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Characterization of t-affine differences and related forms

Andrzej Olbryś

Abstract. In the present paper we are concerned with the problem of characterization of maps which can be expressed as an affine difference i.e. a map of the form

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y),$$

where $t \in (0,1)$ is a given number. We give a general solution of the functional equation associated with this problem.

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1. Introduction

The well-known cocycle functional equation

$$F(x+y,z) + F(x,y) = F(x,y+z) + F(y,z)$$

and its applications have a long history in connection with many areas of mathematics, as discussed for example in [1,5–7]. It has occurred in different fields, including homological algebra, the Dehan theory of polyhedra, statistics and information theory. The general solution of the cocycle functional equation on abelian groups has been known for about half a century (see [1,6,7]). It turns out that a function $F : G \to H$ (where G and H stand for abelian and a divisible abelian group, respectively) is a solution to the system of functional equations

$$F(x+y,z) + F(x,y) = F(x,y+z) + F(y,z), \quad x,y,z \in G$$
$$F(x,y) = F(y,x), \quad x,y \in G$$

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if and only if the function F is representable as a Cauchy difference i.e. it has the form

$$F(x, y) = f(x) + f(y) - f(x + y), \quad x, y \in G$$

for some one-place function $f : G \to H$. The above characterization have been proved first by Erdös in [6] for real functions, then by Jessen, Karpf and Thourup in [7] on abelian groups. An interesting method for finding the symmetric solutions of the cocycle functional equation on commutative semigroups was given by B. Ebanks in [4].

A similar characterization for differences of the form

$$\Delta(x, y) = f(x) + f(y) - \lambda f(\mu(x + y)),$$

where f is arbitrary function and λ and μ are given parameters has been given by Ebanks in [3]. In particular, he has characterized a Jensen differences

$$\Delta(x, y) = f(x) + f(y) - 2f\left(\frac{x + y}{2}\right)$$

as a special case $\lambda = 2, \mu = \frac{1}{2}$. The above form is called Jensen difference because it vanishes exactly when f is a solution of the Jensen functional equation. In our main result we generalize the following statement which is a particular case of result proved by Bruce Ebanks [3, Corollary 7]

Theorem 1. *Let G be a uniquely 2-divisible abelian group, and let X be a rational vector space. Then a map $\Delta : G \times G \to X$ satisfies conditions:*

(a) $\Delta(x, x) = 0, \quad x \in G,$

(b) $\Delta(x, y) = \Delta(y, x), \quad x, y \in G,$

(c) $\Delta(x, y) + \Delta(z, w) + 2\Delta\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = \Delta(x, z) + \Delta(y, w) + 2\Delta\left(\frac{x+z}{2}, \frac{y+w}{2}\right),$ $x, y, w, z \in G,$

if and only if there exists a function $f : G \to X$ such that

$$\Delta(x, y) = f(x) + f(y) - 2f\left(\frac{x + y}{2}\right), \quad x, y \in G.$$

A key tool we are going to use in the proof of our main result is the following theorem from [5, Theorem 3.3]. This theorem gives a general solution of the most general form of the cocycle functional equation.

Theorem 2. *Let G be an abelian group and X a rational vector space. The general solution $F_i : G \times G \to X$ $(i = 1, \ldots, 6)$ of*

$$F_1(x+y, z)+F_2(y+z, x)+F_3(z+x, y)+F_4(x, y)+F_5(y, z)+F_6(z, x) = 0 \quad (1)$$

is given by

$$F_1(x, y) = A_1(x, y) + f_3(x) - (f_5 + f_7)(y) + f_{10}(x + y) - B_2(x + y, y),$$
$$F_2(x, y) = B_1(x, y) + f_6(x) - (f_1 + f_8)(y) + f_{11}(x + y) - B_2(x + y, y),$$
$$F_3(x, y) = C_1(x, y) + f_9(x) - (f_2 + f_4)(y) - (f_{10} + f_{11})(x + y) + B_2(x + y, x),$$

$$F_4(x,y) = -B_1(y,x) - C_1(x,y) + f_1(x) + f_2(y) - f_3(x+y),$$
$$F_5(x,y) = -C_1(y,x) - A_1(x,y) + f_4(x) + f_5(y) - f_6(x+y),$$
$$F_6(x,y) = -A_1(y,x) - B_1(x,y) + f_7(x) + f_8(y) - f_9(x+y),$$

where $A_1, B_1, C_1 : G\times G \to X$ *are additive in the first variable,* $B_2 : G\times G \to X$ *is additive in its second variable, and* $f_i : G \to X$ $(i = 1,\dots,11)$ *is arbitrary.*

2. Results

Let $t \in (0,1)$ be a fixed number. Throughout this paper X and Y stand for linear spaces over the field $\mathbb{K}$ such that $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$, where $\mathbb{Q}(t)$ is the smallest field containing a singleton $\{t\}$. Clearly, $\mathbb{Q} \subseteq \mathbb{Q}(t)$, where $\mathbb{Q}$ denotes the field of rational numbers. The assumptions about X and Y will not be repeated in the sequel. The purpose of the present paper is to characterize a difference of the form

$$a_f(x,y,t) = tf(x) + (1-t)f(y) - f(tx + (1-t)y),$$

where $t \in (0,1)$ is a given number. Let us recall that a function $f : X \to \mathbb{R}$ is said to be t-convex, t-concave, t-affine if

$$a_f(x,y,t) \geq 0, \quad a_f(x,y,t) \leq 0, \quad a_f(x,y,t) = 0, \quad x,y \in X,$$

respectively. The above difference we will call a t-affine difference because it vanishes exactly when f is a t-affine function (see [8] for more information about t-affine and t-convex functions). It can be easily seen that the t-affine difference has the following two properties:

(i) $a_f(x,x,t) = 0, \quad x \in X,$

(ii) $a_f(x,y,t) = a_f(y,x,1-t), \quad x,y \in X.$

Furthermore, let us observe that a_f satisfies the following functional equation:

$$\begin{aligned}&\text{(iii)}\quad ta_f(u,x,t) + (1-t)a_f(v,y,t) - a_f(tu+(1-t)v, tx+(1-t)y, t)\\ &\quad = ta_f(u,v,t) + (1-t)a_f(x,y,t) - a_f(tu+(1-t)x, tv+(1-t)y, t).\end{aligned}$$

Indeed, for arbitrary $u,x,y,v \in X$ we get

$$\begin{aligned}&ta_f(u,x,t) + (1-t)a_f(v,y,t) - a_f(tu+(1-t)v, tx+(1-t)y, t)\\
&\quad = t\big[tf(u) + (1-t)f(x) - f(tu+(1-t)x)\big]\\
&\qquad +(1-t)\big[tf(v) + (1-t)f(y) - f(tv+(1-t)y)\big]\\
&\qquad -tf(tu+(1-t)v) - (1-t)f(tx+(1-t)y)\\
&\qquad +f(t[tu+(1-t)v] + (1-t)[tx+(1-t)y])\\
&\quad = t\big[tf(u) + (1-t)f(v) - f(tu+(1-t)v)\big]\\
&\qquad +(1-t)\big[tf(x) + (1-t)f(y) - f(tx+(1-t)y)\big]\\
&\qquad -tf(tu+(1-t)x) - (1-t)f(tv+(1-t)y)\end{aligned}$$

$$+f(t[tu+(1-t)x]+(1-t)[tv+(1-t)y])$$
$$= ta_f(u,v,t)+(1-t)a_f(x,y,t)-a_f(tu+(1-t)x,tv+(1-t)y,t).$$

In our main result we show that conditions (i)–(iii) characterize exactly those maps $\omega : X\times X\times\{t,1-t\}\to[0,\infty)$ which can be expressed as a t-affine difference a_f for some t-convex function $f : X\to\mathbb{R}$. In [9] we have proved the following result in this spirit.

Theorem 3. *Let D be a t-convex subset of a real linear space i.e. $tD+(1-t)D\subseteq D$ and let the maps $f,g : D\to\mathbb{R}$ and $\omega : D\times D\times[0,1]\to\mathbb{R}$ satisfy the inequalities*

$$f(tx+(1-t)y)-tf(x)-(1-t)f(y)\le\omega(x,y,t)$$
$$\le g(tx+(1-t)y)-tg(x)-(1-t)g(y),\quad x,y\in D,$$

where

$$g(x)\le f(x),\quad x\in D.$$

Then there exists a function $h : D\to\mathbb{R}$ such that

$$\omega(x,y,t)=a_h(x,y,t),\ x,y\in D\quad\text{and}\quad g(x)\le h(x)\le f(x),\quad x\in D,$$

if and only if for all $x,y,u,v\in D$ the map ω satisfies the functional equation:

$$t\omega(u,x,t)+(1-t)\omega(v,y,t)-\omega(tu+(1-t)v,tx+(1-t)y,t)$$
$$= t\omega(u,v,t)+(1-t)\omega(x,y,t)-\omega(tu+(1-t)x,tv+(1-t)y,t).$$

The following theorem gives a general solution of the functional equation corresponding to equation (iii).

Theorem 4. *The general solution $\omega : X\times X\to Y$ of the functional equation*

$$t\omega(u,x)+(1-t)\omega(v,y)-\omega(tu+(1-t)v,tx+(1-t)y)$$
$$= t\omega(u,v)+(1-t)\omega(x,y)-\omega(tu+(1-t)x,tv+(1-t)y) \tag{2}$$

is given by

$$\omega(x,y)=d(x)+r(y)+tn(x)+(1-t)n(y)-n(tx+(1-t)y)+\bar{c},\quad x,y\in X, \tag{3}$$

where $d,r : X\to Y$ are additive functions satisfying the condition

$$(d+r)(tx)=t(d+r)(x),\quad x\in X, \tag{4}$$

$\bar{c}\in Y$ is an arbitrary constant and $n : X\to Y$ is an arbitrary function.

Proof. It is easy to verify that ω given by (3) with (4) is a solution of (2). Conversely, suppose that (2) holds. Put $y=0$ in (2) to get

$$t\omega(u,x)+(1-t)\omega(v,0)-\omega(tu+(1-t)v,tx)$$
$$= t\omega(u,v)+(1-t)\omega(x,0)-\omega(tu+(1-t)x,tv).$$

This equation can be rewritten in the form

$$0=\omega\left((1-t)\left(x+\frac{t}{1-t}u\right),tv\right)+(1-t)\omega(v,0)+t\omega(u,x)$$
$$-\omega\left((1-t)\left(v+\frac{t}{1-t}u\right),tx\right)-(1-t)\omega(x,0)-t\omega(u,v). \quad (5)$$

Putting in (5) $u=\frac{1-t}{t}y$ we get

$$\begin{aligned}0&=\omega((1-t)(x+y),tv)+(1-t)\omega(v,0)+t\omega\left(\frac{1-t}{t}y,x\right)\\&\quad-\omega((1-t)(v+y),tx)-(1-t)\omega(x,0)-t\omega\left(\frac{1-t}{t}y,v\right)\\&=\omega((1-t)(x+y),tv)+(1-t)\omega(v,0)-[\omega((1-t)(v+y),tx)+(1-t)\omega(x,0)]\\&\quad+t\omega\left(\frac{1-t}{t}y,x\right)-t\omega\left(\frac{1-t}{t}y,v\right).\end{aligned}$$

We can rewrite the above equation in the form

$$F_1(x+y,v)+F_2(y+v,x)+F_3(v+x,y)+F_4(x,y)+F_5(y,v)+F_6(v,x)=0,$$

where,

$$\begin{aligned}F_1(x,y)&=\omega((1-t)x,ty)+(1-t)\omega(y,0),\\F_2(x,y)&=-\omega((1-t)x,ty)-(1-t)\omega(y,0),\\F_3(x,y)&=0,\\F_4(x,y)&=t\omega\left(\frac{1-t}{t}y,x\right),\\F_5(x,y)&=-t\omega\left(\frac{1-t}{t}x,y\right),\\F_6(x,y)&=0.\end{aligned}$$

Choosing the fourth line of the solution in Theorem 2, we see that

$$\omega\left(\frac{1-t}{t}y,x\right)=-\frac{1}{t}B_1(y,x)-\frac{1}{t}C_1(x,y)+\frac{1}{t}f_1(x)+\frac{1}{t}f_2(y)-\frac{1}{t}f_3(x+y),$$

where $B_1,C_1:X\times X\to Y$ are additive functions in the first variable and $f_1,f_2,f_3:X\to Y$ are arbitrary. If we replace $-\frac{1}{t}B_1$ by A_1, $-\frac{1}{t}C_1(y,x)$ by $A_2(x,y)$ and $\frac{1}{t}f_1$, $\frac{1}{t}f_2$, $\frac{1}{t}f_3$ by g,f,h, respectively, we get

$$\omega(x,y)=A_1\left(\frac{t}{1-t}x,y\right)+A_2\left(\frac{t}{1-t}x,y\right)+f\left(\frac{t}{1-t}x\right)$$
$$+g(y)-h\left(y+\frac{t}{1-t}x\right),$$

and finally by putting $\alpha:=\frac{t}{1-t}$ we obtain

$$\omega(x,y)=A_1(\alpha x,y)+A_2(\alpha x,y)+f(\alpha x)+g(y)-h(y+\alpha x), \quad (6)$$

where $A_i : X \times X \to Y$ is an additive map with respect to the i-th variable, for $i = 1, 2$ and $f, g, h : X \to Y$ are arbitrary. Now, we substitute (6) into (2) and obtain after rearrangement

$$\begin{aligned}
&t(A_1(\alpha u, x) - A_1(\alpha u, v)) + (1-t)A_1(\alpha[v-x], y)\\
&\quad + A_1(\alpha[tu + (1-t)x], tv + (1-t)y) - A_1(\alpha[tu + (1-t)v], tx + (1-t)y)\\
&\quad + tA_2(\alpha u, x - v) + (1-t)\left(A_2(\alpha v, y) - A_2(\alpha x, y)\right)\\
&\quad + A_2(\alpha[tu + (1-t)x], tv + (1-t)y) - A_2(\alpha[tu + (1-t)v], tx + (1-t)y)\\
&\ = (1-t)(f(\alpha x) - f(\alpha v)) + f(\alpha[tu + (1-t)v]) - f(\alpha[tu + (1-t)x])\\
&\quad + t(g(v) - g(x)) + g(tx + (1-t)y) - g(tv + (1-t)y)\\
&\quad + t(h(x + \alpha u) - h(v + \alpha u)) + (1-t)(h(y + \alpha v) - h(y + \alpha x)). \qquad (7)
\end{aligned}$$

Put $u = 0$ in (7) to obtain

$$\begin{aligned}
&(1-t)A_1(\alpha[v-x], y) + A_1(tx, tv + (1-t)y) - A_1(tv, tx + (1-t)y)\\
&\quad - tA_2(0, x - v) + (1-t)(A_2(\alpha v, y) - A_2(\alpha x, y))\\
&\quad + A_2(tx, tv + (1-t)y) - A_2(tv, tx + (1-t)y)\\
&\ = (1-t)(f(\alpha x) - f(\alpha v)) + f(tv) - f(tx)\\
&\quad + t(g(v) - g(x)) + g(tx + (1-t)y) - g(tv + (1-t)y)\\
&\quad + t\left(h(x) - h(v)\right) + (1-t)\left(h(y + \alpha v) - h(y + \alpha x)\right). \qquad (8)
\end{aligned}$$

Subtracting (8) from (7) we get

$$\begin{aligned}
&t(A_1(\alpha u, x) - A_1(\alpha u, v)) + A_1(\alpha[tu + (1-t)x], tv + (1-t)y)\\
&\quad - A_1(\alpha[tu + (1-t)v], tx + (1-t)y) - A_1(tx, tv + (1-t)y)\\
&\quad + A_1(tv, tx + (1-t)y) + t\left(A_2(\alpha u, x - v) - A_2(0, x - v)\right)\\
&\quad + A_2(\alpha[tu + (1-t)x], tv + (1-t)y) - A_2(tx, tv + (1-t)y)\\
&\quad - A_2(\alpha[tu + (1-t)v], tx + (1-t)y) + A_2(tv, tx + (1-t)y)\\
&\ = f(\alpha[tu + (1-t)v]) - f(tv) - f(\alpha[tu + (1-t)x]) + f(tx)\\
&\quad + t\left(h(x + \alpha u) - h(x)\right) + t\left(h(v) - h(v + \alpha u)\right). \qquad (9)
\end{aligned}$$

We see that the right hand side of (9) is independent of y. Therefore the left-hand side left unchanged upon setting $y = 0$. That is, after some calculation, we get

$$\begin{aligned}
&A_1(\alpha tu, tv + (1-t)y) - A_1(\alpha tu, tx + (1-t)y) - A_1(\alpha tu, tv) + A_1(\alpha tu, tx)\\
&\ = -A_2(\alpha tu + tx, (1-t)y) + A_2(tx, (1-t)y)\\
&\quad + A_2(\alpha tu + tv, (1-t)y) - A_2(tv, (1-t)y).
\end{aligned}$$

Setting u, v, y, x in the palace of $\alpha tu, tv, (1-t)y$, and tx, respectively, we obtain

$$A_1(u, v + y) - A_1(u, x + y) - A_1(u, v) + A_1(u, x)$$

$$= -A_2(u+x,y) + A_2(x,y) + A_2(u+v,y) - A_2(v,y). \tag{10}$$

From this we see that for arbitrary fixed $u, v, x \in X$ the map

$$y \longrightarrow A_1(u, v+y) - A_1(u, x+y) - A_1(u,v) + A_1(u,x)$$

is additive, consequently

$$A_1(u, v+y+z) - A_1(u, x+y+z) + A_1(u, x+y) + A_1(u, x+z)$$
$$-A_1(u, v+y) - A_1(u, v+z) + A_1(u,v) - A_1(u,x) = 0,$$

for all $u, v, x, y, z \in X$. Putting $z := y$ and $v := x+y$ we get

$$A_1(u, x+3y) - 3A_1(u, x+2y) + 3A_1(u, x+y) - A_1(u,x) = 0,$$

for any $x, y, u \in X$, hence the function $x \longrightarrow A_1(u,x)$ is a polynomial function of second order for arbitrary $u \in X$. Therefore, by a result of Djoković [2], this map can be written as the sum of a constant, an additive map and the diagonalization of a symmetric bi-additive map i.e.

$$A_1(u,x) = a_0(u) + a_1(u,x) + a_2(u,x,x), \quad u, x \in X. \tag{11}$$

It is easy to observe that a_0, a_1, a_2 are additive with respect to the first variable. By a similar argument we deduce that

$$A_2(x,y) = b_0(y) + b_1(x,y) + b_2(x,x,y), \quad x, y \in X \tag{12}$$

where $b_0(y)$ is constant (with respect to x), $x \longrightarrow b_1(x,\cdot)$ is additive, and $x \longrightarrow b_2(x,x,\cdot)$ is a diagonalization of a symmetric bi-additive map. It is not hard to check that b_0, b_1, b_2 are additive with respect to y.

Inserting (11) and (12) back into (10) and simplifying we get

$$a_2(u, v-x, y) = b_2(u, v-x, y), \quad u, v, x, y \in X,$$

hence, $a_2 = b_2$, in particular; a_2 is a 3-additive and symmetric function.

Put $v = 0$ in (9) and rewrite this equation using the representations (11) and (12). After rearrangement we get

$$tc(\alpha u, x) - c(\alpha tu, tx) + t[a_2(\alpha u, x, x) + a_2(\alpha u, \alpha u, x)]$$
$$-a_2(\alpha tu, tx, tx) - a_2(\alpha tu, \alpha tu, tx)$$
$$= th(x+\alpha u) - f(t[x+\alpha u]) + th(0) - f(0)$$
$$-(th(\alpha u) - f(\alpha tu)) - (th(x) - f(tx)),$$

where here and in the sequel $c(x,y) := a_1(x,y) + b_1(x,y)$, $x, y \in X$. Replace in the above equation αu by u and define the functions $b, p : X \to Y$ by the formulas

$$p(x) := th(x) - f(tx), \quad x \in X \tag{13}$$

$$b(x,y) := tc(x,y) - c(tx,ty), \quad x, y \in X, \tag{14}$$

$$\gamma(x,y,z) := ta_2(x,y,z) - a_2(tx,ty,tz), \quad x, y, z \in X, \tag{15}$$

to get the form

$$b(u,x)+\gamma(u,x,x)+\gamma(u,u,x)=p(x+u)+p(0)-p(u)-p(x), \quad u,x\in X. \tag{16}$$

Since the right hand side of (16) is symmetric in x and u, the function b is bi-additive and symmetric. Observe that the function $\delta : X \to Y$ given by the formula

$$\delta(x)=p(x)-\frac{1}{2}b(x,x)-\frac{1}{3}\gamma(x,x,x)-p(0), \quad x\in X, \tag{17}$$

is additive. Indeed, for arbitrary $u,x\in X$ by virtue of (16), bi-additivity and symmetry of b, the 3-additivity and symmetry of γ we have

$$\begin{aligned}\delta(u+x)-\delta(u)-\delta(x)&=p(u+x)-\frac{1}{2}b(u+x,u+x)\\&\quad-\frac{1}{3}\gamma(u+x,u+x,u+x)-p(0)\\&\quad-p(u)+\frac{1}{2}b(u,u)+\frac{1}{3}\gamma(u,u,u)+p(0)-p(x)+\frac{1}{2}b(x,x)\\&\quad+\frac{1}{3}\gamma(x,x,x)+p(0)\\&=p(u+x)-p(u)-p(x)+p(0)-b(u,x)-\gamma(u,u,x)-\gamma(u,x,x)=0.\end{aligned}$$

Therefore on account of (13) and (17) f has the form

$$f(x)=th\left(\frac{x}{t}\right)-\delta\left(\frac{x}{t}\right)-\frac{1}{2}b\left(\frac{x}{t},\frac{x}{t}\right)-\frac{1}{3}\gamma\left(\frac{x}{t},\frac{x}{t},\frac{x}{t}\right)+c_1, \quad x\in X, \tag{18}$$

where $c_1=-p(0)=f(0)-th(0)$.

Now, we return to equation (7). Put $y=0$ in (7) to obtain

$$\begin{aligned}&t(A_1(\alpha u,x)-A_1(\alpha u,v))+(1-t)A_1(\alpha(v-x),0)+A_1(\alpha[tu+(1-t)x],tv)\\&\quad-A_1(\alpha[tu+(1-t)v],tx)+tA_2(\alpha u,x-v)+A_2(\alpha[tu+(1-t)x],tv)\\&\quad-A_2(\alpha[tu+(1-t)v],tx)\\&=(1-t)(f(\alpha x)-f(\alpha v))+f(\alpha[tu+(1-t)v])-f(\alpha[tu+(1-t)x])\\&\quad+t(g(v)-g(x))+g(tx)-g(tv)\\&\quad+t(h(x+\alpha u)-h(v+\alpha u))+(1-t)(h(\alpha v)-h(\alpha x)).\end{aligned}$$

Subtracting the resulting equation from (7), we get

$$\begin{aligned}&(1-t)[A_1(\alpha(v-x),y)-A_1(\alpha(v-x),0)]+A_1(\alpha[tu+(1-t)x],tv+(1-t)y)\\&\quad-A_1(\alpha[tu+(1-t)x],tv)-A_1(\alpha[tu+(1-t)v],tx+(1-t)y)\\&\quad+A_1(\alpha[tu+(1-t)v],tx)+A_2(\alpha[tu+(1-t)x],tv+(1-t)y)\\&\quad-A_2(\alpha[tu+(1-t)x],tv)-A_2(\alpha[tu+(1-t)v],tx+(1-t)y)\\&\quad+A_2(\alpha[tu+(1-t)v],tx)+(1-t)(A_2(\alpha v,y)-A_2(\alpha x,y))\\&=g(tx+(1-t)y)-g(tx)-g(tv+(1-t)y)+g(tv)\\&+(1-t)(h(y+\alpha v)-h(\alpha v))+(1-t)(h(\alpha x)-h(y+\alpha x)).\end{aligned} \tag{19}$$

Put $u = v = 0$ in Eq. (19). Having the forms (11) and (12) in mind we obtain

$$\begin{aligned} & c(tx,(1-t)y)-(1-t)c(\alpha x,y)+a_2(tx,(1-t)y,(1-t)y) \\ & \quad +a_2(tx,tx,(1-t)y)-(1-t)[a_2(\alpha x,y,y)+a_2(\alpha x,\alpha x,y)] \\ & \quad = (1-t)h(\alpha x)-g(tx)+(1-t)h(y)-g((1-t)y) \\ & \qquad -[(1-t)h(\alpha x+y)-g((1-t)(\alpha x+y))]-[(1-t)h(0)-g(0)]. \end{aligned}$$

Replace in the above equation αx by x and define the functions $\bar{b}, l, q : X \to Y$ by the formulas

$$\begin{aligned} q(x) &:= (1-t)h(x)-g((1-t)x), \quad x \in X, \\ \bar{b}(x,y) &:= c((1-t)x,(1-t)y)-(1-t)c(x,y), \quad x,y \in X, \\ l(x,y,z) &:= a_2((1-t)x,(1-t)y,(1-t)z)-(1-t)a_2(x,y,z), \quad x,y,z \in X, \end{aligned}$$

to get the form

$$\bar{b}(x,y)+l(x,x,y)+l(x,y,y) = q(x)+q(y)-q(x+y)-q(0).$$

Similarly as before one can check that the function $k : X \to Y$ given by the formula

$$k(x) = q(x)+\frac{1}{2}\bar{b}(x,x)+\frac{1}{3}l(x,x,x)-q(0),$$

is additive, moreover, we get the representation

$$\begin{aligned} g(x) &= (1-t)h\left(\frac{x}{1-t}\right)-k\left(\frac{x}{1-t}\right) \\ & \quad +\frac{1}{2}\bar{b}\left(\frac{x}{1-t},\frac{x}{1-t}\right)+\frac{1}{3}l\left(\frac{x}{1-t},\frac{x}{1-t},\frac{x}{1-t}\right)+c_2, \end{aligned} \tag{20}$$

where $c_2 = g(0)-(1-t)h(0)$.

Now, put (11), (12), (18) and (20) into Eq. (6) to get

$$\begin{aligned} \omega(x,y) &= A_1(\alpha x,y)+A_2(\alpha x,y)+f(\alpha x)+g(y)-h(y+\alpha x) \\ &= a_0(\alpha x)+a_1(\alpha x,y)+a_2(\alpha x,y,y)+b_0(y)+b_1(\alpha x,y) \\ & \quad +a_2(\alpha x,\alpha x,y)-h(y+\alpha x) \\ & \quad +th\left(\frac{x}{1-t}\right)-\delta\left(\frac{x}{1-t}\right)-\frac{1}{2}b\left(\frac{x}{1-t},\frac{x}{1-t}\right) \\ & \quad -\frac{1}{3}\gamma\left(\frac{x}{1-t},\frac{x}{1-t},\frac{x}{1-t}\right)+c_1 \\ & \quad +(1-t)h\left(\frac{y}{1-t}\right)-k\left(\frac{y}{1-t}\right)+\frac{1}{2}\bar{b}\left(\frac{y}{1-t},\frac{y}{1-t}\right) \\ & \quad +\frac{1}{3}l\left(\frac{y}{1-t},\frac{y}{1-t},\frac{y}{1-t}\right)+c_2 \\ &= a_0(\alpha x)-\delta\left(\frac{x}{1-t}\right)+b_0(y)-k\left(\frac{y}{1-t}\right) \end{aligned}$$

$$
\begin{aligned}
&+c(\alpha x, y) + \bar{c}\\
&+a_2(\alpha x, y, y) + a_2(\alpha x, \alpha x, y)\\
&+th\left(\frac{x}{1-t}\right) + (1-t)h\left(\frac{y}{1-t}\right) - h(\alpha x + y)\\
&-\frac{1}{2}\left[tc\left(\frac{x}{1-t}, \frac{x}{1-t}\right) - c(\alpha x, \alpha x)\right]\\
&-\frac{1}{3}\left[ta_2\left(\frac{x}{1-t}, \frac{x}{1-t}, \frac{x}{1-t}\right) - a_2(\alpha x, \alpha x, \alpha x)\right]\\
&+\frac{1}{2}\left[c(y, y) - (1-t)c\left(\frac{y}{1-t}, \frac{y}{1-t}\right)\right]\\
&+\frac{1}{3}\left[a_2(y, y, y) - (1-t)a_2\left(\frac{y}{1-t}, \frac{y}{1-t}, \frac{y}{1-t}\right)\right]\\
={}& a_0(\alpha x) - \delta\left(\frac{x}{1-t}\right) + b_0(y)\\
&-k\left(\frac{y}{1-t}\right) + c(\alpha x, y) + \bar{c}\\
&+a_2(\alpha x, y, y) + a_2(\alpha x, \alpha x, y)\\
&+th\left(\frac{x}{1-t}\right) + (1-t)h\left(\frac{y}{1-t}\right) - h(\alpha x + y)\\
&-t\left[\frac{1}{2}c\left(\frac{x}{1-t}, \frac{x}{1-t}\right)\right.\\
&\left.+\frac{1}{3}a_2\left(\frac{x}{1-t}, \frac{x}{1-t}, \frac{x}{1-t}\right)\right]\\
&-(1-t)\left[\frac{1}{2}c\left(\frac{y}{1-t}, \frac{y}{1-t}\right)\right.\\
&\left.+\frac{1}{3}a_2\left(\frac{y}{1-t}, \frac{y}{1-t}, \frac{y}{1-t}\right)\right]\\
&+\frac{1}{2}c(\alpha x, \alpha x) + \frac{1}{3}a_2(\alpha x, \alpha x, \alpha x)\\
&+\frac{1}{2}c(y, y) + \frac{1}{3}a_2(y, y, y)\\
={}& a_0(\alpha x) - \delta\left(\frac{x}{1-t}\right) + b_0(y)\\
&-k\left(\frac{y}{1-t}\right) + c(\alpha x, y) + \bar{c} + a_2(\alpha x, y, y) + a_2(\alpha x, \alpha x, y)\\
&-\frac{1}{2}c(\alpha x, y) - \frac{1}{2}c(y, \alpha x)
\end{aligned}
$$

$$
\begin{aligned}
&-a_2(\alpha x, \alpha x, y) - a_2(\alpha x, y, y)\\
&+t\left[h\left(\frac{x}{1-t}\right) - \frac{1}{2}c\left(\frac{x}{1-t}, \frac{x}{1-t}\right)\right.\\
&\left.-\frac{1}{3}a_2\left(\frac{x}{1-t}, \frac{x}{1-t}, \frac{x}{1-t}\right)\right]\\
&+(1-t)\left[h\left(\frac{y}{1-t}\right) - \frac{1}{2}c\left(\frac{y}{1-t}, \frac{y}{1-t}\right)\right.\\
&\left.-\frac{1}{3}a_2\left(\frac{y}{1-t}, \frac{y}{1-t}, \frac{y}{1-t}\right)\right]\\
&-\left[h(y+\alpha x) - \frac{1}{2}c(y+\alpha x, y+\alpha x)\right.\\
&\left.-\frac{1}{3}a_2(y+\alpha x, y+\alpha x, y+\alpha x)\right]\\
=\,& d(x) + r(y) + \frac{1}{2}[c(\alpha x, y) - c(y, \alpha x)]\\
&+tn(x) + (1-t)n(y) - n(tx+(1-t)y) + \bar{c}
\end{aligned}
$$

where

$$
\bar{c} := c_1 + c_2, \qquad d(x) := a_0(\alpha x) - \delta\left(\frac{x}{1-t}\right),
$$

$$
r(x) := b_0(x) - k\left(\frac{x}{1-t}\right), \quad x \in X,
$$

and

$$
\begin{aligned}
n(x) := h\left(\frac{x}{1-t}\right) - \frac{1}{2}c\left(\frac{x}{1-t}, \frac{x}{1-t}\right)\\
-\frac{1}{3}a_2\left(\frac{x}{1-t}, \frac{x}{1-t}, \frac{x}{1-t}\right), \quad x \in X.
\end{aligned}
$$

Finally, set $u = y = 0$ in (2) and substitute the new form of ω into Eq. (2) to obtain after rearrangement

$$
\begin{aligned}
d(t(v-x)) - td(v-x) + r(t(v-x)) - tr(v-x)\\
= c(tv, tx) - c(tx, tv), \quad x, v \in X.
\end{aligned}
$$

Put βx and βv in the place of x and v, respectively, for $\beta \in \mathbb{Q} \setminus \{0\}$. Since any additive function is $\mathbb{Q}$-homogeneous we get

$$
\begin{aligned}
d(t(v-x)) - td(v-x) + r(t(v-x)) - tr(v-x)\\
= \beta[c(tv, tx) - c(tx, tv)], \quad x, v \in X,
\end{aligned}
$$

consequently,

$$
(d+r)(tx) = t(d+r)(x), \quad x \in X,
$$

and hence

$$c(x, v) = c(v, x), \quad x, v \in X.$$

This completes the proof of the theorem. □

Remark 1. It follows from the Proof of Theorem 4 that it is also true in the case where $t \in \mathbb{R} \setminus \{0, 1\}$.

As an immediate consequence of the previous theorem we obtain the following corollary.

Corollary 1. *A map $\omega : X \times X \to Y$ satisfies the functional Eq. (2) and vanishes on the diagonal i.e.*

$$\omega(x, x) = 0, \quad x \in X, \tag{21}$$

if and only if it has the form

$$\omega(x, y) = d(x - y) + a_f(x, y, t), \quad x, y \in X,$$

where $d : X \to Y$ is an additive function and $f : X \to Y$ is an arbitrary map.

Proof. Assume that ω satisfies Eq. (2) and vanishes on the diagonal. On account of Theorem 4 ω has the form

$$\omega(x, y) = d(x) + r(y) + a_f(x, y, t) + \bar{c}, \quad x, y \in X,$$

where $\bar{c} \in Y$ is a constant, $d, r : X \to Y$ are additive and $f : X \to Y$ is arbitrary. Since the t-affine difference vanishes on the diagonal,

$$d(x) + r(x) = -\bar{c}, \quad x \in X.$$

Because $d + r$ is an additive map, by putting $x = 0$ we get $\bar{c} = 0$ and consequently

$$r(x) = -d(x), \quad x \in X.$$

□

In order to present our next result we need some kind of symmetry of ω and this leads us to the consideration of ω as a map of three variables. The following theorem generalizes Corollary 8 from [3] in the case where G is a linear space.

Theorem 5. *A map $\omega : X \times X \times \{t, 1-t\} \to Y$ satisfies the functional equation*

$$\begin{aligned} t\omega(u, x, t) + (1-t)\omega(z, y, t) - \omega(tu + (1-t)z, tx + (1-t)y, t) \\ = t\omega(u, z, t) + (1-t)\omega(x, y, t) - \omega(tu + (1-t)x, tz + (1-t)y, t), \end{aligned} \tag{22}$$

vanishes on the diagonal and is symmetric i.e.

$$\omega(x, y, t) = \omega(y, x, 1-t), \quad x, y \in X, \tag{23}$$

if and only if there exist an additive function $d : X \to Y$ and a map $g : X \to Y$ such that

$$\omega(x, y, s) = d((2s - 1)(x - y)) + a_g(x, y, s), \quad x, y \in X, \ s \in \{t, 1 - t\}. \tag{24}$$

Proof. Obviously, the function ω of the form (24) is symmetric, vanishes on the diagonal and satisfies Eq. (22).

Conversely, assume that ω vanishes on the diagonal and satisfies (22) and (23). According to Corollary 1 there exist functions $g, h : X \to Y$ and additive maps $a, b : X \to Y$ such that

$$\omega(x, y, t) = a(x - y) + a_g(x, y, t), \qquad \omega(x, y, 1 - t) = b(x - y) + a_h(x, y, 1 - t),$$

for $x, y \in X$. Since

$$\omega(x, y, t) = \omega(y, x, 1 - t), \quad x, y \in X,$$

then

$$a(x - y) + b(x - y) = a_{h-g}(x, y, t), \quad x, y \in X.$$

By putting

$$m(x) := a(x) + b(x), \quad p(x) := h(x) - g(x), \quad x \in X,$$

we can rewrite the above equation in the form

$$m(x - y) = tp(x) + (1 - t)p(y) - p(tx + (1 - t)y), \quad x, y \in X.$$

It follows from the above identity and the additivity of m that

$$p(tx + (1 - t)z) + p(tz + (1 - t)y) = p(tx + (1 - t)y) + p(z), \quad x, y, z \in X.$$

Putting $z = 0$ and replacing tx by x and $(1 - t)y$ by y we have

$$p(x) + p(y) = p(x + y) + p(0), \quad x, y \in X,$$

so subtracting $2p(0)$ from the both sides of this equation we get

$$(p(x) - p(0)) + (p(y) - p(0)) = (p(x + y) - p(0)), \quad x, y \in X.$$

Therefore, a function $r : X \to Y$ given by the formula

$$r(x) := p(x) - p(0), \quad x \in X,$$

is additive and consequently,

$$m(x - y) = tr(x - y) - r(t(x - y)), \quad x, y \in X,$$

moreover,

$$h(x) = g(x) + r(x) + \bar{c}, \quad x \in X,$$

where $\bar{c} = p(0)$. Hence we have

$$\begin{aligned} \omega(x, y, 1 - t) &= b(x - y) + a_g(x, y, 1 - t) + a_r(x, y, 1 - t) \\ &= b(x - y) + a_g(x, y, 1 - t) + r(t(x - y)) - tr(x - y) \\ &= b(x - y) - a(x - y) - b(x - y) + a_g(x, y, 1 - t) \end{aligned}$$

$$= a(y-x) + a_g(x,y,1-t).$$

This implies that

$$\omega(x,y,s) = \begin{cases} a(x-y) + a_g(x,y,t), & s = t \\ a(y-x) + a_g(x,y,1-t), & s = 1-t. \end{cases}$$

Finally, defining a new additive function $d : X \to Y$ by the formula

$$d(x) = \begin{cases} a\left(\frac{x}{2t-1}\right), & t \neq \frac{1}{2} \\ 0, & t = \frac{1}{2} \end{cases}$$

we get a desired form of ω

$$\omega(x,y,s) = d((2s-1)(x-y)) + a_g(x,y,s), \quad x,y \in X, \ s \in \{t, 1-t\}.$$

The proof of the theorem is completed. □

Now, we present a particular case of Theorem 9 from [10] (for $D = X$) which we are going to use in our last result.

Theorem 6. *Assume that for some point $y \in X$ a map $\omega : X \times X \times \{t, 1-t\} \to [0,\infty)$ satisfies the following three conditions:*

(a) $\omega(y,y,t) = 0,$
(b) $\omega(x,z,t) = \omega(z,x,1-t),$
(c) $s\omega(u,z,s) + (1-s)\omega(v,z,s) - \omega(su + (1-s)v, z, s) \leq s\omega(u,v,s) - \omega(su + (1-s)z, sv + (1-s)z, s),$

for all $x, z \in X$ and $s \in \{t, 1-t\}$. Then for arbitrary $c \in \mathbb{R}$ there exists a t-concave function $g_y : X \to \mathbb{R}$ such that $g_y(y) = c$, $g_y(x) \leq c$, $x \in X$, and

$$\omega(x,z,t) = g_y(tx + (1-t)z) - tg_y(x) - (1-t)g_y(z), \quad x,z \in X.$$

Our main result reads as follows

Theorem 7. *A map $\omega : X \times X \times \{t, 1-t\} \to [0,\infty)$ is symmetric (i.e. satisfies (23)), vanishes on the diagonal and satisfies the functional Eq. (22) if and only if there exists a t-convex function $f : X \to \mathbb{R}$ such that*

$$\omega(x,y,t) = a_f(x,y,t), \quad x,y \in X.$$

Proof. It is easy to see that an affine difference a_f satisfies conditions (a)–(c) from Theorem 6. Conversely, assume that ω is symmetric, vanishes on the diagonal and satisfies the functional Eq. (22). On account of Theorem 5 there exist a function $h : X \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that

$$\omega(x,y,s) = a((2s-1))(x-y)) + a_h(x,y,s), \quad x,y \in X, \ s \in \{t, 1-t\}.$$

As it can be easily checked the function ω of the above form satisfies conditions (a)–(c) from Theorem 6 then there exists a concave function $g : X \to \mathbb{R}$ such that

$$\omega(x,y,t) = g(tx + (1-t)y) - tg(x) - (1-t)g(y), \quad x,y \in X.$$

Finally, by putting $f := -g$ we see that f is t-convex, moreover,

$$\omega(x, y, t) = a_f(x, y, t), \quad x, y \in X.$$

□

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References

[1] Aczél, J.: The general solution of two functional equations by reduction to functions additive in two variables and with the aid of Hamel bases. Glasnik Mat.-Fiz. Astronom (2) **20**, 65–73 (1965)

[2] Djoković, D.Z.: A representation for $(X_1 - 1)(X_2 - 1) \ldots (X_n - 1)$ and its applications. Ann. Polon. Math. **22**, 189–198 (1969)

[3] Ebanks, B.: Characterization of Jensen differences and related forms. Aequationes Math. **76**, 179–190 (2008)

[4] Ebanks, B.: The cocycle equation on commutative semigroups. Results Math. **67**(1–2), 253–264 (2015)

[5] Ebanks, B.R., Ng, C.T.: On generalized cocycle equation. Aequationes Math. **46**, 76–90 (1993)

[6] Erdös, J.: A remark on the paper "On some functional equations" by Kurepa. Glas. Mat. Fiz. Astron. **2**, 3–5 (1959)

[7] Jessen, B., Karpf, J., Thorup, A.: Some functional equations in groups and rings. Math. Scand. **22**, 257–265 (1958)

[8] Kuhn, N.: A note on t-convex functions, General inequalities, 4 (Oberwolfach, 1983), 269–276, Internat. Schriftenreihe Numer. Math., 71, Birkhäuser, Basel (1984)

[9] Olbryś, A.: A sandwich theorem for generalized convexity and applications. J. Convex Anal. **26**(3), 887–902 (2019)

[10] Olbryś, A.: Support theorem for generalized convexity and its applications. J. Math. Anal. Appl. **458**(2), 1044–1058 (2018)

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