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Structured squaring down and zero assignment

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The problem of zero assignment by squaring down is considered for a system of p-inputs, n-outputs and n-states ($m > p$), where not all outputs are free variables for design. We consider the case where a k-subset of outputs is preserved in the new output set, and the rest are recombined to produce a total new set of p-outputs. New invariants for the problem are introduced which include a new class of fixed zeros and the methodology of the global linearization, developed originally for the output feedback pole assignment problem, is applied to this restricted form of the squaring down problem. It is shown that the problem can be solved generically if $(p-k)(m-p) > \delta^*$, where k ($k < p$) is the number of fixed outputs and δ^* is a system and compensation scheme invariant, which is defined as the restricted Forney degree.

1. Introduction

Transfer function models produced at the end of process synthesis and overall instrumentation are usually large dimension models containing many physical variables as inputs and outputs. In general, such models are non-square and have more inputs and outputs than those that can be used for control design and are frequently referred to as progenitor models (Karcanias and Vafiadis 2002a,b). Deriving system models with a smaller set of effective inputs, outputs leads to new transfer functions, which are seen as the results of pre-compensation (input variable reduction) and post-compensation (output variable reduction) (Karcanias 1992, 1994). Such transformations may be assumed to be constant (fast dynamics of the instrumentation scheme) and the resulting models may be square, or non-square. Such models are referred to as input-, output-reduced models and their structure evolves from that of the original progenitor model (containing all available input and output variables) (Karcanias and Vafiadis 2002a). The study of the structure of the input-, output-reduced models as a function of the pre- and post-compensation schemes and the original structure of the progenitor model is one of the central problems of integrated control and instrumentation design and it is frequently referred to as model projection problems (Karcanias 1992, 1994). Such problems are within the general area of structure assignment problems (Loiseau *et al*. 1997, Vardoulakis 1980). This family of problems contains as a sub-problem the standard squaring down problem (Kouvaritakis and MacFarlane 1976, Karcanias and Giannakopoulos 1989) as well as the problems of well conditioning of early system models (Karcanias and Vafiadis 2002a). This paper deals with a restricted version of the standard squaring down problem (fewer degrees of freedom than the general problem), which is more frequently met in applications; this involves the fixing of a number k, $k < p$, of the desirable outputs to be elements of the original set and the use of all m variables to produce p-k linear combinations to define additional outputs, which together with the fixed k variables yield the effective p output set. Deriving the p-k linear combinations involves post-compensation that affects the structure of the resulting square transfer function. The assignment of the zeros of this transfer function is the problem considered here.

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The general squaring down problem (Kouvaritakis and MacFarlane 1976, Karcanias and Giannakopoulos 1989) belongs to the family of determinantal assignment problems (DAP). The DAP approach (Karcanias and Giannakopoulos 1984) has been formulated as a unifying approach for all problems of frequency assignment (pole, zero) and they are problems of multilinear nature and they may be naturally split into a linear and multilinear problem. The final solution is thus reduced to the solvability of a set of linear equations (characterizing the linear problem), together with quadratics (characterizing the multilinear problem of decomposability). The approach heavily relies on exterior algebra (Marcus, 1973) and this has implications on the computability of solutions (reconstruction of solutions whenever they exist) and introduces new sets of invariants, which in turn characterize the solvability of the problem. The reduction of the restricted squaring down to an equivalent free squaring down is an integral part of our approach; this involves the transferring of the restricted structure of the compensation scheme to an appropriate system representation that allows use of the "free DAP" formulation and also leads to the definition of the fixed zeros of the problem. These zeros are those of the original system and new fixed zeros are introduced by the restricted structure of the scheme. The distinct advantages of the restricted DAP approach that is used here are: (i) it provides the means for defining the additional fixed zeros of the problem in an explicit way that also allows the structure redesign to avoid the formation of undesirable invariant features (such as non-minimum phase characteristics). (ii) Enables the use of the "free DAP" approach that can handle both generic and exact solvability investigations (using intersection theory of algebraic varieties), and thus leads to new criteria for the characterization of solvability of different problems. (iii) Provides a systematic procedure for computing the solutions using the new system and compensator structure dependent invariants and the recently developed method of the Global Linearization Framework (Leventides and Karcanias, 1995a) that provides a new powerful technique for establishing solvability conditions of DAP type problems, as well as computing them.

The general problem of squaring down by constant pre-compensation has been introduced in Rosenbrock and Rowe (1970), Kouvaritakis and MacFarlane (1976) and studied using an exterior algebra formulation in Karcanias and Giannakopoulos (1989); state space methods for the study of the problem have been used in Kouvaritakis and MacFarlane (1976), Karcanias and Kouvaritakis (1979) and Saberi and Sannuti (1990). Although conditions for solvability of the "free squaring down" problem have been previously derived, no method for handling solvability and computation of solutions have been derived so far, that can also address the "partially fixed" nature of the compensator structure and its implications. The current approach follows the determinantal assignment problem formulation deployed in Karcanias and Giannakopoulos (1989) and uses the global linearization methodology developed for the pole assignment by output feedback (Leventides and Karcanias 1995a). There are similarities between the general squaring down and the pole assignment by output feedback problems, as far as their mathematical formulation; however, the restricted versions differ significantly due to the nature of the new invariants (expressing the structure of the compensator on the system representation), the possible creation of additional "fixed zeros", the fewer degrees of freedom and their implications for the proof of results in the generic case. This makes the study of the restricted version an interesting problem with practical significance for control design, since the current framework allows the use of the restricted structure as a design parameter. Apart from providing new results for this interesting case, the paper introduces an explicit way for computing the additional "fixed zeros" resulting by the partially fixed structure of the compensator and develops a new computational procedure for finding the zero assigning squaring down compensators based on the tools provided by the global linearization. The current framework enables the use of the partially fixed structure of the compensator as a design parameter, which may be appropriately adjusted to avoid additional undesirable fixed zeros. Regarding the solvability of the zero assignment of the free zeros of the problem, It is shown that the restricted form of the squaring down problem can be solved generically if $(p-k)(m-p) > \delta^*$, where k $(k < p)$ is the number of fixed outputs and δ^* is a new invariant associated with the system and expressing the restricted structure of the problem, which is referred to as the restricted Forney degree. The solvability conditions are based on the properties of the invariants of the exterior algebra framework (Karcanias and Giannakopoulos 1984); the current approach has the potential to handle the generic, as well as the non-generic cases. The results derived for the restricted problem also apply to the case of the full squaring down and then become equivalent to those in Karcanias and Giannakopoulos (1989), while at the same time provide a systematic computational framework for finding solutions (when such solutions exist), which is simplest than the solution of linear and quadratic equations (quadratic Plucker relations) of the mainstream DAP approach. The explicit role of the squaring down structure in the shaping of the solvability conditions provides a framework for deriving a design methodology where the compensator structure becomes a design parameter.

Throughout the paper we shall denote by $R^{n\times p}$ the set of $n\times p$ real matrices and $\mathbb{Z}^+$, the positive integers. The rank of a matrix A is denoted by rank(A) and $\mathcal{N}_r(A)$, $\mathcal{N}_\ell(A)$, denote its right, left nullspace. $R[s]$ denotes the set of polynomials with coefficients from the reals, R, and if $t(s)\in R[s]$, then $\deg[t(s)]$ denotes its degree. Finally, if a property is said to be true for $i\in \underset{\sim}{n}$, $n\in\mathbb{Z}^+$, this means it is true for all $1\le i\le n$.

2. Problem definition and preliminary results

2.1 *The restricted squaring down problem*

Consider the linear system described by

$$S(A,B,C,D):\quad \left.\begin{array}{lll} \underline{\dot{x}}=A\underline{x}+B\underline{u}, & A\in R^{n\times n}, & B\in R^{n\times p} \\ \underline{y}=C\underline{x}+E\underline{u}, & C\in R^{m\times n}, & E\in R^{m\times p}\end{array}\right\} \tag{1}$$

where $(A,\ B)$ is controllable, $(A,\ C)$ is observable. Equivalently, the system is represented by the transfer function matrix $G(s)=C(sI-A)^{-1}B+E$, where $\operatorname{rank}_{R(s)}\{G(s)\}=\min\{m,p\}$. In terms of left, right coprime matrix fraction descriptions (Kailath 1980), $G(s)$ may be represented as

$$G(s)=D_\ell(s)^{-1}N_\ell(s)=N_r(s)D_r(s)^{-1}, \tag{2}$$

where $N_\ell(s)$, $N_r(s)\in R^{m\times p}[s]$, $D_\ell(s)\in R^{m\times m}[s]$ and $D_r(s)\in R^{p\times p}[s]$. The system will be called square if $m=p$ and non-square if $m\neq p$.

For a system with $m>p$ we can expect to have independent control over at most p linear combinations of m outputs. If $\underline{C}\in R^p$ is the vector of the variables which are to be controlled, then $\underline{C}=Ky$, where $K\in R^{p\times m}$ is a squaring down postcompensator, and $G'(s)=KG(s)$ is the squared down transfer function matrix. A right MFD for $G'(s)$ is defined by $G'(s)=KN_r(s)D_r(s)^{-1}$ where $G(s)=N_r(s)D_r(s)^{-1}$. Finding K such that $G'(s)$ has assigned zeros is defined as the zero assignment by squaring down problem. The zero polynomial of $S(A,B,KC,KE)$ is then given by

$$z_k(s)=\det\{KN_r(s)\}. \tag{3}$$

A special form of the general squaring down is the restricted form that preserves a k-subset y_1 of the original set of outputs, $k<p$, and this without loss of generality may be defined by

$$\underline{z}=K_f\underline{y}=\begin{bmatrix} I_k & 0\\ k_1 & k_2\end{bmatrix}\begin{bmatrix}\underline{y}_1\\ \underline{y}_2\end{bmatrix}. \tag{4}$$

Clearly, such compensators have fewer degrees of freedom than the full K compensator; this form of the problem will be referred to as k-restricted squaring down (k-RSD), k referring to the number of fixed variables, and it is the subject of the current investigation.

2.2 *The general constant DAP*

The general squaring down and the restricted version belong to the family of the determinantal assignment problem (DAP) (Karcanias and Giannakopoulos 1984), that is the study of solutions of the equation (5) with respect to a design matrix H. In fact, let $M(s)\in R[s]^{p\times r}[s]$, $r\le p$, be such that $\operatorname{rank}(M(s))=r$ and let $\{H\}$ be a family of full rank $r\times p$ constant matrices having a certain structure. The problem is to solve with respect to $H\in\{H\}$ the equation:

$$f_M(s,H)=\det(HM(s))=f(s), \tag{5}$$

where $f(s)$ is a real polynomial of an appropriate degree d.

Remark (1): The degree of $f(s)$ depends on the degree of $M(s)$ as well as on the structure of H; however, in most cases, the degree of $f(s)$ is equal to the degree of $M(s)$.

The determinantal assignment problem has two main aspects. The first has to do with the solvability conditions for the problem and the second, whenever this problem is solvable, to provide methods for constructing the solutions.

Notation (Marcus and Minc 1964)**:** Let $Q_{k,n}$ denote the set of lexicographically ordered, strictly increasing sequences of k integers from $\{1,\dots,n\}$. If $\{\underline{x}_{i_1},\dots,\underline{x}_{i_k}\}$ is a set of vectors of a vector space V, $\omega=(i_1,\dots,i_k)\in Q_{k,n}$, then we denote by $\underline{x}_{i_1}\wedge\cdots\wedge\underline{x}_{i_k}=\underline{x}_\omega\wedge$ their exterior product. If $H\in F^{m\times n}$ and $r\le\min\{m,n\}$, then by $C_r(H)$ we denote the rth compound matrix of H.

If $\underline{h}_i^t$, $\underline{m}_i(s)$, $i\in r$, denotes the rows of H, columns of $M(s)$ respectively, then

$$\begin{aligned} C_r(H)&=\underline{h}_1^t\wedge\cdots\wedge\underline{h}_r^t=h^t\wedge\in R^{1\times\sigma},\\ C_r(M(s))&=\underline{m}_1(s)\wedge\cdots\wedge\underline{m}_r(s)\\ &=\underline{m}\wedge\in R^\sigma[s],\quad \sigma=\binom{p}{r}\end{aligned} \tag{6}$$

and by the Binet–Cauchy theorem (Marcus and Minc 1964) we have that

$$f_M(s,H) = C_r(H)C_r(M(s)) = <\underline{h}\wedge, \underline{m}(s)\wedge> \\ = \sum_{\omega\in Q_{r,p}} h_\omega m_\omega(s), \quad (7)$$

where $\langle\cdot,\cdot\rangle$ denotes inner product, $\omega = (i_1,\dots,i_r)\in Q_{r,p}$, and h_ω, $m_\omega(s)$ are the coordinates of $\underline{h}\wedge, \underline{m}(s)\wedge$ respectively. Note that h_ω is the $r\times r$ minor of H which corresponds to the ω set of columns of H and thus h_ω is a multilinear alternating function of the entries h_{ij} of H. Thus, the study of the zero structure of the multilinear function $f_M(s,H)$ may be reduced to a linear sub-problem and a standard multilinear algebra problem as it is shown below (Karcanias and Giannakopoulos 1984).

(i) Linear subproblem of DAP: Set $\underline{m}(s)\wedge = \underline{p}(s)\in R^\sigma[s]$. Determine whether there exists a $\underline{k}\in R^\sigma$, $k\neq 0$, such that

$$f_M(s,k) = \underline{k}^t - \underline{p}(s) = \sum k_i p_i(s) = f(s), \\ i\in\sigma, \quad f(s)\in R[s]. \quad (8)$$

(ii) Multilinear subproblem of DAP: Assume that $\{\Delta\}$ is the family of solution vectors k of (7). Determine whether there exists $H^t = [h_1,\dots,h_r]$, where $H^t\in R^{p\times r}$, such that

$$\underline{h}_1\wedge\cdots\wedge\underline{h}_r = h_\wedge = \underline{k}, \quad \underline{k}\in\{\Delta\} \quad (9)$$

Polynomials defined by equation (8) are called polynomial combinants and the zero assignability of them provides necessary conditions for the solution of the DAP. The solution of the exterior equation (9) is a standard problem of exterior algebra and it is known as decomposability of multivectors (Marcus 1973). The multi-vectors $\underline{m}(s)_\wedge$ introduce system invariants (Karcanias and Giannakopoulos 1984) which play a crucial role for the solvability of DAP and for the case of the squaring down problem are defined below.

Column Plucker matrices: For the transfer function $G(s)$, $m\geq p$, we denote by $\underline{n}(s)_\wedge$ the exterior product of the columns of the numerator $N_r(s)$, of a RCMFD and by $P(N)$ the

$$\binom{m}{p}\times(d+1)$$

basis matrix of $\underline{n}(s)\wedge$. Note that $d = \deg\{\underline{n}(s)\wedge\} = \delta$, is the Forney order (Forney, 1975) of the column rational vector space X_g of $G(s)$, if $G(s)$ has no finite zeros and $d = \delta+\kappa$, where κ is the number of finite zeros of $G(s)$, otherwise (Karcanias and Giannakopoulos 1984). If $N_r(s)$ is least degree (has no finite zeros), then $P(N)$ will be called the column space Plucker matrix of the system.

The essence of the DAP approach is projective, that is we use a natural embedding for determinantal problems to embed the space of the unknown, H, of DAP, into an appropriate projective space. In this way we can view DAP as a search for common solutions of some set of linear equations and another set of second order polynomial equations. This study requires to compactify H into $H^\#$ and then use algebraic geometric, or topological intersection theory methods (Leventides and Karcanias 1995b) to determine existence of real solutions for the above sets of equations. The characteristic of the approach is that it allows the use of algebraic geometry and topological methods for the study of solvability conditions but also provides a natural setup for computations. Central to the latter is the solution of the linear system derived by (7) with the quadratics characterizing the solvability of (8), which are known as quadratic Plucker relations (QPR) (Marcus 1973). A new method for the study of DAP has been recently developed based on the linearization of the (5) polynomial combinant. This method is based on special sequences of compensators K which in the limit converge to a so called degenerate squaring down. This technique is referred to as global linearization (Leventides and Karcanias 1995a, 1996) and has the advantage that it asymptotically reduces the multilinear problem to a linear one without reducing the number of free parameters in the compensator. These techniques will be used for the study of the restricted squaring down problem subsequently. The above technique provides an explicit exterior algebra based framework, which may be studied using as an alternative tool the Grobner basis theory (Becker and Weispfenning 1993). This framework is within the overall algebraic geometry approach for control theory (Brockett and Byrnes 1981, Helton *et al.* 1997, Wang 1994).

3. The restricted squaring down problem and its invariants

3.1 *Problem formulation*

Consider a system of p-inputs, m-outputs and n-states with $m > p$. Defining a set of p effective outputs out of the available m outputs, that can be independently controlled, is realized by a transformation that is represented as a static post-compensation. Using such compensators K we obtain a family of systems described by the set of transfer functions:

$$\Omega = \{KG(s)\in R^{p\times p}(s) : K\in R^{p\times m}\}. \quad (10)$$

The post compensator K is assumed to be full rank, but it is otherwise arbitrary. A special subset of the general family of such compensators are those which fix some of the outputs; this corresponds to the design requirement where some variables must be measured and controlled as they appear in the original set and then recombine the rest of the variables such that to produce a number of p outputs. Such restricted forms may be represented by matrices having the following general structure:

$$K = \begin{bmatrix} x & \cdots & x & 0 & x & \cdots & x & 0 & x & \cdots & x & 0 & x & x \\ x & \cdots & x & 0 & x & \cdots & x & 0 & x & \cdots & x & 0 & x & x \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & & \cdots & x & 0 & x & x \\ x & \cdots & x & 0 & x & \cdots & x & 0 & x & \cdots & x & 1 & 0 & 0 \end{bmatrix} \tag{11}$$

and can be represented as

$$K = QK_fR, \tag{12}$$

where Q, R are permutation matrices and K_f is the matrix defined by (4) where K_1, K_2 are arbitrary matrices. Note that given the k-set of output variables which are to be preserved, then the permutation matrices Q and R are defined. The structure of K_f defines a normalized representation of the K-restricted family of compensators. The subfamily of the systems of Ω which correspond to restricted compensators is denoted by Ω_r and the set of compensators having a given fixed structure (as defined above) is denoted by Φ_k. The systems in Ω are square and their zeros are given as the roots of the equation

$$\mathrm{Det}(KN_r(s)) = 0. \tag{13}$$

The representation of the restricted squaring down compensators, given by (12), may be further simplified as shown below:

Proposition (1): *The family of restricted squaring down compensators Φ_k corresponding to a given set of k-fixed outputs may be represented as*

$$K^* = \begin{bmatrix} I_k & 0 \\ 0 & \bar{K} \end{bmatrix} \quad R = KR, \tag{14}$$

where R is a permutation matrix and $\bar{K}$ is an arbitrary matrix.

Proof: It is obvious that for any K given as in (12) we can always find a full rank matrix $\tilde{Q}$ such that $K^* = \tilde{Q}K$. It is clear that since $\det\{\tilde{Q}\} = c \neq 0$, the polynomials $\mathrm{Det}(KN_r(s)) = c\,\mathrm{Det}(K^* N_r(s))$ have the same roots and thus we can use the parametrization in (14) for the study of the restricted squaring down. □

Remark 2: The permutation matrix R in (14) may always be standardized for any k-fixed set of outputs. In fact, using the structure of K as shown in (11) we can define R as the column permutation matrix acting on K that shifts all standard basis column vectors appearing in K to the front and with the order they appear. Such action produces a unique matrix R that is defined from the given fixed set of outputs. The matrix R may be referred as the permutation matrix of the k-fixed given set of outputs.

The above suggest that the zero polynomial under the restricted squaring down may be expressed as

$$z(s) = \mathrm{Det}\{\bar{K}\ R\ N_r(s)\} = \mathrm{Det}\{\bar{K}\ N_r^*(s)\}, \tag{15}$$

where $N_r^*(s)$ is a matrix obtained from the original numerator by the R permutation of its rows. This matrix will be referred as the R-numerator and it is defined from any right numerator and the given matrix R. If we partition the R-numerator as

$$N_r^*(s) = \begin{bmatrix} N_{11}(s) \\ N_{12}(s) \end{bmatrix} \tag{16}$$

then the family of numerators of squared down systems under restricted squaring down may be expressed as

$$\hat{N}(s) = \begin{bmatrix} I & 0 \\ 0 & \bar{K} \end{bmatrix} \begin{bmatrix} N_{11}(s) \\ N_{12}(s) \end{bmatrix} \begin{bmatrix} N_{11}(s) \\ \bar{K}N_{12}(s) \end{bmatrix}. \tag{17}$$

The above expression will be used for the study of the problem, which is now defined as selecting a full rank $\bar{K}$ matrix such that the polynomial z(s) has assignable zeros, where $z(s)$ is defined by:

$$z(s) = \mathrm{Det} \begin{bmatrix} I & 0 \\ 0 & \bar{K} \end{bmatrix} \begin{bmatrix} N_{11}(s) \\ N_{12}(s) \end{bmatrix}. \tag{18}$$

3.2 *Fixed zeros of the restricted squaring down problem*

The partially fixed structure of the compensation scheme may introduce some constraints for the arbitrary zero assignment, which are expressed as "fixed zeros". These fixed zeros are associated with the partially fixed nature of the compensator and are independent from the numerical values in the compensator. The characterization of these "fixed zeros" is examined next. Consider the "squared down" numerator $\hat{N}(s)$ where $\bar{K} \in R^{(p-k)\times(m-k)}$ is arbitrary and $N_r(s) \in R^{m\times p}[s]$ is the original numerator. If $Z_r(s)$ is a right gcd matrix of $N_r(s)$, $Z_r(s) \in R^{p\times p}[s]$, then

$$\hat{N}(s) = \begin{bmatrix} N_{11}(s) \\ \bar{K}N_{12}(s) \end{bmatrix} Z_r(s) \tag{19}$$

and thus

$$z(s) = \left|\hat{N}(s)\right| = \det\left\{\begin{bmatrix} N_{11}(s) \\ \bar{K}N_{12}(s) \end{bmatrix}\right\}.\left|Z_r(s)\right|. \tag{20}$$

The characterization of fixed zeros is given by the following result.

Proposition 2: *For the k-restricted squaring down problem the following properties hold true*:

(i) *The zeros of the original system, as defined by the zeros of $|Z_r(s)|$ are also zeros of all systems in the family Ω of squared down systems.*
(ii) *If $N_{11}(s) \in R^{k \times p}[s]$ is not a least degree matrix and $N_{11}(s) = Z_\ell(s)N_{11}(s)$, where $Z_\ell(s)$ is a left gcd of $N_{11}(s)$, then the zeros of $|Z_\ell(s)|$ are also fixed zeros for all elements in the family Ω.*
(iii) *The overall zero polynomial under all restricted squared down compensators is given by*

$$z(s) = \left|\widehat{N}(s)\right| = \left|Z_\ell(s)\right|\left|\begin{bmatrix} \bar{N}_{11}(s) \\ \bar{K}N_{12}(s) \end{bmatrix}\right| \cdot \left|Z_r(s)\right| \tag{21}$$

where $|Z_\ell(s)|, |Z_r(s)|$ express fixed polynomials and the assignable polynomial is

$$\tilde{z}(s) = \det\left\{\begin{bmatrix} \bar{N}_{11}(s) \\ \bar{K}N_{12}(s) \end{bmatrix}\right\}. \tag{22}$$

□

The study of the equation (21) is facilitated by the reduction of the restricted version to an equivalent general squaring down problem and this reduction introduces some additional fixed zeros. The assignment of zeros of (22) is not in the standard DAP form (Karcanias and Giannakopoulos 1984); however, this extended DAP form may be reduced to the standard form; this requires some manipulation to make $\bar{K}$ to appear explicitly as a design parameter. We examine next the basic properties of the associated zero assignment map and then produce an algebraic formulation that show the equivalence of the restricted squaring down to that of an equivalent free DAP formulation. This analysis introduces new invariants for the restricted problem.

3.3 *Properties of the frequency assignment map*

The frequency assignment map associated with the problem is defined by equation (22) and it is the map ssigning $\overline{\mathrm{K}}$ to the coefficient vector of $\tilde{z}(s) = p(s)$, p, associated with a polynomial $p(s) \in R[s]$, that is:

$$\begin{aligned} &F: R^n \to R^{(p-k)\times(m-k)} \to R^{d+1} \\ &F(\bar{K}) = \varphi \end{aligned} \tag{23}$$

The zero assignment problem is to find $\bar{K}$ such that $F(\bar{K}) = p$ for a given p. Clearly, a system has the arbitrary zero assignment property if F is onto. An important family of compensators, which is crucial for the problem, is the family of the so called degenerate compensators. A compensator $\bar{K}$ is degenerate, if $F(\bar{K}) = 0$, or equivalently

$$\left|\begin{bmatrix} I & 0 \\ 0 & \bar{K} \end{bmatrix} N_r * (s)\right| = 0. \tag{24}$$

In other words, $\bar{K}$ is degenerate if the numerator of the squared down system becomes singular. The notion of degenerate feedback was introduced in Brockett and Byrnes (1981) for the case of output feedback and it is now extended to the case of squaring down. The following result shows the importance of degenerate compensators since it establishes the very important property that if the zero assignment map is locally onto at a degenerate compensator, then the map is globally onto.

Theorem 1: *If there exists a degenerate matrix $\bar{K}_0$ such that the differential $DF_{\bar{K}_0}$ is onto, then any polynomial of degree d can be assigned via some static compensator*.

Proof: The map F comes from a determinantal expansion of $(p-k)\times(p-k)$ determinant and hence has the property that

$$F(\lambda\bar{K}) = \lambda^{p-k}F(\bar{K})$$

Since $DF_{\bar{K}_0}$ is onto and $F(\bar{K}_0) = 0$, the map F is locally onto at a neighbourhood of 0, and therefore the image of F contains a sphere $S(0, \varepsilon)$ for some $\varepsilon > 0$. To prove that F is globally onto we consider any $\underline{p} \in R^{d+1}$ and we construct a K such that $F(\bar{K}) = \underline{p}$. To do so we select a positive λ_0 such that $|\lambda_0| < \sqrt[p-k]{\varepsilon/\|\underline{p}\|}$. For this λ_0 we have that $\|\lambda_0^{p-k}\underline{p}\| < \varepsilon$. The vector $\lambda_0^{p-k}\underline{p}$ constructed this way belongs to $S(0, \varepsilon)$. Since F is onto in the neighborhood $S(0,\epsilon)$, there exists a $\bar{K}_1$ such that $F(\bar{K}_1) = \lambda_0^{p-k}\underline{p}$. For this $\overline{\mathrm{K}}_1$ we have that $F(\lambda_0^{-1}\bar{K}_1) = \underline{p}$, proving that F is onto. □

The above result suggests that degenerate solutions provide the means for developing a sufficient approach for studying zero assignment using special forms of squaring down; such an approach is known as global linearization methodology (Leventides and Karcanias 1995a, 1996) and will be developed for the special case of restricted squaring down here. This solution is based on the construction of a degenerate structured compensator for this specific problem so that the differential of the zero assignment map on this compensator is onto. The specific structure of the problem produces new system invariants, based on the partitioning of the numerator matrix $N_r(s)$ induced by the structure of K_f, as defined in (4).

3.4 *Reduction of partially fixed squaring down to an equivalent free problem*

An alternative more convenient reduction of the original restricted compensator formulation to a standard determinantal assignment is considered next. In fact, if we partition $N_r^*(s)$ as in (16) (conformally to diag$(I_k,\bar{K})$), then the previous problem formulation is reduced to

$$z(s) = \left| \begin{bmatrix} N_{11}(s) \\ \bar{K}N_{12}(s) \end{bmatrix} \right| \tag{25}$$

which leads to an equivalent standard DAP formulation and reveals the nature of invariants characterizing the solvability of the problem.

Theorem 2: *Let $V(s)$ be a least degree polynomial basis for the $p-k$ dimensional right kernel of $N_{11}(s)$ then*

$$z(s) = \begin{vmatrix} N_{11}(s) \\ \bar{K}N_{12}(s) \end{vmatrix} = z_1(s)\left|\bar{K}N_{12}(s)V(s)\right|z_r(s), \tag{26}$$

where $z_1(s)$ is the zero polynomial of $N_{11}(s)$ and $z_r(s)$ is the zero polynomial of the non-square system.

Proof: Consider a unimodular transformation $U(s)$ such that

$$N_{11}(s)U(s) = \begin{bmatrix} Z_\ell(s) & 0 \end{bmatrix},$$

where $Z_\ell(s)$ is the $k \times k$ greatest common left divisor of $N_{11}(s)$. By partitioning the $p \times p$ unimodular matrix $U(s) = [W(s), \quad V(s)]$, where $W(s)$ is $p \times k$ and $V(s)$ is $p \times (p-k)$, then

$$N_{11}(s)W(s) = Z_\ell(s) \quad N_{11}(s)V(s) = 0 \tag{27a}$$

and

$$\begin{aligned} \det\begin{bmatrix} N_{11}(s) \\ KN_{12}(s) \end{bmatrix} &= \det\left(\begin{bmatrix} N_{11}(s) \\ KN_{12}(s) \end{bmatrix} U\right) \\ &= \det\begin{bmatrix} N_{11}(s)W(s) & N_{11}(s)V(s) \\ KN_{12}(s)W(s) & KN_{12}(s)V(s) \end{bmatrix} \\ &= \det\begin{bmatrix} Z_\ell(s) & 0 \\ KN_{12}(s)W(s) & KN_{12}(s)V(s) \end{bmatrix} \\ &= \det\left|KN_{12}(s)V(s)\right| \det\, Z_l(s) \end{aligned} \tag{27b}$$

and this proves the result where $z_\ell(s) = \det Z_\ell(s)$ and $z_r(s) = \det Z_r(s)$. □

The matrix $M(s) = N_{12}(s)V(s)$ defined above provides an equivalent "free" squaring down formulation of the problem and will be referred to as the generator of the restricted squaring down problem. From the above result we have the following obvious remark.

Remark 3: For the restricted squaring down the equivalent formulation of equation (26) indicates the following set of invariants.

(i) The zero structure of $N_{11}(s)$ as defined by the greatest left matrix divisors is a problem invariant.
(ii) The assignability of additional zeros depends on the $R[s]$-column module defined by the matrix

$$M(s) = N_{12}(s)V(s) \in R[s]^{(m-k)\times(p-k)} \tag{28}$$

and its corresponding Plucker matrix, which is also an invariant of the problem.

Corollary 1: *Let $M(s) = M^*(s)Z^*(s)$ be a factorization $(m > p)$, where $M^*(s)$ is a least degree basis for the $R[s]$-column module of $M(s)$, $Z^*(s)$ is a greatest right divisor and let $z^*(s) = |Z^*(s)|$. The zero polynomial of the k-RSD is then given by*

$$z(s) = \det(KM^*(s))z_l(s)zr(s)z^*(s) \tag{29}$$

where $z_f(s) = z_l(s)z_r(s)z^*(s)$ is the fixed zero polynomial of k-RSD.

Remark 4: By inspection of equation (29) we have that the assignable polynomial of the k-RSD is $\tilde{z}(s) = \det(\bar{K}M^*(s))$ and that

$$\deg N_r(s) \geq \deg N_{12}(s)V(s) + \deg Z_1(s) \geq \deg(\tilde{Z}(s)) \tag{30}$$

which indicates that the number of assignable zeros of the restricted squaring down can be less than the same number of the full case.

If $C_{p-k}(M^*(s)) = g_r^*(s) = P^* e_{\delta^*+1}(s)$, then P^* is referred to as the k-RSD Plucker matrix and δ^* is the corresponding *restricted* Forney order. Clearly, the linear part of the problem (condition (8)), yields the following result:

Corollary 2: *Necessary condition for arbitrary zero assignment of the k-RSD is that* rank$(P^*) = \delta^*+1$.

3.5 *Properties of the generator matrix of the restricted squaring down*

The formulation of the restricted problem indicates the significance of the partitioning of the right numerator and the definition of the generator polynomial matrix M(s). We examine next the relationship of the generator matrix M(s) to the state space parameters of the model and establish some properties for this new problem invariant.

Lemma 1 (Karcanias and Mitrouli, 2002)**:** *For the linear system $S(A, B, C, E)$ with a transfer function $G(s)$, let $X(s)$, $U(s)$ be a pair of polynomial matrices defining a minimal basis for $N_r([sI - A \quad -B])$, i.e.*

$$\begin{bmatrix} sI & -A & -B \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = 0 \tag{31}$$

then, a right coprime MFD for $G(s) = N_r(s)D_r(s)^{-1}$ is defined by

$$N_r(s) = CX(s) + EU(s), \quad D_r(s) = U(s) \tag{32}$$

Proposition 3: *For the linear system $S(A, B, C, E)$ consider the restricted squaring down that corresponds to a partition $[N_{11}(s)^t \quad N_{12}(s)^t]^t$ of the right numerator, and let the corresponding partitioning of the output matrix be*

$$\begin{bmatrix} C' & E' \end{bmatrix} = R\begin{bmatrix} C & E \end{bmatrix} = \begin{bmatrix} C_1 & E_1 \\ C_2 & E_2 \end{bmatrix}, \tag{33}$$

where R is the transformation matrix in (15). *The matrix $M(s)$ generating the k-RSD is defined by*

$$\begin{bmatrix} sI - A & -B \\ C_1 & E_1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ U_1(s) \end{bmatrix} = 0 \tag{34}$$

$$M(s) = C_2X_1(s) + E_2U_1(s). \tag{35}$$

Proof: If

$$\begin{bmatrix} X_1(s) \\ U_1(s) \end{bmatrix}$$

is as above, then it belongs to the right kernel of $[sI - A \quad -B]$ and therefore, for some $V(s)$

$$\begin{bmatrix} X_1(s) \\ U_1(s) \end{bmatrix} = \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} V(s), \tag{36}$$

where $X(s)$, $U(s)$ are as in Lemma 1. However, by Lemma 1

$$\begin{aligned} N_{11}(s)V(s) &= \begin{bmatrix} C_1 & E_1 \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} V(s) \\ &= \begin{bmatrix} C_1 & E_1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ U_1(s) \end{bmatrix} = 0 \end{aligned} \tag{37}$$

proving that $V(s)$ is a right kernel for $N_{11}(s)$. By Lemma 1 we may establish the result by calculating:

$$\begin{aligned} C_2X_1(s) + E_2U_1(s) &= [C_2X(s) + E_2U(s)]V(s) \\ &= N_{12}(s)V(s) = M(s). \end{aligned} \tag{38}$$

□

The relationship between the generic degree of $N_r(s)$ and the corresponding matrix for the restricted problem defined by $M(s) = N_{12}(s)V(s)$, is considered next. The correspondence $J(N(s)) = M(s)$ defines a map between two related polynomial modules. If $d = \delta^*$ is the Forney order of any minimal basis of the module generated by $M(s)$ (Forney 1975), $\sum_{j,l}^{d}$ is the set of all ℓ-dimensional polynomial modules in R^{ℓ} $[s]$ whose degree is less than or equal to d, which are represented by basis matrices of $j \times \ell$ dimensions, then J is a map between $\sum_{m,p}^{d} \to \sum_{m-k,p-k}^{d}$. The following result establishes that this map is onto.

Proposition 4: *The map J*: $\sum_{m,p}^{d} \to \sum_{m-k,p-k}^{d}$ *such that* $J(N(s)) = N_{12}(s)V(s)$ *is onto.*

Proof: Let $M(s)$ be an element of $\sum_{m-k,p-k}^{d}$ and consider an $N(s)$ in $\sum_{m,p}^{d}$ of the form

$$N(s) = \begin{bmatrix} I_k & 0 \\ 0 & M(s) \end{bmatrix} \tag{39}$$

Then $N_{11}(s) = [I_k, 0]$ and $N_{12}(s) = [0, M(s)]$. Therefore

$$V(s) = \begin{bmatrix} 0 \\ I_{p-k} \end{bmatrix}$$

and

$$J(N(s)) = N_{12}(s)V(s) = \begin{bmatrix} 0 & M(s) \end{bmatrix} \begin{bmatrix} 0 \\ I_{p-k} \end{bmatrix} = M(s) \tag{40}$$

which proves the result. □

From the above it follows that for a generic $N(s)$ of degree d, $J(N(s))$ has degree d and not less, since the set of all modules of degree less than d is a proper subvariety of $\sum_{m-k,p-k}^{d}$.

The establishment of generic solvability results is based on the number of independent degrees of freedom, which in our case are $(p-k) \times (m-p)$, as well as the number of equations/constraints which are defined by, δ^*, the Forney order of the $R[s]$-module column-span$(N_{12}(s)V(s))$. Central to the current approach is the construction of a degenerate static squaring down compensator on which the differential of the related zero assignment map is onto.

4. Solvability conditions

The work so far has revealed that the restricted squaring down is equivalent to a free squaring down defined on a generator matrix $M(s)$ that has a structure specified from the restricted scheme. Theorem 1 shows that the existence of a degenerate squaring down with certain properties is sufficient to guarantee zero assignability. The existence and construction of degenerate compensators (Brockett and Byrnes 1981), which are instrumental to the current approach is considered next. Both the generic and the exact versions of the problem are considered; the derived results clearly specialize to the full squaring down problem (Karcanias and Giannakopoulos 1989).

4.1 *The generic case and its solvability*

The construction of degenerate K-RSDs is crucial for our methodology and it is considered first.

Proposition 5: *Let d be the least Forney index of the $R[s]$ module generated by $M(s)$ and let $X(s)$ be such a least degree vector with a $(m-k)\times(d+1)$ basis (coefficient) matrix X_r. A sufficient condition for the existence of a degenerate full rank compensator $\bar{K}_0$ is that $m-p>d$. If this condition is satisfied, then $\bar{K}_0$ is defined as a full rank matrix satisfying $\bar{K}_0 X_r=0$.*

Proof: The proof of the above result follows along the lines developed for pole assignment in Leventides and Karcanias 1995a). Since $\det(\bar{K}_0 M(s))=0$ the matrix $\bar{K}_0 M(s)$ is rank deficient. This means that $\bar{K}_0$ is in the left kernel of a polynomial vector $\underline{x}(s)$ of the column span of $M(s)$. For this to be true, if d is the degree of $\underline{x}(s)$, it is sufficient that $m-k-(d+1)\geq p-k$ which is equivalent to $m-p>d$. □

Next we examine the conditions for the generic solvability of the problem.

Theorem 3: *For a generic system of p-inputs, m-outputs and n-states, the problem of arbitrary zero assignment by static k-RSD, K, $K\in\phi_k$ can be solved if*

$$(p-k)(m-p)>\delta^* \tag{41}$$

where k is the number of fixed outputs and δ^* is the Forney order of $M(s)$.

Proof: The proof of the result is based on genericity arguments. In fact the zero assignability property of systems defines a Zarisky open set in the family of systems and proof of the result is equivalent to showing that this set is nonempty. It suffices to prove that we can construct a system which satisfies the conditions of the theorem. To construct such a system we consider the set of the generic Forney dynamical indices corresponding to $M(s)$ which are defined via the Euclidean division of δ^* by $p-k$. Thus, if $\delta*=(p-k)\pi+u$, where $u<p-k$, then $m-p>\pi$. If we now consider as $M(s)$ the matrix as indicated in (42), then we can define a numerator partitioned as shown below in (42) which leads to the above $M(s)$ matrix i.e.,

$$N_r(s)=\begin{bmatrix} I_k & 0_{kx(p-k)} \\ 0_{(m-k)xk} & M(s) \end{bmatrix}=\begin{bmatrix} N_{11}(s) \\ N_{12}(s) \end{bmatrix}. \tag{42}$$

Therefore $N_{12}(s)V(s)=M(s)$. A degenerate full rank compensator is constructed from the last column of $M(s)$ and it is given by the matrix indicated in (44). For this compensator the matrix $\overline{K_0}M(s)$ has the form indicated in (45). Then, by setting $d=\delta^*$ we have that $u(s)=\begin{bmatrix} s^{d-\pi} & -s^{d-2\pi} & \cdots & 1 \end{bmatrix}$ is basis for the left kernel of $\overline{K_0}M(s)$ and $e=[0,0,\ldots,0,1]^{t}$ is the right kernel of $\overline{K_0}M(s)$ giving rise to a vector w(s) as defined in (46). The coefficient matrix of the tensor product is then given by equation (47) and this produces a basis for the differential of zero assignment map (Leventides and Karcanias 1995a, 1996)

$$M(s)=\begin{bmatrix}
s^{\pi+1} & 0 & \vdots & 0 & 0 & \vdots & \vdots & 0 \\
1 & s^{\pi+1} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & \ddots & s^{\pi+1} & 0 & & & \\
\vdots & \vdots & \vdots & 1 & s^{\pi} & \ddots & & \\
\vdots & \vdots & \vdots & & 1 & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & & & \ddots & s^{\pi} & 0 \\
\vdots & \vdots & \vdots & & & & 1 & s^{\pi} \\
\vdots & \vdots & \vdots & & & & 0 & s^{\pi-1} \\
0 & 0 & \vdots & & & & 0 & s^{\pi-2} \\
 & & & & & & \vdots & \vdots \\
 & & & & & & 0 & 1 \\
0 & 0 & \cdots & & & 0 & 1 & 0
\end{bmatrix} \tag{43}$$

$$\overline{K_0}=\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & 1 & 0 & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & 0 & 1 & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix} \tag{44}$$

$$\overline{K_0}M(s)=\begin{bmatrix}
s^{\pi+1} & 0 & \cdots & & 0 & 0 \\
1 & s^{\pi+1} & \cdots & & \vdots & \vdots \\
0 & 1 & \ddots & & & \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & 1 & s^{\pi} & 0 \\
0 & 0 & \cdots & 0 & 1 & 0
\end{bmatrix} \tag{45}$$

$$\underline{w}(s) = M(s)e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ s^{\pi} \\ s^{\pi-1} \\ \vdots \\ 1 \\ 0 \end{bmatrix} \tag{46}$$

$$\underline{u}(s) \otimes \underline{w}(s) = \begin{bmatrix} s^d & s^{d-1} & \cdots & 1 \end{bmatrix} \tag{47}$$

This coefficient matrix contains the identity matrix I_{d+1} and this proves that the differential of the zero assignment map at $\overline{K_0}$ is therefore onto.

Consider now the subset $\mathcal{U}$, of $\sum_{m,p}^{d}$ which is defined as $\mathcal{U} = \{S\colon D_{K0}J(S)$ has full rank$\}$. This is a Zarisky open subset of $\sum_{m,p}^{d}$ which when it is non-empty contains almost all points with the exception of a subvariety of smaller dimension. The example we have constructed above guarantees nonemptiness; therefore, a generic system of p-inputs, m-outputs and n-states has a degenerate compensator for the k-RSD zero assignment problem, which has full rank differential and therefore it is onto. Hence a generic such system has the arbitrary zero assignability property. □

The global linearization framework which has been developed for the pole assignment problem (Leventides and Karcanias 1995a, 1996) can be adjusted to the case of squaring down and k-restricted squaring down and this is described below:

4.2 *Global linearization algorithm for restricted squaring down*

Consider the system described by the right MFD pair $(N_r(s), D_r(s))$ and a family of k-RSD compensators as in (14). Let $N_r(s) = N(s)Z_r(s)$, where $N(s)$ is a least degree basis and $Z_r(s)$ a greatest right divisor. We partition accordingly $N_1(s) = RN(s)$, where R is the permutation matrix in (14), and derive the matrices $N_{11}(s)$, $N_{12}(s)$. If $Z_\ell(s)$ is a greatest left divisor of $N_{11}(s)$ and $V(s)$ is a right annihilator of $N_{11}(s)$, then the matrix generating the k-restricted squaring down is $M(s) = N_{12}(s)V(s)$. The matrix $M(s)$ has dimensions $(m-k)\times(p-k)$ and if $Z^*(s)$ is a greatest right divisor of $M(s)$ then $M(s) = M^*(s)Z^*(s)$ where $M^*(s)$ is a least degree basis for the $R[s]$-module defined by the columns of $M^*(s)$. The zero polynomial under the k-RSD is expressed as

$$z(s, \underline{K}) = \det(\underline{K}M^*(s))z_f(s), \tag{48}$$

where $z_f(s)$ is the fixed polynomial under the k-restricted squaring down and it is expressed as

$$z_f(s) = |Z_l(s)||Z^*(s)||Z_r(s)|. \tag{49}$$

The Algorithm: The algorithm of global linearization aims at assigning the roots of the assignable part of the zero polynomial and uses $M^*(s)$ as the generator of the corresponding determinantal problem. This involves the following steps.

Step 1: Consider a vector $\underline{m}(s)$ in col-span(M^*(s)) such that its degree r satisfies the condition $m - p > r$. Then a basis matrix K_0 for a p-dimensional subspace of the left null space of the coefficient matrix P_m of $\underline{m}(s)$ is a degenerate compensator.

Step 2: Calculate $DF_{\bar{K}_0}$ the differential of the zero assignment map at the specific degenerate compensator. If this map is onto then we have complete zero assignability and we proceed to the next step, otherwise we go to the step 1.

Step 3: Apply the Quasi-Newton algorithm to compute compensators that assign the zero structure and which are at a distance from the degenerate squaring down compensator. We shall denote by $\underline{x}_j = \mathrm{vec}(K_i)$ the vector representation of the matrix K_i and the algorithm is then expressed as

$$\left.\begin{aligned} &\underline{x}_{i+1} = \underline{x}_i - (JF)^{-1}_{x_{n_{k-1}}}(F(\underline{x}_i) - \varepsilon_k \underline{p}) \quad n_{k-1} < i \le n_k \\ &k = 1, \ldots, r, \ldots n_0 = 0, \ldots, \underline{x}_{n_0} = \mathrm{vec}(K_0) \\ &0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_k < \cdots \end{aligned}\right\} \tag{50}$$

where $\underline{p}$ is the coefficient vector of the desired polynomial, F is the zero placement map, JF is the Jacobian matrix representing the differential of the zero assignment map and K_0 is the full degenerate compensator. In other words, starting from $\underline{x}_0 = \mathrm{vec}(K_0)$, the full degenerate compensator, and ε_1 sufficiently small we get a series of compensators represented by the vectors $\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{n_1}$ so that the iteration $\underline{x}_{i+1} = \underline{x}_i - (JF)^{-1}_{x_0}(F(\underline{x}_i) - \varepsilon_1 \underline{p})$ converges. Then we increase ε_1 to ε_2 and get another sequence $\underline{x}_{n_1+1}, \underline{x}_{n_1+2}, \ldots, \underline{x}_{n_2}$ so that the iteration $\underline{x}_{i+1} = \underline{x}_i - (JF)^{-1}_{x_{n_1}}(F(\underline{x}_i) - \varepsilon_2 \underline{p})$, $n_1 < i \le n_2$ converges. We repeat the process for $k = 3, 4\ldots$ until $k = r$ where ε_r is sufficiently high. The final solution is given by $\underline{x}_{n_r}$.

The Jacobian of F (JF) can be easily computed as F is an algebraic polynomial map. The above algorithm is based on the following philosophy: If we denote by $\Delta(\varphi)$ the family of all compensators placing the zeros at the given locations $p(s\mathrm{z})$, then a degenerate compensator, with full rank differential, is a boundary point for all manifolds $\Delta(p)$ corresponding to different p's. Using as a starting point the degenerate perturbation (which can be readily computed as shown before) and selecting ε_1 sufficiently small the Newton-Raphson algorithm produces solution Λ_1 on $\Delta(p)$ which are at the distance from the boundary point. Repeating now the method starting this time from Λ_1 and with a new step ε_2 we produce Λ_2 on $\Delta(p)$ and so on.

Remark 5: An alternative formulation of the above algorithm is to replace step 3 by any homotopy continuation method (Ortega and Rheiboldt 2000, Tsachouridis *et al.* 1980), that is use the homotopy

$$H\colon R^{(p-k)\times(m-k)} \times R \rightarrow R^{\delta^*+1} \tag{51}$$

such that

$$H(K,\varepsilon) = f(K) - \varepsilon \underline{p}, \tag{52}$$

where for $\varepsilon = 0$, $\eta(K,0) = 0$ has an easy solution, namely the degenerate compensator which has to be continuously deformed to a solution of $H(K,1) = 0$ which is the required solution.

Example 1: Consider a system of 5 outputs 3 inputs, whose numerator matrix is given by

$$N(s) = \begin{bmatrix} s+1 & s+2 & s \\ 2s+3 & -s+1 & s \\ s+5 & 2s & -2s+1 \\ 1 & s+1 & s-1 \\ 2 & 2 & s+3 \end{bmatrix}$$

and assume that we would like to square it down to a three output system keeping the first output as it is. In this case $N_{11}(s) = [s+1,\ s+2,\ s]$ and K is of the form

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{bmatrix}$$

the right kernel for $N_{11}(s)$ is given by $V(s)$ as shown below

$$V(s) = \begin{bmatrix} 0 & -2 \\ -s & 1 \\ s+2 & 1 \end{bmatrix}$$

and

$$N_{12}(s)V(s) = \begin{bmatrix} 2s^2+s & -4s-5 \\ -4s^2-3s+2 & -2s-9 \\ -2 & 2s-2 \\ s^2+3s+6 & s+1 \end{bmatrix}.$$

A degenerate compensator for $N_{12}(s)V(s)$ is a static matrix $\overline{K_0}$ which satisfies

$$\overline{K_0}[-4s-5,\ -2s-9, 2s-2, s+1]t = [0, \quad 0]^t.$$

Solving the above we get a degenerate compensator

$$\overline{K_0} = \begin{bmatrix} 1 & 0 & -1/4 & 9/2 \\ 0 & 1 & -7/4 & 11/2 \end{bmatrix}$$

for the problem. Considering now a perturbation $K_1 = (k_{ij})_{i=1,j=1}^{i=2,j=4}$ of $\overline{K_0}$ we calculate the differential of the zero assignment map of this problem by expanding the determinant

$$\begin{aligned} &\left|\left(\overline{K_0} + \varepsilon K_1\right) N_{12}(s) V(s)\right| \\ &\quad = \begin{vmatrix} (1+\varepsilon\kappa_{11})s^2 + \varepsilon\kappa_{12}s + \varepsilon\kappa_{13} & s + \varepsilon\kappa_{12} + \varepsilon\kappa_{14} \\ \varepsilon\kappa_{21}s^2 + \varepsilon\kappa_{22}s + 1 + \varepsilon\kappa_{23} & s\varepsilon\kappa_{22} + \varepsilon\kappa_{24} \end{vmatrix} \\ &\quad = \varepsilon\left(s^2(s\kappa_{22} + \kappa_{24}) + s\kappa_{12} + \kappa_{14}\right) + \varepsilon^2(\cdots) \end{aligned}$$

in this case the differential is given by the limiting polynomial as ε tends to 0 which is

$$\kappa_{22}s^3 + \kappa_{24}s^2 + s\kappa_{12} + \kappa_{14}$$

This means that asymptotically (as $\varepsilon \to 0$) by changing the parameters κ_{22}, κ_{24}, κ_{12}, κ_{14} of the perturbation K_1 we can assign any zero polynomial of degree 3. Therefore this degenerate compensator is regular and one can use it as starting point for a numerical Quasi-Newton method to place the zeroes of the system at any polynomial $p(s)$ of degree 3. In fact, by using the formula

$$\underline{x}_{n+1} = \underline{x}_n - (JF)^{-1}_{\underline{x}_0}(\underline{f} - \varepsilon_1 \underline{p})$$

where $\underline{x} = (x, y, z, w)^T$, $\underline{p} = [1, 9, 2\,27, 27]^T$, $\underline{f} = [f_3, f_2, f_1, f_0]^T$ and $\underline{x}_0 = (-1/4, 9/2, -7/4, 11/2)^T$. Starting with $\varepsilon_1 = 60$ the method converges after 94 ($n_1 = 94$, $r = 1$) iterations to $\underline{x}_{94} = (20{,}3214997,\ -6{,}5221932,\ -58{,}1924582,\ 17{,}1995119)$. Giving rise to the compensator

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 20{,}3215 & -6{,}5222 \\ 0 & 0 & 1 & -58{,}19246 & 17{,}1995 \end{bmatrix}$$

which fixes the first output and transforms the rest four into two new outputs, giving rise to a square system of 3-inputs and 3-outputs, whose zero polynomial is $(s+3)^3$.

4.3 *The full squaring down problem*

The results derived for the restricted version of the problem also apply to the full version of the problem, as this is defined by equation (3). In fact, a right MFD for the squared down transfer function $G'(s)$ is defined by $G'(s) = KN_r(s)D_r\,(s)^{-1}$ where $G(s) = N_r(s)D_r(s)^{-1}$ and if $Z_r(s)$ is a right matrix *gcd* of $N_r(s)$, $Z_r(s) \in R^{p \times p}[s]$, then the zero polynomial of the squared down system is

$$z_k(s) = \det\{K\, N_r(s)\} = \det\{K\, \bar{N}_r(s) Z_r(s)\}$$
$$= \det\{K\, \bar{N}_r(s)\} \det\{Z_r(s)\} \quad (53)$$

Remark 6: The fixed zeros of the full squaring down are defined by the zeros of the original system and the number of additional zeros introduced by full squaring down is defined by δ which is the Forney dynamical order of the column space of $G(s)$ (Forney, 1975). □

Using $\bar{N}_r(s)$ in place of $M(s)$ we can specialize the results of the restricted case to the full case as follows:

Theorem 4: *For a generic system of p-inputs, m-outputs, n-states and with a Forney dynamical order of the column space of $G(s)$, δ, then the problem of arbitrary zero assignment by a full static $p \times m$ squaring down compensator K, can be solved if*

$$p(m-p) > \delta. \quad (54)$$

The computation of solutions is a special case of the algorithm given for the restricted case when $\bar{N}_r(s)$ is in place of $M(s)$. In this case vectors of degrees defined by Forney dynamical orders may be used for the computation of degenerate solutions.

5. Conclusions

The problem of squaring down has been studied so far in the literature for the case where the squaring down compensator is free, as far as the selection of its structure and parameters. The problem of restricted constant squaring down is the more realistic version for control design and an approach has been developed that has the potential to allow the selection of the partially fixed structure (thus avoiding undesirable new fixed zeros and selections that cannot guarantee complete assignment) and also provides a powerful methodology for computing solutions. The current approach contains the essentials of a design framework, by providing: (i) characterization of the structural characteristics that characterize solvability of assignment problems, (ii) an algorithmic procedure for computing solutions, when such solutions exist, (iii) a link between such structural characteristics and the original parameters to the problem and (iv) the means for selecting the structure of the restricted compensator, that enables the design of schemes for which we avoid formation of undesirable zeros and guarantee assignment of the new ones. The paper contributes to both the characterization of existence of solutions and the development of an integrated design approach.

The adopted approach is algebraic and it is based on the method of the global linearization that allows the derivation of new solvability conditions, as well as the development of an algorithm for the computation of solutions. It has been shown that problems with restrictions on the compensator structure introduce fixed zeros (in addition to those associated with the original system). The additional new fixed zeros of the restricted squaring down have been characterized and the invariants characterizing the nature of this new version of determinantal assignment have been determined. The overall approach for study of solvability and computation of solutions is based on the reduction of the restricted problem to an equivalent free squaring down problem, which however has a structure and invariants determined by the original system and the compensation scheme under consideration. Two alternative approaches have been suggested, where the first is based on the use of Sylvester expansion of determinants and the second uses a purely algebraic approach; both lead to new Plücker type invariants which play a crucial role in defining the solvability of exact problems. The development of an efficient methodology for working out solutions away from the singular compensator is an important issue under investigation at the moment. The characterization of the fixed zeros of the restricted problem is done in an explicit way and this provides the means for investigating concrete designs, which avoid the formation of fixed zeros. The framework developed here for the study of the restricted squaring down for zero assignment is rather general and can be used for more general design problems of similar nature, such as the decentralized squaring down, relevant to integrated design during the stage of overall instrumentation (Karcanias 1996) and study of fixed, or reduced dynamics squaring down problems (relevant to cases where sensor dynamics have to be included). The systematic design of such schemes is the subject of future work.

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