Approximate Zeros of Polynomial Matrices and Linear Systems

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Abstract: The aim of this paper is to extend recent results on the approximate GCD of polynomials [1] and approximate zeros to the case of a polynomial matrices within the framework of exterior algebra [2]. The results provide the means to introduce a new characterization of approximate decoupling zeros and measures for approximate controllability and observability for the case of linear systems.

Keywords: Almost zeros, polynomial matrix, Grassmann invariants, Approximate Controllability and Observability.

1. *Introduction*

The notion of almost zeros and almost decoupling zeros for a linear system has been introduced in [4] and their properties have been linked to mobility of poles under compensation. The basis of that definition has been the use of Grassmann polynomial vectors [3] to define system invariant and the definition of "almost zeros" of a set of polynomials as the minima of a function associated with the polynomial vector [2]. In this paper we use the exterior algebra framework introduced in [4], and then use the results on the approximate gcd defined in [1] to define the notion of "approximate input, output decoupling zero polynomials" and "approximate zero polynomial" of a linear system. The current framework allows the characterisation of strength of the given order approximate zero polynomial, as well as permits the characterisation of the optimal approximate solutions of a given order.

The overall aim of the paper is to explore the exterior algebra framework that may lead to a proper definition of the notion of "approximate matrix divisor" of polynomial models. This is done here by deploying the exterior algebra framework and the Plucher embedding [3]. In fact, it is shown that the definition and computation of an "approximate matrix divisor" is equivalent to a distance problem of a general set of polynomials from the intersection of two varieties, the GCD and the Grassmann variety. The results introduce a computational framework that potentially can provide the means for defining new measures of distance of systems from uncontrollability, un-observabillity using the "strength" associated with a given approximate polynomial, and this is another advantage of the current approach. The use of Grassmann vectors, that is polynomial vectors in a projective space, implies that the general results on the "strength" of approximation yield upper bounds for the corresponding approximate polynomials, when these are defined in the affine space set up.

2. *Definitions and Preliminary results.*

Consider the $S(A,B,C,D)$: linear system where $A\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times p}$, $C\in\mathbf{R}^{m\times n}$, $D\in\mathbf{R}^{m\times p}$

$$\dot{\underline{x}} = A\underline{x}+B\underline{u} \quad , \quad \underline{y} = C\underline{x}+D\underline{u} \tag{1}$$

where (A,B) is controllable, (A,C) is observable, or defined by the transfer function matrix represented in terms of left, right coprime matrix fraction descriptions (LCMFD, RCMFD), as

$$G(s)=D_l(s)^{-1}N_l(s) = N_r(s)D_r(s)^{-1} \tag{2}$$

where $N_l(s)$ $N_r(s)\in\mathbf{R}^{m\times p}[s]$, $D_l(s)$ $\in\mathbf{R}^{m\times m}[s]$ and $D_r(s)\in\mathbf{R}^{p\times p}[s]$. The system will be called *square* if $m=p$ and *non-square* if $m\neq p$. We consider the following frequency assignment problems:

(i) Pole assignment by state feedback: The pole assignment by state feedback $L\in\mathbf{R}^{n\times p}$ is reduced to the determinantal problem

$$p_L(s)=\det\{sI-A-BL\}=\det\{B(s)\tilde{L}\} \tag{3}$$

where $B(s)=[sI-A,-B]$ is defined as the system controllability pencil and $\tilde{L}=\left[I_n,L^t\right]^t$. The zeros of *B(s)* are the input decoupling zeros of the system [7].

(ii) Design observers: The design of an n-state observer by an output injection $T\in\mathbf{R}^{n\times m}$ is reduced to the determinantal problem

$$p_T(s)=\det\{sI-A-TC\}=\det\tilde{T}C(s) \tag{4}$$

where $C(s)=\left[sI-A^t,-C^t\right]^t$ is the observability pencil and $\tilde{T}=\left[I_n,T\right]$ represents the output injection. The zeros of *C(s)* define the output decoupling zeros [7].

(iii) Zero assignment by squaring down: Given a system with $m>p$ and $\underline{c}\in\mathbf{R}^p$ the vector of the variables which are to be controlled, then $\underline{c}=Hy$ where $H\in\mathbf{R}^{p\times m}$ is a *squaring down post-compensator*, and $G'(s)=HG(s)$ is the *squared down transfer function matrix* [5] - [7]. A right MFD for $G'(s)$ is defined $G'(s)=HN_r(s)D_r(s)^{-1}$, $G(s)=N_r(s)D_r(s)^{-1}$. Finding H such that $G'(s)$ has assigned zeros is defined as the *zero assignment by squaring down* problem [5], [7] and the zero polynomial of $S(A,B,HC,HD)$ is

$$z_K(s)=\det\{HN_r(s)\} \tag{5}$$

Remark (1): The zeros of *M(s)* are fixed zeros of all polynomial combinants *f(s)*. ■

The above implies that input (output) decoupling zeros are fixed zeros under state feedback (output injection) and nonsquare zeros are fixed zeros under all squaring down compensators. For the case of polynomial matrices, the zeros are expressed as zeros of matrix divisors [9], or as the roots of the GCD of a polynomial multi-vector [3], [4]. It is the latter formulation that allows the development of a framework for defining "almost zeros" in a way that also permits the quantification of the strength of approximation. The results in [1] are deployed here to provide a new characterisation of "approximate" zeros of different type for Linear Systems.

The Abstract Derminantal Assignment Problem (DAP): This problem is to solve equation (7) below with respect to the constant matrix *H*:

$$\det\left(HN(s)\right) = f(s) \tag{6}$$

where $f(s)$ is the polynomial of an appropriate d degree. DAP is a multilinear nature problem of a determinantal character. If, $M(s) \in \mathbb{R}^{p\times r}[s]$, $r \le p$ such that $\operatorname{rank}\{M(s)\} = r$ and let $\mathcal{H}$ be a family of full rank $r \times p$ constant matrices having a certain structure then DAP is reduced to solve equation (7) with respect to $H \in \mathcal{H}$

$$f_M(s,H) = \det\left(H \cdot M(s)\right) = f(s) \tag{7}$$

where $f(s)$ is a real polynomial of some degree *d*.

Notation[10]**:** Let $Q_{k,n}$ be the set of lexicographically ordered, strictly increasing sequences of *k* integers from $1,2,...,n$. If $\{\underline{x}_{i_1},...,\underline{x}_{i_k}\}$ is a set of vectors of a vector space $\mathcal{V}$, $\omega = (i_1,...,i_k) \in Q_{k,n}$ then $\underline{x}_{i_1} \wedge \ldots \wedge \underline{x}_{i_k} = \underline{x}_{\omega} \wedge$ denotes the exterior product and by $\wedge^r \mathcal{V}$ we denote the *r*-th exterior power of $\mathcal{V}$. If $H \in F^{m\times n}$ and $r \le \min\{m,n\}$, then by $C_r(H)$ we denote the *r*-th compound matrix of H [10]. ■

If $\underline{h}_i^t$, $\underline{m}_i(s)$, $i \in \underset{\sim}{r}$, denote the rows of H, columns of $M(s)$ respectively, then

$$C_r(M) = \underline{h}_1^t \wedge \ldots \wedge \underline{h}_r^t = \underline{h}^t \wedge \ \in \mathbf{R}^{l\times\sigma} \tag{8}$$

$$C_r\left(M(s)\right) = \underline{m}_1(s) \wedge \ldots \wedge \underline{m}_r(s) = \underline{m}\wedge \in \mathbf{R}^{\sigma}[s],\ \sigma = \binom{p}{r} \tag{9}$$

and by Binet-Cauchy theorem [10] we have that [4]:

$$\begin{aligned} f_M(s,H) &= C_r(H) \cdot C_r\left(M(s)\right) = \\ &= \left\langle \underline{h}\wedge, \underline{m}(s)\wedge \right\rangle = \sum_{\omega \in Q_{r,p}} h_{\omega} m_{\omega}(s) \end{aligned} \tag{10}$$

$\omega = (i_1,...,i_r) \in Q_{r,p}$,and h_ω, $m_\omega(s)$ are the coordinates of $\underline{h}\wedge$, $m(s)\wedge$ respectively. Note that h_ω is the *r*×*r* minor of H which corresponds to the ω set of columns of H and thus h_ω is a multilinear alternating function [3] of the entries h_{ij} of H.

DAP Linear sub-problem: Set $\underline{m}(s) \wedge \underline{p}(s) \in \mathbf{R}^{\sigma}[s]$, $f(s) \in \mathbb{R}[s]$.Determine the existence of $\underline{k} \in \mathbb{R}^{\sigma}$, $\underline{k} \ne 0$, such that

$$f_M(s,H) = \underline{k}^t \underline{p}(s) = \sum k_i p_i(s) = f(s),\ i \in \underset{\sim}{\sigma} \tag{11}$$

DAP Multilinear sub-problem: Assume that $\mathcal{K}$ is the family of solution vectors $\underline{k}$ of (5). Determine if there exists $H^t = [h_1,...,h_r]$, $H^t \in \mathbb{R}^{p\times r}$, such that

$$\underline{h}_1 \wedge \ldots \wedge \underline{h}_r = \underline{h}\wedge = \underline{k},\ k \in K \tag{12}$$

Lemma (1) [3]**:** Let $\underline{k} \in \mathbf{R}^{\sigma}$, $\sigma = \binom{p}{r}$ and let k_ω, $\omega = (i_1,...,i_r) \in Q_{r,p}$ be the Plücker coordinates of a point in $P_{\sigma-1}(\mathbf{R})$. Necessary and sufficient condition for the existence of $H \in \mathbf{R}^{r\times p}$, $H = \left[\underline{h}_1,...,\underline{h}_r\right]^t$, such that

$$\underline{h}\wedge = \underline{h}_1 \wedge \ldots \wedge \underline{h}_r = \underline{k} = \left[\ldots, k_\omega, \ldots\right]^t \tag{13}$$

is that the coordinates k_ω satisfy the quadratics

$$\sum_{k=1}^{r+1} (-1)^{v-1} k_{i_1,\ldots,i_{r-1},j_v^k j_1,\ldots,j_{v-1},j_{v+1},j_{r+1}} = 0 \tag{14}$$

where $1 \le i_1 < i_2 < \cdots < i_{r-1} \le n$,$1 \le j_1 < j_2 < \cdots < j_{r+1} \le n$ ■

The quadratics defined by Eqn (14) are known as the *Quadratic Plücker Relations* (QPR) [3] and they define the Grassmann variety $\Omega(r,p)$ of $P_{\sigma-1}(\mathbf{R})$.

3. Grassmann Invariants of Linear Systems

Let $T(s) \in \mathbf{R}^{p\times r}[s]$, $T(s) = \left[\underline{t}_1(s),...,\underline{t}_r(s)\right]$, $p \ge r$, $\operatorname{rank}\{T(s)\} = r$, $\mathcal{X}_t = \mathrm{R}_{\mathbf{R}(s)}\left(T(s)\right)$. If $T(s) = M(s)D(s)^{-1}$ is a RCMFD of $T(s)$, then $M(s)$ is a polynomial basis for $\mathcal{X}_t$. If $Q(s)$ is a greatest right divisor of $M(s)$ then $T(s) = \tilde{M}(s)Q(s)D(s)^{-1}$, where $\tilde{M}(s)$ is a least degree polynomial basis of $\mathcal{X}_t$ [9]. A *Grassmann Representative (*GR) for $\mathcal{X}_t$ is defined by [4]

$$\begin{aligned} &\underline{t}(s)\wedge = \underline{t}_1(s) \wedge \ldots \wedge \underline{t}_r(s) = \\ &= \underline{\tilde{m}}_1(s) \wedge \ldots \wedge \underline{\tilde{m}}_r(s) \cdot z_t(s)/p_t(s) \end{aligned} \tag{15}$$

where $z_t(s) = \det\{Q(s)\}$, $p_t(s) = \det\{D(s)\}$ are the zero, pole polynomials of $T(s)$ and $\underline{\tilde{m}}(s) = \underline{m}_1(s) \wedge \ldots \wedge \underline{\tilde{m}}_r(s) \in \mathbf{R}^{\sigma}[s]$, $\sigma = \binom{p}{r}$, is also a GR of $\mathcal{X}_t$.

Since $\tilde{M}(s)$ is a least degree polynomial basis for $\mathcal{X}_t$, the polynomials of $\underline{\tilde{m}}(s)\wedge$ are coprime and $\underline{\tilde{m}}(s)\wedge$ will be referred to as a *reduced polynomial* GR (R -$\mathbf{R}[s]$ - GR) of $\mathcal{X}_t$. If $\delta = \deg\{\underline{\tilde{m}}(s)\wedge\}$, then δ is the Forney dynamical order [11] of $\mathcal{X}_t$. $\underline{\tilde{m}}(s)\wedge$ may be expressed as

$\underline{\tilde{m}}(s)\wedge = p(s) = p_0 + p_1 s + \ldots + p_\delta s^\delta = P_\delta \cdot \underline{e}_\delta(s), P_\delta \in \mathbf{R}^{\sigma\times(\delta+1)}$ (16)

where P_δ is a basis matrix for $\underline{\tilde{m}}(s)\wedge$ and $\underline{e}_\delta(s) = [1, s, ..., s^\delta]^t$. All $\mathbf{R}[s]$-GRs of $\mathcal{X}_t$ differ only by a nonzero scalar factor $a \in \mathbf{R}$ and if $\|\underline{p}_\delta\| = 1$, we define the canonical $\mathbf{R}[s]$ -GR $\underline{g}(\mathcal{X}_t)$ and the basis matrix P_δ is the *Plücker matrix* of $\mathcal{X}_t$ [4].

Theorem (1): $\underline{g}(\mathcal{X}_t)$, or the associated Plücker matrix P_δ, is a complete (basis free) invariant of $\mathcal{X}_t$. ■

If $M(s) \in \mathbf{R}^{p\times r}[s]$, $p \geq r$, $\text{rank}\{M(s)\} = r$, is a polynomial basis of $\mathcal{X}_t$, then $M(s) = \tilde{M}(s)Q(s)$, where $\tilde{M}(s)$ is a least degree basis and $Q(s)$ is a greatest right divisor of the rows of M(s) and thus

$$\underline{m}(s)\wedge = \underline{\tilde{m}}(s)\wedge \cdot \det(Q(s)) = P_\delta \underline{e}_\delta(s) z_m(s) \quad (16)$$

■

For the control problems discussed before the vector $\underline{m}(s)\wedge$ has important properties which stem from the properties of the corresponding system. A number of Plücker type matrices are:

(a) Controllability Plücker Matrix: For the pair (A, B), $\underline{b}(s)^t \wedge$ denotes the exterior product of the rows of $B(s) = [sI - A, -B]$ and $P(A, B)$ is the basis matrix of $\underline{b}(s)^t \wedge$. $P(A, B)$ is the *controllability Plücker matrix* and its rank characterises system controllability.

Theorem (2) [12]: $S(A, B)$ is controllable, iff $P(A, B)$ has full rank. ■

(b) Observability Plücker Matrix: For the pair (A, C), $\underline{c}(s)\wedge$ denotes the exterior product of the columns of $C(s) = \left[sI - A^t, -C^t\right]^t$ and $P(A, C)$ is the basis matrix of $\underline{c}(s)\wedge$. $P(A, C)$ is the *observability Plücker matrix* and its rank characterises system observability.

Theorem (3) [12]: $S(A, C)$ is observable, iff $P(A, C)$ has full rank ■

Remark (2): The properties of the singular values of $P(A, B)$, $P(A, C)$ characterise the degree of controllability, observability respectively and are primary design indicators for state space design. ■

(c) Column Plücker Matrices: For the transfer function $G(s)$, $m \geq p$, $\underline{n}(s)\wedge$ is the exterior product of the columns of the numerator $N_r(s)$, of a RCMFD and $P(N)$ is the basis matrix of $\underline{n}(s)\wedge$. Note that $d = \delta$, the Forney order of $\mathcal{X}_t$, if $G(s)$ has no finite zeros and $d = \delta + k$, where k is the number of finite zeros of $G(s)$, otherwise. If $N_r(s)$ is least degree, then $P_c(N)$ is the *column space Plücker matrix*.

Theorem (3) [13]**:** For a generic system with $m > p$, for which $p(m - p) > \delta + 1$, where δ is the Forney order, $P_c(N)$ has full rank. ■

4. Approximate GCD of Polynomial Sets

Consider a set $\mathcal{P} = \{a(s), b_i(s) \in \mathbf{R}[s], i \in \underset{\sim}{h}\}$ of polynomials which has $h+1$ elements and with the two largest values of degrees (n, p), which is also denoted as $\mathcal{P}_{h+1,n}$. The greatest common divisor (gcd) of $\mathcal{P}$ will be denoted by $\varphi(s)$. For any $\mathcal{P}_{h+1,n}$ we define a vector representative $\underline{p}_{h+1}(s)$ and a basis matrix P_{h+1}. The classical approaches for the study of coprimeness and determination of the GCD makes use of the Sylvester Resultant, $S_\mathcal{P}$ and the gcd properties are summarised below [14], [15] :

Theorem (4): For as set of polynomials $\mathcal{P}_{h+1,n}$ with a resultant $S_\mathcal{P}$ the following properties hold true:

(i) Necessary and sufficient condition for a set of polynomials to be coprime is that:
$rank(S_\mathcal{P}) = n + p$

(ii) Let $\phi(s)$ be the g.c.d. of $\mathcal{P}$. Then:
$rank(S_\mathcal{P}) = n + p - \deg\varphi(s)$

(iii) If we reduce $S_\mathcal{P}$, by using elementary row operations, to its row echelon form, the last non vanishing row defines the coefficients of the g.c.d. ■

The results in [15] establish a matrix based representation of the GCD, which is equivalent to the standard algebraic factorisation of the GCD of polynomials. This new GCD representation provides the means to define the notion of the “approximate GCD” subsequently in a formal way, and thus allows the definition of the optimal solution.

Theorem(5): Consider $\mathcal{P} = \{a(s), b_1(s), \ldots, b_h(s)\}$ $\deg a(s) = n$, $\deg b_i(s) \leq p \leq n$, $i = 1, \ldots, h$ be a polynomial set, $S_\mathcal{P}$ the respective Sylvester matrix, $\varphi(s) = \lambda_k s^k + \cdots + \lambda_1 s + \lambda_0$ be the GCD of the set and let k be its degree. Then there exists transformation matrix $\Phi_\varphi \in \mathbf{R}^{(n+p)\times(n+p)}$ such that:

$$\overline{S}_{\mathcal{P}^*}^{(k)} = S_\mathcal{P}\Phi_\varphi = \left[\mathbf{0}_k \mid \overline{S}_{\mathcal{P}^*}\right] \quad (21)$$

or

$$S_\mathcal{P} = \overline{S}_{\mathcal{P}^*}^{(k)} \hat{\Phi}_\varphi = \left[\mathbf{0}_k \mid \overline{S}_{\mathcal{P}^*}\right] \hat{\Phi}_\varphi \quad (22)$$

where $\Phi_\varphi = \hat{\Phi}_\varphi^{-1}$, $\hat{\Phi}_\varphi$ being the Toeplitz form of $\varphi(s)$ [15] and

$$\bar{S}^{(k)}_{\mathcal{P}^*} = \begin{bmatrix} \mathbf{0} & S^{(k)}_0 \\ \mathbf{0} & S^{(k)}_1 \\ & \vdots \\ \mathbf{0} & S^{(k)}_h \end{bmatrix} = [\mathbf{0} \ \ \tilde{S}^{(k)}_{\mathcal{P}}] \tag{23}$$

where $S_i^{(k)}$, are Toeplitz blocks corresponding to the coprime polynomials obtained from the original set after the division by the gcd.

■

The problem which is addressed next is the formal definition of the notion of the "approximate GCD" [1] and the evaluation of its strength. We shall denote by $\mathcal{T}(n,p;h+1)$ the set of all polynomial sets $\mathcal{P}_{h+1,n}$ with the (n,p) the maximal two degrees and h+1 elements. If $\mathcal{P}_{h+1,n} \in \mathcal{T}(n,p;h+1)$ we can define an (n,p) -ordered perturbed set

$$\mathcal{P}'_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}_{h+1,n} \in \mathcal{T}(n,p;h+1)$$
$$= \left\{ p'_i(s) = p_i(s) - q_i(s) : \deg\{q_i(s)\} \le \deg\{p_i(s)\}\right\} \tag{24}$$

Lemma (2) [1]: For a set $\mathcal{P}_{h+1,n} \in \mathcal{T}(n,p;h+1)$ and an $\omega(s) \in \mathbb{R}[s]$ with $\deg\{\omega(s)\} \le p$, there always exists a family of (n,p) -ordered perturbations $\mathcal{Q}_{h+1,n}$ and for every element of this family $\mathcal{P}'_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}_{h+1,n}$ has a gcd divisible by $\omega(s)$. ■

Definition (2): Let $\mathcal{P}_{h+1,n} \in \mathcal{T}(n,p;h+1)$ and $\omega(s) \in \mathbf{R}[s]$ be a given polynomial with $\deg\{\omega(s)\} = r \le p$. If $\Sigma_\omega = \{\mathcal{Q}_{h+1,n}\}$ is the set of all (n,p)-order perturbations

$$\mathcal{P}'_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}_{h+1,n} \in \mathcal{T}(n,p;h+1) \tag{25}$$

with the property that $\omega(s)$ is a common factor of the elements of $\mathcal{P}'_{h+1,n}$. If $\mathcal{Q}^*_{h+1,n}$ is the minimal norm element of the set Σ_ω, then $\omega(s)$ is referred as an *r-order almost common factor* of $\mathcal{P}_{h+1,n}$, and the norm of $\mathcal{Q}^*_{h+1,n}$, denoted by $\|\mathcal{Q}^*\|$, as the *strength* of $\omega(s)$. If $\omega(s)$ is the gcd of

$$\mathcal{P}^*_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}^*_{h+1,n} \tag{26}$$

then $\omega(s)$ will be called an *r-order almost gcd* of $\mathcal{P}_{h+1,n}$ with strength $\|\mathcal{Q}^*\|$. A polynomial $\hat{\omega}(s)$ of degree r for which the strength $\|\mathcal{Q}^*\|$ is a global minimum will be called the r *-order optimal almost GCD* (OA-GCD) of $\mathcal{P}_{h+1,n}$. ■

The above definition suggests that any polynomial $\omega(s)$ may be considered as an "approximate GCD", as long as $\deg\{\omega(s)\} \le p$. Important issues in the definition of approximate (optimal approximate) GCD are the Parameterisation of the Σ_ω set, the definition of an appropriate metric for $\mathcal{Q}_{h+1,n}$ and the solution of the optimization problem to define $\mathcal{Q}^*_{h+1,n}$. The set of all resultants corresponding to $\mathcal{T}(n,p;h+1)$ set, will be denoted by $\Psi(n,p;h+1)$.

Remark(2): If $\mathcal{P}_{h+1,n}, \mathcal{Q}_{h+1,n}, \mathcal{P}'_{h+1,n} \in \mathcal{T}(n,p;h+1)$ are sets of polynomials and $S_{\mathcal{P}}$, $S_{\mathcal{Q}}$, $\bar{S}_{\mathcal{P}'}$ denote their generalised resultants, then these resultants are elements of $\Psi(n,p;h+1)$ then $S_{\mathcal{P}'} = S_{\mathcal{P}} - S_{\mathcal{Q}}$. ■

Theorem (6): Let $\mathcal{P}_{h+1,n} \in \mathcal{T}(n,p;h+1)$ ■ ,be a set, $S_{\mathcal{P}} \in \Psi(n,p;h+1)$ be the corresponding generalized resultant and let $\upsilon(s) \in \mathbb{R}[s]$, $\deg\{\upsilon(s)\} = r \le p$, $\upsilon(0) \ne 0$. Any perturbation set $\mathcal{Q}_{h+1,n} \in \mathcal{T}(n,p;h+1)$ ie: $\mathcal{P}'_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}_{h+1,n}$, which has $\upsilon(s)$ as common divisor, has a generalized resultant $S_{\mathcal{Q}} \in \Psi(n,p;h+1)$ that is expressed as

$$S_{\mathcal{Q}} = S_{\mathcal{P}} - \bar{S}^{(r)}_{\mathcal{P}^*}\hat{\Phi}_\upsilon = \left[0_r \ \vdots \ \bar{S}_{\mathcal{P}^*}\right]\hat{\Phi}_\upsilon \ (4.1) \tag{27}$$

where $\hat{\Phi}_\upsilon$ is the Toeplitz representation of $\upsilon(s)$ and $\bar{S}_{\mathcal{P}^*} \in \mathbf{R}^{(p+hn)\times(n+p-r)}$ the (n,p) -expanded resultant of a $\mathcal{P}^* \in \mathcal{T}(n-r,p-r;h+1)$. Furthermore, if the parameters of $\bar{S}_{\mathcal{P}^*}$ are such that $\bar{S}_{\mathcal{P}^*}$ has full rank, then $\upsilon(s)$ is a gcd of set $\mathcal{P}'_{h+1,n}$. ■

Remark (3): The result provides a parameterisation of all perturbations $\mathcal{Q}_{h+1,n} \in \mathcal{T}(n,p;h+1)$ which yield sets $\mathcal{P}'_{h+1,n}$ having a gcd with degree at least r and divided by the given polynomial $\upsilon(s)$.The free parameters are the coefficients of the $\mathcal{P}^*_{h+1,n-r} \in \mathcal{T}(n-r,p-r;h+1)$ set of polynomials. For a set of parameters, $\upsilon(s)$ is a divisor of $\mathcal{P}'_{h+1,n}$; for generic sets, $\upsilon(s)$ is a gcd of $\mathcal{P}'_{h+1,n}$. ■

A useful metric for evaluation of strength of "approximate gcd has to relate to the coefficients of the polynomials and the Frobenius norm seems to be an appropriate choice.

Corollary (2): Let $\mathcal{P}_{h+1,n} \in \mathcal{T}(n,p;h+1)$ and $\upsilon(s) \in \mathbf{R}[s]$, $\deg\{\upsilon(s)\} = r \le p$. The polynomial $\upsilon(s)$, $\upsilon(0) \ne 0$ is an r -order almost common divisor of $\mathcal{P}_{h+1,n}$ and its strength is defined as a solution of the following minimization problem:

$$f\left(\mathcal{P},\mathcal{P}^*\right) = \min_{\forall \mathcal{P}^*} \left\| S_{\mathcal{P}} - \left[\mathbf{0}_r \ \vdots \ \bar{S}_{\mathcal{P}^*}\right]\hat{\Phi}_\upsilon \right\|_{\mathrm{F}} \tag{28}$$

where $\mathcal{P}^* \in \mathcal{T}(n,p;h+1)$. Furthermore $\upsilon(s)$ is an r -order almost gcd of $\mathcal{P}_{h+1,n}$ if the minimal corresponds to a coprime set $\mathcal{P}^*$ or to full rank $S_{\mathcal{P}^*}$. ■

The optimization problem defining the strength of any order approximate GCD is now used to investigate the "best" amongst all approximate GCDs of a degree r. We consider polynomials $\upsilon(s)$, $\upsilon(0) \ne 0$. Then $\hat{\Phi}_\upsilon$ is nonsingular and has fixed norm $\|\hat{\Phi}_\upsilon\|_{\mathrm{F}}$ for all $\upsilon(s)$.

Optimisation Problem: This can be expressed as

$$f_1\left(\mathcal{P},\mathcal{P}^*\right) \triangleq \left\|\hat{\Phi}_{\upsilon}\right\|_F \cdot f\left(\mathcal{P},\mathcal{P}^*\right) = \min_{\forall \mathcal{P}^*}\left\{\left\|S_{\mathcal{P}} - \left[\mathbf{0}_r \mid \bar{S}_{\mathcal{P}^*}\right]\hat{\Phi}_{\upsilon}\right\|_F \cdot \left\|\Phi_{\upsilon}\right\|_F\right\} \quad (29)$$
$$= \min_{\forall \mathcal{P}^*}\left\|S_{\mathcal{P}}\Phi_{\upsilon} - \left[\mathbf{0}_r \mid \bar{S}_{\mathcal{P}^*}\right]\right\|_F$$

where $\mathcal{P}$, Φ_{υ} have the structure as previously defined and $\upsilon(s)$ has degree r . ■

We may summarise as shown below [1]:

Theorem (7): Consider the set of polynomials $\mathcal{P} \in \mathcal{T}(n,p;h+1)$ and $S_{\mathcal{P}}$ be its Sylvester matrix. Then the following hold true:

(i) For a certain approximate gcd $\upsilon(s)$ of degree k , the perturbed set $\tilde{\mathcal{P}}$ corresponding to minimal perturbation applied on $\mathcal{P}$, such that $\upsilon(s)$ becomes an exact gcd, defined by:

$$S_{\tilde{\mathcal{P}}} = \tilde{S}''_{\mathcal{P}}\hat{\Phi}_{\upsilon} = \left[\mathbf{0}_k \mid \hat{S}^{(2)}_{\mathcal{P}}\right]\hat{\Phi}_{\upsilon} \quad (30)$$

(ii) The strength of an arbitrary $\upsilon(s)$ of degree k is then given by:

$$f\left(\mathcal{P},\mathcal{P}^*\right) = \min_{\forall \mathcal{P}^*}\left\|\tilde{S}'_{\mathcal{P}}\Phi_{\upsilon}\right\|_F \quad (31)$$

(iii) The *optimal approximate gcd* of degree k is a $\varphi(s)$ defined by solving:

$$f\left(\mathcal{P},\mathcal{P}^*\right) = \min_{\substack{\forall \mathcal{P}^* \\ \deg\{\varphi(s)\}=k}}\left\{\left\|\tilde{S}'_{\mathcal{P}}\Phi_{\varphi}\right\|_F\right\} \quad (32)$$

■

5. *Grassmann invariants and approximate zero polynomials*

An obvious Corollary of Theorem (5) is:

Corollary (3): Let $\mathcal{T}(n,p;h+1)$ be the set of all polynomial sets $\mathcal{P}_{h+1,n}$ with $h+1$ elements and with the two higher degrees (n,p), $n \ge p$ and let $S_{\mathcal{P}}$ be the Sylvester resultant of the general set $\mathcal{P}_{h+1,n}$. The variety of $\mathbf{P}^{N-1}$ which characterise all sets $\mathcal{P}_{h+1,n}$ having a GCD with degree d , $0 < d \le p$ is defined by the set of equations

$$C_{n+p-d+1}\left(S_{\mathcal{P}}\right) = 0 \quad (33)$$

The above is a variety $\Delta_d(n,p;h+1)$ defined by the polynomial equations in the coefficients of the vector $\underline{p}_{h+1,n}$, or the point $P_{h+1,n}$ of $\mathbf{P}^{N-1}$, and will be called the d *-GCD variety* of $\mathbf{P}^{N-1}$. This characterises all sets in $\mathcal{T}(n,p;h+1)$ with a GCD of degree d .The definition of the the "optimal gcd" is thus a problem of finding the distance of a given set $\mathcal{P}_{h+1,n}$ from the variety $\Delta_d(n,p;h+1)$. For any $\mathcal{P}_{h+1,n} \in \mathcal{T}(n,p;h+1)$ this distance is defined by

$$d\left(\mathcal{P},\Delta\right) = \min_{\forall \mathcal{P}^*,\varphi}\left\|S_{\mathcal{P}} - \left[\mathbf{0}_k \mid \bar{S}_{\mathcal{P}^*}\right]\hat{\Phi}_{\varphi}\right\|_F \quad (34)$$

$\varphi(s) \in \mathbf{R}[s]$, $\mathcal{P}^* \in \mathcal{T}(n-k,p-k;h+1)$, $\deg\left(\varphi(s)\right) = k$, the k -distance of $\mathcal{P}_{h+1,n}$ from the the k -gcd variety $\Delta_k(n,p;h+1)$ and $\tilde{\varphi}(s)$ emerges as a solution to an optimisation problem and it is the k *-optimal approximate gcd* and the value $d\left(\mathcal{P},\Delta\right)$ is its k *-strenght*.

For polynomial matrices we can extend the scalar definition of the approximate GCD as follows:

Definition (3): Consider the coprime polynomial matrix $T(s) \in \mathbb{R}^{q\times r}[s]$ and let $\Delta T(s) \in \mathbb{R}^{q\times r}[s]$ be an arbitrary matrix such that

$$T(s) + \Delta T(s) = \hat{T}(s) = \tilde{T}(s)R(s) \quad (35)$$

where $R(s) \in \mathbb{R}^{r\times r}[s]$. Then $R(s)$ will be called an *approximate matrix divisor* of $T(s)$.■

The above definition may be interpreted using exterior products as an extension of the problem defined for polynomial vector sets. The difference between general sets of vectors and those generated from polynomial matrices by taking exterior products is that the latter must satisfy the decomposability conditions [3] and in turn they define another variety of the Grassmann type.

Consider now the set of polynomial vectors $\mathcal{T}(n,p;h+1)$ and let $\mathcal{T}^{\wedge}(n,p;h+1)$ be its subset of the decomposable polynomial vectors $\underline{p}(s) \in \mathbf{R}^{\sigma}[s]$, which correspond to the $q\times r$ polynomial matrices with degree *n*. The set $\mathcal{T}^{\wedge}(n,p;h+1)$ is defined as the Grassmann variety $G\left(q,r;\mathbf{R}[s]\right)$ of the projective space $\mathbf{P}^{\sigma-1}\left(\mathbf{R}[s]\right)$ .The way we can extend the scalar results is based on:

(i) Parameterise the perturbations that move a general set $\mathcal{P}_{\sigma,n}$, to a set $\mathcal{P}'_{\sigma,n} = \mathcal{P}_{\sigma,n} + Q_{\sigma,n} \in \Delta_k(n,p;\sigma)$ where initially $Q_{\sigma,n}$ and $\mathcal{P}'_{\sigma,n}$ are free.

(ii) For the scalar results to be transferred back to the polynomial matrices the sets $\mathcal{P}'_{\sigma,n}$ have to be decomposable multi-vectors which are denoted by $\mathcal{T}^{\wedge}(n,p;\sigma)$. The latter set will be referred to as the *n-order subset* of the Grassmann variety $G(q,r;\mathbb{R}[s])$ and the sets $\mathcal{P}'_{\sigma,n}$ must be such that

$$\mathcal{P}'_{\sigma,n} \in \mathcal{T}(n,p;\sigma) \cap \Delta_k(n,p;\sigma) = \Delta_k^{\wedge}\mathcal{T}(n,p;\sigma) \quad (36)$$

where $\Delta_k^{\wedge}\mathcal{T}(n,p;\sigma)$ is the decomposable subset of $\Delta_k(n,p;\sigma)$. Parameterising all sets $\mathcal{P}'_{\sigma,n}$ provides the means for posing a distance problem as before.This is clearly a constrained distance problem since now we have to consider the intersection variety defined by the corresponding set of QPRs and the equations of the GCD variety. Some preliminary results on this problem are stated below:

Lemma (3): The following properties hold true:

i) $\mathcal{T}^{\wedge}(n,p;h+1)$ is proper subset $\mathcal{T}(n,p;h+1)$ if $r \neq 1$ and $q \neq r-1$.

ii) $\mathcal{T}^{\wedge}(n,p;h+1) = \mathcal{T}(n,p;h+1)$ if either $r=1$ or $q=r-1$.

iii) The set $\Delta_k^{\wedge}\mathcal{T}(n,p;\sigma)$ is always nonempty. ■

The result is a direct implication of the decomposability conditions for multivectors [3].

Theorem (8): Let $\mathcal{P}_{\sigma,n} \in \Pi^{\wedge}(n,p;\sigma)$ and denote by $d\left(\mathcal{P},\Delta_k\right)$, $d\left(\mathcal{P},\Delta_k^{\wedge}\right)$ the distance from $\Delta_k(n,p;\sigma)$ and $\Delta_k^{\wedge}(n,p;\sigma)$ respectively. The following hold true:

i) If $q=r-1$ or $r=1$, then the solutions of the two optimisation problems are identical and $d\left(\mathcal{P},\Delta_k\right) = d\left(\mathcal{P},\Delta_k^{\wedge}\right)$

ii) If $q \neq r-1$ and $r \neq 1$, then $d\left(\mathcal{P},\Delta_k\right) \leq d\left(\mathcal{P},\Delta_k^{\wedge}\right)$ ■

Remark (4): The definition of the almost zero polynomials in the matrix case is clearly a distance problem. For polynomial matrices this distance problem is defined on the set $\mathcal{P}_{h+1,n}$ of $\mathcal{T}(n,p;h+1)$ from the intersection of the varieties $\Delta_d(n,p;h+1)$ and $G\left(q,r;\mathbf{R}[s]\right)$. ■

The above suggests that the Grassmann distance problem has to be considered only when $q \neq r-1$ and $r \neq 1$. The Grassmann distance problem requires the study of some additional topics linked to algebraic geometry and exterior algebra such as: **(i)** Parameterisation of all decomposable sets $\mathcal{P}$ with a fixed order *n*. **(ii)** Characterisation of the set $\Delta_k^{\wedge}(n,p;\sigma)$ and its properties. The above issues are central for the solution of the GDP and are topics for further research. For the special case $r=1$, $q=r-1$ the distance $d\left(\mathcal{P},\Delta_k\right)$ is reduced to that of the polynomial vector case since we guarantee decomposability and this leads to the definition of almost zeros and almost decoupling zeros.

[9] Kailath,

6. *Conclusions*

The extension of the notion of approximate GCD of a set of polynomials has been considered to that of *approximate matrix divisors.* This problem is equivalent to a distance problem from the intersection of two varieties and it is much harder than the polynomial vectors case. Our approach is based on the optimal approximate gcd and when this is applied to linear systems introduces new system invariants with significance in defining system properties under parameter variations on the corresponding model. The natural way for introducing such notions has been the notion of the polynomial Grassmann representative [4], which introduces new sets of polynomials. The nature of the new distance problem stems from the fact that the polynomial vectors are exterior products of the columns of a polynomial matrix and thus the distance has to be considered from a sub-variety of the general k-gcd variety that is the intersection with the Grassmann variety. Computing this distance is a problem of current research.

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