Approximate decomposability in $\wedge^3\mathbb{R}^6$ and the Canonical Decomposition of 3-vectors

John Leventides†, and Nicos Karcanias††

† University of Athens, Department of Economics, Section of Mathematics and Informatics, Pezmazoglou 8, Athens, Greece,
Marie Curie Fellow, City University London, Systems and Control Centre

†† City University London, Systems and Control Centre, Northampton Square, London EC1V OHB, United Kingdom

Key words: Exterior Algebra, Decomposability, Best Decomposable Approximation, Grassmann variety.

Abstract: Given a 3-vector $\underline{z} \in \wedge^3\mathbb{R}^6$ the least distance problem from the Grassmann variety $G_3(\mathbb{R}^6)$ is considered. The solution of this problem is related to a decomposition of $\underline{z}$ into a sum of at most 5 decomposable orthogonal 3-vectors in $\wedge^3\mathbb{R}^6$. This decomposition implies a certain canonical structure for the Grassmann matrix which is a special matrix related to the decomposability properties of $\underline{z}$. This special structure implies the reduction of the problem to a considerably lower dimension tensor space $\otimes^3 R^2$ where the reduced least distance problem can be solved efficiently.

1. Introduction

The Determinantal Assignment Problem (DAP) is an abstract problem formulation unifying the study of frequency assignment problems of linear systems [6]. The solution to this problem is reduced to finding real intersections between the Grassmann variety and a linear variety of a projective space [11]. Computationally, this is an inherently non-linear problem due to its determinantal character, and clearly expresses the significance of exterior algebra and classical algebraic geometry for this family of control problems. The multi-linear nature of DAP has suggested [6] that it may be reduced to a linear problem of zero assignment of polynomial combinants, defining a linear variety, and a standard problem of multi-linear algebra expressed by the additional condition known as decomposability of multi-vectors [12], [13]. The decomposability problem is equivalent to that the multi-vector belongs to the Grassmann variety of the respective projective space [5] and it is thus characterized by the set of Quadratic Plucker Relations (QPR) [12]. An alternative characterisation of decomposability has been introduced by the representation of the decomposable multi-vectors by special structure and properties matrices, the Grassmann Matrices [8], [9].

The DAP framework provides a unifying computational framework for finding the solutions, when such solutions exist, and relies on exterior algebra and on the explicit description of the Grassmann variety in terms of the QPR. This search for exact solutions is equivalent to finding real intersections and this may be interpreted as a zero distance problem distance problem between varieties in the (real) projective e space. Such an interpretation allows the transformation of the exact intersection to a problem of “approximate intersection", i.e., small distance -via a suitable metric- between varieties and transforms the exact DAP from a synthesis method to a DAP design methodology, where approximate solutions to the exact problem are sought. This enables the derivation of solutions, even for non-generic cases and handles problems of model uncertainty, as well as approximate solutions to the cases where generically there is no solution of the exact problem. In [10] the approximate DAP has been considered for the distance from the $G_2(\mathbb{R}^n)$variety and a closed form solution to the distance problem was given based on the skew-symmetric matrix description of multi-vectors via the gap metric. A new algorithm for the calculation of the approximate solution was derived and the stability properties of the approximate DAP solutions were investigated. The study of the general case of distance from the variety $G_m(\mathbb{R}^n)$, $m>n$, is not straightforward; a crucial step to this study is the study of the distance from $G_3(\mathbb{R}^6)$ which is considered here.

In this paper we consider 3-vectors $\underline{z}=\sum_{1\le i<j<k\le 6} z_{ijk}\underline{e}_i\wedge\underline{e}_j\wedge\underline{e}_k\in\wedge^3\mathbb{R}^6$, where $\{\underline{e}_i\}_{i=1}^6$ is an orthonormal basis of $\mathbb{R}^6$. The problem of decomposability of $\underline{z}$ is to find three vectors $\underline{z}_1,\underline{z}_2,\underline{z}_3\in\wedge^3\mathbb{R}^6$ such that $\underline{z}=\underline{z}_1\wedge\underline{z}_2\wedge\underline{z}_3$. If this holds true, the multi-vector is decomposable [12]. Clearly, not all multi-vectors $\underline{z}\in\wedge^3\mathbb{R}^6$ are decomposable and those which are decomposable 3-vectors obey certain algebraic relations the so-called QPR (Quadratic Plucker Relations) which define a projective variety in the projective space $P_1(\wedge^3\mathbb{R}^6)$. This is the Grassmann variety in $P_1(\wedge^3\mathbb{R}^6)=P_1(\mathbb{R}^{20})$ defined as the image of all 3-dimensional subspaces in $\mathbb{R}^6$(the Grasmannian $G(3,\mathbb{R}^6)$) through the Plucker embedding.

When $\underline{z}$ is not decomposable it is desirable in many applications to approximate z by the closest decomposable 3-vector $\underline{z}_0=\underline{z}_1\wedge\underline{z}_2\wedge\underline{z}_3\in\mathbb{R}^6$, i.e. to find $\underline{z}_1,\underline{z}_2,\underline{z}_3\in\wedge^3\mathbb{R}^6$ such that $\left\|\underline{z}-\underline{z}_1\wedge\underline{z}_2\wedge\underline{z}_3\right\|$ is minimized and thus define approximate solutions of the corresponding problem. In the simpler case when $\underline{z}\in\wedge^2\mathbb{R}^n$ the problem has been solved via considering the spectral structure of the matrix $T_{\underline{z}}$ which is the $n\times n$ skew-symmetric matrix representing $\underline{z}_1,\underline{z}_2,\underline{z}_3\in\wedge^3\mathbb{R}^6$ [10]. In the latter case the least distance problem implies a canonical decomposition

$$\underline{z}=\sigma_1\underline{a}_1\wedge\underline{b}_1+\sigma_2\underline{a}_2\wedge\underline{b}_2+\cdots+\sigma_k\underline{a}_k\wedge\underline{b}_k$$

where $k=\left[\frac{n}{2}\right]$ and $\underline{a}_1,\underline{b}_1,\underline{a}_2,\underline{b}_2,\ldots,\underline{a}_k,\underline{b}_k$ is a specially selected orthonormal set. In this paper we consider the case of $\underline{z}\in\wedge^3\mathbb{R}^6$ which is formulated as the optimization problem $\min_{\underline{x}_i\in\mathbb{R}^6}\left\|\underline{z}-\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\right\|$. It is shown that the first order conditions for the related problem $\max\left\langle\underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\right\rangle$, s.t. $\left\|\underline{x}_1\right\|\cdot\left\|\underline{x}_2\right\|\cdot\left\|\underline{x}_3\right\|=1$, imply a selection of an appropriate basis $\underline{x}_1', \underline{x}_2', \underline{x}_3', \underline{y}_1', \underline{y}_2', \underline{y}_3'$ of $\mathbb{R}^6$ such that $\underline{z}$ is written as a sum of at most 5 decomposable multivectors. This decomposition implies a certain diagonal structure for the corresponding Grassmann matrix [8] as well as a certain symmetry of its squared singular values. In fact the squared singular values can be grouped in pairs such that the sum of every pair is the squared norm of the 3-vector $\underline{z}$. Furthermore, it is shown that via this symmetry the problem may be mapped to the lower dimensional vector space $\otimes^3 R^2$ where it can be solved efficiently.

Throughout the paper the following notation is adopted: If $\mathcal{F}$ is a field, or ring then $\mathcal{F}^{m\times n}$ denotes the set of $m\times n$ matrices over $\mathcal{F}$. If H is a map, then $R(H)$, $N_r(H)$, $N_l(H)$ denote the range, right, left nullspaces respectively. $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of k integers from the set $\tilde{n} = \{1, 2, \ldots, n\}$. If V is a vector space and $\{\underline{v}_{i_1}, \ldots, \underline{v}_{i_k}\}$ are vectors of V then $\underline{v}_{i_1} \wedge \ldots \wedge \underline{v}_{i_k} = \underline{v}_{\omega}\wedge,\ \omega = (i_1, \ldots, i_k)$ denotes their exterior product and $\wedge^r V$ the $r-$th exterior power of V [12]. If $H \in \mathcal{F}^{m\times n}$ and $r \le \min\{m, n\}$, then $C_r(H)$ denotes the $r-$th compound matrix of H [13]. In most of the following, we will assume that $\mathcal{F} = \mathbb{R}$.

2. The problem of approximate decomposability

The problem of approximate decomposability (AD) is finding the best approximation of a 3-vector $\underline{z} \in \wedge^3\mathbb{R}^6$ by a decomposable 3-vector $\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3$. This problem has two equivalent formulations in the affine and the projective space settings which are defined below:

Affine space formulation of AD: Solve the optimization problem:

$$\min_{\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3 \in \wedge^3\mathbb{R}^6} \left\| \underline{z} - \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \right\|$$

W

For the Projective space formulation of AD we have to use of a suitable metric in the projective space $P(\wedge^3\mathbb{R}^6)$ such as the gap metric, which is defined as:

$$d([\underline{z}_1],[\underline{z}_2]) = \sqrt{1 - \left(\frac{\langle \underline{z}_1, \underline{z}_2 \rangle}{\|\underline{z}_1\| \|\underline{z}_2\|} \right)^2} \quad [\underline{z}_1],[\underline{z}_2] \in P(\wedge^3\mathbb{R}^6)$$

where $[\underline{z}]$ denotes the line passing through $\underline{z}$ and $\underline{\mathbf{0}}$. Then we may define the projective formulation of AD as

Projective space formulation of AD: Solve the optimization problem:

$$\min_{[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3] \in P(\wedge^3\mathbb{R}^6)} d([\underline{z}],[\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3])$$

Given that the set $\{[\underline{y}]:[\underline{y}]=[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3]\in P(\wedge^3\mathbb{R}^6)\}$ is the image of the Grassmannian $G_3(\mathbb{R}^6)$ through the Plucker embedding [5] and that $G_3(\mathbb{R}^6)$ is compact, we may state the following result:

Theorem(2.1). Let $\underline{z}\in\wedge^3\mathbb{R}^6$ $\underline{z}\neq\underline{0}$ then the projective AD acquires a global minimum which satisfies:

$$\min_{[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3]\in P(\wedge^3\mathbb{R}^6)} d([\underline{z}],[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3])=dist([\underline{z}],G_3(\mathbb{R}^6))$$

Proof:

The distance function $d([\underline{z}],[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3])$ defines a continuous map $G_3(\mathbb{R}^6)\to\mathbb{R}$ and its image is a compact subset of ¡ . Therefore the distance function acquires a global minimum.

W

The relation between the two formulations is now described by the following result:

Proposition(2.1). Let $\underline{z}\in\wedge^3\mathbb{R}^6$ $\underline{z}\neq 0$ and P_{min} be the nonempty set defined by:

$$P_{\min}=\{(m_2,[\underline{x}_1''\wedge\underline{x}_2''\wedge\underline{x}_3'']):(m_2,[\underline{x}_1''\wedge\underline{x}_2''\wedge\underline{x}_3''])\text{ is a global optimum for the projective AD}\}$$

where m_2 is the optimum value of the objective function of the projective AD. Then the set

$$A_{\min}=\{(m_1,\underline{x}_1'\wedge\underline{x}_2'\wedge\underline{x}_3'):(m_1,\underline{x}_1'\wedge\underline{x}_2'\wedge\underline{x}_3')\text{ is a global optimum for the affine AD}\}$$

where m_1 is the optimum value of the objective function of the affine AD, is nonempty. Furthermore, the elements of the two sets P_{min}, A_{min}, $(m_2,[\underline{x}_1''\wedge\underline{x}_2''\wedge\underline{x}_3''])$ and $(m_1,\underline{x}_1'\wedge\underline{x}_2'\wedge\underline{x}_3')$ can be paired so that:

i) $m_1=\|\underline{z}\|m_2$

ii) $\underline{x}_1'\wedge\underline{x}_2'\wedge\underline{x}_3'=\dfrac{\langle\underline{z},\underline{x}_1''\wedge\underline{x}_2''\wedge\underline{x}_3''\rangle}{\|\underline{x}_1''\wedge\underline{x}_2''\wedge\underline{x}_3''\|^2}\underline{x}_1''\wedge\underline{x}_2''\wedge\underline{x}''$

Proof:

We consider the following expansion:

$$\|\underline{z}-\lambda\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|^2=\|\underline{z}\|^2-2\lambda\langle\underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\rangle+\lambda^2\|\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|^2$$

For fixed $\underline{x}_1,\underline{x}_2,\underline{x}_3$ this norm is minimized when $\lambda=\langle\underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\rangle/\|\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|^2$. Therefore the Affine AD minimization problem:

$$\min_{\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\in\wedge^3\mathbb{R}^6}\|\underline{z}-\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|$$

is equivalent to the minimization problem:

$$\min_{\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\in\wedge^3\mathbb{R}^6}\left\|\underline{z}-\frac{\langle \underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\rangle}{\|\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|^2}\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\right\| \qquad (2.1)$$

On the other hand, due to the identity:

$$\left\|\underline{z}-\frac{\langle \underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\rangle}{\|\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|^2}\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\right\|^2=\|\underline{z}\|^2-\left(\frac{\langle \underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\rangle}{\|\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|^2}\right)^2=\|\underline{z}\|^2 d([\underline{z}],[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3])^2$$

The minimization problem (2.1) (which is equivalent to the Affine AD) may be solved via the minimization problem $\min_{[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3]\in P(\wedge^3\mathbb{R}^6)} d([\underline{z}],[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3])$ and as the latter has a global minimum so does the first. This proves that $A_{\min}$ is nonempty. According to the above arguments we may now write:

$$\min_{\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\in\wedge^3\mathbb{R}^6}\|\underline{z}-\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|=\min_{\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\in\wedge^3\mathbb{R}^6}\left\|\underline{z}-\frac{\langle \underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\rangle}{\|\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|^2}\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\right\|= \qquad (2.2)$$
$$=\|\underline{z}\|\min_{[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3]\in P(\wedge^3\mathbb{R}^6)} d([\underline{z}],[\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3])$$

By (2.2), assertion i) is now evident. Furthermore by (2.2), if $[\underline{x}''_1\wedge\underline{x}''_2\wedge\underline{x}''_3]$ is a solution of the projective AD then $\underline{x}'_1\wedge\underline{x}'_2\wedge\underline{x}'_3=\frac{\langle \underline{z},\underline{x}''_1\wedge\underline{x}''_2\wedge\underline{x}''_3\rangle}{\|\underline{x}''_1\wedge\underline{x}''_2\wedge\underline{x}''_3\|^2}\underline{x}''_1\wedge\underline{x}''_2\wedge\underline{x}''_3$ is a solution of the affine AD which is the assertion ii).

W

Definition(2.1) Let $\underline{z}\in\wedge^3\mathbb{R}^6$, $\underline{z}\neq\underline{0}$ and $(m_1,\underline{x}'_1\wedge\underline{x}'_2\wedge\underline{x}'_3)$, $(m_2,[\underline{x}''_1\wedge\underline{x}''_2\wedge\underline{x}''_3])$ be solutions (global optima) of the affine and the projective AD respectively, we call $\underline{x}'_1\wedge\underline{x}'_2\wedge\underline{x}'_3$ the best decomposable approximation of $\underline{z}$ and m_1, m_2 the respective affine and projective distances of $\underline{z}$ from the set of decomposable vectors in $\wedge^3\mathbb{R}^6$ that is the Grassmann variety of the projective space $P_1(\wedge^3\mathbb{R}^6)$.

W

Remark(2.1) The best decomposable approximation may not be unique for example if $\underline{z}=3\underline{e}_1\wedge\underline{e}_2\wedge\underline{e}_3+3\underline{e}_4\wedge\underline{e}_5\wedge\underline{e}_6$, then $P_{\min}=\left\{(\frac{1}{\sqrt{2}},[\underline{e}_1\wedge\underline{e}_2\wedge\underline{e}_3]),(\frac{1}{\sqrt{2}},[\underline{e}_4\wedge\underline{e}_5\wedge\underline{e}_6])\right\}$ and $A_{\min}=\left\{(3,3\underline{e}_1\wedge\underline{e}_2\wedge\underline{e}_3),(3,3\underline{e}_4\wedge\underline{e}_5\wedge\underline{e}_6)\right\}$.

W

In other words there are two decomposable approximations of $\underline{z}$ namely: $3\underline{e}_1\wedge\underline{e}_2\wedge\underline{e}_3$ and $3\underline{e}_4\wedge\underline{e}_5\wedge\underline{e}_6$ which equally approximate $\underline{z}$. Now the projective AD formulation suggests that we may equivalently maximize $\langle \underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\rangle$ given that $\|\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|=1$. Thus we may define the three following constrained maximization problems:

M(1): $\max_{\underline{x}_i\in\mathbb{R}^6}\langle \underline{z},\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\rangle$ subject to $\|\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\|=1$.

M(2): $\max_{\underline{x}_i \in \mathbb{R}^6} \langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle$ subject to $[\underline{x}_1 : \underline{x}_2 : \underline{x}_3]' [\underline{x}_1 : \underline{x}_2 : \underline{x}_3] = I_3$

M(3): $\max_{\underline{x}_i \in \mathbb{R}^6} \langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle$ subject to $\|\underline{x}_1\| \cdot \|\underline{x}_2\| \cdot \|\underline{x}_3\| = 1$

Where $[\underline{x}_1 : \underline{x}_2 : \underline{x}_3]$ is the 3x6 matrix having as columns the vectors $\underline{x}_1, \underline{x}_2, \underline{x}_3$. As explained previously, the maximization problem M(1) is derived from the least distance projective formulation. The problems M(1) and M(3) are both relaxations of problem M(2). More importantly they all share a common solution. Therefore one may prefer to solving M(2) since it is computationally more robust given that we use orthogonal bases.

Proposition(2.2) The three maximization problems M(1),M(2),M(3) attain the same global maximum value m which satisfies:

i) $m_2 = \sqrt{1 - m^2 / \|\underline{z}\|^2}$

ii) $m_1 = \sqrt{\|z\|^2 - m^2}$

Proof: Due to the inequality:

$$\|\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3\| \le \|\underline{x}_1\| \cdot \|\underline{x}_2\| \cdot \|\underline{x}_3\|$$

the problem M(4) defined as:

M(4): $\max_{\underline{x}_i \in \mathbb{R}^6} \langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle$ subject to $\|\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3\| \le 1$

is a relaxation of all M(1), M(2) and M(3). M(4) can be equivalently expressed as

M(5): $\max_{\underline{y} \in \mathbb{R}^{20}} \langle \underline{z}, \underline{y} \rangle$ subject to $\|\underline{y}\| \le 1$ and $QPR(\underline{y}) = 0$.

where QPR($\underline{y}$)=0 denotes the quadratic equations defining decomposability of $\underline{y}$. Problem M(5) is defined on a compact set therefore it attains a global maximum m. We will prove that this is also a global maximum for all M(1), M(2) and M(3). Indeed by a rescaling argument a maximiser of M(4) must be located on the boundary $\|\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3\| = 1$. Furthermore, given a maximizer of M(4), expressed in matrix form by X_0, then $X_0 (X_0' X_0)^{-1/2} = [\underline{x}_1' : \underline{x}_2' : \underline{x}_3']$ satisfies all constraints for M(1),M(2),M(3) and attains the same value m for all objective functions, therefore it is also a maximizer for the three problems. Taking into account we have that:

$$\min_{[\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3] \in P(\wedge^3 \mathbb{R}^6)} d([\underline{z}], [\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3]) = \sqrt{\left(1 - 1/\|\underline{z}\|^2\right) \left(\max_{[\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3] \in P(\wedge^3 \mathbb{R}^6)} \langle z, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle^2 / \|\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3\|^2 \right)}$$

we deduce that the maximum m of M(1) is related to m_2 by i). Furthermore, by Proposition(2.1) and part i) of the current proposition, part ii) readily follows.

The above optimization problems may be solved utilizing known optimization algorithms such as those in [1], [3], or specialized methods for tensor problems as those described in [2], [4], [11], [12]. Additionally since all problems are defined on smooth varieties (constraints) and the objective functions are polynomial (multi-linear), it suffices to solve simultaneously the first order conditions and select those solutions that assign the highest value to the objective function as we know that the global optimum exist for all the problems. However the special skew symmetric and multilinear structure of the problems suggests that we may simplify it by contracting the 3-vector with one of the 1-vectors $\{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$ and then consider an equivalent problem in $\wedge^2\mathbb{R}^6$ which can be solved using matrix theory. In fact the least distance problem in $\wedge^2\mathbb{R}^6$ may be solved by the following theorem:

Theorem(2.2) Let $\underline{z} \in \wedge^2\mathbb{R}^6$ then $\underline{z}$ can always expressed as:

$$\underline{z} = \sigma_1 \underline{a}_1 \wedge \underline{b}_1 + \sigma_2 \underline{a}_2 \wedge \underline{b}_2 + \sigma_3 \underline{a}_3 \wedge \underline{b}_3$$

Where $\{\underline{a}_1, \underline{b}_1, \underline{a}_2, \underline{b}_2, \underline{a}_3, \underline{b}_3\}$ is an orthonormal basis of $\mathbb{R}^6$ and the coefficients s_1, s_2, s_3 are three nonnegative numbers satisfying s_1 ³ s_2 ³ s_3.These numbers are derived from the 3 imaginary eigenvalue pairs $\pm is_1, \pm is_2, \pm is_3$ of the skew symmetric matrix T_z (this matrix has its ij-th element to be z_{ij} if i<j, - z_{ij} if i>j and 0 otherwise) representing the 2-vector $\underline{z}$. In this setting the closest decomposable vector to $\underline{z}$ is given by $\sigma_1 \underline{a}_1 \wedge \underline{b}_1$. Furthermore, the vectors $\underline{a}_1$, $\underline{b}_1$ maximize the bilinear form $< \underline{z}, \underline{x} \wedge \underline{y} >$ with $\|\underline{x}\| . \|\underline{y}\| = 1$ and the maximum value is s_1.

W

The above theorem can be found in [10] as a consequence of Lemma 2.2 p 145 and Corollary 2.2 p.148.

Based on the following definition of the Hodge*-operator which defines a duality in the exterior algebra and can be used as a generalisation of Kernel spaces or as in the present paper for contraction purposes so that the elements of $\wedge^3\mathbb{R}^6$ are viewed as parameterised elements of $\wedge^2\mathbb{R}^6$:

Definition(2) [5]: The Hodge *-operator, for a oriented n-dimensional vector space U equipped with an inner product <.,.>, is an operator defined as: $* : \wedge^m \mathcal{U} \to \wedge^{n-m} \mathcal{U}$ such that $\underline{a}$ $(\underline{b}*)$=<$\underline{a}$,$\underline{b}$> $\underline{w}$ where $\underline{a}$, $\underline{b}$ $\in \wedge^m U$, $\underline{w} \in \wedge^n U$ defines the orientation on U and m<n.

we may apply Theorem (2.2) Using the identity

$$\left\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \right\rangle = \left\langle (\underline{x}_1 \wedge \underline{z}^*)^*, \underline{x}_2 \wedge \underline{x}_3 \right\rangle$$

and we may use Theorem(2.2) for the parameterized 2-vector $z_{\underline{x}_1} = (\underline{x}_1 \wedge \underline{z}^*)^* \in \wedge^2\mathbb{R}^6$

Theorem (2.3) The optimization problem M(2) may be reduced to the following lower dimensional maximization problem:

$$\max_{\underline{x}_1 \in \mathbb{R}^6} x \text{ such that } x^2 - f_2(\underline{x}_1)x + f_4(\underline{x}_1)/4, \quad \|\underline{x}_1\| = 1$$

where $f_2(\underline{x}_1)=\left\|z_{\underline{x}_1}\right\|^2$, $f_4(\underline{x}_1)=\left\|z_{\underline{x}_1}\wedge z_{\underline{x}_1}\right\|^2$. If $(x,\underline{x}_1')$ is a solution of the above then

- **i.** $m=\sqrt{x}$
- **ii.** $(\underline{y}_1',\underline{z}_1')$ can be taken from the complex eigenvector $\underline{y}_1'\pm i\underline{z}_1'$ corresponding to the maximum eigenvalue pair $\pm im$ of $T_{z_{x_1'}}$. The solution of M(2) is then $(m,\underline{x}_1',\underline{y}_1',\underline{z}_1')$.

Proof: Since

$$\underline{x}_1^T T_{z_{\underline{x}_1}}\underline{f}=\left\langle z_{\underline{x}_1},\underline{x}_1\wedge\underline{f}\right\rangle=(\underline{x}_1\wedge\underline{f}\wedge\underline{x}_1\wedge\underline{z}_1^*)^*=0\quad,\forall\underline{f}\in\mathbb{R}^6$$

the matrix $T_{z_{x_1}}$ has non-trivial kernel and as its spectrum is purely imaginary this kernel is at least two dimensional. Therefore the spectrum of $T_{z_{x_1}}$ is of the form $(\pm is_1,\pm is_2,0,0)$. Then according to theorem (2.2)

$$\underline{z}_{\underline{x}_1}=\sigma_1(\underline{x}_1)\underline{\alpha}_1\wedge\underline{b}_1+\sigma_2(\underline{x}_1)\underline{\alpha}_2\wedge\underline{b}_2 \text{ with } s_1 \geq s_2 \geq 0$$

and $\{\underline{a}_1,\underline{b}_1,\underline{a}_2,\underline{b}_2\}$ are orthonormal. Then,

$$\left\|z_{\underline{x}_1}\right\|^2=\sigma_1^2(\underline{x}_1)+\sigma_2^2(\underline{x}_1) \text{ and } \left\|\underline{z}_{\underline{x}_1}\wedge\underline{z}_{\underline{x}_1}\right\|^2=4\sigma_1^2(\underline{x}_1)\sigma_2^2(\underline{x}_1).$$

Therefore $s_1^2(\underline{x}_1)$, $s_2^2(\underline{x}_1)$ satisfy the equation

$$x^2-f_2(\underline{x}_1)x+f_4(\underline{x}_1)/4 \tag{2.3}$$

For a fixed $\underline{x}_1$ the solution of M(2) (according to Theorem 1) is $(\sigma(\underline{x}_1),\underline{x}_1,\underline{\alpha}_1,\underline{\beta}_1)$. Therefore M(2) is equivalent to maximizing $s(\underline{x}_1)$ when $\underline{x}_1\in S^5$ where S^n is the n-dimensional sphere. As $s(\underline{x}_1)$ satisfies (2.3) the result is proved.

W

Corollary (2.1): The study of the problem M(2) can be reduced to

$$\max_{\underline{x}\in\mathbb{R}^6} F(\underline{x})=\left(f_2+\sqrt{f_2^2-f_4}\right)/2 \quad \text{subject to } \|\underline{x}\|=1 \tag{2.4}$$

W

The quadruple $\left\{m,\underline{x}_1',\underline{y}_1',\underline{z}_1'\right\}$ defined in Theorem (2.3) provides the solution to the problem where m is the maximum value of the objective function and $\left\{\underline{x}_1',\underline{y}_1',\underline{z}_1'\right\}$ are the three vectors realizing this maximum value ie the vectors forming the best decomposable approximation which is $m\underline{x}_1'\wedge\underline{y}_1'\wedge\underline{z}_1'$.

Corollary (2.2) The optimal $\underline{x}\in S^5$ may be calculated via a Krylov type of iteration

$$\underline{x}_{n+1}=\nabla F(\underline{x}_n)/\left\|\nabla F(\underline{x}_n)\right\|$$

W

The iteration converges to a vector $\underline{x}'_1$ which is one of the three vectors constituting the best decomposable approximation of $\underline{z}$. The other two are found by applying Theorem (2.3) to the multi-vector $z_{\underline{x}'_1} = (\underline{x}'_1 \wedge \underline{z}^*)^* \in \wedge^2 \mathbb{R}^6$.

Next section describes the first order conditions for M(3) and its implication for a canonical decomposition of a 3-vector.

3. The first order conditions and a canonical decomposition of a 3-vector

Here we will consider the first order conditions for the optimization problem M(3) ie

$$\max_{\underline{x}_i \in \mathbb{R}^6} \left\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \right\rangle \text{ subject to } \|\underline{x}_1\| \cdot \|\underline{x}_2\| \cdot \|\underline{x}_3\| = 1$$

The Lagrangian of this problem is given by

$$L = \left\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \right\rangle - \lambda \left(\|\underline{x}_1\| \cdot \|\underline{x}_2\| \cdot \|\underline{x}_3\| - 1 \right)$$

Theorem(3.1): The First Order Conditions (FOC) for M(3) are given by:

$$\begin{aligned} \nabla_{\underline{x}_1}(L) &: -\left(\underline{z}^* \wedge \underline{x}_2 \wedge \underline{x}_3\right)^* - \lambda (\|\underline{x}_2\| \cdot \|\underline{x}_3\| / \|\underline{x}_1\|).\, \underline{x}_1 = 0 \\ \nabla_{\underline{x}_2}(L) &: \left(\underline{z}^* \wedge \underline{x}_1 \wedge \underline{x}_3\right)^* - \lambda (\|\underline{x}_1\| \cdot \|\underline{x}_3\| / \|\underline{x}_2\|).\, \underline{x}_2 = 0 \qquad (3.1) \\ \nabla_{\underline{x}_3}(L) &: -\left(\underline{z}^* \wedge \underline{x}_1 \wedge \underline{x}_2\right)^* - \lambda (\|\underline{x}_1\| \cdot \|\underline{x}_2\| / \|\underline{x}_3\|).\underline{x}_3 = 0,\ \|\underline{x}_1\| \cdot \|\underline{x}_2\| \cdot \|\underline{x}_3\| = 1 \end{aligned}$$

Proof : These conditions may by deduced from the identities:

$$\left\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \right\rangle = \left\langle -\left(\underline{z}^* \wedge \underline{x}_2 \wedge \underline{x}_3\right)^*, \underline{x}_1 \right\rangle = \left\langle \left(\underline{z}^* \wedge \underline{x}_1 \wedge \underline{x}_3\right)^*, \underline{x}_2 \right\rangle = \left\langle -\left(\underline{z}^* \wedge \underline{x}_1 \wedge \underline{x}_2\right)^*, \underline{x}_3 \right\rangle$$

and taking into account that $\nabla_{\underline{x}_i}(\|\underline{x}_i\|) = \underline{x}_i / \|\underline{x}_i\|$. For example $\nabla_{\underline{x}_1}(L)$ may be calculated as follows:

$$\nabla_{\underline{x}_1}(L) = \nabla_{\underline{x}_1}(\left\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \right\rangle - \lambda \left(\|\underline{x}_1\| \cdot \|\underline{x}_2\| \cdot \|\underline{x}_3\| - 1 \right)) =$$

$$= \nabla_{\underline{x}_1}(\left\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \right\rangle) - \lambda \|\underline{x}_2\| \cdot \|\underline{x}_3\| \nabla_{\underline{x}_1}(\|\underline{x}_1\|) = -\left(\underline{z}^* \wedge \underline{x}_2 \wedge \underline{x}_3\right)^* - \lambda (\|\underline{x}_2\| \cdot \|\underline{x}_3\| / \|\underline{x}_1\|).\, \underline{x}_1$$

□

Proposition(3.1): If l , $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ satisfy the FOC for M3 then we have:

i) $\lambda = \left\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \right\rangle$

ii) $\left\langle \underline{x}_i, \underline{x}_j \right\rangle = 0$ when i ¹ j and when $\lambda \neq 0$

Proof: i) If we apply the inner product by $\underline{x}_1$ both sides of the first FOC we get:

$$\left\langle -\left(\underline{z}^* \wedge \underline{x}_2 \wedge \underline{x}_3\right)^*, \underline{x}_1 \right\rangle = \lambda \|\underline{x}_2\| \cdot \|\underline{x}_3\| / \|\underline{x}_1\| . \left\langle \underline{x}_1, \underline{x}_1 \right\rangle$$

which is equivalent to:

$\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle = \lambda \|\underline{x}_2\| \cdot \|\underline{x}_3\| / \|\underline{x}_1\| . \|\underline{x}_1\|^2$
and this proves i).

ii) If we apply the inner product by $\underline{x}_2$ both sides of the first FOC we get:

$$\left\langle -\left(\underline{z}^* \wedge \underline{x}_2 \wedge \underline{x}_3\right)^*, \underline{x}_2 \right\rangle = \lambda \|\underline{x}_2\| \cdot \|\underline{x}_3\| / \|\underline{x}_1\| . \langle \underline{x}_1, \underline{x}_2 \rangle$$

which is equivalent to:

$$\langle \underline{z}, \underline{x}_2 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle = 0 = \lambda \|\underline{x}_2\| \cdot \|\underline{x}_3\| / \|\underline{x}_1\| . \langle \underline{x}_1, \underline{x}_2 \rangle$$

As $\lambda \neq 0$ we must have $\langle \underline{x}_1, \underline{x}_2 \rangle = 0$. Similarly the other two inner products $\langle \underline{x}_1, \underline{x}_3 \rangle, \langle \underline{x}_2, \underline{x}_3 \rangle$ are zero.

□

Remark (3.1): Based on Proposition (3.1) if 1, $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ satisfy the FOC for M(3) then $\underline{x}_1 / \|\underline{x}_1\|$, $\underline{x}_2 / \|\underline{x}_2\|$, $\underline{x}_3 / \|\underline{x}_3\|$ satisfy the constraints of M(2) and the simplified first order conditions:

$$-\left(\underline{z}^* \wedge \underline{x}_2 \wedge \underline{x}_3\right)^* - \lambda \underline{x}_1 = 0,\ \left(\underline{z}^* \wedge \underline{x}_1 \wedge \underline{x}_3\right)^* - \lambda \underline{x}_2 = 0 \quad -\left(z^* \wedge x_1 \wedge x_2\right)^* - \lambda x_3 = 0 \qquad (3.20)$$

These new conditions are not the FOC for M(2), however the solutions corresponding to the global maximum must coincide. From now on we will consider that $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ are orthonormal and satisfy the above simplified FOC.

W

If we define the annihilator set of an $\underline{a} \in \wedge^3 \mathbb{R}^6$ as $\mathcal{N}_{\underline{a}} = \left\{ \underline{n} \in \wedge^3 \mathbb{R}^6 : \underline{n} \wedge \underline{a} = 0 \right\}$, then

Proposition (3.2) Let $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ be orthonormal vectors that satisfy the simplified FOC (3.2) and let $\underline{y}_1$, $\underline{y}_2$, $\underline{y}_3$ be orthonormal vectors that extend $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ to an oriented orthonormal basis of ¡ ^{6}, and let $\mathcal{N}$ denotes the annihilator set. Then,

$$\left(\underline{z}^* - \lambda \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3\right) \in \mathcal{N}_{\underline{x}_2 \wedge \underline{x}_3} \cap \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_3} \cap \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_2}$$

Proof: Applying the Hodge star operator to the simplified FOC (3.2) we get:

$$\underline{z}^* \wedge \underline{x}_2 \wedge \underline{x}_3 = \lambda \underline{x}_1^* = \lambda \underline{x}_2 \wedge \underline{x}_3 \wedge \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3, \quad -\underline{z}^* \wedge \underline{x}_1 \wedge \underline{x}_3 = \lambda \underline{x}_2^* = -\lambda \underline{x}_1 \wedge \underline{x}_3 \wedge \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

$$\underline{z}^* \wedge \underline{x}_1 \wedge \underline{x}_2 = \lambda \underline{x}_3^* = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

Therefore,

$$\left(\underline{z}^* - \lambda \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3\right) \wedge \underline{x}_2 \wedge \underline{x}_3 = 0,\ \left(\underline{z}^* - \lambda \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3\right) \wedge \underline{x}_1 \wedge \underline{x}_3 = 0$$

$$\left(\underline{z}^* - \lambda \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3\right) \wedge \underline{x}_1 \wedge \underline{x}_2 = 0$$

which proves the result.

□

Lemma (3.1): Let $\underline{a} \in \wedge^m \mathbb{R}^n$ and $\underline{x}_1$, $\underline{x}_2 \in \mathbb{R}^n : \langle \underline{x}_1, \underline{x}_2 \rangle = 0$. Then the following hold true:

i) If $a \wedge x_1 = 0$, then,

$$\underline{a} = \underline{a}_1 \wedge \underline{x}_1, \text{ where, } \underline{a}_1 \in \wedge^{m-1} span(\underline{x}_1)^{\perp}$$

ii) If $\underline{a} \wedge \underline{x}_1 \wedge \underline{x}_2 = 0$, then,

$$\underline{a} = \underline{a}_1 \wedge \underline{x}_1 + \underline{a}_2 \wedge \underline{x}_2, \quad \text{where, } \underline{a}_1 \in \wedge^{m-1} span(\underline{x}_1)^{\perp}, \ \underline{a}_2 \in \wedge^{m-1} span(\underline{x}_2)^{\perp}$$

□

Proof:
Without loss of generality we assume that $\|\underline{x}_2\| = \|\underline{x}_1\| = 1$

i) Consider an orthonormal basis for $\mathbb{R}^n$ whose first vector is $\underline{x}_1$. Expand $\underline{a} \in \wedge^m \mathbb{R}^n$ with respect to this basis. Then
$\underline{a} = \underline{a}_1 \wedge \underline{x}_1 + \underline{a}_2$, where $\underline{a}_1 \in \wedge^{m-1} span(\underline{x}_1)^{\perp}$ and $\underline{a}_2 \in \wedge^m span(\underline{x}_1)^{\perp}$
Since $\underline{a}_1 \wedge \underline{x}_1 = 0$ we must have $\underline{a}_2 \wedge \underline{x}_1 = 0$ implying $\underline{a}_2 = 0$ and hence the result.

ii) Consider an orthonormal basis for $\mathbb{R}^n$ whose first two vectors are x_1, x_2.
Expand $\underline{a} \in \wedge^m \mathbb{R}^n$ with respect to this basis then $\underline{a} = \underline{a}_1 \wedge \underline{x}_1 + \underline{a}_2 \wedge \underline{x}_2 + \underline{a}_3$, where $\underline{a}_1 \in \wedge^{m-1} span(\underline{x}_1)^{\perp}$, $\underline{a}_2 \in \wedge^{m-1} span(\underline{x}_2)^{\perp}$ and $\underline{a}_3 \in \wedge^m span(\underline{x}_1, \underline{x}_2)^{\perp}$

Since $\underline{a} \wedge \underline{x}_1 \wedge \underline{x}_2 = 0$ we must have $\underline{a}_3 \wedge \underline{x}_1 \wedge \underline{x}_2 = 0$ implying $\underline{a}_3 = 0$ and hence the result.

□

Proposition (3.3): The set $\mathcal{N}_{\underline{x}_2 \wedge \underline{x}_3} \cap \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_3} \cap \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_2}$ defined in Proposition(3.2) is given by:

$$\mathcal{N}_{\underline{x}_2 \wedge \underline{x}_3} \cap \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_3} \cap \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_2} =$$
$$\left\{ \underline{y} \wedge \underline{x}_2 \wedge \underline{x}_3 + y' \wedge \underline{x}_1 \wedge \underline{x}_3 + y'' \wedge \underline{x}_1 \wedge \underline{x}_2 + \rho \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 : \underline{y}, \underline{y}', \underline{y}'' \in span\{\underline{y}_1, \underline{y}_2, \underline{y}_3\} \text{ and } \rho \in \mathbb{R} \right\}$$

Proof

Let $\underline{a} \in \wedge^3 \mathbb{R}^n : \underline{a} \in \mathcal{N}_{\underline{x}_2 \wedge \underline{x}_3} \cap \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_3} \cap \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_2}$, then since $\underline{a} \in \mathcal{N}_{\underline{x}_1 \wedge \underline{x}_2}$ by Lemma(3.1)

$$\underline{a} = \underline{a}_1 \wedge \underline{x}_1 + \underline{a}_2 \wedge \underline{x}_2, \quad \underline{a}_1 \in \wedge^2 span(\underline{x}_1)^{\perp}, \ \underline{a}_2 \in \wedge^2 span(\underline{x}_2)^{\perp} \qquad (3.3)$$

Furthermore, we also have that $\underline{a} \wedge \underline{x}_1 \wedge \underline{x}_3 = \underline{0}$; by Lemma(3.1) the following also holds true

$$\underline{a}_2 = \underline{b}_1 \wedge \underline{x}_1 + \underline{b}_2 \wedge \underline{x}_3, \quad \underline{b}_1 \in span(\underline{x}_1, \underline{x}_2)^{\perp}, \quad \underline{b}_2 \in span(\underline{x}_2, \underline{x}_3)^{\perp}$$

Since also $\underline{a} \wedge \underline{x}_2 \wedge \underline{x}_3 = 0$, by Lemma(3.1) we must have

$$a_1 = c_1 \wedge x_1 + c_2 \wedge x_3, \quad c_1 \in span(x_1, x_2)^{\perp}, \quad c_2 \in span(x_2, x_3)^{\perp}$$

Therefore,

$$\underline{b}_1 = d_1 \underline{x}_3 + \underline{y}_1, \ \underline{b}_2 = d_2 \underline{x}_1 + \underline{y}_2, \ \underline{c}_1 = d_3 \underline{x}_3 + \underline{y}_3, \ \underline{c}_2 = d_4 \underline{x}_2 + \underline{y}_4, \quad \underline{y}_1, \underline{y}_2, \underline{y}_3, \underline{y}_4 \in span(\underline{y}_1, \underline{y}_2, \underline{y}_3)$$

Substituting all the above to (3.3) we get that

$$\underline{a} = \underline{y} \wedge \underline{x}_2 \wedge \underline{x}_3 + \underline{y}' \wedge \underline{x}_1 \wedge \underline{x}_3 + \underline{y}'' \wedge \underline{x}_1 \wedge \underline{x}_2 + \rho \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3$$

Conversely if $\underline{a}$ is an element of

$$\left\{\underline{y}\wedge\underline{x}_2\wedge\underline{x}_3+\underline{y}'\wedge\underline{x}_1\wedge\underline{x}_3+\underline{y}''\wedge\underline{x}_1\wedge\underline{x}_2+\rho\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3:\underline{y},\underline{y}',\underline{y}''\in span\{\underline{y}_1,\underline{y}_2,\underline{y}_3\}\text{ and }\rho\in\mathbb{R}\right\}$$

we can easily deduce that $\underline{a}\wedge\underline{x}_1\wedge\underline{x}_2=\underline{a}\wedge\underline{x}_1\wedge\underline{x}_3=\underline{a}\wedge\underline{x}_2\wedge\underline{x}_3=0$

□

Definition(3.1): We define the dot-exterior product "$\dot{\wedge}$" as:

$\left(\wedge^m\mathbb{R}^n\right)^p\times\left(\wedge^k\mathbb{R}^n\right)^p\xrightarrow{\dot{\wedge}}\wedge^{m+k}\mathbb{R}^n$, where

$$(\underline{a}_1,...\underline{a}_p)\dot{\wedge}(\underline{b}_1,...,\underline{b}_p)=\sum_{i=1}^{p}\underline{a}_i\wedge\underline{b}_i$$

and the elements $(\underline{a}_1,...\underline{a}_p)$, $(\underline{b}_1,...,\underline{b}_p)$, $(\underline{a}_1\wedge\underline{b}_1,...\underline{a}_p\wedge\underline{b}_p)$ may be considered as $\begin{pmatrix}n\\m\end{pmatrix}\times p$, $\begin{pmatrix}n\\k\end{pmatrix}\times p$ and $\begin{pmatrix}n\\m+k\end{pmatrix}\times 1$ matrices respectively.

□

With the above definition we may state the following result.

Theorem(3.2): Let $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ and $\underline{y}_1$, $\underline{y}_2$, $\underline{y}_3$ as in Proposition(3.2) then $\underline{z}\in\wedge^3\mathbb{R}^6$ can be expressed as:

$$\underline{z}=\lambda\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3+C_2(Y)\dot{\wedge}(XA)+\rho'\underline{y}_1\wedge\underline{y}_2\wedge\underline{y}_3 \tag{3.4}$$

where $C_2(Y)=[\underline{y}_1\wedge\underline{y}_2,\underline{y}_1\wedge\underline{y}_3,\underline{y}_2\wedge\underline{y}_3]$, $X=[\underline{x}_1:\underline{x}_2:\underline{x}_3]$ and A is a real 3x3 matrix. Furthermore if $\underline{z}$ is expressed as in (3.4) with $\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{y}_1, \underline{y}_2, \underline{y}_3\}$ being an oriented orthonormal basis for $\mathbb{R}^6$, then $\{1, \underline{x}_1, \underline{x}_2, \underline{x}_3\}$ satisfy the simplified FOC for the problem.

Proof: Using Propositions(3.2) and (3.3) , $\underline{z}^*$ can be written as:

$$\begin{aligned}\underline{z}^*=\lambda\underline{y}_1\wedge\underline{y}_2\wedge\underline{y}_3+\underline{x}_1\wedge\underline{x}_2\wedge(a_{11}\underline{y}_1+a_{12}\underline{y}_2+a_{13}\underline{y}_3)+\underline{x}_1\wedge\underline{x}_3\wedge(a_{21}\underline{y}_1+a_{22}\underline{y}_2+a_{23}\underline{y}_3)+\\+\underline{x}_2\wedge\underline{x}_3\wedge(a_{31}\underline{y}_1+a_{32}\underline{y}_2+a_{33}\underline{y}_3)+\rho\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3\end{aligned}$$

Applying the Hodge-star operator both sides and rearranging the terms accordingly

$$\begin{aligned}\underline{z}=\lambda\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3+\underline{y}_1\wedge\underline{y}_2\wedge(a'_{11}\underline{x}_1+a'_{21}\underline{x}_2+a'_{31}\underline{x}_3)+\underline{y}_1\wedge\underline{y}_3\wedge(a'_{12}\underline{x}_1+a'_{22}\underline{x}_2+a'_{32}\underline{x}_3)\\+\underline{y}_2\wedge\underline{y}_3\wedge(a'_{11}\underline{x}_1+a'_{23}\underline{x}_2+a'_{33}\underline{x}_3)+\rho'\underline{y}_1\wedge\underline{y}_2\wedge\underline{y}_3\end{aligned}$$

which in the formulation of definition(3.1), it may be written as the theorem states. Conversely if z can be written as eq(3.4) then it is easy to verify $\{1, \underline{x}_1, \underline{x}_2, \underline{x}_3\}$ satisfy the FOC.

W

The results so far indicate that the first order conditions imply a certain decomposition of $\underline{z}$ in terms of an orthonormal basis of $\mathbb{R}^6$ as a sum of 11 decomposable vectors with coefficients $\lambda,\rho\in\mathbb{R}$ and $A\in\mathbb{R}^{3x3}$. The following results simplify this decomposition into a sum of five decomposable vectors, by transforming appropriately the basis $\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{y}_1, \underline{y}_2, \underline{y}_3\}$. First we state the following Lemmas:

Lemma(3.2): Let A ,B be $\begin{pmatrix} n \\ m \end{pmatrix} \times p$, $\begin{pmatrix} n \\ k \end{pmatrix} \times p$ matrices representing two elements of (Ù^{m} ¡ ^{n})^{p} ,(Ù^{k} ¡ ^{n})^{p} respectively and T be a pxp matrix then the following identity holds true:

$$A \dot{\wedge} (BT) = (AT') \dot{\wedge} B$$

□

Lemma(3.3): Let $U \in O(3)$ be the group of 3x3 orthogonal matrices SO(3) be the group of 3x3 special orthogonal matrices, C_2 be the second order compound matrix and C_3 be the third order compound matrix. Then the following hold true:

i) $C_2(U) \in SO(3)$

ii) $C_2(C_2(U)) = \det(U).U$

Proof:

i) Based on Sylvester-Franke theorem we have $\det(C_2(U)) = \det(U)^2 = 1$

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

ii) $C_2(U) = \det(U).JU^T J$ where

$$C_2(C_2(U)) = \det(C_2(U)).JC_2(U^T)J = JC_2(U^T)J = \det(U)J^2UJ^2 = \det(U).U$$

□

Theorem (3.3). Let $\underline{z} \in \wedge^3 \mathbb{R}^6$ which can be expressed as in theorem (3.2):

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + C_2(Y) \dot{\wedge} (XA) + \rho \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

If $U, V \in O(3)$ then $\underline{z}$ can also be written as:

$$\underline{z} = \lambda \det(U) \underline{x}_1' \wedge \underline{x}_2' \wedge \underline{x}_3' + \det(V) C_2(Y') \dot{\wedge} (X'A') + \rho' \underline{y}_1' \wedge \underline{y}_2' \wedge \underline{y}_3'$$

where

$X' = [\underline{x}_1', \underline{x}_2', \underline{x}_3'] = [\underline{x}_1, \underline{x}_2, \underline{x}_3]\mathrm{U} = \mathrm{XU}$, $\mathrm{Y}' = [\underline{y}_1', \underline{y}_2', \underline{y}_3'] = [\underline{y}_1, \underline{y}_2, \underline{y}_3] C_2(V) = YC_2(\mathrm{V})$ and $A' = U^t AV$

Proof: Note that

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + C_2(Y) \dot{\wedge} (XUU^T AVV^T) + \rho' \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

by Lemma (3.2) we have:

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + C_2(Y)V \dot{\wedge} (XUU^T AV) + \rho' \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

by Lemma (3.3) we get:

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \det(V) C_2(Y) C_2(C_2(V)) \dot{\wedge} (XUU^T AV) + \rho' \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3 =$$
$$= \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \det(V) C_2(Y C_2(V)) \dot{\wedge} (XUU^T AV) + \rho' \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3 =$$
$$= \lambda \det(U) \underline{x}_1' \wedge \underline{x}_2' \wedge \underline{x}_3' + \det(V) C_2(Y') \dot{\wedge} (X'A') + \rho' \underline{y}_1' \wedge \underline{y}_2' \wedge \underline{y}_3'$$

□

As a result of the previous theorem and the singular value decomposition of the matrix A we may simplify the decomposition of $\underline{z}$ as a sum of 5 decomposable vectors as shown below.

Theorem(3.4) For any $\underline{z} \in \wedge^3 \mathbb{R}^6$ there exists an oriented orthonormal basis of $\mathbb{R}^6$ $\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{y}_1, \underline{y}_2, \underline{y}_3\}$ such that

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \lambda_2 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + \lambda_3 \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 + \lambda_4 \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 + \lambda_5 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3 \quad (3.5)$$

Proof: We start with a decomposition $\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + C_2(Y) \dot{\wedge} (XA) + \rho' \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$ arising from the FOC. Consider now the singular value decomposition of A: $A = U \cdot \Sigma \cdot V^T$ and denote the sign of $\det(A)$ as $\text{sign}(A)$. Then A can be written as:

$$A = U_1 \cdot (sign(A)\Sigma) \cdot V_1^T$$

where U_1, V_1 orthogonal and $\det(U_1) = \det(V_1) = 1$. By applying Theorem (3.3) we get:

$$\underline{z} = \lambda \underline{x}_1' \wedge \underline{x}_2' \wedge \underline{x}_3' + sign(A) C_2(Y') \dot{\wedge} (X'\Sigma) + \rho' \underline{y}_1' \wedge \underline{y}_2' \wedge \underline{y}_3'$$

Where $X' = [\underline{x}_1', \underline{x}_2', \underline{x}_3'] = [\underline{x}_1, \underline{x}_2, \underline{x}_3] U_1 = XU_1$, $Y' = [\underline{y}_1', \underline{y}_2', \underline{y}_3'] = [\underline{y}_1, \underline{y}_2, \underline{y}_3] C_2(V_1) = Y C_2(V_1)$

Note that

$$sign(A) C_2(Y') \dot{\wedge} (X'\Sigma) = sign(A) \left(\sigma_3 \underline{y}_1' \wedge \underline{y}_2' \wedge \underline{x}_1' + \sigma_2 \underline{y}_1' \wedge \underline{y}_3' \wedge \underline{x}_2' + \sigma_1 \underline{y}_2' \wedge \underline{y}_3' \wedge \underline{x}_3' \right)$$

Therefore $\underline{z}$ is written as:

$$\underline{z} = \lambda \underline{x}_1' \wedge \underline{x}_2' \wedge \underline{x}_3' + sign(A) \left(\sigma_3 \underline{y}_1' \wedge \underline{y}_2' \wedge \underline{x}_1' + \sigma_2 \underline{y}_1' \wedge \underline{y}_3' \wedge \underline{x}_2' + \sigma_1 \underline{y}_2' \wedge \underline{y}_3' \wedge \underline{x}_3' \right) + \rho' \underline{y}_1' \wedge \underline{y}_2' \wedge \underline{y}_3'$$

□

Remark(3.2) This decomposition of z into 5 decomposable vectors is not unique as it can be formed for every $\lambda, \underline{x}_1, \underline{x}_2, \underline{x}_3$ satisfying the FOC. Furthermore if we fix one solution of FOC then the SVD of the matrix A may not be unique (the case of multiple singular values). To get a canonical decomposition we impose two requirements, first to fix one solution of FOC (this can be done be choosing the global maximizer, provided it is unique) and second the corresponding matrix A to have a unique SVD (there are no multiplicities); Conditions for this to happen in terms of z are derived in the following chapters.

□

Corollary(3.1): For any $\underline{z} \in \wedge^3\mathbb{R}^6, \underline{z} \neq 0$ there exists an oriented orthonormal basis of $\mathbb{R}^6$ $\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{y}_1, \underline{y}_2, \underline{y}_3\}$such that

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \lambda_2 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + \lambda_3 \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 + \lambda_4 \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 + \lambda_5 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3 \quad (3.6)$$

with $\lambda_1^2 \geq \lambda_2^2 + \lambda_5^2$ and $|\lambda_2| \geq |\lambda_3| \geq |\lambda_4|$.

Proof: Construct $l_i, \underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{y}_1, \underline{y}_2, \underline{y}_3$ as previously for the solution of the FOC corresponding to the global maximum of M(3). This way $\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3$ has been constructed so λ is the maximum of the objective function $\langle \underline{z}, \underline{k}_1 \wedge \underline{k}_2 \wedge \underline{k}_3 \rangle$ with $\|\underline{k}_1\|\|\underline{k}_2\|\|\underline{k}_3\| = 1$. Thus we have that

$$\lambda_1 = \langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle \geq \left| \frac{\langle \underline{z}, \underline{y}_1 \wedge \underline{y}_2 \wedge (\lambda_2 \underline{x}_1 + \lambda_5 \underline{y}_3) \rangle}{\sqrt{\lambda_2^2 + \lambda_5^2}} \right| = \sqrt{\lambda_2^2 + \lambda_5^2}$$

And this proves the first inequality. The second inequality is proved by the fact that the three numbers λ_2, λ_3, λ_4 are related to the three singular values of the matrix A in descending order.

□

A consequence of Corollary(3.1) is the following:

Corollary(3.2) For any $\underline{z} \in \wedge^3\mathbb{R}^6, \underline{z} \neq 0$ there exists an oriented orthonormal basis of $\mathbb{R}^6$ $\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{y}_1, \underline{y}_2, \underline{y}_3\}$ such that

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \lambda_2 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + \lambda_3 \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 + \lambda_4 \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 + \lambda_5 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

with $\lambda_1 \geq \max\{|\lambda_2|, |\lambda_3|, |\lambda_4|, |\lambda_5|\}$ and $|\lambda_2| \geq |\lambda_3| \geq |\lambda_4|$.

□

Proof:
As $\lambda_1^2 \geq \lambda_2^2 + \lambda_5^2$, $|\lambda_2| \geq |\lambda_3| \geq |\lambda_4|$ we can easily deduce that $\lambda_1 \geq \max\{|\lambda_2|, |\lambda_3|, |\lambda_4|, |\lambda_5|\}$.

□

The above discussion and results imply the following algorithm for the construction of a canonical decomposition:

Construction of Canonical Decomposition

- **i)** Construct the global maximize λ, $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ of M(3)
- **ii)** Complete $\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{y}_1, \underline{y}_2, \underline{y}_3\}$ to be an oriented orthonormal basis for $\mathbb{R}^6$.
- **iii)** Develop an expression of z of the form

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + C_2(Y) \dot{\wedge} (XA) + \rho' \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

This can be done by multiplying $\underline{z}^t$ by $C_3([X, Y])$. The eleven nonzero coordinates are the coefficients of the above expression.

iv) Calculate the SVD of A: $A = U \cdot \Sigma \cdot V^T$

v) Calculate a new basis $[X', Y']$ such that $X' = \det(U) XU$, $Y' = YC_2(V)$

vi) Calculate an expression of the form

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \lambda_2 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + \lambda_3 \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 + \lambda_4 \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 + \lambda_5 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

This can be done by the five non zero terms of the expression $\underline{z}^t C_3([X', Y'])$

□

Example(3.1): Let

$\underline{z} = [5,-3,-8,-7,-4,-10,4,8,-2,-4,-7,-9,-4,2,-5,-3,4,-2,-9,6]^T$. Then solving the maximization problem: $\max \langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle$ s.t. $\|\underline{x}_1\| \cdot \|\underline{x}_2\| \cdot \|\underline{x}_3\| = 1$, we get:

$$X = [\underline{x}_1 : \underline{x}_2 : \underline{x}_3] = \begin{bmatrix} 0.3628 & 0.461 & 0.556 & -0.4727 & -0.3517 & 0.00032 \\ 0.4426 & -0.1127 & -0.5356 & 0.0458 & -0.5957 & 0.384 \\ 0.4645 & 0.4637 & -0.0463 & 0.3374 & 0.5630 & 0.3691 \end{bmatrix}^T$$

with maximum value $\lambda = 23.0209$. Construct Y^t as the left null space of X ie

$$Y = [\underline{y}_1 : \underline{y}_2 : \underline{y}_3] = \begin{bmatrix} 0.133 & -0.153 & 0.4892 & 0.773 & -0.3292 & -0.1186 \\ 0.492 & -0.726 & 0.2606 & -0.2494 & 0.3056 & 0.0868 \\ -0.444 & -0.1012 & 0.3078 & -0.0327 & -0.0521 & 0.8333 \end{bmatrix}^T$$

then $\underline{z}$ is written as

$$\begin{aligned} \underline{z} = {} & 23.0209 \cdot \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + 5.811 \cdot \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + 0.483 \cdot \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_2 - 2.664 \cdot \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_3 \\ & -0.446 \cdot \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_1 + 1.818 \cdot \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 - 1.063 \cdot \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_3 \\ & -3.087 \cdot \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_1 + 4.3934 \cdot \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_2 + 7.643 \cdot \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 \\ & -4.589 \cdot \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3 \end{aligned}$$

Next, we calculate a Singular Value Decomposition for the matrix A:

$$A = \begin{bmatrix} 5.811 & 0.483 & -2.664 \\ -0.446 & 1.818 & -1.063 \\ -3.087 & 4.393 & 7.643 \end{bmatrix}$$

$$A = U \cdot D \cdot V^T = \begin{bmatrix} -0.48 & -0.873 & 0.079 \\ 0.005 & -0.094 & -0.996 \\ 0.877 & -0.478 & 0.05 \end{bmatrix} \cdot \begin{bmatrix} 10.348 & 0 & 0 \\ 0 & 4.627 & 0 \\ 0 & 0 & 2.12 \end{bmatrix} \cdot \begin{bmatrix} -0.532 & -0.769 & 0.355 \\ 0.351 & -0.582 & -0.734 \\ 0.771 & -0.265 & 0.579 \end{bmatrix}^T$$

We change the basis for $\text{colspan}(X)$, $\text{colspan}(Y)$ to X', Y' as follows:

$$X' = X\cdot V = \begin{bmatrix} 0.3203 & 0.073 & -0.519 & 0.528 & 0.412 & 0.418 \\ -0.660 & -0.412 & -0.1035 & 0.247 & 0.467 & -0.324 \\ 0.0731 & 0.515 & 0.564 & -0.0062 & 0.638 & -0.067 \end{bmatrix}^{\mathrm{T}} = [\underline{x}_1' : \underline{x}_2' : \underline{x}_3']$$

$$Y = Y\cdot C_2(U) = \begin{bmatrix} 0.461 & -0.738 & 0.308 & -0.213 & 0.284 & 0.147 \\ -0.37 & -0.0936 & 0.478 & 0.364 & -0.231 & 0.663 \\ 0.327 & -0.081 & 0.28 & 0.695 & -0.265 & -0.505 \end{bmatrix}^{\mathrm{T}} = [\underline{y}_1' : \underline{y}_2' : \underline{y}_3']$$

With respect to the new basis, $\underline{z}$ can be written as

$$\underline{z} = 23.0209\cdot \underline{x}_1' \wedge \underline{x}_2' \wedge \underline{x}_3' + 10.348\cdot \underline{y}_1' \wedge \underline{y}_2' \wedge \underline{x}_1' + 4.627\cdot \underline{y}_1' \wedge \underline{y}_3' \wedge \underline{x}_2' + 2.12\cdot \underline{y}_2' \wedge \underline{y}_3' \wedge \underline{x}_3' \\ -4.589\cdot \underline{y}_1' \wedge \underline{y}_2' \wedge \underline{y}_3'$$

□

4. The structure of the Grassmann matrix

The Grassmann matrix is the matrix representation of the multiplication operator ($\wedge$) [8], [9] that is: $\Phi_6^3(\underline{z}):\mathbb{R}^6 \to \wedge^4\mathbb{R}^6$ defined by $\Phi_6^3(\underline{z})\underline{u} = \underline{u}\wedge\underline{z}$. The transpose of this matrix $\mathrm{F}_6^3(\underline{z})^{\mathrm{T}}$ is the matrix representation of the operator

$$\Phi_6^3(\underline{z})^{\mathrm{T}} : \wedge^4\mathbb{R}^6 \to \mathbb{R}^6 \text{such that} \quad \Phi_6^3(\underline{z})^{\mathrm{T}}\underline{y} = -(\underline{y}^* \wedge \underline{z})^*$$

The singular values of this matrix are related to the decomposability properties of $\underline{z}$ [10]. To establish this we will examine the spectral properties of the operator

$$\Phi = (\Phi_6^3(\underline{z}))^{\mathrm{T}}\Phi_6^3(\underline{z}):\mathbb{R}^6 \to \mathbb{R}^6 \text{ where } \Phi_6^3(\underline{z})^{\mathrm{T}}\Phi_6^3(\underline{z})\underline{u} = -((\underline{u}\wedge\underline{z})^*\wedge\underline{z})^*$$

Theorem(4.1): Let

$$\underline{z} = \lambda\underline{x}_1\wedge\underline{x}_2\wedge\underline{x}_3 + \lambda_2\underline{y}_1\wedge\underline{y}_2\wedge\underline{x}_1 + \lambda_3\underline{y}_1\wedge\underline{y}_3\wedge\underline{x}_2 + \lambda_4\underline{y}_2\wedge\underline{y}_3\wedge\underline{x}_3 + \lambda_5\underline{y}_1\wedge\underline{y}_2\wedge\underline{y}_3$$

be the decomposition of $\underline{z}$ as in Theorem (3.4). Then consider the 2-dim spaces:

$$V_1 = \mathrm{span}(\underline{x}_1, \underline{y}_3),\ V_2 = \mathrm{span}(\underline{x}_2, \underline{y}_2),\ V_3 = \mathrm{span}(\underline{x}_3, \underline{y}_1)$$

These three subspaces are Φ-invariant and $\mathcal{V}_1\oplus\mathcal{V}_2\oplus\mathcal{V}_3$ is an orthogonal decomposition of $\mathbb{R}^6$. The restriction on F on these subspaces has the following representation:

$$\Phi/\mathcal{V}_1 = \begin{bmatrix} \lambda_3^2+\lambda_4^2+\lambda_5^2 & -\lambda_2\lambda_5 \\ -\lambda_2\lambda_5 & \lambda_1^2+\lambda_2^2 \end{bmatrix},\ \Phi/\mathcal{V}_2 = \begin{bmatrix} \lambda_2^2+\lambda_4^2+\lambda_5^2 & \lambda_3\lambda_5 \\ \lambda_3\lambda_5 & \lambda_1^2+\lambda_3^2 \end{bmatrix},$$

$$\Phi/\mathcal{V}_3 = \begin{bmatrix} \lambda_2^2+\lambda_3^2+\lambda_5^2 & -\lambda_4\lambda_5 \\ -\lambda_4\lambda_5 & \lambda_1^2+\lambda_4^2 \end{bmatrix}$$

□

Proof: We calculate the value of F on the basis of $\mathcal{V}_1$ ie $\{\underline{x}_1, \underline{y}_3\}$

$$(\underline{x}_1 \wedge \underline{z})^* = \lambda_3(\underline{x}_1 \wedge \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2)^* + \lambda_4(\underline{x}_1 \wedge \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3)^* + \lambda_5(\underline{x}_1 \wedge \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3)^*$$
$$= -\lambda_3 \underline{x}_3 \wedge \underline{y}_2 - \lambda_4 \underline{x}_2 \wedge \underline{y}_1 + \lambda_5 \underline{x}_2 \wedge \underline{x}_3$$

$$((\underline{x}_1 \wedge \underline{z})^* \wedge \underline{z})^* = \lambda_3^2(\underline{x}_3 \wedge \underline{y}_2 \wedge \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2)^* - \lambda_4^2(\underline{x}_2 \wedge \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3)^*$$
$$+\lambda_5^2(\underline{x}_3 \wedge \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_2)^* + \lambda_5\lambda_2(x_2 \wedge x_3 \wedge \underline{y}_1 \wedge \underline{y}_2 \wedge x_1)^*$$
$$= -\lambda_3^2 \underline{x}_1 - \lambda_4^2 \underline{x}_1 - \lambda_5^2 \underline{x}_1 + \lambda_2\lambda_5 \underline{y}_3$$

Therefore,

$$\Phi \cdot \underline{x}_1 = (\lambda_3^2 + \lambda_4^2 + \lambda_5^2)\underline{x}_1 - \lambda_2\lambda_5 \underline{y}_3$$

Additionally we have,

$$(\underline{y}_3 \wedge \underline{z})^* = \lambda_1(\underline{y}_3 \wedge \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3)^* + \lambda_2(\underline{y}_3 \wedge \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1)^*$$
$$= -\lambda_1 \underline{y}_1 \wedge \underline{y}_2 - \lambda_2 \underline{x}_2 \wedge \underline{x}_3$$

$$((y_3 \wedge z)^* \wedge z)^* = \lambda_1^2(y_1 \wedge y_2 \wedge x_1 \wedge x_2 \wedge x_3)^* - \lambda_2^2(x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge x_1)^*$$
$$-\lambda_2\lambda_5(x_1 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3)^*$$
$$= -\lambda_1^2 y_3 - \lambda_2^2 y_3 + \lambda_2\lambda_5 x_1$$

Therefore,

$$\Phi \cdot \underline{y}_3 = (\lambda_1^2 + \lambda_2^2)\underline{y}_3 - \lambda_2\lambda_5 \underline{x}_1$$

Proving that V_1 is Φ-invariant and

$$\Phi / \mathcal{V}_1 = \begin{bmatrix} \lambda_3^2 + \lambda_4^2 + \lambda_5^2 & -\lambda_2\lambda_5 \\ -\lambda_2\lambda_5 & \lambda_1^2 + \lambda_2^2 \end{bmatrix}$$

The proof is similar for the other two subspaces.

□

Corollary(4.2): If we denote by T the matrix

$$T = \left[\ \underline{x}_1 \mid \underline{y}_3 \mid \underline{x}_2 \mid \underline{y}_2 \mid \underline{x}_3 \mid \underline{y}_1\ \right]$$

then Φ can be diagonalized as follows

$$T^{T}\Phi T=\left[\begin{array}{cc|cc|cc}
\lambda_3^2+\lambda_4^2+\lambda_5^2 & -\lambda_2\lambda_5 & & & & \\
-\lambda_2\lambda_5 & \lambda_1^2+\lambda_2^2 & & & & \\
\hline
 & & \lambda_2^2+\lambda_4^2+\lambda_5^2 & \lambda_3\lambda_5 & & \\
 & & \lambda_3\lambda_5 & \lambda_1^2+\lambda_3^2 & & \\
\hline
 & & & & \lambda_2^2+\lambda_3^2+\lambda_5^2 & -\lambda_4\lambda_5 \\
 & & & & -\lambda_4\lambda_5 & \lambda_1^2+\lambda_4^2
\end{array}\right]=\left[\begin{array}{c|c|c}
A_1 & & \\
\hline
 & A_2 & \\
\hline
 & & A_3
\end{array}\right]$$

Proof:
As $V_1 = \mathrm{span}(\underline{x}_1,\underline{y}_3), V_2 = \mathrm{span}(\underline{x}_2,\underline{y}_2), V_3 = \mathrm{span}(\underline{x}_3,\underline{y}_1)$ and by Theorem(4.1) all three spaces are Φ invariant subspaces, using a change of basis by T the matrix Φ can be block diagonalised with the blocks defined by Theorem(4.1).

□

Corollary(4.3): As $\|\underline{z}\|^2=\sum\lambda_i^2=tr(A_1)=tr(A_2)=tr(A_3)$, the six eigenvalues, $\{\mu_i\}$, of Φ can be paired so that the sum of every pair is $\|z\|^2$.

Proof:
The restriction of Φ on every space $\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3$ is given by the 2x2 matrices A_1, A_2, A_3 respectively. The three pairs of eigenvalues of these three matrices are eventually the six eigenvalues of the matrix Φ. As the trace of each of the three matrices equals to $\|\underline{z}\|^2$ the result readily follows.

□

Corollary(4.4) :The singular values of $\Phi_6^3(\underline{z})$ can be grouped into three pairs such that the 2-norm of each pair is $\|\underline{z}\|$.

Proof:
As the singular values of $\Phi_6^3(\underline{z})$ are the positive square roots of the eigenvalues of the matrix Φ the result readily follows from Corollary(4.3).

□

Theorem(4.2): Assume that the three blocks A_i have the following three pairs of eigenvalues

$$eig(A_i)=\left\{\mu_i^2,\|\underline{z}\|^2-\mu_i^2\right\},\ \mu_i^2\geq\frac{\|\underline{z}\|^2}{2}\quad i=1,2,3$$

Then $\mu_i^2\neq\mu_j^2\ i,j=1,2,3$ iff $|\lambda_i|\neq|\lambda_j|\ \ i,j=1,2,3,4$

Proof: As the traces of the three blocks are equal, to prove that $\mu_i^2\neq\mu_j^2\ i,j=1,2,3$ it is equivalent to prove that $\det(A_i)\neq\det(A_j)$ i, j = 1,2,3. One can easily calculate the following:

$$\begin{aligned}
\det(A_1)-\det(A_2)&= -(\lambda_1^2-\lambda_4^2)(\lambda_2^2-\lambda_3^2)\\
\det(A_1)-\det(A_3)&= -(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)\qquad(4.1)\\
\det(A_2)-\det(A_3)&= -(\lambda_1^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)
\end{aligned}$$

which proves the result.

□

Theorem(4.3) Let $\underline{z} \in \wedge^3\mathbb{R}^6$ and $\{\lambda, \underline{x}_1, \underline{x}_2, \underline{x}_3\}$ be the global optimum solution for M(3). Additionally let the related decomposition to this solution be:

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \lambda_2 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + \lambda_3 \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 + \lambda_4 \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 + \lambda_5 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

with the additional generic relation $\lambda_1 > |\lambda_2| > |\lambda_3| > |\lambda_4|$ to hold true. Then the matrices A_1, A_2, A_3 defined as in Corollary (4.2) satisfy $\det(A_1) < \det(A_2) < \det(A_3)$. Furthermore, the invariant subspaces $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$ where the three blocks are defined can be uniquely associated with the eigenstructure of Φ.

□

Proof: According to Corollary(3.1) the above decomposition of $\underline{z}$ satisfies the following property: $\lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4|$. For a generic $\underline{z}$ the above inequalities are strict. In this case taking into account equations (4.1) in the proof of theorem(4.2) we have $\det(A_1) < \det(A_2) < \det(A_3)$. If

$$eig(\Phi) = \left\{\sigma_6^2, \sigma_5^2, \sigma_4^2,\ \sigma_3^2 = \|\underline{z}\|^2 - \sigma_4^2,\ \sigma_2^2 = \|\underline{z}\|^2 - \sigma_5^2,\ \sigma_1^2 = \|\underline{z}\|^2 - \sigma_6^2\right\}$$

are the eigenvalues of Φ in descending order with corresponding eigenvectors

$$\left\{\underline{u}_6, \underline{u}_5, \underline{u}_4, \underline{u}_3, \underline{u}_2, \underline{u}_1\right\}$$

then the eigenvalue pair that corresponds to A_1 is the one whose product of elements attains the lowest value ie $\left\{\sigma_6^2, \sigma_1^2 = \|\underline{z}\|^2 - \sigma_6^2\right\}$, as well as the corresponding 2-dim invariant subspace V_1 is given by $span\left\{u_6, u_1\right\}$, similarly A_2 corresponds to the eigenvalue pair $\left\{\sigma_5^2, \sigma_2^2 = \|\underline{z}\|^2 - \sigma_5^2\right\}$ and $\mathcal{V}_2 = span\left\{\underline{u}_5, \underline{u}_2\right\}$ and finally A_3 corresponds to the eigenvalue pair $\left\{\sigma_4^2, \sigma_3^2 = \|\underline{z}\|^2 - \sigma_4^2\right\}$ and $\mathcal{V}_3 = span\left\{\underline{u}_4, \underline{u}_3\right\}$.

□

The above result implies that the space $\mathcal{V}_1$ and the corresponding block A_1 correspond to the pair of the highest and the lowest eigenvalues; the space $\mathcal{V}_2$ and the corresponding block A_2 correspond to the pair of the second lowest and second highest eigenvalues of Φ and the space $\mathcal{V}_3$ and the corresponding block A_3 correspond to the pair of the third and the fourth eigenvalues of Φ.

Corollary (4.4): Let $\underline{z} \in \wedge^3\mathbb{R}^6$. When $F_6^3(\underline{z})$ has 6 distinct singular values the assumptions and the implications of Theorem(4.3) hold true.

□

Corollary (4.5): Let $\underline{z} \in \wedge^3\mathbb{R}^6$ when $\Phi_6^3(\underline{z})$ has 4 distinct singular values $\neq \|\underline{z}\|/\sqrt{2}$ and a double singular value $= \|\underline{z}\|/\sqrt{2}$ the assumptions and the implications of Theorem(4.3) hold true.

□

If we define the characteristic polynomial of $\Phi_6^3(\underline{z})^T \Phi_6^3(\underline{z})$ as:

$$\chi(\lambda,\underline{z}) = \det(\lambda I_6 - \Phi_6^3(\underline{z})^T \Phi_6^3(\underline{z}))$$

and the resultant:

$$\mathrm{R}(\underline{z}) = \mathrm{Re}s_{6,5}(\chi(\lambda,\underline{z}), \frac{\partial\chi(\lambda,\underline{z})}{\partial\lambda}, \lambda)$$

Then the set $\mathcal{E} = \{\underline{z} \in \wedge^3\mathbb{R}^6, R(\underline{z}) \neq 0\}$ contains all three-vectors whose Grassmann matrix has distinct singular values. This is a Zarisky open set in $\underline{z} \in \wedge^3\mathbb{R}^6$ by its definition. Let $\underline{z} \in \mathcal{E}$, then according to Corollary(4.4) we may apply theorem(4.3) so that if

$$\underline{z} = \lambda \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \lambda_2 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + \lambda_3 \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 + \lambda_4 \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 + \lambda_5 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

is a decomposition of $\underline{z}$ and if $\{\underline{u}_6, \underline{u}_5, \underline{u}_4, \underline{u}_3, \underline{u}_2, \underline{u}_1\}$ are the right singular vectors of $\mathrm{F}_6^3(\underline{z})$ (corresponding to singular values in descending order) then:

$$\mathrm{span}\{\underline{u}_6, \underline{u}_1\} = \mathrm{span}\{\underline{x}_1, \underline{y}_3\} \quad \mathrm{span}\{\underline{u}_5, \underline{u}_2\} = \mathrm{span}\{\underline{x}_2, \underline{y}_2\} \quad \mathrm{span}\{\underline{u}_4, \underline{u}_3\} = \mathrm{span}\{\underline{x}_3, \underline{y}_1\}$$

If we relabel $\{\underline{u}_6, \underline{u}_5, \underline{u}_4, \underline{u}_3, \underline{u}_2, \underline{u}_1\}$ by $\{\underline{e}_1, \underline{e}_3, \underline{e}_5, \underline{e}_6, \underline{e}_4, \underline{e}_2\}$ we may derive the following theorem:

Theorem(4.4): Let $\underline{z} \in \mathcal{E}$, then $\underline{z}$ can be written as

$$\underline{z} = \sum_{i,j,k=1,2} w_{i,j,k}\, \underline{e}_i \wedge \underline{e}_{j+2} \wedge \underline{e}_{k+4}$$

where $\underset{i,j,k=1,2}{w_{i,j,k}} = \left\langle \underline{z}, \underline{e}_i \wedge \underline{e}_{j+2} \wedge \underline{e}_{k+4} \right\rangle$.

Proof: To prove the result it is equivalent to prove that

$$\left\langle \underline{z}, \underline{e}_i \wedge \underline{e}_j \wedge \underline{e}_l \right\rangle = 0$$

when the set of two out of the three indices i,j,l equals to one of the sets {1,2},{3,4} or {5,6}. Assume that i=1, j=2 then $\underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_l = \underline{x}_1 \wedge \underline{y}_3 \wedge \underline{e}_l$ up to sign. Then

$$\left\langle \underline{z}, \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_l \right\rangle =$$
$$\left\langle \lambda_1 \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + \lambda_2 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + \lambda_3 \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 + \lambda_4 \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 + \lambda_5 \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3, \underline{x}_1 \wedge \underline{y}_3 \wedge \underline{e}_l \right\rangle$$
$$= \lambda_1 < \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3, \underline{x}_1 \wedge \underline{y}_3 \wedge \underline{e}_l > + \lambda_2 < \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1, \underline{x}_1 \wedge \underline{y}_3 \wedge \underline{e}_l > +$$
$$+ \lambda_3 < \underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2, \underline{x}_1 \wedge \underline{y}_3 \wedge \underline{e}_l > + \lambda_4 < \underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3, \underline{x}_1 \wedge \underline{y}_3 \wedge \underline{e}_l > + \lambda_5 < \underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3, \underline{x}_1 \wedge \underline{y}_3 \wedge \underline{e}_l >$$
$$= 0 + 0 + 0 + 0 + 0 = 0$$

We may perform similar calculations for the other combinations and end up with the same result. As $\left\{\underline{e}_i \wedge \underline{e}_j \wedge \underline{e}_l\right\}_{1 \leq i < j < l \leq 6}$ is a basis for $\wedge^3\mathbb{R}^6$ the only possible combinations with nonzero coefficients in the expansion of $\underline{z}$ are the ones that the present theorem states.

□

Corollary (4.5): There is a map $T:\mathcal{E}\to\otimes^3 R^2$ where $\mathrm{T}(\underline{z})=\left(\mathrm{w}_{\mathrm{i,j,k}}\right)_{\mathrm{i,j,k=1,2}}$ and $w_{i,j,k}=\left\langle \underline{z},\underline{\mathrm{e}}_{\mathrm{i}}\wedge\underline{\mathrm{e}}_{\mathrm{j+2}}\wedge\underline{\mathrm{e}}_{k+4}\right\rangle$ where $\underline{\mathrm{e}}_{\mathrm{i}}$ are the right singular vectors of $\Phi_6^3(\underline{z})$ as in Theorem(4.4).

□

Corollary (4.6): All $\underline{z}$ in $\mathcal{E}$ acquire a decomposition in eight decomposable vectors if expanded on a special basis of $\wedge^3\mathbb{R}^6$ in terms of the right singular vectors of $\Phi_6^3(\underline{z})$

□

Next theorem states that the map T is not onto but the tensor W defined $W=\left(w_{i,j,k}\right)_{i,j,k=1,2}$ where $w_{i,j,k}=\left\langle \underline{z},\underline{\mathrm{e}}_{\mathrm{i}}\wedge\underline{\mathrm{e}}_{\mathrm{j+2}}\wedge\underline{\mathrm{e}}_{k+4}\right\rangle$ defined by Corollary (4.5) satisfies certain conditions:

Theorem(4.5): The tensor W satisfies the following conditions:

$$\sum_{j,k=1,2} w_{1,j,k}w_{2,j,k}=0,\quad \sum_{\mathrm{i},k=1,2} w_{\mathrm{i},1,k}w_{\mathrm{i},2,k}=0,\quad \sum_{i,j=1,2} w_{\mathrm{i},j,1}w_{\mathrm{i},j,2}=0$$

Proof: Consider the expansion of $\underline{z}$ as in theorem(4.4) ie

$$\underline{z}=\sum_{i,j,k=1,2} w_{i,j,k}\,\underline{\mathrm{e}}_{\mathrm{i}}\wedge\underline{\mathrm{e}}_{\mathrm{j+2}}\wedge\underline{\mathrm{e}}_{k+4}$$

Then as $\underline{\mathrm{e}}_{\mathrm{i}}$'s are the right singular vectors of $\mathrm{F}_6^3(\underline{\mathrm{z}})$ we have that:

$$\underline{e}_1^T\Phi_6^3(\underline{z})^T\Phi_6^3(\underline{z})\underline{e}_2=0,\quad \underline{e}_3^T\Phi_6^3(\underline{z})^T\Phi_6^3(\underline{z})\underline{e}_4=0,\quad \underline{e}_5^T\Phi_6^3(\underline{z})^T\Phi_6^3(\underline{z})\underline{e}_6=0$$

Writing the above three equalities in terms of the expansion of $\underline{z}$ we get

$$\left\langle \sum_{i,j,k=1,2} w_{i,j,k}\,\underline{e}_1\wedge\underline{e}_i\wedge\underline{e}_{j+2}\wedge\underline{e}_{k+4},\ \sum_{i,j,k=1,2} w_{i,j,k}\,\underline{e}_2\wedge\underline{e}_i\wedge\underline{e}_{j+2}\wedge\underline{e}_{k+4}\right\rangle=\sum_{j,k=1,2} w_{1,j,k}w_{2,j,k}=0$$

$$\left\langle \sum_{i,j,k=1,2} w_{i,j,k}\,\underline{e}_3\wedge\underline{e}_i\wedge\underline{e}_{j+2}\wedge\underline{e}_{k+4},\ \sum_{i,j,k=1,2} w_{i,j,k}\,\underline{e}_4\wedge\underline{e}_i\wedge\underline{e}_{j+2}\wedge\underline{e}_{k+4}\right\rangle=\sum_{\mathrm{i},k=1,2} w_{\mathrm{i},1,k}w_{\mathrm{i},2,k}=0$$

$$\left\langle \sum_{i,j,k=1,2} w_{i,j,k}\,\underline{e}_5\wedge\underline{e}_i\wedge\underline{e}_{j+2}\wedge\underline{e}_{k+4},\ \sum_{i,j,k=1,2} w_{i,j,k}\,\underline{e}_6\wedge\underline{e}_i\wedge\underline{e}_{j+2}\wedge\underline{e}_{k+4}\right\rangle=\sum_{\mathrm{i,j}=1,2} w_{\mathrm{i},j,1}w_{\mathrm{i},j,2}=0$$

which proves the result.

□

The above results allow us to establish an effective algorithm to solve the minimum distance problem initially defined.

5. An efficient computation of decomposition in $\wedge^3\mathbb{R}^6$

Both the closest decomposable vector and the decompositions of multivectors rely on the calculation of an appropriate orthonormal basis $\{\underline{x}_1,\underline{x}_2,\underline{x}_3,\underline{y}_1,\underline{y}_2,\underline{y}_3\}$ of $\mathbb{R}^6$. Due to Theorem(4.3) we may group the eigenvectors of F into pairs $\{\underline{e}_1,\underline{e}_2\}$, $\{\underline{e}_3,\underline{e}_4\}$, $\{\underline{e}_5,\underline{e}_6\}$ so that the corresponding eigenvalues are complementary. In this case

$$\mathcal{V}_1=span\{\underline{e}_1,\underline{e}_2\}=span\{\underline{x}_1,\underline{y}_3\},\ \mathcal{V}_2=span\{\underline{e}_3,\underline{e}_4\}=span\{\underline{x}_2,\underline{y}_2\},$$
$$\mathcal{V}_3=span\{\underline{e}_5,\underline{e}_6\}=span\{\underline{x}_3,\underline{y}_1\}$$

Although this correspondence is unique when the singular values of $\Phi_6^3(\underline{z})$ are distinct, it can still be carried out when there are multiplicities.
Therefore, the maximizer $\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3$ of $\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle$ may be written as

$$\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 = (a_1\underline{e}_1 + a_2\underline{e}_2) \wedge (b_1\underline{e}_3 + b_2\underline{e}_4) \wedge (c_1\underline{e}_5 + c_2\underline{e}_6), \quad a_1^2 + a_2^2 = b_1^2 + b_2^2 = c_1^2 + c_2^2 = 1.$$

Therefore, the original maximization problem $\langle \underline{z}, \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 \rangle$ in $\wedge^3 \mathbb{R}^6$ is transformed into a much simpler maximization problem in $\otimes^3 \mathbb{R}^2$, i.e.

$$\max \sum_{i,j,k=1,2,\ldots} a_i b_j c_k \langle z, \underline{e}_i \wedge \underline{e}_{j+2} \wedge \underline{e}_{k+4} \rangle = \langle W, \underline{a} \otimes \underline{b} \otimes \underline{c} \rangle$$

where W is the tensor $W = (w_{i,j,k})$ defined above, $\underset{i,j,k=1,2}{w_{i,j,k}} = \langle \underline{z}, \underline{e}_i \wedge \underline{e}_{j+2} \wedge \underline{e}_{k+4} \rangle$,
$\underline{a} = (a_1, a_2)$, $\underline{b} = (b_1, b_2)$, $\underline{c} = (c_1, c_2)$, $\|\underline{a}\| = \|\underline{b}\| = \|\underline{c}\| = 1$.

This may be rewritten as:

$$\max_{\underline{a},\underline{b},\underline{c}} [b_1, b_2] \begin{bmatrix} a_1 w_{111} + a_2 w_{211} & a_1 w_{112} + a_2 w_{212} \\ a_1 w_{121} + a_2 w_{221} & a_1 w_{122} + a_2 w_{222} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ subject to } \|\underline{a}\| = \|\underline{b}\| = \|\underline{c}\| = 1$$

For a fixed $\underline{a}$, this is the maximum singular value $\sigma_{\max} A(\underline{a})$ of the matrix:

$$A(\underline{a}) = \begin{bmatrix} a_1 w_{111} + a_2 w_{211} & a_1 w_{112} + a_2 w_{212} \\ a_1 w_{121} + a_2 w_{221} & a_1 w_{122} + a_2 w_{222} \end{bmatrix}$$

And the optimal vectors $\underline{b}$, $\underline{c}$ correspond to the left and right singular vectors for this singular value. In this setting our optimization problem is reduced to:

$$\max \ \sigma_{\max} A(\underline{a}) \quad \text{subject to } \|\underline{a}\| = 1 \tag{5.1}$$

The square $x=\sigma^2$ of the maximum singular value σ, satisfies the equation:

$$|AA' - xI_2| = x^2 - tr(AA')x + |A|^2 = 0$$

As the function $x: S^1 \to \mathbb{R}$, S^1 is the circle defined by $\|\underline{a}\| = 1$ defined by:

$$x(\underline{a}) = \{ tr(A(\underline{a})A(\underline{a})') + \sqrt{(tr(A(\underline{a})A(\underline{a})'))^2 - 4|A(\underline{a})|^2} \} / 2$$

is a (generically[1]) smooth real function defined on a compact and connected set so its image must be a closed finite interval. This function acquires a global maximum which must satisfy the first order conditions. Therefore we may calculate the solutions of the first order conditions and select the one that corresponds to the maximum x. One way to accomplish this, is to parametrise the circle $\|\underline{a}\|=1$ as:

$$S^1 = S_+^1 \cup S_-^1 \cup S_\infty^1$$

[1] The quantity, d, under the root is a sum of squares and thus the points of the circle that d=0 are defined by more than two equations giving generically an empty set.

where

$$S_{+}^{1}=\{(\frac{1}{\sqrt{1+y^{2}}}\ ,\ \frac{y}{\sqrt{1+y^{2}}}),y\in\mathbb{R}\},\ \ S_{-}^{1}=\{(-\frac{1}{\sqrt{1+y^{2}}}\ ,\ -\frac{y}{\sqrt{1+y^{2}}}),y\in\mathbb{R}\}$$

$$S_{\infty}^{1}=\{(0,1),(0,-1)\}$$

As $x(-\underline{a}) = x(\underline{a})$ we need only to consider $\underline{a}\in S_{+}^{1}\cup\{(0,1)\}$. After the substitution

$$\underline{a} = \left(1/\sqrt{1+y^2}\ ,\ y/\sqrt{1+y^2}\right)$$

We may rewrite the optimization problem (5.1) in terms of (x,y) and obtain the following equivalent problem:

$$\max_{x,y}(f) \text{ where } \ f(x,y)=(1+y^{2})^{2}x^{2}-(1+y^{2})tr(A(y)A(y)^{t})x+\left|A(y)\right|^{2}=0$$

The first order conditions of this problem are given by the following system of polynomial equations:

$$\frac{\P f(x,y)}{\P y} = 0, \quad f(x,y) = 0$$

whose solutions (x,y) along with the solutions at infinity form a finite set Σ of candidates for the global optimum for our problem. The optimum solution (x_0,y_0) is selected from Σ, as the pair (x,y) with the maximum x coordinate. The optimum singular value is given by:

$$s_0 = \sqrt{x_0}$$

and the optimum pair $\underline{a}_0 = (a_1,a_2)$ is given by:

$$a_1 = 1/\sqrt{1+y_0^{\ 2}}\ ,\ \ a_2 = y_0/\sqrt{1+y_0^{\ 2}}$$

using this pair $\underline{a}_0$ we calculate the singular value decomposition of the matrix $A(\underline{a}_0)$. Its maximum singular value is by construction the σ_0 and the corresponding left and right singular vectors are the optimal vectors $\underline{b}_0$, $\underline{c}_0$. The best decomposable 3-vector approximating $\underline{z}$ is given by:

$$\sigma_0(a_1\underline{e}_1+a_2\underline{e}_2)\wedge(b_1\underline{e}_3+b_2\underline{e}_4)\wedge(c_1\underline{e}_5+c_2\underline{e}_6)$$

Furthermore the basis of $\mathbb{R}^6$ for the five vector decomposition of z in $\wedge^3\mathbb{R}^6$ is given by:

$$\{x_1,x_2,x_3,y_1,y_2,y_3\} =$$
$$\{a_1\underline{e}_1 + a_2\underline{e}_2, b_1\underline{e}_3 + b_2\underline{e}_4, c_1\underline{e}_5 + c_2\underline{e}_6, -c_2\underline{e}_5 + c_1\underline{e}_6, -b_2\underline{e}_3 + b_1\underline{e}_4, -a_2\underline{e}_1 + a_1\underline{e}_2\}$$

We may summarise the process of decomposition as it is indicated below:

Steps

1. Given a multi-vector $\underline{z}$ Calculate the Grassmann matrix then the matrix Φ and its eigenvalues and eigenvectors
2. With the help of this eigenframe calculate the tensor W
3. With the help of W calculate the parametrised matrix $A(\underline{a})$ and solve the maximisation problem (5.1)

4. The solution of (5.1) leads to the calculation of the optimal vectors $\underline{a}$,$\underline{b}$,c.
5. Using the vectors a,b,c and the eigenvectors of step 1 form an new basis $\{a_1\underline{e}_1 + a_2\underline{e}_2, b_1\underline{e}_3 + b_2\underline{e}_4, c_1\underline{e}_5 + c_2\underline{e}_6, -c_2\underline{e}_5 + c_1\underline{e}_6, -b_2\underline{e}_3 + b_1\underline{e}_4, -a_2\underline{e}_1 + a_1\underline{e}_2\}$ for ¡ ^{6} .

The expansion of $\underline{z}$ in this basis gives us its optimal 5 decomposable vector decomposition, where the best decomposable approximation corresponds to

$(a_1\underline{e}_1 + a_2\underline{e}_2)\wedge(b_1\underline{e}_3 + b_2\underline{e}_4)\wedge(c_1\underline{e}_5 + c_2\underline{e}_6)$

□

Example (5.1): Consider $\underline{z}$ in $\wedge^3\mathbb{R}^6$ as in Example (3.1). The Grassmann matrix of $\underline{z}$ which is given by:

$$\Phi_6^3(z) = \begin{bmatrix} -7 & 4 & -3 & -5 & 0 & 0 \\ -9 & 10 & -8 & 0 & -5 & 0 \\ -4 & -4 & -7 & 0 & 0 & -5 \\ -2 & 8 & 0 & 8 & -3 & 0 \\ 5 & -2 & 0 & 7 & 0 & -3 \\ -3 & 4 & 0 & 0 & -7 & 8 \\ 4 & 0 & -8 & -10 & 4 & 0 \\ -2 & 0 & 2 & 4 & 0 & 4 \\ -9 & 0 & 4 & 0 & 4 & 10 \\ 6 & 0 & 0 & 4 & -2 & -8 \\ 0 & -4 & 2 & 9 & -7 & 0 \\ 0 & 2 & -5 & 4 & 0 & -7 \\ 0 & -9 & 3 & 0 & -4 & 9 \\ 0 & 6 & 0 & 3 & -5 & -2 \\ 0 & 0 & 6 & 9 & -2 & -4 \end{bmatrix}$$

Its squared singular values are given by:

[640.845, 552.444, 534.691, 149.309, 131.556, 43.155]

We can pair the 1st with the 6th, the 2nd with the 5th and the 3rd with the 4th so that their sum is equal to 684 the square norm of z. The corresponding 2-dimensional subspaces of $\mathbb{R}^6$ formed by the related eigenvectors can be given (in terms of their basis matrices):

$$\begin{pmatrix} -0.4631 & 0.0614733 \\ 0.724921 & 0.533318 \\ -0.322316 & 0.555798 \\ 0.212711 & -0.000833998 \\ -0.299763 & 0.630776 \\ -0.145039 & -0.0707275 \end{pmatrix}, \begin{pmatrix} -0.351728 & 0.293279 \\ 0.0751618 & 0.079233 \\ -0.237349 & -0.539815 \\ -0.734598 & 0.47046 \\ 0.231609 & 0.431891 \\ 0.470135 & 0.456559 \end{pmatrix}, \begin{pmatrix} 0.336006 & -0.677594 \\ 0.0726397 & -0.415762 \\ -0.482902 & -0.0791934 \\ -0.351291 & 0.265293 \\ 0.2548 & 0.455157 \\ -0.678455 & -0.290152 \end{pmatrix}$$

Based on this decomposition, the 3rd order homogeneous polynomial to be maximised is given by:

$$F(\underline{a},\underline{b},\underline{c}) = 3.455a_1b_1c_1 \quad 10.670a_2b_1c_1 + 4.838a_1b_2c_1 + 0.329a_2b_2c_1$$
$$2.289a_1b_1c_2 + 0.722a_2b_1c_2 \quad 1.604a_1b_2c_2 + 22.942a_2b_2c_2$$

Subject to ||a||=||b||=||c||=1. This is reduced to finding the matrix A(a) with the maximum possible singular value from the family of matrices:

$$\begin{pmatrix} -3.45509a_1 - 10.6705a_2 & -2.28881a_1 + 0.721638a_2 \\ 4.8378a_1 + 0.329252a_2 & -1.60439a_1 + 22.9425a_2 \end{pmatrix}$$

subject to ||a||=1. This squared singular value $x=\sigma^2$, satisfies the equation:

$$\left|AA^t - xI_2\right| = x^2 - tr(AA^t)x + \left|A\right|^2 = 0$$

After the substitution:

$$a_1 = 1/\sqrt{1+y^2}\ ,\ a_2 = y/\sqrt{1+y^2}$$

and taking the first order conditions, we obtain the following system of polynomial equations:

$$f_1(x,y) = 276.095 - 43.1547x + x^2 - 2156.32y - 3933.13y^2 - 684.xy^2 + \\ +2.x^2y^2 + 31800.1y^3 + 60046.8y^4 - 640.845xy^4 + x^2y^4 = 0$$

$$f_2(x,y) = -2156.32 - 7866.26y - 1368.xy + 4.x^2y + 95400.3y^2 + \\ +240187.y^3 - 2563.38xy^3 + 4x^2y^3 = 0$$

The real solutions (x,y) of this system of equations are: (529.961.-12.506), (124.818,3.613), (33.603,−0.0419), (11.381,−0.075), (0,−0.425), (0,0.160) and the solutions at infinity are: (113.967,∞), (526.878, ∞). The maximum squared singular value of A corresponds to the maximum x appearing to the set of solutions ie $s_{max} = \sqrt{529.961} = 23.0209$. The optimal values for a are the ones corresponding to y= -12.506 ie

$$a_1 = 1/\sqrt{1+(-12.506)^2} = 0.08\ ,\quad a_2 = -12.506/\sqrt{1+(-12.506)^2} = -0.997$$

The optimal A(0.08,-0.997), is given by:

$$\begin{pmatrix} 10.3611 & -0.901772 \\ 0.0573917 & -22.9974 \end{pmatrix}$$

The optimal b, c^t are the right and left singular vectors of A for σ_{max} ie.

$$b_1 = -0.051,\ b_2 = -0.999,\ c_1 = -0.0252,\ c_2 = 0.999$$

The basis matrix for the canonical decomposition [X,Y]=[$\underline{x}_1,\underline{x}_2,\underline{x}_3,\underline{y}_1,\underline{y}_2,\underline{y}_3$] is given by:

$$[a_1\underline{v}_1 + a_2\underline{v}_6, b_1\underline{v}_2 + b_2\underline{v}_5, c_1\underline{v}_3 + c_2\underline{v}_4, -c_2\underline{v}_3 + c_1\underline{v}_4, -b_2\underline{v}_2 + b_1\underline{v}_5, -a_2\underline{v}_1 + a_1\underline{v}_6]$$

Giving rise to the basis matrices:

$$X^T = \begin{pmatrix} -0.320381 & -0.0729902 & 0.51918 & -0.527515 & -0.412056 & -0.417634 \\ 0.659757 & 0.411562 & 0.103485 & -0.24721 & -0.467446 & 0.324052 \\ 0.073135 & 0.514863 & 0.563751 & -0.00619919 & 0.638137 & -0.0670465 \end{pmatrix}$$

$$Y^T = \begin{pmatrix} 0.461402 & -0.738143 & 0.308194 & -0.212623 & 0.283757 & 0.146777 \\ 0.369803 & 0.093548 & -0.478285 & -0.364243 & 0.231484 & -0.662932 \\ -0.327233 & 0.0812379 & -0.27962 & -0.694763 & 0.265296 & 0.505029 \end{pmatrix}$$

Using these matrices we may decompose $\underline{z}$ by multiplying $\underline{z}$ by $C_3([X,Y])$. This way, we get only five non zero entries which induce the following decomposition:

$$\underline{z} = 23.0209\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 + 10.35\underline{y}_1 \wedge \underline{y}_2 \wedge \underline{x}_1 + 4.627\underline{y}_1 \wedge \underline{y}_3 \wedge \underline{x}_2 + 2.12\underline{y}_2 \wedge \underline{y}_3 \wedge \underline{x}_3 - 4.59\underline{y}_1 \wedge \underline{y}_2 \wedge \underline{y}_3$$

Furthermore the best decomposable approximation of z is the term $23.0209\underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3$.

□

Conclusions

The approximate decomposability of 3-vectors in $\wedge^3\mathbb{R}^6$ was considered. The first order conditions of the optimization problem, imply a decomposition of 3-vectors in five orthogonal decomposable 3-vectors. Utilizing the Grassmann matrix [8] of $\underline{z}$, the problem can be reduced into a similar problem in the tensor space $\otimes^3\mathbb{R}^2$. The results leads to a computationally efficient method to calculate the best decomposable approximation of a 3-vector in $\wedge^3\mathbb{R}^6$ which then can be utilized to solve approximate frequency assignment problems. For this method to be applicable to such problems has to be modified so that it calculates best decomposable approximation of three-vectors parameterised by a linear variety. Such approach is currently under investigation.

References

[1] P.-A. Absil, R. Mahony, R. Sepulchre (2008), Optimization Algorithms on Matrix Manifolds, Princeton Univ. Press..

[2] Brett W. Bader and Tamara G. Kolda. Algorithm 862: MATLAB tensor classes for fast algorithm prototyping. ACM Transactions on Mathematical Software, 32(4):635-653,December 2006.

[3] D.Bertsekas(1982),Constrained Optimization and Lagrange Multiplier Methods, Academic press.

[4] L. De Lathauwer, B. De Moor, J.Vandewalle, 2000, A multilinear Singular Value Decomposition, Siam J. Matrix Analysis, Vol. 21, No. 4, pp. 1253–1278

[5] W.V.D. Hodge and P.D. Pedoe: Methods of Algebraic Geometry, Volume 2. Cambridge University Press. (1952)

[6] N. Karcanias and C. Giannakopoulos: Grassmann invariants, almost zeros and the determinantal pole-zero assignment problems of linear systems. International Journal of Control. . 673–698. (1984)

[7] J. Leventides and N. Karcanias, 1995. "Global Asymptotic Linearisation of the pole placement map: A closed form solution for the constant output feedback problem". Automatica, Vol. 31, No. 9, pp 1303-1309.

[8] N. Karcanias and C. Giannakopoulos, 1988. "Grassmann Matrices, Decomposability of Multivectors and the Determinantal Assignment Problem", in Linear Circuit, Systems and Signal Processing: Theory and Applications, Ed. C. I. Byrnes etc. North Holland, 307-312.

[9] N. Karcanias and J. Leventides, 2007. "Grassman Matrices, Determinantal Assignment Problem and Approximate Decomposability". Proceedings of 3rd IFAC Symposium on Systems Structure and Control Symposium (SSSC 07), 17-19 October, Foz do Iguacu, Brazil

[10] J.Leventides, G.Petroulakis, N.Karcanias (2014), The approximate Determinantal Assignment Problem, Linear Algebra and its Applications, Vol. 461, p 139-162.

[11] J. Leventides and N. Karcanias 1992. "A new Sufficient Condition for Arbitrary Pole Placement by Real Constant Output Feedback" Systems and Control Letters, Vol. 18. No. 3, pp 191-200.

[12] M. Marcus: Finite Dimensional Multilinear Algebra (Parts 1 and 2). Marcel Deker. New York. (1973)
[13] M. Marcus and H. Minc: A Survey of Matrix Theory and Matrix Inequalities. Allyn and Bacon. Boston. (1964)
[14] Rasmus Bro. Parafac. tutorial and applications. Chemomemcs and Intelligent LaboratorySystems, 38:149-171, 1997.
[15] Tamara G. Kolda and Brett W. Bader. Tensor decompositions and applications. SIAM Review, 51(3):455-500, September 2009.