A Path to the Epistemology of Mathematics: Homotopy Theory

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> . . . the history of topology provides, . . . , a typical shortcut of the history of mathematics. Algebraic topology finds its origin, on the one hand, from the examination of problems arising in other parts of mathematics, notably functions of one complex variable and algebraic geometry, and on the other hand, from its own developments (like the *Hauptvermutung*); their solution requires new methods and techniques, and the reflection on these methods leads to new kinds of problems (theory of algorithms or decidability, for instance) or sparks the creation of new notions or theories (such as categories or homological algebra) which will allow instructive synthesis by shedding a new light on questions found in other chapters of mathematics, or that will even provide tools that will lead to significant progress in the study of other types of problems. (Hirsch, 1978, 260–261; our translation)

1 Introduction

Algebraic topology is indisputably one of the greatest achievements of twentieth-century mathematics. If, as Hisrch suggests, algebraic topology is a typical shortcut of the history of—and we would add here *twentieth-century*—mathematics, homotopy theory is a shortcut to the history of algebraic topology itself. The notion of a homotopy between maps has its roots in the late eighteenth century, appeared implicitly in the nineteenth century in the theory of functions of a complex variable, the theory of algebraic functions and the calculus of variations, was used informally by Poincaré in his papers entitled *Analysis Situs* that mark the birth of algebraic topology, and was finally explicitly defined as we know it by Brouwer in 1912. Homotopy *theory* came into existence in the 1930s, after Hopf's introduction of the fibrations that now bear his name and Hurewicz's introduction of the higher homotopy groups together with some of their fundamental properties. From this point on, homotopy theory interacted strongly with the other tools of algebraic topology, e.g. homology theory, cohomology theory, spectral sequences, it moved slowly to the forefront of algebraic topology in general, led to new synthesis in the form of homotopical algebra and is now being applied in a wide variety of fields, e.g. Voevodski's application of homotopical methods in algebraic geometry, for which he obtained the Fields Medal in 2002.

If Hirsch is correct, this is a typical evolution of a successful mathematical field: a notion appears in a given context or given contexts as being part of the solution to a problem or a class of problems, it is then clarified, cleaned up of extraneous elements, developed to a certain extent autonomously and, either at the same time or soon after, it is applied to other, unexpected, problems and fields and, in the best cases, it leads to the development of new notions, new tools, new theories that are then applied to a variety of contexts. This suggests that there is a pattern to the development of mathematics, at least in the twentieth century, or perhaps, going in a slightly different direction, it suggests that there are distinctive elements to mathematics of the late nineteenth and twentieth centuries. The elements mentioned by Hirsch are of course too broad and vague to be of any real value. A more detailed analysis of the various steps, moments, moves and periods is required. Of course, only a start can be made on that here. We do believe that algebraic topology in general, and homotopy theory in particular, do indeed provide a rich and fertile ground for philosophical reflection on the nature of mathematical knowledge and its development.

We will concentrate in this chapter on one specific epistemological element that can be extracted from the history of homotopy theory. Our main objective is to show that a typical component of twentieth century mathematics is the emergence, proliferation and establishment of *systematic mathematical technologies* within mathematics and that this development is not unlike the emergence, proliferation and establishment of scientific technologies in general. We believe that we can see within the history of homotopy theory such examples of these technologies. Furthermore, in the same way that the shift of attention towards the experimental and technological aspects of scientific research in philosophy of science is giving rise to an epistemology of scientific instrumentation and thus a more faithful epistemology of scientific knowledge[1], we claim that similarly in philosophy of mathematics, the recognition and analysis of mathematical technologies and instrumentations should lead us to a modification or a more adequate epistemology of contemporary mathematics. We suggest that *parts* of mathematical knowledge should be thought of as a form of conceptual engineering and that, therefore, mathematical knowledge is as complex and as messy as scientific knowledge in general. If this is correct, the picture of mathematical knowledge and of its development we end up with is radically different from the standard 'axioms-definitions-theorems-proofs of truths picture' of mathematical knowledge we often find in the literature.

[1] See, for instance, Galison 1997, Baird 2004.

2 Forms of mathematical knowledge

> The equivalence of all these infinite loop space machines was later proved by May and Thomason… (Kriz, 2001, xxiii)

Mathematical knowledge is a fabulously intricate mixture of know that and know how. In order to prove certain results, to construct specific counterexamples, to compute or solve certain equations, to define a new concept, to transfer various constructions from one field to another, one has to *know how* to do certain things and *know that* properties hold of the objects and procedures one is using and one is working with. Various periods and various people have often insisted more on the know how, the *technè* aspect of mathematics, presenting the latter as an art, others have underlined the know that, the *episteme* aspect of mathematics, presenting it as a science. But as Polanyi has already observed '… mathematics can be equally well affiliated either to natural science or technology.' (Polanyi, 1958, 184) It should be obvious to everyone that the *practice* of mathematics involves a lot of technical expertise[2] and that the *results* of mathematical practice are often considered the epitome of scientific knowledge. Furthermore, it would not be such a great exaggeration to claim that mathematical knowledge is characterized by the continual transformation of know how into forms of know that. This simply means that the methods, techniques and tools developed by mathematicians become *objects* of knowledge themselves.

But between scientific knowledge and technological knowledge, we find intermediate forms of knowledge. Here is how Polanyi puts it:

> We have, correspondingly, two forms of enquiry that lie between science and technology. Technologies founded on an application of science may form a scientific system of their own. Electrotechnics and the theory of aerodynamics are examples of *systematic technology* which *can be cultivated in the same way as pure science.* Yet their technological character is apparent in the fact that they might lose all interest and fall into oblivion, if a radical change of economic relationships were to destroy their practical usefulness. On the other hand, it may happen that some parts of pure science offer such exceptionally ample sources of technically useful information that they are thought worth cultivation for this reason, though they would otherwise lack sufficient interest. The scientific study of coal, metals, wool, cotton, etc. are branches of such *technically justified science.* (Polanyi, 1958, 179. See also Polanyi, 1960–61, 405.)

We submit that it is reasonable to *transpose*, with appropriate adjustments, Polanyi's classification of forms of knowledge to mathematics. We claim that these distinctions can and should be introduced *within pure mathematics* itself. More precisely, we believe that twentieth-century mathematics in general and algebraic

[2] It is customary among mathematicians to qualify some mathematical work as being a technical prowess.

topology in particular are marked, on the one hand, by the appearance of *systematic conceptual technologies*, and not just techniques and methods, and, on the other hand, by *technically justified mathematics*. For the purposes of this chapter, we will not distinguish these two forms from now on[3]. We believe, moreover, that historically these developments parallel the developments seen in the natural sciences and technologies and, conceptually, they have much in common.

What is, informally, a systematic conceptual or mathematical technology in pure mathematics? It is a conceptual technology, that is a specific conceptual know how with a specific epistemic goal. Mathematics is filled with these conceptual know hows. But it is systematic in as much as it rests upon a whole mathematical theory or a collection of mathematical theories for its design, definition and applications. The remaining sections of this chapter will hopefully illuminate these claims, as well as illustrate and provide evidence for them.

If there are pieces of mathematics that are viewed as systematic conceptual technologies or technically justified mathematics, we should observe differences in the way a piece of work is *valued* by mathematicians, depending on whether the work is seen as a piece of science or a piece of technology (or, in the case of mathematics, both, something that might be a distinctive feature of mathematics itself). We will use as a springboard a simple list of values proposed by Polanyi for sciences and technologies. According to Polanyi, … 'a statement is of value to natural science if it (1) corresponds to the facts, (2) is relevant to the system of science and (3) bears on a subject matter which is not without intrinsic interest;' and 'a statement is of value in technology (1) if it reveals an effective and ingenious operational principle which (2) achieves, in existing circumstances, a substantial material advantage.' (Polanyi, 1953, 187.) We should add here that an additional element is that technologies can fall into oblivion simply because they are replaced by other technologies. It is easy to give examples of mathematical knowledge that are valued because (1) they are relevant to the system of mathematics and (2) bear on a subject matter that is not without intrinsic interest. We leave aside the question of relevance to facts, since it is clearly more controversial in the case of mathematical knowledge. We submit that algebraic topology and, in particular, homotopy theory are filled with statements and, more generally, forms of knowledge, that (1) reveal an effective and ingenious operational principle that (2) achieves, in existing circumstances, a substantial *conceptual* advantage. But what is more, these same forms of knowledge are *also* relevant to the system of mathematics, for they often reveal *how* various pieces of mathematical knowledge *are* related to one another, and they certainly bear on a subject matter that is not without intrinsic interest.

[3] As Polanyi himself observed, the distinction might in practice be merely rhetorical: Systematic technology and technically justified science are two fields of study lying between pure science and pure technology. But the two fields may overlap completely. (Polanyi, 1958, 179.)

Mathematicians themselves regularly talk about parts of algebraic topology and homotopy theory in terms of technologies, machines, tools and instruments[4]. We will now take a close look at what seems to us to be a representative sample of what can be found in the literature. The list could be extended indefinitely. Needless to say, the fact that mathematicians talk in that way does not constitute a conclusive argument in favour of our claim, but the following quotes provide powerful evidence in support of our thesis. We will comment on the quotes as we go along.

> In this chapter we obtain some results about the homotopy groups of spheres. The method we follow is due to Serre and uses the *technical tool* known as a spectral sequence. This algebraic concept is introduced for the study of the homology and cohomology properties of arbitrary fibrations, but it has other important applications in algebraic topology, and the number of these is constantly increasing. Some indication of the *power* of spectral sequences will be apparent from the results obtained by its use here. (Spanier, 1966, 465) [our emphasis]

Let us immediately underline the elements that stand out. Spanier identifies a method as a technical tool, namely the method of spectral sequences. This is clearly a case of know how. In the next sentence, he claims that it is a concept and states its purpose: although we are talking about a technical tool, it is in the end a form of knowledge. To learn and understand spectral sequences is to *know how* to use spectral sequences. Finally, Spanier argues in favour of the *power* of the technology on the basis of the results obtained with its help. It is not so much the quantity of results that is at stake here, but the conceptual importance of the results and the fact that they cannot be obtained otherwise. Spectral sequences are *valued* because of their power and this, despite the fact that they are extraordinarily complicated and difficult to use. The next quote goes exactly in the same direction and does not require any further comments:

> The book might well end at this point. However, having eschewed the use of the heavy *machinery of modern homotopy*, I owe the reader a sample of things to come. Therefore a final chapter is devoted to the Leray–Serre spectral sequence and its generalization to non-standard homology theories. . . . . Some applications are given and the book ends by demonstrating the *power* of the machinery with some qualitative results on the homology of fibre spaces and on homotopy groups. (Whitehead, 1978, xv)[our emphasis]

At least one algebraic topologist has used explicitly the analogy between components of algebraic topology and components of the natural sciences.

> Despite the large amount of information and *techniques* currently available, stable homotopy is still very mysterious. Each new computational

4 One mathematician once told me that his introductory (graduate) course in algebraic topology was all about machinery. I suspect that most mathematicians teaching the subject would say something similar.

> breakthrough heightens our appreciation of the difficulty of the problem. The subject has a highly *experimental* character. One computes as many homotopy groups as possible with *existing machinery*, and the resulting *data* form the basis for new conjectures and new theorems, which may lead to better methods of computation. In contrast with physics, in this case the experimentalists who gather data and the theoreticians who interpret them are the same individuals. (Ravenel, 1986, xvi)[our emphasis]

We have here a specific admission that methods of computation are replaced by new, more powerful, methods. We will see in the next sections what Ravenel has in mind when he talks about *existing machinery*. But it follows immediately that we should take seriously the idea that mathematicians build conceptual machinery, evaluate the quality of this machinery and they replace existing machinery by new ones. The next quote is even more explicit.

> The study of the homotopy groups of spheres can be compared with astronomy. The groups themselves are like distant stars waiting to be discovered by the determined observer, who is *constantly building better telescopes* to see further into the distant sky. The telescopes are spectral sequences and other algebraic constructions of various sorts. *Each time a better instrument is built* new discoveries are made and our perspective changes. The more we find the more we see how complicated the problem really is. We can distinguish three levels in the subject. The first (comparable to observational astronomy) is the collection of *data* about homotopy groups by *various computational devices* (...). While this aspect of the subject is not fashionable and is seldom discussed in public, it is vital to the subject. Without *experimental data* there can be no valid theories. (...) The second level of ideas in homotopy theory is the identification of certain patterns known as periodic families. This may be compared to the discoveries of Kepler and Halley. (...) The third level (comparable to cosmology) is the formulation of general theories about the mechanisms which produce the observed phenomena. (...) As in theoretical physics one can make various models of the universe based on certain oversimplification or idealizations. While these constructs have obvious limitations, their study is instructive as it leads to some insight into the nature of the real world. We will discuss several of these models now. (Ravenel, 1987, 175–176)[our emphasis]

Gathering data about homotopy groups of spheres is a highly technical endeavour. It is very hard. Conceptual machines, instruments, probes and tools have to be built and used properly. This is the kind of highly systematic know how that we want to focus on.

A few words about algebraic topology might help illuminate these quotes further. As its name indicates, algebraic topology is the study of topological spaces and continuous transformations by algebraic means. The Graal of algebraic topology is the classification of spaces under continuous deformations. The general strategy is to associate to a space various algebraic structures, e.g. groups, modules, rings, algebras, etc., in such a way that a continuous map of spaces is transformed into a homomorphism of the appropriate kind, e.g. a homomorphism of groups, or modules or rings, etc. and homeomorphisms of spaces are transformed into isomorphisms of the associated algebraic structures. In other words, the algebraic structures associated to a space are invariant. Thus, one tries to encode topological properties by algebraic means in such a way that whenever there is a difference between the corresponding algebraic structures associated to two spaces, then one can conclude that the spaces are different. We submit that finding systematic ways of encoding topological properties in algebraic structures is a form of conceptual engineering and what is elaborated to do the encoding constitute examples of systematic conceptual technologies. Thus, homology theories, cohomology theories, homotopy groups, spectral sequences, fibrations, etc. are all instances of systematic conceptual technologies.

These technologies are valued when they reveal an effective and ingenious operational principle and they achieve a substantial conceptual advantage. Mathematicians rarely praise purely *ad hoc* solutions, no matter how clever these solutions are.

In the remaining parts of this chapter, we will concentrate on one specific concept in the history of homotopy theory, namely the concept of fibration that has an interesting history of its own and, furthermore, illustrates some of the key features of these systematic technologies.

3 Forms of mathematical knowledge: fibrations

> The concept of fibration has been one of the most important mathematical tools in the twentieth century; born in geometry and topology, it has gradually invaded many other parts of mathematics. (Dieudonné, 1989, 383)

A brief history of homotopy theory with fibrations in mind

Homotopy theory starts from an extremely simple geometric idea: the continuous deformation of a curve into a curve, or a path into a path. It is extraordinarily easy to give a vivid illustration of a specific homotopy between two curves. But it is a different matter to know *how* to *define* the notion precisely and it is even less clear *why* such a definition ought to be given. For one thing, the notion is so intuitively clear that it does not seem necessary to provide a precise formal definition. Furthermore, once a rigorous definition has been provided, it is not clear what has been gained thereby, apart from rigour for its own sake. Even when someone understands the precise definition clearly, it does not mean that one understands the point of the notion, *why* it *is* an important notion. The latter makes sense only when the *role* played by that notion in a broader context is understood. We believe that this is true of many other similarly simple mathematical notions: to understand a mathematical notion in a given context, one has to understand its *function* in that context. This means that for many mathematical notions, to understand that notion, it is irrelevant to specify what it is 'made of', or its underlying 'ontology' but rather what it is *used for*: when, why and how.

It is certainly not our goal to sketch the whole history of homotopy theory. This would be a daunting task that would require a book, perhaps many books. With the concept of fibration in mind, we can roughly distinguish the following periods in the history of homotopy theory:

1. The prehistory: from Lagrange until and including Poincaré;
2. The introduction of the concept and its first uses: Brouwer's explicit definition of a homotopy of paths in 1912 and its applications by Brouwer himself;
3. The birth of homotopy *theory* as such: from Hopf's study of maps between spheres between 1926 and 1935, the definition of higher homotopy groups by Hurewicz in 1935 to Serre's computations of classes of homotopy groups of spheres using fibrations and spectral sequences in 1951;
4. The development of simplicial homotopy theory by Kan and others in the mid-1950s until Quillen's introduction of homotopical algebra with its underlying notion of model categories published in 1967.

Again, this is extremely schematic and does not do justice to the extraordinarily complex development of the field. A history of computations of homotopy groups of spheres, for instance, would be divided differently[5]. But our goal here is to provide the general background in which the notion of fibration appeared and played a key role. Let us now turn to some of the details of this history and its key developments, especially the first three phases.

The concept of a homotopy of paths appears implicitly in the works of Lagrange, Cauchy, Riemann, Puiseux, Jordan, Klein and Poincaré[6]. Poincaré's work has to be set apart, for although he does not define formally a homotopy of paths and for this reason has to be put in the first period of the history, he is the first one to see how the concept can be intrinsically useful to reveal important properties of a manifold. Before Poincaré, the notion appeared in specific (non-topological) contexts, e.g. the calculus of variations, integration of a complex function of a complex variable, algebraic functions, etc. In these contexts, the focus of attention of the mathematicians was, for instance, certain specific functions and their integration and a homotopy of paths was simply an obvious requirement that had to be met by those functions. There was no reason to define the notion of homotopy precisely, since it did not play any mathematical role in these contexts. It may very well also be that the idea of invariance under a change as a *general* and significant method was being assimilated slowly in the nineteenth century. Even Klein, who introduced the idea of invariance of geometric property via

[5] Toda's paper, precisely on the topic of the history of computations of homotopy groups of sphere, cuts the history into slices of ten years. This is as if the field had no internal conceptual dynamics.Whitehead's historical paper does not have much more conceptual perspective. See Toda 1982 and Whitehead 1983.

[6] We rely essentially on Vanden Eynde 1999 for this period.

transformation groups in elementary geometry, does not make the notion of homotopy precise and, like many of his contemporaries, confused homotopy with homology[7]. Furthermore, it is far from clear that mathematicians of that period would have had the means, that is the concepts and appropriate language, to define the concept of homotopy explicitly.

Some of these remarks apply to Poincaré as well. Although Poincaré gave birth to algebraic topology in his paper *Analysis Situs* and its five complements, published between 1895 and 1904, and although Poincaré is certainly the first mathematician to see that continuous deformations can actually reveal properties of a manifold in which they are defined, it is clear that Poincaré's focus of attention is on the concept of homology. Indeed, in the 1895 paper, Poincaré defines what he calls the fundamental group of a manifold, what will later become the first homotopy group, but as he says clearly in §13 of that paper, the information obtained from the fundamental group is used to determine what he calls the 'fundamental homologies' and it is not considered intrinsically. Furthermore, it is not before 1904, in the fifth complement, that Poincaré shows that the fundamental group can be different from the first homology group, thereby showing that the two concepts differ. Adding to these ingredients the fact that general topology was still not available as a language to define this concept in all its generality and Poincaré's own informal style, we can see why the concept was not made explicit by him, despite the fact that it was used explicitly for topological reasons[8].

Brouwer was the first mathematician to give a precise formal definition of a homotopy of paths. His definition is almost identical to the one we find in contemporary algebraic topology textbooks. However, Brouwer thought it was sufficient to give the definition *in a footnote* of his paper on continuous transformations of spheres in themselves. Here is Brouwer's definition:

> By a continuous modification of a univalent continuous transformation we understand in the following always the construction of a continuous series of univalent continuous transformations, i.e. a series of transformations depending in such a manner on a parameter, that the position of an arbitrary point is a continuous function of its initial position and the parameter. [Brouwer, 1912, 1976, 527, ft 4]

The process of continuous deformation is now made explicit. The definition tells us how an arbitrary point 'moves' from its initial position. Nowadays, this is stated thus: Let X and Y be topological spaces, I be the standard unit interval $[0, 1]$ and f and g be continuous maps $X \to Y$. The map f is said to be *homotopic* to g, denoted by $f \simeq g$, if there exists a *homotopy* of f to g, that is, a map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. In the body

[7] Vanden Eynde attributes this confusion to the fact that in the context of Riemann's work, the distinction does not have to be made. See Vanden Eynde, 1999, pp. 75–76.

[8] The *name* 'homotopy' was introduced by Dehn and Heegaard in their article on Analysis Situs published in the German encyclopedia of mathematics in 1907. However, the name does not designate the concept we now know.

of the same paper, Brouwer defines what we now call the *homotopy class* of a map: 'we shall say that two transformations *belong to the same class* if they can be transformed continuously into each other.' (Brouwer, 1912, 1976, 528.) In other words, the existence of a homotopy defines an equivalence relation between maps. The notion of homotopy class of a map was in itself very important, since it provided a novel classification of maps, different from the classification that homology was about to deliver.

Why did Brouwer define the concepts of homotopy of paths and of homotopy class of a map? First, it is interesting to note, as is emphasized by Freudenthal in his commentaries to Brouwer's topological papers (see Freudenthal 1976, 436), what is absent in Brouwer's work: there is no mention and no use whatsoever of the tools of homology[9]. Thus, Brouwer's focus of attention is radically different from Poincaré's. Second, the concept of homotopy played a key role in his proofs of some of his infamous theorems, e.g. the fixed-point theorem, the invariance of dimension, etc. Brouwer's main tool, together with simplicial approximation, is the notion of degree of a map[10]. Informally, the degree of a map $f : S^1 \to S^1$ of the circle into itself, denoted by $\deg(f)$, is the number of times $f(z)$ turns around S^1 when z turns once around S^1. The concept can be defined for any map $f : S^n \to S^n$ of the n-sphere into itself[11]. The crucial property of the notion of degree of a map is that it is homotopy invariant, that is, if $f \simeq g : S^n \to S^n$, then $\deg(f) = \deg(g)$. Furthermore, homotopies of maps play an essential role in the method of simplicial approximation. We now see that the focus of attention is on the homotopy class of maps and not, as in the case of Poincaré, the group of such homotopy classes. Furthermore, in contrast with his predecessors, the deformation *is* the key property and not simply an obvious condition in the background of the problem. Nonetheless, Brouwer did not deem it necessary to include the definition in the main part of his paper.

The notion of homotopy remained a footnote until Hopf made essential use of it in his work on continuous mappings of spheres from 1925 until 1935. We have seen earlier that the notion of the degree of a map is homotopy invariant. In 1912, Brouwer conjectured that the converse of this statement was also true, that is, for any $f, g : S^n \to S^n$ such that $\deg(f) = \deg(g)$, $f \simeq g$, but sketched a proof only for $n = 2$. Hopf proved the conjecture in 1925. Furthermore, Hopf's proof yields an

[9] 'In retrospect, it therefore seems legitimate to consider Brouwer as the cofounder, with Poincaré, of simplicial topology. More precisely, it may be said that Poincaré defined the *objects* of that discipline, but it is Brouwer who imagined *methods* by which theorems about these objects could be *proved*, something Poincaré had been unable to do. (...) It is all the more surprising then that Brouwer did not attempt to use his techniques in order to put Poincaré's 'theorems' in simplicial homology on less shaky foundations. (...) At any rate, Brouwer never showed any interest for homological concepts in his '*n*-dimensional manifolds.' '(Dieudonné, 1989, 168)

[10] Hopf gave the actual definition of degree of a map with the help of homology groups in 1930. See, for instance, Whitehead 1978, 13 or Spanier 1966, 196.

[11] Brouwer defines the notion of degree for any continuous map $f : M \to N$, where M and N are compact, connected, oriented n-dimensional 'manifolds' (in a restricted sense).

isomorphism between the n-th homotopy group of the n-sphere and the integers, i.e. in Hurewicz's notation $\pi_n(S^n) \approx \mathrm{Z}$. Brouwer had worked with maps between manifolds of the same dimension. For $n < m$, it was known that the homotopy groups $\pi_n(S^m)$ are trivial, in other words, for any continuous map $f : S^n \to S^m$, f is homotopic to a constant map. Before 1930, almost nothing was known about continuous maps $f : S^m \to S^n$ for $m > n$. What was known is that homology was *useless* in that context. To use homological information, one would look at the induced homomorphism $f_* : H_\bullet(S^m) \to H_\bullet(S^n)$ between homology groups. But for $p > 0$, either $H_p(S^m) = 0$ or $H_p(S^n) = 0$ (for $H_p(S^n) \neq 0$ if and only if $p = n$), and thus, in both cases, f_* is a trivial group homomorphism. Something else has to be used.

This is where the notion homotopy class of a map turned out to be informative and, thus, played a crucial role in our understanding of the situation. In 1930, Hopf proved that there are infinitely many homotopy classes of maps from S^3 to S^2. This was to be interpreted shortly after as saying that $\pi_3(S^2) \approx \mathbb{Z}$. Hopf's proof came as a total surprise. In particular, Hopf defined a continuous map $f : S^3 \to S^2$ that is not homotopy equivalent to a constant map, now known as the *Hopf fibration* or the *principal Hopf bundle*. As its name already indicates, it will play a role in our story since it is an early example of a fibration. (For a detailed description of the map, see Aguilar *et al.* 2002, 129–130 or Hatcher 2002, 377–378.) Then, in 1935, Hopf generalized his results to maps $f : S^{2n-1} \to S^n$. Hopf showed that for $n = 4$ and $n = 8$, the maps $f : S^7 \to S^4$ and $f : S^{15} \to S^8$ are what we now call fibrations. These yield the isomorphisms $\pi_7(S^4) \approx \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $\pi_{15}(S^8) \approx \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ (these are taken from Toda 1962). In fact, Hopf did much more than give these specific maps, for he obtained results for n even, introducing along the way a construction that was going to be extremely influential and important afterwards. It is certainly fair to say that Hopf showed that homotopic methods could provide important information about spaces that seemed to be inaccessible otherwise and, in this sense, launched homotopy theory. Hopf's work convinced mathematicians that homotopy classes of maps could be used effectively to obtain information about various spaces. A technology was on its way. It had to be developed systematically. The Polish mathematician Witold Hurewicz took care of that.

Before we move to Hurewicz, let us briefly go back to the fundamental group introduced by Poincaré, for we now can define it. Let (X, x_0) be a *pointed space*, that is a space with a privileged point $x_0 \in X$ and $(S^1, *)$ be the circle (as a subspace of $\mathbb{R}^2$) with privileged point $* = (1, 0)$. A *loop* in X at x_0 is a continuous mapping $\alpha : S^1 \to X$ such that $\alpha(*) = x_0$. Poincaré showed how loops α_1 and α_2 can be composed: first go around α_1 and then around α_2[12]. This gives a law of composition for loops, denoted by $\alpha_1 \vee \alpha_2$. This law is *not* commutative: for given a $z \in S^1$, $(\alpha_1 \vee \alpha_2)(z)$ will not, in general, be equal to $(\alpha_2 \vee \alpha_1)(z)$. Clearly, there is a *constant loop* $\alpha_0 : S^1 \to X$, defined by $\alpha_0(z) = x_0$ and given a loop α_1,

[12] We will leave it to the reader to provide the formal details. This is another case of an extraordinarily simple geometric idea that has to be turned into a genuine mathematical concept.

we can define the inverse loop α_1^{-1} of α_1 as the loop going exactly along the same path but in the direction opposite of α_1. These data do *not* yield a group, however. It is precisely at this point that the notion of homotopy of loops enters the scene and plays a key role. A homotopy between loops $\alpha_1, \alpha_2 : S^1 \to X$ is a continuous map $F : (S^1, *) \times I \to (X, x_0)$ such that $F(z, 0) = \alpha_1$, $F(z, 1) = \alpha_2$ and $F(*, t) = x_0$ for all $t \in [0, 1]$. We can consider the set of equivalence classes of loops, denoted by $[S^1, *; X, x_0]$. A tedious but straightforward verification shows that the equivalence class $[\alpha_1 \vee \alpha_2]$ depends solely on the classes $[\alpha_1]$ and $[\alpha_2]$. Therefore, we can define a product between equivalence classes of loops by putting $[\alpha_1] \cdot [\alpha_2] = [\alpha_1 \vee \alpha_2]$. It can be verified that the product thus defined does indeed yield a group, named by Poincaré the *fundamental group* of the space and it is denoted by $\pi_1(X, x_0)$[13]. Thus, in this context, the notion of homotopy allows one to define a group structure on a space. It is again by moving to the homotopy classes of maps that we succeed in obtaining relevant and useful information about a space.

It seems entirely natural and a promising idea to generalize Poincaré's construction by considering the set $[S^n, *; X, x_0]$ of equivalence classes of maps $(S^n, *) \to (X, x_0)$ for $n > 1$. This is precisely what Čech did and presented in a very short note in 1932 at the International Congress of Mathematicians. Čech showed that the higher-dimensional homotopy groups, as they are called, are abelian. Because of that, no one thought they would be of any use and Čech himself abandoned this line of research. It was expected that only non-abelian groups would provide genuinely new information, that is information going beyond what homology groups revealed. This expectation was based on what Poincaré had already shown: the abelianization of the first homotopy group (of a variety in the case of Poincaré) is isomorphic to the first homology group of that space.

When Hurewicz turned his attention to the topology of deformations, he had already done important work in dimension theory and descriptive set theory. In particular, he had assimilated various concepts and methods of point set topology, the most important being that of a function space and its topology[14]. Indeed, the very first sentence of his first paper on homotopy groups sets the stage:

> When investigating continuous mappings of a space X into a space Y, it proves very useful to interpret the collection of those mappings as a topological space in its own right. In the most important cases, the components of this function space coincide with the Brouwer classes of mappings that are continuously deformable into each other (homotopic). (Hurewicz 1935, in Kuperberg, 1995, 350)

[13] In fact, more is true. One can thus define, as it is now done, the *fundamental groupoid* of a space. The latter notion could have appeared much earlier and naturally in the history of mathematics, but it did not.

[14] It is worth noting that Hurewicz is using Fréchet's work on functional spaces. As he indicates himself in a footnote, the functional spaces Y^X are usually *metric* spaces, the resulting metric depending on the choice of a metric in Y. But he indicates immediately that when X is compact, the resulting topology of Y^X does not depend on the choice of the metric in Y.

Hurewicz made essential use of function spaces in his definition of the higher-dimensional homotopy groups. Given two spaces X and Y, it is possible to define a topology on the set X^Y of functions $Y \to X$. In 1935, Hurewicz had to assume that X was a metric space and Y was compact. This is not as such a considerable restriction, but one of the first questions left open by Hurewicz's work was whether the homotopy groups could be defined for any topological space X and Y. The answer was given in the early 1940s when the compact-open topology was introduced[15].

Instead of starting with the set $[S^1, *; X, x_0]$ of homotopy classes of loops, Hurewicz started with the *loop space* $\Omega(X, x_0)$, i.e. the function space $(X, x_0)^{(S^1,*)}$ with the appropriate topology. A *path* in the loop space $\Omega(X, x_0)$ is a continuous map $I \to (X, x_0)^{(S^1,*)}$. With the appropriate topology, each path is equivalent to a homotopy $(S^1, *) \times I \to (X, x_0)$ between loops at x_0. Hurewicz observed that the components of the space $(X, x_0)^{(S^1,*)}$ are therefore the same as the homotopy classes of maps from $(S^1, *) \to (X, x_0)$. The loop space is itself a pointed space: it is the space $(\Omega(X, x_0), \alpha_0)$ where $\alpha_0 : S^1 \to \{x_0\}$ is the constant loop. We can therefore consider its fundamental group $\pi_1((\Omega(X, x_0), \alpha_0)$. The beauty of this construction is that it can be repeated inductively: $\Omega^n(X, x_0) = \Omega(\Omega^{n-1}(X, x_0), \alpha_{n-1})$, where $\alpha_{n-1} : S^1 \to \{\alpha_{n-2}\}$ is the obvious constant map. The *n-th homotopy group* $\pi_n(X, x_0)$ is then the *fundamental group* of the $n-1$ loop space of (X, x_0), i.e. $\pi_n(X, x_0) = \pi_1(\Omega^{n-1}(X, x_0), \alpha_{n-1})$.

With this definition in hand, Hurewicz stated without proofs various important properties of the homotopy groups and applying these results to the homotopy groups of topological groups, he obtained a new proof of Hopf's result on $\pi_3(S^2)$ as well as many others.

Hurewicz published three more papers on homotopy groups, all of which established important properties (with proofs in these cases) of homotopy groups, for instance their connections with homology groups in the second note, and also using homotopy groups to prove properties of spaces and even to define classes of spaces, e.g. the aspherical spaces in the fourth note. But the next important concept was to appear *at the very end* of the third note. It is the notion of *homotopy type*[16]. Two spaces X and Y are said to have the same *homotopy type*, or to be *homotopy equivalent*, if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$. Before the introduction of this definition, homotopic information was used to classify *maps*, now, it is used to classify *spaces*. Hurewicz used the notion in the fourth paper on homotopy theory to classify aspherical spaces.

With the publications of these four papers, homotopy theory was now on firm grounds and could start a life of its own. The basic definitions and their relevance to algebraic topology were explicit and clear; some of the information it delivered

[15] See, for instance, Aguilar *et al.* Chap. 1 for the definition of the compact-open topology.

[16] Hurewicz defined it for compact spaces only.

was only available through its channels. There was only one fundamental glitch: homotopy groups were extraordinarily hard to compute. Freudenthal took an important step in 1937 with the introduction of the notion of the suspension of a space. The next step would await the clarification and the use of fibrations together with spectral sequences. A detailed and illuminating discussion of spectral sequences would require too much space, and we will therefore limit ourselves to fibrations. But this is no great loss, since fibrations occupy a central role in contemporary homotopy theory, as we will see.

Fibrations: an historical sketch

The history of fibrations intertwines with the history of fibre bundles (vector bundles, sphere bundles) and fibre spaces. We will leave the history of fiber bundles aside since they are related more to differential geometry than to algebraic topology, and concentrate on the homotopical aspects of the story. (The interested reader should consult Dieudonné 1989 and Zisman 1999 for the history of fiber bundles.)

A special case of fibration appeared, according to Zisman 1999, as early as 1879 in a note published by Emile Picard. Other specific cases showed up again in the work of Seifert in 1931, 1935, Hurewicz in 1935, Hopf in 1935 and Borsuk in 1937. Then, in 1940 and 1941, five mathematicians, namely Hurewicz and Steenrod working together, Ehresmann and Feldbau also working together and Eckmann, identified a property, namely the *homotopy lifting property* (HLP)[17], that allowed them to obtain new and interesting results about homotopy groups. With these results in hand, it seemed reasonable to define a new structure by a property general enough to 1) include the spaces they were interested in as well as others that seemed important *and* 2) that would yield a simple proof of the HLP for a large class of spaces. Mathematicians had found a property that played a key role in the proofs of important results but that did not seem to characterize an entity as such. The search for a general property that would fit the bill was launched. Hurewicz and Steenrod in 1940, published in 1941, were the first to introduce *fibre spaces* with these goals in mind.

Before we look at fibre spaces, let us state the homotopy lifting property. Let $p : E \to B$ be a continuous map and $\mathcal{C}$ a class of topological spaces. Then p is said to satisfy the *homotopy lifting property (HLP) with respect to* $\mathcal{C}$, if for every $X \in \mathcal{C}$, every map $f : X \to E$ and every homotopy $H : X \times I \to B$ such that $H(x, 0) = (p \circ f)(x)$, there is a homotopy $\tilde{H} : X \times I \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H}(x, 0) = f(x)$. A simple diagram, when read properly, allows us to grasp the whole definition at a glance: given all the data, p satisfied the HLP with respect to

[17] Hurewicz and Steenrod (1941) called it the *covering homotopy property (CHP)* and it is sometimes called this in various books and articles.

$\mathcal{C}$ if the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ j\downarrow & \tilde{H}\nearrow & \downarrow p \\ X\times I & \xrightarrow[H]{} & B \end{array},$$

where $j : X \to X \times I$ is the inclusion $j(x) = (x, 0)$.

Notice that the HLP is a property of a *map*. As such, it is a simple property and it is hard to see *why* it is important. To state this, we need to introduce a bit of terminology: let $b_0 \in B$, then $F = p^{-1}(b_0)$ is said to be the *fiber* above b_0; E is called the *total space* and B is called the *base space*[18]. Fibres in this context customarily have additional structure, e.g. F is itself a topological space. Clearly, we have an inclusion $i : F \to E$. The crucial fact is that when a map p satisfies the HLP with respect to a class $\mathcal{C}$, it is possible to construct isomorphisms $\pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0)$ for any $x_0 \in F$[19]. Furthermore, the HLP allows defining a homomorphism $\partial : \pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$, which in turns yields the so-called homotopy exact sequence

$$\cdots \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, x_0) \to \cdots .$$

In other words, the homotopy groups of the fibres are systematically connected to the homotopy groups of the total space and the base space all the way down to the path-components π_0. When E, B or F satisfy further specific conditions, one uses these connections to establish other useful isomorphisms, e.g. $\pi_n(E) \approx \pi_n(B) \oplus \pi_n(F)$ for $n \geq 2$. Thus, the HLP plays a crucial role in the construction of various maps that in turn allows one to obtain significant results about homotopy groups.

Hurewicz and Steenrod stipulated that E is a *fibre space over B relative to p*, where $p : E \to B$ is a continuous map, E is a topological space and (B, δ) is a *metric* space, if there exists an $\varepsilon_0 > 0$ such that for all open subsets $U_{\varepsilon_0} = \{(e, b) \in E \times B : \delta(e, b) < \varepsilon_0\}$ there is a continuous function $\phi : U_{\varepsilon_0} \to E$ such that for all $(e, b) \in U_{\varepsilon_0}$

$$p \circ \phi(e, b) = b \text{ and } \phi(e, p(e)) = e.$$

A map ϕ is called a *slicing function* (nowadays, we would say that it is a local section). Hurewicz and Steenrod then gave a list of examples of fibre spaces: these include all the fibre bundles as defined then by Whitney (sphere bundles), product spaces, covering spaces, the Hopf maps and projection maps of a Lie group onto a quotient by a closed subgroup. Theorem 1 of their paper is the HLP

[18] This terminology comes from the theory of fibre bundles and was introduced by Whitney in 1937.

[19] In their paper, Hurewicz and Steenrod construct an isomorphism between the *relative* homotopy group $\pi_n(E, F, x_0)$ and $\pi_n(B, x_0)$, for any $x_0 \in F$. Since we haven't said a word about relative homotopy, we refrain from formulating these results in those terms.

with respect to *all* topological spaces, which is achieved by assuming that the homotopy $H : X \times I \to B$ is *uniformly* continuous. They indicate after the proof that the uniformity requirement is unnecessary if X is assumed to be a compact metric space. They then proceed to use the HLP to prove important properties of homotopy groups, now defined for arbitrary topological spaces[20]. They also show, for the first time, that if the base space is arcwise connected, then the fibres all have the same homotopy type.

At the same time, unknowingly of Hurewicz and Steenrod's work because of the Second World War, Ehresmann and Fledbau were defining fibre bundles more or less as we know them now. They then proved the HLP for bundles with respect to finite complexes and then used the HLP to deduce various isomorphisms. Thus, their goal was not to define a structure whose main purpose is to capture the HLP, but they are well aware of its importance and the fact that it ought to be proved for the class of structures they are interested in.

Meanwhile in Zurich, Eckmann defined what he called *retrahierbare Zerlegungen*, retractable partition. We will not give Eckmann's definition here, for it is given under more restrictive assumptions than Hurewicz and Steenrod's and can be shown to be a special case of theirs. But the general strategy is the same: after presenting his definition, Eckmann proceeds to prove the HLP with respect to compact spaces, followed by a proof (of what will become) the homotopy exact sequence and obtains various results about homotopy groups.

The situation did not change much during the 1940s. In 1943, Ralph Fox generalized Hurewicz and Steenrod's definition by removing the restriction on the base space B. Then, Jean-Pierre Serre shocked the world of homotopy theory with the publication of his thesis in 1951.

Serre's attitude towards fibre spaces is remarkable and left an ineffaceable imprint: he says explicitly that since the only thing he needs to establish his results is the HLP, he *defines* a *fibre space* as a map $p : E \to B$ that satisfies the HLP with respect to *finite polyhedra*. Serre immediately points out that fibre spaces in this sense include (locally trivial) fibre bundles, Hurewicz and Steenrod's fibre spaces, principal G-bundles and the class of spaces he is going to use later in his paper, namely *path spaces*. The latter, which have become important in their own right, are defined as follows (Serre, 1951, 479): let X be an arcwise connected space and $A, B \subset X$; then the function space $E_{A,B} = \{f : I \to X : f(0) \in A \wedge f(1) \in B\}$ with the compact-open topology is called a *path space*. Define $p_{A,B} : E_{A,B} \to A \times B$ by $p_{A,B}(f) = (f(0), f(1))$. Serre showed that this map satisfies the HLP with respect to *all* spaces. Then, Serre considered the special case when $A = \{x_0\}$ and $B = X$. In this case, the map $p_{A,B} : E_{A,B} \to A \times B$ becomes $p_{x_0,X} : E_{x_0,X} \to X$ with fibres $F = \Omega(X, x_0)$, the loop space at x_0. The fibration $p_{x_0,X} : E_{x_0,X} \to X$ occupies a key role in the whole work: by applying a spectral sequence to it, it is possible to find the homology groups of the loop space $\Omega(X, x_0)$ from the homology groups of the

[20] As underlined by Zisman 1999, the restrictions imposed on the spaces in the original definition of homotopy groups presented by Hurewicz five years earlier are lifted without a single comment.

base space X. Furthermore, since $p_{x_0,X} : E_{x_0,X} \to X$ is a fibration, it is possible to construct the homotopy exact sequence, i.e. we get

$$\cdots \to \pi_n(\Omega(X)) \xrightarrow{i_*} \pi_n(E_{x_0,X}) \xrightarrow{p_*} \pi_n(X) \xrightarrow{\partial} \pi_{n-1}(\Omega(X)) \to \cdots .$$

This automatically yields the following connection between the homotopy groups and the loop space of a space:

$$\pi_n(X, x_0) \approx \pi_{n-1}(\Omega(X), \alpha_1) \approx \cdots \approx \pi_{n-p}(\Omega^p(X), \alpha_p) \approx \cdots \approx \pi_0(\Omega^n(X)).$$

Using spectral sequences and knowledge of homology groups, Serre then proceeds to prove general results about homotopy groups of spheres, for instance:

1. for all $i > n$, if n is odd, the groups$\pi_i(S^n)$ are finite;
2. if n is even and $i = 2n - 1$, then $\pi_i(S^n)$ is the direct sum of $\mathbb{Z}$ and a finite group.

As we have just seen, for Serre, a fibre space is a map $p : E \to B$ satisfying the HLP with respect to finite polyhedra (it was soon shown afterwards to be equivalent to satisfying the HLP with respect to all CW-complexes, a large and useful category of topological spaces, especially in homotopy theory). Curtis and Hurewicz soon after independently gave new but equivalent definitions of fibre spaces for which they proved that a map $p : E \to B$ was a fibre space in this new sense if and only if it satisfies the HLP with respect to *all* topological spaces. (See Curtis 1956 and Hurewicz 1955[21].)

But Serre's point of view prevails to this day. A *Serre-fibration* is defined to be a map $p : E \to B$ satisfying the HLP with respect to hypercubes I^n, whereas a *Hurewicz-fibration* is a map $p : E \to B$ satisfying the HLP with respect to all topological spaces. Other types of fibrations have been defined, e.g. weak fibrations and quasifibrations. The latter are quite interesting in themselves: a *quasifibration* is a continuous surjective map $p : E \to B$ such that for each point $b \in B$:

1) the map $\pi_i(E, F, y) \to \pi_i(B, b)$ is an isomorphism, for any $i \geq 1$ and any $y \in F = p^{-1}(b)$;
2) $\pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0$ has to be exact as a sequence of pointed sets.

Thus, one retains only what one was able to prove from the HLP and use *that* as the defining property. The surprising fact is that there are non-trivial quasifibrations.

But this is not the whole story, far from it. In 1967, Quillen changed the scenery completely, a development that led to what deserves to be called *abstract homotopy theory*. The nature and the consequences of this radical shift will be briefly explored in the next and last section.

[21] A terminological remark: Hurewicz's 1955 paper defines *fibre* spaces, whereas he previously used *fibre* spaces. Both terms are found in the literature right from the beginning.

Fibrations: their form and functions

We will confine ourselves in this section to the essential elements required for our analysis, leaving the historical details and most of the mathematical background behind.

With hindsight and from a purely conceptual point of view, homotopy theory becomes relevant in a mathematical situation whenever either one is not interested so much in a specific map but rather in the class of homotopically equivalent maps, or one is not interested so much in a specific space or any homeomorphically equivalent space but only in the homotopy type of the space. Thus an abstract homotopy theory should allow us, first, to define appropriate equivalence relations between maps of objects and appropriate equivalence relations between objects themselves and, second, apply the tools of homotopy theory, e.g. homotopy groups, the homotopy exact sequence, etc. to these objects and maps. What is 'appropriate' here is in a sense dictated by the topological case: whatever abstract sense one gives to homotopically equivalent maps and homotopy type, one should be able to recover the standard topological meaning of these expressions. This is precisely what Quillen succeeded in doing in 1967. Using Quillen's framework, it is possible to define a homotopy theory in various contexts and to compare homotopy theories, e.g. state precisely when two homotopy theories are in fact the same.

Quillen made essential use of category theory in his work, in particular the idea of model category (see Quillen 1967, 1969) which there is not space to define properly here. The axioms of a model category have two functions[22]. First, and this is clearly a standard feature of the axiomatic method, if a property of a model category can be proved from the axioms, then it holds of any model category and therefore they can be applied directly to any specific case. Second, and this is perhaps more peculiar, the axioms ought to be thought of as a *conceptual design* for a homotopy theory. They stipulate the conditions under which a homotopy theory can be built. It is the whole point of the definition: to be able to define the notion of homotopy in that context and then use all the machinery of homotopy theory to obtain significant results. Thus, although one might want to say that a property is true of model categories if it follows from the axioms, I seriously doubt that anyone would want to claim that the axioms *themselves* are true. What matters most or what is valued most, I believe, is rather the fact that the axioms, in the words of Dwyer and Spalanski,

> give a reasonably general context in which it is possible to set up the basic machinery of homotopy theory. The machinery can then be used

[22] This is certainly true of other axiomatic definitions, e.g. Eilenberg and Steenrod's axioms for homology and cohomology. I do believe that the attitude towards the axiomatic method as a method of capturing essential ingredients present in various contexts instead as a way of presenting intuitive truths about a fixed domain of objects emerged during the last quarter of the nineteenth and the beginning of the twentieth century. It is thus a characteristic feature of twentieth-century mathematics.

> immediately in a large number of different settings, as long as the axioms are checked in each case. . . . Certainly each setting has its own technical and computational peculiarities, but the advantage of an abstract approach is that they can all be studied with the same tools and described in the same language. (Dwyer and Spalinski, 1995, 75)

Given a model category C, it is possible to construct the *homotopy category* Ho(C) associated to C. Presenting the construction would require introducing many other concepts and thus, considerably more space. Suffice it to say that, from the conceptual point of view, the construction of the homotopy category guarantees that the weak equivalences of C are turned into genuine isomorphisms in Ho(C).

Furthermore, Quillen proposed a criterion to determine when two homotopy categories, say Ho(C) and Ho(D), are the same, that is equivalent as categories. Such an equivalence is now called a *Quillen equivalence.* Quillen equivalence determines a *homotopy theory*, i.e. a homotopy theory is more or less all the homotopical information preserved by a Quillen equivalence. Using this terminology, Quillen has shown, for instance, that the homotopy category of the category of simplicial sets and the homotopy category of topological spaces with the appropriate model structure are equivalent, thus they are models of the same homotopy theory. Another interpretation of this result if given by Dwyer and Spalanski: 'this shows that the category of simplicial sets is a good category of algebraic or combinatorial 'models' for the study of ordinary homotopy theory.' (Dwyer and Spalanski, 1995, 122.)

Let us come back to fibrations. In the context of model categories, we have left behind the topological setting. It is one setting among many others. Fibrations are defined together with cofibrations in the axioms. Their properties are stipulated by the axioms. The HLP still plays a key role: faithful to Serre's approach, it is used to define model categories. It is not given with respect to a class of objects, but with respect to the class of cofibrations (and with respect to fibrations). This should be no surprise by now. But the exact role of fibrations in homotopy theory is still being clarified:

> A closer look at the notion of a model category reveals that the weak equivalences already determine its 'homotopy theory', while the cofibrations and the fibrations provide additional structure which enables one to 'do' homotopy theory, in the sense that, while many homotopy notions involved in doing homotopy theory can be defined in terms of the weak equivalences, the verification of many of their properties (e.g. their existence) requires the cofibrations and/or the fibrations. (Dwyer *et al.* 2004)

Here lies the divide: a systematic technology, in contrast to a technique, depends upon scientific knowledge for its design. Homotopy theory rests upon a portion that should be qualified as being 'scientific', certain fundamental mathematical laws captured by what Dwyer *et al.* call 'homotopical categories', which are defined by a class of weak equivalences satisfying some simple properties. But to actually carry on homotopy theory, to construct the various structures required, one needs some machinery and this is precisely where fibrations (cofibrations) come in.

4 Concluding remarks

Fibrations are sometimes introduced as the appropriate homotopic generalization of the concept of fibre bundle. Here is a typical example:

> Of course, from a homotopy viewpoint, having homeomorphic fibres and the rest of the rigid structure of a fibre bundle is overkill. In this section we will study a generalization (and its dual) of the concept of fibre bundle in which the fibres over points in a common path component are not homeomorphic but merely homotopy equivalent,... (Selick, 1997, 53.)

The generalization in question is the concept of fibration. This certainly suggests that this is how one should view or understand what fibrations are about: they constitute the generalization of the concept of fibre bundle with the right homotopy theoretic property. Although fibrations do indeed have that property and it is indeed homotopically important, anyone who thinks that this is the point of fibrations would miss the crucial element. It is worth reading the whole quote from Selick's book for he himself is entirely aware of this point:

> One of the features of a fibre bundle $p : X \to B$ (...) is that the 'fibres', $F_b = p^{-1}(b)$, are homeomorphic for all points b in a common path component of B. From the homotopy point of view, the key property of a fibre bundle is that for any pointed space W, there is an exact sequence $[W, F_b] \to [W, X] \to [W, B]$, where b is the base point of B. Of course, from a homotopy viewpoint, having homeomorphic fibres and the rest of the rigid structure of a fibre bundle is overkill. In this section we will study a generalization (and its dual) of the concept of fibre bundle in which the fibres over points in a common path component are not homeomorphic but merely homotopy equivalent, and although the overall structure is much less rigid than that of a fibre bundle, it is still sufficient to give the exact sequence $[W, F_b] \to [W, X] \to [W, B]$. (Selick, 1997, 53)

Selick is in fact clear: from a homotopical point of view, the important property of fibre bundles is that they are fibrations, i.e. one can construct the desired exact sequence. The fact that fibers over points in a common path component are homotopically equivalent is a crucial *theoretical* indication that the concept is just right, that it captures the right kind of information, but it would be wrong to conclude that from an epistemic point of view, fibrations are *merely* generalizations of fibre bundles.

The concept of fibration is essentially a *relational* concept. What characterizes a fibration is the *class* of spaces with respect to which it satisfies the HLP. Although it is possible, as we have seen, to define a fibre space as an object with an intrinsic property and prove that such spaces satisfy the HLP, mathematicians prefer to define fibrations directly by specifying the class of spaces with respect to which their maps satisfy the HLP. Thus, the concept is tailored to their specific *needs*. If you want more maps as fibrations, use Serre-fibrations (e.g. if you are interested in locally trivial bundles); if you need fewer maps as fibrations, use Hurewicz-fibrations; if you can't use the HLP but the basic isomorphism of the homotopy exact sequence is available, use quasifibrations.

Within the world of mathematical concepts, fibrations have a different epistemological status than, say, fibre bundles or principal bundles, but also different from the homotopy groups. Fibre bundles are fundamentally geometric and, as such, model various properties of what one might think of as space. Homotopy groups should be thought of as *measuring instruments* since they provide information about certain crucial aspects of spaces. Although the latter are groups in the standard axiomatic sense, they are epistemologically radically different from the groups of the nineteenth century. Groups in the nineteenth century were always acting on something, either a set or a space; they were *transformation* groups of a space or *permutation* groups of a set of roots of a polynomial equation. Homotopy groups (and here we might as well mention homology and cohomology groups) do not act on anything. They are not *defined* in the same way nor are they *used* in the same way. The purpose of these geo*metric* devices is to classify spaces in different *homotopy types*. Many concepts and methods of point-set topology, e.g. compactness, are simply irrelevant to homotopy types (compactness is not an homotopy invariant notion). Homotopy theory contributed to a large extent to the sharp separation between algebraic topology and point-set topology in the 1950s. Points of spaces, as defined in the usual set-theoretical way, do not play an essential role in homotopy types. This is in sharp contrast to the role they have in homeomorphism types. Fibrations also play a role in the separation between algebraic topology and point-set topology, since, as we have seen, they can be defined in various contexts, e.g. categories, and used to develop homotopy theory and homotopy types in these contexts. But in contrast to homotopy groups, fibrations *cannot* be thought of as *measuring instruments*. Fibrations are devices that make it possible to apply the measuring instruments and other devices; they have to be seen as an ingenious and extraordinarily useful tool for the construction of informative structures.

Knowledge of fibrations is clearly knowledge of their *usage* and that knowledge resembles more technological knowledge than scientific knowledge. It should be clear at this stage that fibrations reveal, to paraphrase Polanyi, an effective and ingenious operational principle that achieves, in existing circumstances, a substantial conceptual advantage.

Furthermore, fibrations are not merely a technique, but a set of rules one follows to solve a problem or compute a certain quantity. We can talk about the 'concept' of fibration and fibrations are thought of as a certain structure. They certainly deserve the label 'systematic technology' since, in the case of a fibration of a space, fundamental concepts of topology have to be applied properly and, in the more general case of model categories, fundamental concepts of category theory have to be applied properly.

If fibrations are to be thought of as tools, then there is no point in thinking about them in terms of truth, but rather in terms of efficiency. As such, like any technology and, more to the point, like any technological knowledge, it might very well become useless or obsolete, although it might be hard to imagine how this could be now.

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