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Abelian oil and water dynamics does not have an absorbing-state phase transition

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Abstract

The *oil and water* model is an interacting particle system with two types of particles and a dynamics that conserves the number of particles, which belongs to the so-called class of *Abelian networks.* Widely studied processes in this class are *sandpiles models* and *activated random walks*, which are known (at least for some choice of the underlying graph) to undergo an *absorbing-state phase transition.* This phase transition characterizes the existence of two regimes, depending on the particle density: a regime of *fixation* at low densities, where the dynamics converges towards an absorbing state and each particle jumps only finitely many times, and a regime of *activity* at large densities, where particles jump infinitely often and activity is sustained indefinitely. In this work we show that the oil and water model is substantially different than sandpiles models and activated random walks, in the sense that it does not undergo an absorbing-state phase transition and is in the regime of fixation at *all* densities. Our result works in great generality: for *any* graph that is vertex transitive and for a large class of initial configurations.

1 Introduction

We consider an interacting particle system called *oil and water*, which is defined as follows. There are two types of particles, which we call *oils* and *waters.* Take $G = (V(G), E(G))$ to be an infinite graph, and let ν be a probability measure on the set of non-negative integers $\mathbb{N}$. The initial configuration of particles is distributed as a product of independent random variables distributed as ν; that is, at each vertex $x \in V(G)$ place a random number of oils and independently a random number of waters, both values sampled from the distribution ν. We denote by $\mu = \mu(\nu)$ the expected number of particles at a given vertex; thus $\mu/2$ is the expectation of a random variable distributed as ν. We shall refer to this initial configuration as *oil and water at density μ.*

Starting from the above configuration, particles move according to the following dynamics. Each vertex of G has an independent Poisson clock of rate 1. Whenever the clock

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of a vertex x rings, if x hosts at least one oil and one water then it *fires* an oil-water pair: one water and one oil jump independently according to one step of simple random walk on G (that is, each of the two chooses independently a neighbor of x uniformly at random and jumps there). On the other hand, if at the time the Poisson clock of x rings, x has no particles or hosts only particles of one type (either oil or water), then x does not fire; in this case we say that x is *stable*. Note that x may host arbitrarily many particles, but as long as they are all of the same type, x is stable and none of its particles are allowed to jump. Note also that if we reach a configuration where every vertex of G is stable, which we refer to as a *stable configuration*, then no vertex fires from that time onwards. Thus stable configurations are absorbing states for the dynamics.

Oil and water has the so-called *Abelian property* [BL16], which states that the final configuration of the system does not depend on the order at which vertices fire. This gives that the times at which the Poisson clocks ring are irrelevant.

There are two possible outcomes of the system: either it is *active*, in which each vertex fires infinitely many times, or it *fixates*, that is each vertex fires finitely many times. It is easy to check that if one vertex fires infinitely many times, then all vertices do as well. We show two fundamental properties in Section 2. The first one is *monotonicity* (Lemma 2.2), which gives that if oil and water at density μ fixates, then it also fixates for all densities $\mu' \in [0, \mu]$. The second property is a 0-1 law (Lemma 2.3), which states that given μ the probability that the system fixates is either 0 or 1.

With the above two properties, and inspired by results for other models having the Abelian property and also satisfying those properties, such as *stochastic sandpiles* and *activated random walks* [RS12, ST17], one may conjecture that the oil and water model undergoes a phase transition between *activity* and *fixation* at some critical density μ_c. (A more thorough discussion of the relation between oil and water and other models with the Abelian property is given in Section 1.1 below.)

The main result of our paper is to show that the above conjecture is not true, for *any* graph G, with the only requirement that G is vertex transitive. Let $\mathbb{P}_\nu$ denote the probability law of the oil and water dynamics starting from a configuration of density $\mu = \mu(\nu)$ as above.

Theorem 1.1. *Let G be an infinite, vertex-transitive and finite-degree graph. Then, for any ν with $\mu = \mu(\nu) < \infty$,*

$$\mathbb{P}_\nu\big[\textit{oil and water fixates}\big] = 1.$$

A natural question is whether vertex transitivity is a necessary property. We expect this not to be the case and that Theorem 1.1 holds even in greater generality. In fact, it would be interesting to know whether it is possible to engineer a finite-degree graph for which oil and water is active for some $\mu < \infty$.

Oil and water was introduced in [BL16] as an example of an Abelian network that is not unary (that is, which has more than one type of particles); see Section 1.1 below. The oil and water model was analyzed in [CGHL17] in a different setting. They consider the one-dimensional lattice $\mathbb{Z}$, and let the initial configuration be given by N oil-water pairs at the origin, with all other vertices initially unoccupied. Then the oil and water dynamics is run until a stable configuration is obtained; this occurs in finite time, almost surely, since the number of particles is finite. In this setting, [CGHL17] investigated several statistics of the model, including how long it takes for the process to stop and how far from the origin particles spread as a function of N.

1.1 Related models

Oil and water was introduced in [BL16] within the more general framework of Abelian networks, which was introduced by Bond and Levine [BL16] building on the work of Dhar on sandpile models [Dha99]. This framework was created with the goal of defining a general concept that includes several widely studied processes, such as *Abelian and stochastic sandpiles*, *bootstrap percolation*, *rotor-router networks*, *internal DLA* and *activated random walks*. Informally speaking, a particle system (or, more generally, a cellular automaton) is considered an Abelian Network if it satisfies the so-called *Abelian property*, which gives that the final configuration of the system does not depend on the order of the interactions. In other words, the final configuration is invariant to changes in the order at which vertices fire.

Abelian networks have been widely studied in several disciplines. For example, in computer science, they are a fundamental model in distributed systems, as they do not require any central synchronization or shared memory, see [BL16] for more details. In mathematics and physics, several types of Abelian networks have been investigated, an archetypal example being *sandpile models* [Jár18]. The study of sandpile models was initiated in [BTW87, BTW88] motivated by the observation that they present characteristics of *self-organized criticality*. This means that as the process evolves, the system drives itself to a "critical state" without having to tune any parameter. Here, "critical state" means that after a long time the configuration shows characteristics that are common to systems at criticality. Refer to [Jár18] for more information about self-organized criticality and sandpile models.

There have been several works in the physics literature to understand self-organized criticality. One approach has been to relate this phenomenon to the more classical one of phase transitions, called *aborbing-state phase transition* [MDPS$^+$01]. This corresponds to a phase transition between a regime of fixation (where for a small density of particles the system moves towards an absorbing state) and a regime of activity (where for a large density of particles the activity is sustained indefinitely). Physicists believe that the presence of an *absorbing-state phase transition* is intrinsically connected to the phenomenon of *self-organized criticality* [MDPS$^+$01], and even defines a new universality class [RPSV00]. In particular, physicists studied several systems with a conserved number of particles which are connected to systems from self-organized criticality, and showed non-rigorously that such systems undergo an absorbing-state phase transition. Examples of such systems include stochastic sandpiles, fixed energy sandpiles, conserved threshold transfer processes, and activated random walks [MDPS$^+$01, RPSV00, PSV00].

In the mathematics literature, results in this area are much more scarce. Ingenious proofs have been developed to show that stochastic sandpiles and activated random walks undergo an absorbing-state phase transition in some graphs [RS12, ST17, ST18, BGH18, Tag19], and it is expected that such a result should be true for any vertex-transitive graph. In this paper, we show that the same is not true for the oil and water model, for any vertex-transitive graph. In some sense, the strong interactions between the particles in the oil and water dynamics cause the particles to organize themselves in order to achieve fixation. To the best of our knowledge, this is the first time that a natural model of an Abelian network (with a conserved number of particles) is shown not to undergo an absorbing-state phase transition. Another additional feature of our result is that our proof is not engineered for a specific graph, but works in any vertex transitive graph and any initial configuration of particles that is obtained from a product measure.

1.2 Proof overview

Two fundamental properties that will be heavily employed in the proof are the *Abelian property* and the *0-1 law*. A popular strategy to analyze Abelian networks [RS12, ST17, BGH18, CGHL17] is to devise a so-called *stabilization algorithm*. For example, if one wants to show fixation (resp., activity), this strategy consists of choosing a smart order to fire the vertices, exploiting the Abelian property, in order to obtain that a given vertex *does not fire at all* (resp., fires infinitely many times) with positive probability, which by the 0-1 law implies almost surely fixation (resp., activity). Usually, the stabilization algorithm exploits the structure of the graph (which, in all the aforementioned papers, was always a grid such as $\mathbb{Z}^d$, $d \geq 1$), making such proofs very much graph dependent. Moreover, in some models, such as stochastic sandpiles and activated random walks, where an absorbing-state phase transition takes place, one also uses *monotonicity*; that is, it suffices to show fixation for some small enough μ, and to show activity for some large enough μ.

The oil and water model gives rise to different challenges, since we need to show that the process fixates for *all* μ, no matter how large it may be, and for *all* transitive graphs. In order to do this, we had to develop a new proof strategy. Before describing it, we fix some terminology. For any vertex x, if x has k_o oils and k_w waters, we say that x has $k_o \wedge k_w$ oil-water pairs, where we view each such pair as a matching between an oil particle and a water particle from x. So, each vertex x may only have unpaired particles of at most one type (either oil or water).

Now suppose that vertex $x \in V(G)$ is unstable, thus x has at least one oil-water pair. If all neighbors of x have nonzero unpaired oils, when we fire x, the water particle that gets to jump from x will be paired to one of the unpaired oils located at the neighbors of x (or to the oil particle that jumped from x, if both oil and water jump to the same neighbor). As a consequence, the number of oil-water pairs in the system does not change. In fact, even if the water jumping from x gets paired to a different oil particle, we observe that the firing of x effectively causes an oil-water pair to do a step of a *simple random walk* from x. The same occurs if all neighbors of x have nonzero unpaired waters.

On the other hand, suppose that $d_w \geq 1$ neighbors of x have unpaired waters, $d_o \geq 1$ neighbors of x have unpaired oils, and that $d_w + d_o = \mathbf{d}$ with $\mathbf{d}$ denoting the degree of each vertex of G (that is, each neighbor of x has at least one unpaired particle). In this case, the number of oil-water pairs changes by either -1, 0 or 1. For example, it changes by -1 (resp., $+1$) if the water jumps from x to a neighbor with unpaired waters (resp., oils), and the oil jumps from x to a neighbor with unpaired oils (resp., waters); in other cases the number of oil-water pairs does not change. We can readily see that

$$\text{the number of oil-water pairs changes} \begin{cases} \text{by } 0 \text{ with probability } &= 1 - 2\dfrac{d_w d_o}{\mathbf{d}^2}, \\ \text{by } 1 \text{ with probability } &= \dfrac{d_w d_o}{\mathbf{d}^2}, \\ \text{by } -1 \text{ with probability } &= \dfrac{d_w d_o}{\mathbf{d}^2}. \end{cases}$$

The above gives that, in this case, the configuration of oil-water pairs behaves as a *critical branching random walk* on G. Suppose now that x has at least one neighbor with no unpaired particles (such neighbors are called holes), then we have that the configuration of oil-water pairs behaves as a *subcritical* branching random walk.

Putting all these cases together, when a vertex x fires, the configuration of oil-water pairs behaves either as a simple random walk, as a critical branching random walk, or as a subcritical branching random walk, depending on the environment of unpaired particles at the neighbors of x. Moreover, it behaves as a subcritical branching random walk only when x is the neighbor of a hole.

Intuitively, since oil-water pairs cause a vertex to fire, in order to show fixation we need to show that the number of oil-water pairs decreases quickly. Thus, we want to show that for a large enough number of steps we fire a vertex that neighbors a hole.

The proof works by contradiction. We assume that the system is active, which implies that each vertex fires a very large number of times. Now consider a vertex x that fires k times, and let y be a neighbor of x which, for instance, has unpaired oils. Then, we can show that y will be a hole for a number of times that increases with k. This is because each time x fires, conditioning on x sending exactly one particle to y, with equal probability this particle is an oil or a water. So the number of unpaired particles at y behaves as a simple random walk on $\mathbb{N}$, reflected at the origin, which is recurrent. Developing this argument we will obtain that a very large number of holes will be created during this process. At those times, the number of oil-water pairs behaves as a supermartingale. Hence, it decreases quickly.

In order to implement this strategy, we need to control the evolution of the locations of the oil-water pairs. The challenge is that they behave as a mix of simple random walk, critical branching random walk and subcritical branching random walk, depending on (and affecting) the environment of the unpaired particles. We are able to control this by defining a suitable martingale, which depends on the configuration of the particles. This martingale allows us to relate the expected number of oil-water pairs to the Green's function of simple random walk on G. This step, which is at the core of our proof, is given in Lemma 3.4; see also Remark 3.5.

2 Graphical representation and properties

In this section we introduce a graphical representation for the model. Via this representation we can prove a 0-1 law for the probability of fixation, and the Abelian property, where the latter informally states that the number of firings at a given vertex does not depend on the temporal order of firings of the system and was proved in [BL16]. The structure of this section is inspired by [RS12], where the authors prove a 0-1 law for two models which are strictly related to the present one, namely *stochastic sandpiles* and *activated random walks*.

Notation. The graph G is infinite, vertex-transitive with finite degree, and it is fixed along the whole proof. We fix an arbitrary reference vertex and call it *origin* $o \in V(G)$. When considering two vertices $x, y \in V(G)$, we denote by $d(x, y)$ the graph distance between x and y, namely the length of the shortest path from x to y. As a shorthand we also write $x \sim y$ when $d(x, y) = 1$.

2.1 Definitions

The space of possible configurations will be denoted by $\Omega := \mathbb{N}^{V(G)} \times \mathbb{N}^{V(G)}$. We shall denote an element of Ω by

$$\eta = \big(\eta^o(x), \eta^w(x)\big)_{x \in V(G)},$$

where $\eta^o(x)$ (resp., $\eta^w(x)$) corresponds to the number of oils (resp., waters) at x. Also, recall that $\mu > 0$ is the expected number of particles at each site in the starting configuration, that is

$$\mu = \mathbb{E}\big(\, \eta^o(o) + \eta^w(o)\,\big),$$

where $o \in V(G)$ denotes a reference vertex that we call the origin. When investigating the long-time behavior of this model we might expect two possible outcomes, which can depend on μ and on the properties of the graph G, namely *fixation* or *activity*, which we describe below. For all $x \in V(G)$ and all $t \geq 0$, let $u_t(x)$ denote the number of firings occurred at x by time t; we say that the process *fixates* when for any finite set of vertices $A \subset V(G)$ there is a (random) time $\mathbf{t}_A < \infty$ for which

$$\forall x \in A, \text{ for all } t > \mathbf{t}_A \quad u_t(x) = u_{\mathbf{t}_A}(x).$$

In other words, no vertex of A fires after time $\mathbf{t}_A$. On the other hand, we say that the process is *active* if it does not fixate.

Given a configuration $\eta \in \Omega$, a vertex $x \in V(G)$ is called *stable* if $\eta^o(x) \wedge \eta^w(x) = 0$ and it is called *unstable* otherwise.

For any $x \in V(G)$ and any pair of vertices $y_o, y_w \sim x$, we define a pair of instructions $(\tau^o_{x,y_o}, \tau^w_{x,y_w})$ as an operator acting on configurations $\eta = (\eta^o, \eta^w) \in \Omega$ which are unstable at x. Given such a configuration η as input, the operator returns a configuration $\eta_1 = (\eta^o_1, \eta^w_1) \in \Omega$ such that, for $q \in \{o, w\}$,

$$\eta^q_1(z) := \begin{cases} \eta^q(z) - 1 & \text{if } z = x, \\ \eta^q(z) + 1 & \text{if } z = y_q, \\ \eta^q(z) & \text{otherwise.} \end{cases}$$

In words, the operator $(\tau^o_{x,y_o}, \tau^w_{x,y_w})$ makes one oil jump from x to y_o and one water jump from x to y_w.

Now we fix an *array* $\tau = \{\tau^{x,j} : x \in V(G),\, j \in \mathbb{N}\}$, where each element $\tau^{x,j}$ is a *pair of instructions* of the form $\tau^{x,j} = (\tau^{x,j,o}, \tau^{x,j,w})$; in particular, each such a pair is an element of the set $\{(\tau^o_{x,y_o}, \tau^w_{x,y_w}) \ : \ y_o \sim x, y_w \sim x\}$.

We also need to define a function $\mathbf{h} = (h(x) : x \in V(G))$ that counts the number of pairs of instructions used at each vertex. Given the counter $\mathbf{h}$, we say that x *fires* (or that we *topple* x, borrowing the notation from the abelian sandpiles setting) when we act on the pair $(\eta, \mathbf{h})$ through an operator Φ_x which is defined as,

$$\Phi_x(\eta, \mathbf{h}) = (\tau^{x,h(x)+1}\, \eta,\, \mathbf{h} + \delta_x), \tag{2.1}$$

where $\delta_x(y) = 1$ if $y = x$ and $\delta_x(y) = 0$ otherwise. In words, the operator Φ_x makes one oil and one water jump from x simultaneously and then it updates the counter $\mathbf{h}$. The operation Φ_x is said to be *legal* for $(\eta, \mathbf{h})$ if x is unstable in η, otherwise it is *illegal.*

2.2 Properties

We now describe the properties of this representation. For a sequence of vertices $\alpha = (x_1, x_2, \ldots, x_k)$, we write $\Phi_\alpha = \Phi_{x_k}\Phi_{x_{k-1}}\ldots\Phi_{x_1}$ and we say that Φ_α is *legal* for η if Φ_{x_ℓ} is legal for $\Phi_{(x_{\ell-1},\ldots,x_1)}(\eta, \mathbf{0})$ for all $\ell \in \{2, \ldots, k\}$, where $\mathbf{0}$ is the counter which equals zero at every vertex. Given a particle configuration $\eta \in \Omega$, a legal sequence α and a fixed array of instructions τ, we write $\Phi_\alpha \eta \in \Omega$ for the particle configuration of the pair $\Phi_\alpha(\eta, \mathbf{0})$. In other words, $\Phi_\alpha \eta$ is the particle configuration which is obtained from η when we topple the vertices according to the sequence α. Let $m_\alpha = \{m_\alpha(x) : x \in V(G)\}$ be given by

$$m_\alpha(x) = \sum_\ell \mathbb{1}\{x_\ell = x\}, \tag{2.2}$$

that is the number of times the vertex x appears in the firing sequence α.

We write $m_\alpha \geq m_\beta$ if $m_\alpha(x) \geq m_\beta(x)$ for all $x \in V(G)$. We write $\eta_1 \geq \eta_2$ if $\eta_1^q(x) \geq \eta_2^q(x)$ for $q \in \{o, w\}$ and $x \in V(G)$. We also write $(\eta', \mathbf{h}') \geq (\eta, \mathbf{h})$ if $\eta' \geq \eta$ and $\mathbf{h}' \geq \mathbf{h}$.

Let η, η' be two configurations, let $x \in V(G)$, let τ be an array of instructions, and let K be a finite subset of $V(G)$. A configuration η is said to be *stable* in K if all the vertices $x \in K$ are stable. We say that a sequence α is contained in K if all its elements are in K, and we say that α *stabilizes* η in K if $\Phi_\alpha \eta$ is stable in K. The following property was proved by Bond and Levine.

Lemma 2.1 (Abelian Property, [BL16])**.** *Let $K \subset V(G)$ be a finite set. If α and β are both legal sequences for η that are contained in K and stabilize η in K, then $m_\alpha = m_\beta$ and $\Phi_\alpha \eta = \Phi_\beta \eta$.*

For any finite subset $K \subset V(G)$, any $x \in V(G)$, any particle configuration η, and any array of instructions τ, we denote by $m_{K,\eta,\tau}(x)$ the number of times that x fires in the stabilization of K starting from η and using the instructions in τ. Note that by Lemma 2.1, we have that $m_{K,\eta,\tau}$ is well defined. The following fact is a direct consequence of the Abelian property.

Lemma 2.2 (Monotonicity)**.** *For finite subsets $K \subset K' \subset V(G)$ and particle configurations $\eta \leq \eta'$, we have that,*

$$m_{K,\eta,\tau} \leq m_{K',\eta',\tau}.$$

Proof. Fix an array τ, and let $\beta := (x_1, x_2, \ldots, x_k)$ be a legal sequence that stabilizes η in K; then, by Lemma 2.1 we have that any other legal sequence stabilizing K will use the same number of firings as β. By definition, this sequence has not yet stabilized any vertex in the set $K' \setminus K$. Since the set K' cannot be stable until the set K is stable, the claim follows from (2.2). $\square$

By monotonicity, given any growing sequence of subsets $V_1 \subseteq V_2 \subseteq \cdots \subseteq V(G)$ such that $\lim_{t\to\infty} V_t = V(G)$, the limit

$$m_{\eta,\tau} := \lim_{t\to\infty} m_{V_t,\eta,\tau} \tag{2.3}$$

exists and does not depend on the particular sequence $\{V_t\}_t$.

So far we have fixed a deterministic array τ and a particle configuration η. We now introduce a probability measure on the space of instructions and particle configurations. We denote by $\mathcal{P}$ the probability measure according to which the pairs of instructions $\tau^{x,j} := (\tau^{x,j,o}, \tau^{x,j,w})$ are independent across different values of x, j and $\{o, w\}$, and by $\mathbf{d}_x$ the degree of vertex $x \in V(G)$. Moreover, the two elements $\tau^{x,j,o}$ and $\tau^{x,j,w}$ are independent and have distribution

$$\mathcal{P}\big(\tau^{x,j,q} = \tau^q_{x,y_q}\big) := \frac{1}{\mathbf{d}_x},$$

for any $y_q \sim x$, $q \in \{o, w\}$. Roughly speaking, under the measure $\mathcal{P}$ the instructions induce any particle that uses them to perform a step of independent simple random walk.

Finally, we denote by $\mathcal{P}_\nu = \mathcal{P} \otimes \nu$ the joint law of η and τ. We shall often omit the dependence on ν by writing $\mathcal{P}$ instead of $\mathcal{P}_\nu$. The following lemma relates the dynamics of the oil-water model to the stability property of the representation. Recall that $\mathbb{P}_\nu$ denotes the law of the oil-water dynamics under the assumption that the initial configuration was distributed according to a product of measures ν.

Lemma 2.3 (0-1 law). *Let $m_{\eta,\tau}$ be as in (2.3). Then*

$$\mathbb{P}_\nu(\textit{oil and water fixates}) = \mathcal{P}_\nu(m_{\eta,\tau}(o) < \infty) \in \{0, 1\}. \tag{2.4}$$

Lemma 2.3 was proved in [RS12] for two models which are related to oil and water, activated random walk and the stochastic sandpile model. Here we present the main steps of the proof and we refer to [RS12] for the complete argument.

Sketch of the proof of Lemma 2.3. The 0-1 law follows from the fact that the event $\{$*oil and water fixates*$\}$ is automorphism invariant and, since the process is determined by i.i.d. variables at the vertices (initial configuration, Poisson clocks and jump instructions), it is ergodic. We now sketch the proof of the identity in (2.4). The proof consists in coupling the quantities $\lim_{t\to\infty} u_t(x)$ and $m_{\eta,\tau}(x)$. More precisely, let $u_{t,M}(x)$ denote the number of firings that occurred at x before time t when no particle is allowed to jump from vertices outside B_M, the ball of radius M centered at o (the firings at such vertices are "frozen"). The proof consists in two main steps.

In the first step, one constructs a natural coupling between the variables $m_{B_M,\eta,\tau}(x)$ and $u_{\infty,M}(o) := \lim_{t\to\infty} u_{t,M}(o)$ as follows. Recall that $\mathcal{P}_\nu$ is the joint law of the variables $\eta \in \Omega$ and τ, under which they are independent. On the other hand, $\mathbb{P}_\nu$ is the law of the oil and water dynamics, given by $\mathcal{P}_\nu$ together with the law of the sequence of random variables $\boldsymbol{t} = \{t_{i,x}\}_{i\in\mathbb{N}, x\in V(G)}$, where $\{t_{i,x}\}_{i\in\mathbb{N}}$ are i.i.d. exponential random variables with rate 1, and the sequences $\{t_{i,x}\}_{i\in\mathbb{N}}$ are independent across x. The elements of the sequence $\{t_{i,x}\}_{i\in\mathbb{N}}$ represent the times between consecutive attempts for firing x. When such an attempt happens, if x is unstable, one oil and one water perform a simple random walk step from x using the next couple of instructions at x of the array τ. Thus, by this construction, the random variable $u_{t,M}(x)$ is a deterministic function of the random variables $\boldsymbol{t}$, η and τ. Since $u_{t,M}(x)$ is a monotone function in t for every x and every M fixed, the limit $u_{\infty,M}(x) = \lim_{t\to\infty} u_{t,M}(x)$ exists. Now we observe that on a finite set the system fixates within an almost surely finite time and by Lemma 2.1, $m_{B_M,\eta,\tau}(o)$ does not depend on the order according to which the instructions τ are used (provided that only legal instructions are used). Thus, we deduce from this construction that,

$$\forall r, M \in \mathbb{N}, \quad \mathbb{P}_\nu\big(u_{\infty,M}(o) > r\big) = \mathcal{P}_\nu\big(m_{B_M,\eta,\tau}(o) > r\big). \tag{2.5}$$

The second step of the proof consists in showing that the limits over $M \to \infty$ and $t \to \infty$ commute, i.e,

$$\forall r \in \mathbb{N}, \quad \mathbb{P}_\nu\big(\lim_{t\to\infty}\lim_{M\to\infty} u_{t,M}(o) > r\big) = \mathbb{P}_\nu\big(\lim_{M\to\infty}\lim_{t\to\infty} u_{t,M}(o) > r\big), \tag{2.6}$$

and that a blow up does not occur in finite time, i.e,

$$\forall t \in \mathbb{R}_{\geq 0}, \quad \lim_{r\to\infty} \mathbb{P}_\nu\big(u_t(o) > r\big) = 0. \tag{2.7}$$

Equations (2.6) and (2.7) and the fact that,

$$\forall t \in \mathbb{R}_{\geq 0},\ \forall r \in \mathbb{N}, \quad \mathbb{P}_\nu\big(u_t(o) > r\big) = \lim_{M\to\infty} \mathbb{P}_\nu\big(u_{t,M}(o) > r\big) \tag{2.8}$$

imply (2.4). The proof of (2.7) is standard and follows from the fact that, since the jump rates are bounded, particles starting at an infinite distance from o cannot reach the origin within finite time. We refer to [RS12] for the proof of (2.6) given (2.5) and of how the equality in (2.4) follows from these statements. □

From now on, when this is not generating any confusion, we will write m_K instead of $m_{K,\eta,\tau}$, and $m(x)$ instead of $m_{\eta,\tau}(x)$ in order to make the paper more readable.

2.3 Green's function of simple random walk

In this section we recall some classical facts concerning the simple random walk and we provide some definitions. We let $X(t)$ denote a simple random walk in G, and P_x denote its law when $X(0) = x \in V(G)$. We let E_x denote the corresponding expectation. Given a set $Z \subset V(G)$ we define $\tau_Z := \inf\{t \geq 0 \ :\ X(t) \in Z\}$ and $\tau_Z^+ := \inf\{t > 0 \ :\ X(t) \in Z\}$. If $Z = \{y\}$, we write τ_y and τ_y^+ instead of τ_Z and τ_Z^+. For any $x, y \in V(G)$, we define the Green's function,

$$G_Z(x,y) := E_x\left[\sum_{t=0}^{\tau_{Z^c}} \mathbb{1}\{X(t) = y\}\right],$$

where $Z^c := V(G) \setminus Z$. In words, $G_Z(x,y)$ denotes the expected number of visits to a vertex y performed by a simple random walk started at x and killed upon exiting the set Z.

Given a function $g : V(G) \to \mathbb{R}$, $g = (g_x)_{x\in V(G)}$, we let $\triangle g : V(G) \to \mathbb{R}$ denote the discrete Laplacian, that is, for every $x \in V(G)$ we set

$$(\triangle g)_x := \frac{1}{\mathbf{d}_x} \sum_{y\sim x} (g_y - g_x),$$

where we recall that $\mathbf{d}_x$ denotes the degree of x. We say that g is *harmonic* in a set $K \subset V(G)$ if for any $x \in K$, $(\triangle g)_x = 0$. The next proposition states some classical facts and its proof can be found, for example, in [LP16, Chapter 2].

Proposition 2.4. *Consider a finite set $K \subset V(G)$ and a vertex $y \in K$. Let $g : V(G) \to \mathbb{R}$ be a function which is harmonic in $K \setminus \{y\}$ and such that $g_y = 1$, $g_z = 0$ for any $z \in K^c$. Then the function g is unique and satisfies*

$$g_w = P_w(\tau_y < \tau_{K^c}), \quad \forall w \in K, \tag{2.9}$$

$$-(\triangle g)_y = 1 - P_y(\tau_y^+ < \tau_{K^c}). \tag{2.10}$$

Moreover, for all $x, y \in K$ the Green's function satisfies,

$$G_K(y,y) = \frac{1}{1 - P_y(\tau_y^+ < \tau_{K^c})}, \tag{2.11}$$

$$G_K(x,y) = P_x(\tau_y < \tau_{K^c})\, G_K(y,y) = G_K(y,x). \tag{2.12}$$

3 Diffusive fluctuations and number of visits

Now we are able to introduce the following terminology.

Definition 3.1. Given a particle configuration $\eta = (\eta^o, \eta^w)$, let $\eta^o(x) \wedge \eta^w(x)$ be the number of *pairs* at x, and $\eta^o(x) \vee \eta^w(x) - \eta^o(x) \wedge \eta^w(x)$ be the number of *unpaired* particles at x. We say that η has a *hole* at x if the number of oils and waters at x is the same (or, equivalently, if the number of unpaired particles at x is zero). When we refer to a *pair*, we always refer to two particles of different type.

This section is divided into two subsections. In Section 3.1 we show that, if we assume that the system is active and we stabilize some arbitrarily chosen finite set K, then at any vertex $x \in K$ which is far enough from the boundary of K, we will observe a hole at x *many* times during the stabilization. As we pointed out in the proof overview in Section 1.2, the occurrence of holes is helpful to make the number of oil-water pairs decrease over time. In Section 3.2 we introduce a Markov chain which describes an inductive procedure to stabilize K starting from an arbitrary particle configuration. Such a procedure is defined in an enlarged probability space where some virtual particles, called *ghosts*, are added to the system whenever a water jumps into a hole. We will refer to this procedure as the *ghost-pair stabilization*. The ghosts will play a fundamental role at the end of the proof, in Section 4.

3.1 Number of waters falling into holes

We start by stabilizing an arbitrary finite set $K \subset V(G)$ following some legal ordering, which we shall determine through a *strategy*. A strategy for stabilizing K is a function $F_K : \Omega \to V(G) \cup \emptyset$ that acts as follows. Given a particle configuration η, F_K outputs an arbitrary vertex of K that is currently unstable. If η is stable in K then $F_K(\eta) = \emptyset$.

Let $K \subset V(G)$ be a finite set and F_K a strategy. We say that we *stabilize* η *in* K *following strategy* F_K when we perform a sequence of firings as follows. Start by setting $\eta_0 := \eta \in \Omega$ and apply F_K to η_0. If $F_K(\eta_0) = \emptyset$, then we are done as this means that η_0 is stable. If $F_K(\eta_0) \neq \emptyset$, then we topple the vertex $F_K(\eta_0)$, and denote by $\eta_1 \in \Omega$ the resulting configuration. If η_1 is stable then we are done, if not, then we proceed by applying F_K again. Thus, if η_1 is unstable, then we proceed to topple $F_K(\eta_1)$, obtaining a new particle configuration which we call $\eta_2 \in \Omega$. We continue inductively until we reach a random time T_{F_K} at which we have stabilized K. More formally, we set

$$T_{F_K} := \inf \big\{ i \in \mathbb{N}_{\geq 0} \;:\; F(\eta_i) = \emptyset \big\}.$$

For any $x \in V(G)$, we define the number of times a water falls into a hole at x while following the strategy F_K starting from the particle configuration η_0,

$$H_{K,F_K}(x) := \Big| \{ 0 \leq i \leq T_{F_K} - 1 \;:\; \eta_i^w(x) = \eta_i^o(x) \text{ and } \eta_{i+1}^w(x) = \eta_{i+1}^o(x) + 1 \} \Big|. \tag{3.1}$$

We emphasize that this variable also depends on η_0 and on the chosen array τ, however we will omit this dependency to simplify the notation.

This procedure defines the sequence of particle configurations $(\eta_i)_{i \in [0, T_{F_K}]}$, where the last step, $i = T_{F_K}$, is the step at which the set K is stable. In the proof of the next proposition, we will need to introduce some variables which depend also on the instructions

τ which are not "used" for the stabilization of the initial particle configuration in K. For this reason, we will now define also the steps $i > T_{F_K}$ of the stabilization procedure. This will allow to define such variables. Since the set K is stable at step $i = T_{F_K}$, in order to perform some firings we will need to add new pairs to the stable configuration, making it unstable. More precisely, for any step $i > T_{F_K}$, we proceed as follows.

- If $F_K(\eta_i) = \emptyset$ (i.e, η_i is stable in K), then we add one pair at the origin, obtaining the new particle configuration η_{i+1}, which is unstable in K, and we move to the next step $i+1$. In this case no vertex fires at step i.
- If $F_K(\eta_i) \neq \emptyset$ (i.e, η_i is unstable in K), then the vertex $F_K(\eta_i) \in K$ fires, and we obtain a new particle configuration η_{i+1}, which might be stable or unstable in K. We move to the step $i+1$.

Thus, at any step $i > T_{F_K}$ either one unstable vertex fires or a pair is added at the origin. In this way the infinite sequence of random variables $(\eta_i)_{i\in\mathbb{N}}$ is well defined.

Lemma 3.2. *Assume that the system starting from a particle configuration which is distributed as a product of measure ν is almost surely active. Then, for any $\epsilon > 0$ and $M \in \mathbb{N}$, there exists $D = D(\nu, \epsilon, M) < \infty$ large enough such that,*

$$\inf_{\substack{K\subset V(G):\\ d(o,K^c)>D}} \ \inf_{\substack{F_K:\Omega\to V(G)\cup\emptyset:\\ F_K \text{ is a strategy}}} \mathcal{P}_\nu\big(\ H_{K,F_K}(o) > M\ \big) \ \geq\ 1-\epsilon.$$

Before proceeding to the formal proof we present the main idea behind it, which consists in showing that the value of $H_{K,F_K}(o)$ (defined in (3.1)) can be associated to the number of visits to zero of a lazy simple random walk on $\mathbb{Z}$. Once we have established this, classical results give that the number of returns to the origin of a simple random walk on the integers is, with high probability, comparable to the square root of the number of steps performed. The last step of the proof consists in showing that we can in fact let the walk run for as many steps as we need, in order to deduce the claim.

Proof of Lemma 3.2. To begin, we fix a finite set $K \subset V(G)$ such that $B_D \subset K$, where B_D is the ball of radius D centered at o. Then we stabilize the set K following an arbitrary strategy F_K, as defined before the statement of Lemma 3.2. For any $j \in \mathbb{N}_{>0}$, we let t_j be the j-th time a neighbor of the origin fires. More precisely, let

$$\mathcal{N}_o := \{x \in V(G)\ :\ x \sim o\}$$

denote the set of neighbors of o, and set $t_0 := 0$. Thus we define for any $j \in \mathbb{N}_{>0}$,

$$t_j := \inf\{i > t_{j-1}\ :\ F_K(\eta_{i-1}) \in \mathcal{N}_o\}.$$

In words, t_j denotes the first time after t_{j-1} at which a firing occurs at $\mathcal{N}_o$. We let

$$N_K := \sum_{x\in\mathcal{N}_o} m_K(x) \tag{3.2}$$

be the number of times that, during the stabilization of K, there is a firing from a nearest neighbor of the origin. We now define a sequence of random variables $\{R_j\}_{j\geq 0}$, which keeps track of the difference between the number of oils and waters at the origin whenever a firing occurs inside $\mathcal{N}_o$. Subsequently, we show that these random variables are distributed like the steps of a lazy simple random walk on $\mathbb{Z}$. More precisely, first we set

$$R_0 := \eta^w_{t_0}(o) - \eta^o_{t_0}(o),$$

that is, R_0 is the difference between the number of waters and the number of oils at vertex o in the initial configuration. Secondly, for all integers $j \in \mathbb{N}_{>0}$, we define

$$R_j := \eta^w_{t_j}(o) - \eta^o_{t_j}(o). \tag{3.3}$$

Let $\mathbf{d}$ denote the degree of any vertex of G, which is vertex-transitive. Since the difference between the number of oils and waters at o can only change when a neighbor of o fires, it immediately follows that the transition probabilities of the walk are given by the following formulas. The probability to increase of 1 unit is given by

$$\mathcal{P}_\nu\left[R_{j+1} = R_j + 1 \mid R_j\right] = \mathcal{P}_\nu[\eta^w_{t_{j+1}}(o) = \eta^w_{t_j}(o) + 1,\ \eta^o_{t_{j+1}}(o) = \eta^o_{t_j}(o)] = \frac{\mathbf{d}-1}{\mathbf{d}^2}.$$

Symmetrically, we have

$$\mathcal{P}_\nu\left[R_{j+1} = R_j - 1 \mid R_j\right] = \mathcal{P}_\nu[\eta^w_{t_{j+1}}(o) = \eta^w_{t_j}(o),\ \eta^o_{t_{j+1}}(o) = \eta^o_{t_j}(o) + 1] = \frac{\mathbf{d}-1}{\mathbf{d}^2},$$

and finally

$$\mathcal{P}_\nu\left[R_{j+1} = R_j \mid R_j\right] = 1 - 2\frac{(\mathbf{d}-1)}{\mathbf{d}^2}.$$

At this point, it is clear that $\{R_j\}_{j\in\mathbb{N}}$ is distributed as the steps of a symmetric lazy random walk on the integers with a given starting value R_0. For any $j \in \mathbb{N}$, let $\mathcal{J}(j)$ be the number of times the random walk jumps from 0 to $+1$ in the first j steps, i.e,

$$\mathcal{J}(j) := \Big|\big\{k \in [0, j)\ :\ R_k = 0 \text{ and } R_{k+1} = +1\,\big\}\Big|.$$

By definition, for N_K as in (3.2) we have that,

$$H_{K,F_K}(o) = \mathcal{J}(N_K). \tag{3.4}$$

We deduce that, for any $M \in \mathbb{N}$ and $\varphi \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{P}_\nu\big(H_{K,F_K}(o) > M\big) &\geq \mathcal{P}_\nu\big(H_{K,F_K}(o) > M,\ N_K > \varphi\big)\\
&= \mathcal{P}_\nu\big(\mathcal{J}(N_K) > M, N_K > \varphi\big)\\
&\geq \mathcal{P}_\nu\big(\mathcal{J}(\varphi) > M, N_K > \varphi\big)\\
&\geq \mathcal{P}_\nu\big(\mathcal{J}(\varphi) > M\big) - \mathcal{P}_\nu\big(N_K \leq \varphi\big)\\
&\geq \mathcal{P}_\nu\big(\mathcal{J}(\varphi) > M\big) - \mathcal{P}_\nu\big(N_{B_D} \leq \varphi\big),
\end{aligned}$$

where in the last step we used the fact that $B_D \subset K$ and applied Lemma 2.2. Recall that the starting value $R_0 = \eta^w_0(o) - \eta^o_0(o)$ is finite almost surely since ν has finite expectation. Since the lazy random walk on $\mathbb{Z}$ is recurrent, we deduce that for any $\epsilon \in (0, 1)$ and any $M \in \mathbb{N}$ and any ν with finite expectation, we can choose a value $\varphi = \varphi(\nu, \epsilon, M)$ large enough such that

$$\mathcal{P}_\nu(\mathcal{J}(\varphi) > M) \geq 1 - \frac{\epsilon}{2}.$$

Since the system is active by assumption, we deduce that there exists D large enough depending on ϵ and φ such that

$$\mathcal{P}_\nu(N_{B_D}(o) \leq \varphi) \leq \frac{\epsilon}{2},$$

where, by Lemma 2.1 (Abelian property), the previous estimate holds uniformly in the strategy F_K. Combining the previous estimates, we obtain that for any ϵ and M we can set $D = D(\nu, \epsilon, M)$ large enough such that, uniformly in $K \supset B_D$ and in the strategy F_K,

$$\mathcal{P}_\nu\big(H_{K,F_K}(o) > M\big) \geq 1 - \epsilon.$$

This concludes the proof. □

3.2 Ghost-pair stabilization

In this section we define a stabilization procedure where we introduce some auxiliary (virtual) particles, which we will call *ghosts*. These auxiliary particles do not interact with oils nor waters and perform independent simple random walks. Each step of the procedure corresponds either to an oil-water pair performing a simple random walk step from an unstable vertex, or a ghost performing a simple random walk step and, at any given step of the procedure, at most one ghost is created. We will refer to this stabilization procedure as *ghost-pair stabilization*. The procedure is defined in an augmented set of configurations, which we denote by

$$\widetilde{\Omega} := \mathbb{N}^{V(G)} \times \mathbb{N}^{V(G)} \times \mathbb{N}^{V(G)},$$

where $(\tilde{\eta}^o, \tilde{\eta}^w, \tilde{\eta}^g) \in \widetilde{\Omega}$ is a triplet such that $\tilde{\eta}^q(x)$ denotes the number oils, waters or ghosts which are located at $x \in V(G)$ when $q = o$, $q = w$, $q = g$ respectively. As before, Ω will continue to denote the set of configurations of (only) oil and water particles.

Definition 3.3 (Ghost-pair stabilization)**.** Let $K \subset V$ be a finite set, let $\sigma \in \Omega$ denote an unstable particle configuration (consisting only of oils and waters, but no ghosts). At time zero, we start from a configuration $\tilde{\eta}_0 \in \widetilde{\Omega}$ such that oils and waters are placed according to σ, that is $\sigma = (\tilde{\eta}_0^o, \tilde{\eta}_0^w) \in \Omega$ and, moreover, no ghost is present, i.e., $\tilde{\eta}_0^g(z) = 0$ for all $z \in V(G)$. We let $\boldsymbol{\delta}_x \in \mathbb{N}^{V(G)}$ be the vector which equals one at $x \in V(G)$ and zero everywhere else. Inductively, for every integer $t \geq 0$, we first follow (i) and then (ii) described below.

(i) Either a ghost or an oil-water pair in $\tilde{\eta}_t$ which are located on a vertex of K perform a simple random walk step, where the latter means that an oil and a water which are located at the same vertex take one independent step according to simple random walk. This leads to a new particle configuration which we call $\theta_t \in \tilde{\Omega}$.

(ii) If during (i) a water falls into a vertex $x \in K$ which is hosting a hole (i.e., $\tilde{\eta}_t^o(x) = \tilde{\eta}_t^w(x)$ and $\theta_t^w(x) = \theta_t^o(x) + 1$), then a ghost is added at that vertex, that is,

$$\tilde{\eta}_{t+1}^g := \theta_t^g + \boldsymbol{\delta}_x, \quad \text{and} \quad \tilde{\eta}_{t+1}^q := \theta_{t+1}^q, \quad q \in \{o, w\},$$

otherwise nothing happens, (i.e, $\tilde{\eta}_{t+1} := \theta_t$). This defines $\tilde{\eta}_{t+1}$.

Since K is finite, after an almost surely finite number of steps no pair and no ghost is present in K and the procedure stops. We define

$$T = T(K) := \inf\{s \geq 0 \ : \ K \text{ is stable with respect to } (\tilde{\eta}_s^o, \tilde{\eta}_s^w) \text{ and } \tilde{\eta}_s^g(y) = 0 \ \forall y \in K\},$$

and for every $t \geq T$ we set $\tilde{\eta}_t := \tilde{\eta}_T$. In the following we set, for any $y \in K$,

$$\tilde{m}(y) := \#\{\text{times that either a ghost or an oil-water pair jumps from } y\},$$

and we denote by $\tilde{P}_{K,\sigma}$ the law of the ghost-pair stabilization.

The lemma below is the main step in the proof of our main result. It shows that, during the stabilization procedure started from an arbitrary (unstable) configuration σ, the expected value of $\tilde{m}(y)$, for any fixed $y \in K$, can be estimated in terms of the Green's function of simple random walk and of the number of pairs in the initial configuration.

Lemma 3.4. *For any finite set $K \subset V$, any vertex $y \in K$, and any unstable particle configuration $\sigma := (\tilde{\eta}_0^o, \tilde{\eta}_0^w) \in \Omega$,*

$$\tilde{E}_{K,\sigma}\big(\tilde{m}(y)\big) = \sum_{x \in K} \big(\tilde{\eta}_0^o(x) \wedge \tilde{\eta}_0^w(x)\,\big)\, G_K(x,y), \tag{3.5}$$

where $\tilde{E}_{K,\sigma}$ denotes the expectation with respect to $\tilde{P}_{K,\sigma}$.

Proof. Let $K \subset V$ be a finite set, fix one vertex $y \in K$. Let $g : V(G) \mapsto \mathbb{R}$ be the function which is harmonic in $K \setminus \{y\}$ and such that $g_y = 1$, $g_z = 0$ for any $z \in K^c$. Recall that $\tilde{\eta}_t$ denotes the state of the process (cf. Definition 3.3) at time t. For convention, we refer to as *step* t the transition from $\tilde{\eta}_{t-1}$ to $\tilde{\eta}_t$, and let x_t denote the vertex from which a pair or a ghost jumps at step t. For each $t \in \mathbb{N}_{\geq 0}$ define

$$M_t := \sum_{x \in K} \Big(\tilde{\eta}_t^o(x) \wedge \tilde{\eta}_t^w(x) + \tilde{\eta}_t^g(x)\Big)\, g_x - (\triangle g)_y \sum_{i=1}^{t} \mathbb{1}\{x_i = y\}. \tag{3.6}$$

Let $(\tilde{\Sigma}, \tilde{\mathcal{F}}, \tilde{P}_{K,\sigma})$ be the probability space where the process $\{\tilde{\eta}_t\}_t$ is defined; the proof of the proposition will follow from the fact that M_t is a martingale, namely

$$\tilde{E}_{K,\sigma}[\,M_t \,\big|\, \mathcal{F}_{t-1}\,] = M_{t-1}. \tag{3.7}$$

We will now prove (3.7) considering different cases.

In the first case, consider that at step t a ghost jumps from $x_t = b \in K$. Then, in this case,

$$\tilde{E}_{K,\sigma}\,[\,M_t \,\big|\, \mathcal{F}_{t-1}\,] = M_{t-1} \; - g_b \; + \frac{1}{\mathbf{d}_b}\Big(\sum_{z \sim b} g_z\Big) - \mathbb{1}\{b = y\}(\triangle g)_y = M_{t-1}, \tag{3.8}$$

where the last identity holds since g is harmonic in $K \setminus \{y\}$.

In the second case, consider that at step t an oil and water pair jumps from some vertex $x_t = b \in K$. Let $\mathcal{N}_{b,t}^{oe}$ (resp. $\mathcal{N}_{b,t}^{w}$) be the set of vertices $z \in V(G)$ such that $z \sim b$ and $\tilde{\eta}_{t-1}^o(z) - \tilde{\eta}_{t-1}^w(z) \geq 0$ (resp. $\tilde{\eta}_{t-1}^o(z) - \tilde{\eta}_{t-1}^w(z) < 0$). Note that $\mathcal{N}_{b,t}^{oe}$ and $\mathcal{N}_{b,t}^{w}$ are measurable with respect to $\mathcal{F}_{t-1}$. Then, denoting by z_o (resp. z_w) the destination of the oil (resp. water) in the next sum,

$$\begin{aligned}
\tilde{E}_{K,\sigma}[\, M_t \;\big|\; \mathcal{F}_{t-1}\,] &= M_{t-1} \; - g_b \; + \frac{1}{\mathbf{d}_b^2}\Big(\sum_{z_w \in \mathcal{N}^{oe}_{b,t}, z_o \in \mathcal{N}^{oe}_{b,t}} g_{z_w} \Big) + \\
&\quad + \frac{1}{\mathbf{d}_b^2}\Big(\sum_{z_w \in \mathcal{N}^{oe}_{b,t}, z_o \in \mathcal{N}^{w}_{b,t}} (g_{z_o} + g_{z_w}) \Big) + \frac{1}{\mathbf{d}_b^2}\Big(\sum_{z_w \in \mathcal{N}^{w}_{b,t}, z_o \in \mathcal{N}^{w}_{b,t}} g_{z_o} \Big) - \mathbb{1}\{b = y\}(\triangle g)_y \\
&= M_{t-1} \; - \; g_b \; + \frac{|\mathcal{N}^w_{b,t}| + |\mathcal{N}^{oe}_{b,t}|}{\mathbf{d}_b^2} \Big(\sum_{z \sim b} g_z \Big) \; - \; \mathbb{1}\{b = y\}(\triangle g)_y \\
&= M_{t-1} \; - g_b \; + \frac{1}{\mathbf{d}_b}\Big(\sum_{z \sim b} g_z \Big) - \mathbb{1}\{b = y\}(\triangle g)_y \\
&= M_{t-1},
\end{aligned} \tag{3.9}$$

where the last identity follows from the fact that g is harmonic in $K \setminus \{y\}$. This concludes the proof of (3.7).

Now we prove the lemma using (3.7). Recall that T is the first time at which the set K is stable and no ghost is present in K. Since K is finite, $\mathbb{E}T < \infty$ almost surely, furthermore M_t has bounded increments, thus, the conditions of the optional stopping theorem are fulfilled and we deduce that

$$\tilde{E}_{K,\sigma}\,[\, M_T\,] = \tilde{E}_{K,\sigma}\,[\, M_0\,] = \sum_{x \in K} \big(\tilde{\eta}^o_0(x) \wedge \tilde{\eta}^w_0(x)\big)\, g_x,$$

recalling that $\tilde{\eta}^o_0 \wedge \tilde{\eta}^w_0$ corresponds to the number of pairs at x in the initial configuration σ and that we start with no ghost at time zero. This leads to,

$$-\,(\triangle g)_y\, \tilde{E}_{K,\sigma}\big(\, \tilde{m}(y)\,\big) = -(\triangle g)_y\, \tilde{E}_{K,\sigma}\Big(\sum_{t=1}^{\infty} \mathbb{1}\{x_t = y\} \Big) = \sum_{x \in K} \big(\tilde{\eta}^o_0(x) \wedge \tilde{\eta}^w_0(x)\big)\, g_x. \tag{3.10}$$

Using Proposition 2.4, we obtain

$$\begin{aligned}
\tilde{E}_{K,\sigma}\big(\, \tilde{m}(y)\,\big) &= \sum_{x \in K} \big(\tilde{\eta}^o_0(x) \wedge \tilde{\eta}^w_0(x)\big)\, g_x G_K(y,y) \\
&= \sum_{x \in K} \big(\tilde{\eta}^o_0(x) \wedge \tilde{\eta}^w_0(x)\big)\, P_x(\tau_y < \tau_{K^c}) G_K(y,y) \\
&= \sum_{x \in K} \big(\tilde{\eta}^o_0(x) \wedge \tilde{\eta}^w_0(x)\big)\, G_K(x,y).
\end{aligned}$$

This finishes the proof. □

Remark 3.5. In the overview in Section 1.2, we noticed that the oil-water pairs move as a mix of simple random walk, critical branching random walk, and subcritical branching random walk, depending on the environment. In particular, the total number of pairs which are present in the oil and water system (with no introduction of ghosts) is a super-martingale. In fact, if we fire a vertex that does not neighbor a hole, then the number of oil-water pairs behaves as a martingale; otherwise, the expected number of pairs strictly decreases. It is extremely hard to control the evolution of the system consisting exclusively of oil-water pairs, because this requires controlling the evolution of the configuration of holes and of pairs at the same time, which are strongly correlated. The introduction of ghosts compensates the pairs that are lost when we fire a vertex neighboring a hole. In particular, if we were to define M_t as simply $\sum_{x\in K}(\tilde{\eta}_t^o(x) \wedge \tilde{\eta}_t^w(x) + \tilde{\eta}_t^g(x))$, we would be able to show that M_t is a super-martingale (where it would not be a martingale only due to particles or ghosts jumping out of K). The introduction in M_t of the function g, which is harmonic everywhere in K but at y, is to make each firing at y give an extra contribution. This allowed us to add the negative term at the end of (3.6), which counts the number of times that a pair or a ghost jumps from y; that is, it allows us to estimate $\tilde{m}(y)$. Both ghosts and pairs contribute to the total number of jumps $\tilde{m}(y)$, and to show fixation we actually need to control only the contribution given by oil-water pairs. In Section 4, we will isolate the two contributions and compare them.

4 Proof of Theorem 1.1

In this section we present the proof of our main theorem, which works by contradiction and uses the ghost-pair stabilization (recall Definition 3.3). As explained in Section 3.2, the expected number of pairs which are present in the system when a firing occurs at a nearest neighbor of a hole is strictly decreasing. Ghosts are introduced to compensate the loss of pairs, in such a way that the total number of pairs and ghosts which are present at any step of the ghost-pair stabilization is a martingale. The proof of the theorem is based on the following idea. Suppose the system is active. Then, Lemma 3.2 implies that a *large* number of ghosts is produced at most vertices; but ghosts are produced to compensate the decrease in the number of pairs. Thus if many ghosts are produced, that means that a *large* number of pairs was lost. The proof consists in showing that it is not possible to produce so many ghosts if we start with a finite density of pairs, leading to the desired contradiction. To show this fact we will exploit the Green's function of a suitably defined random walk to relate the expectation of three different quantities, namely the number of particles which start from every vertex, the number of ghosts which are produced at every vertex and the number of times a ghost or a pair visit the origin.

To begin, we state an auxiliary result. From now on, fix an arbitrary sequence of finite sets, namely the sequence of balls centered at the origin and of radius $L \geq 1$, which we denote by $\{B_L\}_{L\in\mathbb{N}}$.

Lemma 4.1. *For any $D \in \mathbb{N}$ there exists $L_0 = L_0(D)$ large enough such that, for any $L > L_0$,*

$$\sum_{x\in B_L} G_{B_L}(x,o) \leq 2 \sum_{\substack{x\in B_L:\\ B(x,D)\subset B_L}} G_{B_L}(x,o),$$

where $B_L^c := V(G) \setminus B_L$, and $B(x,D)$ is the ball of radius D centered at x.

We will now prove Theorem 1.1 using Lemma 4.1. The proof of Lemma 4.1 will be presented afterwards.

Proof of Theorem 1.1. To begin, for any L fixed and arbitrarily large, consider the following procedure. Stabilize the set B_L following the ghost-pair stabilization: while stabilizing the set B_L, every time a water falls into a hole, a ghost is created at that vertex. Ghosts perform independent simple random walks until they leave B_L. For any $x \in B_L$ we define,

$$\begin{aligned}
\widetilde{m}_L(x) &:= \text{number of } \textit{pairs} \text{ or } \textit{ghosts} \text{ that jump from } x \text{ during the stabilization of } B_L,\\
m_L(x) &:= \text{number of firings at } x \text{ during the stabilization of } B_L,\\
\mathrm{w}_L(x) &:= \text{number of } \textit{ghosts} \text{ that jump from } x \text{ during the stabilization of } B_L,\\
\mathrm{H}_L(x) &:= \text{number of ghosts } \textit{started} \text{ (created) at } x \text{ during the stabilization of } B_L.
\end{aligned}$$

Recall that $\mu = \mu(\nu) \in (0, \infty)$ is the expected number of particles which are present at each vertex in the starting configuration. We claim that, for any $L \in \mathbb{N}$,

$$\tilde{\mathbb{E}}_\nu\big(\widetilde{m}_L(x)\big) \leq \sum_{y \in B_L} \mu\, G_{B_L}(y, x); \tag{4.1}$$

$$\tilde{\mathbb{E}}_\nu\left(\mathrm{w}_L(x)\right) = \sum_{y \in B_L} \tilde{\mathbb{E}}_\nu\big(\mathrm{H}_L(y)\big) G_{B_L}(y, x), \tag{4.2}$$

where $\tilde{\mathbb{E}}_\nu$ denotes the expectation of the measure which is defined in the enlarged probability space of oils, waters and ghosts. Equation (4.1) follows from Lemma 3.4 by averaging over the initial particle configuration and observing that the expected number of pairs of the initial configuration at every vertex cannot be larger than the expected number of particles. Equation (4.2) follows from linearity of expectation and from the fact that every ghost performs an independent simple random walk until it leaves B_L. We also claim that, if we assume that the system starting with initial particle distribution ν is almost surely active, then there is a large enough $D = D(\nu)$ such that for any $L > D$, and for any $x \in B_L$ such that $B(x, D) \subset B_L$,

$$\tilde{\mathbb{E}}_\nu\big[\mathrm{H}_L(x)\big] \geq 3\mu. \tag{4.3}$$

Indeed, equation (4.3) follows from Lemma 3.2 and from the fact that G is vertex-transitive, since, by definition, a ghost is produced at x every time a water falls into a hole and the estimate in Lemma 3.2 holds uniformly over all strategies.

For the rest of the proof, we will keep assuming that the system is almost surely active and we will look for a contradiction. We will also keep the value D fixed as above. By definition, $m_L(x)$ is the number of times that a pair jumps from x, and this number equals the number of times that a ghost or a pair jump from x minus the number of times a ghost jumps from x, that is,

$$m_L(x) = \widetilde{m}_L(x) - \mathrm{w}_L(x).$$

It follows from the linearity of expectation and from (4.1), (4.2), and (4.3), that

$$\begin{aligned}
\tilde{\mathbb{E}}_\nu\big(m_L(x)\big) &\leq \sum_{y \in B_L} \mu G_{B_L}(y, x) - \sum_{y \in B_L} \tilde{\mathbb{E}}_\nu\big(\mathrm{H}_L(y)\big) G_{B_L}(y, x) && (4.4)\\
&\leq \sum_{y \in B_L} \mu G_{B_L}(y, x) - \sum_{\substack{y \in B_L:\\ B(y,D) \subset B_L}} \tilde{\mathbb{E}}_\nu\big(\mathrm{H}_L(y)\big) G_{B_L}(y, x)\\
&\overset{(4.3)}{\leq} \mu \Big(\sum_{y \in B_L} G_{B_L}(y, x) - 3 \sum_{\substack{y \in B_L:\\ B(y,D) \subset B_L}} G_{B_L}(y, x) \Big). && (4.5)
\end{aligned}$$

From Lemma 4.1, we conclude that $\tilde{\mathbb{E}}_\nu\big(m_L(o)\big) < 0$ for large enough L. Since the number of firings cannot be negative, the above leads to the desired contradiction. We conclude that the probability that the system is active is strictly smaller than 1. By Lemma 2.3 (the 0-1 law), we deduce that the system fixates almost surely, concluding the proof. $\square$

It remains to prove Lemma 4.1.

Proof of Lemma 4.1. Pick L_0 very large such that

$$L_0 \geq D\left(1 + \mathbf{d}^D\right). \tag{4.6}$$

For $L \geq L_0$, define the annulus

$$A_{L,D} := \bigcup_{x \in \partial B_L} B(x, D),$$

and the set $A := A_{L,D} \cap B_L$. Moreover, for any $x \in V(G)$, define $Q_x = \sum_{z \in A} G_{B_L}(x, z)$ and $Q := \max_{x \in A} Q_x$. Now note that,

$$\forall x \in A \quad Q_x \leq D + (1 - \mathbf{d}^{-D})Q,$$

which follows from the fact that, for every $x \in A$, the simple random walk starting at x has probability at least $\mathbf{d}^{-D}$ to exit B_L within D steps. From this we deduce that $Q \leq D + (1 - \mathbf{d}^{-D})Q$, which immediately implies that

$$Q \;\leq\; D\,\mathbf{d}^D.$$

Next, recall from Section 2.3 that $X(t)$ denotes the simple random walk on G and τ_Z its hitting time for a set Z, then define

$$S := \sum_{z \in B_L \setminus A} G_{B_L}(o, z)$$

and observe that,

$$Q_o = \sum_{y \in A} P_o\big(\{X(\tau_A) = y\} \cap \{\tau_A < \tau_{B_L^c}\}\big)\, Q_y \;\leq\; Q,$$

where we used the Markov property. Thus, by our assumption on L and by the fact that $B_{L-D} \subset B_L \setminus A$, we deduce that,

$$Q_o \leq Q \leq D\,\mathbf{d}^D \;\leq L_0 - D \leq L - D \leq S.$$

Finally, using the symmetry of the Green's function and the previous inequality we obtain that,

$$\sum_{z \in B_L} G_{B_L}(z, o) = \sum_{z \in B_L} G_{B_L}(o, z) \leq Q_0 + S \leq 2S = 2 \sum_{\substack{z \in B_L: \\ B(z,D) \subset B_L}} G_{B_L}(z, o),$$

concluding the proof. $\square$

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