e_5^2 + e_6^2) + f^2 e_7^2. \tag{4.2.7}$$

We say that the metric is $\mathrm{SU}(3) \times \mathrm{U}(1)$-invariant. The space of $\mathrm{SU}(3) \times \mathrm{U}(1)$-invariant metrics on $N(1,-1)$ is 4-dimensional.

Remark 4.2.8. Besides the Aloff–Wallach spaces, three further homogeneous spaces with a transitive action of SU(3) are relevant to us. $F_3 = \mathrm{U}(3)/\mathrm{U}(1)^3 = \mathrm{SU}(3)/\mathrm{U}(1)^2$ is the manifold of complete flags in $\mathbb{C}^3$. If we embed SU(2) in SU(3) as

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

then $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ is the 5-sphere and $\mathbb{C}P^2 = \mathrm{SU}(3)/(\mathrm{SU}(2) \times \mathrm{U}(1))$, where U(1) denotes the subgroup (4.2.6) of SU(3). Any Aloff–Wallach space $N(k,l)$ is a circle bundle over F_3. For example, the bundle structure of $N(1,-1)$ over F_3 is given by right multiplication with the circle (4.2.6) and thus is generated by $2E_7$. The flag manifold F_3 is the twistor space of $\mathbb{C}P^2$, and in particular an S^2-bundle over $\mathbb{C}P^2$. This leads to a fibration of each Aloff–Wallach space $N(k,l)$ over $\mathbb{C}P^2$. As discussed in Remark 4.2.2, permutations of the triple $(k, l - k - l)$ lead to isomorphic Aloff–Wallach spaces. However, the fibration structure over $\mathbb{C}P^2$ depends on the choice of a particular triple in an equivalent class. The fibres of $N(k,l)$ over $\mathbb{C}P^2$ are given by the lens spaces $L(1, |k+l|)$ (for more details we refer to [GS02, Section 4.1]). Here for convenience $L(1,0)$ is defined to be $S^1 \times S^2$. The equivalence class of the Aloff–Wallach space $N(k,l)$ therefore gives rise to $L(1,|k|), L(1,|l|)$ and $L(1,|k+l|)$-bundles over $\mathbb{C}P^2$. The equivalence classes of the exceptional Aloff–Wallach spaces $N(1,-1)$ and $N(1,1)$ give rise to two different bundle structures while the equivalence classes of the generic Aloff–Wallach spaces give rise to three different bundle structures.

4.2.2 Spin(7)-structures with principal orbit $N(1,-1)$

We now want to describe $\mathrm{SU}(3) \times \mathrm{U}(1)$-invariant cohomogeneity one Spin(7)-structures with principal orbit $N(1,-1)$, where the extra U(1)-factor acts as described in Remark 4.2.5. We adopt the viewpoint from Section 4.1.1 that outside the singular orbit a torsion-free cohomogeneity one Spin(7)-structure is a solution to the evolution equation (4.1.5) in the space of co-closed invariant G_2-structures on $N(1,-1)$. In Remark 4.2.8 we have explained that $N(1,-1)$ is a circle bundle over the flag manifold F_3. In the introduction we have noted the motivation for this work: torsion-free ALC Spin(7)-structures with principal orbit $N(1,-1)$ collapsing to the SU(3)-invariant Bryant–Salamon AC G_2-metric on $\Lambda^2_- \mathbb{C}P^2$, which is asymptotic to the cone over the homogeneous nearly Kähler structure on the flag manifold F_3. We want to describe $\mathrm{SU}(3) \times \mathrm{U}(1)$-invariant Spin(7)-structures with principal orbit $N(1,-1)$ in such a way that we can easily read off this fibration.

We start with the base F_3. The circle bundle structure of $N(1,-1)$ over F_3 is generated by $2E_7$. Therefore $T_{[\mathrm{Id}]}F_3 \cong U_1 \oplus U_2 \oplus U_3 = \mathrm{span}\{E_1, \cdots, E_6\}$. We denote $e_{i_1} \wedge \cdots \wedge e_{i_k}$ by $e_{i_1 \dots i_k}$. The SU(3)-invariant nearly Kähler structure on F_3 is given by (see [MS10, Section

6])

$$\omega_0 = e_{12} + e_{43} + e_{56}, \tag{4.2.9a}$$

$$\Omega_0 = e_{136} + e_{246} + e_{235} - e_{145} + i(e_{236} - e_{146} - e_{135} - e_{245}). \tag{4.2.9b}$$

To determine the space of SU(3)-invariant G_2-structures on $N(1,-1)$ we need to compute the other invariant 3-forms. In the course of this computation we need the following

Lemma 4.2.10. *Let* $\mathbb{C}_k = \mathrm{Span}\{v_1, v_2\}$ *be an oriented* U(1)*-module of weight* k *and* $\mathbb{C}_l = \mathrm{Span}\{v_1', v_2'\}$ *be an oriented* U(1)*-module of weight* l*. Then as oriented* U(1)*-modules we have*

$$\mathbb{C}_k \otimes \mathbb{C}_l = \mathbb{C}_{k+l} \oplus \mathbb{C}_{k-l},$$

where

$$\mathbb{C}_{k+l} = \mathrm{Span}\{v_1 \otimes v_1' - v_2 \otimes v_2', v_1 \otimes v_2' + v_2 \otimes v_1'\},$$
$$\mathbb{C}_{k-l} = \mathrm{Span}\{v_1 \otimes v_2' - v_2 \otimes v_1', v_1 \otimes v_1' + v_2 \otimes v_2'\}.$$

Lemma 4.2.11. *(i) The space of* SU(3)*-invariant 1-forms on* $N(1,-1)$ *is spanned by* e_7*.*

(ii) The space of SU(3)*-invariant 2-forms on* $N(1,-1)$ *is five dimensional and spanned by*

$$e_{12},\ e_{34},\ e_{56},\ e_{35} - e_{46},\ e_{36} + e_{45}.$$

(iii) The space of SU(3)*-invariant 3-forms on* $N(1,-1)$ *is seven dimensional and spanned by*

$$e_{127},\ e_{347},\ e_{567},\ e_{357} - e_{467},\ e_{367} + e_{457},$$
$$\mathrm{Re}\ \Omega_0 = e_{136} - e_{145} + e_{235} + e_{246},\ \mathrm{Im}\ \Omega_0 = -e_{146} - e_{135} + e_{236} - e_{245}.$$

Proof. (i) follows immediately from (4.2.3).
(ii) As U(1)-modules we have

$$U_1^* \cong U_1 \cong \mathbb{C}_2, \quad U_2^* \cong U_2 \cong \mathbb{C}_1, \quad U_3^* \cong U_3 \cong \mathbb{C}_{-1}, \quad U_4^* \cong U_4 \cong \mathbb{R}.$$

Using Lemma 4.2.10 we compute the invariant 2-forms:

$$\begin{aligned}\Lambda^2\mathfrak{m}^* &= \Lambda^2 U_1 \oplus \Lambda^2 U_2 \oplus \Lambda^2 U_3 \oplus (U_1 \oplus U_2 \oplus U_3) \otimes U_4 \oplus (U_1 \otimes U_2) \oplus (U_1 \otimes U_3) \oplus (U_2 \otimes U_3) \\ &\cong \mathbb{R}^3 \oplus \mathbb{C}_2 \oplus \mathbb{C}_1 \oplus \mathbb{C}_{-1} \oplus (\mathbb{C}_3 \oplus \mathbb{C}_1) \oplus (\mathbb{C}_3 \oplus \mathbb{C}_1) \oplus (\mathbb{C}_2 \oplus \mathbb{R}^2).\end{aligned}$$

If we write $U_2^* = \operatorname{span}\{e_3, e_4\}$ and $U_3^* = \operatorname{span}\{e_5, e_6\}$ as oriented U(1)-modules of weight 1 and -1, respectively, then the trivial $\mathbb{R}^2 \subset U_2^* \otimes U_3^*$ is spanned by

$$e_{35} - e_{46}, \quad e_{36} + e_{45}.$$

Hence the space of invariant 2-forms is 5-dimensional and spanned by the claimed forms.
(iii) The space of 3-forms decomposes as

$$\begin{aligned}\Lambda^3\mathfrak{m}^* =& \Lambda^2 U_1 \otimes (U_2 \oplus U_3 \oplus \mathbb{R}) \oplus \Lambda^2 U_2 \otimes (U_1 \oplus U_3 \oplus \mathbb{R}) \oplus \Lambda^2 U_3 \otimes (U_1 \oplus U_2 \oplus \mathbb{R}) \\ & \oplus (U_1 \otimes U_2 \otimes U_3) \oplus (U_1 \otimes U_2 \otimes \mathbb{R}) \oplus (U_1 \otimes U_3 \otimes \mathbb{R}) \oplus (U_2 \otimes U_3 \otimes \mathbb{R}).\end{aligned}$$

We have

$$U_1 \otimes U_2 \otimes U_3 = \mathbb{C}_2 \otimes \mathbb{C}_1 \otimes \mathbb{C}_{-1} = \mathbb{C}_4 \oplus 2\mathbb{C}_2 \oplus \mathbb{R}^2.$$

The $\mathbb{C}_1$ part in $U_1 \otimes U_2$ is spanned by

$$e_{14} - e_{23}, \quad e_{13} + e_{24}.$$

Hence the invariant part of $U_1 \otimes U_2 \otimes U_3$ is spanned by

$$(e_{14} - e_{23}) \wedge e_5 - (e_{13} + e_{24}) \wedge e_6, \quad (e_{14} - e_{23}) \wedge e_6 + (e_{13} + e_{24}) \wedge e_5.$$

We conclude that the space of invariant 3-forms is 7-dimensional and spanned by the claimed forms. □

Using the Maurer–Cartan equation and the structural constants we can compute the exterior derivatives of some of the invariant forms computed in Lemma 4.2.11.

Lemma 4.2.12. *We have*

$$
\begin{aligned}
de_7 &= -2e_{43} + 2e_{56},\\
de_{12} = de_{43} = de_{56} &= \text{ Re } \Omega_0,\\
de_{127} &= \text{ Re } \Omega_0 \wedge e_7 - 2e_{1243} + 2e_{1256},\\
de_{437} &= \text{ Re } \Omega_0 \wedge e_7 + 2e_{4356},\\
de_{567} &= \text{ Re } \Omega_0 \wedge e_7 - 2e_{4356},\\
d\text{Im } \Omega_0 &= -2\omega_0^2,\\
d(e_{357} - e_{467}) &= d(e_{367} + e_{457}) = 0.
\end{aligned}
$$

Now we are ready to describe $\mathrm{SU}(3) \times \mathrm{U}(1)$-invariant $\mathrm{Spin}(7)$-structures on $(0, \infty) \times N(1, -1)$ in a way such that the asymptotic behaviour can be conveniently read off the coefficient functions. Starting with the homogeneous nearly Kähler structure (4.2.9) on F_3, we can scale U_1, U_2, U_3 respectively by non-zero a, b, c to get the invariant $\mathrm{SU}(3)$-structure

$$
\begin{aligned}
\omega &= a^2 e_{12} + b^2 e_{43} + c^2 e_{56},\\
\Omega &= abc\ \Omega_0.
\end{aligned}
$$

On $(0, \infty) \times F_3$ we evolve such $\mathrm{SU}(3)$-structures to get the G_2-structure

$$
\begin{aligned}
\tilde{\varphi} &= dt \wedge \omega + \text{Re } \Omega = a^2 dt \wedge e_{12} + b^2 dt \wedge e_{43} + c^2 dt \wedge e_{56} + abc \text{ Re } \Omega_0,\\
*\tilde{\varphi} &= \frac{1}{2}\omega^2 - dt \wedge \text{Im } \Omega = a^2 b^2 e_{1243} + b^2 c^2 e_{4356} + c^2 a^2 e_{1256} - abc\ dt \wedge \text{Im } \Omega_0.
\end{aligned}
$$

If we now consider $(0, \infty) \times N(1, -1)$ as a circle bundle over $(0, \infty) \times F_3$, this G_2-structure together with the multiple $-fe_7$ of the invariant connection gives the $\mathrm{Spin}(7)$-structure

$$
\begin{aligned}
\psi =& (-fe_7) \wedge \tilde{\varphi} + *\tilde{\varphi}\\
=& (-fe_7) \wedge \left(a^2 dt \wedge e_{12} + b^2 dt \wedge e_{43} + c^2 dt \wedge e_{56} + abc \text{ Re } \Omega_0\right)\\
& + \left(a^2 b^2 e_{1243} + b^2 c^2 e_{4356} + c^2 a^2 e_{1256} - abc\ dt \wedge \text{Im } \Omega_0\right)\\
=& a^2 f dt \wedge e_{127} + b^2 f dt \wedge e_{437} + c^2 f dt \wedge e_{567} - abc\ dt \wedge \text{Im } \Omega_0\\
& + abcf \text{ Re } \Omega_0 \wedge e_7 + a^2 b^2 e_{1243} + b^2 c^2 e_{4356} + c^2 a^2 e_{1256}.
\end{aligned}
\tag{4.2.13}
$$

By the formulas (2.1.15) and (2.1.16) the $\mathrm{Spin}(7)$-structure (4.2.13) induces the metric

$$
g = dt^2 + a^2(e_1^2 + e_2^2) + b^2(e_3^2 + e_4^2) + c^2(e_5^2 + e_6^2) + f^2 e_7^2. \tag{4.2.14}
$$

Remark 4.2.15. As promised the choice of parameters a, b, c, f easily allows to read off the asymptotic behaviour. Because the nearly Kähler structure on F_3 is given by $a = b = c = 1$ and the coefficient f describes the length of the circle fibres of the circle bundle $(0,\infty) \times N(1,-1) \to (0,\infty) \times F_3$, ψ is an ALC Spin(7)-structure asymptotic to a circle bundle with fibre length ℓ over the G_2-cone over the homogeneous nearly Kähler structure on F_3 if

$$a(t)/t \to 1, \quad b(t)/t \to 1, \quad c(t)/t \to 1, \quad f(t) \to \ell \quad \text{as } t \to \infty.$$

While the above construction of the Spin(7)-structure is helpful in reading off the asymptotic behaviour, it is not compatible with the viewpoint from Section 4.1.1 that cohomogeneity one Spin(7)-metrics correspond to an evolution of G_2-structures. However, alternatively we can consider $N(1,-1)$ as a circle bundle over F_3 now equipped with the rotated SU(3)-structure $(\omega, \widetilde{\Omega}) = (\omega, i\Omega)$. Then on $N(1,-1)$ we get the G_2-structure

$$\begin{aligned}
\varphi =&(fe_7) \wedge \omega + \mathrm{Re}\ \widetilde{\Omega} \\
=&f\omega \wedge e_7 - abc\ \mathrm{Im}\ \Omega_0 \\
=&a^2 f e_{127} + b^2 f e_{437} + c^2 f e_{567} - abc\ \mathrm{Im}\ \Omega_0, \\
*\varphi =&\frac{1}{2}\omega^2 - (fe_7) \wedge \mathrm{Im}\ \widetilde{\Omega} \\
=&a^2b^2 e_{1243} + b^2c^2 e_{4356} + c^2a^2 e_{1256} + abcf\ \mathrm{Re}\ \Omega_0 \wedge e_7.
\end{aligned}$$

This G_2-structure induces on $(0,\infty) \times N(1,-1)$ the Spin(7)-structure

$$\psi = dt \wedge \varphi + *\varphi,$$

which coincides with (4.2.13).

Remark 4.2.16. In Remark 4.2.5 we showed that any SU(3) $\times$ U(1)-invariant metric on $N(1,-1)$ is purely diagonal. Furthermore, a direct computation shows that the additional U(1)-action also preserves the G_2-structure φ. Therefore, we are really studying SU(3) $\times$ U(1)-invariant Spin(7)-structures.

The next Lemma shows that the static part of the torsion-free condition (4.1.4a), i.e. that φ is coclosed is always satisfied.

Lemma 4.2.17. *The* G_2*-structure* φ *is coclosed.*

Proof. Using Lemma 4.2.12 we get

$$\begin{aligned}
d * \varphi = &a^2b^2 \mathrm{Re}\ \Omega_0 \wedge (e_{12} + e_{43}) + b^2c^2 \mathrm{Re}\ \Omega_0 \wedge (e_{43} + e_{56}) + c^2a^2 \mathrm{Re}\ \Omega_0 \wedge (e_{56} + e_{12}) \\
&+ abcf(d\mathrm{Re}\ \Omega_0 \wedge e_7 - \mathrm{Re}\ \Omega_0 \wedge (-2e_{43} + 2e_{56})).
\end{aligned}$$

The result follows because

$$\text{Re } \Omega_0 \wedge e_{12} = \text{Re } \Omega_0 \wedge e_{43} = \text{Re } \Omega_0 \wedge e_{56} = 0$$

and Re Ω_0 is closed by the nearly Kähler condition (2.2.6). □

Remark 4.2.18. With (4.2.13) we have constructed one $SU(3) \times U(1)$-invariant Spin(7)-structure which induces the metric (4.2.14). To see if there are others, we can ask equivalently what $SU(3) \times U(1)$-invariant G_2-structures other than φ induce the metric (4.2.7) on $N(1,-1)$. Reidegeld [Rei08, Lemma 3.1.50] has shown that the set of all SU(3)-invariant G_2-structures on $N(1,-1)$ which induce the metric (4.2.7) is parametrised by $N_{SO(7)}U(1)_{1,-1}/N_{G_2}U(1)_{1,-1}$ and that the connected component of the identity is isomorphic to U(1) [Rei10, (42) on p. 22]. Furthermore, he has shown that ψ is up to discrete symmetries the only $SU(3) \times U(1)$-invariant Spin(7)-structure inducing the metric g which can be torsion-free [Rei10, Theorem 4.4 (2)]. The reason is that the other invariant G_2-structures in the connected component of φ are not coclosed, i.e. fail to solve the static condition (4.1.4a).

The evolution equation (4.1.5) given by $d\varphi = \partial_t * \varphi$ is equivalent to an ODE system for the coefficient functions a, b, c, f.

Proposition 4.2.19. *The Spin(7)-structure* (4.2.13) *on* $I \times N(1,-1)$*, where* $I \subset \mathbb{R}_t$ *is some interval, is torsion-free if and only if* (a, b, c, f) *is a solution of the ODE system*

$$\frac{\dot{a}}{a} = \frac{b^2 + c^2 - a^2}{abc}, \tag{4.2.20a}$$

$$\frac{\dot{b}}{b} = \frac{c^2 + a^2 - b^2}{abc} - \frac{f}{b^2}, \tag{4.2.20b}$$

$$\frac{\dot{c}}{c} = \frac{a^2 + b^2 - c^2}{abc} + \frac{f}{c^2}, \tag{4.2.20c}$$

$$\frac{\dot{f}}{f} = \frac{f}{b^2} - \frac{f}{c^2}. \tag{4.2.20d}$$

The holonomy of the associated metric is all of Spin(7).

Proof. ψ is torsion-free if and only if φ solves the system (4.1.4). By Lemma 4.2.17 the static equation (4.1.4a) is always satisfied. The evolution equation (4.1.4b) is equivalent to a system of ODEs, which we now derive using Lemma 4.2.12.

$$\begin{aligned}d\varphi =&a^2 f(\text{Re }\Omega_0 \wedge e_7 - 2e_{1243} + 2e_{1256}) + b^2 f(\text{Re }\Omega_0 \wedge e_7 + 2e_{4356})\\&+ c^2 f(\text{Re }\Omega_0 \wedge e_7 - 2e_{4356}) + 2abc\,\omega_0^2\\=&(a^2+b^2+c^2)f\ \text{Re }\Omega_0 \wedge e_7\\&+ (-2a^2 f + 4abc)e_{1243} + (2a^2 f + 4abc)e_{1256} + (2b^2 f - 2c^2 f + 4abc)e_{4356}.\end{aligned}$$

Equating this with $\partial_t * \varphi$ leads to the system

$$\begin{aligned}\partial_t(a^2b^2) &= -2a^2 f + 4abc,\\\partial_t(b^2c^2) &= 2b^2 f - 2c^2 f + 4abc,\\\partial_t(c^2a^2) &= 2a^2 f + 4abc,\\\partial_t(abcf) &= (a^2+b^2+c^2)f.\end{aligned}$$

Denoting differentiation with respect to t by a dot, we can simplify the above system to get

$$\begin{aligned}\frac{\dot a}{a} + \frac{\dot b}{b} &= -\frac{f}{b^2} + 2\frac{c^2}{abc}\\\frac{\dot b}{b} + \frac{\dot c}{c} &= \frac{f}{c^2} - \frac{f}{b^2} + 2\frac{a^2}{abc}\\\frac{\dot c}{c} + \frac{\dot a}{a} &= \frac{f}{c^2} + 2\frac{b^2}{abc}\\\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c} + \frac{\dot f}{f} &= \frac{a^2+b^2+c^2}{abc}.\end{aligned}$$

This finally gives (4.2.20). The statement about the holonomy group follows from [Rei10, Theorem 4.4] □

Remark 4.2.21. The system (4.2.20) is compatible with $f \equiv 0, b \equiv c$. It reduces to

$$\dot a = 2 - \frac{a^2}{b^2}, \tag{4.2.22a}$$

$$\dot b = \frac{a}{b}. \tag{4.2.22b}$$

Because we are interested in positive solutions, by equation (4.2.22b) we can invert the function b on a domain of interest, and therefore we can reparametrise that interval with a coordinate r such that $b(r) = r$. With respect to r the general solution is given by

$$a(r) = r \left(1 + \frac{C}{r^4}\right)^{\frac{1}{2}}.$$

This is the Bryant–Salamon solution on $\Lambda^2_- \mathbb{C}P^2$ asymptotic to the cone over the homogeneous nearly Kähler structure on F_3.

Remark 4.2.23. As explained in remark 4.2.15 ALC asymptotics can be easily read off from the coefficient functions a, b, c, f. The same is true for an AC Spin(7)-structure asymptotic to the cone over the diagonal SU(3)-invariant nearly parallel G_2-structure on $N(1, -1)$. Substituting the coefficients $a(t) = a_c\, t, b(t) = b_c\, t, c(t) = c_c\, t, f(t) = f_c\, t$ of the conical Spin(7)-structure in the system (4.2.20) gives

$$a_c = \frac{2}{\sqrt{5}} \approx 0.89, \quad b_c = \sqrt{\frac{2}{15}(5 - \sqrt{5})} \approx 0.61,$$
$$c_c = \sqrt{\frac{2}{15}(5 + \sqrt{5})} \approx 0.98, \quad f_c = \frac{4}{3\sqrt{5}} \approx 0.60.$$

Remark 4.2.24. More generally, SU(3)-invariant torsion-free Spin(7)-structures with principal orbit a generic Aloff–Wallach space $N(k, l)$ are characterised by the ODE system

$$\frac{\dot{a}}{a} = \frac{b^2 + c^2 - a^2}{abc} + \frac{m}{\Delta}\frac{f}{a^2}, \tag{4.2.25a}$$
$$\frac{\dot{b}}{b} = \frac{c^2 + a^2 - b^2}{abc} + \frac{l}{\Delta}\frac{f}{b^2}, \tag{4.2.25b}$$
$$\frac{\dot{c}}{c} = \frac{a^2 + b^2 - c^2}{abc} + \frac{k}{\Delta}\frac{f}{c^2}, \tag{4.2.25c}$$
$$\frac{\dot{f}}{f} = -\frac{m}{\Delta}\frac{f}{a^2} - \frac{l}{\Delta}\frac{f}{b^2} - \frac{k}{\Delta}\frac{f}{c^2}. \tag{4.2.25d}$$

Here $m = -k - l$ and $\Delta = k^2 + kl + l^2$. Besides $N(1, -1)$ we are also interested in the principal orbit $N(1, 0)$, which is equivariantly diffeomorphic to $N(1, -1)$. Note that the system (4.2.25) for $(k, l, m) = (1, 0, -1)$ coincides with the system for $(k, l, m) = (1, -1, 0)$ after swapping a and b. For us it will be convenient to consider cohomogeneity one torsion-free Spin(7)-structures with principal orbit $N(1, 0)$ as solutions of the system (4.2.20) after exchanging the initial conditions for a and b.

4.2.3 Preservation laws and a coordinate change on projective space

To understand the long-time behaviour of local solutions of the system (4.2.20) it is crucial to understand preserved orderings of the functions a, b, c and f. The following Lemma is an elementary yet important observation.

Lemma 4.2.26. *Assume that a (local) solution* (a, b, c, f) *of the system (4.2.20), where* a, b, c, f *are positive functions, satisfies both*

(i) $b < c$ *and*

(ii) $a < c$

at some time t_0*. This set of conditions is preserved forward as long as the solution exists, and* f *is strictly monotone increasing from then onwards.*

Proof. As long as the solution exists all functions stay positive.
(i) Assume $b(t_1) = c(t_1)$ for some $t_1 > t_0$. Then at time t_1

$$\dot{b} = \frac{a}{b} - \frac{f}{b},$$
$$\dot{c} = \frac{a}{b} + \frac{f}{b}.$$

Because $f > 0$ we get $\dot{b}(t_1) < \dot{c}(t_1)$, which is a contradiction if t_1 is the smallest $t_1 > t_0$ such that $b(t_1) = c(t_1)$.
(ii) Assume $a(t_1) = c(t_1)$ for some $t_1 > t_0$. Then at time t_1

$$\dot{c} = \frac{b}{a} + \frac{f}{c} > \frac{b}{a} = \dot{a}.$$

The monotonicity of f is a direct consequence of (i) as

$$\dot{f} = \frac{f^2}{b^2} - \frac{f^2}{c^2}.$$

□

The previous Lemma suggests that the quotients a/c and b/c are well-behaved. Because the right-hand side of the ODE system (4.2.20) is homogeneous we can consider the system in the projective coordinates

$$A = \frac{a}{c}, \quad B = \frac{b}{c}, \quad F = \frac{f}{c}. \tag{4.2.27}$$

A similar use of projective coordinates was made by Atiyah–Hitchin [AH88, Chapter 9].

In the following we derive the evolution equations in these coordinates.

$$\begin{aligned}
&\frac{d}{dt}\log a - \frac{d}{dt}\log c = 2\frac{c^2-a^2}{abc} - \frac{f}{c^2}, \\
&\frac{d}{dt}\frac{a}{c} = \frac{2}{b}\left(1-\left(\frac{a}{c}\right)^2\right) - \frac{af}{c^3},
\end{aligned} \tag{4.2.28}$$

$$\begin{aligned}
&\frac{d}{dt}\log b - \frac{d}{dt}\log c = 2\frac{c^2-b^2}{abc} - f\left(\frac{1}{b^2}+\frac{1}{c^2}\right), \\
&\frac{d}{dt}\frac{b}{c} = \frac{2}{a}\left(1-\left(\frac{b}{c}\right)^2\right) - \frac{f}{bc}\left(1+\left(\frac{b}{c}\right)^2\right),
\end{aligned} \tag{4.2.29}$$

$$\begin{aligned}
&\frac{d}{dt}\log f - \frac{d}{dt}\log c = \frac{f}{b^2} - 2\frac{f}{c^2} - \frac{a^2+b^2-c^2}{abc}, \\
&\frac{d}{dt}\frac{f}{c} = \frac{f^2}{b^2c} - 2\frac{f^2}{c^3} + \frac{f}{c}\frac{c^2-a^2-b^2}{abc}.
\end{aligned} \tag{4.2.30}$$

Changing the parameter by $dt = \frac{ab}{c}ds$ (4.2.28)-(4.2.30) becomes

$$\begin{aligned}
\frac{d}{ds}A = \frac{d}{ds}\frac{a}{c} &= 2\frac{a}{c}\left(1-\left(\frac{a}{c}\right)^2\right) - \frac{a^2bf}{c^4} \\
&= 2A(1-A^2) - A^2BF,
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}B = \frac{d}{ds}\frac{b}{c} &= 2\frac{b}{c}\left(1-\left(\frac{b}{c}\right)^2\right) - \frac{af}{c^2}\left(1+\left(\frac{b}{c}\right)^2\right) \\
&= 2B(1-B^2) - AF(1+B^2),
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}F = \frac{d}{ds}\frac{f}{c} &= \frac{af^2}{bc^2} - 2\frac{abf^2}{c^4} + \frac{f}{c}\frac{c^2-a^2-b^2}{c^2} \\
&= \frac{AF^2}{B} - 2ABF^2 + F(1-A^2-B^2).
\end{aligned}$$

To sum up, if we denote differentiation with respect to s by a dot, then the system (4.2.20) takes the form

$$\dot{A} = A\left(2-2A^2-ABF\right), \tag{4.2.31a}$$

$$\dot{B} = B\left(2-2B^2-ABF-\frac{AF}{B}\right), \tag{4.2.31b}$$

$$\dot{F} = F\left(1-A^2-B^2-2ABF+\frac{AF}{B}\right). \tag{4.2.31c}$$

The main difficulty in the analysis of the ODE system (4.2.20) is that apart from monotonicity under the conditions (i) and (ii) in Lemma 4.2.26 nothing can be said about the behaviour of f in relation to any of the other functions. In particular, it is of concern that f blows up in finite time. The lack of control of f is reflected by the fact that for the system (4.2.31) no bounds can be derived for F. A key observation is that the controlled quantities a/c and b/c dominate the ill-behaved quantity f/c. To be more precise, set

$$X = A^2, \quad Y = B^2, \quad Z = ABF. \tag{4.2.32}$$

Still denoting differentiation with respect to the variable s by a dot, the ODE system takes the form

$$\dot{X} = 2X(2 - 2X - Z), \tag{4.2.33a}$$

$$\dot{Y} = 4Y - 4Y^2 - 2YZ - 2Z, \tag{4.2.33b}$$

$$\dot{Z} = Z(5 - 3X - 3Y - 4Z). \tag{4.2.33c}$$

Remark 4.2.34. Let $(a(t), b(t), c(t), f(t))$ be positive functions which solve the system (4.2.20) for t in the interval (T_1, T_2). Then there exists a corresponding solution $(X(s), Y(s), Z(s))$ of (4.2.33) defined on the interval (S_1, S_2), where $S_1 \in \{-\infty\} \cup \mathbb{R}$, $S_2 \in \mathbb{R} \cup \{\infty\}$. After choosing $s(t_0)$ arbitrarily for some $t_0 \in (T_1, T_2)$, because of $dt = \frac{ab}{c} ds$ the s-parameter is given by

$$s(t) = \int_{t_0}^{t} \frac{c(\tilde{t})}{a(\tilde{t})b(\tilde{t})} d\tilde{t} + s(t_0).$$

This is well-defined because a, b, c are positive functions. We will say that the solution $(X(s), Y(s), Z(s))$ is *associated* to $(a(t), b(t), c(t), f(t))$.

All of the information on f is contained in Z. We are finally able to control this quantity.

Lemma 4.2.35. *Assume that a (local) solution (X, Y, Z) of the system (4.2.33) satisfies all of the three conditions*

(i) $0 < X < 1$,

(ii) $Y < 1$,

(iii) $0 < Z < \kappa, \quad \kappa \geq \frac{5}{4}$,

at some time s_0. Then this set of conditions is preserved forward as long as $Y > 0$.

Proof. $0 < X, Z$ is preserved as the system (4.2.33) is compatible with $X \equiv 0$ and $Z \equiv 0$.

(i) Assume $X(s_1) = 1$ for some $s_1 > s_0$. Then at time s_1

$$\dot{X} = -2Z < 0.$$

(ii) Assume $Y(s_1) = 1$ for some $s_1 > s_0$. Then at time s_1

$$\dot{Y} = -4Z < 0.$$

(iii) Assume $Z(s_1) = \kappa$ with $\kappa \geq \frac{5}{4}$ for some $s_1 > s_0$. Then at time s_1

$$\dot{Z} = 4Z(5/4 - Z) - 3Z(X+Y) \leq -3Z(X+Y) < 0.$$

All cases lead to a contradiction. □

Besides controlling f we also got rid of all singularities on the right-hand side of the ODE system. This means that a local solution (X, Y, Z) can only develop a singularity by shooting off to infinity in finite time. If we start with the conditions in Lemma 4.2.35 the solution is contained in a compact cube until it hits the hypersurface $Y = 0$. If (X, Y, Z) is a solution associated with a solution (a, b, c, f) of the system (4.2.20), $Y = 0$ implies $b = 0$, i.e. the original solution already develops a singularity at $Y = 0$. To sum up, we have enough preservation laws such that the long-time behaviour of any solution to the system (4.2.20) is encoded only in the ratio b/c. More precisely we get

Lemma 4.2.36. *Let (a, b, c, f) be a (local) solution of the system (4.2.20), where a, b, c, f are positive functions satisfying $a, b < c$. If for the associated solution (X, Y, Z) of the system (4.2.33) given by Remark 4.2.34 the function Y stays bounded away from zero, then the solution (X, Y, Z) is forward complete, i.e. it exists for all large s. Moreover, (a, b, c, f) itself is forward complete, i.e. it exists for all large t.*

Proof. Because a, b, c, f are positive and we have $a, b < c$, the conditions of Lemma 4.2.35 are satisfied for some κ. As they are preserved and we assume that Y stays bounded away from zero the solution (X, Y, Z) is contained in a compact region and is therefore forward complete and positive for all s. To obtain (a, b, c, f) from (X, Y, Z) we need to make one more integration. With $a\sqrt{Y}\, ds = dt$ we can reformulate the evolution equation (4.2.20a) for a as

$$\frac{d}{ds} \log a = \frac{1}{a}\frac{da}{ds} = \sqrt{Y}\frac{da}{dt} = Y - X + 1.$$

We already know that a exists for some $s_0 = s_0(t_0)$. Then we recover a by

$$\log a(s) = \log a(s_0) + \int_{s_0}^{s} (Y - X + 1) d\hat{s}. \tag{4.2.37}$$

Because $X < 1$ is preserved the integrand is always positive and hence a is positive and uniformly bounded from below. Because a, X, Y, Z are all positive this gives (a, b, c, f). Finally we recover the t-parameter as

$$t(s) = t(s_0) + \int_{s_0}^{s} a\sqrt{Y}\ d\hat{s}.$$

We know that a is bounded away from zero and the same is true for Y by assumption. Therefore $t \to \infty$ as $s \to \infty$. We conclude that (a, b, c, f) extends to a forward complete solution of (4.2.20). □

Remark 4.2.38. To conclude this section we discuss the fixed points of the dynamical system (4.2.33) and their geometric interpretation. As we consider solutions with positive coefficients we only list critical points with non-negative coordinates. These are given by

$$(0,0,0),(1,0,0),(0,1,0),(1,1,0),\left(\frac{15-3\sqrt{5}}{10},\frac{3-\sqrt{5}}{2},\frac{3\sqrt{5}-5}{5}\right). \qquad (4.2.39)$$

Before we move on to describe these in more detail, we quickly review the theory of *hyperbolic* fixed points. For details we refer to [Per96, Chapter 2.7]. A fixed point p of a dynamical system $\dot{x} = \Phi(x)$ is called hyperbolic if the real parts of all eigenvalues of the linearisation $d\Phi|_p$ of the system at the fixed point are non-zero. If the system is n-dimensional and $d\Phi|_p$ has k eigenvalues with negative real part and $(n-k)$-eigenvalues with positive real part, then there is a k-dimensional submanifold, the *stable manifold* at p, of trajectories converging towards p, and a $(n-k)$-dimensional submanifold, the *unstable manifold* at p, of trajectories emanating from p. Moreover, by the Hartman–Grobman theorem [Per96, Chapter 2.8] the dynamical system in a neighbourhood of p is equivalent to the linearised system.

All of the fixed points (4.2.39) are hyperbolic:

- $(0,1,0)$ has a 1-dimensional stable manifold and a 2-dimensional unstable manifold. In Section 4.3 we describe up to scale a 1-parameter family Ψ_μ of smooth cohomogeneity one Spin(7)-structures with principal orbit $N(1,-1)$ closing smoothly on a S^5. The trajectories of the associated solutions originate in this critical point and sweep out an open subset of the unstable manifold. Therefore, this fixed point can be thought off as the singular orbit S^5.

- The dynamics around $(1,0,0)$ are the same as around $(0,1,0)$ and correspond to the singular orbit $\mathbb{C}P^2$. A 1-parameter family Υ_τ of smooth cohomogeneity one Spin(7)-structures with principal orbit $N(1,0)$ closing smoothly on a $\mathbb{C}P^2$ is described in Section 4.3. One of the two trajectories which compromise the 1-dimensional stable manifold is the explicit solution

$$X(s) = \frac{e^{4s}}{1+e^{4s}}, \quad Y(s) = 0, \tag{4.2.40}$$

which emanates from $(0,0,0)$.

- $(0,0,0)$ is a source. We have not found a geometric interpretation of this fixed point.
- $(1,1,0)$ is a sink. Geometrically this fixed point can be interpreted as an ALC end. An ALC Spin(7)-structure as described in Remark 4.2.15 in terms of (X,Y,Z) will converge to this critical point as $s \to \infty$.
- The fixed point

$$(X_c, Y_c, Z_c) := \left(\frac{15-3\sqrt{5}}{10}, \frac{3-\sqrt{5}}{2}, \frac{3\sqrt{5}-5}{5} \right) \approx (0.83, 0.38, 0.34) \tag{4.2.41}$$

corresponds to the Spin(7)-cone C over the unique $\mathrm{SU}(3) \times \mathrm{U}(1)$-invariant nearly parallel G_2-structure on $N(1,-1)$. Indeed, the associated solution of the cone solution described in Remark (4.2.23) is this fixed point. The linearisation of the system (4.2.33) at (X_c, Y_c, Z_c) is given by

$$\begin{pmatrix} -6+\frac{6}{\sqrt{5}} & 0 & -3+\frac{3}{\sqrt{5}} \\ 0 & -6+\frac{14}{\sqrt{5}} & -5+\sqrt{5} \\ 3-\frac{9}{\sqrt{5}} & 3-\frac{9}{\sqrt{5}} & 4-\frac{12}{\sqrt{5}} \end{pmatrix}.$$

The eigenvalues rounded to one digit after the decimal point are $-4.1, -1.7, 1.4$. By the discussion above there is a 2-dimensional stable manifold and a 1-dimensional unstable manifold. In Section 4.4 we construct up to scale two cohomogeneity one Spin(7)-metrics with principal orbit $N(1,-1)$ and an isolated conical singularity modelled on the Spin(7)-cone C. The two trajectories of the associated solutions constitute the unstable manifold at (X_c, Y_c, Z_c). Furthermore, we show that the 2-dimensional stable manifold is made up of a 2-parameter family $\Psi^{\mathrm{ac}}_{\alpha,\beta}$ of AC ends.

4.3 Local solutions around the singular orbits S^5 and $\mathbb{C}P^2$

In Remark 4.2.8 we have explained that $N(k,l)$ is a $L(1,|k+l|)$-bundle over $\mathbb{C}P^2$. For $N(1,-1)$, the fibre $L(1,0) = S^1 \times S^2$ is not a sphere. In particular, there is no cohomogeneity one space with principal orbit $N(1,-1)$ and singular orbit $\mathbb{C}P^2$ (see Section 4.1.1 for details). However, $N(1,-1)$ is an S^2-bundle over the 5-sphere $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ (see

Remark 4.2.8). Indeed, the adjoint bundle M_{S^5} of the principal SU(2)-bundle SU(3) $\to$ SU(3)/SU(2) is a cohomogeneity one space with principal orbit $N(1,-1)$ and singular orbit S^5. The group diagram is given by $\mathrm{U}(1)_{1,-1} \subset \mathrm{SU}(2) \subset \mathrm{SU}(3)$. The extra $U(1)$-factor (4.2.6) also is in the normalizer of SU(2), and therefore gives a global symmetry of M_{S^5}.

As explained in Section 4.1.1, we want to approach the construction of Spin(7)-metrics on M_{S^5} by first considering local invariant Spin(7)-structures closing smoothly on the singular orbit S^5 and then decide which of these extend to complete Spin(7)-structures to all of M_{S^5}. Local cohomogeneity one Spin(7)-structures around the singular orbit have been investigated by Reidegeld [Rei10]. He proves

Theorem 4.3.1. *[Rei10, Theorem 6.1] For any $\mu \in (0,\infty)$ there exists a unique* $\mathrm{SU}(3) \times \mathrm{U}(1)$*-invariant torsion-free Spin(7)-structure Ψ_μ in a neighbourhood of the singular orbit S^5 in M_{S^5} with*

$$a(0) = 0, \quad b(0) = c(0) = 1, \quad f(0) = \mu.$$

The holonomy of the associated metric is all of Spin(7). Ψ_μ depends continuously on μ.

The asymptotic expansion of Ψ_μ is given by

$$a(t) = 2t - \frac{4}{27}(9-\mu^2)t^3 + \mathcal{O}(t^5), \tag{4.3.2a}$$

$$b(t) = 1 - \frac{1}{3}\mu t + \left(1 - \frac{5}{18}\mu^2\right)t^2 + \frac{1}{810}\mu(126 - 167\mu^2)t^3 + \mathcal{O}(t^4), \tag{4.3.2b}$$

$$c(t) = 1 + \frac{1}{3}\mu t + \left(1 - \frac{5}{18}\mu^2\right)t^2 - \frac{1}{810}\mu(126 - 167\mu^2)t^3 + \mathcal{O}(t^4), \tag{4.3.2c}$$

$$f(t) = \mu + \frac{2}{3}\mu^3 t^2 + \mathcal{O}(t^4). \tag{4.3.2d}$$

In the coordinates (X, Y, Z) the short-distance asymptotic expansion takes the form

$$X(t) = 4t^2 - \frac{8}{3}\mu t^3 + \mathcal{O}(t^4), \tag{4.3.3a}$$

$$Y(t) = 1 - \frac{4}{3}\mu t + \frac{8}{9}\mu^2 t^2 - \frac{8}{405}\mu(83\mu^2 - 99)t^3 + \mathcal{O}(t^4), \tag{4.3.3b}$$

$$Z(t) = 2\mu t - \frac{8}{3}\mu^2 t^2 + \frac{4}{27}\mu(31\mu^2 - 36)t^3 + \mathcal{O}(t^4). \tag{4.3.3c}$$

Remark 4.3.4. $\mu = 0$ gives the Bryant–Salamon AC G_2 holonomy metric on $\Lambda^2_-\mathbb{C}P^2$ with $f \equiv 0$ and $b \equiv c$ described in Remark 4.2.22. The continuous dependence of the functions (a, b, c, f) on μ extends to $\mu = 0$.

Remark 4.3.5. By Remark 4.2.34 each Ψ_μ gives rise to an associated solution of the system (4.2.33). By abuse of notation we will denote them by the same symbol Ψ_μ. Let us determine the range of parameters s for which these are defined. It follows from the asymptotic expansion (4.3.2) that we can find a positive constant C and a small time t_0 such that for all $t \in (0, t_0)$

$$Ct^{-1} < \frac{c}{ab}.$$

Set $s(t_0) = s_0$ where s_0 is an arbitrary constant of integration. Then

$$s(t) = -\int_t^{t_0} \frac{c}{ab} d\hat{t} + s(t_0) < -C \int_t^{t_0} \hat{t}^{-1} d\hat{t} + s(t_0) = C\log(t) - C\log(t_0) + s(t_0).$$

Hence $s \to -\infty$ as $t \to 0$. Therefore, there exists some $S \in \mathbb{R}$ such that Ψ_μ is defined for $s \in (-\infty, S)$. As $s \to -\infty$, for each μ the solution (X, Y, Z) converges to the critical point $(0, 1, 0)$. Hence this critical point corresponds to a singular orbit S^5.

To use $\mathbb{C}P^2$ as the singular orbit, we need to use $N(1, 0)$ instead of $N(1, -1)$ as the principal orbit. Indeed, by Remark 4.2.8 $N(1, 0)$ is a $L(1, 1) = S^3$ bundle over $\mathbb{C}P^2$, and the universal quotient bundle $M_{\mathbb{C}P^2}$ is a cohomogeneity one space with principal orbit $N(1, 0)$ and singular orbit $\mathbb{C}P^2$ (see [GST03]). The extra U(1)-factor (4.2.6) also is a subgroup of the normalizer of $U(1)_{1,0}$ and of the isotropy group of $\mathbb{C}P^2$. Therefore, the extra symmetry from Remark 4.2.5 acts globally on $M_{\mathbb{C}P^2}$.

As mentioned in Remark 4.2.24, an SU(3)-invariant torsion-free Spin(7)-structure with principal orbit $N(1, 0)$ is still characterised as a solution of the system (4.2.20). We only need to swap the roles of a and b in the discussion of smooth extension over the singular orbit. Taking this into account, Reidegeld proves

Theorem 4.3.6. *[Rei10, Theorem 7.1] For any $\tau \in \mathbb{R}$ there exists a unique* $\mathrm{SU}(3) \times \mathrm{U}(1)$*-invariant torsion-free Spin(7)-structure Υ_τ in a neighbourhood of the singular orbit $\mathbb{C}P^2$ in $M_{\mathbb{C}P^2}$ with the asymptotic expansion*

$$a(t) = 1 + \frac{2}{3}t^2 + \frac{-104 - \tau}{288}t^4 + \mathcal{O}(t^5), \tag{4.3.7a}$$

$$b(t) = t - \frac{12 + \tau}{24}t^3 + \mathcal{O}(t^5), \tag{4.3.7b}$$

$$c(t) = 1 + \frac{5}{6}t^2 + \frac{-140 + \tau}{288}t^4 + \mathcal{O}(t^5), \tag{4.3.7c}$$

$$f(t) = t + \frac{\tau}{12}t^3 + \mathcal{O}(t^5). \tag{4.3.7d}$$

The holonomy of the associated metric is all of Spin(7). Υ_τ depends continuously on τ.

In (X, Y, Z) coordinates the short-distance expansion takes the form

$$X(t) = 1 - \frac{1}{3}t^2 - \frac{-40+\tau}{72}t^4 + \mathcal{O}(t^5), \tag{4.3.8a}$$

$$Y(t) = t^2 - \frac{32+\tau}{12}t^4 + \mathcal{O}(t^5), \tag{4.3.8b}$$

$$Z(t) = t^2 + \frac{-56+\tau}{24}t^4 + \mathcal{O}(t^5). \tag{4.3.8c}$$

Remark 4.3.9. Using the asymptotic expansion (4.3.7), as in Remark 4.3.5 we can show that for every $\tau \in \mathbb{R}$ there exists some $S > 0$ such that the solution of the system 4.2.33 associated with Υ_τ is defined for $s \in (-\infty, S)$. As $s \to -\infty$, the solution (X, Y, Z) converges to the critical point $(1, 0, 0)$. Hence this critical point corresponds to the singular orbit $\mathbb{C}P^2$.

Remark 4.3.10. Scaling a Ricci-flat metric by a non-zero positive constant gives another Ricci-flat metric. In the situation of the Spin(7)-structure (2.1.16), replacing the Spin(7)-form $\psi = dt \wedge \varphi + *\varphi$ by $\hat{\psi} = \kappa^4\psi$ scales the associated metric to $\kappa^2 g = \kappa^2 dt^2 + \kappa^2 h$. Now $\hat{t} = \kappa t$ is the arc-length parameter meeting the principal orbits orthogonally. The scaled Spin(7)-structure $\hat{\psi}$ is represented by the coefficient functions $(\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{f}(t)) = (\kappa\ a(t/\kappa), \kappa\ b(t/\kappa), \kappa\ c(t/\kappa), \kappa\ f(t/\kappa))$. We are only interested in solutions to (4.2.20) up to scale. In Theorem 4.3.1 we chose the scale for the family Ψ_μ such that $b(0) = 1$, and in Theorem 4.3.6 we chose the scale for the family Υ_τ such that $a(0) = 1$.

4.4 CS and AC ends

In this section we will construct families of local $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant CS and AC Spin(7)-metrics with principal orbit $N(1,-1)$. In both cases the asymptotic cone is the cone over the unique $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant nearly parallel G_2-structure on $N(1,-1)$. As in [FHN18] this will be achieved by considering a singular initial value problem around the conical singularity in the CS case and at infinity of the asymptotic cone in the AC case. The following statement can be found in [FHN18, Theorem 5.1]. A proof can be found in Picard's treatise [Pic28, Chapter I, §13].

Theorem 4.4.1. *Consider the singular initial value problem*

$$t\dot{y} = \Phi(y), \quad y(0) = y_0, \tag{4.4.2}$$

where y takes values in $\mathbb{R}^k$ and $\Phi\colon : \mathbb{R}^k \to \mathbb{R}^k$ is a real analytic function in a neighbourhood of y_0 with $\Phi(y_0) = 0$. After possibly a change of basis, assume that $d\Phi|_{y_0}$ contains a diagonal block $\mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ in the upper-left corner. Furthermore assume that the eigenvalues $\lambda_1, \ldots, \lambda_m$ satisfy:

(i) $\lambda_1, \dots, \lambda_m > 0$;

(ii) for every $\boldsymbol{h} = (h_1, \dots, h_m) \in \mathbb{Z}^m$ *with* $|\boldsymbol{h}| = h_1 + \cdots + h_m \geq 2$ *the matrix*

$$(\boldsymbol{h} \cdot \boldsymbol{\lambda})\mathrm{Id} - d\Phi|_{y_0}$$

is invertible. Here $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ *and* $\boldsymbol{h} \cdot \boldsymbol{\lambda} = \sum_{i=1}^m h_i \lambda_i$.

Then for every $(u_1, \dots, u_m) \in \mathbb{R}^m$ *there exists a unique solution* $y(t)$ *of (4.4.2) given as a convergent generalised power series*

$$y(t) = y_0 + (u_1 t^{\lambda_1}, \dots, u_m t^{\lambda_m}, 0 \dots 0) + \sum_{|\boldsymbol{h}| \geq 2} y_{\boldsymbol{h}} t^{\boldsymbol{h} \cdot \boldsymbol{\lambda}}.$$

Furthermore, the solutions depend real analytically on $u_1, \dots, u_m$.

In the following, denote by ν_1, ν_2, ν_3 the ordered roots of the cubic equation

$$x^3 + 8x^2 - 4x - 60 = 0.$$

The numerical values, rounded to two digits after the decimal point, are given by

$$\nu_0 \approx -7.46, \quad \nu_1 \approx -3.12, \quad \nu_2 \approx 2.58. \tag{4.4.3}$$

Proposition 4.4.4. *Let C be the Spin(7)-holonomy cone over $N(1, -1)$.*

(i) For every $\lambda \in \mathbb{R}$ there is some $\varepsilon > 0$ such that on $(0, \varepsilon) \times N(1, -1)$ there exists a torsion-free CS Spin(7)-structure $\Psi^{\mathrm{cs}}_\lambda$ asymptotic to C which has the asymptotic expansion

$$\frac{\sqrt{5}}{2} t^{-1} a(t) \approx 1 - 0.25\lambda\, t^{\nu_2} + \mathcal{O}(t^{2\nu_2}), \tag{4.4.5a}$$

$$\sqrt{\frac{15}{2(5 - \sqrt{5})}} t^{-1} b(t) \approx 1 - 4.84\lambda\, t^{\nu_2} + \mathcal{O}(t^{2\nu_2}), \tag{4.4.5b}$$

$$\sqrt{\frac{15}{2(5 + \sqrt{5})}} t^{-1} c(t) \approx 1 + 0.09\lambda\, t^{\nu_2} + \mathcal{O}(t^{2\nu_2}), \tag{4.4.5c}$$

$$\frac{3\sqrt{5}}{4} t^{-1} f(t) \approx 1 + 10\lambda\, t^{\nu_2} + \mathcal{O}(t^{2\nu_2}). \tag{4.4.5d}$$

Here all coefficients of (4.4.5a)-(4.4.5c) *have been rounded to two digits after the decimal point.*

(ii) For every $(\alpha, \beta) \in \mathbb{R}^2$ there is some $T > 0$ such that on $(T, \infty) \times N(1, -1)$ there exists a torsion-free AC Spin(7)-structure $\Psi^{\mathrm{ac}}_{\alpha,\beta}$ asymptotic to C which has the asymptotic

expansion

$$\frac{\sqrt{5}}{2}t^{-1}a(t) \approx 1 - 10.6\alpha\ t^{\nu_1} + 3.6\beta\ t^{\nu_0} + \sum_{k,l\geq 0, k+l\geq 2} a_{kl}\, t^{k\nu_1 + l\nu_0}, \quad (4.4.6a)$$

$$\sqrt{\frac{15}{2(5-\sqrt{5})}}t^{-1}b(t) \approx 1 + 10.8\alpha\ t^{\nu_1} + 0.8\beta\ t^{\nu_0} + \sum_{k,l\geq 0, k+l\geq 2} b_{kl}\, t^{k\nu_1 + l\nu_0}, \quad (4.4.6b)$$

$$\sqrt{\frac{15}{2(5+\sqrt{5})}}t^{-1}c(t) \approx 1 - 5.1\alpha\ t^{\nu_1} - 4.8\beta\ t^{\nu_0} + \sum_{k,l\geq 0, k+l\geq 2} c_{kl}\, t^{k\nu_1 + l\nu_0}, \quad (4.4.6c)$$

$$\frac{3\sqrt{5}}{4}t^{-1}f(t) \approx 1 + 10\alpha\ t^{\nu_1} + \beta\ t^{\nu_0} + \sum_{k,l\geq 0, k+l\geq 2} f_{kl}\, t^{k\nu_1 + l\nu_0}. \quad (4.4.6d)$$

Here the leading coefficients of (4.4.6a)-(4.4.6c) *have been rounded to one digit after the decimal point and the higher coefficients* $a_{kl}, b_{kl}, c_{kl}, f_{kl}$ *are determined by* (α, β)*. If* $\alpha = 0$*,* $\Psi^{\mathrm{ac}}_{\alpha,\beta}$ *has decay rate* ν_0*, otherwise it has decay rate* ν_1*.*

Proof. Recall from Remark 4.2.23 that the cone over the $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant nearly parallel G_2-structure on $N(1,-1)$ is given by

$$a = \frac{2}{\sqrt{5}}\,t, \quad b = \sqrt{\frac{2}{15}(5-\sqrt{5})}\,t, \quad c = \sqrt{\frac{2}{15}(5+\sqrt{5})}\,t, \quad f = \frac{4}{3\sqrt{5}}\,t.$$

Therefore, any deformation of the conical Spin(7)-structure on $(0,\infty)\times N(1,-1)$ can be described as

$$t^{-1}a = \frac{2}{\sqrt{5}}(1+X_1), \quad t^{-1}b = \sqrt{\frac{2}{15}(5-\sqrt{5})}(1+X_2),$$
$$t^{-1}c = \sqrt{\frac{2}{15}(5+\sqrt{5})}(1+X_3), \quad t^{-1}f = \frac{8}{3\sqrt{5}}(1+X_4).$$

Setting $(X_1, X_2, X_3, X_4) = (0,0,0,0)$ recovers the Spin(7)-cone. The system (4.2.20) becomes

$$\begin{aligned}
t\dot{X}_1 &= -X_1 + \frac{5-\sqrt{5}}{4}\frac{1+X_2}{1+X_3} + \frac{5+\sqrt{5}}{4}\frac{1+X_3}{1+X_2} - \frac{3}{2}\frac{1+X_1}{1+X_2}\frac{1+X_1}{1+X_3} - 1,\\
t\dot{X}_2 &= -X_2 + \frac{5+\sqrt{5}}{4}\frac{1+X_3}{1+X_1} + \frac{3}{2}\frac{1+X_1}{1+X_3} - \frac{5-\sqrt{5}}{4}\frac{1+X_2}{1+X_3}\frac{1+X_2}{1+X_1} - \frac{2}{\sqrt{5}-1}\frac{1+X_4}{1+X_2} - 1,\\
t\dot{X}_3 &= -X_3 + \frac{3}{2}\frac{1+X_1}{1+X_2} + \frac{5-\sqrt{5}}{4}\frac{1+X_2}{1+X_1} - \frac{5+\sqrt{5}}{4}\frac{1+X_3}{1+X_1}\frac{1+X_3}{1+X_2} + \frac{2}{\sqrt{5}+1}\frac{1+X_4}{1+X_3} - 1,\\
t\dot{X}_4 &= -X_4 + \frac{2}{\sqrt{5}-1}\frac{(1+X_4)^2}{(1+X_2)^2} - \frac{2}{\sqrt{5}+1}\frac{(1+X_4)^2}{(1+X_3)^2} - 1.
\end{aligned}$$

The linearisation L of the right-hand side at $(0,0,0,0)$ is given by

$$L = \begin{pmatrix} -4 & \frac{-\sqrt{5}+3}{2} & \frac{\sqrt{5}+3}{2} & 0 \\ \frac{-\sqrt{5}+3}{2} & \sqrt{5}-3 & 1 & -\frac{2}{\sqrt{5}-1} \\ \frac{\sqrt{5}+3}{2} & 1 & -\sqrt{5}-3 & \frac{2}{\sqrt{5}+1} \\ 0 & -\sqrt{5}-1 & \sqrt{5}-1 & 1 \end{pmatrix}.$$

The eigenvalues of L are given by ν_0, ν_1, ν_2 and -1. Writing $y = (X_1, X_2, X_3, X_4)$, this is a system of the form (4.4.2) with $y_0 = (0,0,0,0)$.

We will first construct the family of CS solutions. The numerical values (4.4.3) show that condition (ii) in Theorem 4.4.1 is satisfied if we set $m = 1$ and $\lambda_1 = \nu_2$. The eigenspace of L associated with ν_2 is spanned by $(-0.25, -4.84, 0.09, 10)$, where all components are rounded to two digits after the decimal point. The existence of the 1-parameter family $\Psi_\lambda^{\mathrm{cs}}$ follows from Theorem 4.4.1.

We have to replace t by $1/t$ to construct the AC ends with Theorem 4.4.1. Then the linearisation is given by $-L$. By using the numerical approximations (4.4.3), one can see that the non-resonance condition (ii) of Theorem 4.4.1 is satisfied if we set $m = 2$ and $\lambda_1 = -\nu_0$, $\lambda_2 = -\nu_1$. Rounded to one digit after the decimal point, the eigenspaces associated with ν_0 and ν_1 are spanned by the vectors $(3.6, 0.8, -4.8, 1)$ and $(-10.6, 10.8, -5.1, 10)$, respectively. The statement follows with Theorem 4.4.1. $\square$

The solution Ψ_0^{cs} is the Spin(7)-cone itself. All solutions $\Psi_\lambda^{\mathrm{cs}}$, $\lambda > 0$, are related by scaling, as are all solutions $\Psi_\lambda^{\mathrm{cs}}$, $\lambda < 0$. By Remark 4.2.34 each $\Psi_\lambda^{\mathrm{cs}}$ gives rise to an associated solution of the system (4.2.33). Because by passing to (X, Y, Z) coordinates Spin(7)-structures related by scaling are identified, we only get three distinct solutions and different choices of λ of the same sign merely correspond to a shift in the s-parameter. The associated solution of the Spin(7)-cone Ψ_0^{cs} is the fixed point (X_c, Y_c, Z_c), which we have described in Remark 4.2.38. To determine the remaining two trajectories corresponding to the conically singular solutions $\Psi_\lambda^{\mathrm{cs}}$, we use the asymptotic expansion (4.4.5) to argue as in Remark 4.3.5 that for each $\Psi_\lambda^{\mathrm{cs}}$ we can find an $S \in \mathbb{R}$ such that the associated solution (X, Y, Z) is defined for $s \in (-\infty, S)$. As $s \to -\infty$, the solution (X, Y, Z) converges to the fixed point (X_c, Y_c, Z_c), which corresponds to the Spin(7)-cone. Thus, the two trajectories associated with the families $\Psi_\lambda^{\mathrm{cs}}$, $\lambda > 0$, and $\Psi_\lambda^{\mathrm{cs}}$, $\lambda < 0$, are precisely the two branches of the 1-dimensional unstable manifold at (X_c, Y_c, Z_c).

It will still be useful to us to compute the asymptotic expansion with respect to the

t-parameter of the associated solution of $\Psi^{\mathrm{cs}}_\lambda$ as $t \to 0$, which is given by

$$X(t) \approx X_c(1 - 0.68\lambda t^{\nu_2} + \mathcal{O}(t^{2\nu_2})), \tag{4.4.7a}$$

$$Y(t) \approx Y_c(1 - 9.86\lambda t^{\nu_2} + \mathcal{O}(t^{2\nu_2})), \tag{4.4.7b}$$

$$Z(t) \approx Z_c(1 + 4.46\lambda t^{\nu_2} + \mathcal{O}(t^{2\nu_2})). \tag{4.4.7c}$$

Again all coefficients are rounded to two digits after the decimal point.

The solution $\Psi^{\mathrm{ac}}_{0,0}$ is the Spin(7)-cone itself. Next we determine which ones of the AC ends $\Psi^{\mathrm{ac}}_{\alpha,\beta}$, $(\alpha,\beta) \neq (0,0)$, are related by scaling. In Remark 4.3.10 we have described how the Spin(7)-structures scale. After rescaling by $\kappa > 0$, the AC end $\kappa\Psi^{\mathrm{ac}}_{\alpha,\beta}$ is described by the functions

$$(\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{f}(t)) = (\kappa\ a(t/\kappa), \kappa\ b(t/\kappa), \kappa\ c(t/\kappa), \kappa\ f(t/\kappa)).$$

This corresponds to replacing the functions $(X_1(t), X_2(t), X_3(t), X_4(t))$ from the proof of Proposition 4.4.4 by $(X_1(t/\kappa), X_2(t/\kappa), X_3(t/\kappa), X_4(t/\kappa))$. By the asymptotic expansion (4.4.6), the latter quadruple can also be obtained by replacing the parameters (α, β) by $(\kappa^{-\nu_1}\alpha, \kappa^{-\nu_0}\beta)$. Hence the orbits of the action of $\mathbb{R}_+$ on $\mathbb{R}^2 - \{(0,0)\}$ given by $\kappa \circ (\alpha,\beta) = (\kappa^{-\nu_1}\alpha, \kappa^{-\nu_0}\beta)$ consist precisely of parameters corresponding to AC ends which are related by scaling. The quotient of $\mathbb{R}^2 - \{(0,0)\}$ by this action is homeomorphic to S^1 and each orbit has a unique representative on $S^1 \subset \mathbb{R}^2$.

Using the asymptotic expansion (4.4.6), again we can argue similarly as in Remark 4.3.5 to deduce that for each (α,β) there exists $S \in \mathbb{R}$ such that the solution $(X_{\alpha,\beta}(s), Y_{\alpha,\beta}(s), Z_{\alpha,\beta}(s))$ associated with $\Psi^{\mathrm{ac}}_{\alpha,\beta}$ this time is defined for $s \in (S, \infty)$. As $s \to \infty$, the solution approaches the Spin(7)-cone (X_c, Y_c, Z_c). Furthermore, because S^1 is compact, we can find $s_0 \in \mathbb{R}$ independent of $(\alpha,\beta) \in S^1 \subset \mathbb{R}^2$ such that the associated solution of $\Psi^{\mathrm{ac}}_{\alpha,\beta}$ for any $(\alpha,\beta) \in S^1$ is defined for $s \in (s_0, \infty)$. The map

$$S^1 \to \mathbb{R}^3,$$
$$(\alpha,\beta) \mapsto (X_{\alpha,\beta}(s_0), Y_{\alpha,\beta}(s_0), Z_{\alpha,\beta}(s_0)),$$

is an embedding. As we increase the choice of s_0, this embedded circle sweeps out a punctured embedded 2-ball centred at (X_c, Y_c, Z_c). Thus, the 2-dimensional stable manifold at the fixed point (X_c, Y_c, Z_c) corresponds precisely to the trajectories associated with the AC ends $\Psi^{\mathrm{ac}}_{\alpha,\beta}$. This proves

Lemma 4.4.8. *Let (X, Y, Z) be a forward complete solution of the system (4.2.33) which converges to the critical point (X_c, Y_c, Z_c). Then there exist $\alpha_0, \beta_0 \in \mathbb{R}$ such that (X, Y, Z) is associated to the AC end $\Psi^{\mathrm{ac}}_{\alpha_0,\beta_0}$.*

In the construction of the AC ends in the proof of Proposition 4.4.4, the linearisation $-L$ also has the positive eigenvalue 1. Deformations given by this eigenvalue correspond to translations of the t-variable, and therefore do not give new solutions. In particular we get

Corollary 4.4.9. *There exists a gauge such that any* $\mathrm{SU}(3) \times \mathrm{U}(1)$*-invariant AC Spin(7) metric which is asymptotic to the cone over the unique* $\mathrm{SU}(3) \times \mathrm{U}(1)$*-invariant nearly parallel* G_2*-structure on* $N(1,-1)$ *has decay rate equal to* ν_0 *or* ν_1*.*

4.5 Analysis of the ODE system

In the remainder of the paper we want to investigate which members of the families Ψ_μ, Υ_τ and $\Psi^{\mathrm{cs}}_\lambda$ give rise to forward complete Spin(7)-holonomy metrics and determine the asymptotic type of complete solutions. As discussed in Section 4.2.3, solutions of the ODE system (4.2.20) are best studied by looking at their associated solutions of the system (4.2.33) described in Remarks 4.3.5, 4.3.9 and at the end of section 4.4.

The following Lemma will allow us to compare the local solutions for different parameters.

Lemma 4.5.1. *Suppose* (X_1, Y_1, Z_1) *and* (X_2, Y_2, Z_2) *are two solutions of the system (4.2.33), where all functions are positive. Furthermore, suppose at some time* $s_0 \in \mathbb{R}$ *we have*

$$X_1 > X_2, \quad Y_1 > Y_2, \quad Z_1 < Z_2.$$

Then this condition is forward preserved as long as all functions stay positive.

Proof. We start by looking at the quantity Z. Assume all three inequalities are preserved until some time $s_1 > s_0$ when we have $Z_1(s_1) = Z_2(s_1) = \alpha > 0$. Note that at s_1 there must be a strict inequality for either $X_1 > X_2$ or $Y_1 > Y_2$. Otherwise the solutions would be the same. Furthermore at s_1 we have

$$\dot{Z}_2 - \dot{Z}_1 = 3\alpha((X_1 - X_2) + (Y_1 - Y_2)) > 0.$$

Therefore, $Z_1 < Z_2$ is strictly preserved as long as $X_1 \geq X_2$ and $Y_1 \geq Y_2$. Given that, suppose that at $s_1 > s_0$ we have $X_1(s_1) = X_2(s_1) = \alpha > 0$. Then at the same time

$$\dot{X}_2 - \dot{X}_1 = 2\alpha(Z_1 - Z_2) < 0.$$

If we suppose that at $s_1 > s_0$ we have $Y_1(s_1) = Y_2(s_1) = \alpha > 0$, then at the same time

$$\dot{Y}_2 - \dot{Y}_1 = -2(1+\alpha)(Z_2 - Z_1) < 0.$$

All cases lead to a contradiction. □

As an immediate application of Lemma 4.5.1 we obtain

Lemma 4.5.2. *Denote by* (X_1, Y_1, Z_1) *and* (X_2, Y_2, Z_2) *the two solutions of the system (4.2.33) corresponding to*

- Ψ_{μ_1} *and* Ψ_{μ_2} *for* $0 < \mu_1 < \mu_2$*, respectively, or*
- Υ_{τ_1} *and* Υ_{τ_2} *for* $\tau_1 < \tau_2$*, respectively, or*
- $\Psi^{\mathrm{cs}}_{\lambda_1}$ *and* $\Psi^{\mathrm{cs}}_{\lambda_2}$ *for* $\lambda_1 < \lambda_2$*, respectively.*

In all three cases we have

$$X_1 > X_2, \quad Y_1 > Y_2, \quad Z_1 < Z_2,$$

as long as the solutions exist.

Proof. By the short distance asymptotic expansions (4.3.3), (4.3.8), (4.4.7) all three statements are true for small times. By Lemma 4.5.1 this is preserved as long as all functions stay positive. Any of the (X, Y, Z) coordinates becoming zero means that one of the corresponding functions a, b, c, f must be zero, and thus that the respective solution develops a singularity. □

A simple consequence of the above comparison argument is that the families Ψ_μ and Υ_τ contain at most one AC space.

Lemma 4.5.3. *Suppose* (X_1, Y_1, Z_1) *and* (X_2, Y_2, Z_2) *are two complete solutions of the system (4.2.33), where all functions are positive and satisfy*

$$X_1 > X_2, \quad Y_1 > Y_2, \quad Z_1 < Z_2.$$

for all times. Then not both solutions can converge to the cone (X_c, Y_c, Z_c) *as* $s \to \infty$*.*

Proof. The positivity and the given ordering of the two solutions imply

$$\begin{aligned}\dot{Y}_1 - \dot{Y}_2 &= 4(Y_1 - Y_2) - 4(Y_1^2 - Y_2^2) - 2Z_1(1+Y_1) + 2Z_2(1+Y_2)\\ &> 4(Y_1 - Y_2) - 4(Y_1^2 - Y_2^2) - 2Z_1(1+Y_1) + 2Z_1(1+Y_2)\\ &= (Y_1 - Y_2)(4 - 4Y_1 - 4Y_2 - 2Z_1).\end{aligned}$$

If both solutions converge to the cone, for $\epsilon > 0$ arbitrarily small and all sufficiently large times we get with (4.2.41)

$$4 - 4Y_1 - 4Y_2 - 2Z_1 > 4 - 8Y_c - 2Z_c - \varepsilon = \frac{14}{\sqrt{5}} - 6 - \varepsilon \approx 0.26 - \varepsilon.$$

Therefore, the function $Y_1 - Y_2$ is monotone increasing for large times. In particular, Y_1 and Y_2 cannot have the same limit which is a contradiction. $\square$

4.5.1 Extrema of Y

By the asymptotic expansions (4.3.3) and (4.3.8) for Ψ_μ and Υ_τ, respectively, and the fact that all Ψ_λ^{cs} originate in the cone (X_c, Y_c, Z_c), all these solutions initially satisfy the conditions of Lemma 4.2.35. With these preservation laws our main insight in Section 4.2.3 was Lemma 4.2.36: it is sufficient to bound the function Y of the associated solution away from zero to obtain forward completeness for the original solution (a, b, c, f) of the system (4.2.20). In this section we approach the problem of bounding Y from below by studying minima of Y. Further strong motivation for this strategy comes from the geometry of the family Ψ_μ. In Remark 4.2.38 we have noted that the local solutions Ψ_μ originate in the critical point $(0, 1, 0)$ and ALC solutions converge toward the critical point $(1, 1, 0)$. Because for all parameters μ the corresponding function Y is initially decreasing, for any complete ALC solution Y necessarily has a minimum at some time. If however Y never has a minimum and eventually reaches 0, the solution is incomplete. A further possibility is that Y is monotone decreasing with $Y \to Y_c$ which corresponds to an AC solution. Therefore, studying extremal points of the function Y should allow us to make statements about completeness and asymptotic behaviour of complete solutions.

This approach has two further important advantages. First, Y having a minimum is an open condition. In this chapter we will show that we can characterise forward complete ALC solutions among the family Ψ_μ by Y having a minimum. Hence the subset of parameters μ which give rise to ALC solutions is an open subset. This will allow us later to deduce the existence of an AC solution. Secondly, the next Lemma shows that the study of extrema of Y is essentially a 2-dimensional problem. This will allow us to project the trajectories on the (X, Y)-plane and we obtain a much more tractable problem.

Lemma 4.5.4. *(i) For the growth of Y we have*

$$\dot{Y} < (>, =)\ 0 \quad \Leftrightarrow \quad Z > (<, =)\ 2Y\frac{1 - Y}{1 + Y}.$$

(ii) In the following we set

$$Q(X,Y) := -3X + \frac{5Y^2 - 6Y + 5}{1+Y}.$$

If Y has a minimum at time s_0 with $Y(s_0) \geq 0$, then $Q(X(s_0), Y(s_0)) \leq 0$. If Y has a maximum at time s_0 with $Y(s_0) \geq 0$, then $Q(X(s_0), Y(s_0)) \geq 0$.

Proof. The first statement follows easily from the evolution equation for Y. To determine the nature of critical points we compute the second derivative of Y at a critical point:

$$\frac{d^2}{ds^2}Y = \frac{d}{ds}\left(4Y - 4Y^2 - 2YZ - 2Z\right) = -2(1+Y)\dot{Z}.$$

Hence, at a critical point of Y with $Y \geq 0$ the second derivative of Y has the opposite sign as the first derivative of Z. We can use the equation $\dot{Y} = 0$ to solve for Z and obtain

$$\begin{aligned}\frac{d}{ds}\log(Z) &= 5 - 3X - 3Y - 4Z \\ &= 5 - 3X - 3Y - 8Y\frac{1-Y}{1+Y} = -3X + \frac{5Y^2 - 6Y + 5}{1+Y}.\end{aligned}$$

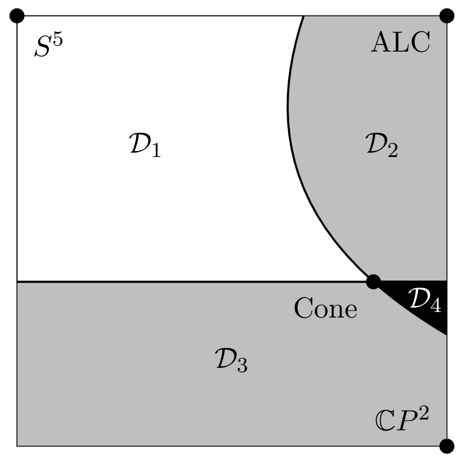
We see that a critical point of Y is a minimum if $Q(X,Y) \leq 0$. The statement for maxima of Y follows analogously. □

The above Lemma suggests that that we can study extrema of Y by ignoring Z and considering the projection of the trajectory on the (X,Y)-plane. More specifically, the curve $Q(X,Y) = 0$ partitions the plane in two disjoint regions such that Y can have a minimum only in one of them and a maximum only in the other one. As explained in the beginning of this section, all solutions of our interest satisfy the bounds $0 < X < 1$ and $Y < 1$. Because the Spin(7)-structure associated with the solution (X,Y,Z) becomes singular at $Y = 0$, we only partition the unit square. Finally, to extract information on the asymptotic geometry of complete solutions we should compare the evolution of Y with Y_c, the Y-coordinate of the critical point corresponding to the conical solution. In fact, the intersection of the curves $Q(X,Y) = 0$ and $Y = Y_c$ is precisely the projection (X_c, Y_c) of the cone (X_c, Y_c, Z_c). We partition the unit square as

$$\begin{aligned}\mathcal{D}_1 &= \{(X,Y) \in [0,1]^2 |\ Q(X,Y) > 0,\ Y > Y_c\}, \\ \mathcal{D}_2 &= \{(X,Y) \in [0,1]^2 |\ Q(X,Y) \leq 0,\ Y > Y_c\}, \\ \mathcal{D}_3 &= \{(X,Y) \in [0,1]^2 |\ Q(X,Y) > 0,\ Y \leq Y_c\}, \\ \mathcal{D}_4 &= \{(X,Y) \in [0,1]^2 |\ Q(X,Y) \leq 0,\ Y \leq Y_c\}.\end{aligned}$$

Note that $Q(X,Y)=0$ is equivalent to

$$X = \frac{1}{3}\frac{5Y^2 - 6Y + 5}{1+Y}. \tag{4.5.5}$$



We can now reformulate Lemma 4.5.4 as

Lemma 4.5.6. *Let (X,Y,Z) be a local solution contained inside the unit square. The function Y can have a minimum only in $\mathcal{D}_2 \cup \mathcal{D}_4$ and a maximum only in $\mathcal{D}_1 \cup \mathcal{D}_3$.*

To proceed from here, we again look at the family Ψ_μ for motivation. The solutions Ψ_μ start out in $\mathcal{D}_1$ and Y is initially decreasing. The next Lemma is the key technical insight of this section. It essentially says that the regions $\mathcal{D}_2$ and $\mathcal{D}_3$ are traps. If Y decreases enough and the solution enters $\mathcal{D}_3$, then it is trapped there and later we will see that in this case the solution is doomed to develop a singularity at $Y=0$. If on the contrary Y has a minimum inside $\mathcal{D}_2$, then it is trapped there being drawn towards the critical point $(1,1,0)$, the ALC end.

Lemma 4.5.7. *Let $(X(s),Y(s),Z(s))$ be a solution of the system* (4.2.33) *satisfying the conditions of Lemma 4.2.35.*

(i) Assume that at some time we have $\dot{Y} \geq 0$ and the solution projects to the interior of the common boundary of $\mathcal{D}_1$ and $\mathcal{D}_2$. Then at this time we have

$$\frac{d}{ds}Q(X,Y) < 0.$$

In particular, if at some time the solution is in $\mathcal{D}_2$ while $\dot{Y} \geq 0$, from then onwards, it is trapped in $\mathcal{D}_2$ and Y is monotone increasing.

(ii) Assume that at some time we have $\dot{Y} \leq 0$ and the solution projects to the interior of

the common boundary of $\mathcal{D}_3$ and $\mathcal{D}_4$. Then at this time we have

$$\frac{d}{ds}Q(X,Y) > 0.$$

In particular, if at some time the solution is in $\mathcal{D}_3$ while $\dot{Y} \le 0$, from then onwards as long as $Y \ge 0$, it is trapped in $\mathcal{D}_3$ and Y is monotone decreasing.

Proof. **(i):** We have

$$\nabla Q(X,Y) = \left(-3, \frac{5Y^2 + 10Y - 11}{(1+Y)^2}\right).$$

Because $\dot{Y} \ge 0$, by Lemma 4.5.4 (i) we have $Z \le 2Y\frac{1-Y}{1+Y}$. Combining this with (4.5.5) we get

$$\begin{aligned}
\partial_X Q(X,Y)\dot{X} &= -6X(2 - 2X - Z) \\
&\le -2\frac{5Y^2 - 6Y + 5}{1+Y}\left(2 - \frac{2}{3}\frac{5Y^2 - 6Y + 5}{1+Y} - 2Y\frac{1-Y}{1+Y}\right) \\
&= \frac{8}{3}\frac{(Y^2 - 3Y + 1)(5Y^2 - 6Y + 5)}{(1+Y)^2}. \qquad (4.5.8)
\end{aligned}$$

This function is negative if $Y > Y_c = \frac{3-\sqrt{5}}{2}$ and positive if $Y < Y_c = \frac{3-\sqrt{5}}{2}$.

Because $\partial_X Q(X,Y)\dot{X}$ is negative on the common boundary between $\mathcal{D}_1$ and $\mathcal{D}_2$ and $\dot{Y} \ge 0$ by assumption, we can assume $\partial_Y Q(X,Y)$ to be non-negative. This allows the estimate

$$\begin{aligned}
\partial_Y Q(X,Y)\dot{Y} &= \partial_Y Q(X,Y)(4Y - 4Y^2 - 2YZ - Z) \\
&\le \partial_Y Q(X,Y)4Y(1-Y) = \frac{5Y^2 + 10Y - 11}{(1+Y)^2}4Y(1-Y). \qquad (4.5.9)
\end{aligned}$$

Combining (4.5.8) and (4.5.9) we get

$$\begin{aligned}
\frac{d}{ds}Q(X,Y) &= \nabla Q(X,Y) \cdot (\dot{X}, \dot{Y}) \\
&\le \frac{8}{3}\frac{(Y^2 - 3Y + 1)(5Y^2 - 6Y + 5)}{(1+Y)^2} + \frac{5Y^2 + 10Y - 11}{(1+Y)^2}4Y(1-Y) \\
&= -\frac{4}{3}\frac{5Y^4 + 57Y^3 - 119Y^2 + 75Y - 10}{(1+Y)^2},
\end{aligned}$$

which is negative for $Y > \frac{1}{5}$. By Lemma 4.5.6 the function Y cannot have a maximum as long as the solution is in $\mathcal{D}_2$. Because the conditions $X < 1$ and $Y < 1$ are preserved by Lemma 4.2.35 and $\dot{Y} \ge 0$ as long as the solution is in $\mathcal{D}_2$, it can exit $\mathcal{D}_2$ only along the common boundary of $\mathcal{D}_1$ and $\mathcal{D}_2$. This was shown to be impossible.

(ii): Because now $\dot{Y} \leq 0$, we get (4.5.8) with reversed inequality sign. This estimates $\partial_X Q(X,Y)\dot{X}$ from below by a function which is positive if $Y < Y_c = \frac{3-\sqrt{5}}{2}$. Furthermore $\partial_Y Q(X,Y)$ is negative if $Y < Y_c = \frac{3-\sqrt{5}}{2}$. Because we assume $\dot{Y} \leq 0$ we get

$$\frac{d}{ds}Q(X,Y) = \nabla Q(X,Y) \cdot (\dot{X},\dot{Y}) = \partial_X Q(X,Y)\dot{X} + \partial_Y Q(X,Y)\dot{Y} > 0.$$

By Lemma 4.5.6 the function Y cannot have a minimum in $\mathcal{D}_3$. As $0 < X < 1$ is preserved by Lemma 4.2.35 and $\dot{Y} \leq 0$ by assumption, the solution can exit $\mathcal{D}_3$ only at $Y = 0$ or on the common boundary with $\mathcal{D}_4$. The latter was shown to be impossible. □

The following partition of the unit square gives another trapping argument.

$$\begin{aligned}
\mathcal{Q}_1 &= \{(X,Y) \in [0,1]^2 |\ 0 < X \leq X_c, Y_c \leq Y < 1\},\\
\mathcal{Q}_2 &= \{(X,Y) \in [0,1]^2 |\ X_c \leq X < 1, Y_c \leq Y < 1\},\\
\mathcal{Q}_3 &= \{(X,Y) \in [0,1]^2 |\ X_c \leq X < 1, 0 \leq Y \leq Y_c\},\\
\mathcal{Q}_4 &= \{(X,Y) \in [0,1]^2 |\ 0 < X \leq X_c, 0 \leq Y \leq Y_c\}.
\end{aligned}$$

Lemma 4.5.10. *Let (X,Y,Z) be a solution of the system (4.2.33).*

(i) Assume the solution is in $\mathcal{Q}_1$ at some time and then enters the interior of $\mathcal{Q}_2$ at time s_0. Then the solution is trapped in the interior of $\mathcal{Q}_2$.

(ii) The solution cannot enter $\mathcal{Q}_3$ straight from $\mathcal{Q}_1$.

(iii) Assume the solution is in $\mathcal{Q}_3$ at some time and then enters the quadrant $\mathcal{Q}_4$ at time s_0. Then the solution is trapped in the interior of $\mathcal{Q}_4$ as long as it is contained in the unit square.

(iv) The solution cannot enter $\mathcal{Q}_1$ straight from $\mathcal{Q}_3$.

Proof. **(i):** If the trajectory enters $\mathcal{Q}_2$ from $\mathcal{Q}_1$ at time s_0, then, in particular, we have $X(s_0) = X_c$, $Y(s_0) \geq Y_c$ and $\dot{X}(s_0) \geq 0$. This implies

$$0 \leq \frac{1}{2}\frac{d}{ds}\bigg|_{s=s_0} \log X \leq 2 - 2X_c - Z(s_0). \tag{4.5.11}$$

From this we deduce $Z(s_0) \leq Z_c$. Because $X(s_0) = X_c$ and the solution is not the Spin(7)-cone, one of the inequalities $Y(s_0) \geq Y_c$ and $Z(s_0) \leq Z_c$ is strict. Assume that $Z(s_0) = Z_c$. Then at time s_0 we have $Y(s_0) > Y_c$ and

$$\frac{d}{ds}\log Z < 5 - 3X_c - 3Y_c - 4Z_c = 0. \tag{4.5.12}$$

Hence for small $\epsilon > 0$ we have $Z(s_0+\epsilon) < Z_c$. By the same reasoning the condition $Z < Z_c$ is preserved as long as the solution is contained in $\mathcal{Q}_2$.

For $s > s_0$ on the common boundary of $\mathcal{Q}_2$ and $\mathcal{Q}_3$, we have

$$\frac{d}{ds}Y = 4Y_c - 4Y_c^2 - 2Y_cZ - 2Z > 4Y_c - 4Y_c^2 - 2Y_cZ_c - 2Z_c = 0. \tag{4.5.13}$$

On the common boundary of $\mathcal{Q}_2$ and $\mathcal{Q}_1$ we have

$$\frac{1}{2}\frac{d}{ds}\log X = 2 - 2X_c - Z > 2 - 2X_c - Z_c = 0. \tag{4.5.14}$$

In both cases the solution cannot cross the boundary and is trapped in $\mathcal{Q}_2$.

(ii): Assume that the solution enters $\mathcal{Q}_3$ from $\mathcal{Q}_1$ at time s_0. Then, in particular, $(X(s_0), Y(s_0)) = (X_c, Y_c)$, $\dot{X}(s_0) \geq 0$ and $\dot{Y}(s_0) \leq 0$. By (4.5.11) we again get $Z(s_0) \leq Z_c$. Because (X, Y, Z) is not the constant cone solution the inequality is strict. We get a contradiction by (4.5.13).

(iii) and **(iv)** follow as (i) and (ii), respectively, by reversing all inequalities. □

We are now ready to make the above mentioned intuition for the family Ψ_μ precise.

Proposition 4.5.15. *Suppose (X, Y, Z) is the (local) solution of (4.2.33) associated to Ψ_μ.*

(i) If Y attains a minimum in $\mathcal{D}_2$, then the solution is trapped in $\mathcal{D}_2$, Y is monotone increasing from then onwards and in particular Ψ_μ is complete.

(ii) Ψ_μ can never enter $\mathcal{D}_4$.

(iii) Y can have a minimum only in $\mathcal{D}_2$.

(iv) If Ψ_μ enters $\mathcal{D}_3$, then it is trapped there as long as it exists and Y is monotone decreasing.

(v) Y is monotone after some time.

Proof. **(i):** This follows from Lemma 4.5.7 (i). Completeness follows with Lemma 4.2.36.

(ii): Because Y can have a minimum only in $\mathcal{D}_2$ and $\mathcal{D}_4$, by (i) we know that if the solution enters $\mathcal{D}_4$ the first time, then $\dot{Y} \leq 0$. By Lemma 4.5.7 (ii) this cannot happen on the interior of the common boundary with $\mathcal{D}_3$. By Lemma 4.5.10 (i) it cannot enter $\mathcal{D}_4$ in the interior of the common boundary with $\mathcal{D}_2$ as this is the same as exiting $\mathcal{Q}_2$ after entering it from $\mathcal{Q}_1$. By Lemma 4.5.10 (ii) it cannot enter $\mathcal{D}_4$ via (X_c, Y_c).

(iii) is a consequence of (ii) and Lemma 4.5.6.

(iv): By (i) and (iii) we have $\dot{Y} \leq 0$ if Ψ_μ enters $\mathcal{D}_3$. The statement follows from Lemma 4.5.7 (ii).

(v): Initially Y is decreasing. If it has a minimum, by (iii) this can only happen in $\mathcal{D}_2$ and by (i) it is monotone increasing from then onwards. Otherwise Y is monotone decreasing. $\square$

We can draw analogous conclusions for the family Υ_τ.

Proposition 4.5.16. *Suppose (X, Y, Z) is the (local) solution of (4.2.33) associated to Υ_τ.*

(i) If the solution enters $\mathcal{D}_2$, then the solution is trapped in $\mathcal{D}_2$, Y is monotone increasing from then onwards and in particular Υ_τ is complete.

(ii) If the solution enters $\mathcal{Q}_4$ then it is trapped there as long as it exists.

(iii) Υ_τ can never enter $\mathcal{D}_1$.

(iv) Y is monotone after some time.

Proof. **(i):** The solution cannot enter $\mathcal{D}_1$ before entering $\mathcal{D}_2$ by Lemma 4.5.10 (iii), (iv). Therefore, if s_0 is the first time the solution enters $\mathcal{D}_2$, this must happen along the common boundary with $\mathcal{Q}_3$. Then $\dot{Y}(s_0) \geq 0$. Because Y cannot have a maximum in $\mathcal{D}_2 \cup \mathcal{D}_4$, Y is monotone increasing as long as the solution is in $\mathcal{D}_2$. By Lemma 4.5.7 (i) it is trapped in $\mathcal{D}_2$ and complete by Lemma 4.2.36.

(ii): If the solution enters $\mathcal{Q}_4$ by (i) and Lemma 4.5.10 (iv) it has to get there via $\mathcal{Q}_3$. The statement follows from Lemma 4.5.10 (iii).

(iii): follows from (i), (ii) and Lemma 4.5.10 (iv).

(iv): If the solution enters $\mathcal{D}_2$ at some time, the statement follows from (i). Because $\mathcal{Q}_4 \subset \mathcal{D}_3$, Y can have only a maximum in $\mathcal{Q}_4$. Therefore, if the solution ever enters $\mathcal{Q}_4$, the statement follows from (ii). We are left to deal with the case in which the solution is contained in $\mathcal{Q}_3$ as long as it exists. If Y ever has a maximum, then by Lemma 4.5.6 this must occur in $\mathcal{Q}_3 - \mathcal{D}_4$. Then by Lemma 4.5.7 (ii) the solution is trapped in $\mathcal{Q}_3 - \mathcal{D}_4$ as long as it exists and Y is monotone decreasing from then onwards. If Y never has a maximum, then it is monotone increasing. $\square$

4.5.2 Convergence to a critical point

In Propositions 4.5.15 (v) and Proposition 4.5.16 (iv) we have seen that for Ψ_μ and Υ_τ the function Y is monotone after some time. In accordance with our philosophy that the behaviour of the solution (X, Y, Z) is encoded solely in the function Y, this is enough information to deduce that all three functions become monotone after some time and the trajectory of (X, Y, Z) converges to one of the critical points listed in Remark 4.2.38.

Proposition 4.5.17. *Let (X, Y, Z) be a solution of the system (4.2.33) satisfying the conditions from Lemma 4.2.35, and suppose that Y stays positive and after some time is monotone. Then the solution is complete and converges to a critical point as $s \to \infty$.*

Proof. Completeness follows from the fact that by the assumption that Y stays positive the conditions from Lemma 4.2.35 are preserved. Because Y is monotone and bounded, it has to converge to some Y_∞ as $s \to \infty$. Using this we will first show that the same is true for X. At a critical point of X or Z, the respective second derivative is given by

$$\frac{1}{2}\frac{d^2}{ds^2}\log X = -\dot{Z}, \tag{4.5.18}$$

$$\frac{d^2}{ds^2}\log Z = -3\dot{X} - 3\dot{Y}. \tag{4.5.19}$$

First assume that Y is monotone decreasing after some time. If for example we denote by $(+-+)$ the chamber where X is increasing, Y decreasing and Z increasing, we get the following diagram:

$$\begin{array}{ccc} (+-+) & \longrightarrow & (--+) \\ \updownarrow & & \uparrow \\ (+--) & \longleftarrow & (---) \end{array}$$

Here an arrow between two chambers indicates that a solution of (4.2.33) can transition from one chamber to the other in the direction of the arrow. Because $(+-+) \to (+--) \to (+-+)$ is the only cycle, we see that X eventually has to become monotone and only Z can possibly oscillate.

If we assume that Y is monotone increasing after some time, we get the diagram

$$\begin{array}{ccc} (-++) & \longleftarrow & (+++) \\ \updownarrow & & \downarrow \\ (-+-) & \longrightarrow & (++-) \end{array}$$

Again X becomes monotone after a finite time. Because it is additionally bounded, we can conclude $X \to X_\infty$ as $s \to \infty$ for some X_∞.

If we set $L = \frac{1}{4}(5 - 3X_\infty - 3Y_\infty)$, then after a finite time

$$4Z(L - \varepsilon - Z) < \dot{Z} < 4Z(L + \varepsilon - Z)$$

for arbitrarily small $\varepsilon > 0$. If at a sufficiently large time $Z < L - \varepsilon$, then either Z is monotone increasing from then on or $Z > L - \varepsilon$ after some time, which is preserved. If at a sufficiently large time $Z > L + \varepsilon$, then either Z is monotone decreasing from then on or $Z < L - \varepsilon$ after some time, which is preserved. We can conclude that either Z becomes monotone or converges to L. Because Z is globally bounded, in either case Z converges to some Z_∞ as $s \to \infty$. It is clear that $(X_\infty, Y_\infty, Z_\infty)$ is a critical point of the system (4.2.33). □

4.5.3 Summary

We summarise the results of this section for the 1-parameter families Ψ_μ and Υ_τ.

Proposition 4.5.20. *(i) Y has a minimum at some time if and only if Ψ_μ is forward complete with (X, Y, Z) converging to $(1, 1, 0)$.*

(ii) Ψ_μ is incomplete if and only if the solution enters the interior of $\mathcal{D}_3$ at some time.

Proof. **(i):** If Y has a minimum at some time, by Proposition 4.5.15 (i), (iii) the solution is trapped in $\mathcal{D}_2$ and forward complete. Furthermore, Y is monotone increasing from then onwards. By Proposition 4.5.17 the trajectory has to converge to a critical point which projects to $\mathcal{D}_2$ while Y is monotone increasing. This excludes all but $(1, 1, 0)$ from the list given in Remark 4.2.38.

If Ψ_μ is forward complete and converges to $(1, 1, 0)$, then Y has a minimum at some time because the trajectory originates in $(0, 1, 0)$ and Y is initially decreasing.

(ii): It follows immediately from Lemma 4.2.36 that Ψ_μ can only develop a singularity at the hypersurface $Y = 0$, and thus must enter the interior of $\mathcal{D}_3$ after some time.

To prove the reverse direction, we argue by contradiction. Suppose that Ψ_μ enters the interior of $\mathcal{D}_3$ at some time and is complete. By Proposition 4.5.15 (iv) the solution is trapped in $\mathcal{D}_3$ and Y is monotone decreasing. Then its trajectory must converge to a critical point by Proposition 4.5.17. The only critical points from the list in Remark 4.2.38 which project to $\mathcal{D}_3$ are $(0, 0, 0)$ and $(1, 0, 0)$. However, $(0, 0, 0)$ is a source and the branch with $X < 1$ of the 1-dimensional stable manifold at $(1, 0, 0)$ is given by the explicit solution (4.2.40) along the X-axis. Thus Ψ_μ cannot approach either of them which is a contradiction. We conclude that Ψ_μ is incomplete. □

Proposition 4.5.21. *(i) The set $\mathfrak{X}_{\text{alc}} := \{\mu \mid \Psi_\mu$ is complete and $\lim_{s\to\infty}(X, Y, Z) = (1, 1, 0)\}$ is open.*

(ii) The set $\mathfrak{X}_{\text{ac}} := \{\mu \mid \Psi_\mu$ is complete and $\lim_{s\to\infty}(X, Y, Z) = (X_c, Y_c, Z_c)\}$ is closed.

(iii) The set $\mathfrak{X}_{\text{inc}} := \{\mu \mid \Psi_\mu$ is incomplete$\}$ is open.

The aforementioned sets are disjoint and the union is $(0, \infty)$.

Proof. By Proposition 4.5.20 (i) μ lies in $\mathfrak{X}_{\text{alc}}$ if and only if Y has a minimum at some time. This is an open condition as Ψ_μ depends continuously on μ. By Proposition 4.5.20 (ii) μ lies in $\mathfrak{X}_{\text{inc}}$ if and only if Ψ_μ enters the interior of $\mathcal{D}_3$ which again by continuity is an open condition. It is clear that the three sets are disjoint. (ii) follows once we show that the union is $(0, \infty)$.

Assume that μ lies neither in $\mathfrak{X}_{\text{alc}}$ nor in $\mathfrak{X}_{\text{inc}}$. By Proposition 4.5.20 (ii) Y never falls below the threshold Y_c. Therefore, Ψ_μ is complete by Lemma 4.2.36. Because $\mu \notin \mathfrak{X}_{\text{alc}}$, by Proposition 4.5.20 (i) Y is monotone decreasing. By Proposition 4.5.17 (X, Y, Z) converges to some critical point $(X_\infty, Y_\infty, Z_\infty)$ with non-negative coordinates. By assumption $Y_c \leq Y_\infty$. Furthermore $Y_\infty < 1$ as Y is monotone decreasing. The only such critical point is the Spin(7)-cone (X_c, Y_c, Z_c) and we get $\mu \in \mathfrak{X}_{\text{ac}}$. □

Next we treat the family Υ_τ in an analogous way.

Proposition 4.5.22. *(i) Y has a maximum at some time if and only if Υ_τ is incomplete.*

(ii) The solution enters the interior of $\mathcal{D}_2$ if and only if it is complete with (X, Y, Z) converging to $(1, 1, 0)$.

Proof. **(i):** Suppose Y has a maximum at some time. By Proposition 4.5.16 (iii) the solution can never enter $\mathcal{D}_1$. Hence by Lemma 4.5.6 Y can have a maximum only in $\mathcal{D}_3$. By Lemma 4.5.7 (ii) Y is monotone decreasing after attaining a maximum and the solution is trapped in $\mathcal{D}_3$. Υ_τ is incomplete by the same argument which we have used in the proof of Proposition 4.5.20 (ii).

By Lemma 4.2.36 Υ_τ can become incomplete only at the edge $Y = 0$. Therefore, if Υ_τ is incomplete, Y needs to have a maximum as it starts out at 0 and is initially increasing.

(ii): If the solution enters $\mathcal{D}_2$, by Proposition 4.5.16 (i) it is trapped there from then onwards, complete and Y is monotone increasing. By Proposition 4.5.17 the solution has to converge to a critical point which projects onto $\mathcal{D}_2$ with increasing Y. This excludes all but $(1, 1, 0)$ from the list given in Remark 4.2.38.

The reverse direction is clear. □

Proposition 4.5.23. *(i) The set $\mathfrak{Y}_{\text{alc}} := \{\tau \mid \Upsilon_\tau$ is complete and $\lim_{s\to\infty}(X, Y, Z) = (1, 1, 0)\}$ is open.*

(ii) The set $\mathfrak{Y}_{\text{ac}} := \{\tau \mid \Upsilon_\tau$ is complete and $\lim_{s\to\infty}(X, Y, Z) = (X_c, Y_c, Z_c)\}$ is closed.

(iii) The set $\mathfrak{Y}_{\text{inc}} := \{\tau \mid \Upsilon_\tau$ is incomplete$\}$ is open.

The aforementioned sets are disjoint and the union is $\mathbb{R}$.

Proof. $\mathfrak{Y}_{\text{alc}}$ is open because by Proposition 4.5.22 (ii) $\tau \in \mathfrak{Y}_{\text{alc}}$ if and only if it enters $\mathcal{D}_2$, which is an open condition. $\mathfrak{Y}_{\text{inc}}$ is open because by Proposition 4.5.22 (i) $\tau \in \mathfrak{Y}_{\text{inc}}$ if and only if the function Y of Υ_τ has a maximum, which is an open condition. The sets are clearly disjoint. (ii) follows once we have shown that the union is all of $\mathbb{R}$.

Assume $\tau \notin \mathfrak{Y}_{\text{inc}}$. In particular, Υ_τ is complete and by Proposition 4.5.22 (i) Y is monotone increasing. By Proposition 4.5.17 the solution has to converge to a critical point. Because it cannot enter $\mathcal{D}_1$ by Proposition 4.5.16 (iii), the only critical points left are the cone and $(1, 1, 0)$. Therefore, τ must lie in $\mathfrak{Y}_{\text{ac}}$ or $\mathfrak{Y}_{\text{alc}}$. □

Remark 4.5.24. By Lemma 4.4.8 all members of the sets $\mathfrak{X}_{\mathrm{ac}}$ and $\mathfrak{Y}_{\mathrm{ac}}$ are AC. In section 4.6 we will make rigorous the intuition from Remark 4.2.38: forward complete solutions converging to $(1,1,0)$ are ALC. Therefore the sets $\mathfrak{X}_{\mathrm{alc}}$ and $\mathfrak{Y}_{\mathrm{alc}}$ corresponds precisely to complete ALC solutions. Our roadmap to prove the existence of at least one complete AC metric in each of the families Ψ_μ and Υ_τ, which then must be unique by Lemma 4.5.3, is now clear:

- Show that $\mathfrak{X}_{\mathrm{alc}}$ and $\mathfrak{Y}_{\mathrm{alc}}$ are non-empty.
- Show that $\mathfrak{X}_{\mathrm{inc}}$ and $\mathfrak{Y}_{\mathrm{inc}}$ are non-empty.

4.6 Complete ALC metrics

4.6.1 ALC Asymptotics

In this section we are going to show that any forward complete solution (a,b,c,f) of the ODE system (4.2.20) describes an ALC Spin(7)-structure if and only if the associated solution (X,Y,Z) of the system (4.2.33) converges to the critical point $(1,1,0)$ as $s\to\infty$.

Lemma 4.6.1. *Let (a,b,c,f) be a solution of the system (4.2.20), where a,b,c,f are positive functions satisfying $a,b<c$. If the associated solution (X,Y,Z) of the system (4.2.33) is forward complete with $\lim_{s\to\infty}(X,Y,Z)=(1,1,0)$, then (a,b,c,f) is forward complete and there exists $\ell>0$ such that*

$$\lim_{t\to\infty}\frac{a(t)}{t}=1,\quad \lim_{t\to\infty}\frac{b(t)}{t}=1,\quad \lim_{t\to\infty}\frac{c(t)}{t}=1,\quad \lim_{t\to\infty}f(t)=\ell. \tag{4.6.2}$$

Proof. (a,b,c,f) is forward complete by Lemma 4.2.36. The relation

$$(X,Y,Z)=(a^2/c^2,b^2/c^2,abf/c^3)$$

allows us to substitute $\lim_{s\to\infty}(X,Y,Z)=(1,1,0)$ into the right-hand side of the ODE system (4.2.20) to obtain

$$\lim_{t\to\infty}\dot a(t)=1,\quad \lim_{t\to\infty}\dot b(t)=1,\quad \lim_{t\to\infty}\dot c(t)=1.$$

This proves the assertion for a,b and c.

Next we show that f is bounded. By assumption

$$Z=\frac{abf}{c^3}=\left(\frac{a}{t}\right)\left(\frac{b}{t}\right)\left(\frac{t}{c}\right)^3\frac{f}{t}$$

converges to zero. By the already established limiting behaviour on a, b, c, we see that f grows at most as $o(t)$. In particular, there exists some $\kappa > 0$ such that for sufficiently large times $f(t) < t/\kappa$. To see that f is bounded we write

$$\dot{f} = \frac{f^2}{t^2}\frac{t^2}{b^2}(1 - Y).$$

Hence because $\lim_{t\to\infty} Y(t) = 1$ and $\lim_{t\to\infty} b(t)/t = 1$ we see that for a sufficiently large times $f(t) < t/\kappa$ and

$$\dot{f} < \kappa \frac{f^2}{t^2}.$$

The boundedness of f follows from Lemma 4.6.3. Because we have $b < c$, f is monotone increasing by Lemma 4.2.26. Since f is bounded, monotone increasing and positive, it will converge to some positive constant ℓ. □

We have used the following comparison principle.

Lemma 4.6.3. *Let $f \in \mathcal{C}^1([T, \infty))$ with $T > 0$. If there exist $t_0 > T$ and $C > 0$ such that $\dot{f} < Cf^2/t^2$ for all $t \in [t_0, \infty)$ and $f(t_0) < t_0/C$, then f is bounded from above.*

Proof. The solution of the model equation $\dot{h} = Ch^2/t^2$ is

$$h_\alpha(t) = \frac{t}{C - \alpha t}$$

where α is the constant of integration. For us only $\alpha < 0$ is relevant. In this case h_α has a singularity at $t^* = C/\alpha < 0$ with $\lim_{t\nearrow t^*} h(t) = \infty$ and $\lim_{t\searrow t^*} h(t) = -\infty$ and is asymptotic to $1/|\alpha|$ for $t \to \pm\infty$.

If $f(t_0) < t_0/C$ for some $t_0 > T$, we can find an $\alpha < 0$ to make this inequality slightly stronger:

$$f(t_0) < \frac{t_0}{C - \alpha t_0} = h_\alpha(t_0).$$

By the above discussion h_α is smooth for $t \geq 0$ because $\alpha < 0$ and bounded from above by $1/|\alpha|$. Hence for all $t \geq t_0$ we have the bound $f(t) < h_\alpha(t) < 1/|\alpha|$. It follows that f is bounded. □

Proposition 4.6.4. *Assume that (a, b, c, f) is a forward complete solution of the system* (4.2.20) *which satisfies* (4.6.2). *Write*

$$\tilde{a}(t) = t^{-1}a(t) - 1, \quad \tilde{b}(t) = t^{-1}b(t) - 1, \quad \tilde{c}(t) = t^{-1}c(t) - 1, \quad \tilde{f}(t) = \frac{1}{\ell}f(t) - 1.$$

Then there exists $\gamma > 0$ such that $\tilde{a}^{(k)}(t), \tilde{b}^{(k)}(t), \tilde{c}^{(k)}(t), \tilde{f}^{(k)}(t)$ behave like $\mathcal{O}(t^{-k-\gamma})$ as $t \to \infty$ for $k \geq 0$. Here $\tilde{a}^{(k)}(t)$ denotes the k-th derivative of $\tilde{a}(t)$.

Proof. Set

$$a(t) = t(1 + X_1(t)), \quad b(t) = t(1 + X_2(t)), \quad c(t) = t(1 + X_3(t)), \quad f(t) = \ell(1 + X_4(t)).$$

The assumption (4.6.2) is equivalent to

$$\lim_{t\to\infty} X_i(t) = 0, \quad \text{for } i = 1, \dots, 4$$

After the change of variable $e^\tau = t$ the system (4.2.20) becomes

$$\begin{aligned}
\frac{dX_1}{d\tau} &= -X_1 + \frac{(1+X_2)^2 + (1+X_3)^2 - (1+X_1)^2}{(1+X_2)(1+X_3)} - 1, \\
\frac{dX_2}{d\tau} &= -X_2 + \frac{(1+X_3)^2 + (1+X_1)^2 - (1+X_2)^2}{(1+X_3)(1+X_1)} - \ell\, e^{-\tau}\frac{1+X_4}{1+X_2} - 1, \\
\frac{dX_3}{d\tau} &= -X_3 + \frac{(1+X_1)^2 + (1+X_2)^2 - (1+X_3)^2}{(1+X_1)(1+X_2)} + \ell\, e^{-\tau}\frac{1+X_4}{1+X_3} - 1, \\
\frac{dX_4}{d\tau} &= \ell\, e^{-\tau}\frac{(1+X_4)^2}{(1+X_2)^2} - \ell\, e^{-\tau}\frac{(1+X_4)^2}{(1+X_3)^2}.
\end{aligned}$$

Setting $X_5 = e^{-\tau}$ and $X = (X_1, X_2, X_3, X_4, X_5)$, we get a system of equations of the form $\frac{dX}{d\tau} = \Phi(X)$, where $X(0) = 0$ and the linearisation of Φ at 0 is given by

$$d\Phi|_{U=0} = \begin{pmatrix} -3 & 1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & -\ell \\ 1 & 1 & -3 & 0 & \ell \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

$d\Phi_{U=0}$ has a 1-dimensional kernel spanned by $(0,0,0,1,0)$ and four negative eigenvalues. Moreover, $\{(0,0,0,c,0) \,|\, c \in \mathbb{R}\}$ is the center manifold of the system. The center manifold equation is

$$\frac{dX_4}{d\tau} = 0.$$

Hence by [Car81, Theorem 2] for any solution X converging to the stationary point $X = 0$ as in our hypothesis there exists $\gamma > 0$ such that

$$(X_1, X_2, X_3, X_4, X_5) = (0,0,0,0,0) + \mathcal{O}(e^{-\gamma\tau}).$$

The polynomial decay follows by switching back to the variable t. The argument for the derivatives of (X_1, X_2, X_3, X_4) follows from a bootstrap argument. □

The results in this section prove

Proposition 4.6.5. *Let (a, b, c, f) be a solution of the system (4.2.20), where a, b, c, f are positive functions satisfying $a, b < c$. Suppose the associated solution (X, Y, Z) of the system (4.2.33) is forward complete with*

$$\lim_{s\to\infty}(X, Y, Z) = (1, 1, 0).$$

Then (a, b, c, f) defines an $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant ALC Spin(7) metric on $(T, \infty)\times N(1, -1)$ for some $T > 0$.

4.6.2 Existence of ALC solutions

In this section we address the first part of the strategy outlined in Remark 4.5.24, i.e. the existence of ALC metrics among the families Ψ_μ and Υ_τ. Cvetič–Gibbons–Lü–Pope [CGLP02a], Gukov–Sparks [GS02] and Kanno–Yasui [KY02] found an explicit ALC member of the family Υ_τ. This shows that $\mathfrak{Y}_{\mathrm{alc}}$ is non-empty. In his treatment of highly collapsed ALC Spin(7)-manifolds, Foscolo [Fos19] showed that Ψ_μ is ALC if μ is very small. This proves that $\mathfrak{X}_{\mathrm{alc}}$ is non-empty. Alternatively, this result can be recovered using ODE methods.

Proposition 4.6.6. *If $\mu > 0$ is sufficiently small, then Ψ_μ is complete and ALC.*

Proof. By Proposition 4.6.5 it is enough to show that the solution (X, Y, Z) of the system (4.2.33) associated with Ψ_μ converges to the critical point $(1, 1, 0)$ as $s \to \infty$ if μ is sufficiently small. The linearisation of the system (4.2.33) around $(1, 1, 0)$ is given by

$$\begin{pmatrix} -4 & 0 & -2 \\ 0 & -4 & -4 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore, the critical point $(1, 1, 0)$ is a sink. As explained in Remark 4.3.4, $\mu = 0$ gives the Bryant–Salamon G_2 holonomy metric on $\Lambda^2_-\mathbb{C}P^2$, of which the associated solution (X, Y, Z) converges to $(1, 1, 0)$. Because Ψ_μ depends continuously on μ, if μ is sufficiently small, Ψ_μ at some time will be close to the sink $(1, 1, 0)$ and then sucked towards it. The statement follows from Proposition 4.6.5. □

4.7 Existence of incomplete solutions

In this section we carry out the second step in the strategy outlined in Remark 4.5.24. We show that Ψ_μ and Υ_τ are incomplete if μ and τ, respectively, are sufficiently large. This shows that the sets $\mathfrak{X}_{\text{inc}}$ and $\mathfrak{Y}_{\text{inc}}$ are non-empty.

We start by considering the family Ψ_μ. Looking at the asymptotics (4.3.2) suggests to rescale time by $\hat{t} = \mu t$ and consider $\widehat{\Psi}_\mu = \{\hat{a}, \hat{b}, \hat{c}, \hat{f}\} = \{\mu a, b, c, f/\mu\}$. This prevents a from collapsing and f from exploding in the limit. We are now going to derive the ODE system for these functions with respect to the new parameter. We will write $\epsilon = 1/\mu$.

$$\frac{d}{d\hat{t}}\hat{a} = \frac{\hat{b}}{\hat{c}} + \frac{\hat{c}}{\hat{b}} - \epsilon^2 \frac{\hat{a}^2}{\hat{b}\hat{c}}, \tag{4.7.1a}$$

$$\frac{d}{d\hat{t}}\hat{b} = \frac{\hat{c}}{\hat{a}} + \epsilon^2 \frac{\hat{a}}{\hat{c}} - \frac{\hat{b}^2}{\hat{c}\hat{a}} - \frac{\hat{f}}{\hat{b}}, \tag{4.7.1b}$$

$$\frac{d}{d\hat{t}}\hat{c} = \epsilon^2 \frac{\hat{a}}{\hat{b}} + \frac{\hat{b}}{\hat{a}} - \frac{\hat{c}^2}{\hat{a}\hat{b}} + \frac{\hat{f}}{\hat{c}}, \tag{4.7.1c}$$

$$\frac{d}{d\hat{t}}\hat{f} = \frac{\hat{f}^2}{\hat{b}^2} - \frac{\hat{f}^2}{\hat{c}^2}. \tag{4.7.1d}$$

The key insight is that this system of equations is well-defined for $\epsilon = 0$. This allows us to make sense of the limit of $\widehat{\Psi}_\mu$ as $\mu \to \infty$. However, because Ψ_μ is only defined for finite values of μ, we need to prove the short time existence of $\widehat{\Psi}_\epsilon$ for $\epsilon = 0$ and show that $\widehat{\Psi}_\epsilon$ depends continuously on ϵ. The continuity in the parameter ϵ will allow us to reduce the study of $\widehat{\Psi}_\mu$ for large μ to the study of the system (4.7.1) in the limit $\epsilon = 0$. To carry out this program, we need to consider a singular initial value problem of the following form:

Theorem 4.7.2. *[FHN18, Theorem 4.3] Consider the initial value problem*

$$\dot{y} = \frac{1}{t} M_{-1}(y) + M(y,t), \quad y(0) = y_0, \tag{4.7.3}$$

where y takes values in $\mathbb{R}^k$, $M_{-1} \colon \mathbb{R}^k \to \mathbb{R}^k$ is a smooth function of y in a neighbourhood of y_0 and $M \colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is smooth in t, y in a neighbourhood of $(0, y_0)$. Assume that

(i) $M_{-1}(y_0) = 0$;

(ii) $h\mathrm{Id} - d_{y_0} M_{-1}$ is invertible for all $h \in \mathbb{N}, h \geq 1$.

Then there exists a unique solution $y(t)$ of (4.7.3) which is smooth up to $t = 0$. Furthermore y depends continuously on y_0 satisfying (i) and (ii).

This allows us to prove

Proposition 4.7.4. *For every $\epsilon \geq 0$, there exists a local solution $\widehat{\Psi}_\epsilon$ of the rescaled system (4.7.1), such that after a coordinate and parameter change $\widehat{\Psi}_\epsilon$ corresponds to $\Psi_{1/\epsilon}$ for $\epsilon > 0$. Moreover, $\widehat{\Psi}_\epsilon$ depends continuously on ϵ.*

Proof. Using the relation $\widehat{\Psi}_\mu = \{\hat{a}, \hat{b}, \hat{c}, \hat{f}\} = \{\mu a, b, c, f/\mu\}$ and the asymptotic expansion (4.3.2) allows us to write

$$\hat{a} = 2\hat{t} + x_1\hat{t}^3, \quad \hat{b} = 1 - \frac{1}{3}\hat{t} + x_2\hat{t}^2, \quad \hat{c} = 1 + \frac{1}{3}\hat{t} + x_3\hat{t}^2, \quad \hat{f} = 1 + x_4\hat{t}^2$$

with

$$x_1(0) = -\frac{4}{27}(9\epsilon^2 - 1), \quad x_2(0) = x_3(0) = \epsilon^2 - \frac{5}{18}, \quad x_4(0) = \frac{2}{3}. \tag{4.7.5}$$

Denoting differentiation with respect to $\hat{t}$ by a dot, the ODE system (4.7.1) becomes

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{\hat{t}}(-3x_1 - 4\epsilon^2 + 4/9) + F_1(x, \hat{t}, \epsilon),\\
\dot{x}_2 &= \frac{1}{\hat{t}}\left(-3x_2 + x_3 + 2\epsilon^2 - 5/9\right) + F_2(x, \hat{t}, \epsilon),\\
\dot{x}_3 &= \frac{1}{\hat{t}}\left(x_2 - 3x_3 + 2\epsilon^2 - 5/9\right) + F_3(x, \hat{t}, \epsilon),\\
\dot{x}_4 &= \frac{1}{\hat{t}}(-2x_4 + 4/3) + F_4(x, \hat{t}, \epsilon).
\end{aligned}$$

Here $F_1(x, \hat{t}, \epsilon), F_2(x, \hat{t}, \epsilon), F_3(x, \hat{t}), F_4(x, \hat{t}, \epsilon)$ are functions which depend smoothly on $(\hat{t}, x)$. This is a system of the form (4.7.3). M_{-1} vanishes at the initial condition (4.7.5). The linearisation L of M_{-1} is given

$$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

We have

$$\det(n\mathrm{Id} - L) = (n+2)(n+3)(n^2 + 6n + 8),$$

which is positive for $n \geq 0$. The statement follows with Theorem 4.7.2. □

To study long time behaviour of the local solutions $\widehat{\Psi}_\epsilon$ for the rescaled system (4.7.1), analogously to the treatment of the original ODE system (4.2.20) we switch to coordinates

$(\hat{X}, \hat{Y}, \hat{Z}) = (\hat{a}^2/\hat{c}^2, \hat{b}^2/\hat{c}^2, \hat{a}\hat{b}\hat{f}/\hat{c}^3)$. The rescaled analogue of (4.2.33) is the system

$$\frac{d}{d\hat{s}}\hat{X} = 2\hat{X}(2 - 2\epsilon^2\hat{X} - \hat{Z}), \tag{4.7.6a}$$
$$\frac{d}{d\hat{s}}\hat{Y} = 4\hat{Y} - 4\hat{Y}^2 - 2\hat{Y}\hat{Z} - 2\hat{Z}, \tag{4.7.6b}$$
$$\frac{d}{ds}\hat{Z} = \hat{Z}(5 - 3\epsilon^2\hat{X} - 3\hat{Y} - 4\hat{Z}). \tag{4.7.6c}$$

The analysis of this system is much easier than that of (4.2.33) because the system simplifies significantly if we pass to the limit $\epsilon = 0$:

$$\frac{d}{d\hat{s}}\hat{X} = 2\hat{X}(2 - \hat{Z}), \tag{4.7.7a}$$
$$\frac{d}{d\hat{s}}\hat{Y} = 4\hat{Y} - 4\hat{Y}^2 - 2\hat{Y}\hat{Z} - 2\hat{Z}, \tag{4.7.7b}$$
$$\frac{d}{ds}\hat{Z} = \hat{Z}(5 - 3\hat{Y} - 4\hat{Z}). \tag{4.7.7c}$$

In particular, we have the subsystem

$$\frac{d}{d\hat{s}}\hat{Y} = 4\hat{Y} - 4\hat{Y}^2 - 2\hat{Y}\hat{Z} - 2\hat{Z}, \tag{4.7.8a}$$
$$\frac{d}{ds}\hat{Z} = \hat{Z}(5 - 3\hat{Y} - 4\hat{Z}). \tag{4.7.8b}$$

for the evolution of (Y, Z). The fist step in the analysis of the system (4.7.8) is to derive the same preservation laws as in Lemma 4.2.35.

Lemma 4.7.9. *Assume that a (local) solution $(\hat{Y}, \hat{Z})$ of the system (4.7.8) satisfies*

(i) $\hat{Y} < 1$,

(ii) $0 < \hat{Z} < \frac{5}{4}$.

Then this set of conditions is preserved as long as $\hat{Y}$ is positive.

This allows us to prove

Lemma 4.7.10. *Assume $(\hat{Y}, \hat{Z})$ is a (local) solution of (4.7.8) satisfying the conditions from Lemma 4.7.9 and additionally $\frac{d}{d\hat{s}}\hat{Y} < 0$ and $\frac{d}{d\hat{s}}\hat{Z} > 0$. Then this set of conditions is preserved and after a finite time $\hat{Y}$ attains the value 0.*

Proof. By Lemma 4.7.9 $(\hat{Y}, \hat{Z}) \in (0,1) \times (0, 5/4)$ is preserved unless $\hat{Y}$ attains the value 0. Extremal points of $\hat{Y}$ are characterised by the equation

$$\hat{Z} = 2\hat{Y}\frac{1 - \hat{Y}}{1 + \hat{Y}}. \tag{4.7.11}$$

and extremal points for $\hat{Z}$ is characterised by the equation

$$\hat{Y} = \frac{5}{3} - \frac{4}{3}\hat{Z}. \tag{4.7.12}$$

In particular, there are no critical points in $[0,1] \times [0,5/4]$. At an extremal point of $\hat{Y}$ the second derivative is given by

$$\frac{d^2}{d\hat{s}^2}\hat{Y} = -2(1+\hat{Y})\frac{d}{d\hat{s}}\hat{Z}.$$

This is negative by assumption. Hence, under our assumptions as long as $\hat{Y}$ is non-negative it cannot have a minimum and is monotone decreasing.

The second derivative of $\hat{Z}$ at an extremal point is given by

$$\frac{d^2}{d\hat{s}^2}\hat{Z} = -3\hat{Z}\frac{d}{d\hat{s}}\hat{Y},$$

which is positive by assumption. Hence under our assumptions $\hat{Z}$ cannot have a maximum and is monotone increasing.

If the solution exists for all times with $\hat{Y} > 0$, then it has to converge to a critical point $(\hat{Y}_\infty, \hat{Z}_\infty) \in [0,1] \times [0,5/4]$, which doesn't exist. Therefore, $\hat{Y}$ has to cross 0 after a finite time. □

We get

Proposition 4.7.13. *For μ sufficiently large, the function Y of the associated solution of Ψ_μ reaches 0 after a finite time and therefore Ψ_μ is incomplete. In particular, the set $\mathfrak{X}_{\text{inc}}$ is non-empty.*

Proof. Because $\widehat{\Psi}_\epsilon$ depends continuously on ϵ and $\hat{Y}$ reaches the value 0 if $\epsilon = 0$, the same is true for ϵ small, i.e. μ large. In particular, if μ is sufficiently large $Y = \hat{Y}$ attains 0 after a finite time and therefore Ψ_μ is incomplete. □

Next we consider the family Υ_τ. Here the rescaling

$$\hat{t} = \sqrt{\tau}t, \quad (\hat{a}, \hat{b}, \hat{c}, \hat{f}) = (a, \sqrt{\tau}b, c, \sqrt{\tau}f)$$

is helpful. Writing $\epsilon = 1/\sqrt{\tau}$ the ODE system becomes

$$\frac{d}{d\hat{t}}\hat{a} = \epsilon^2 \frac{\hat{b}}{\hat{c}} + \frac{\hat{c}}{\hat{b}} - \frac{\hat{a}^2}{\hat{b}\hat{c}}, \tag{4.7.14a}$$

$$\frac{d}{d\hat{t}}\hat{b} = \frac{\hat{c}}{\hat{a}} + \frac{\hat{a}}{\hat{c}} - \epsilon^2 \frac{\hat{b}^2}{\hat{c}\hat{a}} - \frac{\hat{f}}{\hat{b}}, \tag{4.7.14b}$$

$$\frac{d}{d\hat{t}}\hat{c} = \frac{\hat{a}}{\hat{b}} + \epsilon^2 \frac{\hat{b}}{\hat{a}} - \frac{\hat{c}^2}{\hat{a}\hat{b}} + \epsilon^2 \frac{\hat{f}}{\hat{c}}, \tag{4.7.14c}$$

$$\frac{d}{d\hat{t}}\hat{f} = \frac{\hat{f}^2}{\hat{b}^2} - \epsilon^2 \frac{\hat{f}^2}{\hat{c}^2}. \tag{4.7.14d}$$

Proposition 4.7.15. *For every $\epsilon \geq 0$, there exists a local solution $\widehat{\Upsilon}_\epsilon$ of the rescaled system (4.7.14), such that after a coordinate and parameter change $\widehat{\Upsilon}_\epsilon$ corresponds to Υ_{1/ϵ^2} for $\epsilon > 0$. Moreover, $\widehat{\Upsilon}_\epsilon$ depends continuously on ϵ.*

Proof. The short distance asymtptotic expansion (4.3.7) allows us to write

$$\hat{a}(\hat{t}) = 1 + x_1(\hat{t})\hat{t}^2, \quad \hat{b}(\hat{t}) = \hat{t} + x_2(\hat{t})\hat{t}^3, \quad \hat{c}(\hat{t}) = 1 + x_3(\hat{t})\hat{t}^2, \quad \hat{f}(\hat{t}) = \hat{t} + x_4(\hat{t})\hat{t}^3,$$

with

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = \left(\frac{2}{3}\epsilon^2, -\frac{1}{2}\epsilon^2 - \frac{1}{24}, \frac{5}{6}\epsilon^2, \frac{1}{12}\right).$$

For $x = (x_1, x_2, x_3, x_4)$ we get the system

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{\hat{t}}(-4x_1 + 2x_3 + \epsilon^2) + F_1(x, \hat{t}, \epsilon), \\
\dot{x}_2 &= \frac{1}{\hat{t}}(-2x_2 - x_4 - \epsilon^2) + F_2(x, \hat{t}, \epsilon), \\
\dot{x}_3 &= \frac{1}{\hat{t}}(2x_1 - 4x_3 + 2\epsilon^2) + F_3(x, \hat{t}, \epsilon), \\
\dot{x}_4 &= \frac{1}{\hat{t}}(-2x_2 - x_4 - \epsilon^2) + F_4(x, \hat{t}, \epsilon),
\end{aligned}$$

where $F_1(x, \hat{t}, \epsilon), F_2(x, \hat{t}, \epsilon), F_3(x, \hat{t}, \epsilon), F_4(x, \hat{t}, \epsilon)$ are smooth functions of x and $\hat{t}$. Hence, it is of the form (4.7.3). M_{-1} vanishes at $(x_1(0), x_2(0), x_3(0), x_4(0))$, and the linearisation is given by

$$L = \begin{pmatrix} -4 & 0 & 2 & 0 \\ 0 & -2 & 0 & -1 \\ 2 & 0 & -4 & 0 \\ 0 & -2 & 0 & -1 \end{pmatrix}.$$

We have

$$\det(n\mathrm{Id} - L) = n^4 + 11n^3 + 36n^2 + 36n,$$

which is positive for $n \geq 1$. The statement follows from Theorem 4.7.2. □

Again transforming to (X, Y, Z) coordinates, the system (4.7.14) becomes

$$\frac{d}{d\hat{s}}\hat{X} = 2\hat{X}(2 - 2\hat{X} - \epsilon^2\hat{Z}), \tag{4.7.16a}$$
$$\frac{d}{d\hat{s}}\hat{Y} = 4\hat{Y} - \epsilon^2 4\hat{Y}^2 - \epsilon^2 2\hat{Y}\hat{Z} - 2\hat{Z}, \tag{4.7.16b}$$
$$\frac{d}{d\hat{s}}\hat{Z} = \hat{Z}(5 - 3\hat{X} - \epsilon^2 3\hat{Y} - \epsilon 4\hat{Z}). \tag{4.7.16c}$$

In the limit $\epsilon = 0$ this becomes

$$\frac{d}{d\hat{s}}\hat{X} = 4\hat{X}(1 - \hat{X}), \tag{4.7.17a}$$
$$\frac{d}{d\hat{s}}\hat{Y} = 4\hat{Y} - 2\hat{Z}, \tag{4.7.17b}$$
$$\frac{d}{d\hat{s}}\hat{Z} = \hat{Z}(5 - 3\hat{X}). \tag{4.7.17c}$$

The asymptotic expansion of $\widehat{\Upsilon}_\tau$ as $\hat{t} \to 0$ is given by

$$\begin{aligned}
\hat{a}(\hat{t}) &= 1 + \frac{2}{3}\epsilon^2\hat{t}^2 + \frac{-104\epsilon^2 - 1}{288}\epsilon^2\hat{t}^4 + \mathcal{O}(\hat{t}^5),\\
\hat{b}(\hat{t}) &= \hat{t} - \frac{12\epsilon^2 + 1}{24}\hat{t}^3 + \mathcal{O}(\hat{t}^5),\\
\hat{c}(\hat{t}) &= 1 + \frac{5}{6}\epsilon^2\hat{t}^2 + \frac{-140\epsilon^2 + 1}{288}\epsilon^2\hat{t}^4 + \mathcal{O}(\hat{t}^5),\\
\hat{f}(\hat{t}) &= \hat{t} + \frac{1}{12}\hat{t}^3 + \mathcal{O}(\hat{t}^5).
\end{aligned}$$

In $(\hat{X}, \hat{Y}, \hat{Z})$ coordinates we obtain the asymptotic expansion

$$\hat{X}(\hat{t}) = 1 - \frac{\epsilon^2}{3}\hat{t}^2 + \frac{40\epsilon^2 - 1}{72}\epsilon^2\hat{t}^4 + \mathcal{O}(\hat{t}^5), \tag{4.7.18a}$$
$$\hat{Y}(\hat{t}) = \hat{t}^2 + \frac{-64\epsilon^2 - 2}{24}\hat{t}^4 + \mathcal{O}(\hat{t}^5), \tag{4.7.18b}$$
$$\hat{Z}(\hat{t}) = \hat{t}^2 + \frac{-56\epsilon^2 + 1}{24}\hat{t}^4 + \mathcal{O}(\hat{t}^5). \tag{4.7.18c}$$

Proposition 4.7.19. *For τ sufficiently large, Y reaches 0 after a finite time and therefore Υ_τ is incomplete. In particular, $\mathfrak{Y}_{\mathrm{inc}}$ is non-empty.*

Proof. We first study the limit system 4.7.17. The general solution of the first equation is $\hat{X}(\hat{s}) = e^{4\hat{s}}/(C + e^{4\hat{s}})$. Because we know from the asymptotic expansion (4.7.18) that $\hat{X}(\hat{s}) = X(s) \to 1$ as $\hat{s} \to -\infty$, we can conclude that $C = 0$ and $\hat{X} \equiv 1$. The remaining system for $(\hat{Y}, \hat{Z})$ solves as

$$\hat{Y}(\hat{s}) = C_2 e^{4\hat{s}} + C_1 e^{2\hat{s}}, \quad \hat{Z}(\hat{s}) = C_1 e^{2\hat{s}},$$

where C_1, C_2 are constants. From the asymptotic expansion, we can see that $\hat{Z} > 0$, which implies $C_1 > 0$, and that $\hat{Y} < \hat{Z}$, which implies $C_2 < 0$. Then clearly in finite time $\hat{Y} = 0$. Because $\widehat{\Upsilon}_\epsilon$ depends continuously on ϵ, the same is true for ϵ small, i.e. τ large. From $\hat{Y} = \tau^2 Y$, we can conclude the same for the function Y of Υ_τ for τ sufficiently large. □

4.8 Proofs of Theorems B, C and D

Proof of Theorem B. By the Propositions 4.5.21, 4.6.6 and 4.7.13 the sets $\mathfrak{X}_{\text{alc}}$ and $\mathfrak{X}_{\text{inc}}$ are open and non-empty. Proposition 4.5.21 and Lemmas 4.5.2 and 4.5.3 imply that there exists a unique parameter $\mu_{\text{ac}} \in \mathfrak{X}_{\text{ac}}$, which gives rise to an AC space.

Suppose that $\mu \in (0, \mu_{\text{ac}})$. Then by Lemma 4.5.2 and Proposition 4.5.20 we have $Y_\mu(s) > Y_{\mu_{\text{ac}}}(s) > Y_c$ as long as Ψ_μ exists. With Proposition 4.5.21 we get $\mu \in \mathfrak{X}_{\text{alc}}$, and by Proposition 4.6.5 Ψ_μ is ALC.

Suppose that $\mu > \mu_{\text{ac}}$. With Lemma 4.5.2 we get $Y_\mu(s) < Y_{\mu_{\text{ac}}}(s)$ as long as Ψ_μ exists. Because $Y_{\mu_{\text{ac}}}$ converges to $Y_c < 1$ we have $\mu \notin \mathfrak{Y}_{\text{alc}}$. With Proposition 4.5.21 and the uniqueness of μ_{ac} we get $\mu \in \mathfrak{X}_{\text{inc}}$. □

Proof of Theorem C. Proposition 4.5.23, the discussion at the beginning of section 4.6.2, and Proposition 4.7.19 show that both $\mathfrak{Y}_{\text{alc}}$ and $\mathfrak{Y}_{\text{inc}}$ are open and non-empty. Proposition 4.5.23 and Lemmas 4.5.2 and 4.5.3 imply that there exists a unique parameter $\tau_{\text{ac}} \in \mathfrak{Y}_{\text{ac}}$, which gives rise to an AC space.

Suppose that $\tau < \tau_{\text{ac}}$. By Lemma 4.5.2 as long as Υ_τ exists we have $Y_\tau(s) > Y_{\tau_{\text{ac}}}(s)$. Because the latter converges to Y_c we get $\tau \notin \mathfrak{Y}_{\text{inc}}$. By the uniqueness of τ_{ac} we get $\tau \in \mathfrak{Y}_{\text{alc}}$.

Suppose that $\tau > \tau_{\text{ac}}$. Then $Y_\tau(s) < Y_{\tau_{\text{ac}}}(s) < Y_c$ as long as the solution exists. The uniqueness of τ_{ac} implies $\tau \in \mathfrak{Y}_{\text{inc}}$.

Formulas (4.2.13) and 4.3.7 show that $\Upsilon_\tau|_{\mathbb{C}P^2} = e_{4356}$ (with the original use of the functions a and b). This is the volume form of $\mathbb{C}P^2$ with respect to the induced metric and the appropriate orientation. Therefore, the zero section is a Cayley submanifold with respect to Υ_τ for all τ. Because the $\mathbb{C}P^2$ is a generator of $H_4(M_{\mathbb{C}P^2})$ and its volume with respect to Υ_τ is positive, the cohomology class of Υ_τ is non-trivial. □

Proof of Theorem D. It is clear that $\lambda = 0$ gives the Spin(7)-cone. The short distance asymptotic expansion (4.4.7) for $\Psi_\lambda^{\mathrm{cs}}$ gives

$$Y(t) \approx Y_c(1 - 9.86\lambda t^{\nu_2} + \mathcal{O}(t^{2\nu_2})),$$
$$Q(X(t), Y(t)) \approx 14.41\lambda t^{\nu_2} + \mathcal{O}(t^{2\nu_2}).$$

If $\lambda < 0$, this implies that $\Psi_\lambda^{\mathrm{cs}}$ enters the region $\mathcal{D}_2$ with $\dot{Y} > 0$. By Lemma 4.5.7 (i) it follows that the solution is trapped in $\mathcal{D}_2$ for all times and Y is monotone increasing. By Lemma 4.2.36 it is forward complete and by Proposition 4.5.17 it has to converge to a critical point $(X_\infty, Y_\infty, Z_\infty)$ which projects onto $\mathcal{D}_2$ with $Y_\infty > Y_c$. The only such critical point is $(1, 1, 0)$ and hence $\Psi_\lambda^{\mathrm{cs}}$ is ALC by Proposition 4.6.5.

If $\lambda > 0$ the solution enters $\mathcal{D}_3$ with $\dot{Y} < 0$. By Lemma 4.5.7 (ii) it is trapped there as long as $Y \geq 0$ and Y is monotone decreasing. If the solution is forward complete, then by Proposition 4.5.17 it converges to some critical point $(X_\infty, Y_\infty, Z_\infty)$. By Lemma 4.5.2 we can compare it with the cone to get $X_\infty \leq X_c, Y_\infty \leq Y_c, Z_\infty \geq Z_c$. As Y is monotone decreasing the inequality for the Y-coordinate is strict. No such critical point exists and therefore the solution develops a singularity at $Y = 0$ in finite time. □

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