Zero assignment of matrix pencils by additive structured transformations

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A B S T R A C T

Matrix pencil models are natural descriptions of linear networks and systems. Changing the values of elements of networks, that is redesigning them, implies changes in the zero structure of the associated pencil and this is achieved by structured additive transformations. The paper examines the problem of zero assignment of regular matrix pencils by a special type of structured additive transformations. For a certain family of network redesign problems the additive perturbations may be described as diagonal perturbations and such modifications are considered here. This problem has certain common features with the pole assignment of linear systems by structured static compensators and thus the new powerful methodology of global linearization [J. Leventides, N. Karcanias, Sufficient conditions for arbitrary Pole assignment by constant decentralised output feedback, Mathematics of Control for Signals and Systems 8 (1995) 222–240; J. Leventides, N. Karcanias, Global asymptotic linearisation of the pole placement map: A closed form solution for the constant output feedback problem, Automatica 31 (1995) 1303–1309] can be used. For regular pencils with infinite zeros, families of structured degenerate additive transformations are defined and parameterized and this lead to the derivation of conditions for zero structure assignment, as well as methodology for computing such solutions. The case of regular pencils with no infinite zeros is also considered and conditions of zero assignment are developed. The results here provide the means for studying problems of linear network redesign by modification of the non-dynamic elements.

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1. Introduction

The problem of redesigning passive electric networks [16] involves the selection of alternative values for dynamic (inductances, capacitances) and non-dynamic (resistances) elements within a fixed interconnection topology and/or alteration of the interconnection topology and possible evolution of the network (increase of elements, branches). The general redesign problem is much more complex than the subclass of problems considered here, which may be described as transformations on network based matrix pencil models [17]. The problem considered here is within the general class of redesign problems and it is reduced to the zero assignment of regular matrix pencils. In fact, the problem considered is the assignment of zeros of $sF + G + H$, where $sF + G$ expresses the internal dynamics matrix of a system and $H = U\Lambda V$ represents a static structural change; the matrices U, V are known graph incidence matrices (they express a topology modification) and Λ is a diagonal matrix of continuous design parameters. In reality, the three matrices U, V, Λ are design parameters. We shall assume that the incidence matrices U, V are fixed and thus only the diagonal matrix Λ is free for the assignment of zeros $sF + G + U\Lambda V$. A large family of such problems can be reduced to the case of diagonal additive perturbations and this is the problem considered here in some detail. The paper is within the area of matrix pencils and linear systems [23] and deals with both the study of solvability conditions, as well as the derivation of solutions, whenever such solutions exist. The work deals with properties of matrix pencils [4], it is within the general area dealing with problems for assigning invariants [5–12,26]; the methodology also relates to the intersection theory of varieties [3].

The general properties of the frequency assignment map are considered first and the notion of degenerate transformation, i.e. those making the pencil $sF + G + H$ singular is defined. For the case of pencils with infinite zeros, a parameterization of the set of degenerate transformations H is given based on the nature of the resulting singularity of the pencil. The significance of degenerate solutions is emphasized by establishing the property that if the differential of the frequency assignment map at a degenerate point H_0 is onto, then this implies assignability of the zero structure of the pencil by some appropriate H. The explicit form of the differential at a degenerate point is computed and it is shown that for a generic pencil there exist degenerate points H_0 such that the corresponding differential is onto. Using as the starting point such degenerate solutions, it is shown that transformations H, which are non-degenerate, may be constructed to assign the zeros of $sF + G + H$ in the neighbourhood of any arbitrary symmetric set of complex numbers. The results are developed for pencils $sA + B \in \mathbb{R}^{n\times n}[s]$ for which $\mathrm{rank}(A) = n - 1$, whereas the more general case $\mathrm{rank}(A) < n - 1$ is not considered here. The proposed methodology for zero assignment uses a Quasi-Newton type numerical approach to define perturbations that assign the zeros and which are at a distance from the degenerate perturbation that is the starting point of the algorithm; the convergence properties of the scheme are also examined. The methodology for computing solutions in this case is similar to that developed for the static decentralised output feedback [13,1] case, based on the degenerate compensator methodology [2,29]. Finally, the case of pencils with no infinite zeros, that is $\mathrm{rank}(A) = n$, is considered and conditions for the complex zero assignment are derived in terms of invariants associated with the pencil; the latter case is considered separately since it requires a different methodology, given that no degenerate perturbations exist for this case. Here, if we allow complex solutions, then the Dominant Morphism Theorem for complex varieties [27,28] can be applied simplifying the problem to that of finding one point such that the differential of the Frequency Assignment Map, is onto.

The results in this paper provide a methodology for solving an important passive design problem in circuit theory when the topology and the nature of the elements are given, some elements have specified values and the values of the remaining elements are to be determined.

2. Problem formulations and background results

Passive electrical network models give rise to appropriate matrix pencil descriptions [17] and a number of problems that may be defined on them may be addressed within the framework of matrix pencil structure modifications. Here we consider a family of problems which are of “open loop” nature and deal with the change of the basic characteristics of the network by changing the parameters and/or the topology of the network without deploying feedback compensation. Such problems are referred to as *network redesign* [16] and aim at improving the dynamics of the network, as these are expressed by the natural frequencies defined by the roots of the finite elementary divisors of an associated pencil. The natural frequencies of an electrical network depend on two factors:

(a) The topology of the network.
(b) The nature and the values of the elements of the network.

For the designer, it is important to be able to assign these frequencies at specific locations so that the network has certain desirable characteristics. The designer can exploit the available degrees of freedom and in this area we may distinguish the following clusters of problems:

Case 1. Both the topology of the network and the elements are design parameters.

This is the general synthesis problem of the network theory that can be formulated as: Given a rational matrix determine the conditions under which it can be realized as an RLC network. This problem is the classical problem of network synthesis [17] and it is not considered here. When the topology is fixed and the nature of the elements is given, but not their values, then we have a general problem of assignment of impedance and admittance matrices, which is not a standard network theory problem, since the topology, and location of elements are not any more free parameters for design and corresponds to:

Case 2. The topology of the network and the nature of the elements are given, but their values are free parameters to be determined.

A more restricted version of the above case corresponds to:

Case 3. The topology and the nature of the elements are given, some elements have specified values and the values of the remaining elements are to be determined.

The last case represents typical problems of network redesign, where given the system we have to change the least number of elements to improve the zero structure and thus improve the resulting system performance. Two special cases of this version that can be readily handled within the Determinantal Assignment Problem (DAP) framework developed for control [18] are considered next. These are:

(i) *Determination of resistors in an RL network*: Assume that the branch impedance matrix [17] is given by

$$Z(s) = \begin{bmatrix} sL + R & 0 \\ 0 & D \end{bmatrix},$$

where $sL + R$ is a known diagonal matrix, D is diagonal static matrix to be determined (characterising variable resistances) and B is the network matrix which can be partitioned as $B = [B_1, B_2]$. Then the loop impedance matrix is given by [17]

$$BZ(s)B^t = B_1(sL + R)B_1^t + B_2DB_2^t.$$

If B_2 is non-singular, then the zeros of BZB^t are defined as the zeros of the pencil

$$(B_2)^{-1}B_1(sL + R)B_1^t \left(B_2^t\right)^{-1} + D.$$

The problem in this case is reduced to constructing a diagonal perturbation D such that the above matrix has predefined zeros.

(ii) *Determination of resistors in an RC network*: This problem is dual to that of determining the values of the resistors required for tuning the zeros of the admittance matrix AYA^t in an RC network. In this case, the previous expression becomes

$$(A_2)^{-1}A_1(sC + G)A_1^t \left(A_2^t\right)^{-1} + D.$$

The common mathematical formulation of the above problems is expressed as follows:

Problem formulation. Given a square pencil $sA+B$ such that $A, B \in \mathbb{R}^{n\times n}$, rank $A = n_1 \leqslant n$ the problem to be examined refers to the investigation of the solvability of the equation:

$$\det(sA + B + \Lambda) = \phi(s) \tag{2.1}$$

with respect to $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ when $\phi(s)$ is a given polynomial of n_1 degree where λ_i is either real, or complex.

Notation. If m, n are two integers $m \leqslant n$, then $Q_{m,n}$ is the set of lexicographically ordered sequences (see [24]) of m integers from the set $\{1, 2, \ldots, n\}$ and D_n is any sequence of n integers from $\{1, 2, \ldots, n\}$ with possible repetition and any order [24]. If X is an $m \times n$ matrix and $r \leqslant \min(m, n)$ then we shall denote by $C_r(X)$ the rth compound matrix of X, which is a matrix made up of all $r \times r$ minors of X lexicographically ordered [24]. Note $C_r(X)$ is a matrix with $\binom{m}{r}$ rows and $\binom{n}{r}$ columns. Each row of $C_r(X)$ ia associated with a sequence $\theta = (i_1, i_2, \ldots, i_r) \in Q_{r,m}$ and each column of $C_r(X)$ is associated with a sequence $\rho = (i_1, i_2, \ldots, i_r) \in Q_{r,n}$. The elements of $C_r(X)$ are minors parameterized by the pair of sequences (θ, ρ). If $r = \min(m, n)$ then $C_r(X)$ is a vector (row or column respectively) referred to as the exterior product of rows (columns). In this case if $r = m < n$, $C_r(X)$ is a row vector and its elements are simply parameterized by the sequences θ only.

Definition 2.1. For the matrix $[I_n, \Lambda] \in \mathbb{R}^{n\times 2n}$, the nth compound $C_n([I_n, \Lambda]) \in \mathbb{R}^{1\times\binom{2n}{n}}$ is a row vector and its elements are defined by the sequences $\omega = (i_1, i_2, \ldots, i_n) \in Q_{n,2n}$. The minors of the compound matrix (row vector) are simply denoted by α_ω. For such sequences ω, we define the following:

(a) The operation π on $\omega \in Q_{n,2n}$ as:

$$\pi(\omega) \triangleq (\pi(i_1), \pi(i_2), \ldots, \pi(i_n)) = (j_1, \ldots, j_n),$$

where

$$\pi(i_k) = \begin{cases} i_k & \text{if } i_k \leqslant n, \\ \hat{i}_k = i_k - n & \text{if } i_k > n. \end{cases}$$

(b) A sequence $\omega = (i_1, i_2, \ldots, i_n) \in Q_{n,2n}$ is called *degenerate*, if $\pi(\omega) = (j_1, j_2 \ldots, j_n)$ has at least two equal elements (ie $j_l = j_k$) and it is *non-degenerate*, if $\pi(\omega) = (j_1, j_2 \ldots, j_n)$ has distinct elements. In the latter case $\pi(\omega) = (j_1, j_2 \ldots, j_n)$ is a permutation of n distinct elements from $\{1, 2, \ldots, n\}$ and thus its sign, sign $(j_1, j_2 \ldots, j_n)$, as a permutation is defined.

(c) For a sequence $\omega \in Q_{n,2n}$, which is non-degenerate we define as the *sign* of ω as:

$$\operatorname{sgn}(\omega) \triangleq \sigma(\omega) = \operatorname{sign}(j_1, j_2 \ldots, j_n)$$

and as the *trace* of ω, the subset of the elements of $\pi(\omega) = (j_1, j_2 \ldots, j_n)$ which correspond to $i_k > n$ and thus it is the set $\langle\omega\rangle = \{j_{k_1}, j_{k_2}, \ldots, j_{k_\mu}\}$, $\mu \leqslant n$.

Proposition 2.1. *Let* $[I_n, \Lambda] \in \mathbb{R}^{n\times 2n}$ *and define* $C_n([I_n, \Lambda]) = [\ldots, a_\omega, \ldots] \in \mathbb{R}^{1\times\partial}$, $\partial = \binom{2n}{n}$, $\omega \in Q_{n,2n}$. *Then the coordinates* a_ω *are defined as*: $a_\omega = 0$, *if* ω *is degenerate and* $a_\omega \neq 0$, *if* ω *is non-degenerate. Furthermore, if* ω *is non-degenerate*, $\sigma(\omega)$ *is the sign of* ω *and* $\langle\omega\rangle = \{j_{k_1}, j_{k_2}, \ldots, j_{k_\mu}\}$ *is the trace of* ω, *then* $a_\omega = \sigma(\omega)\lambda_{j_{k_1}}\lambda_{j_{k_2}}\cdots\lambda_{j_{k_\mu}}$.

The set of $Q_{n,2n}$ sequences may thus be divided into two disjoint sets, the set $Q^D_{n,2n}$ of degenerate sequences and the set $Q^{nD}_{n,2n}$ of non-degenerate sequences. Both subsets of sequences are assumed to be lexicographically ordered. Consider now the characteristic polynomial of the redesigned system

$$\phi(s) = \det(sA + B + \Lambda) \equiv \phi(A, B, \Lambda).$$

By the Binet–Cauchy theorem [24] we have that:

$$\det[sA+B+\Lambda] = \det\left([I_n, \Lambda_n] \cdot \left[sA^t + B^t, I_n\right]^t\right)$$
$$= C_n([I_n, \Lambda_n])C_n\left(\left[sA^t + B^t, I_n\right]^t\right) = \phi(s). \tag{2.2}$$

Definition 2.2. Let $Q^D_{n,2n}$, $Q^{nD}_{n,2n}$ be the ordered subsets of degenerate and nondegenerate sequences of $Q_{n,2n}$ associated with the $[I_n, I_n]$ structure. We shall denote by $\widetilde{C}_n([I_n, \Lambda])$ the sub-vector of $C_n([I_n, \Lambda])$ obtained by omitting all zero coordinates corresponding to $Q^D_{n,2n}$ degenerate sequences (indices) and retaining the order of the rest. Similarly we shall denote by $\widetilde{C}_n([sA^t + B^t, I_n])$ the reduced dimension sub-vector of $C_n([sA^t + B^t, I_n])$ derived by deleting the $Q^D_{n,2n}$ set of coordinates retaining the order of the rest. Note that the remaining set has 2^n elements. The sub-vectors $\widetilde{C}_n([I_n, \Lambda])$, $\widetilde{C}_n([sA^t + B^t, I_n])$ will be referred to as $[I_n, I_n]$-structured projections.

Note that

$$C_n[I_n, \Lambda]C_n\left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix}\right) = \widetilde{C}_n([I_n, \Lambda])\widetilde{C}_n\left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix}\right) = \phi(s) \tag{2.3}$$

and given that $\widetilde{C}_n([I_n, \Lambda]) = [\ldots, a_\omega, \ldots]$ where $\omega \in Q^{nD}_{n,2n}$, then

$$\widetilde{C}_n([I_n, \Lambda]) = \left[\ldots, \sigma(\omega) \cdot \lambda_{j_{k_1}} \cdot \lambda_{j_{k_2}} \cdots \lambda_{j_{k_{\mu-1}}} \cdot \lambda_{j_{k_\mu}}, \ldots\right]$$
$$= \left[\ldots, \lambda_{j_{k_1}} \cdot \lambda_{j_{k_2}} \cdots \lambda_{j_{k_{\mu-1}}} \cdot \lambda_{j_{k_\mu}}, \ldots\right] \mathrm{diag}\{\ldots, \sigma(\omega), \ldots\}$$
$$= \widehat{C}_n([I_n, \Lambda])D(\sigma(\omega)) \tag{2.4}$$

then

$$\phi(s) = \widehat{C}_n([I_n, \Lambda])D\{\sigma(\omega)\}\widetilde{C}_n\left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix}\right) = \widehat{C}_n([I_n, \Lambda])\widehat{C}_n\left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix}\right) \tag{2.5}$$

The vectors

$$\widehat{C}_n([I_n, \Lambda]) \triangleq \widetilde{C}_n[I_n, \Lambda]D\{\sigma(\omega)\} \in \mathbb{R}^{1\times\widehat{\partial}}, \quad \widehat{\partial} = 2^n, \tag{2.6}$$
$$\widehat{C}_n\left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix}\right) \triangleq D\{\sigma(\omega)\}\widetilde{C}_n\begin{bmatrix} sA+B \\ I_n \end{bmatrix} = \underline{\hat{p}}(s) \in \mathbb{R}^{\widehat{\partial}}[s]$$

will be referred to as *normalized* $[I_n, I_n]$*-structured projections* of $C_n([I_n, \Lambda])$, $C_n([sA^t + B^t, I_n])^t$ respectively. In particular, $\widehat{C}_n([sA^t + B^t, I_n])^t = \underline{\hat{p}}(s)$, will be called the $[I_n, I_n]$-*Grassmann representative* of the system.

Proposition 2.2. *The normalized* $[I_n, I_n]$*-structured projection of* $\widehat{C}_n([I_n, \Lambda])$ *may be expressed using tensor products* $\otimes$ *as*:

$$\widehat{C}_n([I_n, \Lambda]) = (1, \lambda_1) \otimes (1, \lambda_2) \otimes \cdots \otimes (1, \lambda_n). \tag{2.7}$$

The above result follows by inspection of the expression of $\widehat{C}_n([I_n, \Lambda])$. The characteristic polynomial is expressed as in (2.5) and it is generated by the $[I_n, I_n]$-Grassmann representative of the system ie

$$\underline{\hat{p}}(s) = \widehat{C}_n\left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix}\right). \tag{2.8}$$

Remark 2.1. For any $sA+B, C_n\left([sA^t+B^t, I_n]\right)$ is a polynomial vector $\underline{\widehat{p}}(s)$, the components of which are not necessarily coprime.

Definition 2.3. The greatest common divisor of the entries of $\underline{\hat{p}}(s)$ will be denoted by $\phi_{A,B}(s)$ and this will be referred to as the $[I_n, I_n]$-*fixed polynomial* of the system. A system for which $\phi_{A,B}(s)=1$ will be called $[I_n, I_n]$-*irreducible*; otherwise, it will be called $[I_n, I_n]$-*reducible*.

The following result can be readily established (see also [22]):

Theorem 2.1. *The fixed zeros of the redesigned polynomial $\phi(A, B, \Lambda)$ for all possible Λ are defined by the roots of the polynomial $\phi_{A,B}(s)$.*

If $\phi_{A,B}(s)$ is nontrivial, we can factorize $\underline{\hat{p}}(s)$ as $\underline{\hat{p}}(s)=\underline{\tilde{p}}(s)\phi_{A,B}(s)$ and then use the vector $\underline{\tilde{p}}(s)$ as the generator of the assignable zeros. In the following we assume that $\phi_{A,B}(s)=1$ and thus $\underline{\hat{p}}(s)$ is the generator of the assignable zeros. We can now easily establish that

$$\begin{aligned}\det(sA+B+\Lambda) &= \widehat{C}_n([I_n, \Lambda])D\{\sigma(\omega)\}\widetilde{C}_n\left(\begin{bmatrix} sA+B \\ I_n\end{bmatrix}\right)\\ &= (1,\lambda_1)\otimes(1,\lambda_2)\otimes\cdots\otimes(1,\lambda_n)\cdot\underline{\widehat{p}}(s).\end{aligned} \tag{2.9}$$

The polynomial vector $\underline{\widehat{p}}(s)$ has dimension $\widehat{\partial}=2^n$ and degree $n_1=\text{rank}(A)$. The coefficient matrix of $\underline{\widehat{p}}(s)$ is a matrix of dimension $\widehat{\partial}\times(n_1+1)$ and it is called the reduced Plucker matrix [18,30], P, for the pencil $\left[sA^t+B^t,\ I_n\right]^t$ with reference to the diagonal problem. By equating the coefficients of equal powers of s in (2.9) we get

$$(1,\lambda_1)\otimes(1,\lambda_2)\cdots\otimes(1,\lambda_n)\cdot P=\underline{\phi}, \tag{2.10}$$

where $\underline{\phi}$ is the coefficient vector of $\phi(s)$.

Example 2.1. Let a system matrix of an RL circuit be:

$$sA+B=\begin{bmatrix} s+5 & s-1 & s \\ 2s & s & s+3 \\ 1 & 2 & -1\end{bmatrix}.$$

In this case the $C_3([I_3, \Lambda_3])$ matrix is

$$\begin{aligned}&C_3[I_3, \Lambda_3]\cdot C_3\begin{bmatrix} 1 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda_3\end{bmatrix}\\ &\quad=(1,0,0,\lambda_3,0,-\lambda_2,0,0,0,\lambda_2\lambda_3,\lambda_1,0,0,0,-\lambda_1\lambda_3,0,\lambda_1\lambda_2,0,0,\lambda_1\lambda_2\lambda_3)\end{aligned}$$

The $[sA+B, I_n]^t$ matrix is then expressed as:

$$\begin{bmatrix} sA+B \\ I_3\end{bmatrix}\equiv\begin{bmatrix} s+5 & s-1 & s \\ 2s & s & s+3 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$$

The nonzero elements of $C_3([I_3, \Lambda_3])$ are $(1,\lambda_3,-\lambda_2,\lambda_2\lambda_3,\lambda_1,-\lambda_1\lambda_3,\lambda_1\lambda_2,\lambda_1\lambda_2\lambda_3)$ and the corresponding of $C_3([sA+B, I_3]^t)$ are $(3s^2-21s-33,-s^2+7s,2s+5,s+5,-3s-6,-s,-1,1)$. Therefore:

$$\det(sA+B+\Lambda)=[1 \quad \lambda_3 \quad \lambda_2 \quad \lambda_2\lambda_3 \quad \lambda_1 \quad \lambda_1\lambda_3 \quad \lambda_1\lambda_3 \quad \lambda_1\lambda_2\lambda_3] \\ \times [3s^2-21s-33, -s^2+7s, 2s+5, s+5, -3s-6, -s, -1, 1]^t.$$

The problem described above involves the solution of a set of nonlinear algebraic equations. When the number of solutions is finite, this number is combinatorially large (one can prove that the degree is $n!$) and this makes the problem difficult to be investigated via the standard Groebner basis tools [25], especially when n is large. To define solutions to the problem we will follow the methodology in [2] by studying the local properties of degenerate solutions.

3. Frequency assignment map and degeneracy

Consider the matrix pencil $sA+B$ where $\text{rank}(A)=n_1$. The *Frequency Assignment Map* associated with the problem is the map assigning to the n free elements of the diagonal matrix $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$, the coefficient vector $\underline{\phi}$, i.e.

$$F:\mathbb{R}^n \to \mathbb{R}^{n_1} : F(\Lambda)=\underline{\phi} \tag{3.1a}$$

such that

$$\det(sA+B+\Lambda)=\phi(s). \tag{3.1b}$$

The problem of arbitrary frequency assignment can be formulated in terms of the map F. This is equivalent to proving that the map F is onto. The study of the properties of this map contains two distinct cases. The first corresponds to case where $\text{rank}\,A=n$ and the second is when $\text{rank}\,A<n$. The full rank case is considered in Section 6, whereas the rank deficient case is considered below. In fact when $\text{rank}\,A<n$ there exist a special class of matrices which play a crucial role for the problem of assignment, and this is the set of degenerate matrices. A diagonal matrix Λ_0 is *degenerate* iff:

$$F(\Lambda_0)=0 \Leftrightarrow \det(sA+B+\Lambda_0)=0. \tag{3.2}$$

In other words, Λ_0 is degenerate if the pencil $sA+B+\Lambda_0$ becomes singular. In the following, for the sake of simplicity of the presentation, we concentrate to the case of $\text{rank}\,A=n-1$ which is representative of the more general case of $\text{rank}\,A<n$. The following theorem demonstrates the significance of degenerate matrices.

Theorem 3.1. *If there exists a degenerate matrix Λ_0 such that the differential DF_{Λ_0} is onto, then any set of $n-1$ frequencies can be assigned via some diagonal perturbation.*

Proof. Since the differential DF_{Λ_0} is onto and $F(\Lambda_0)=0$ there is a ball, $B(0,\varepsilon)$, such that $F(\mathbb{R}^n)\supset B(0,\varepsilon)$. For any set of frequencies $s_1, s_2, \ldots, s_{n-1}$, there exists a polynomial $\phi(s)=r(s-s_1)(s-s_2)\cdots(s-s_{n-1})$ whose coefficient vector $\underline{\phi}$ is in the ball $B(0,\varepsilon)$. If we now consider $\Lambda\in F^{-1}(B(0,\varepsilon))$ such that $F(\Lambda)=\underline{\phi}$, the result is established. $\square$

For a generic $n\times n$ pencil when n is small the set of all degenerate matrices may be constructed via the Groebner Basis methodology as this is demonstrated below.

Example 3.1. Consider the pencil

$$sA+B=\begin{bmatrix} -3s & 2+4s & -1-s \\ -3+4s & 5+s & -1-2s \\ -4+s & 6+5s & -1-3s \end{bmatrix}$$

then the set of equations defining all the degenerate matrices $\text{diag}\{x,y,z\}$ is given by:

$$x-4y-xy+6z+5xz+xyz=0, \quad -5+x-3xy-11z+xz-3yz=0, \\ -2+7x+10y-19z=0.$$

The above set of equations defining the degenerate compensator is an algebraic set in three unknowns and may be solved by methods such as Groebner Basis techniques. Simple calculations using MATHEMATICA reduces the system to the following equivalent system:

$$
\begin{aligned}
&480 + 5312x + 16433x^2 + 21474x^3 + 15452x^4 + 5726x^5 + 147x^6 = 0,\\
&1579680 - 10392988x - 18923271x^2 - 12885549x^3\\
&\quad -3302425x^4 - 81879x^5 + 2714400y = 0,\\
&2122560 - 12293068x - 18923271x^2 - 12885549x^3\\
&\quad -3302425x^4 - 81879x^5 + 5157360z = 0.
\end{aligned}
$$

In this example, the number of solutions is defined by the degree of the first equation 6 (=3!) and these are four real and two complex.

One can calculate the number of degenerate matrices for a generic pencil as shown by the following result.

Theorem 3.2. *For a generic $n \times n$ pencil $sA + B$ such that* $\operatorname{rank}(A) = n - 1$ *the number of degenerate diagonal matrices is finite and equal to $n!$.*

Proof. Consider the pencil

$$
sA + B = \begin{bmatrix} s & r_1 - s & 0 & 0 \\ 0 & s & \ddots & 0 \\ 0 & \ddots & \ddots & r_{n-1} - s \\ r_n - s & 0 & 0 & s \end{bmatrix}, \tag{3.3}
$$

where $r_1, r_2, \ldots, r_n$ are distinct numbers. Then it can be readily verified that

$$
\det(sA + B - \Lambda) = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n) - (s - r_1)(s - r_2)\cdots(s - r_n). \tag{3.4}
$$

Therefore the number of degenerate compensators Λ for this pencil is equal to the number of permutations of $(r_1, r_2, \ldots, r_n)$, i.e. $n!$. The differential of F at a degenerate point Λ_0 (see Lemma 5.1 later on for its computation) is given by the coefficient matrix of the polynomial vector formed by the diagonal elements of $\operatorname{adj}(sA + B - \Lambda_0)$. These elements can be calculated as follows: By omitting the first row and column of $sA + B - \Lambda_0$ we calculate the determinant to be $(s - \lambda_2)(s - \lambda_3)\cdots(s - \lambda_n)$. Similarly, by omitting the second row and column and then calculate the determinant we get $(s - \lambda_1)(s - \lambda_3)\cdots(s - \lambda_n)$. The process continues until the nth determinant is computed which is equal to $(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_{n-1})$. Therefore, the differential of F at that degenerate point is equal to the coefficient matrix of

$$
(s - r_1)(s - r_2)\cdots(s - r_n)\begin{bmatrix} \frac{1}{s-\lambda_1} & \frac{1}{s-\lambda_2} & \cdots & \frac{1}{s-\lambda_n} \end{bmatrix}.
$$

To calculate the rank of the coefficient matrix G of the above polynomial vector we consider a vector $\underline{v} = [v_1, \ldots, v_n]$ in its left Kernel. This vector must satisfy

$$
\frac{v_1}{s - \lambda_1} + \frac{v_2}{s - \lambda_2} + \cdots + \frac{v_n}{s - \lambda_n} = 0 \quad \forall s,
$$

which leads clearly to that $\underline{v} = \underline{0}$; this implies that G and therefore $DF_{\Lambda 0}$ has full rank. The above proves that the differential has rank full and it is equal to n. Therefore the degenerate compensators of the above pencil are all permutations of $(r_1, r_2, \ldots, r_n)$ and each one of them has multiplicity one. Counting them we can prove that the number of degenerate solutions for this pencil is $n!$, and this is as many as the permutations of n objects.

To establish the result for real pencils we extend the set and consider now the set Σ of all complex pencils $sA + B$ with $\operatorname{rank}(A) = n - 1$ and the subset:

$$
\Sigma_1 = \{sA + B \in \Sigma : \text{ if } \det(sA + B + \Lambda) = 0 \text{ then } \det(DF_\Lambda) \neq 0\}. \tag{3.5}
$$

The above is a Zarisky open subset of Σ; furthermore, for every $sA+B$ in Σ_1, the set of its degenerate compensators is finite (since the differential of F has full rank, the set $F^{-1}(0)$ is zero dimensional). Consider now the map that assigns to every pencil the number of its degenerate perturbations, that is

$$g : \Sigma_1 \to N : g(sA+B) = \#\{\Lambda : \det(sA+B+\Lambda) = 0\}. \tag{3.6}$$

Clearly, the above map as being from a connected set to a discrete set and being continuous, cannot be multivalued, that is g is the constant map. Furthermore, since the value of g on the pencil we constructed (Eq. (3.3)) is $n!$, then $g(sA+B) = n!$ for every $sA+B$ in Σ_1. This also holds true for the restriction of Σ_1 on the real pencils. This proves that the number of degenerate compensators for a generic pencil $sA+B$ such that $\operatorname{rank} A = n-1$ is $n!$. □

4. Classification of degenerate compensators

We may classify the degenerate matrices Λ for a pencil $sA+B$ according to the values of row or column minimal indices of $sA+B-\Lambda$ [20].

Definition 4.1. A *degenerate matrix* Λ for a pencil $sA+B$ is of degree k, if the polynomial module that spans the right Kernel of $sA+B-\Lambda$ has Forney dynamical order k [19].

Remark 4.1. For the pencil $sA+B$ ($\operatorname{rank} A = n-1$), a degenerate matrix Λ of degree k can be constructed by searching for a polynomial vector $\underline{u}(s) = \underline{u}_k s^k + \underline{u}_{k-1}s^{k-1} + \cdots + \underline{u}_0$ in the right null space of $sA+B-\Lambda$, that is

$$(sA+B-\Lambda)\left(\underline{u}_k s^k + \underline{u}_{k-1}s^{k-1} + \cdots + \underline{u}_0\right) = \underline{0} \tag{4.1}$$

or equivalently

$$\begin{bmatrix} A & & & & 0 \\ B-\Lambda & A & & & \\ & B-\Lambda & O & & \\ & & O & A & \\ 0 & & & B-\Lambda \end{bmatrix} \begin{bmatrix} \underline{u}_k \\ \underline{u}_{k-1} \\ \vdots \\ \underline{u}_1 \\ \underline{u}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{4.2}$$

Examination of small dimension cases leads to a conjecture on the number of degenerate perturbations of degree d, which is stated as a conjecture below. However although this conjecture is of general theoretical interest it does not play a role in the subsequent developments.

Conjecture 4.1. *For a generic $n \times n$ pencil $sA+B$ such that* $\operatorname{rank}(A) = n-1$, *the number of degenerate diagonal matrices of degree d, B_d, $(0 \leqslant d \leqslant n-1)$ is finite and it is equal to:*

$$B_d = \begin{cases} \binom{n}{d+1} A_{d+1} & \text{if } d > 0, \\ 1 \quad \text{if } d = 0, \end{cases} \tag{4.3}$$

where A_{d+1} is the number of permutations of $d+1$ objects with no fixed points.

Although the construction of degenerate matrices looks as though it has the same complexity to the problem we have started, there are certain degenerate matrices that can be easily constructed via linear equations. These are the degenerate diagonal matrices of degree 0 and $n-1$.

Proposition 4.1. *Consider the $n \times n$ pencil $sA+B$ such that rank$(A) = n-1$ and let $\underline{v}^t, \underline{w}$ be vectors such that*:

$$\underline{v}^t A = \underline{0}, \quad A\underline{w} = \underline{0} \tag{4.4}$$

then, the diagonal matrices defined by

$$\Lambda_0 = -\mathrm{diag}\left\{\frac{\underline{v}^t\underline{b}_1}{v_1}, \ldots, \frac{\underline{v}^t\underline{b}_n}{v_n}\right\}, \ldots, \Lambda_{n-1} = -\mathrm{diag}\left\{\frac{\underline{b}_1^t\underline{w}}{w_1}, \ldots, \frac{\underline{b}_n^t\underline{w}}{w_n}\right\}, \tag{4.5}$$

where $\underline{b}_i$, $(\underline{b}_i^t)$ are the columns (rows) of B and v_i (w_i) are the coordinates of $\underline{v}$ $(\underline{w})$, are degenerate matrices of degree 0.

Proof. By Remark 4.1 we have that for a degree one degenerate matrix we have that

$$\begin{bmatrix} A \\ B - \Lambda \end{bmatrix} \underline{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.6}$$

or equivalently

$$A\underline{w} = \underline{0} \quad \text{and} \quad B\underline{w} = \Lambda\underline{w}. \tag{4.7}$$

By solving with respect to Λ the result follows. □

Another classification of the degenerate matrices may be given in terms of infinite and finite gain properties, as shown below:

Definition 4.2. A degenerate solution matrix is referred to as *infinite*, if they are defined as limits of sequences of matrices $\{\Lambda_n\}$ where at least one element tends to infinity.

In order to include infinity in the set of diagonal matrices we have to compactify the set of diagonal matrices and this is done by representing this set as a product of one-dimensional projective spaces.

Remark 4.2. Any finite matrix Λ can be embedded in $P^1(\mathbb{R}) \times P^1(\mathbb{R}) \times \cdots \times P^1(\mathbb{R})$ by using the representation $[I_n, \Lambda]$, that is by introducing the function $f : \mathbb{R}^n \to (P^1(\mathbb{R}))^n$ such that

$$f(\lambda_1, \lambda_2, \ldots, \lambda_n) = [(1, \lambda_1), (1, \lambda_2), \ldots, (1, \lambda_n)]. \tag{4.8}$$

In this setting, if some entry λ_j is a sequence $\lambda_j\,(\varepsilon) = a/\varepsilon$ tending to infinity as ε tends to 0, then it is represented by the pair $(1, a/\varepsilon)$ in $P^1(\mathbb{R})$, which is equal to the pair (ε, a) which tends to $(0, a)$.

We may state the result:

Corollary 4.1. *The product of projective spaces $(P^1(\mathbb{R}))^n = P^1(\mathbb{R}) \times P^1(\mathbb{R}) \times \cdots \times P^1(\mathbb{R})$ is the parameter space of all diagonal matrices that includes finite and infinite elements. The finite matrices are represented by elements of the type $[I_n, \Lambda]$, whereas the infinite matrices by*

$$[A, \Lambda], A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\} \tag{4.9}$$

with at least one of a_i entries zero.

Taking into account the above formulation, the degenerate matrices constructed in Proposition 4.1 are finite iff $v_i \neq 0$, $\forall j$. In the case where $\mathrm{rank}(A) = n - k < n - 1$, then if V is the basis matrix of the left kernel of A, the following result characterizes the existence of at least one finite degenerate matrix.

Proposition 4.2. *If $V = [\underline{v}_1 \quad \cdots \quad \underline{v}_k]$ is a basis matrix of the left kernel of A, then there exists a $\underline{v} \in V$ such that the corresponding degenerate matrix produced by $\underline{v}$ is finite, iff $\forall j : 1 \leqslant j \leqslant k$, $\exists i$ such that $\underline{v}_{ji} \neq 0$.*

Proof. If there is an index j such that for all basis vectors the corresponding coordinate is 0, then every $\underline{v}$ in V will have the same coordinate zero and thus it will give rise to an infinite degenerate matrix. □

5. Genericity results and construction of solutions

The differential of the frequency assignment map F associated with our problem, plays a very important role in the determination of the onto properties of the map and it has thus a crucial role in the solvability of the problem. This differential can be calculated in many ways and for a general square and rank deficient polynomial matrix $A(s)$ the following result can be proved:

Lemma 5.1. *If* $\det(A(s)) = 0$ *then*

$$\det(A(s) + xB(s)) = x \times \operatorname{trace}(\operatorname{adj}(A(s))B(s)) + O(x^2). \tag{5.1}$$

Proof. If we expand $\det(A(s) + xB(s))$, then the coefficient of x will be the sum of all determinants of matrices having $n-1$ columns from $A(s)$ and 1 column from $B(s)$. By expanding these determinants with respect to the columns coming from $B(s)$ and rewriting their sum, the result is established. □

Corollary 5.1. *If* $\operatorname{adj}(sA + B - \Lambda_0) = \underline{v}(s) \cdot \underline{g}^t(s)$ *and* $g_i(s), v_i(s)$ *are the coordinates of these vectors, then* $DF_{\Lambda 0}$ *can be represented by the coefficient matrix of the polynomial vector* $(g_1(s)v_1(s), \ldots, g_n(s)v_n(s))$.

Proof. By Lemma 5.1 the differential of F at Λ_0 is given by the coefficient of x which is the $\operatorname{trace}(\operatorname{adj}(A(s))B(s))$. Setting now $A(s) = sA + B - \Lambda_0$ and $B(s) = \Lambda$ we get

$$\begin{aligned} DF_{\Lambda 0}(\Lambda) &= \operatorname{coefVec}[trace(\underline{v}(s) \cdot \underline{g}^t(s)\Lambda)] \\ &= \operatorname{coefVec}[g_1(s)v_1(s)\lambda_1 + \cdots + g_n(s)v_n(s)\lambda_n] \end{aligned} \tag{5.2}$$

and this readily proves the result. □

Using the above we may now establish the following result:

Theorem 5.1. *For a generic pencil* $sA + B$, $\operatorname{rank} A = n - 1$, *the degenerate diagonal matrix* Λ_0 *in the map of the zero assignment problem, satisfies the condition:* $\operatorname{rank} DF_{\Lambda_0} = n$.

Proof. Let us consider the matrix

$$K(s) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -s & \ddots & & \vdots \\ & \ddots & 1 & \\ & & -s & 0 \end{bmatrix} \tag{5.3}$$

and a pencil defined by $K(s)$ as: $sA + B = K(s)U + I$, where U is a full rank square matrix such that the rightmost column $\underline{v}$ of U^{-1} has all its entries nonzero. Then it can be easily proved that the following properties hold true:

- $A\underline{v} = \underline{0}$.
- The identity matrix I is a degenerate diagonal compensator.

Since the vector $\underline{g}^t(s) = [s^{n-1}, s^{n-2}, \ldots, 1]$ is the basis for the left Kernel of $K(s)U$, its adjoint (which is a representation of the differential of F at the degenerate compensator) can be written as $\underline{v}\underline{g}^t(s)$. By Corollary 5.1, $DF_{\Lambda 0}$ can then be represented by the diagonal matrix

$$\begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ & \ddots & v_{n-1} & 0 \\ 0 & & 0 & v_n \end{bmatrix},$$

which has full rank, since by construction all entries of $\underline{v}$ are nonzero. □

Corollary 5.2. *For a generic pencil $sA + B$, with* $\text{rank}A = n - 1$, *any zero polynomial of degree $n - 1$ can be assigned via diagonal perturbations*.

Proof. Due to Theorem 5.1 there exists a diagonal degenerate perturbation Λ_0 such that DF_{Λ_0} has full rank. As a consequence of Theorem 3.1 any set of $n - 1$ frequencies can be assigned by an appropriate selection of some diagonal perturbation. $\square$

The above establishes the existence of perturbations that assign any $n - 1$ set of frequencies. However, the solutions which are produced this way have little practical value, since they are based on the degenerate perturbation which for the problem we examine creates very large sensitivity norms. An algorithm that assigns the required zeros, based on perturbations of reduced sensitivity norm, may be developed using a Quasi-Newton type procedure that starts from a degenerate diagonal matrix and gradually leading diagonal matrices which assign the desired frequencies and are at a distance from the degenerate solution.

5.1. Frequency assignment algorithm via diagonal perturbations

The algorithm for computing Λ placing the zeros of $sA + B + \Lambda$ to the required location expressed by the polynomial $\phi(s)$, with coefficient vector $\underline{\phi}$, is based on the *Global Linearization* methodology established in [2] and it is described below:

Step 1: Calculate a degenerate diagonal matrix Λ_0 as in Remark 4.1 or by using Proposition 4.1.

Step 2: Calculate the differential DF_{Λ_0} of the Frequency assignment map at the specific degenerate compensator. If this map is onto, then we have complete frequency assignability and we may proceed to the next step; otherwise we go back to the step (1).

Step 3: Apply the Quasi-Newton algorithm to compute perturbations that assign the zero structure and which are at a distance from the degenerate point. In the following let us denote by $x_i = vec(X_i)$, where X_i is a matrix perturbation of the appropriate dimensions. The algorithm may be expressed as shown below:

$$\underline{x}_{i+1} = \underline{x}_i - (JF)^{-1}_{\underline{x}_{n_{k-1}}} (F(\underline{x}_i) - \varepsilon_k \underline{\phi}), \quad n_{k-1} < i \leqslant n_k,$$
$$k = 1, \ldots, r, \; n_0 = 0, \; \underline{x}_{n_0} = \underline{\lambda}_0, \underline{\lambda}_0 = vec(\Lambda_0) \; 0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_k < \cdots,$$

where $\underline{\phi}$ is the coefficient vector of the desired polynomial, F is the frequency placement map, $J\overline{F}$ is the Jacobian matrix representing the differential of the zero assignment map and Λ_0 is the degenerate matrix for which the differential DF_{Λ_0} has full rank.

Remark 5.1. The Jacobian of $F(JF)$ can be easily computed as F is an algebraic polynomial map. An easier computation of JF is indicated in the following section based on the decomposition of F to a product of a multilinear and a linear map.

The above algorithm is based on the following philosophy: If we denote by $\Omega(\phi)$ the family of all perturbations placing the zeros as the roots of the polynomial $\phi(s)$, then a degenerate perturbation, with full rank differential, is a boundary point for all manifolds $\Omega(\phi)$ corresponding to different ϕ's. Using as a starting point the degenerate perturbation (which can be readily computed as in Proposition 4.1) and selecting ε_1 sufficiently small the Newton–Raphson algorithm produces a solution Λ_1 on $\Omega(\phi)$ which is at a small distance from the boundary point. Repeating now the method starting this time from Λ_1 and with a new step ε_2 we produce Λ_2 on $\Omega(\phi)$ and so on.

Example 5.1. Consider a network whose system matrix $T_1(s)$ is defined by:

$$T_1(s) = \begin{bmatrix} G_1 + G_2 + sC & -G_2 & 0 \\ -G_2 & G_2 + G_3 & 1 \\ 0 & 1 & -sL - (1/G_4) \end{bmatrix}$$

when the values are: $C = 1, L = 1, G_1 = 4, G_2 = 1, G_3 = 0, G_4 = \infty$, then the system matrix becomes:

Table 1
Summary of the Quasi-Newton computations Algorithm for Example (5.1).

Iterations	ε	G_2	G_3	G_4	Dist from deg perturbation
$n_0 = 0$	0	−2	1	−3	0
$n_1 = 60$	$\varepsilon_1 = 0.5$	−2.55	1.050	−2.741	0.610
$n_2 = 100$	$\varepsilon_2 = 1.2$	−3.325	1.125	−2.652	1.375
$n_3 = 195$	$\varepsilon_3 = 2.5$	−4.706	1.206	−2.611	2.741
$n_4 = 330$	$\varepsilon_4 = 5$	−7.278	1.278	−2.594	5.301
$n_5 = 580$	$\varepsilon_5 = 10$	−12.33	1.333	−2.588	10.34
$n_6 = 660$	$\varepsilon_6 = 18$	−20.36	1.365	−2.586	18.37

$$T_1(s) = \begin{bmatrix} s+5 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -s \end{bmatrix}.$$

Assuming that we would like to change the natural frequencies of the above system by tuning the values of G_2, G_3, G_4, we define the following perturbation:

$$\begin{bmatrix} G_2 & -G_2 & 0 \\ -G_2 & G_2+G_3 & 0 \\ 0 & 0 & G_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_2 & 0 & 0 \\ 0 & G_3 & 0 \\ 0 & 0 & G_4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U\Lambda U^t,$$

which is equivalent to applying a diagonal perturbation $\Lambda = \mathrm{diag}(G_2, G_3, G_4)$ to the modified system

$$U^{-1}T_1(s)(U^t)^{-1} = \begin{bmatrix} s+5 & s+4 & 0 \\ s+4 & s+4 & 1 \\ 0 & 1 & -s \end{bmatrix}$$

The equations defining the degenerate perturbations are:

$$\begin{aligned} f_2(G_2, G_3, G_4) &= -1 - G_2 - G_3 = 0 \\ f_1(G_2, G_3, G_4) &= -5 - 4G_2 - 5G_3 - G_2G_3 + G_4 + G_2G_4 + G_3G_4 = 0 \\ f_0(G_2, G_3, G_4) &= -5 - G_2 + 4G_4 + 4G_2G_4 + 5G_3G_4 + G_2G_3G_4 = 0 \end{aligned}$$

and the finite solutions of these equations are given by:

(**a**) $G_2 = -2, G_3 = 1, G_4 = -3$; (**b**) $G_2 = 0, G_3 = -1, G_4 = -5$.

Note that both of the above solutions are full (regular) solutions, and thus both can be used as staring points for a numerical Quasi-Newton method to place the characteristic polynomial at any given second order one, $\phi(s)$ using the iterative procedure [21]:

$$\underline{x}_{n+1} = \underline{x}_n - (JF)^{-1}_{\underline{x}_0}(\underline{f} - \varepsilon_k \underline{\phi}),$$

where $\underline{x} = (G_2, G_3, G_4)^t$, $\underline{\phi} = [1, 8, 15]^t$, $\underline{f} = [f_2, f_1, f_0]^t$ and $\underline{x}_0 = (-2, 1, -3)^t$. Starting with a value $\varepsilon_1 = 0.5$, the method converges after about 60 iterations to $\underline{x}_{60} = (-2, 5507, 1, 050697, -2, 74137)^t$. Taking now this as a starting point we repeat the method for $\varepsilon_2 = 1.2$ and so on. Table 1 displays the various solutions we obtain through this algorithm the last column being the Euclidean distance of the solution from the degenerate one. The final row refers to the solution $G_2 = -20, 36, G_3 = 1, 365, G_4 = -2, 586$ which is the furthest away from the degenerate compensator. This compensator is achieved after 660 iterations, it assigns the zeros −2, 99915 and −4, 99982 (−3 and −5 were the required ones) and its distance from the degenerate compensator is 18, 37.

6. Necessary and sufficient condition for arbitrary assignment in the complex domain: the case rank(A) = n

So far we have examined the surjectivity property of the frequency assignment map F in terms of degenerate compensators and their properties, when rank $A = n_1 < n$. For the case where rank

$(A) = n$, there are no degenerate compensators and the above approach cannot be deployed. In this case we will follow the approach in [15], which is based on the examination of the rank properties of the differential of the map F. In fact, the rank of the differential of a complex algebraic map, although it is a local invariant, it may determine global properties of this map. The following lemma explains how this rank is related with the (almost) surjectivity property of F. Note that by "almost onto" it is meant that all polynomials, but a negligible set can be assigned.

Lemma 6.1 (*Dominant Morphism* [14,15]). *If F is an algebraic map between two complex varieties X, Y for which* $\dim X \geqslant \dim Y$, *then there exists x in X such that* $\operatorname{rank} DF_X = \dim Y$, *iff F is* (*almost*) *onto*.

This shows that the invariant that characterizes the onto property of the map F is the nth exterior product of its differential DF_X. In the case we examine, this invariant is the determinant of the Jacobian of F, i.e. $\det(J(F)_X)$. As F can be factored as [15]:

$$F : \mathbb{C}^n \xrightarrow{T} \mathbb{C}^\sigma \xrightarrow{P} \mathbb{C}^n, \tag{6.1}$$

where

$$T(\underline{x}) = \bigotimes_{i=1}^{n} (1, x_i), \quad F(\underline{x}) = T(\underline{x}) \cdot P \quad \text{and} \quad \sigma = 2^n \tag{6.2}$$

the Jacobian of the zero assignment map can be calculated in terms of the Jacobian of T and the Plucker matrix P [18]. By the Binet Cauchy theorem $\det(J(F))$ can be factored as a product of two compounds as follows:

$$\det(J(F)_x) = C_n(J(T)) \cdot C_n(P). \tag{6.3}$$

The compound $C_n(P)$ is a vector containing the system parameters, whereas the vector $C_n(J(T))$ contains the structure implied by the diagonal structure of the controller. The calculation of $\det(J(F)_x)$ is thus reduced to calculating $C_n(J(T))$. The calculation of $J(T)$ can be easily performed as shown by the following lemma:

Lemma 6.2 [15]. *The partial derivative of T with respect to x_i, is given by*:

$$\partial T / \partial x_i = \otimes(1, x_1) \otimes \cdots \otimes (1, x_{I-1}) \otimes (0, 1) \otimes (1, x_{I+1}) \otimes \cdots \otimes (1, x_n). \tag{6.4}$$

Select n entries of the vector $T(x)$ say $\underline{a} = [a_l, a_2, \ldots, a_n]$, and call the Jacobian of the function $\underline{a}$, $J(\underline{a})$; then this is a square $n \times n$ matrix whose determinant is one of the coordinates of the vector $C_n(J(T))$; conversely, all the coordinates $C_n(J(T))$ are of the form $\det(J(\underline{a}))$ for some $\underline{a}$, i.e.

$$C_n(J(T)) = (\det(J(\underline{a}))_a.$$

The following result is central in providing an expression for the Jacobian $J(\underline{a})$ needed for the description for the compound $C_n(J(T))$.

Proposition 6.1 [15]. *The Jacobian $J(\underline{a})$ is given by*:

$$J(\underline{a}) = \operatorname{diag}\left(x_l^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) I(\underline{a}) \operatorname{diag}(a_l, a_2, \ldots, a_n), \tag{6.5}$$

where the ij entry of $I(\underline{a})$ is 1 if a_j contains the x_i and it is 0 otherwise. Therefore the determinant of $J(\underline{a})$ is expressed as:

$$\det(J(\underline{a})) = I(\underline{a}) a_1, a_2 \ldots a_n / x_1 x_2 \cdots x_n. \tag{6.6}$$

From the above we have the following description of $C_n(J(T))$.

Corollary 4.1 [15]. *The compound $C_n(J(T))$ is given by*

$$C_n(J(T)) = (\det(I(\underline{a})) a_1 a_2 \cdots a_n / x_1 x_2 \cdots x_n)_a.$$

Now every selection of n monomials $\underline{a} = [a_1 a_2 \cdots a_n]$ correspond to a minor M_a of P. For a given monomial m consider the sum

$$P_m = \Sigma \det(I(\underline{a}))M_a, \tag{6.7}$$

where the sum is defined over all terms such that

$$a_1 a_2 \cdots a_n / x_1 x_2 \cdots x_n = m \quad \text{and} \quad \det(I(\underline{a})) \neq 0. \tag{6.8}$$

The determinant of the Jacobian JF is then given as a sum $\sum_m P_m m$, i.e.

$$\det(JF) = \sum_m P_m m, \tag{6.9}$$

where the P_m are numbers calculated as in (6.7) and they are related to the minors of the Plucker matrix [18] and m are linearly independent monomials arising from the diagonal structure of the controller and defined by (6.8). As the monomials m are linearly independent over, the collection of all P_m constitutes a system invariant characterizing the onto properties of the pole placement map and this leads to the following result:

Theorem 6.1 [15]. *The complex pole placement map is (almost) onto, if there exists m such that* $P_m \neq 0$.

Indeed as the determinant of the Jacobian $J(F)$ is a linear combination of linearly independent monomials m and it is zero, iff all the coefficients P_m are zero. This way the determination of the rank properties of $J(F)$ reduces to examining whether the set of numbers P_m, which are calculated in terms of the minors of the Plucker matrix, are zero or not.

Example 6.1. The construction of the sets of invariants related to our problem is explained below. Consider the matrix pencil

$$sA + B = \begin{bmatrix} s & 1 & 0 \\ 1 & s & 1 \\ 0 & 1 & s \end{bmatrix}.$$

The map F can be factored as $T \cdot P$ where T is given by

$$T(x_1, x_2, x_3) = [1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3]$$

and P is the reduced Plucker matrix. For the sake of simplicity we may omit the first entry (1) of T, then $J(T)$ is the collection of the three partial derivatives (with respect to (x_1, x_2 and x_3) and it is given by

$$J(T) = \begin{bmatrix} 1 & 0 & 0 & x_2 & x_3 & 0 & x_2x_3 \\ 0 & 1 & 0 & x_1 & 0 & x_3 & x_1x_3 \\ 0 & 0 & 1 & 0 & x_1 & x_2 & x_1x_2 \end{bmatrix}.$$

Then the determinant of $J(F)$ is given as the product of the 3rd compounds of $J(T)$ and P. The 3rd compound of $J(T)$ is given by

$$\begin{aligned} C_3(J(T)) = [&1, 0, x_1, x_2, x_1x_2, -x_1, 0, -x_3, -x_1x_3, x_1^2, x_1x_2, x_1^2x_2 - x_1x_3, -x_1^2x_3, 0, x_2x_3, \\ &- x_1x_2, -x_2^2, -x_1x_2^2, -x_2x_3, 0, x_2^2x_3, -x_1x_3, x_2x_3, 0, x_3^2, x_1x_3^2, -x_2x_3^2, -2x_1x_2x_3, \\ &- x_1^2x_2x_3, x_1x_2^2x_3, -x_1x_2x_3^2] \end{aligned}$$

This vector has to be multiplied by the compound vector of P ie the vector of all 3×3 minors $M[i]$ of the Plucker matrix. If we perform this multiplication and we collect together all the monomials m appearing in $C_3(J(T))$, we obtain $\det(J(F))$ as the linear combination $\sum_m P_m m$. The coefficients of this expression in terms of the minors of the Plucker matrix $M[i]$ are given in Table 2. Since there exist at least one nonzero P_m, the map F is (almost) onto. Furthermore the Jacobian of F has full rank and its determinant is given by

Table 2
Computation of the determinant of the Jacobian det($J(F)$).

$.m$	P_m	$.m$	P_m
1:	$M[1] = 0$	$.x_1x_3$:	$M[19] - M[23] + M[27] = 0$
x_1:	$M[3] - M[6] = 1$	$.x_1x_3x_2^2$:	$M[34] = 0$
x_1^2:	$M[10] = 0$	$.x_2x_3$:	$M[19] - M[23] + M[27] = 0$
x_2:	$M[4] + M[16] = 0$	$.x_2x_3x_1^2$:	$M[33] = 0$
x_2^2:	$M[21] = 0$	$.x_1^2x_2$:	$M[12] = -1$
x_3:	$M[17] - M[8] = -1$	$.x_1^2x_3$:	$M[14] = 1$
x_3^2:	$M[29] = 0$	$.x_2^2x_3$:	$M[25] = -1$
x_1x_2:	$M[5] + M[11] - M[20] = 0$	$.x_2^2x_1$:	$M[22] = 1$
$x_1x_2x_3$:	$M[32] = 0$	$.x_3^2x_1$:	$M[30] = -1$
$x_1x_2x_3^2$:	$M[35] = 0$	$.x_3^2x_2$:	$M[31] = 1$

$$\det(J(F)) = \sum_m P_m m = x_1 - x_1^2x_2 + x_1x_2^2 - x_3 + x_1^2x_3 - x_2^2x_3 - x_1x_3^2 + x_2x_3^2$$

It is apparent that Det(JF) is zero only in a proper subvariety of C^3. This means that in almost all points of $\mathbb{C}^3$, JF has full rank. The Dominant morphism theorem states that even if there exists one point such that JF has full rank then F is almost onto.

The dominant morphism theorem proves existence, but it is not appropriate for the construction of solution. An efficient method for the construction of solutions for the case rank$(A) = n$ is still an open problem which the methodology of the present paper does not address. For this case, the computation of solutions may follow the following alternative routes: We can use the usual methods based on the multi-linear/determinantal formulation and then solving the set of algebraic equations using Groebner basis methods. Alternatively, the Newton–Raphson method can be employed; in fact, the dominant morhism theorem guarantees that the Jacobian matrix is invertible for almost all diagonal controllers and therefore. The starting point of the iterative method may be any random point and the method guarantees convergence, if this point is close enough to the solution. We should point out, however, that the method does not exploit the special structure of the problem and convergence to the solution is not always guaranteed.

7. Conclusions

The problem of zero assignment for matrix pencil $sA + B$, for the special case of diagonal type perturbations, has been considered and a number of results have been established characterizing the solvability of this problem. These results cover the case of generic and nongeneric problems and extensive use was made of the determinantal formulation of the problem. For the cases where rank$(A) \leqslant n - 1$, degenerate perturbations are defined and the global linearization provides a framework for establishing results and computing assigning perturbations; although here we have considered for the sake of simplicity the case of rank$(A) = n - 1$ (one degenerate perturbation), the results can be extended to the rank$(A) < n - 1$ where more than one degenerate solutions exist. The interest of such cases is that we can deploy global linearization ideas, which permit establishment of solvability conditions, as well as development of an algorithm for the computation of the zero assigning perturbations. The analysis has made a distinction between the case of regular pencils with infinite zeros and that of regular pencils with no infinite zeros, which require alternative methods for study. For the rank$(A)) \leqslant n - 1$ cases, methodologies for computing the solutions have been given. The proposed method for computing solutions when rank$(A) = n$ has the usual advantages and disadvantages of the Newton method for a set of equations. It is clear that this general method needs to exploit the special features of the problem, if we are to improve the chances for convergence. The results here provide the means for studying problems of linear network redesign by modification of the non-dynamic elements. The general network redesign may still be formulated in terms of matrix pencils, but more general transformations than the additive diagonal transformations are involved.

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