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A v-ANALOGUE OF PEEL'S THEOREM

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ABSTRACT. We compute the v-decomposition numbers $d_{\lambda\mu}(v)$ for λ being a hook partition, and μ e-regular.

1. INTRODUCTION

Throughout we fix an integer $e \geq 2$. Lascoux, Leclerc, and Thibon [8] used the representation theory of the quantum affine algebra $U_v(\widehat{\mathfrak{sl}}_e)$ to introduce for every pair of partitions λ and σ, with σ e-regular, a polynomial $d_{\lambda\sigma}(v)$ with integer coefficients (which depends on e). They conjectured these polynomials to be v-analogues of decomposition numbers for Hecke algebras at complex e-th root of unity (hence the term 'v-decomposition numbers'); this conjecture was proved later by Ariki [1]. These v-decomposition numbers are also known to be parabolic affine Kazhdan-Lusztig polynomials.

Leclerc's lectures [9] are a good introduction to this subject as well as a convenient reference for the results we need here.

The purpose of this note is to prove the following theorem, which describes the v-decomposition numbers corresponding to rows labelled by hook partitions:

Theorem 1. *For $0 \leq i \leq n-1$, let $\alpha_i^n = (n-i, 1^i)$ and denote its 'e-regularised' partition by $(\alpha_i^n)^R$ (see §2.1).*

(1) *If $e \geq 3$, then*

$$d_{\alpha_i^n(\alpha_i^n)^R}(v) = \begin{cases} v^{\lfloor i/e \rfloor}, & \text{if } e \nmid n, \text{ or both } e \mid n \text{ and } i < n - \frac{n}{e}, \\ v^{\lfloor i/e \rfloor + 1}, & \text{if } e \mid n \text{ and } i \geq n - \frac{n}{e}; \end{cases}$$

$$d_{\alpha_i^n(\alpha_{i-1}^n)^R}(v) = v^{\lfloor i/e \rfloor + 1}, \qquad \text{if } e \mid n \text{ and } 1 \leq i < n - \tfrac{n}{e};$$

$$d_{\alpha_i^n(\alpha_{i+1}^n)^R}(v) = v^{\lfloor i/e \rfloor}, \qquad \text{if } e \mid n \text{ and } n - \tfrac{n}{e} \leq i < n;$$

$$d_{\alpha_i^n\sigma}(v) = 0, \qquad \text{for all other } e\text{-regular } \sigma\text{'s}.$$

(2) *If $e = 2$, then*

$$d_{\alpha_i^n(\alpha_j^n)^R}(v) = \begin{cases} v^{\lfloor i/2 \rfloor}, & \text{if } j \leq i < n-j \text{ and } i-j \text{ even}, \\ v^{(i+1)/2}, & \text{if } 2 \mid n,\ j < i < n-j,\ i \text{ odd and } j \text{ even}, \\ v^{i/2+1}, & \text{if } 2 \mid n,\ j < i < n-j,\ i \text{ even and } j \text{ odd}; \end{cases}$$

$$d_{\alpha_i^n\sigma}(v) = 0, \qquad \text{for all other } e\text{-regular } \sigma\text{'s}.$$

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Peel [13] initiated the study of the corresponding decomposition numbers of symmetric group algebras in odd characteristic, and this is continued by James [4, Theorem 6.22] and James-Mathas [6, Theorem 7.6] for Hecke algebras at complex e-th root of unity with $e \geq 2$. When we use these theorems of James and James-Mathas together with Ariki's theorem [1], we can then conclude that these v-decomposition numbers, when non-zero, are monic monomials. However, we do not make use of this fact here and work entirely in the context of the basic $U_v(\widehat{\mathfrak{sl}}_e)$-module (or the Fock space), thereby providing another proof of these corresponding decomposition numbers of Hecke algebras when we evaluate these v-decomposition numbers at $v = 1$ and use Ariki's theorem.

This paper is organised as follows: in section 2, we introduce the background theory and obtain some useful preliminary results. We then prove part (1) and (2) of Theorem 1 in sections 3 and 4 respectively.

2. Background

2.1. **Partitions.** A partition is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers. We write $|\lambda| = \sum_i \lambda_i$. If $|\lambda| = n$, we say that λ is a partition of n. We denote the set of partitions of n by $\mathcal{P}_n$, and write $\mathcal{P} = \bigcup_n \mathcal{P}_n$ for the set of all partitions. A partition λ is e-regular if and only if there is no i such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+e-1} \neq 0$. We identify a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with its Young diagram

$$\left\{(j,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid 1 \leq k \leq \lambda_j\right\}.$$

The standard lexicographic and dominance ordering on $\mathcal{P}_n$ are denoted by $>$ and $\rhd$ respectively, and we introduce a total ordering $\succ$ on $\mathcal{P}$ as follows: $\lambda \succ \mu$ if, and only if, either $|\lambda| > |\mu|$ or both $|\lambda| = |\mu|$ and $\lambda > \mu$.

Given any integer j, we write $\overline{j}$ for its residue class modulo e. The residue of a node (j,k) in a Young diagram μ is $\overline{k-j}$. If (j,k) has residue i, we say that (j,k) is an i-node. If in removing (j,k) from μ, we obtain a Young diagram λ then we call (j,k) a removable i-node of μ or an indent i-node of λ.

A ladder $\ell = \ell_r$ is a set of nodes of the form

$$\{(j,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid k = (1-e)j + r\}.$$

All nodes in ℓ_r have residue $\overline{r}$. The intersection of a ladder with the Young diagram λ is a ladder of λ. If we replace each ladder of λ by the same number of nodes as high up as possible in the same ladder, then we obtain an e-regular partition which is labelled λ^R [5, 6.3.48].

2.2. **The algebra $U_v(\widehat{\mathfrak{sl}}_e)$ and its basic module.** The algebra $U = U_v(\widehat{\mathfrak{sl}}_e)$ is the associative algebra over $\mathbb{C}(v)$ with generators e_i, f_i, k_i, k_i^{-1} $(0 \leq i \leq e-1)$, d, d^{-1} subject to certain relations for which the interested reader may refer to, for example, [9, §4]. An important U-module is the Fock space representation $\mathcal{F}$ [3, 12], which as a $\mathbb{C}(v)$-vector space has a basis $\{s(\lambda)\}_{\lambda \in \mathcal{P}}$. For our purposes an explicit description of the action of just the f_i's on $\mathcal{F}$ will suffice.

Let λ be a partition with m indent i-nodes $(j_1,k_1), (j_2,k_2), \ldots, (j_m,k_m)$, and write μ for the partition obtained by adding these m indent nodes to λ.

Let N_r be the number of indent i-nodes of λ not equal to (j_s, k_s) for all s that are situated to the right of (j_r, k_r) minus the the number of removable i-nodes of λ situated to the right of (j_r, k_r). Let $N(\lambda, \mu) = \sum_{r=1}^{m} N_r$. We have

$$f_i^{(m)} s(\lambda) = \sum_{\mu} v^{N(\lambda,\mu)} s(\mu),$$

where $f_i^{(m)} = f_i^m / (\prod_{r=1}^{m} \frac{v^r - v^{-r}}{v - v^{-1}})$, and the sum is over all Young diagrams μ obtained from λ by adding m indent i-nodes.

We identify the basic U-module $M(\Lambda_0)$ with the U-submodule of $\mathcal{F}$ generated by $s(\emptyset)$. This is an irreducible highest weight module for U, and has a distinguish basis $\{G(\sigma)\}$, called the canonical basis or lower global crystal basis, which is indexed by e-regular partitions σ [7]. Let $\langle -, - \rangle$ denote the inner product on $\mathcal{F}$ for which $\{s(\lambda) \mid \lambda \in \mathcal{F}\}$ is orthonormal. Then the v-decomposition number $d_{\lambda\sigma}(v)$ is defined as $\langle G(\sigma), s(\lambda) \rangle$, the coefficient of $s(\lambda)$ in $G(\sigma)$. These v-decomposition numbers are shown to have the following properties:

Theorem 2 ([8, Theorem 6.8], [14]; see also [11, Theorem 6.28])**.** *We have*

$$d_{\sigma\sigma}(v) = 1,$$
$$d_{\lambda\sigma}(v) \in v\mathbb{N}[v] \quad \textit{for all } \lambda \neq \sigma.$$

Furthermore, $d_{\lambda\sigma}(v) \neq 0$ only if $\lambda \trianglelefteq \sigma$, and λ and σ have the same e-core.

Lascoux, Leclerc and Thibon provided a recursive combinatorial algorithm to calculate $G(\sigma)$'s. Now commonly known as the LLT algorithm, it is based on the following principle:

Theorem 3. *Let σ be an e-regular partition, and let $\widetilde{\sigma}$ be the partition obtained by removing the rightmost ladder of σ, which has residue r and size k. Then*

$$f_r^{(k)}(G(\widetilde{\sigma})) = G(\sigma) + \sum_{\substack{\mu \lhd \sigma \\ \mu \ e\text{-regular}}} L_\mu(v) G(\mu),$$

with $L_\mu(v) = L_\mu(v^{-1}) \in \mathbb{N}[v, v^{-1}]$.

2.3. **Notations.** We adopt the following notations in this paper:

(1) If $x, y, x - y \in \bigoplus_{\lambda \in \mathcal{P}} \mathbb{N}[v, v^{-1}] s(\lambda)$, then we write $y \mid x$.
For example, $d_{\lambda\mu}(v) s(\lambda) \mid G(\mu)$ by Theorem 2, and, keeping the notation of Theorem 3, $G(\sigma), L_\lambda(v) G(\lambda) \mid f_r(G(\widetilde{\sigma}))$.

(2) If $x \in \mathcal{F}$, then $[x]_{hook} = \sum_n \sum_{i=0}^{n-1} \langle x, s(\alpha_i^n) \rangle\, s(\alpha_i^n)$, the 'hook-part' of x.

(3) If $k \in \mathbb{Z}$, then $\overline{k}$ denotes the residue class of k modulo e, and

$$\delta_k = \begin{cases} 1, & \text{if } e \mid k; \\ 0, & \text{otherwise.} \end{cases}$$

(4) We denote the hook partition $(n - i, 1^i)$ (for $0 \leq i < n$) by α_i^n.

2.4. **Some useful results.** We collate together some results which we shall require.

Proposition 4. *Keep the notations of Theorem 3, and let* $\left\langle f_r^{(k)}(G(\widetilde{\sigma})), s(\lambda)\right\rangle = p(v)$ $(\lambda \lhd \sigma)$.

(1) *If* $p(v) = 1$, *then* λ *is* e*-regular,* $L_\lambda(v) = 1$ *and* $d_{\lambda\sigma}(v) = 0$.
(2) *If* $p(v) = v^m$ *with* $m > 0$, *and* $d_{\lambda\mu}(v) \neq v^m$ *for all* e*-regular* μ *with* $\lambda \lhd \mu \lhd \sigma$, *then* $d_{\lambda\sigma}(v) = v^m$.
(3) *If* $p(v) \in \mathbb{N}[v]$, *and* $\left\langle f_r^{(k)}(G(\widetilde{\sigma})), s(\mu)\right\rangle \in v\mathbb{N}[v]$ *for all* e*-regular* μ *with* $\lambda \lhd \mu \lhd \sigma$, *then* $L_\lambda(v) = p(0)$ *and* $d_{\lambda\sigma}(v) = p(v) - p(0)$.

Proof. By Theorem 3, $p(v) = d_{\lambda\sigma}(v) + \sum_\mu L_\mu(v) d_{\lambda\mu}(v)$, where the sum runs over e-regular partitions $\mu \lhd \sigma$.

If $p(v) = v^m$ with $m \geq 0$, then since $L_\mu(v) \in \mathbb{N}[v, v^{-1}]$, $d_{\lambda\mu}(v), d_{\lambda\sigma}(v) \in \mathbb{N}[v]$, we have either

- $d_{\lambda\sigma}(v) = v^m$, or
- $d_{\lambda\sigma}(v) = 0$, and $L_{\mu_0}(v) = 1$, $d_{\lambda\mu_0}(v) = v^m$ for some e-regular $\mu_0 \lhd \sigma$ while $L_\mu(v) d_{\lambda\mu}(v) = 0$ for all other e-regular $\mu \lhd \sigma$.

Thus, (2) follows since $d_{\lambda\mu_0}(v) \neq v^m$ with $m > 0$ implies $\lambda \lhd \mu_0$ by Theorem 2, while (1) follows since $d_{\lambda\nu}(v) \neq 1$ unless $\lambda = \nu$ by Theorem 2.

For (3), if $L_\mu(v) \neq 0$ for some $\lambda \lhd \mu \lhd \sigma$, let μ_0 be maximal in the dominance order among these. Then $\left\langle f_r^{(k)}(G(\widetilde{\sigma})), s(\mu_0)\right\rangle = d_{\mu_0\sigma}(v) + L_{\mu_0}(v)$ by Theorems 2 and 3. Thus,

$$L_{\mu_0}(v) = \left\langle f_r^{(k)}(G(\widetilde{\sigma})), s(\mu_0)\right\rangle - d_{\mu_0\sigma}(v) \in v\mathbb{N}[v]$$

by our hypothesis and Theorem 2. But this contradicts $L_\mu(v) = L_\mu(v^{-1})$. Thus, $L_\mu(v) = 0$ for all $\lambda \lhd \mu \lhd \sigma$, so that $p(v) = d_{\lambda\sigma}(v) + L_\lambda(v)$. If $p(v) = a_0 + a_1 v + \cdots + a_m v^m$, then the fact that $L_\lambda(v) = L_\lambda(v^{-1}) \in \mathbb{N}[v, v^{-1}]$, as well as $d_{\lambda\sigma}(v) \in v\mathbb{N}[v]$, forces $L_\lambda(v) = a_0 = p(0)$ and $d_{\lambda\sigma}(v) = a_1 v + \cdots + a_m v^m = p(v) - p(0)$. □

The following lemma follows immediately from the Young diagram of a hook partition.

Lemma 5. *We have* $[f_r(s(\lambda))]_{hook} = 0$ *unless* $\lambda = \alpha_i^n$ *for some* $i, n \in \mathbb{N}$ *with* $i < n$, *and*

$$[f_r(s(\alpha_i^n))]_{hook} = \delta_{n-i-r} s(\alpha_i^{n+1}) + \delta_{r+i+1} v^{\delta_{n+1} - \delta_n + \delta_r} s(\alpha_{i+1}^{n+1}).$$

In particular, if $x \in \mathcal{F}$, *then* $[f_r(x)]_{hook} = [f_r([x]_{hook})]_{hook}$.

The following v-decomposition numbers are computed by Lyle in her Ph.D. thesis.

Proposition 6 ([10, Theorem 2.2.3])**.**

(1) *If* $e \geq 3$ *and* $e \mid n$, *then* $d_{(n-1,1),(n)}(v) = v = d_{(n-1,2),(n+1)}(v)$.
(2) *If* $e = 2$ *and* $0 \leq i < n/2$, *then*

$$d_{(n-i,i)\sigma}(v) = \begin{cases} 1, & \text{if } \sigma = (n-i, i); \\ v, & \text{if } \sigma = (n-i+1, i-1) \text{ and } 2 \mid n; \\ 0, & \text{otherwise.} \end{cases}$$

3. Part (1) of Theorem 1

Throughout this section, we assume $e \geq 3$. For each $i \in \mathbb{Z}$, we denote its quotient and remainder, when i is divided by $e-1$, by q_i and s_i respectively. If $0 \leq i < n$, let

$$\beta_i^n = \begin{cases} (n-i, (q_i+1)^s, q_i^{e-1-s_i}), & \text{if } n-i > q_i+1; \\ ((q_i+1)^{s_i+1}, q_i^{e-2-s_i}, n-i-1), & \text{if } n-i \leq q_i+1. \end{cases}$$

Then $\beta_i^n = (\alpha_i^n)^R$. It is not difficult to check that the condition $n-i > q_i+1$ is equivalent to $i \leq (n-1)(1-\frac{1}{e})$. Furthermore, if $j = \lfloor (n-1)(1-\frac{1}{e}) \rfloor$, then

$$\beta_0^n > \beta_1^n > \cdots > \beta_j, \qquad \beta_{j+1} < \beta_{j+2} < \cdots < \beta_{n-1},$$

and $\beta_j \geq \beta_{j+1}$ with equality if and only if $e \mid n$.

Before we state the main theorem of this section, we prove the following proposition, which helps to take care of a special case later:

Proposition 7. *Suppose $i, n \in \mathbb{Z}^+$ satisfy $e \mid n$, $(e-1) \mid i$ and $0 < n-i < i/(e-1)$. Then $d_{\beta_{i-1}^n \beta_i^n}(v) = v$ and $d_{\beta_{i-2}^n \beta_i^n}(v) = 0$.*

Proof. The rightmost ladder of $\beta_i^n = (q_i+1, q_i^{e-2}, n-i-1)$ has size 1 and residue $\overline{q_i}$, and removing it produces $\beta_{i-1}^{n-1} = (q_i^{e-1}, n-i-1)$. Since $\beta_{i-1}^n = (q_i^{e-1}, n-i)$ has a unique removable $\overline{q_i}$-node (removal of this node produces β_{i-1}^{n-1}), so that $\langle f_{\overline{q_i}}(s(\mu)), s(\beta_{i-1}^n) \rangle = 0$ for all $\mu \neq \beta_{i-1}^{n-1}$, we have

$$\langle f_{\overline{q_i}}(G(\beta_{i-1}^{n-1})), s(\beta_{i-1}^n) \rangle = \langle f_{\overline{q_i}}(s(\beta_{i-1}^{n-1})), s(\beta_{i-1}^n) \rangle = v.$$

Now if $\beta_{i-1}^n \lhd \lambda \lhd \beta_i^n$, then $\lambda = (q_i+1, q_i^{e-3}, q_i-1, n-i)$, so that λ has a unique removable $\overline{q_i}$-node, and its removal produces $\widetilde{\lambda} = (q_i^{e-2}, q_i-1, n-i)$. Thus,

$$\langle f_{\overline{q_i}}(G(\beta_{i-1}^{n-1})), s(\lambda) \rangle = \left\langle f_{\overline{q_i}}(d_{\widetilde{\lambda}\beta_{i-1}^{n-1}}(v) s(\widetilde{\lambda})), s(\lambda) \right\rangle = d_{\widetilde{\lambda}\beta_{i-1}^{n-1}}(v),$$

so that $d_{\beta_{i-1}^n \beta_i^n}(v) = v$ by Theorem 2 and Proposition 4(3) .

For $d_{\beta_{i-2}^n \beta_i^n}(v)$, note that $\beta_{i-2}^n = (q_i^{e-2}, q_i-1, n-i+1)$ has a unique removable $\overline{q_i}$-node, and removing it produces $\nu = (q_i^{e-2}, q_i-2, n-i+1)$. We have $d_{\nu\beta_{i-1}^{n-1}}(v) = d_{(m-2,2),(m)}(v)$ where $m = q_i - n - i + 1$, by [2, Theorem 1], which in turn equals v by Proposition 6(1). Hence

$$\langle f_{\overline{q_i}}(G(\beta_{i-1}^{n-1})), s(\beta_{i-2}^n) \rangle = \langle f_{\overline{q_i}}(vs(\nu), s(\beta_{i-2}^n) \rangle = 1.$$

Thus, by Proposition 4(1), $d_{\beta_{i-2}^n \beta_i^n}(v) = 0$. □

The following is a reformulation of Part (1) of Theorem 1.

Theorem 8. *Suppose $e \geq 3$. Then*

$$[G(\beta_i^n)]_{hook} = \begin{cases} v^{\lfloor i/e \rfloor} s(\alpha_i^n) + \delta_n v^{\lfloor (i+1)/e \rfloor + 1} s(\alpha_{i+1}^n), & \text{if } i \leq (n-1)(1-\frac{1}{e}); \\ v^{\lfloor i/e \rfloor + \delta_n} s(\alpha_i^n) + \delta_n v^{\lfloor (i-1)/e \rfloor} s(\alpha_{i-1}^n), & \text{if } i > (n-1)(1-\frac{1}{e}). \end{cases}$$

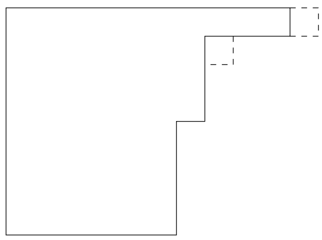
Furthermore, $[G(\sigma)]_{hook} = 0$ for all other e-regular σ's.

Note. When $e \mid n$ and $i = \lfloor (n-1)(1-\frac{1}{e}) \rfloor$, then the formulas for $[G(\beta_i^n)]_{hook}$ and $[G(\beta_{i+1}^n)]_{hook}$ as stated in Theorem 8 coincide as expected, since $G(\beta_i^n) = G(\beta_{i+1}^n)$.

Proof. We prove by induction. For $n = 0, 1$, the theorem is trivially true. Let σ be an e-regular partition of n such that $[G(\sigma)]_{hook} \neq 0$, and assume the theorem holds for all e-regular partitions $\lambda \prec \sigma$. Suppose the rightmost ladder of σ has residue r and size k, and removing this ladder produces $\widetilde{\sigma}$. By Theorem 3 and Lemma 5, $[G(\widetilde{\sigma})]_{hook} \neq 0$, so that by induction hypothesis, $\widetilde{\sigma} = \beta_i^{n-k}$ for some i. Let $q, s \in \mathbb{Z}$ such that $i = q(e-1) + s$, with $0 \leq s < e - 1$.

We have the following two cases to consider:

Case 1. $i \leq (n-k-1)(1 - \frac{1}{e})$**:**



$$\widetilde{\sigma} = \beta_i^{n-k} = (n-k-i, (q+1)^s, q^{e-1-s})$$

From the above Young diagram of $\widetilde{\sigma}$, we see that the rightmost ladder of σ has size 1 (i.e. $k = 1$), containing either the indent node in the first or second row of $\widetilde{\sigma}$, the latter only if $n - 1 - i = q + 2$ and $s \geq 1$. These two sub-cases correspond to $r = \overline{n-1-i}$ and $r = \overline{q}$ respectively, and will be considered separately. We note that by induction hypothesis, we have

$$(*) \qquad [G(\widetilde{\sigma})]_{hook} = v^{\lfloor i/e \rfloor} s(\alpha_i^{n-1}) + \delta_{n-1} v^{\lfloor (i+1)/e \rfloor + 1} s(\alpha_{i+1}^{n-1}).$$

Case 1a. $r = \overline{n-1-i}$**:** In this subcase, $\sigma = \beta_i^n$. From $(*)$, Theorem 3 and Lemma 5, we see that

$$\begin{aligned}
[f_r(G(\widetilde{\sigma}))]_{hook} &= [v^{\lfloor i/e \rfloor} f_r(s(\alpha_i^{n-1})) + \delta_{n-1} v^{\lfloor (i+1)/e \rfloor + 1} f_r(s(\alpha_{i+1}^{n-1}))]_{hook} \\
&= v^{\lfloor i/e \rfloor} s(\alpha_i^n) + \delta_n v^{\lfloor i/e \rfloor + \delta_{i+1} + 1} s(\alpha_{i+1}^n) \\
&= v^{\lfloor i/e \rfloor} s(\alpha_i^n) + \delta_n v^{\lfloor (i+1)/e \rfloor + 1} s(\alpha_{i+1}^n).
\end{aligned}$$

By induction hypothesis, if $\lambda < \sigma$, $d_{\alpha_i^n \lambda}(v) = 0$, and $d_{\alpha_{i+1}^n \lambda}(v) = 0$ unless $\lambda = \beta_{i+1}^n$, in which case $d_{\alpha_{i+1}^n \lambda}(v) = v^{\lfloor (i+1)/e \rfloor}$. Thus, by Proposition 4(2),

$$[G(\sigma)]_{hook} = v^{\lfloor i/e \rfloor} s(\alpha_i^n) + \delta_n v^{\lfloor (i+1)/e \rfloor + 1} s(\alpha_{i+1}^n)$$

as required.

Case 1b. $r = \overline{q}$**:** Here, $s \geq 1$ and $n - 1 - i = q + 2$. By Lemma 5, $f_r(s(\alpha_i^{n-1})) = 0$. Since $[G(\sigma)]_{hook} \neq 0$, we see from $(*)$, Theorem 3 and Lemma 5 that $[f_r(\delta_{n-1} s(\alpha_{i+1}^{n-1}))]_{hook} \neq 0$. This forces $e \mid (n-1)$ and $s = e - 2$. Thus $\widetilde{\sigma} = (q+2, (q+1)^{e-2}, q)$

and $\sigma = ((q+2)^2, (q+1)^{e-3}, q) = \beta^n_{i+2}$ and

$$\begin{aligned}[f_r(G(\widetilde{\sigma}))]_{hook} &= [f_r(v^{\lfloor (i+1)/e \rfloor + 1} s(\alpha^{n-1}_{i+1}))]_{hook} \\ &= v^{\lfloor (i+1)/e \rfloor + \delta_r} s(\alpha^n_{i+2}) \\ &= v^{\lfloor (i+2)/e \rfloor} s(\alpha^n_{i+2})\end{aligned}$$

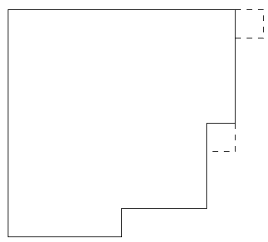
By induction hypothesis and Proposition 4(2),

$$[G(\sigma)]_{hook} = [f_r(G(\widetilde{\sigma}))]_{hook} = v^{\lfloor (i+2)/e \rfloor} s(\alpha^n_{i+2})$$

as required.

Case 2. $i > \lfloor (n-k-1)(1-\frac{1}{e}) \rfloor$**:** By induction hypothesis, we have

$$(**) \qquad [G(\widetilde{\sigma})]_{hook} = v^{\lfloor i/e \rfloor + \delta_{n-k}} s(\alpha^{n-k}_i) + \delta_{n-k} v^{\lfloor (i-1)/e \rfloor} s(\alpha^{n-k}_{i-1}).$$



$$\widetilde{\sigma} = \beta^{n-k}_i = ((q+1)^{s+1}, q^{e-2-s}, n-k-i-1)$$

From the above Young diagram of $\widetilde{\sigma}$, we see that the rightmost ladder of σ contains either of the first two indent nodes of $\widetilde{\sigma}$, or both. These three sub-cases correspond to (a) $k=1$, $r=\overline{q+1}$, (b) $k=1$, $r=\overline{-i-1}$ and (c) $k=2$, $r=\overline{q+1}=\overline{-i-1}$ respectively:

Case 2a. $k=1$, $r=\overline{q+1}$**:** Since $[G(\sigma)]_{hook} \neq 0$, we see from $(**)$, Theorem 3 and Lemma 5(1) that either $[f_r(s(\alpha^{n-1}_i))]_{hook}$ or $[f_r(\delta_{n-1} s(\alpha^{n-1}_{i-1}))]_{hook} \neq 0$. Suppose that $[f_r(\delta_{n-1} s(\alpha^{n-1}_{i-1}))]_{hook} \neq 0$. Then $e \mid (n-1)$, and $s=0$ by Lemma 5(1). If $n-i-1=q$, then $\widetilde{\sigma} = \beta^n_{i-1}$, and we have dealt with this in subcase 1b. If $n-i-1<q$, then by Proposition 7, we have $d_{\beta^{n-1}_{i-1}\widetilde{\sigma}}(v) = v$ and $d_{\beta^{n-1}_{i-2}\widetilde{\sigma}}(v) = 0$. As the two removable nodes of $\beta^n_{i-1} = (q^{e-1}, n-i)$ have residue r, and removing them in turn produces β^{n-1}_{i-1} $(= (q^{e-1}, n-i-1))$ and β^{n-1}_{i-2} $(= (q^{e-2}, q-1, n-i))$, we see that

$$\begin{aligned}\langle f_r(G(\widetilde{\sigma})), s(\beta^n_{i-1})\rangle &= \langle f_r(vs(\beta^{n-1}_{i-1})), s(\beta^n_{i-1})\rangle \\ &= 1.\end{aligned}$$

Thus $G(\beta^n_{i-1}) \mid f_r(G(\widetilde{\sigma}))$ by Proposition 4(1). Since

$$\begin{aligned}[f_r(v^{\lfloor (i-1)/e \rfloor} s(\alpha^{n-1}_{i-1}))]_{hook} &= v^{\lfloor (i-1)/e \rfloor} s(\alpha^n_{i-1}) \\ &= d_{\alpha^n_{i-1}\beta^n_{i-1}}(v) s(\alpha^n_{i-1})\end{aligned}$$

from induction hypothesis, we see that $[f_r(v^{\lfloor (i-1)/e \rfloor} s(\alpha^{n-1}_{i-1}))]_{hook}$ gives zero contribution to $[G(\sigma)]_{hook}$.

We thus conclude that $[f_r(s(\alpha^{n-1}_i))]_{hook} \neq 0$. Consequently, $s = \overline{n-2}$ (equivalently, $r = \overline{n-i-1}$) or $s = e-2$. If $s =$

$\overline{n-2} \neq e-2$, then since $\beta_i^n = ((q+1)^{s+1}, q^{e-2-s}, n-i-1)$ has two removable r-nodes, and removing them in turn produces $\widetilde{\sigma}$ and $\mu = ((q+1)^{s+1}, q^{e-3-s}, q-1, n-i-1)$, we have

$$\begin{aligned}\langle f_r(G(\widetilde{\sigma})), s(\beta_i^n)\rangle &= \left\langle f_r(s(\sigma) + d_{\mu\widetilde{\sigma}}(v)s(\mu)), s(\beta_i^n)\right\rangle \\ &= 1 + v d_{\mu\widetilde{\sigma}}(v)\end{aligned}$$

Note that by [2, Theorem 1], $d_{\mu\widetilde{\sigma}}(v) = d_{(m-1,1),(m)}(v)$, where $m = q-(n-i-2)$, which in turn equals v by Lemma 5(2). Thus $\langle f_r(G(\widetilde{\sigma})), s(\beta_i^n)\rangle = 1 + v^2$. Now, if $\beta_i^n \lhd \lambda \lhd \sigma$, then

$$\lambda = ((q+1)^{s+2}, q^{e-3-s}, n-i-2),$$

which has a unique removable r-node that upon removal produces $\widetilde{\lambda} = ((q+1)^{s+2}, q^{e-4-s}, q-1, n-i-2) \ntrianglelefteq \widetilde{\sigma}$. Thus,

$$\langle f_r(G(\widetilde{\sigma})), s(\lambda)\rangle = d_{\widetilde{\lambda}\widetilde{\sigma}}(v) \left\langle f_r(s(\widetilde{\lambda})), s(\lambda)\right\rangle = 0$$

by Theorem 2, so that $G(\beta_i^n) \mid f_r(G(\widetilde{\sigma}))$ (and $d_{\beta_i^n\sigma}(v) = v^2$) by Proposition 4(3). But

$$\begin{aligned}\langle f_r(G(\widetilde{\sigma})), s(\alpha_i^n)\rangle &= \left\langle f_r(v^{\lfloor i/e\rfloor+\delta_{n-1}} s(\alpha_i^{n-1}), s(\alpha_i^n)\right\rangle \\ &= v^{\lfloor i/e\rfloor+\delta_{n-1}} \\ &= d_{\alpha_i^n\beta_i^n}(v),\end{aligned}$$

so that $[f_r(v^{\lfloor i/e\rfloor+\delta_{n-1}} s(\alpha_i^{n-1}))]_{hook}$ gives zero contribution to $[G(\sigma)]_{hook}$, a contradiction. Thus, $s = e-2$, and hence $\sigma = \beta_{i+1}^n$ and

$$\begin{aligned}[f_r(G(\widetilde{\sigma}))]_{hook} &= [f_r(v^{\lfloor i/e\rfloor+\delta_{n-1}} s(\alpha_i^{n-1}))]_{hook} \\ &= \delta_n v^{\lfloor i/e\rfloor+\delta_{n-1}} s(\alpha_i^n) + v^{\lfloor i/e\rfloor+\delta_r+\delta_n} s(\alpha_{i+1}^n) \\ &= \delta_n v^{\lfloor i/e\rfloor} s(\alpha_i^n) + v^{\lfloor (i+1)/e\rfloor+\delta_n} s(\alpha_{i+1}^n)\end{aligned}$$

By induction hypothesis and Proposition 4(2) (similar to that used in Case 1a), we have $[G(\sigma)]_{hook} = [f_r(G(\widetilde{\sigma}))]_{hook}$ as required.

Case 2b. $k = 1$**,** $r = \overline{-i-1}$**:** Here, $\sigma = \beta_{i+1}^n$, and

$$\begin{aligned}[f_r(G(\widetilde{\sigma})]_{hook} &= [f_r(v^{\lfloor i/e\rfloor+\delta_{n-1}} s(\alpha_i^{n-1}) + \delta_{n-1} v^{\lfloor (i-1)/e\rfloor} s(\alpha_{i-1}^{n-1}))]_{hook} \\ &= \delta_n v^{\lfloor i/e\rfloor} s(\alpha_i^n) + v^{\lfloor i/e\rfloor+\delta_n+\delta_r} s(\alpha_{i+1}^{n-1}) \\ &= \delta_n v^{\lfloor i/e\rfloor} s(\alpha_i^n) + v^{\lfloor (i+1)/e\rfloor+\delta_n} s(\alpha_{i+1}^{n-1}).\end{aligned}$$

By induction hypothesis and Proposition 4(2) (similar to that used in Case 1a), we have $[G(\sigma)]_{hook} = [f_r(G(\widetilde{\sigma}))]_{hook}$ as required.

Case 2c. $k = 2$**,** $r = \overline{q+1}$**:** Here, $\widetilde{\sigma} = ((q+1)^{e-1}, q)$, $\sigma = (q+2, (q+1)^{e-1}) = \beta_i^n$, and

$$\begin{aligned}[f_r^{(2)}(G(\widetilde{\sigma}))]_{hook} &= [f_r^{(2)}(v^{\lfloor i/e\rfloor} s(\alpha_i^{n-2}))]_{hook} \\ &= v^{\lfloor i/e\rfloor+\delta_r} s(\alpha_{i+1}^n) \\ &= v^{\lfloor (i+1)/e\rfloor} s(\alpha_{i+1}^n)\end{aligned}$$

By induction hypothesis and Proposition 4(2), we have $[G(\sigma)]_{hook} = [f_r^{(2)}(G(\widetilde{\sigma}))]_{hook}$ as required.

□

4. Part (2) of Theorem 1

In this section, we deal with the case of $e = 2$. Let $\beta_i^n = (n-i, i)$ for $i < n/2$. Then $\beta_i^n = (\alpha_i^n)^R = (\alpha_{n-1-i}^n)^R$.

The following is just a reformulation of part (2) of Theorem 1.

Theorem 9. *For $0 \le i < n/2$,*

$$[G(\beta_i^n)]_{hook} = \sum_{\substack{i \le j < n-i \\ j \equiv i \ (\mathrm{mod}\ 2)}} \left(v^{\lfloor j/2 \rfloor} s(\alpha_j^n) + \delta_n v^{\lfloor j/2 \rfloor + 1 + \delta_{i+1}} s(\alpha_{j+1}^n) \right).$$

Furthermore, $[G(\sigma)]_{hook} = 0$ for all other 2-regular σ's.

Proof. We prove by induction on n. The theorem is trivially true for $n = 0, 1$. Now suppose that $[G(\sigma)]_{hook} \neq 0$, and that the theorem holds for all 2-regular $\mu \prec \sigma$. Let $\widetilde{\sigma}$ be the partition obtained by removing the last ladder of σ, which has size k and residue r. Then $[G(\widetilde{\sigma}))]_{hook} \neq 0$ by Theorem 3 and Lemma 5. By inductive hypothesis, $\widetilde{\sigma} = \beta_i^{n-k} = (n-k-i, i)$ for some $i < (n-k)/2$. Thus, $1 \le k \le 3$.

Case 1: $k = 1$**:** In this case, $\sigma = \beta_i = (n-i, i)$ with $i < (n-1)/2$, and $r = \overline{n-i-1}$. By inductive hypothesis,

$$[G(\widetilde{\sigma}))]_{hook} = \sum_{\substack{i \le j < n-1-i \\ j \equiv i \ (\mathrm{mod}\ 2)}} \left(v^{\lfloor j/2 \rfloor} s(\alpha_j^{n-1}) + \delta_{n-1} v^{\lfloor j/2 \rfloor + 1 + \delta_{i+1}} s(\alpha_{j+1}^{n-1}) \right).$$

Thus, by Lemma 5, we have

$$\begin{aligned}
[f_r(G(\widetilde{\sigma}))]_{hook} &= \sum_{\substack{i \le j < n-1-i \\ j \equiv i \ (\mathrm{mod}\ 2)}} \left(v^{\lfloor j/2 \rfloor} [f_r(s(\alpha_j^{n-1}))]_{hook} + \delta_{n-1} v^{\lfloor j/2 \rfloor + 1 + \delta_{i+1}} [f_r(s(\alpha_{j+1}^{n-1}))]_{hook} \right) \\
&= \sum_{\substack{i \le j < n-1-i \\ j \equiv i \ (\mathrm{mod}\ 2)}} \left(v^{\lfloor j/2 \rfloor} s(\alpha_j^n) + \delta_n v^{\lfloor j/2 \rfloor + 1 + \delta_{i+1}} s(\alpha_{j+1}^n) + \delta_{n-1} v^{\lfloor j/2 \rfloor + 1} s(\alpha_{j+2}^n) \right) \\
&= \sum_{\substack{i \le j < n-1-i \\ j \equiv i \ (\mathrm{mod}\ 2)}} \left(v^{\lfloor j/2 \rfloor} s(\alpha_j^n) + \delta_n v^{\lfloor j/2 \rfloor + 1 + \delta_{i+1}} s(\alpha_{j+1}^n) \right) + \delta_{n-1} [G(\beta_{i+2}^n)]_{hook},
\end{aligned}$$

by inductive hypothesis. Now by Proposition 6(2),

$$d_{\beta_j^{n-1} \widetilde{\sigma}}(v) = \begin{cases} 1, & \text{if } j = i; \\ v, & \text{if } j = i+1 \text{ and } 2 \mid (n-1); \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, $\left\langle f_r(s(\lambda)), s(\beta_j^n)\right\rangle = 0$ unless $\lambda = \beta_{j-1}^{n-1}$ (when $r = \overline{j}$) or β_j^{n-1} (when $r = \overline{n-j-1}$). As such, we have

$$\begin{aligned}\langle f_r(G(\widetilde{\sigma})), s(\beta_j^n)\rangle &= \sum_\lambda d_{\lambda\widetilde{\sigma}}(v)\,\langle f_r(s(\lambda)), s(\beta_j^n)\rangle \\ &= \delta_{r-j} v^{\delta_n-\delta_{n-1}} d_{\beta_{j-1}^{n-1}\widetilde{\sigma}}(v) + \delta_{r-n+j+1} d_{\beta_j^{n-1}\widetilde{\sigma}}(v) \\ &= \begin{cases} 1, & \text{if } j = i; \\ 1, & \text{if } j = i+2 \text{ and } 2 \mid (n-1); \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Thus, keeping the notations of Theorem 3, we have in fact $L_{\beta_{i+2}^n}(v) = \delta_{n-1}$, and $L_{\beta_j^n}(v) = 0$ for all $j \neq i, i+2$ by Proposition 4(3). Hence, together with induction hypothesis, we have

$$[G(\sigma)]_{hook} = \sum_{\substack{i \leq j < n-i \\ j \equiv i \ (\mathrm{mod}\ 2)}} \left(v^{\lfloor j/2 \rfloor} s(\alpha_j^n) + \delta_n v^{\lfloor j/2 \rfloor + 1 + \delta_{i+1}} s(\alpha_{j+1}^n)\right)$$

as claimed.

Case 2: $k = 2$**:** In this case, $\sigma = (a+1, a) = \beta_a^n$ and $\widetilde{\sigma} = (a, a-1) = \beta_{a-1}^{n-2}$, where $n = 2a+1$, and $r = \overline{a}$. By induction hypothesis, $[G(\widetilde{\sigma})]_{hook} = v^{\lfloor (a-1)/2 \rfloor} s(\alpha_{a-1}^{n-2})$, so that by Lemma 5, we have

$$\begin{aligned}[f_r^{(2)}(G(\widetilde{\sigma}))]_{hook} &= v^{\lfloor (a-1)/2 \rfloor + \delta_a} s(\alpha_a^n) \\ &= v^{\lfloor a/2 \rfloor} s(\alpha_a^n).\end{aligned}$$

Since $d_{\alpha_a^n \mu}(v) = 0$ for all $\mu \lhd \sigma$ by induction hypothesis, we have $[G(\sigma)]_{hook} = v^{\lfloor a/2 \rfloor} s(\alpha_a^n)$ by Proposition 4(2) as claimed.

Case 3: $k = 3$**:** In this case, $\sigma = (3,2,1)$. As σ is a 2-core partition, we have $G(\sigma) = \sigma$ by Theorem 2, contradicting $[G(\sigma)]_{hook} \neq 0$.

□

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