6. ON A SOLUTION METHOD FOR THE RIEMANN PROBLEM WITH TWO PAIRS OF UNKNOWN FUNCTIONS

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The solution of the Riemann problem with a piecewise constant matrix is constructed. The obtained result is expressed in terms of solutions of a differential equation of Fuchs class. To construct the corresponding differential equation a method of logarithmization of the matrix product is proposed.

KEY WORDS: Riemann problem, Piecewise constant matrix, Differential equation of Fuchs type, Monodromy group, Logarithmization method
MSC (2010): primary 34M35, 15A23, secondary 30E25, 34M50

1 INTRODUCTION

In 1857, B. Riemann (see, e.g. [25]) posed the following problem: to find a system of functions $Y(z) = (y_1(z), \ldots, y_m(z))$, with 3 properties:

1. functions are analytic everywhere in $\widehat{\mathbb{C}}$ except of a finite number of points $a_1, a_2, \ldots, a_n$;

2. the function $Y(z)$ possesses a linear transformation with a non-singular constant matrix V_k whenever z is traversing around each singular point a_k $(k = 1, 2, \ldots, n)$, i.e. $Y \rightarrowtail V_k Y$, such that $V_1 \cdot V_2 \cdot \ldots \cdot V_n = E$;

3. at each singular point a_k $(k = 1, 2, \ldots, n)$ functions $y_1(z), y_2(z), \ldots, y_m(z)$ can turn to infinity of a finite order.

Matrices $V_1, V_2, \ldots, V_n$ form a monodromy group. Riemann showed that, in the neighborhood of each particular point a_k the solution of the problem is

$$Y(z) = D_k \begin{pmatrix} (z-a_k)^{\mu_{1k}} & u_1(z) \\ \cdots\cdots & \cdots\cdots \\ (z-a_k)^{\mu_{mk}} & u_m(z) \end{pmatrix},$$

where $u_1(z), \ldots, u_m(z)$ are analytic functions, $u_j(a_k) \neq 0$ $(j = 1, \ldots m)$, $\mu_{jk} = \frac{1}{2\pi i} \ln \lambda_{jk}$, $\lambda_{1k}, \ldots, \lambda_{mk}$ are the characteristic numbers of the matrix V_k $(k = 1, \ldots n)$, D_k are matrices transforming the matrices V_k to a Jordan form. Riemann also pointed out that the functions $y_1, \ldots, y_m$ will be solutions of an m-th order complex differential equation with rational coefficients (see, e.g., [2]). In 1900, Hilbert included the problem of construction of the differential equation of Fuchsian type as 21-st into the list of the mathematical problems for XX century. This problem is known nowadays as the Riemann-Hilbert problem, see [3], [7].

The Riemann problem can be formulated as the Riemann boundary value problem for analytic functions (see [22], and known monographs [20], [27]). Let's draw a simple closed loop through singular points. Then bypassing the point a_k the following transformation yields $Y^+ \rightarrowtail V_1 \cdot V_2 \cdot \ldots \cdot V_k \cdot Y^+ = Y^-$. Hence we arrive at the boundary condition

$$Y^+(t) = A_k \cdot Y^-(t)\,, \quad t \in (a_k, a_{k+1})\,, \quad k = 1, \ldots n, \quad a_{n+1} = a_1, \tag{1}$$

where $A_k = (V_1 \cdots V_k)^{-1}\,, \quad A_n = E$. The vector-matrix Riemann boundary value problem with piece-wise continuous algebraic coefficients was first solved by [10] by using Green's function method. The solution of the Riemann-Hilbert problem using its reduction to the Riemann boundary value problem for analytic functions was proposed in 1908 by Plemelj [23] (see also [24]).

For a long time it was thought that Plemelj had found a complete and positive answer to the question of existence of the complex differential equation with a given monodromy group. Therefore the interest in this problem was moved into the area of the effective construction of its solution. We have to mention here the results by Lappo-Danilevsky, Röhrl and Erugin. N.P. Erugin [6] considered the case of four singular points and showed, in particular, that the Riemann-Hilbert problem is related to the Penleve type differential equations.

In the late 1980s, Bolibrukh (see e.g. [2]) showed that the proof of Plemelj is incomplete and that the negative answer is also possible. An extended description of the modern state of the Riemann-Hilbert problem as well as the presentation of the main results by A.Bolibruch is presented in [2] and [3]. We also have to mention the paper [7] which is devoted to the connection between the factorization of piece-wise constant $n \times n$ matrix functions with

m jumps and the Riemann-Hilbert problem. In studying these related problems, some results for the partial indices for general n and m were obtained, including complete answers for $n = 2, m = 4$ and for $n = m = 3$. In some cases, the partial indices can be determined explicitly, while in the remaining cases, there remain two possibilities. The determination of the correct possibility is equivalent to the description of the monodromy of n-th order linear Fuchsian differential equations with m singular points.

Despite the fact that more than 160 years have passed since the statement of the Riemann problem, it has not been completely solved. This paper presents one of possible directions in the study of the Riemann problem. In order not to be too cumbersome we limit our attention to the case of $m = 2$, i.e. for vector functions $Y(z) = (y_1, y_2)$.

2 CONSTRUCTION OF THE CANONICAL MATRIX FOR THE RIEMANN PROBLEM WITH THREE SINGULAR POINTS

Before proceeding with the presentation of a solution scheme for the Riemann problem with an arbitrary number of singular points, let us show how its solutions are constructed in the case of 3 singular points a_1, a_2, a_3. Without loss of generality, we assume that $a_3 = \infty$. The question is to find the solution of the homogeneous Riemann boundary value problem with boundary conditions

$$\begin{cases} Y^+(t) = A_1 \cdot Y^-(t), & t \in (a_1, a_2), \\ Y^+(t) = A_2 \cdot Y^-(t), & t \in (a_2, \infty), \end{cases} \tag{2}$$

where $A_1,\ A_2$ are constant non-singular matrices of the 2nd order. The solution to problem (2) will be sought in the class of functions that are integrable as $z \to a_1$ and $z \to a_2$, but almost bounded (i.e. bounded or admitting a logarithmic singularity) as $z \to \infty$. In the selected class, the homogeneous boundary value problem (2) is always solvable.

Denote by $\alpha_k,\ \beta_k$ the characteristic numbers of matrices $V_k = A_{k-1}A_k^{-1}$, $k = 1,2,3$, $A_0 = A_3 = E$, respectively. For each $k = 1,2,3$ find the numbers

$$\rho_k = \frac{1}{2\pi i}\ln\alpha_k,\ -1 < Re\rho_k \le 0,\ \sigma_k = \frac{1}{2\pi i}\ln\beta_k,\ \ -1 < Re\sigma_k \le 0,$$

$$\Delta = \sum_{k=1}^{3}\left(\rho_k + \sigma_k\right),\ \ \Delta \text{ is an integer},\ -5 \le \Delta \le 0.$$

We look for the solution of problem (2) in the neighborhood of each singular point in the form

$$Y(z) = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = D_k \begin{pmatrix} (z-a_k)^{\rho_k} & u_k(z) \\ (z-a_k)^{\sigma_k} & v_k(z) \end{pmatrix}, \quad k = 1,2, \quad (3)$$

$$Y(z) = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = D_3 \begin{pmatrix} z^{-\rho} & u_3(z) \\ z^{-\sigma} & v_3(z) \end{pmatrix}, \qquad (4)$$

where $D_k\,(k = 1,2,3)$ transform monodromy matrices V_k to a normal Jordan form, the functions $u_k(z)$ are analytic in a neighborhood of points a_k, and the functions $v_k(z)$ are either analytic if $\rho_k \ne \sigma_k$, or have the form

$$v_k(z) = \frac{1}{2\pi i}\ln(z-a_k)\,u_k(z) + w_k(z) \text{ at } \rho_k = \sigma_k,\ k = 1,2, \quad (5)$$

$$v_3(z) = \frac{1}{2\pi i}\ln z \cdot u_3(z) + w_3(z) \text{ at } \rho_3 = \sigma_3, \qquad (6)$$

with $w_k(z)$ being analytic in the neighborhood of the points a_k, $u_k(a_k) \ne 0$, $w_k(a_k) \ne 0$, $k = 1,2,3$; $\rho = \rho_3 + \kappa_1$, $\sigma = \sigma_3 + \kappa_2$, and integers κ_1 and κ_2 determine the order of the solution at infinity, which will be maximal if $|Re\,(\rho - \sigma)| < 1$.

In order to solve the homogeneous Riemann boundary value problem, it is necessary to find the canonical matrix $X\,(z)$ of this problem (see [20]). The columns of the matrix $X\,(z)$ satisfy the boundary conditions and the matrix $X\,(z)$ has the following properties [18]:

1. $\det X\,(z) \neq 0$ for $\forall z \neq a_k$ $(k = 1, 2, 3)$;

2. the columns of the matrix $X\,(z)$ belong to the selected class of functions;

3. the order of the determinant $X\,(z)$ is equal to the sum of the orders of its columns.

The first two conditions are satisfied by the matrix

$$X_0\,(z) = \begin{pmatrix} y_1 & (z-a_1)\,(z-a_2)\,y_1' \\ y_2 & (z-a_1)\,(z-a_2)\,y_2' \end{pmatrix}.$$

Taking into account representation (4) we establish that the order p of the determinant of the matrix $X_0\,(z)$ is equal $p = Re\,(\rho+\sigma) - 1$, and the orders p_1 and p_2 its columns are equal $p_1 = \min\,(Re\rho, Re\sigma)$, $p_2 = p_1 - 1$. Therefore, $p = p_1 + p_2$ only in the case $\rho = \sigma$.

Let us consider the matrix

$$X\,(z) = X_0\,(z) \begin{pmatrix} 1 & \varepsilon z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_1 & \varepsilon z y_1 + (z-a_1)\,(z-a_2)\,y_1' \\ y_2 & \varepsilon z y_2 + (z-a_1)\,(z-a_2)\,y_2' \end{pmatrix}, \tag{7}$$

where $\varepsilon = \begin{cases} \rho, & \text{if } Re\rho \leq Re\sigma, \\ \sigma, & \text{if } Re\rho \geq Re\sigma. \end{cases}$

The matrix $X\,(z)$ has the property 3 of canonical matrices, i.e. $p = p_1 + p_2$. Indeed, let $Re\rho < Re\sigma$. Then $\varepsilon = \rho$ and the order p_1 of the first column of the matrix $X\,(z)$ at infinity will be equal to

$p_1 = \min(Re\rho, Re\sigma) = Re\rho$. The order p_2 of the second column of the matrix $X(z)$ at infinity is $p_2 = \min(Re\rho, Re\sigma - 1) = Re\sigma - 1$. Consequently, $p = Re\rho + Re\sigma - 1 = p_1 + p_2$ and the matrix $X(z)$ is the canonical matrix of the problem (2).

Let us find the differential equation to which the matrix satisfies $X_0(z)$. By construction, the matrix $X_0(z)$ is a solution to the boundary value problem (2), i.e.

$$X_0^+(t) = A_k \cdot X_0^-(t), \quad t \in (a_k, a_{k+1}), \ k = 1, 2. \tag{8}$$

where

$$A_k = X_0^+(t) \cdot \left[X_0^-(t)\right]^{-1}. \tag{9}$$

Differentiating both parts of the boundary condition (8) and taking into account (9), we arrive at the boundary condition or $\frac{dX_0^+}{dt} = X_0^+ \cdot \left[X_0^-\right]^{-1} \frac{dX_0^-}{dt}$ or

$$\left[X_0^{-1}\frac{dX_0}{dt}\right]^+ = \left[X_0^{-1}\frac{dX_0}{dt}\right]^-. \tag{10}$$

Let us denote $p(z) = (z - a_1)(z - a_2)$ and consider the matrix

$$X_0^{-1}(z) \cdot \frac{dX_0(z)}{dz} = \frac{1}{p(z)(y_1 y_2' - y_2 y_1')} \cdot \begin{pmatrix} p(z) y_2' & -p(z) y_1' \\ -y_2 & y_1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} y_1' & p'(z) y_1' + p(z) y_1'' \\ y_2' & p'(z) y_2' + p(z) y_2'' \end{pmatrix} = \begin{pmatrix} 0 & \varphi_1(z) \\ \frac{1}{p(z)} & \varphi_2(z) \end{pmatrix}, \tag{11}$$

where the following notation is used $\varphi_1(z) := p(z) \frac{y_2' y_1'' - y_1' y_2''}{y_1 y_2' - y_2 y_1'}$, $\varphi_2(z) := \frac{p'(z)}{p(z)} + \frac{y_1 y_2'' - y_2 y_1''}{y_1 y_2' - y_2 y_1'}$.

In order to apply to (11) the theorem on analytic continuation and the generalized Liouville's theorem we find the main parts of the decomposition of the elements of this matrix in the vicinity of singular points $a_1, a_2, a_3 = \infty$, using formulas (3) – (6).

If $\rho_k \neq \sigma_k$, $k = 1, 2, 3$, then in a neighborhood of the point a_1 there are representations

$$\varphi_1(z) = \frac{(a_2 - a_1)\,\rho_1\sigma_1}{z - a_1} + \ldots, \quad \varphi_2(z) = \frac{\rho_1 + \sigma_1}{z - a_1} + \ldots.$$

Similar representations take place in a neighborhood of the point $z = a_2$

$$\varphi_1(z) = \frac{(a_1 - a_2)\,\rho_2\sigma_2}{z - a_2} + \ldots, \quad \varphi_2(z) = \frac{\rho_2 + \sigma_2}{z - a_2} + \ldots.$$

In a neighborhood of a point $a_3 = \infty$ the functions $\varphi_1(z)$ and $\varphi_2(z)$ admit the following representation

$$\varphi_1(z) = -\rho\sigma + \ldots, \quad \varphi_2(z) = \frac{1 - \rho - \sigma}{z} + \ldots.$$

Given that $\underset{a_1}{res}\,\varphi_2 + \underset{a_2}{res}\,\varphi_2 + \underset{\infty}{res}\,\varphi_2 = 0$, we get the following relation $\rho_1 + \sigma_1 + \rho_2 + \sigma_2 - (1 - \rho - \sigma) = 0$ or

$$\rho_1 + \sigma_1 + \rho_2 + \sigma_2 + \rho + \sigma = 1, \tag{12}$$

which is called *the Fuchs relation.*

Since $\rho = \rho_3 + \kappa_1$, $\sigma = \sigma_3 + \kappa_2$, then we have $\sum_{k=1}^{3} (\rho_k + \sigma_k) + \kappa_1 + \kappa_2 = 1$, and thus $\kappa_1 + \kappa_2 = 1 - \Delta$. From the last equality it follows that the solution $Y(z)$ will have the maximum possible order at infinity, if we choose the numbers κ_1 and κ_2 in such a way that $k_1 = \left[\frac{2-\Delta}{2}\right], k_2 = \left[\frac{1-\Delta}{2}\right]$ at $Re\rho_3 \leq Re\sigma_3$, $k_1 = \left[\frac{1-\Delta}{2}\right], k_2 = \left[\frac{2-\Delta}{2}\right]$ at $Re\rho_3 \leq Re\sigma_3$. So we have found that

$$X_0^{-1}(z) \cdot \frac{dX_0(z)}{dz} = \begin{pmatrix} 0 & \frac{(a_2-a_1)\rho_1\sigma_1}{z-a_1} + \frac{(a_1-a_2)\rho_2\sigma_2}{z-a_2} - \rho\sigma \\ \frac{1}{p(z)} & \frac{\rho_1+\sigma_1}{z-a_1} + \frac{\rho_2+\sigma_2}{z-a_2} \end{pmatrix},$$

and the matrix $X_0(z)$ satisfies the differential equation

$$\frac{dX_0}{dz} = X_0 \left[\frac{U_1^0}{z-a_1} + \frac{U_2^0}{z-a_2} + \begin{pmatrix} 0 & -\rho\sigma \\ 0 & 0 \end{pmatrix} \right] =: X_0(z)\,U(z)\,, \tag{13}$$

where $U(z) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$, and the matrix coefficients U_1^0, U_2^0 are defined by the relations $U_1^0 = \begin{pmatrix} 0 & (a_2-a_1)\,\rho_1\sigma_1 \\ (a_1-a_2)^{-1} & \rho_1+\sigma_1 \end{pmatrix}$, $U_2^0 = \begin{pmatrix} 0 & (a_1-a_2)\,\rho_2\sigma_2 \\ (a_2-a_1)^{-1} & \rho_2+\sigma_2 \end{pmatrix}$. We now construct a differential equation for the canonical matrix (7). Substituting $X_0(z) = X(z)\cdot\begin{pmatrix} 1 & -\varepsilon z \\ 0 & 1 \end{pmatrix}$, where $\varepsilon = \begin{cases} \rho, & \text{if } Re\rho \le Re\sigma, \\ \sigma, & \text{if } Re\rho \ge Re\sigma, \end{cases}$ into equation (13) we obtain

$$\frac{dX}{dz}\cdot\begin{pmatrix} 1 & -\varepsilon z \\ 0 & 1 \end{pmatrix} + X(z)\begin{pmatrix} 0 & -\varepsilon \\ 0 & 0 \end{pmatrix} = X(z)\cdot\begin{pmatrix} 1 & \varepsilon z \\ 0 & 1 \end{pmatrix} U(z),$$

or

$$\frac{dX}{dz} = X(z)\left[\begin{pmatrix} 1 & -\varepsilon z \\ 0 & 1 \end{pmatrix} U(z) \begin{pmatrix} 1 & \varepsilon z \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \varepsilon z \\ 0 & 1 \end{pmatrix}\right].$$

Direct calculations show that

$$\begin{pmatrix} 1 & -\varepsilon z \\ 0 & 1 \end{pmatrix} U_1^0 \begin{pmatrix} 1 & \varepsilon z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon a_1}{a_2-a_1} & \frac{(\rho_1(a_1-a_2)+\varepsilon a_1)(\sigma_1(a_1-a_2)+\varepsilon a_1)}{a_2-a_1} \\ \frac{1}{a_1-a_2} & \rho_1+\sigma_1+\frac{\varepsilon a_1}{a_1-a_2} \end{pmatrix} +$$

$$+\left(\frac{z-a_1}{a_1-a_2}\right)\cdot\begin{pmatrix} -\varepsilon & -\varepsilon^2(z-a_1) - 2a_1\varepsilon^2 + \varepsilon(\rho_1+\sigma_1)(a_2-a_1) \\ 0 & \varepsilon \end{pmatrix} =$$

$$= U_1 + (z-a_1)\,U_1^*,$$

$$\begin{pmatrix} 1 & -\varepsilon z \\ 0 & 1 \end{pmatrix} U_2^0 \begin{pmatrix} 1 & \varepsilon z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon a_2}{a_1-a_2} & \frac{(\rho_2(a_2-a_1)+\varepsilon a_2)(\sigma_2(a_2-a_1)+\varepsilon a_2)}{a_1-a_2} \\ \frac{1}{a_2-a_1} & \rho_2+\sigma_2+\frac{\varepsilon a_2}{a_2-a_1} \end{pmatrix} +$$

$$+\left(\frac{z-a_2}{a_2-a_1}\right)\cdot\begin{pmatrix} -\varepsilon & -\varepsilon^2(z-a_2)-2a_2\varepsilon^2+\varepsilon(\rho_2+\sigma_2)(a_1-a_2) \\ 0 & \varepsilon \end{pmatrix} =$$

$$= U_2 + (z-a_2)\, U_2^*,$$

$$\begin{pmatrix} 1 & -\varepsilon z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\rho\sigma \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \varepsilon z \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \varepsilon z \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \varepsilon-\rho\sigma \\ 0 & 0 \end{pmatrix} = U_3^*.$$

Therefore, it follows from (12) that

$$U_1^* + U_2^* + U_3^* = \begin{pmatrix} 0 & -\varepsilon^2+\varepsilon(\rho+\sigma)-\rho\sigma \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -(\varepsilon-\rho)(\varepsilon-\sigma) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

since either $\varepsilon=\rho$, or $\varepsilon=\sigma$. Consequently, the canonical matrix $X(z)$ of the boundary value problem (2) is a solution of the Fuchs's class equation (a regular system of differential equations) of the form

$$\frac{dX}{dz} = X\left[\frac{U_1}{z-a_1}+\frac{U_2}{z-a_2}\right] \tag{14}$$

with matrix coefficients U_1, U_2:

$$U_1 = \frac{1}{a_2-a_1}\begin{pmatrix} \varepsilon a_1 & (\rho_1(a_1-a_2)+\varepsilon a_1)\cdot(\sigma_1(a_1-a_2)+\varepsilon a_1) \\ -1 & (\rho_1+\sigma_1)(a_2-a_1)-\varepsilon a_1 \end{pmatrix},$$

$$U_2 = \frac{1}{a_1-a_2}\begin{pmatrix} \varepsilon a_2 & (\rho_2(a_2-a_1)+\varepsilon a_2)\cdot(\sigma_2(a_2-a_1)+\varepsilon a_2) \\ -1 & (\rho_2+\sigma_2)(a_1-a_2)-\varepsilon a_2 \end{pmatrix},$$

where

$$\varepsilon = \begin{cases} \rho, & \text{if } Re\rho \leq Re\sigma, \\ \sigma, & \text{if } Re\rho \geq Re\sigma. \end{cases}$$

From the matrix equation (13) we obtain a second-order differential equation, whose solutions are components $y_1(z)$ and $y_2(z)$ of the vector $Y(z)$. The functions of the first and second rows of the matrix $X_0(z)$ in (13) satisfy the relation

$$\begin{pmatrix} y' & (p(z)\, y')' \end{pmatrix} = \begin{pmatrix} y & p(z)\, y' \end{pmatrix} \begin{pmatrix} 0 & u_{12}(z) \\ u_{21}(z) & u_{22}(z) \end{pmatrix},$$

which yields the system of two equations

$$\begin{cases} y' = p(z)\, y' \cdot u_{21}(z), \\ p'(z)\, y' + p(z)\, y'' = u_{12}(z)\, y + p(z)\, u_{22}(z) \cdot y'. \end{cases}$$

The first equation is satisfied identically, and the second equation can be rewritten as

$$y'' + \left(\frac{p'(z)}{p(z)} - u_{22}(z)\right) y' - \frac{u_{12}(z)}{p(z)} y = 0.$$

Substituting into the last equation the values of the functions $p(z)$, $u_{12}(z)$, $u_{22}(z)$, we obtain the differential Riemann equation with singular points a_1, a_2, and ∞, whose solutions are functions $y_1(z)$ and $y_2(z)$:

$$y'' + \left(\frac{1-\rho_1-\sigma_1}{z-a_1} + \frac{1-\rho_2-\sigma_2}{z-a_2}\right) y' - \tag{15}$$

$$-\frac{1}{(z-a_1)(z-a_2)}\left(\rho\sigma + \frac{(a_1-a_2)\,\rho_1\sigma_1}{z-a_1} + \frac{(a_2-a_1)\,\rho_2\sigma_2}{z-a_2}\right) y = 0.$$

By linear fractional transformation mapping the points a_1, a_2, ∞ to 0, 1, ∞, we arrive at the hypergeometric equation

$$u''+\left(\frac{1+\rho_1-\sigma_1}{z}+\frac{1+\rho_2-\sigma_2}{z-1}\right)u'-\frac{(\rho+\rho_1+\rho_2)\,(\sigma+\rho_1+\rho_2)}{z\,(z-1)}u=0$$

or

$$z\,(1-z)\;u''+(c-(a+b+1)\,z)\;u'-ab\,u=0, \tag{16}$$

where $a=\rho+\rho_1+\rho_2$, $\;b=\sigma+\rho_1+\rho_2$, $\;c=1+\rho_1-\sigma_1$.

The following systems of functions [1] can be taken as a fundamental system of solutions $u_k\,(z)$, $v_k\,(z)$, $k=1,2,3$, to equation (16) in the vicinity of points $a_1=0$, $a_2=1$, $a_3=\infty$:

$$\begin{aligned}
&u_1\,(z)=F\,(a,b;c;z)\,,\\
&v_1\,(z)=z^{1-c}F\,(a-c+1,b-c+1;2-c;z)\,,\quad \text{if a }\; c\neq 1\;,\\
&v_1\,(z)=\tfrac{1}{2\pi i}\left(-u_1\,(z)\ln z+\sum_{n=0}^{\infty}z^n\psi_n\right),\quad \text{if } c=1\;,
\end{aligned} \tag{17}$$

where $\psi_n=\frac{(a)_n(b)_n}{(n!)^2}\,[2\psi\,(n+1)-\psi\,(a+n)-\psi\,(b+n)]$, $\psi\,(z)$ is the Euler psi function; $F\,(a,b;c;z)$ is the hypergeometric function;

$$\begin{aligned}
&u_2\,(z)=F\,(a,b;a+b+1-c;1-z)\,,\\
&v_2\,(z)=(1-z)^{c-a-b}\,F\,(c-a,c-b;c+1-a-b;1-z)\,, \text{ if } c\neq a+b,\\
&v_2\,(z)=\tfrac{1}{2\pi i}\left(-u_2\,(z)\ln(1-z)+\sum_{n=0}^{\infty}(1-z)^n\,\psi_n\right), \text{ if } c=a+b,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&u_3\,(z)=(-z)^{-a}\,F\left(a,a+1-c;a+1-b;\tfrac{1}{z}\right),\\
&v_3\,(z)=(-z)^{-b}\,F\left(b+1-c,b;b+1-a;\tfrac{1}{z}\right), \text{ if } c\neq b,
\end{aligned} \tag{19}$$

$$v_3\,(z)=\frac{1}{2\pi i}[u_3\,(z)\ln(-z)+(-z)^{-a}\sum_{n=0}^{\infty}\frac{(a)_n\,(a+1-c)_n}{z^n\,(n!)^2}\,(2\psi\,(n+1)-$$

$$-\psi\left(a+n\right)-\psi\left(a+1-c+n\right))],\ \text{if}\ a=b.$$

Analytic continuation u_1, v_1 to the entire complex plane can be carried out according to formulas [1]:

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \Lambda_1 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \Lambda_1\Lambda_2 \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}, \tag{20}$$

with $\Lambda_1 = \begin{pmatrix} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} & \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \end{pmatrix}$, if $\rho_1 \neq \sigma_1,\ \rho_2 \neq \sigma_2$,

$\Lambda_1 = \begin{pmatrix} \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)} & \frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)} \\ \frac{\Gamma(a)\Gamma(b)}{2\pi i\Gamma(a+b)} & 0 \end{pmatrix}$, if $\rho_1 = \sigma_1,\ \rho_2 \neq \sigma_2$,

$\Lambda_1 = \begin{pmatrix} 0 & \frac{2\pi i\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \\ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b-1)} & \frac{2i\sin\pi a\cdot\sin\pi b}{\sin\pi(a+b)} \end{pmatrix}$, if $\rho_1 \neq \sigma_1,\ \rho_2 = \sigma_2$,

$\Lambda_1 = \begin{pmatrix} 0 & 2i\sin\pi a \\ \frac{1}{2i\sin\pi a} & 0 \end{pmatrix}$, if $\rho_1 = \sigma_1,\quad \rho_2 = \sigma_2$.

Λ_2 is constructed similarly.

We replace the corresponding entry by zero if the argument in Gamma-factors in the denominator is a nonpositive integer number.

Thus, the canonical matrix of the problem (2) in the neighborhood of each singular point $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$ has the form

$$X\left(z\right) = z^{\rho_1}\left(1-z\right)^{\rho_2} D_k \begin{pmatrix} u_k & \varepsilon z u_k + z\left(z-1\right)u_k^{'} \\ v_k & \varepsilon z v_k + z\left(z-1\right)v_k^{'} \end{pmatrix}, \tag{21}$$

where D_k is the matrix transforming the matrix $V_k = A_{k-1}A_k^{-1}$ to the normal Jordan form, $k = 1, 2, 3$, $A_0 = A_3 = E$, and u_k,v_k are the functions defined in (17) – (19). The total index æ of the problem, partial indices $æ_1, æ_2$ and the number ℓ of linearly independent solutions are determined by the formulas

$$æ = -\Delta, \quad æ_1 = \left[\frac{1-\Delta}{2}\right], \quad æ_2 = \left[\frac{-\Delta}{2}\right], \quad \ell = æ + 1. \qquad (22)$$

In order for local solutions (21) to be analytic continuations of each other, it is necessary to fulfill the conditions $D_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} = D_{k+1} \begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix}$, or $D_k \Lambda_k = D_{k+1}$, $k = 1, 2$. Matrices D_k $(k = 1, 2, 3)$ have the form $D_k = \tilde{D}_k T_k$, where $T_k = \begin{pmatrix} \gamma_k & 0 \\ 0 & \delta_k \end{pmatrix}$, if $\alpha_k \neq \beta_k$ and $T_k = \begin{pmatrix} \gamma_k & 0 \\ \delta_k & \frac{\gamma_k}{\alpha_k} \end{pmatrix}$, if $\alpha_k = \beta_k$, and $\tilde{D}_k$ is any fixed matrix performing transformation of the matrix V_k to the normal Jordan form. At last, the numbers γ_k, δ_k are subject to further definition. Denoting $S_k = (s_{ij}) = \tilde{D}_k^{-1} \cdot \tilde{D}_k$, we find γ_k and δ_k from the system of equations $T_k \Lambda_k = S_k T_{k+1}$, $k = 1, 2$.

If $\rho_1 \neq \sigma_1$ and $\rho_2 \neq \sigma_2$, then this system will take the form

$$\begin{pmatrix} \gamma_1 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} \gamma_2 & 0 \\ 0 & \delta_2 \end{pmatrix}. \qquad (23)$$

System (23) is solvable under the following condition

$$\lambda_{11}\lambda_{22}s_{12}s_{21} = \lambda_{12}\lambda_{21}s_{11}s_{22}. \qquad (24)$$

If the matrices A_1 and A_2 are reduced by a similarity transformation to the triangular form, then the matrix Λ_1 has the same form and condition (24) is satisfied. If the matrices A_1 and A_2 are not reduced by a similarity transformation to a triangular shape, then condition (24) can be written as $\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} = \frac{s_{11}s_{22}}{s_{12}s_{21}}$ or

$$\frac{\sin\pi\,(c-a)\ \sin\pi\,(c-b)}{\sin\pi a\ \sin\pi b} = \frac{\alpha_3+\beta_3-(\alpha_1\beta_2+\beta_1\alpha_2)\,\alpha_3\beta_3}{\alpha_3+\beta_3-(\alpha_1\alpha_2+\beta_1\beta_2)\,\alpha_3\beta_3}. \quad (25)$$

Direct verification ensures that equality (25) is an identity.

From system (23) we find that as γ_1, δ_1 and γ_2, δ_2 we can take any pairs of numbers that satisfy the relations $\frac{\gamma_1}{\delta_1} = \frac{s_{11}}{s_{21}} \cdot \frac{\lambda_{21}}{\lambda_{11}} = \frac{s_{12}}{s_{22}} \cdot \frac{\lambda_{22}}{\lambda_{12}}$, $\frac{\gamma_2}{\delta_2} = \frac{s_{12}}{s_{11}} \cdot \frac{\lambda_{11}}{\lambda_{12}} = \frac{s_{22}}{s_{21}} \cdot \frac{\lambda_{21}}{\lambda_{22}}$. Similarly, we find the relations between γ_k and δ_k in cases when any of the matrices V_k $(k=1,2)$ is reduced to a triangular Jordan form. For example,
$\gamma_1 = \frac{(s_{11}-\beta_2)\Gamma(2-c)}{\Gamma(1-a)\Gamma(1-b)}$, $\delta_1 = \frac{s_{21}\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}$, if $\rho_1 \neq \sigma_1$, $\rho_2 \neq \sigma_2$,
$\gamma_1 = \Gamma^2(a)\,\Gamma^2(b)\,S_{12}$, $\delta_1 = \frac{2\pi i}{a+b}\Gamma^2(a+b)\,\alpha_2$, if $\rho_1 \neq \sigma_1$, $\rho_2 = \sigma_2$,
$\gamma_1 = s_{11}-\alpha_2$, $\delta_1 = s_{12}$ or $\gamma_1 = s_{12}$, $\delta_1 = s_{22}-\alpha_2$ if $\rho_1 = \sigma_1$.

3 TRANSFORMATION OF DIFFERENTIAL MATRICES AND THE METHOD OF LOGARITHMIZATION OF MATRIX PRODUCT

Let V_1, V_2 be constant non-degenerate matrices of the 2nd order. Equality $\ln(V_1V_2) = \ln V_1 + \ln V_2$ is valid only for transitive matrices. We derive a formula connecting the logarithms of matrices V_1, V_2 and $V_3 = V_1\,V_2$ in the case of non-commuting matrices. Denote by α_k, β_k the characteristic numbers of matrices V_k, and by $\rho_k = \frac{1}{2\pi i}\ln\alpha_k$, $\sigma_k = \frac{1}{2\pi i}\ln\beta_k$ the characteristic numbers of matrices $W_k = \frac{1}{2\pi i}\ln U_k$, $k = 1,2,3$. Fix any branches of logarithms ρ_1, σ_1, ρ_2, σ_2 so that $|Re\,(\rho_k-\sigma_k)| < 1$, $k=1,2$. Then the branches of logarithms for ρ_3, σ_3 should be consistent with, and selected from, the condition $\rho_1+\sigma_1+\rho_2+\sigma_2 = \rho_3+\sigma_3$.

Let $\rho_3 \neq \sigma_3$. Imagine the matrix $S = \begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$ as the sum of two matrices $S = S_1 + S_2$, where $S_k \sim W_k$, $k=1,2$. The last equality can be written as

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} s_1 & c \\ d & \rho_1+\sigma_1-s_1 \end{pmatrix} + \begin{pmatrix} s_2 & -c \\ -d & \rho_2+\sigma_2-s_2 \end{pmatrix},$$

which is equivalent to a system of 4 equations:

$$s_1 + s_2 = \rho_3, \quad \rho_1+\sigma_1+\rho_2+\sigma_2-s_1-s_2 = \sigma_3,$$

$$s_1\left(\rho_1+\sigma_1-s_1\right) - cd = \rho_1\sigma_1, \quad s_2\left(\rho_2+\sigma_2-s_2\right) - cd = \rho_2\sigma_2.$$

Subtracting from the 3rd equation of the 4th, we obtain a system for determining s_1 and s_2:

$$\begin{cases} s_1\left(\rho_1+\sigma_1\right) - s_1^2 - s_2\left(\rho_2+\sigma_2\right) + s_2^2 = \rho_1\sigma_1 - \rho_2\sigma_2, \\ s_1 + s_2 = \rho_3, \end{cases}$$

or

$$\begin{cases} s_1\left(\rho_1+\sigma_1-\rho_3\right) - s_2\left(\rho_2+\sigma_2-\rho_3\right) = \rho_1\sigma_1 - \rho_2\sigma_2, \\ s_1 + s_2 = \rho_3, \end{cases}$$

Thus

$$s_1 = \frac{\rho_1\sigma_1 - \left(\rho_3-\rho_2\right)\left(\rho_3-\sigma_2\right)}{\sigma_3-\rho_3}, s_2 = \frac{\rho_2\sigma_2 + \left(\rho_3-\rho_1\right)\left(\rho_3-\sigma_1\right)}{\sigma_3-\rho_3}.$$

Therefore

$$c\cdot d = s_1\left(\rho_1+\sigma_1-s_1\right) - \rho_1\sigma_1 = -\left(s_1-\rho_1\right)\left(s_1-\sigma_1\right) =$$

$$= -\frac{\left[\left(\rho_3-\rho_1\right)\left(\sigma_3-\sigma_1\right)-\rho_2\sigma_2\right]\left[\left(\sigma_3-\rho_1\right)\left(\rho_3-\sigma_1\right)-\rho_2\sigma_2\right]}{\left(\sigma_3-\rho_3\right)^2}.$$

Thus, the following matrix representation is obtained:

$$S = \begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \frac{\rho_1\sigma_1-(\rho_3-\rho_2)(\rho_3-\sigma_2)}{\sigma_3-\rho_3} & \frac{(\rho_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{\sigma_3-\rho_3}A \\ \frac{\rho_2\sigma_2-(\rho_3-\sigma_1)(\sigma_3-\rho_1)}{A(\sigma_3-\rho_3)} & \frac{(\sigma_3-\rho_2)(\sigma_3-\sigma_2)-\rho_1\sigma_1}{\sigma_3-\rho_3} \end{pmatrix} +$$

$$+\begin{pmatrix} \frac{\rho_2\sigma_2+(\rho_3-\rho_1)(\rho_3-\sigma_1)}{\sigma_3-\rho_3} & \frac{\rho_2\sigma_2-(\rho_3-\rho_1)(\sigma_3-\sigma_1)}{\sigma_3-\rho_3}A \\ \frac{(\rho_3-\sigma_1)(\sigma_3-\rho_1)-\rho_2\sigma_2}{A(\sigma_3-\rho_3)} & \frac{(\sigma_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{\sigma_3-\rho_3} \end{pmatrix} \tag{26}$$

moreover, this representation, up to a similarity transformation using a diagonal matrix, is unique (c - an arbitrary constant, $c \neq 0$), if $\rho_3 = \sigma_3$. Then the matrix representation $S = \begin{pmatrix} \rho_3 & 0 \\ 1 & \rho_3 \end{pmatrix}$ as the sum of two matrices similar W_k, $k = 1, 2$, given in [11]. Later it turned out that we only need to use the decomposition (26).

Let us return to equation (14) and show another way of constructing differential matrices U_1, U_2. For definiteness, we assume $\varepsilon = \rho$ and find the sum of matrices U_1 and U_2:

$$U_1 + U_2 = \begin{pmatrix} -\rho & \mu \\ 0 & 1-\sigma \end{pmatrix}, \tag{27}$$

$\mu = -a_1\left[(\rho+\rho_1)(\rho+\sigma_1) - \rho_2\sigma_2\right] - a_2\left[(\rho+\rho_2)(\rho+\sigma_2) - \rho_1\sigma_1\right]$. Due to the choice of logarithms for numbers ρ and σ the elements standing on the main diagonal of the matrix (27), are different. From formulas (19), in particular, it follows that even in the case of $\alpha_3 = \beta_3$ equality is valid $\rho = \sigma$, otherwise function $v_3(z)$ when traversing around a point $a_3 = \infty$ will be not transformed linearly. Therefore, the matrix (26) can always be diagonalized $\tilde{S} = \begin{pmatrix} -\rho & 0 \\ 0 & 1-\sigma \end{pmatrix}$ using similarity transformation $D = \begin{pmatrix} 1 & \mu/(\sigma-\rho-1) \\ 0 & c \end{pmatrix}$. Matrix representation $\tilde{S}$ we obtain from formula (26), replacing it ρ_3 by $-\rho$, σ_3 by $1-\sigma$:

$$\tilde{S} = \tilde{S}_1 + \tilde{S}_2, \tag{28}$$

$$\tilde{S}_1 = \begin{pmatrix} \frac{\rho_1\sigma_1-(\rho+\rho_2)(\rho+\sigma_2)}{1+\rho-\sigma} & \frac{(\rho+\rho_1)(\sigma+\sigma_1-1)-\rho_2\sigma_2}{1+\rho-\sigma}A \\ \frac{\rho_2\sigma_2-(\rho+\sigma_1)(\sigma+\rho_1-1)}{A(1+\rho-\sigma)} & \frac{(\sigma+\rho_2-1)(\sigma+\sigma_2-1)-\rho_1\sigma_1}{1+\rho-\sigma} \end{pmatrix}, \tilde{S}_1 \sim W_1,$$

$$\tilde{S}_2 = \begin{pmatrix} \frac{\rho_2\sigma_2-(\rho+\rho_1)(\rho+\sigma_1)}{1+\rho-\sigma} & \frac{(\rho+\rho_2)(\sigma+\sigma_2-1)-\rho_1\sigma_1}{1+\rho-\sigma}A \\ \frac{\rho_1\sigma_1-(\rho+\sigma_2)(\sigma+\rho_2-1)}{A(1+\rho-\sigma)} & \frac{(\sigma+\rho_1-1)(\sigma+\sigma_1-1)-\rho_2\sigma_2}{1+\rho-\sigma} \end{pmatrix}, \tilde{S}_2 \sim W_2.$$

Multiplying by the matrix D both sides of equation (14) on the left, we come to the equation $\frac{d}{dz}(YD) = (YD)\left[\frac{D^{-1}U_1D}{z-a_1} + \frac{D^{-1}U_2D}{z-a_2}\right]$ or

$$\frac{d\tilde{Y}}{dz} = \tilde{Y}\left[\frac{\tilde{U}_1}{z-a_1} + \frac{\tilde{U}_2}{z-a_2}\right], \tag{29}$$

where $\tilde{Y} = YD$, $\tilde{U}_1 = \tilde{S}_1$, $\tilde{U}_2 = \tilde{S}_2$. Thus, the right side of (28) is the sum of the differential matrices $\tilde{U}_1$ and $\tilde{U}_2$ of equation (29).

In what follows, we use formula (28) to solve the Riemann problem with four or even more singular points.

4 LOGARITHMIZATION OF THE PRODUCT OF THREE MATRICES OF THE SECOND ORDER

Let $V_1,\ V_2,\ V_3$ be constant nonsingular matrices of the 2nd order, $V_4 = V_1V_2V_3$. Denote by $\alpha_k,\ \beta_k$ the characteristic numbers of matrices V_k, and by $\rho_k = \frac{1}{2\pi i}\ln\alpha_k$, $\sigma_k = \frac{1}{2\pi i}\ln\beta_k$ the characteristic numbers of matrices $W_k = \frac{1}{2\pi i}\ln U_k$, $k = 1, \ldots, 4$, where the branches of logarithms satisfy the conditions $|Re\,(\rho_k - \sigma_k)| < 1$ and $\sum_{k=1}^{3}(\rho_k + \sigma_k) = \rho_4 + \sigma_4$.

Let $\rho_4 \neq \sigma_4$. Then the matrix W_4 is reduced to a diagonal Jordan form $S_4 = \begin{pmatrix} \rho_4 & 0 \\ 0 & \sigma_4 \end{pmatrix}$. Let the matrix S_4 be a sum of three matrices

$$S_4 = S_1 + S_2 + S_3 = \sum_{k=1}^{3} \begin{pmatrix} s_k & c_k \\ d_k & s'_k \end{pmatrix}, \tag{30}$$

where $S_k \sim W_k$, $s'_k = \rho_k + \sigma_k - s_k$, $c_k d_k = s_k s'_k - \rho_k \sigma_k = -(s_k - \rho_k)(s_k - \sigma_k)$, $k = 1, 2, 3$, $\sum_{k=1}^{3} s_k = \rho_4$, $\sum_{k=1}^{3} s'_k = \sigma_4$, $\sum_{k=1}^{3} c_k = 0$, $\sum_{k=1}^{3} d_k = 0$.

We write the product of matrices $V_1 \cdot V_2 \cdot V_3$ in the form of multiplication of two matrices as follows:

$$V_4 = V_1 \cdot V_2 \cdot V_3 = V_1 \cdot (V_2 \cdot V_3) = V_1 \cdot V_{23}, \tag{31}$$

$$V_4 = V_1 \cdot V_2 \cdot V_3 = (V_1 \cdot V_2) \cdot V_3 = V_{12} \cdot V_3. \tag{32}$$

Denote by α_{12}, β_{12} and α_{23}, β_{23} the characteristic numbers of matrices V_{12} and V_{23}, $\rho_{12} = \frac{1}{2\pi i} \ln \alpha_{12}$, $\sigma_{12} = \frac{1}{2\pi i} \ln \beta_{12}$, $\rho_{23} = \frac{1}{2\pi i} \ln \alpha_{23}$, $\sigma_{23} = \frac{1}{2\pi i} \ln \beta_{23}$, whose branches are chosen according to the relations

$$\rho_{12} + \sigma_{12} = \rho_1 + \sigma_1 + \rho_2 + \sigma_2, |Re\,(\rho_{12} - \sigma_{12})| < 1,$$

$$\rho_{23} + \sigma_{23} = \rho_2 + \sigma_2 + \rho_3 + \sigma_3, |Re\,(\rho_{23} - \sigma_{23})| < 1.$$

Applying to (31) and (32) to formula (26) for the logarithm of two matrices, we obtain two representations of the matrix S_4:

$$S_4 = S_1 + S_2 + S_3 = S_1 + S_{23} = \begin{pmatrix} s_1 & c_1 \\ d_1 & s'_1 \end{pmatrix} + \begin{pmatrix} \rho_4 - s_1 & -c_1 \\ -d_1 & \sigma_4 - s'_1 \end{pmatrix}, \tag{33}$$

$$S_4 = S_1 + S_2 + S_3 = S_{12} + S_3 = \begin{pmatrix} \rho_4 - s_3 & -c_3 \\ -d_3 & \sigma_4 - s'_3 \end{pmatrix} + \begin{pmatrix} s_3 & c_3 \\ d_3 & s'_3 \end{pmatrix}, \tag{34}$$

where

$$s_1 = \frac{1}{\sigma_4 - \rho_4}\left[\rho_1\sigma_1 - (\rho_4 - \rho_{23})(\rho_4 - \sigma_{23})\right] =$$

$$= \frac{1}{\sigma_4 - \rho_4}\left[\rho_4(\sigma_4 - \rho_1 - \sigma_1) + \rho_1\sigma_1 - \rho_{23}\sigma_{23}\right],$$

$$s_3 = \frac{1}{\sigma_4 - \rho_4}\left[\rho_3\sigma_3 - (\rho_4 - \rho_{12})(\rho_4 - \sigma_{12})\right] =$$

$$= \frac{1}{\sigma_4 - \rho_4}\left[\rho_4(\sigma_4 - \rho_3 - \sigma_3) + \rho_3\sigma_3 - \rho_{12}\sigma_{12}\right],$$

$S_{12} \sim \frac{1}{2\pi i}\ln(V_1 \cdot V_2)$, $\quad S_{23} \sim \frac{1}{2\pi i}\ln(V_2 \cdot V_3)$. From (33) and (34) it follows that $S_1 + S_2 = S_{12}$ and $S_2 + S_3 = S_{23} \;\Rightarrow\; S_2 = S_{12} - S_1 = S_{23} - S_3$ or $\begin{pmatrix} s_2 & c_2 \\ d_2 & s_2' \end{pmatrix} = \begin{pmatrix} \rho_4 - s_3 - s_1 & -c_3 - c_1 \\ -d_3 - d_1 & \sigma_4 - s_3' - s_1' \end{pmatrix}$. Hence

$$s_2 = \rho_4 - s_1 - s_3 = \frac{1}{\sigma_4 - \rho_4}\left[\rho_4(\sigma_4 - \rho_4) - \right.$$

$$\left. -\rho_4(2\sigma_4 - \rho_1 - \sigma_1 - \rho_3 - \sigma_3) - \rho_1\sigma_1 - \rho_3\sigma_3 + \rho_{12}\sigma_{12} + \rho_{23}\sigma_{23}\right] =$$

$$= \frac{1}{\sigma_4 - \rho_4}\left[-\rho_4(\rho_2 - \sigma_2) - \rho_1\sigma_1 - \rho_3\sigma_3 + \rho_{12}\sigma_{12} + \rho_{23}\sigma_{23}\right], \quad (35)$$

$$s_2 s_2' - (c_1 + c_3)(d_1 + d_3) = \rho_2\sigma_2. \quad (36)$$

Let us transform formula (35) using the identity for the determinant of the sum of three matrices of the 2nd order: $D(S_1 + S_2 + S_3) = D(S_1 + S_2) + D(S_1 + S_3) + D(S_2 + S_3) - D(S_1) - D(S_2) - D(S_3)$ or $\rho_4\sigma_4 = \rho_{12}\sigma_{12} + \rho_{13}\sigma_{13} + \rho_{23}\sigma_{23} - \rho_1\sigma_1 - \rho_2\sigma_2 - \rho_3\sigma_3$. Denote

$$\tau_3 = \rho_{12}\sigma_{12}, \quad \tau_1 = \rho_{23}\sigma_{23}, \tau_2 = \rho_{13}\sigma_{13} = \sum_{k=1}^{4} \rho_k\sigma_k - \tau_1 - \tau_3. \quad (37)$$

Given (37), formula (35) can be written as

$$s_2 = \frac{1}{\sigma_4 - \rho_4}\left[-\rho_4\left(\rho_2 - \sigma_2\right) + \rho_1\sigma_4 + \rho_2\sigma_2 - \rho_{13}\sigma_{13}\right] =$$

$$= \frac{1}{\sigma_4 - \rho_4}\left[\rho_4\left(\sigma_4 - \rho_2 - \sigma_2\right) + \rho_2\sigma_2 - \tau_2\right].$$

Let us transform formula (36), linking constants $c_1,\ c_3,\ d_1,\ d_3$. Denoting $\gamma_k = -\left(s_k - \rho_k\right)\left(s_k - \sigma_k\right)$, and taking into account $c_k d_k = s_k s'_k - \rho_k\sigma_k = -\left(s_k - \rho_k\right)\left(s_k - \sigma_k\right) = \gamma_k$, we rewrite (36) in the form $\gamma_2 - \gamma_1 - c_3 d_1 - c_1 d_3 - \gamma_3 = 0 \quad \Rightarrow$

$$\gamma_2 - \gamma_1 - \gamma_3 - c_3\frac{\gamma_1}{c_1} - c_1\frac{\gamma_3}{c_3} = 0 \quad \left(c_1 \neq 0,\ c_3 \neq 0\right), \tag{38}$$

$$\gamma_2 - \gamma_1 - \gamma_3 - \frac{\gamma_3}{d_3}d_1 - \frac{\gamma_1}{d_1}d_3 = 0 \quad \left(d_1 \neq 0,\ d_3 \neq 0\right). \tag{39}$$

Denoting in (38) $\tau = \frac{c_3}{c_1}$ or in (39) $\tau = \frac{d_3}{d_1}$, we arrive at a quadratic equation with respect to τ:

$$\gamma_1\,\tau^2 + \left(\gamma_1 + \gamma_3 - \gamma_2\,\right)\tau + \gamma_3 = 0. \tag{40}$$

Hence

$$\tau = \frac{1}{2\gamma_1}\left(\gamma_2 - \gamma_1 - \gamma_3 \pm \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 2\left(\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3\right)}\right),$$

for $\gamma_1 \neq 0$ and $\tau = \frac{\gamma_3}{\gamma_2 - \gamma_3}$ for $\gamma_1 = 0$.

Knowing the parameter τ, we can find items $c_k,\ d_k$ matrices $S_k,\ k = 1, 2, 3$.

If $\tau = \frac{c_3}{c_1}$, then $c_1 = c, \quad c_3 = \tau c, \quad c_2 = -\left(c_1 + c_3\right) = -\left(1 + \tau\right)c$, $d_1 = \frac{\gamma_1}{c}, \quad d_2 = -\frac{\gamma_2}{(1+\tau)c}, \quad d_3 = \frac{\gamma_3}{c}$, where c – arbitrary constant, $c \neq 0$.

If $\tau = \frac{d_3}{d_1}$, then $d_1 = d$, $d_3 = \tau d$, $d_2 = -(d_1 + d_3) = -(1+\tau)\, d$, $c_1 = \frac{\gamma_1}{d}$, $c_2 = -\frac{\gamma_2}{(1+\tau)d}$, $d_3 = \frac{\gamma_3}{d}$, when d – arbitrary constant, $d \neq 0$.

Thus, we have the representations of the Jordan form of the logarithm of the product of three non-singular 2nd order matrices:

$$\begin{pmatrix} \rho_4 & 0 \\ 0 & \sigma_4 \end{pmatrix} = \begin{pmatrix} s_1 & c \\ \frac{\gamma_1}{c} & \rho_1 + \sigma_1 - s_1 \end{pmatrix} + \begin{pmatrix} s_2 & -(1+\tau)\, c \\ \frac{-\gamma_2}{(1+\tau)c} & \rho_2 + \sigma_2 - s_2 \end{pmatrix} + \tag{41}$$

$$+ \begin{pmatrix} s_3 & \tau c \\ \frac{\gamma_3}{\tau c} & \rho_3 + \sigma_3 - s_3 \end{pmatrix},$$

$$\begin{pmatrix} \rho_4 & 0 \\ 0 & \sigma_4 \end{pmatrix} = \begin{pmatrix} s_1 & \frac{\gamma_1}{d} \\ d & \rho_1 + \sigma_1 - s_1 \end{pmatrix} + \begin{pmatrix} s_2 & \frac{-\gamma_2}{(1+\tau)d} \\ -(1+\tau)\, d & \rho_2 + \sigma_2 - s_2 \end{pmatrix} + \tag{42}$$

$$+ \begin{pmatrix} s_3 & \frac{\gamma_3}{\tau d} \\ \tau d & \rho_3 + \sigma_3 - s_3 \end{pmatrix},$$

$s_k = \frac{1}{\sigma_4 - \rho_4}\left[\rho_4\left(\sigma_4 - \rho_k - \sigma_k\right) + \rho_k \sigma_k - \tau_k\right]$, $\gamma_k = -\left(s_k - \rho_k\right)\left(s_k - \sigma_k\right)$, τ_k are defined by the formulas (37), τ is determined from equation (40), c and d are arbitrary constants.

Representations (41) and (42), are, in general, equivalent. If the matrix D_4, transforms V_4 to the Jordan form, and any of the matrices V_k $(k = 1, 2, 3)$ are transformed to the triangular form, then $\gamma_k = 0$ and we can choose one of the representations that corresponds to the form of a triangular matrix (upper or lower triangular form).

Both (41) and (42) represent the logarithm of the product of three matrices of the second order as a sum of matrices similar to the logarithms of the matrices of factors. It is unique up to a similarity transformation by a diagonal matrix.

If $\gamma_1 = -(s_1 - \rho_1)(s_1 - \sigma_1) = 0$, then $c_1 d_1 = 0$. The following simpler matrix representations are possible S_4:

$$S_4 = \begin{pmatrix} \rho_1 & 0 \\ \frac{(\gamma_2-\gamma_3)}{c} & \sigma_1 \end{pmatrix} + \begin{pmatrix} s_2 & -c \\ \frac{-\gamma_2}{c} & \rho_2+\sigma_2-s_2 \end{pmatrix} + \begin{pmatrix} s_3 & c \\ \frac{\gamma_3}{c} & \rho_3+\sigma_3-s_3 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} \rho_1 & \frac{\gamma_2-\gamma_3}{d} \\ 0 & \sigma_1 \end{pmatrix} + \begin{pmatrix} s_2 & \frac{-\gamma_2}{d} \\ -d & \rho_2+\sigma_2-s_2 \end{pmatrix} + \begin{pmatrix} s_3 & \frac{\gamma_3}{d} \\ d & \rho_3+\sigma_3-s_3 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} \rho_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} + \begin{pmatrix} s_2 & \frac{-\gamma_3}{d} \\ -d & \rho_2+\sigma_2-s_2 \end{pmatrix} + \begin{pmatrix} s_3 & \frac{\gamma_3}{d} \\ d & \rho_3+\sigma_3-s_3 \end{pmatrix}.$$

Similar representations occur in cases $\gamma_2 = 0$, $\gamma_3 = 0$.

5 CONSTRUCTION OF THE CANONIC MATRIX FOR THE RIEMANN PROBLEM WITH FOUR SINGULAR POINTS

A differential equation of Fuchs class with differential matrices (42) has the form

$$\frac{dY}{dz} = Y \sum_{k=1}^{3} \frac{S_k}{z-a_k} \quad \text{or} \quad \frac{dY}{dz} = Y\, S(z)\,, \tag{43}$$

where $Y(z) = (y_{ij})$, $S(z) = (s_{ij})$ is the 2nd order matrix with elements $s_{11} = \sum_{k=1}^{3} \frac{s_k}{z-a_k}$, $s_{12} = \sum_{k=1}^{3} \frac{c_k}{z-a_k}$, $s_{22} = \sum_{k=1}^{3} \frac{s'_k}{z-a_k} = \sum_{k=1}^{3} \frac{\rho_k-\sigma_k-s_k}{z-a_k}$, $s_{21} = \sum_{k=1}^{3} \frac{d_k}{z-a_k} = \frac{d(a_1-a_2+\tau(a_3-a_2))(z-b)}{\prod_{k=1}^{3}(z-a_k)}$, if $d_k \neq 0$, $k = 1, 2, 3$, and

$$b := \frac{a_3(a_1-a_2) + \tau\, a_1(a_3-a_2)}{a_1 - a_2 + \tau(a_3-a_2)}, \tag{44}$$

where $\tau \neq \frac{a_1-a_2}{a_2-a_3}$.

If $d_1 = 0$, then $d_2 + d_3 = 0$ and $s_{21} = \frac{d_2}{z-a_2} - \frac{d_2}{z-a_3} = \frac{d_2(a_2-a_3)}{(z-a_2)(z-a_3)}$.
If $d_2 = 0$, then $d_1 + d_3 = 0$ and $s_{21} = \frac{d_1(a_1-a_3)}{(z-a_1)(z-a_3)}$.
If $d_3 = 0$, then $d_1 + d_2 = 0$ and $s_{21} = \frac{d_1(a_1-a_2)}{(z-a_1)(z-a_2)}$.
If $\tau = \frac{a_1-a_2}{a_2-a_3}$, then $s_{21} = \frac{d(a_2-a_1)(a_3-a_1)}{(z-a_1)(z-a_2)(z-a_3)}$.

Equation (43) for matrix $Y(z)$ leads to two systems of equations

$$\begin{cases} y'_{11} = s_{11}y_{11} + s_{21}y_{12}, \\ y'_{12} = s_{12}y_{11} + s_{22}y_{12}. \end{cases} \tag{45}$$

$$\begin{cases} y'_{21} = s_{11}y_{21} + s_{21}y_{22}, \\ y'_{22} = s_{12}y_{21} + s_{22}y_{22}. \end{cases} \tag{46}$$

By expressing y_{12} from the first equation of system (45)

$$y_{12} = \frac{1}{s_{21}}\left(y'_{11} - s_{11}y_{11}\right) \tag{47}$$

and substituting it into the second equation, we obtain the differential equation of the 2nd order

$$y'' - \left(s_{11} + s_{22} + \frac{s'_{21}}{s_{21}}\right)y' + \left(s_{11}s_{22} - s_{12}s_{21} - s'_{11} + \frac{s'_{21}}{s_{21}}s_{11}\right)y = 0. \tag{48}$$

From system (46) we conclude that y_{21} is also a solution to equation (48).

If $b \neq a_k$, then $\frac{s'_{21}}{s_{21}} = (\ln s_{21})' = \frac{1}{z-b} - \sum\limits_{k=1}^{3}\frac{1}{z-a_k}$,

$$s_{11}+s_{22}+\frac{s'_{21}}{s_{21}} = \sum_{k=1}^{3}\frac{s_k + s'_k - 1}{z-a_k} + \frac{1}{z-b} = \sum_{k=1}^{3}\frac{\rho_k + \sigma_k - 1}{z-a_k} + \frac{1}{z-b},$$

$$s_{11}s_{22} - s_{12}s_{21} - s'_{11} + \frac{s'_{21}}{s_{21}}s_{11} = \frac{1}{\prod\limits_{k=1}^{3}(z-a_k)}\Big[\rho_4\left(\sigma_4 - 1\right)z + \rho_4 b +$$

$$+\sum_{k=1}^{3}(\rho_k\sigma_k-\tau_k)\,a_k+\frac{q}{z-b}+\sum_{k=1}^{3}\frac{\rho_k\sigma_k}{z-a_k}\prod_{\substack{j=1\\ j\neq k}}^{3}(a_k-a_j)\Big],$$

$q=s_1\,(b-a_2)\,(b-a_3)+s_2\,(b-a_1)\,(b-a_3)+s_3\,(b-a_1)\,(b-a_2)=$ $\frac{1}{\sigma_4-\rho_4}\sum_{k=1}^{3}\left(\rho_4\,(\sigma_4-\rho_k-\sigma_k)+\rho_k\sigma_k-\tau_k\right)\prod\limits_{\substack{j=1\\ j\neq k}}^{3}(b-a_j)\,.$

So the functions y_{11} and y_{12} are the fundamental system of solutions of a differential equation of the Fuchs class of the 2nd order:

$$y''+\left(\sum_{k=1}^{3}\frac{1-\rho_k-\sigma_k}{z-a_k}-\frac{1}{z-b}\right)y'+\frac{1}{\prod\limits_{k=1}^{3}(z-a_k)}\Big[\rho_4\,(\sigma_4-1)\,z+b\rho_4+$$

$$+\sum_{k=1}^{3}(\rho_k\sigma_k-\tau_k)\,a_k+\frac{q}{z-b}+\sum_{k=1}^{3}\frac{\rho_k\sigma_k}{z-a_k}\prod_{\substack{j=1\\ j\neq k}}^{3}(a_k-a_j)\Big]y=0. \quad (49)$$

(49) corresponds to the symbol $P=\begin{pmatrix} a_1 & a_2 & a_3 & b & \infty & \\ \rho_1 & \rho_2 & \rho_3 & 0 & -\rho_4 & z \\ \sigma_1 & \sigma_2 & \sigma_3 & 2 & 1-\sigma_4 & \end{pmatrix}$ (so called Riemann symbo). Denote $\rho=-\rho_4$, $\sigma=1-\sigma_4$, rewrite equation (50) in the "standard" form

$$y''+\left(\sum_{k=1}^{3}\frac{1-\rho_k-\sigma_k}{z-a_k}-\frac{1}{z-b}\right)y'+\frac{1}{\prod\limits_{k=1}^{3}(z-a_k)}\Big[\rho\sigma z-b\rho+$$

$$+\sum_{k=1}^{3}(\rho_k\sigma_k-\tau_k)\,a_k+\frac{q}{z-b}+\sum_{k=1}^{3}\frac{\rho_k\sigma_k}{z-a_k}\prod_{\substack{j=1\\ j\neq k}}^{3}(a_k-a_j)\Big]y=0, \quad (50)$$

$$q = \frac{1}{1+\rho-\sigma} \sum_{k=1}^{3} \left(\rho\left(\sigma+\rho_k+\sigma_k-1\right)+\rho_k\sigma_k-\tau_k\right)\prod_{\substack{j=1\\ j\neq k}}^{3}\left(b-a_j\right), \tag{51}$$

$\rho_1+\sigma_1+\rho_2+\sigma_2+\rho_3+\sigma_3+\rho+\sigma=1$ is the Fuchs relation. As in the case of three singular points, it is proved that all the formulas and equations obtained here remain valid regardless of the form of the Jordan form of the matrix V_4.

Consider some particular cases of equation (50).

If $d_1=0$ ($\gamma_1=0$ and matrix S_1 is upper triangular), then $\frac{s_{21}'}{s_{21}} = \frac{1}{z-a_2}+\frac{1}{z-a_3}$, $s_{11}+s_{22}+\frac{s_{21}'}{s_{21}} = \sum\limits_{k=1}^{3}\frac{\rho_k+\sigma_k}{z-a_k}-\frac{1}{z-a_2}-\frac{1}{z-a_3} = \frac{\rho_1+\sigma_1}{z-a_1}+\frac{\rho_2+\sigma_2-1}{z-a_2}+\frac{\rho_3+\sigma_3-1}{z-a_3}$, i.e. in equation (50) should be put $b=a_1,\ q=\rho_1\left(a_1-a_2\right)\left(a_1-a_3\right)$.

If $d_2=0$ ($\gamma_2=0$), then in equation (50) we assume $b=a_2,\ q=\rho_2\left(a_2-a_1\right)\left(a_2-a_3\right)$.

If $d_3=0$ ($\gamma_3=0$), then in equation (50) we assume $b=a_3,\ q=\rho_3\left(a_3-a_1\right)\left(a_3-a_2\right)$.

If $\tau=\frac{a_1-a_2}{a_2-a_3}$, it is enough to go to the limit in equation (50) at $b\to\infty$, and the differential equation takes the form

$$y''+\sum_{k=1}^{3}\frac{1-\rho_k-\sigma_k}{z-a_k}\,y'+\frac{1}{\prod\limits_{k=1}^{3}\left(z-a_k\right)}\Big[\rho\left(\sigma+1\right)z+$$

$$+\sum_{k=1}^{3}\left(\rho_k\sigma_k-\rho-\tau_k\right)a_k+\sum_{k=1}^{3}\frac{\rho_k\sigma_k}{z-a_k}\prod_{\substack{j=1\\ j\neq k}}^{3}\left(a_k-a_j\right)\Big]y=0.$$

In the neighborhood of each singular point a_k, $k = 1, \ldots, 4$, equation (50) has two linearly independent solutions

$$u_k(z) = (z - a_k)^{\rho_k} \sum_{n=0}^{\infty} B_n^{(k)} (z - a_k)^n, \tag{52}$$

$v_k(z) = (z - a_k)^{\sigma_k} \sum\limits_{n=0}^{\infty} C_n^{(k)} (z - a_k)^n$ at $\rho_k \neq \sigma_k$ or $v_k(z) = \frac{1}{2\pi i} \ln(z - a_k)\ u_k(z) + w_k(z)$ at $\rho_k = \sigma_k$ $(k = 1, 2, 3)$,

$$u_4(z) = z^{-\rho} \sum_{n=0}^{\infty} B_n^{(4)} z^{-n}, \tag{53}$$

$v_4(z) = z^{-\sigma} \sum\limits_{n=0}^{\infty} C_n^{(4)} z^{-n}$ at $\rho \neq \sigma$ or $v_4(z) = -\frac{1}{2\pi i} \ln z\ u_4(z) + w_4(z)$ at $\rho = \sigma$, $w_k(z)$ $(k = 1, \ldots, 4)$ – holomorphic in neighborhoods of points a_k. Coefficients $B_n^{(k)}$, $C_n^{(4)}$ he series (41), (42) after substituting these series in equation (40) are found from the recurrence relations. In the vicinity of the point $z = b$ the equation has 2 linearly independent solutions of the form $u_b(z) = \sum\limits_{n=0}^{\infty} B_n^{(b)} (z - b)^n$ and $v_b(z) = (z - b)^2 \sum\limits_{n=0}^{\infty} C_n^{(b)} (z - b)^n$.

Substituting s_{21} and s_{11} in (47), we obtain $y_{12} = \frac{1}{d\,r} \prod\limits_{k=1}^{3} (z - a_k) \left(y'_{11} - \sum\limits_{k=1}^{3} \frac{s_k}{z - a_k}\, y_{11} \right)$, with $r = a_1 - a_2 + \tau(a_3 - a_2)$. Similarly, $y_{22} = \frac{1}{d\,r} \frac{\prod\limits_{k=1}^{3} (z - a_k)}{z - b} \left(y'_{21} - \sum\limits_{k=1}^{3} \frac{s_k}{z - a_k}\, y_{21} \right)$.

Therefore, if $y_1,\ y_2$ is the fundamental system of solutions to (50), then the solution $Y(z)$ to system (43) can be written as

$$Y(z)=\left(\begin{array}{cc} y_1 & \frac{\prod\limits_{k=1}^{3}(z-a_k)}{z-b}\left(y_1'-\sum\limits_{k=1}^{3}\frac{s_k}{z-a_k}\,y_1\right)\\ y_2 & \frac{\prod\limits_{k=1}^{3}(z-a_k)}{z-b}\left(y_2'-\sum\limits_{k=1}^{3}\frac{s_k}{z-a_k}\,y_2\right)\end{array}\right)\left(\begin{array}{cc}1 & 0\\ 0 & d\,r\end{array}\right)= \tag{54}$$

$$=X(z)\left(\begin{array}{cc}1 & 0\\ 0 & d\,r\end{array}\right).$$

Order p matrix determinant $X(z)$ equals $p=Re\rho+Re\sigma-1$. Order p_1 the first column of the matrix $X(z)$ equals $p_1=\min\{Re\rho,Re\sigma\}=Re\rho$. Because the $s_1+s_2+s_3=\rho_4=-\rho$, that order p_2 second column matrix $X(z)$ equals $p_2=\min\{Re\rho,Re\sigma-1\}=Re\sigma-1$, i.e. $p=p_1+p_2$. Therefore, the matrix $X(z)$ is the canonical matrix of the boundary value problem

$$Y^+(t)=A_kY^-(t)\,,\quad t\in(a_k,a_{k+1})\,,\quad k=1,2,3,\quad a_4=\infty, \tag{55}$$

$$A_1=V_1^{-1},\quad A_2=(V_1\cdot V_2)^{-1}\,,\quad A_3=(V_1\cdot V_2\cdot V_3)^{-1}\,.$$

Using formula (54) we rewrite the differential equation (43) in the form

$$\frac{dX}{dz}=X\,\sum_{k=1}^{3}\frac{U_k}{z-a_k}, \tag{56}$$

$$U_1=\left(\begin{array}{cc}s_1 & \gamma_1 r\\ 1/r & \rho_1+\sigma_1-s_1\end{array}\right),\ U_2=\left(\begin{array}{cc}s_2 & -\gamma_2 r/(1+\tau)\\ -(\tau+1)/r & \rho_2+\sigma_2-s_2\end{array}\right),$$
$$U_3=\left(\begin{array}{cc}s_3 & \gamma_3 r/\tau\\ \tau/r & \rho_3+\sigma_3-s_3\end{array}\right),\quad r=a_1-a_2+\tau(a_3-a_2)\,,\quad r\neq 0.$$

In [12] the matrix $X^*(z) = \begin{pmatrix} y_1 & \frac{\prod_{k=1}^{3}(z-a_k)}{z-b} y_1' + \left(\rho z - \frac{q}{z-b} y_1\right) \\ y_2 & \frac{\prod_{k=1}^{3}(z-a_k)}{z-b} y_2' + \left(\rho z - \frac{q}{z-b} y_1\right) \end{pmatrix}$ is taken as the canonical matrix for the differential equation

$$\frac{dX^*}{dz} = X^* \sum_{k=1}^{3} \frac{U_k^*}{z-a_k} \tag{57}$$

with $U_k^* = \begin{pmatrix} -\eta_k & \frac{(\eta_k+\rho_k)(\eta_k+\sigma_k)}{(b-a_k)\omega_k} \\ (a_k-b)\,\omega_k & \eta_k+\rho_k+\sigma_k \end{pmatrix}$, $\omega_k = \frac{1}{\prod\limits_{\substack{j=1 \\ j\neq k}}^{3}(a_k-a_j)}$, $\eta_k = [\rho a_k(a_k-b)-q]\,\omega_k$, $k=1,2,3$, $U_1^*+U_2^*+U_3^* = \begin{pmatrix} -\rho & \theta \\ 0 & 1-\sigma \end{pmatrix}$, $\theta = \sum_{k=1}^{3} \frac{(\eta_k+\rho_k)(\eta_k+\sigma_k)}{(b-a_k)\,\omega_k}$.

A differential equation of the form (with unknown (accessory) parameters b and q)

$$\begin{gathered} y'' + \left(\sum_{k=1}^{3}\frac{1-\rho_k-\sigma_k}{z-a_k} - \frac{1}{z-b}\right) y' + \\ + \frac{1}{\prod_{k=1}^{3}(z-a_k)}\Bigg[\frac{q}{z-b} - \frac{q^2}{\prod_{k=1}^{3}(b-a_k)} + q\sum_{k=1}^{3}\frac{\rho_k+\sigma_k}{b-a_k} \\ + \sum_{k=1}^{3}\frac{\rho_k\sigma_k}{a_k-b}\prod_{\substack{j=1\\ j\neq k}}^{3}(a_k-a_j) + \sum_{k=1}^{3}\frac{\rho_k\sigma_k}{z-a_k}\prod_{\substack{j=1\\ j\neq k}}^{3}(a_k-a_j)\Bigg] y = 0, \end{gathered} \tag{58}$$

was obtained for functions $y_1,\ y_2$. In [16] formulas are proposed for determining the parameters b and q, which coincide with formulas (44) and (51). Direct calculations show that when replacing $X^*(z) = X(z)\cdot\begin{pmatrix} 1 & \frac{\theta}{\sigma-\rho-1} \\ 0 & 1 \end{pmatrix}$ we arrive at the equation for the function $X(z)$, coinciding with equation (56). Substituting the values q from (51) in equation (58), we obtain equation (50), which confirms the previously obtained results.

Equation (50) can be simplified by replacing $y = \prod\limits_{k=1}^{3}(z-a_k)^{\rho_k} w$. Then the function w will satisfy an equation of the form (50), where it should be replaced

$$\begin{array}{l} \rho_k \to 0,\ \ \sigma_k \to \sigma_k - \rho_k,\ \ \rho \to \rho + \sum\limits_{k=1}^{3} \rho_k = \rho^*,\ \ \sigma \to \sigma + \sum\limits_{k=1}^{3} \rho_k = \sigma^*, \\ \tau_1 \to \tau_1 - (\rho_2+\rho_3)(\sigma_2+\sigma_3) = \tau_1^*,\ \ \tau_2 \to \tau_2 - (\rho_1+\rho_3)(\sigma_1+\sigma_3) = \tau_2^*, \\ \tau_3 \to \tau_3 - (\rho_1+\rho_2)(\sigma_1+\sigma_2) = \tau_3^*, \\ q \to \frac{1}{1+\rho-\sigma} \sum\limits_{k=1}^{3} \left[(\rho+\rho_k)(\sigma-\sigma_k-1) - \tau_k\right] \prod\limits_{\substack{j=1 \\ j\neq k}}^{3} (b-a_j) \ = q^*. \end{array} \tag{59}$$

This equation has then the form

$$w'' + \left(\sum_{k=1}^{3} \frac{1+\rho_k-\sigma_k}{z-a_k} - \frac{1}{z-b}\right) w' + \frac{1}{\prod\limits_{k=1}^{3}(z-a_k)} \Big(\rho^*\sigma^* z - b\,\rho^* - \tag{60}$$

$$- \sum_{k=1}^{3} \tau_k^* a_k + \frac{q^*}{z-b}\Big) w = 0.$$

Therefore, as the canonical matrix of problem (55) in the neighborhood of each singular point $a_k, k = 1, ..., 4$, we can take the matrix

$$X(z) = \prod_{k=1}^{3} (z - a_k)^{\rho_k} D_k \begin{pmatrix} u_k^* & \frac{\prod\limits_{k=1}^{3}(z-a_k)}{z-b} \left(u_k^{*\prime} - \sum\limits_{k=1}^{3} \frac{s_k^*}{z-a_k} u_k^* \right) \\ v_k^* & \frac{\prod\limits_{k=1}^{3}(z-a_k)}{z-b} \left(v_k^{*\prime} - \sum\limits_{k=1}^{3} \frac{s_k^*}{z-a_k} v_k^* \right) \end{pmatrix},$$

where D_k is the matrix transforming V_k to normal Jordan form, u_k^*, v_k^* is the fundamental solution system of equation (60) in the neighborhood of a singular point a_k, whose parameters are found by formulas (59), $s_k^* = \frac{1}{1+\rho-\sigma}\left[(\rho+\rho_k)(\sigma+\sigma_k-1)-\tau_k\right], k = 1, 2, 3$. The homogeneous boundary value problem (55) will be unconditionally solvable if the branches of logarithms ρ_k, σ_k characteristic numbers α_k, β_k matrices $V_k = A_{k-1}A_k^{-1}$, $k = 1, ..., 4$ choose from conditions $-1 < Re\rho_k \leq 0$, $-1 < Re\sigma_k \leq 0$. Then, denoting $\Delta = \sum\limits_{k=1}^{4}(\rho_k + \sigma_k)$ ($\Delta -$ integer, $-7 \leq \Delta \leq 0$), numbers ρ and σ more convenient to find the formula $\rho = \rho_4 + \left[\frac{1-\Delta}{2}\right], \sigma = \sigma_4 + \left[\frac{2-\Delta}{2}\right]$, if $Re\rho_4 \geq Re\sigma_4$,$\rho = \rho_4 + \left[\frac{2-\Delta}{2}\right], \sigma = \sigma_4 + \left[\frac{1-\Delta}{2}\right]$, if $Re\rho_4 \leq Re\sigma_4$. The total index χ of the problem (55), partial indices χ_1, χ_2 and the number ℓ of linearly independent solutions are found by the formulas (22). The gluing together of local solutions is performed in the same way as in the case of three singular points [12].

The method of logarithmization of the product of matrices that we considered was applied in solving one hydrodynamic problem [14] and solving the integral equation of Carleman type on the pair disjoint segments [15]. The scheme of applying this method for the case of five singular points is given in [16], [17].

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