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Elementary doctrines as coalgebras^{*}

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Abstract

Lawvere’s hyperdoctrines mark the beginning of applications of category theory to logic. In particular, existential elementary doctrines proved essential to give models of non-classical logics. The clear connection between (typed) logical theories and certain **Pos**-valued functors is exemplified by the embedding of the category of elementary doctrines into that of primary doctrines, which has a right adjoint given by a completion which freely adds quotients for equivalence relations.

We extend that result to show that, in fact, the embedding is 2-functorial and 2-comonadic. As a byproduct we apply the result to produce an algebraic way to extend a first order theory to one which eliminates imaginaries, discuss how it relates to Shelah’s original, and show how it works in a wider variety of situations.

Keywords: elementary doctrine, 2-comonad, quotient completion, elimination of imaginaries

2020 MSC: 03G30, 18C50, 18C20, 03B10, 03B20, 03C45

1. Introduction

Lawvere’s hyperdoctrines mark the beginning of applications of category theory in logic, and they provide a very clear algebraic tool to work with syntactic theories and their extensions in logic, see [12, 10]. Lawvere’s basic intuition in the categorical approach to logic was to rely on the notion of fibration.

As a very basic instance of this, recall that a *primary doctrine* is a functor $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ such that (i) the base category \mathcal{C} has finite products, (ii) for every object A of \mathcal{C} , the poset $P(A)$ has finite meets and, (iii) for every arrow $f: A \rightarrow B$, the monotone function $f^* = P(f): P(B) \rightarrow P(A)$ preserves finite meets. Indeed, these data amount to the same as a faithful fibration with

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In Section 2 we recall the main definitions and fix notation. We also provide a simple characterization of elementary doctrines which is useful for the proof of the main theorem. Section 3 contains the statement of the main theorem about the comonadicity of \mathcal{R} and its proof. In Section 4 we present an application of the main theorem to the elimination of imaginaries. This is compared to Shelah's T^{eq} from [24] in Section 5.

2. Preliminaries

We recall some notions and constructions from [19] following very closely the notations introduced there: a **primary doctrine** is a functor $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ from a category \mathcal{C} with finite products into the category of posets such that, for every object A in the base category \mathcal{C} , the fibre $P(A)$ is an inf-semilattice, and for every arrow $f: A \rightarrow A'$, the reindexing $f^*: P(A') \rightarrow P(A)$ preserves finite meets. This amounts to the same data as a functor $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{ISL}$ like in [19].

When necessary to avoid confusion, we may decorate with an index A the order \leq , the meet \wedge and the top element \top of the fibre $P(A)$.

There is a large variety of examples for which we refer the reader to [7, 15, 20]. Throughout the section we shall exemplify definitions considering the following one.

Example 2.1. Given an inf-semilattice \mathbf{H} consider the functor $\mathbf{H}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$ that sends a set A to the poset \mathbf{H}^A of functions from A to \mathbf{H} with the pointwise order, and a function $f: A \rightarrow B$ to $\mathbf{H}^f := - \circ f$. The contravariant functor $\mathbf{H}^{(-)}$ is a primary doctrine.

For \mathbf{H} the two-element boolean algebra \mathbf{B} , the doctrine $\mathbf{B}^{(-)}$ is isomorphic to the contravariant powerset functor $\mathcal{P}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$.

Primary doctrines are the objects of the 2-category \mathbf{PD} where

the 1-cells are pairs $\mathbf{f} = (\bar{\mathbf{f}}, \hat{\mathbf{f}})$ where $\bar{\mathbf{f}}: \mathcal{C} \rightarrow \mathcal{D}$ is a product-preserving functor and $\hat{\mathbf{f}}: P \rightarrow R \circ (\bar{\mathbf{f}})^{\text{op}}$ is a natural transformation as in the diagram

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & \xrightarrow{P} & \mathbf{Pos} \\
 \bar{\mathbf{f}}^{\text{op}} \downarrow & \hat{\mathbf{f}} \downarrow & \nearrow R \\
 \mathcal{D}^{\text{op}} & \xrightarrow{R} & \mathbf{Pos}
 \end{array}$$

and, for every object A in the base \mathcal{C} , the monotone function $\hat{\mathbf{f}}_A: P(A) \rightarrow R(\bar{\mathbf{f}}(A))$ preserves finite meets.

the 2-cells $\theta: \mathbf{f} \Rightarrow \mathbf{g}$ are natural transformations $\theta: \bar{\mathbf{f}} \rightarrow \bar{\mathbf{g}}$ such that

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & & \\
 \bar{\mathbf{f}}^{\text{op}} \swarrow & \xrightarrow{P} & \searrow \\
 \mathcal{D}^{\text{op}} & & \mathbf{Pos} \\
 \bar{\mathbf{g}}^{\text{op}} \searrow & \xrightarrow{R} & \swarrow \\
 & &
 \end{array}
 \quad
 \begin{array}{c}
 \hat{\mathbf{f}} \cdot (\leq) \cdot \hat{\mathbf{g}} \\
 \downarrow \\
 R
 \end{array}$$

so that, for every A in \mathcal{C} and every α in $P(A)$, one has $\hat{\mathbf{f}}_A(\alpha) \leq R_{\theta_A}(\hat{\mathbf{g}}_A(\alpha))$.

A primary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is *elementary* when, for every object A in \mathcal{C} , there is an object δ_A^P in $P(A \times A)$ such that, for every object X in \mathcal{C} , the functor

$$\begin{array}{ccc}
 P(X \times A) & \xrightarrow{\mathcal{E}_{X,A}^P} & P(X \times A \times A) \\
 \alpha \longmapsto & \longrightarrow & \langle \text{pr}_1, \text{pr}_2 \rangle^*(\alpha) \wedge \langle \text{pr}_2, \text{pr}_3 \rangle^*(\delta_A^P)
 \end{array}$$

is left adjoint to the reindexing $\langle \text{pr}_1, \text{pr}_2, \text{pr}_2 \rangle^*: P(X \times A \times A) \rightarrow P(X \times A)$ along the arrow $\langle \text{pr}_1, \text{pr}_2, \text{pr}_2 \rangle: X \times A \rightarrow X \times A \times A$.

We shall drop the superscript from δ_A^P when the doctrine P is clear from the context.

Example 2.2. Given an inf-semilattice \mathbf{H} , the primary doctrine $\mathbf{H}^{(-)}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$ is elementary if and only if \mathbf{H} has a least element. Indeed, if \mathbf{H} has a least element \perp , the object $\delta_A \in \mathbf{H}^{A \times A}$ can be taken as the function

$$(x, x') \longmapsto \begin{cases} \top, & \text{if } x = x' \\ \perp, & \text{otherwise.} \end{cases}$$

Conversely suppose $\mathbf{H}^{(-)}$ is elementary and consider the set $A := \{0, 1\}$. For every $h \in \mathbf{H}$ let $\alpha_h \in \mathbf{H}^A$ be the function that maps 0 to \top and 1 to h . Then

$$\delta_A(0, 1) = \top \wedge \delta_A(0, 1) = \alpha_h \circ \text{pr}_1(0, 1) \wedge \delta_A(0, 1) \leq \alpha_h \circ \text{pr}_2(0, 1) = h$$

showing that $\delta_A(0, 1)$ is a bottom element. In particular, the powerset functor \mathcal{P} is elementary.

The 2-category **ED** is the 2-full subcategory of **PD** on the elementary doctrines where a 1-cell $\mathbf{f}: P \rightarrow R$ of **PD** is in **ED** if the b_A 's commute with the left adjoints, in the sense that there are commutative diagrams

$$\begin{array}{ccccc}
 P(X \times A) & \xrightarrow{\hat{\mathbf{f}}_{X \times A}} & R(\bar{\mathbf{f}}(X \times A)) & \xrightarrow{\sim} & R(\bar{\mathbf{f}}X \times \bar{\mathbf{f}}A) \\
 \downarrow \mathcal{E}_{X,A}^P & & \downarrow R(\langle \bar{\mathbf{f}}\text{pr}_1, \bar{\mathbf{f}}\text{pr}_2 \rangle^{-1}) & & \downarrow \mathcal{E}_{\bar{\mathbf{f}}X, \bar{\mathbf{f}}A}^R \\
 P(X \times A \times A) & \xrightarrow{\hat{\mathbf{f}}_{X \times A \times A}} & R(\bar{\mathbf{f}}(X \times A \times A)) & \xrightarrow{\sim} & R(\bar{\mathbf{f}}X \times \bar{\mathbf{f}}A \times \bar{\mathbf{f}}A) \\
 & & \downarrow R(\langle \bar{\mathbf{f}}\text{pr}_1, \bar{\mathbf{f}}\text{pr}_2, \bar{\mathbf{f}}\text{pr}_3 \rangle^{-1}) & &
 \end{array}$$

We shall say that a primary doctrine P is *first-order* if each fibre is a Heyting algebra, reindexing preserves the structure and reindexing along a product projection has left and right adjoints satisfying the Beck-Chevalley condition. First-order doctrines are the objects of the category \mathbf{FOD} , which is the 2-full subcategory of \mathbf{PD} on those 1-cells \mathbf{f} such that every component of $\widehat{\mathbf{f}}$ is a homomorphism of Heyting algebras commuting with the left and right adjoints. Elementary first-order doctrines are what Pitts in [21] calls first order hyperdoctrines. We shall use the two names interchangeably and denote the category of (first order) hyperdoctrines as \mathbf{HD} , which is the pullback of \mathbf{FOD} and \mathbf{ED} in \mathbf{PD} .

Remark 2.3. In an elementary first-order doctrine $P: \mathcal{C}^{op} \rightarrow \mathbf{Pos}$, for every arrow $f: A \rightarrow B$ in the base category, the reindexing functor $f^*: P(B) \rightarrow P(A)$ has a left adjoint $\mathcal{E}_f^P: P(A) \rightarrow P(B)$ which is obtained from those for projections and for parametrised diagonals, see [21, Remark 4.6].

Example 2.4. For a given inf-semilattice H , the doctrine $H^{(-)}$ from Example 2.1 is elementary first-order if and only if H is complete.

It is well-known, see [7, 25], that the 2-category of primary doctrines is equivalent to the 2-category of faithful fibrations with fibred products, and that the 2-category of elementary doctrines is equivalent to the 2-category of faithful fibrations with equality. Yet we have no reference for the following characterization of elementary doctrines which will be useful for our future purposes.

Proposition 2.5. *Let $P: \mathcal{C}^{op} \rightarrow \mathbf{Pos}$ be a primary doctrine. The following are equivalent:*

- (i) P is elementary.
- (ii) for each object C in \mathcal{C} , there is an object \mathfrak{d}_C in $P(C \times C)$ such that
 - (a) $\text{pr}_1^*(\beta) \wedge \mathfrak{d}_C \leq \text{pr}_2^*(\beta)$ for every C and every β in $P(C)$;
 - (b) $\top_C \leq \langle \text{id}_C, \text{id}_C \rangle^*(\mathfrak{d}_C)$ for every C ;
 - (c) $\langle \text{pr}_1, \text{pr}_3 \rangle^*(\mathfrak{d}_C) \wedge \langle \text{pr}_2, \text{pr}_4 \rangle^*(\mathfrak{d}_D) \leq \mathfrak{d}_{C \times D}$ for every C and D .

Proof. (i) \Rightarrow (ii) is well-known choosing δ_C for \mathfrak{d}_C .

(ii) \Rightarrow (i) We want to show that choosing the object δ_A^P to be \mathfrak{d}_A in $P(A \times A)$ makes the primary doctrine P elementary, in other words we must check that, for every objects X in \mathcal{C} , the functor

$$\begin{array}{ccc} P(X \times A) & \xrightarrow{\quad\quad\quad} & P(X \times A \times A) \\ \alpha \mapsto & \xrightarrow{\quad\quad\quad} & \langle \text{pr}_1, \text{pr}_2 \rangle^*(\alpha) \wedge \langle \text{pr}_2, \text{pr}_3 \rangle^*(\mathfrak{d}_A) \end{array}$$

is left adjoint to reindexing along the arrow $\langle \text{pr}_1, \text{pr}_2, \text{pr}_2 \rangle: X \times A \rightarrow X \times A \times A$ of \mathcal{C} . Note first that, for γ in $P(C \times C)$,

$$\langle \text{pr}_1, \text{pr}_1 \rangle^*(\gamma) \wedge \mathfrak{d}_C \leq \gamma \tag{1}$$

since, using first (c), then (a), in $P(C \times C \times C \times C)$ one has that

$$\begin{aligned} \langle \text{pr}_1, \text{pr}_2 \rangle^*(\gamma) \wedge \langle \text{pr}_1, \text{pr}_3 \rangle^*(\mathfrak{d}_C) \wedge \langle \text{pr}_2, \text{pr}_4 \rangle^*(\mathfrak{d}_C) &\leq \langle \text{pr}_1, \text{pr}_2 \rangle^*(\gamma) \wedge \mathfrak{d}_{C \times C} \\ &\leq \langle \text{pr}_3, \text{pr}_4 \rangle^*(\gamma). \end{aligned}$$

By reindexing that inequality along $\langle \text{pr}_1, \text{pr}_1, \text{pr}_1, \text{pr}_2 \rangle: C \times C \rightarrow C \times C \times C \times C$, condition (b) yields (1). For the left adjoint to reindexing along the diagonal $\langle \text{id}_C, \text{id}_C \rangle: C \rightarrow C \times C$, consider the assignment $\alpha \mapsto \text{pr}_1^*(\alpha) \wedge \mathfrak{d}_C$ where $\text{pr}_1: C \times C \rightarrow C$. If $\alpha \leq \langle \text{id}_C, \text{id}_C \rangle^*(\gamma)$, then by (1)

$$\text{pr}_1^*(\alpha) \wedge \mathfrak{d}_C \leq \text{pr}_1^*(\langle \text{id}_C, \text{id}_C \rangle^*(\gamma)) \wedge \mathfrak{d}_C \leq \langle \text{pr}_1, \text{pr}_1 \rangle^*(\gamma) \wedge \mathfrak{d}_C \leq \gamma.$$

If $\text{pr}_1^*(\alpha) \wedge \mathfrak{d}_C \leq \gamma$, then (b) yields

$$\alpha \leq \alpha \wedge \langle \text{id}_C, \text{id}_C \rangle^*(\mathfrak{d}_C) = \langle \text{id}_C, \text{id}_C \rangle^*(\text{pr}_1^*(\alpha) \wedge \mathfrak{d}_C) \leq \langle \text{id}_C, \text{id}_C \rangle^*(\gamma).$$

Taking C as $X \times A$, it follows that $\mathfrak{d}_{X \times A} \leq \langle \text{pr}_1, \text{pr}_3 \rangle^*(\mathfrak{d}_X)$ since by (b)

$$\top \leq \langle \text{id}_X, \text{id}_X \rangle^*(\mathfrak{d}_X) = \langle \text{id}_{X \times A}, \text{id}_{X \times A} \rangle^*(\langle \text{pr}_1, \text{pr}_3 \rangle^*(\mathfrak{d}_X)).$$

Similarly, $\mathfrak{d}_{X \times A} \leq \langle \text{pr}_2, \text{pr}_4 \rangle^*(\mathfrak{d}_A)$. So the left adjoint to reindexing along the diagonal $\langle \text{pr}_1, \text{pr}_2, \text{pr}_1, \text{pr}_2 \rangle: X \times A \rightarrow X \times A \times X \times A$ sends an object γ in $P(X \times A \times X \times A)$ to $\langle \text{pr}_1, \text{pr}_2 \rangle^*(\gamma) \wedge \langle \text{pr}_1, \text{pr}_3 \rangle^*(\mathfrak{d}_X) \wedge \langle \text{pr}_2, \text{pr}_4 \rangle^*(\mathfrak{d}_A)$. The conclusion follows easily. \square

Remark 2.6. It follows from [14, Remark 2.2] that, once a product

$$A \xleftarrow{\text{pr}_1} A \times A \xrightarrow{\text{pr}_2} A$$

is chosen, there is a unique object δ_A in $P(A \times A)$ which satisfies conditions (a)-(c) in Proposition 2.5.

Corollary 2.7. *Let P and R be elementary doctrines. A 1-cell $\mathbf{f}: P \rightarrow R$ of \mathbf{PD} is in \mathbf{ED} if and only if*

$$\widehat{\mathbf{f}}_{A \times A}(\delta_A^P) = \langle \bar{\mathbf{f}}_{\text{pr}_1}, \bar{\mathbf{f}}_{\text{pr}_2} \rangle^*(\delta_{\bar{\mathbf{f}}A}^R)$$

for every object A .

In [19] the second author proves that the inclusion $\mathbf{ED} \hookrightarrow \mathbf{PD}$, whose functor we shall denote by \mathcal{L} when needed, has a right adjoint; for the sake of completeness, in the following we give a brief summary of that construction. For $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ a primary doctrine, let \mathcal{E}_P be the category determined by the following data.

An object in \mathcal{E}_P is a pair (A, ρ) where A is an object in \mathcal{C} and ρ is in $P(A \times A)$ such that

- (a) $\top_A \leq \langle \text{id}_A, \text{id}_A \rangle^*(\rho)$;
- (b) $\rho \leq \langle \text{pr}_2, \text{pr}_1 \rangle^*(\rho)$;

$$(c) \langle \text{pr}_1, \text{pr}_2 \rangle^*(\rho) \wedge \langle \text{pr}_2, \text{pr}_3 \rangle^*(\rho) \leq \langle \text{pr}_1, \text{pr}_3 \rangle^*(\rho).$$

An arrow in \mathcal{E}_P $f: (A, \rho) \rightarrow (B, \sigma)$ is an arrow $f: A \rightarrow B$ in \mathcal{C} such that $\rho \leq (f \times f)^*(\sigma)$.

It is customary to refer to conditions (a), (b), and (c) above as reflexivity, symmetry and transitivity, respectively, and to say that the ρ -component of an object of \mathcal{E}_P is a ***P-equivalence relation over A***.

Remark 2.8. We haste to note that the usual construction of a “category of partial equivalence relations” can be obtained as an appropriate quotient of a category of the form \mathcal{E}_P which forces the equality of arrows in \mathcal{C} to coincide with the equality of the primary doctrine P , see [16].

A terminal object in \mathcal{E}_P can be computed as the pair $(T, \top_{T \times T})$ on a(ny) terminal object T in \mathcal{C} ; a product of (A, ρ) and (B, σ) in \mathcal{E}_P can be taken as

$$(A, \rho) \xleftarrow{\text{pr}_1} (A \times B, \rho \boxtimes \sigma) \xrightarrow{\text{pr}_2} (B, \sigma)$$

where $\rho \boxtimes \sigma := \langle \text{pr}_1, \text{pr}_3 \rangle^*(\rho) \wedge \langle \text{pr}_2, \text{pr}_4 \rangle^*(\sigma)$.

The category \mathcal{E}_P is the base of a primary doctrine $P^{\mathcal{R}}: \mathcal{E}_P^{\text{op}} \rightarrow \mathbf{Pos}$ determined as follows: the poset $P^{\mathcal{R}}(A, \rho)$ is the sub-poset of $P(A)$ on the ***descent data*** for ρ , *i.e.*

$$P^{\mathcal{R}}(A, \rho) = \{\alpha \in P(A) \mid \text{pr}_1^*(\alpha) \wedge \rho \leq \text{pr}_2^*(\alpha)\} \subseteq P(A).$$

It is easy to check that $P^{\mathcal{R}}(A, \rho)$ is a sub-inf-semilattice and that, for $f: (A, \rho) \rightarrow (B, \sigma)$ in \mathcal{E}_P , the function f^* maps $P^{\mathcal{R}}(A, \rho)$ into $P^{\mathcal{R}}(B, \sigma)$. So $P^{\mathcal{R}}: \mathcal{E}_P^{\text{op}} \rightarrow \mathbf{Pos}$ is indeed a primary doctrine, and it is elementary with $\delta_{(A, \rho)}^{P^{\mathcal{R}}} = \rho$ by Proposition 2.5.

For the same reason the construction extends to a 2-functor¹ $\mathcal{R}: \mathbf{PD} \rightarrow \mathbf{ED}$ since, for a 1-cell $\mathbf{f}: P \rightarrow R$ of elementary doctrines, each functor

$$P(A \times A) \xrightarrow{\widehat{\mathbf{f}}_{A \times A}} R(\overline{\mathbf{f}}(A \times A)) \xrightarrow{R((\overline{\mathbf{f}}\text{pr}_1, \overline{\mathbf{f}}\text{pr}_2)^{-1})} R(\overline{\mathbf{f}}A \times \overline{\mathbf{f}}A)$$

turns P -equivalence relations into R -equivalence relations and preserves descent data; the action of \mathcal{R} on a 1-cell $\mathbf{f} = (\overline{\mathbf{f}}, \widehat{\mathbf{f}})$ is $\mathbf{f}^{\mathcal{R}} = (\overline{\mathbf{f}}^{\mathcal{R}}, \widehat{\mathbf{f}}^{\mathcal{R}})$ where $\overline{\mathbf{f}}^{\mathcal{R}}$ is

$$\begin{array}{ccc} \mathcal{E}_P & \xrightarrow{\overline{\mathbf{f}}^{\mathcal{R}}} & \mathcal{E}_R \\ (A, \rho) \vdash & \longrightarrow & (\overline{\mathbf{f}}A, R((\overline{\mathbf{f}}\text{pr}_1, \overline{\mathbf{f}}\text{pr}_2)^{-1})(\widehat{\mathbf{f}}_{A \times A}(\rho))) \\ \downarrow g & \longrightarrow & \downarrow \overline{\mathbf{f}}g \\ (A', \rho') \vdash & \longrightarrow & (\overline{\mathbf{f}}A', R((\overline{\mathbf{f}}\text{pr}_1, \overline{\mathbf{f}}\text{pr}_2)^{-1})(\widehat{\mathbf{f}}_{A' \times A'}(\rho'))) \end{array}$$

¹The value $\mathcal{R}(P) = P^{\mathcal{R}}$ is denoted as $P_{\mathcal{D}}$ in [19]. In fact, in the following we use the two notations $P^{\mathcal{R}}$ and $\mathcal{R}(P)$ for the action of a 2-functor interchangeably, with the hope to improve readability.

and, for (A, ρ) in \mathcal{E}_P , the (A, ρ) component of $\widehat{\mathbf{f}^{\mathcal{R}}}$ is

$$\begin{array}{ccc} P^{\mathcal{R}}(A, \rho) & \xrightarrow{\left(\widehat{\mathbf{f}^{\mathcal{R}}}\right)_{(A, \rho)}} & R^{\mathcal{R}}(\overline{\mathbf{f}^{\mathcal{R}}}(A, \rho)) \\ \alpha \dashv & \xrightarrow{\quad\quad\quad} & \widehat{\mathbf{f}}_A(\alpha). \end{array}$$

The action of \mathcal{R} on a 2-cell $\theta: \mathbf{f} \Rightarrow \mathbf{g}: P \rightarrow R$ is simply $(\theta^{\mathcal{R}})_{(A, \rho)} = \theta_A$.

Examples 2.9. The following examples are from [11, 23].

(a) Consider the positive real line $[0, \infty)$ with the opposite of the natural order, so 0 is the top element and there is no bottom element. Hence the primary doctrine $[0, \infty)^{(-)}$, as in Example 2.1, is not elementary by Example 2.2. The base category $\mathcal{E}_{[0, \infty)^{(-)}}$ of the elementary doctrine

$$\left([0, \infty)^{(-)}\right)^{\mathcal{R}} : \mathcal{E}_{[0, \infty)^{(-)}}^{\text{op}} \rightarrow \mathbf{Pos}$$

is the category of pseudo ultrametric spaces; it consists of pairs (X, d) where $d: X \times X \rightarrow [0, \infty)$ satisfies all the conditions of an ultrametric space but for the identity of indiscernibles

$$d(x_1, x_2) = 0 \quad \text{if and only if} \quad x_1 = x_2$$

weakened to just $d(x_1, x_1) = 0$.

(b) The closed unit interval $[0, 1]$ with the opposite of the natural order is a complete Heyting algebra. Hence the doctrine $[0, 1]^{(-)}$ is elementary first-order and the category $\mathcal{E}_{[0, 1]^{(-)}}$ is the category of 1-bounded pseudo ultrametric spaces.

As announced previously, there is an adjunction $\mathbf{ED} \xleftarrow{\mathcal{R}} \mathbf{PD} \xrightarrow{\mathcal{T}}$. The component of the unit \mathbf{j} on the elementary doctrine $E: \mathcal{B}^{\text{op}} \rightarrow \mathbf{Pos}$ is $\mathbf{j}_E = (\overline{\mathbf{j}}_E, \widehat{\mathbf{j}}_E)$

$$\begin{array}{ccc} \mathcal{B}^{\text{op}} & \xrightarrow{E} & \mathbf{Pos} \\ (\overline{\mathbf{j}}_E)^{\text{op}} \downarrow & \widehat{\mathbf{j}}_E \downarrow \cdot & \downarrow \\ \mathcal{E}_E^{\text{op}} & \xrightarrow{E^{\mathcal{R}}} & \mathbf{Pos} \end{array}$$

where $\overline{\mathbf{j}}_E(f: A \rightarrow A') = f: (A, \delta_A^E) \rightarrow (A', \delta_{A'}^E)$ and $\widehat{\mathbf{j}}_E : E(A) \rightarrow E^{\mathcal{R}}(\overline{\mathbf{j}}_E(A))$ is the identity since $E^{\mathcal{R}}(A, \delta_A^E) = E(A)$. The component of the counit \mathbf{u} on a primary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is $\mathbf{u}_P = (\overline{\mathbf{u}}_P, \widehat{\mathbf{u}}_P)$

$$\begin{array}{ccc} \mathcal{E}_P^{\text{op}} & \xrightarrow{P^{\mathcal{R}}} & \mathbf{Pos} \\ (\overline{\mathbf{u}}_P)^{\text{op}} \downarrow & \widehat{\mathbf{u}}_P \downarrow \cdot & \downarrow \\ \mathcal{C}^{\text{op}} & \xrightarrow{P} & \mathbf{Pos} \end{array}$$

where $\overline{\mathbf{u}}_P: \mathcal{E}_P \rightarrow \mathcal{C}$ is the first projection mapping $f: (A, \rho) \rightarrow (B, \sigma)$ to $f: A \rightarrow B$ and $(\widehat{\mathbf{u}}_P)_A: P^{\mathcal{R}}(A, \rho) \hookrightarrow P(A)$ is the inclusion.

3. Elementary doctrines are coalgebras

In this section we prove the main result of the paper: elementary doctrines are coalgebras for the comonad associated to the 2-adjunction in the following proposition.

Proposition 3.1. *The adjunction $\mathbf{ED} \begin{array}{c} \xleftarrow{\mathcal{R}} \\ \xrightarrow{\mathcal{T}} \end{array} \mathbf{PD}$ is a 2-adjunction.*

Proof. We must show that, for every elementary doctrine $E: \mathcal{B}^{\text{op}} \rightarrow \mathbf{Pos}$ and every primary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$, the functor

$$\mathbf{ED}(E, P^{\mathcal{R}}) \xrightarrow{\mathbf{u}_P \circ -} \mathbf{PD}(E, P)$$

is an isomorphism. Since one already knows from [19] that the bijection on the objects sends $\mathbf{f}: E \rightarrow P^{\mathcal{R}}$ to $\mathbf{u}_P \circ \mathbf{f}: E \rightarrow P$, it follows that it is an isomorphism because it is fully faithful. \square

Write $\mathcal{T}: \mathbf{PD} \rightarrow \mathbf{PD}$ for the composite $\mathcal{L}\mathcal{R}$ so that the 2-comonad induced by the 2-adjunction is $(\mathcal{T}, \mathbf{u}, \mathbf{j}\mathcal{R})$.

Theorem 3.2. *The canonical comparison*

$$\begin{array}{ccc} \mathbf{ED} & \xrightarrow{\mathcal{K}} & \mathbf{PD}^{\mathcal{T}} \\ & \searrow \mathcal{R} & \swarrow \mathcal{T} \\ & \mathbf{PD} & \end{array}$$

is a 2-isomorphism.

The comparison 2-functor $\mathcal{K}: \mathbf{ED} \rightarrow \mathbf{PD}^{\mathcal{T}}$ maps an elementary doctrine E to the coalgebra $\mathbf{j}_E = (\overline{\mathbf{j}}_E, \widehat{\mathbf{j}}_E): E \rightarrow \mathcal{T}E$.

The rest of the section is devoted to proving the theorem.

Lemma 3.3. *The comparison 2-functor $\mathcal{K}: \mathbf{ED} \rightarrow \mathbf{PD}^{\mathcal{T}}$ is surjective on objects.*

Proof. Consider an object (P, \mathbf{g}) in $\mathbf{PD}^{\mathcal{T}}$. We first show that the fact that the coalgebra map $\mathbf{g}: P \rightarrow P^{\mathcal{T}}$ is a section of the counit $\mathbf{u}_P: P^{\mathcal{T}} \rightarrow P$ suffices to conclude that P is elementary. Since $\mathbf{u}_P \mathbf{g} = \text{id}_P$, for every object A one has $\overline{\mathbf{g}}A = (A, \delta_{\overline{\mathbf{g}}A}^{P^{\mathcal{R}}})$ for some P -equivalence relation $\delta_{\overline{\mathbf{g}}A}^{P^{\mathcal{R}}} \in (A \times A)$, and also

$$(\widehat{\mathbf{u}}_P)_{\overline{\mathbf{g}}A} \widehat{\mathbf{g}}A = \text{id}_{PA}. \quad (2)$$

For every $f: A \rightarrow B$ in \mathcal{C} , it is $\overline{\mathbf{u}_P \mathbf{g}}(f) = f$; hence

$$f: (A, \delta_{\overline{\mathbf{g}}A}^{P\mathcal{R}}) \longrightarrow (B, \delta_{\overline{\mathbf{g}}B}^{P\mathcal{R}}). \quad (3)$$

Since $(\widehat{\mathbf{u}_P})_{\overline{\mathbf{g}}A}: P^{\mathcal{R}}(A, \delta_{\overline{\mathbf{g}}A}^{P\mathcal{R}}) \longrightarrow PA$ is the inclusion, it follows from (2) that

$$P^{\mathcal{R}}(A, \delta_{\overline{\mathbf{g}}A}^{P\mathcal{R}}) = PA \quad (4)$$

and

$$\widehat{\mathbf{g}}_A = \text{id}_{PA}. \quad (5)$$

The identity (4) amounts to say that every object in PA is descent data for $\delta_{\overline{\mathbf{g}}A}^{P\mathcal{R}}$, establishing condition (a) in Proposition 2.5. Condition (b) holds since $\delta_{\overline{\mathbf{g}}A}^{P\mathcal{R}}$ is a P -equivalence relation. Condition (c) is a consequence of the fact that $\overline{\mathbf{g}}$ preserves products. Indeed, for A and B in \mathcal{C} , the underlying arrow of the iso $(\overline{\mathbf{g}}\text{pr}_1, \overline{\mathbf{g}}\text{pr}_2): \overline{\mathbf{g}}(A \times B) \rightarrow \overline{\mathbf{g}}A \times \overline{\mathbf{g}}B$ is the identity on $A \times B$ because of (3). Hence the underlying arrow of its inverse is the identity too, that is to say $\delta_{\overline{\mathbf{g}}(A \times B)}^{P\mathcal{R}} \geq \delta_{\overline{\mathbf{g}}A}^{P\mathcal{R}} \boxtimes \delta_{\overline{\mathbf{g}}B}^{P\mathcal{R}}$. Hence P is elementary by Proposition 2.5, and $\delta_A^P = \delta_{\overline{\mathbf{g}}A}^{P\mathcal{R}}$ by Remark 2.6. So $\overline{\mathbf{g}} = \overline{\mathbf{j}_P}$ which, together with (5), yields $\mathbf{g} = \mathbf{j}_P$ as required. \square

Lemma 3.4. *For elementary doctrines E, E' , the functor*

$$\mathcal{K}_{E, E'}: \mathbf{ED}(E, E') \longrightarrow \mathbf{PD}^{\mathcal{T}}(\mathbf{j}_E, \mathbf{j}_{E'}).$$

is an isomorphism.

Proof. The functor $\mathcal{K}_{E, E'}$ is clearly faithful. It is also full since \mathbf{ED} is a 2-full subcategory of \mathbf{PD} . For any 1-cell $\mathbf{f}: E \rightarrow E'$, it is $\mathcal{K}\mathbf{f} = \mathbf{f}$, hence \mathcal{K} is clearly 1-faithful and we are left to show that a homomorphism of coalgebras $\mathbf{f}: \mathbf{j}_E \rightarrow \mathbf{j}_{E'}$ is also a 1-cell $\mathbf{f}: E \rightarrow E'$ in \mathbf{ED} . By Corollary 2.7, it is enough to show that, for every object A ,

$$\widehat{\mathbf{f}}_{A \times A}(\delta_A^E) = \langle \overline{\mathbf{f}}\text{pr}_1, \overline{\mathbf{f}}\text{pr}_2 \rangle^* (\delta_{\overline{\mathbf{f}}A}^{E'}).$$

Since \mathbf{f} is a homomorphism of coalgebras we have

$$\overline{\mathbf{f}}^{\mathcal{T}} \circ \overline{\mathbf{j}}_E = \overline{\mathbf{j}}_{E'} \circ \overline{\mathbf{f}} \quad \text{and} \quad \widehat{\mathbf{f}}_A = \left(\widehat{\mathbf{j}}_{E'} \right)_{\overline{\mathbf{f}}A} \circ \widehat{\mathbf{f}}_A = \left(\widehat{\mathbf{f}}^{\mathcal{T}} \right)_{\overline{\mathbf{j}}_{E'}A} \circ \left(\widehat{\mathbf{j}}_E \right)_A = \left(\widehat{\mathbf{f}}^{\mathcal{T}} \right)_{\overline{\mathbf{j}}_{E'}A}.$$

Notice that $\mathbf{f}^{\mathcal{T}}$ is a 1-cell in \mathbf{ED} and that $\delta_{\overline{\mathbf{j}}_{E'}(A)}^{E'} = \left(\widehat{\mathbf{j}}_E \right)_{A \times A}(\delta_A^E) = \delta_A^E$, and

similarly $\delta_{\overline{\mathbf{j}_E}(\overline{\mathbf{f}_A})}^{E'\mathcal{R}} = \delta_{\overline{\mathbf{f}_A}}^{E'}$. Hence

$$\begin{aligned} \widehat{\mathbf{f}}_{A \times A}(\delta_A^E) &= \left(\widehat{\mathbf{f}}^{\mathcal{T}} \right)_{\overline{\mathbf{j}_E}A \times \overline{\mathbf{j}_E}A} \left(\delta_{\overline{\mathbf{j}_E}(A)}^{E'\mathcal{R}} \right) \\ &= \langle \overline{\mathbf{f}}^{\mathcal{T}}(\overline{\mathbf{j}_E}\text{pr}_1), \overline{\mathbf{f}}^{\mathcal{T}}(\overline{\mathbf{j}_E}\text{pr}_2) \rangle^* \left(\delta_{\overline{\mathbf{j}_E}(\overline{\mathbf{f}_A})}^{E'\mathcal{R}} \right) \\ &= \langle \overline{\mathbf{f}}^{\mathcal{T}}(\overline{\mathbf{j}_E}\text{pr}_1), \overline{\mathbf{f}}^{\mathcal{T}}(\overline{\mathbf{j}_E}\text{pr}_2) \rangle^* \left(\delta_{\overline{\mathbf{f}_A}}^{E'} \right) \\ &= \langle \overline{\mathbf{f}}_{\text{pr}_1}, \overline{\mathbf{f}}_{\text{pr}_2} \rangle^* \left(\delta_{\overline{\mathbf{f}_A}}^{E'} \right) \end{aligned}$$

where in the first and the last steps we used the fact that $\overline{\mathbf{j}_E}$ preserves chosen products since $\delta_{A \times A}^E = \delta_A^E \boxtimes \delta_A^E$ by Remark 2.6. \square

This concludes the proof of Theorem 3.2.

Remark 3.5. It is also possible to prove that the comonad \mathcal{T} is KZ, see [9].

4. Elimination of imaginaries

Recall from Poizat [22] that a structure \mathcal{A} in a possibly multi-sorted language \mathbb{L} has *uniform elimination of imaginaries* if, for every formula ρ with at most the free variables x and x' , which we may write $\rho(x, x')$, such that

$$\mathcal{A} \models \text{'}\rho \text{ is symmetric and transitive'}$$

there is a formula $\phi(x, y)$ such that

$$\begin{aligned} \mathcal{A} \models \text{'}\phi \text{ is a functional relation' } \\ \mathcal{A} \models \forall x:A \forall x':A [\rho(x, x') \leftrightarrow \exists y:B (\phi(x, y) \wedge \phi(x', y))]. \end{aligned}$$

A theory T in a possibly multi-sorted language \mathbb{L} with equality has *uniform elimination of imaginaries* if every model of T has uniform elimination of imaginaries. A result in [6, Theorem 4.4.2] ensures that T has uniform elimination of imaginaries precisely when, for every formula $\sigma(x, y)$ such that

$$T \vdash \text{'}\sigma \text{ is reflexive, symmetric and transitive' } \quad (6)$$

there is a formula $\phi(x, z)$ such that

$$T \vdash \forall y:A \exists! z:B \forall x:A (\sigma(x, y) \leftrightarrow \phi(x, z)). \quad (7)$$

The syntactic characterization gives an interesting extension of the notion of uniform elimination of imaginaries to any theory in intuitionistic first order logic.

Definition 4.1. Let T be an intuitionistic first order theory. We say that T has *uniform elimination of imaginaries* if, for every formula $\sigma(x, y)$ such that (6) holds, there is a formula $\phi(x, z)$ such that (7) holds.

We can apply the results in the previous section to complete any intuitionistic theory T to one with uniform elimination of imaginaries as follows.

Recall from [15, 20, 21] how theories and doctrines are related. As detailed in [15, Example 2.2], for a given theory T , consider the first-order doctrine \mathbf{P}_T *associated* to T whose base category consists of contexts and term substitutions, and whose fibre over a context is the Lindenbaum-Tarski algebra of well-formed formulas in that context. As in [4, Section 8.2.1], consider also the category $\mathbf{Mod}(T)$ of models \mathfrak{M} of T and elementary homomorphisms $f: \mathfrak{M} \rightarrow \mathfrak{N}$, *i.e.* f is a homomorphism on the underlying algebras such that for each well-formed formula α in \mathbf{L} , the map f preserves the interpretation of α , *i.e.* for every $\langle m_1, \dots, m_i \rangle$,

$$\text{if } \langle m_1, \dots, m_i \rangle \in \alpha^{\mathfrak{M}}, \text{ then } \langle f(m_1), \dots, f(m_i) \rangle \in \alpha^{\mathfrak{N}}.$$

It is easy to see that the category $\mathbf{Mod}(T)$ is equivalent to the hom-category $\mathbf{FOD}(\mathbf{P}_T, \mathscr{P})$ on the 1-cell into the first-order doctrine $\mathscr{P}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$

$$\mathbf{Mod}(T) \cong \mathbf{FOD}(\mathbf{P}_T, \mathscr{P}).$$

Conversely, for a first-order doctrine P , let \mathbf{Th}_P be the internal language of P as in [21]. Similarly, $\mathbf{Mod}(\mathbf{Th}_P) \cong \mathbf{FOD}(P, \mathscr{P})$.

One word of warning: when T has equality, one can consider models where the equality predicate is interpreted as the diagonal. If $\mathbf{Mod}_=(T)$ denotes the full subcategory of $\mathbf{Mod}(T)$ on models where the equality predicate is interpreted as the diagonal, and if P is an elementary first-order doctrine (*i.e.* a first order hyperdoctrine in the sense of [21]), then the isomorphisms above restrict to isomorphisms

$$\mathbf{Mod}_=(T) \cong \mathbf{HD}(\mathbf{P}_T, \mathscr{P}) \quad \mathbf{Mod}_=(\mathbf{Th}_P) \cong \mathbf{HD}(P, \mathscr{P}).$$

Theorem 4.2. *Given an intuitionistic theory T , consider the doctrine $(\mathbf{P}_T)^{\mathcal{R}}$ and write \bar{T} for the theory $\mathbf{Th}_{(\mathbf{P}_T)^{\mathcal{R}}}$ associated to the doctrine $(\mathbf{P}_T)^{\mathcal{R}}$. Then \bar{T} has uniform elimination of imaginaries in the sense of Definition 4.1.*

Proof. The sorts of \bar{T} are of the form (a, ρ) where $a:A$ and ρ is a formula in the variables a, a' with also $a':A$. So a formula $\sigma(x, x')$, where $x, x':(a, \rho)$, is such that

$$\bar{T} \vdash \text{'}\sigma \text{ is reflexive, symmetric and transitive'}$$

precisely when $[\sigma(x, x')] \in (\mathbf{P}_T)^{\mathcal{R}}((a, \rho) \times (a, \rho))$ is a $(\mathbf{P}_T)^{\mathcal{R}}$ -equivalence relation. Hence σ is descent data for ρ and we can pick (a, σ) as B and σ as ϕ in (7). So the consequence in (7) becomes

$$\bar{T} \vdash \forall y:(a, \rho) \exists ! z:(a, \sigma) \forall x:(a, \rho) (\sigma(x, y) \leftrightarrow \sigma(x, z)) \quad (8)$$

which is clearly provable—note that the two occurrences of σ in (8) are in different fibres of $(\mathbf{P}_T)^{\mathcal{R}}$. \square

Recall that, for a set A and an equivalence relation R on A , the inf-semilattice $\mathcal{P}^{\mathcal{R}}(A, R)$ consists of those subsets S of A which are invariant with respect to R , *i.e.*

$$\text{if } a' R a \in S, \text{ then } a' \in S.$$

The base category $\mathcal{E}_{\mathcal{P}}$ of the doctrine $\mathcal{P}^{\mathcal{R}}$ is the category of equivalence relations and relation-preserving functions. Consider the 1-cell $\mathbf{q} = (\bar{\mathbf{q}}, \hat{\mathbf{q}}): \mathcal{P}^{\mathcal{R}} \rightarrow \mathcal{P}$ in \mathbf{HD} whose functor $\bar{\mathbf{q}}: \mathcal{E}_{\mathcal{P}} \rightarrow \mathbf{Set}$ takes $f: (A, R) \rightarrow (A', R')$ to the induced function $f': A/R \rightarrow A'/R'$ on the quotient sets, while the monotone function $(\hat{\mathbf{q}})_{(A,R)}: \mathcal{P}^{\mathcal{R}}(A, R) \rightarrow \mathcal{P}(A/R)$ sends $S \in \mathcal{P}^{\mathcal{R}}(A, R)$ to the set $\{[a] \in A/R \mid a \in S\}$. It is easy to check that $(\hat{\mathbf{q}})_{(A,R)}$ is an isomorphism.

Proposition 4.3. *For every first-order doctrine P , the functor*

$$\begin{array}{ccc} \mathbf{FOD}(P, \mathcal{P}) & \xrightarrow{\mathcal{R}_{P, \mathcal{P}}} & \mathbf{HD}(P^{\mathcal{R}}, \mathcal{P}^{\mathcal{R}}) & \xrightarrow{\mathbf{q} \circ -} & \mathbf{HD}(P^{\mathcal{R}}, \mathcal{P}) \\ \wr \parallel & & & & \parallel \wr \\ \mathbf{Mod}(\mathbf{Th}_P) & \xrightarrow{\quad \quad \quad} & & & \mathbf{Mod}_=(\mathbf{Th}_{P^{\mathcal{R}}}) \end{array} \quad (9)$$

applies the category of models of P into the category of models of $P^{\mathcal{R}}$. Moreover, if P is elementary, then the functor precomposition with $\mathbf{j}_P: P \rightarrow P^{\mathcal{R}}$

$$\begin{array}{ccc} \mathbf{HD}(P^{\mathcal{R}}, \mathcal{P}) & \xrightarrow{- \circ \mathbf{j}_P} & \mathbf{HD}(P, \mathcal{P}) \\ \wr \parallel & & \parallel \wr \\ \mathbf{Mod}_=(\mathbf{Th}_{P^{\mathcal{R}}}) & \xrightarrow{\quad \quad \quad} & \mathbf{Mod}_=(\mathbf{Th}_P) \end{array} \quad (10)$$

is an equivalence of categories.

Proof. Let P be a hyperdoctrine. Since every 1-cell $\mathbf{f}: P \rightarrow \mathcal{P}$ in \mathbf{ED} factors as

$$\begin{array}{ccccc} & & \mathbf{f} & & \\ & & \curvearrowright & & \\ P & \xrightarrow{\quad \mathbf{f} \quad} & \mathcal{P} & \xrightarrow{\quad \text{id}_{\mathcal{P}} \quad} & \mathcal{P} \\ \mathbf{j}_P \downarrow & & \mathbf{j}_{\mathcal{P}} \downarrow & & \nearrow \mathbf{q} \\ P^{\mathcal{R}} & \xrightarrow{\quad \mathbf{f}^{\mathcal{R}} \quad} & \mathcal{P}^{\mathcal{R}} & & \end{array}$$

the composition

$$\mathbf{HD}(P, \mathcal{P}) \hookrightarrow \mathbf{FOD}(P, \mathcal{P}) \xrightarrow{(\mathbf{q} \circ -)\mathcal{R}_{P, \mathcal{P}}} \mathbf{HD}(P^{\mathcal{R}}, \mathcal{P}) \xrightarrow{- \circ \mathbf{j}_P} \mathbf{HD}(P, \mathcal{P})$$

is naturally isomorphic to the identity. This shows that $- \circ \mathbf{j}_P$ is full and essentially surjective. It remains to prove its faithfulness. Let $\mathbf{f}, \mathbf{g}: P^{\mathcal{R}} \rightarrow \mathcal{P}$ be in \mathbf{HD} , consider two parallel 2-cells $\theta, \gamma: \mathbf{f} \Rightarrow \mathbf{g}$ and suppose $\theta \circ \mathbf{j}_P = \gamma \circ \mathbf{j}_P$. So

the following diagram of sets and functions commutes

$$\begin{array}{ccc}
& \bar{\mathbf{f}}(A, \delta_A) & \xrightarrow{\theta_{(A, \delta_A)} = \gamma_{(A, \delta_A)}} & \bar{\mathbf{g}}(A, \delta_A) \\
& \swarrow \bar{\mathbf{f}}(\pi) & & \searrow \bar{\mathbf{g}}(\pi) \\
& \bar{\mathbf{f}}(A, \rho) & \xrightarrow{\gamma_{(A, \rho)}} & \bar{\mathbf{g}}(A, \rho) \\
& \swarrow \bar{\mathbf{f}}(\pi) & & \searrow \bar{\mathbf{g}}(\pi) \\
& \bar{\mathbf{f}}(A, \rho) & \xrightarrow{\theta_{(A, \rho)}} & \bar{\mathbf{g}}(A, \rho)
\end{array}$$

where clearly $\pi = \text{id}_A: (A, \delta_A) \rightarrow (A, \rho)$ is such that $\top_{(A, \rho)} \leq \mathcal{A}_\pi^{P^\mathcal{R}}(\top_{(A, \delta_A)})$ where $\mathcal{A}_\pi^{P^\mathcal{R}}$ is the left adjoint to $\pi^* = P^\mathcal{R}(\pi)$ as in Remark 2.3. Since \mathbf{f} is in **HD**, the functor $\widehat{\mathbf{f}}_{(A, \rho)}$ preserves the top and left adjoints. Hence

$$\bar{\mathbf{f}}(A, \rho) = \top_{\bar{\mathbf{f}}(A, \rho)} \subseteq \mathcal{A}_{\bar{\mathbf{f}}(\pi)}^{\mathcal{P}}(\top_{\bar{\mathbf{f}}(A, \delta_A)}) = \bar{\mathbf{f}}(\pi)[\bar{\mathbf{f}}(A, \delta_A)]$$

which proves that $\bar{\mathbf{f}}(\pi)$ is surjective. Therefore $\theta_{(A, \rho)} = \gamma_{(A, \rho)}$. \square

Reading Proposition 4.3 for a doctrine of the form \mathbf{P}_T where T is a multi-sorted theory (not necessarily with equality), one sees that the functor (9) ensures that every model of the original theory T can be turned functorially into a model of the theory $\bar{T} = \text{Th}_{(\mathbf{P}_T)^\mathcal{R}}$, which uniformly eliminates imaginaries by Theorem 4.2. Moreover, when T has equality, the functor in (10) ensures that every model of T can be expanded to a model of \bar{T} , and every model of \bar{T} is completely determined by its reduct to T .

5. Comparison with Shelah's T^{eq}

The construction of \bar{T} in Section 4 is a radically different, simpler characterisation of Shelah's $(-)^{\text{eq}}$ in [24] than the one given in [5]. To see this, let T be an intuitionistic theory in a possibly multi-sorted language \mathbf{L} . In [5] it is proved that, if T is classical (*i.e.* $T \vdash \alpha \vee \neg\alpha$ for all well-formed formulas α in \mathbf{L}), has equality and

$$\begin{aligned}
T \vdash \exists x:A x = x & \quad \text{for every sort } A \text{ in } \mathbf{L} \\
T \vdash \exists x:A_0 \exists y:A_0 x \neq y & \quad \text{for some sort } A_0 \text{ in } \mathbf{L}
\end{aligned} \tag{11}$$

then T^{eq} coincides with the theory associated to the pretopos completion of the syntactic category of the theory T as in [18]. Recall from [18, Section 8.2] that the syntactic category \mathbf{R}_T of T consists of

objects: pairs $\langle \vec{x}, \phi \rangle$ where \vec{x} is a context in \mathbf{L} and ϕ is a well-formed formula in context \vec{x} ;

arrows: an arrow $[\theta]: \langle \vec{x}, \phi \rangle \rightarrow \langle \vec{y}, \psi \rangle$ is an equivalence class of formulas in a context \vec{x}', \vec{y}' with appropriate distinct variables, such that θ is a functional relation.

It is denoted as \mathbb{T} in [5, Section 4] and computed as $\mathcal{EF}_{(\mathbf{P}_T)_c}$ in [13, Section 3].

From Theorem 4.2, we can obtain a similar result for general intuitionistic first order theories. First we need a strengthening of [1, Lemma 2.2(i)] when an ex/reg completion produces a pretopos. In order to state the result, recall that an object B in a regular category is **well-supported** if the unique arrow $B \rightarrow 1$ is regular epic.

Proposition 5.1. *Let \mathcal{A} be a coherent category, that is, a regular category with pullback-stable unions of subobjects. Suppose that*

- (i) *for every object A in \mathcal{A} , there is a mono $m: A \rightarrow B$ into a well-supported object B ;*
- (ii) *there is a decidable object D in \mathcal{A} such that the complement of its diagonal is well-supported.*

Then the ex/reg completion of \mathcal{A}

$$\mathcal{A} \xrightarrow{\Gamma_{ex/reg}} \mathcal{A}_{ex/reg}$$

is also the pretopos completion of \mathcal{A} as a coherent category.

Proof. Is in the Appendix. □

Remark 5.2. The two conditions in the hypotheses of Proposition 5.1 easily compare with those in (11): condition 5.1(i) is just a categorical reformulation of the first in (11) read in the syntactic category. More interestingly, condition 5.1(ii) has a global requirement about the object D that cannot be spotted in the classical case.

As a consequence of Proposition 5.1, we know that the theory T^{eq} for T a theory in classical first order logic is the theory associated to the subobject doctrine of $(\mathbf{R}_T)_{ex/reg}$. In turn the ex/reg completion has a neat algebraic description in terms of doctrines as $\mathcal{C}_{ex/reg} = \mathcal{EF}_{(\mathbf{Sub}_c)^{\mathcal{R}}}$, see [16, Example 3.2].

In conclusion, the characterisation of T^{eq} in [5] is the theory of the subobject doctrine of the category

$$\mathcal{EF} \left(\mathbf{Sub}_{(\mathcal{EF}_{(\mathbf{P}_T)_c})} \right)^{\mathcal{R}}.$$

But there is an equivalence between $(\mathbf{P}_T)^{\mathcal{R}}$ and a subobject doctrine if and only if \mathbf{P}_T validates a rule of choice, see [17, Proposition 4.11].

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A. Appendix

The appendix is devoted to the proof of Proposition 5.1 which relies on a technical result about coherent categories.

Proposition A.1. *Let \mathcal{A} be a coherent exact category. Suppose that*

- (i) *for every object A in \mathcal{A} , there is a mono $m: A \twoheadrightarrow B$ into a well-supported object B ;*
- (ii) *there is a decidable object D in \mathcal{A} such that the complement of its diagonal is well-supported.*

Then \mathcal{A} is a pretopos.

We defer the proof of the technical result to after the proof of Proposition 5.1.

Proof of Proposition 5.1. It is well-known that the ex/reg completion of a coherent category is coherent and the embedding $\Gamma_{\text{ex/reg}}: \mathcal{A} \rightarrow \mathcal{A}_{\text{ex/reg}}$ is a coherent functor, see [8, Corollary 3.3.10]. We are left to show that, when \mathcal{A} satisfies conditions 5.1(i)-(ii), the exact category $\mathcal{A}_{\text{ex/reg}}$ is in fact a pretopos, as the required universal property follows from the fact that $\Gamma_{\text{ex/reg}}$ is universal among regular functors into exact categories. To this aim, it is enough to show that the ex/reg completion preserves conditions 5.1(i) and (ii) and apply Proposition A.1.

Recall from [3, Section 2.3] that objects of $\mathcal{A}_{\text{ex/reg}}$ are equivalence relations $R \twoheadrightarrow A \times A$ in \mathcal{A} and that an arrow from $r: R \twoheadrightarrow A \times A$ to $s: S \twoheadrightarrow B \times B$ is a relation $f: F \twoheadrightarrow A \times B$ such that

$$f \cdot r = f = s \cdot f, \quad r \leq f^\circ \cdot f \quad \text{and} \quad f \cdot f^\circ \leq s,$$

where f° is the converse of the relation f and $f \cdot g$ denotes relational composition. The embedding $\Gamma_{\text{ex/reg}}: \mathcal{A} \rightarrow \mathcal{A}_{\text{ex/reg}}$ maps an object A in \mathcal{A} to the identity relation $\Delta_A := \langle \text{id}_A, \text{id}_A \rangle$ and an arrow $f: A \rightarrow B$ to its graph $\langle \text{id}_A, f \rangle: A \twoheadrightarrow A \times B$. We know it preserves the regular structure; in particular, it takes a well-supported object to a well-supported object.

Ad (i). Let $r: R \twoheadrightarrow A \times A$ be an arbitrary object in $\mathcal{A}_{\text{ex/reg}}$. By hypothesis, there is a mono $m: A \twoheadrightarrow B$ into a well-supported object B in \mathcal{A} . In \mathcal{A} consider the join $s: S \twoheadrightarrow B \times B$ of the subobjects $(m \times m)r: R \twoheadrightarrow B \times B$ and

$\Delta_A: B \twoheadrightarrow B \times B$ which is clearly an equivalence relation, say $j_1: R \twoheadrightarrow S$ and $j_2: B \twoheadrightarrow S$ are the two inclusions into the join. Moreover the diagram

$$\begin{array}{ccccc}
 R & \xrightarrow{j_1} & S & \xleftarrow{j_2} & B \\
 \downarrow r & & \downarrow s & & \downarrow \Delta_B \\
 A \times A & \xrightarrow{m \times m} & B \times B & \xleftarrow{\text{id}_B \times \text{id}_B} & B \times B
 \end{array}$$

produces a mono from r into s and a factor from Δ_B into s of the terminal arrow which ensures that s is well-supported in $\mathcal{A}_{\text{ex/reg}}$.

Ad (ii). Immediate since $\Gamma_{\text{ex/reg}}: \mathcal{A} \rightarrow \mathcal{A}_{\text{ex/reg}}$ preserves finite unions and finite intersections of subobjects. \square

Proof of Proposition A.1. To see that \mathcal{A} is extensive, consider first that, in a coherent category, any disjoint sum that happens to exist is universal in the sense of [2, Definition 2.10] because a disjoint sum is also a disjoint union of subobjects, and these are stable under pullback. So, by [2, Proposition 2.14] it is enough to prove that \mathcal{A} has disjoint sums. And, by hypothesis (i) it suffices to do so just for well-supported objects. So let A and B be well-supported objects. Let D be a decidable object such that, in the complement $\neg\Delta_D: D^c \twoheadrightarrow D \times D$ of the diagonal $\Delta_D: D \twoheadrightarrow D \times D$, the object D^c is well-supported. In the diagram

$$\begin{array}{ccccc}
 K_{\text{pr}_A} & \twoheadrightarrow & K_{\text{pr}_A} \cup K_{\text{pr}_B} & \xleftarrow{\quad} & K_{\text{pr}_B} \\
 \downarrow k_1 \downarrow k_2 & & \downarrow s_1 \downarrow s_2 & & \downarrow k'_1 \downarrow k'_2 \\
 D \times A \times B & \xrightarrow{\Delta_D \times \text{id}_A \times \text{id}_B} & D \times D \times A \times B & \xleftarrow{\neg\Delta_D \times \text{id}_A \times \text{id}_B} & D^c \times A \times B \\
 \downarrow \text{pr}_A & & \downarrow q & & \downarrow \text{pr}_B \\
 A & \dashrightarrow^{i_A} & Q & \dashleftarrow^{i_B} & B
 \end{array}$$

where the side columns are kernel pairs of regular epis because D , D^c , A and B are well-supported. The join is taken in the poset of subobjects of $(D \times D \times A \times B)^2$; it is an equivalence relation as the (partial) equivalence relations $K_{\text{pr}_A} \twoheadrightarrow (D \times D \times A \times B)^2$ and $K_{\text{pr}_B} \twoheadrightarrow (D \times D \times A \times B)^2$ are disjoint. So, taking the coequalizer of $K_{\text{pr}_A} \cup K_{\text{pr}_B} \twoheadrightarrow D \times D \times A \times B$, all columns are exact; now one easily checks that the bottom row is a disjoint sum. \square

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