Karcanias, N., Halikias, G. & Papageorgiou, A. (2010). Strong stability of internal system descriptions. INTERNATIONAL JOURNAL OF CONTROL, 83(1), pp. 182-205. doi: 10.1080/00207170903124669

Strong stability of internal system descriptions

N. Karcanias[1], G. Halikias[1] and A. Papageorgiou[1]

Abstract: The paper introduces a new notion of stability for internal autonomous system descriptions that is referred to as "strong stability" and characterizes the case of avoiding overshoots for all initial conditions within a given sphere. This is a stronger notion of stability compared to alternative definitions (asymptotic, Lyapunov), which allows the analysis and design of control systems described by natural coordinates to have no overshooting response for arbitrary initial conditions. For the case of LTI systems necessary and sufficient conditions for strong stability are established in terms of the negative definiteness of the symmetric part of the state matrix. The invariance of strong stability under orthogonal transformations is established and this enables the characterisation of the property in terms of the invariants of the Schur form of the state matrix. Some interesting relations between strong stability and special forms of coordinate frames are examined and links between the skewness of the eigenframe and the violation of strong stability are derived. The characterisation of properties of strong stability given here provides the basis for the development of feedback theory for strong stabilisation.

Keywords: Strong stability, non-overshooting transients, eigen-frame skewness, Schur form.

1. Introduction

Stability is a crucial system property that has been extensively studied from many aspects [1], [9], [15], [13], [7], [6]. Here we examine a refined form of stability of internal (state-space) autonomous system descriptions that depends on the selection of the state coordinate frame and which is important for system descriptions where the states are physical variables and are referred to as physical system representations. The importance of such descriptions is that we are interested in the behaviour of the physical states. Asymptotic (or Lyapunov) stability is clearly necessary for boundness in some sense of these variables but does not guarantee that such physical variables do not overshoot. We define as overshooting the case where for some initial state vector the corresponding physical variable exceeds its initial value. Non overshooting behaviour is a desirable property in certain applications and can be considered as a special case of constrained control. **In many practical applications, classical notions of stability (such as asymptotic or Lyapunov stability) may be too weak for characterizing satisfactorily operation of systems under feedback control. The new notion of strong stability introduced here is relevant to many real-time applications where a human operator may interpret overshoots as early indications of instability and this may result in actions that can be wrong for the process and may have catastrophic consequences. A non-overshooting response separates stable and unstable behaviours, if we base their diagnosis on the finite time early observation of time responses.**

[1]Control Engineering Research Centre, School of Engineering and Mathematical Sciences, City University, Northampton Square, London EC1V 0HB, U.K.

The paper introduces the concept of "strong stability" for autonomous internal LTI system descriptions. This is a stronger version of stability compared to the standard definitions of asymptotic and Lyapunov stability and characterises the special case where there is no overshooting transient response for all arbitrary initial conditions taken from a given sphere. **A fundamental assumption behind this work is that the system model is characterized by physical variables and thus imposing constraints on the states makes sense.** This new notion of stability is relevant to nonlinear system descriptions having physical state variables. In this paper we restrict ourselves to the linear time invariant autonomous case and necessary and sufficient conditions are established in terms of the negative definiteness (semi-definiteness) of the symmetric part of the state matrix. The dependence of the strong stability property on general coordinate transformations is noted and the existence of special coordinate systems for which we cannot have strong stability is established. It is shown that the property is invariant under orthogonal transformations and this leads to the use of the Schur canonical form, established under orthogonal transformations, as the basis for investigating further the parametrisation of strongly stable state matrices. A number of interesting properties for the family of strongly stable matrices are derived and criteria establishing this property based on the model parameters are obtained. Finally, we examine the role of the skewness of the eigenframe on the violation of the strong stability property and bounds on the eigenframe skewness that lead to the presence of overshoots are established. The latter property indicates that there is a link between loss of strong stability due to eigenframe skewness and reduced robustness of stability to parameter variations.

Stability is of course one of the most important system properties. Apart from the classical notions (e.g. stability in the sense of Lyapunov, asymptotic stability, global vs local stability) many alternative definitions have been proposed in the literature (e.g. see [1], [2], [7], [15]) with specific relevance to linear, non-linear, time-varying or stochastic systems. In the linear time-invariant case considered in this work, refined stability notions which have been proposed include qualitative (sign) stability, D-stability, total stability and R-stability (see [1], [12] for a survey of these notions and their inter-relations). The definition of "strong stability" introduced here (in its three variants) is particularly relevant to issues related to the transient response of a system (e.g. its overshooting behaviour or its transient energy) [7], [16] and could also prove useful for analysing stability properties of systems under switching regimes [14].

The paper is organised as follows: Section 2 introduces the problem, provides links to standard stability notions and introduces the required definitions. In section 3 we establish the necessary and sufficient conditions for different types of strong stability. Section 4 deals with a range of properties of strongly stable matrices. The invariance of strong stability under orthogonal transformations is established in Section 5, where the Schur form is used for defining parameter dependent conditions for strong stability. Finally, the link of skewness of the eigenframe to the violation of strong stability is derived in Section 6.

This new notion is directly related to the absence of state-space overshoots for systems described by physical variables and can be applied, if required, to limit the exponential growth of the system's response [7], [8], [16], or its transient energy and to address objectives related to energy dissipation [18]. Such designs may be relevant to applications involving the human operator as an overall controller, who can intervene in real time

and take actions on the basis of responses over an initial time horizon. Such actions may have catastrophic consequences for the overall performance. Applications where this concept is relevant include cases involving the human operator, such as process-control, economic operations, etc. This new notion has some additional benefits when it is used for studying stabilization problems under state or output feedback. Early results in the area of strong stabilization [5] suggest that problems of this type are easily solvable via convex programming techniques and, further, that closed-form parametrizations of the optimal solution sets can be obtained.

The notation used in the paper is standard and is summarized here for convenience. $\mathcal{R}$ and $\mathcal{C}$ denote the fields of real and complex numbers, respectively. The set of complex numbers with negative real part is denoted by $\mathcal{C}_-$ and is referred to as the open-left-half-plane (OLHP). The set of complex numbers with non-positive real part is denoted by $\bar{\mathcal{C}}_-$ and is referred to as the closed-left-half-plane (CLHP). $\mathcal{R}^{m\times n}$ ($\mathcal{C}^{m\times n}$) denotes the space of all $m \times n$ real (complex) matrices. For a real or complex matrix A, A' denotes the transpose of A and A^* the complex conjugate transpose of A. For a square invertible matrix A, A^{-1} is the inverse of A and $A^{-\prime} = (A^{-1})' = (A')^{-1}$. If A is a square matrix, then $\lambda(A)$ denotes the spectrum of A, i.e. the set of its eigenvalues, $|A|$ is the determinant of A and $\rho(A)$ is the spectral radius of A. If $x \in \mathcal{R}^n$ or $x \in \mathcal{C}^n$, then $\|x\|$ denotes the Euclidian norm of x. For a real or complex matrix A, $\|A\|$ is the induced 2-norm (largest singular value). For a Hermitian or symmetric matrix A, $\lambda_{max}(A)$ denotes the largest eigenvalue of A and $\lambda_{min}(A)$ the smallest eigenvalue of A. For a real square matrix A, the symmetric part of A, $\frac{1}{2}(A + A')$, is denoted by $\bar{A}$. A positive definite matrix A (positive semidefinite, negative definite, negative semi-definite) is denoted as $A < 0$ ($A \geq 0$, $A < 0$, $A \leq 0$, respectively). Finally, the kernel (null-space) of A is denoted as $\mathrm{Ker}(A)$.

2. Problem Statement and Basic Results

We consider the LTI autonomous system:

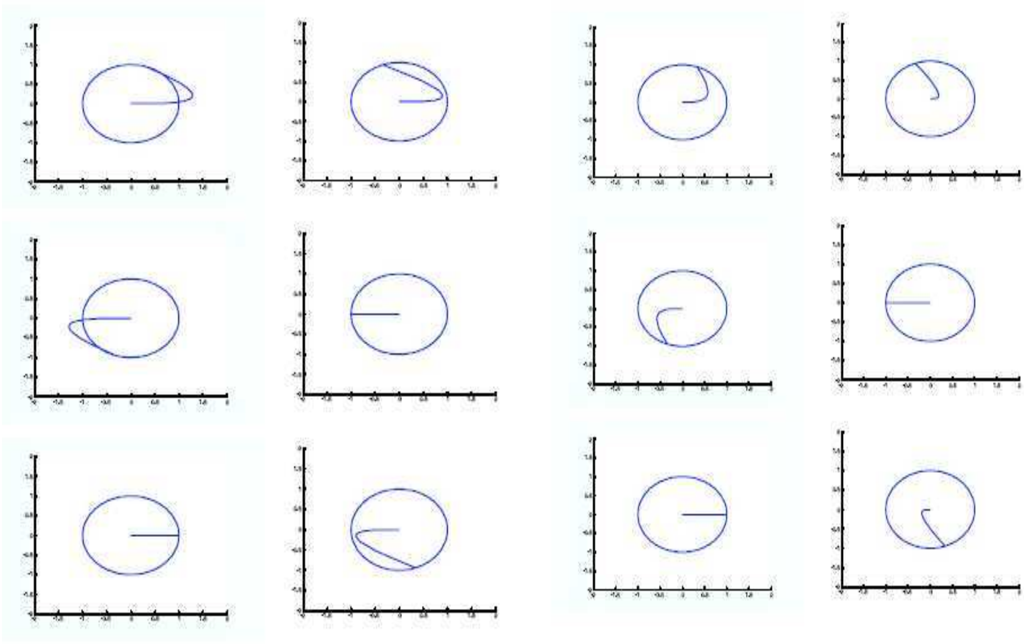
$$\mathcal{S}(A): \quad \dot{x} = Ax, \;\; A \in \mathcal{R}^{n\times n} \tag{1}$$

where we assume that A has the eigenvalue-eigenvector decomposition $A = UJV$ where J is in Jordan form of A and U, V are the generalised right and left eigenvector matrices, respectively. The basic notions of asymptotic and Lyapunov stability for such a system are well established and the eigenvalues of A provide a simple characterisation of such properties, whereas the properties of the eigenframe have no influence. Within this framework of stability we consider some refined aspects of dynamic response linked to the existence of "overshoots" in the free and stable motion as shown by the following example.

Example 2.1: Consider the matrices:

$$A_1 = \begin{pmatrix} -1 & 6 \\ 0 & -3 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$$

Both A_1 and A_2 are clearly asymptotically stable. Figure 1(a) and 1(b) shows the trajectories of the corresponding systems for various initial conditions. Note the existence of overshooting trajectories for the first system (corresponding to A_1) and the absence of overshooting trajectories for the second



(a) (b)

Figure 1: Example 2.1

system (corresponding to A_2). Note that overshooting trajectories are denoted by deviations from the unit disc. $\square$

This simple example demonstrates that we may have state overshoots in the free response of a system. For the case of systems having physical state variables such overshoots may lead to values of the state which exceed permissible limits. It is not difficult to infer from the above example that coordinate transformations that diagonalise a stable matrix A having real eigenvalues lead to a form where there are no overshoots for any initial condition. This suggests that the study of overshoots is significant for physical system descriptions, where it makes sense to have constraints on the permissible values of the state. Finding out the reasons behind overshoots in internal stable behaviours may help to illuminate the role of other structural features, such as the role of eigenstructure, in shaping the fine features of internal system behaviours. Designing systems to avoid overshoots is clearly a sufficient (but conservative) approach to constrained internal control. Another motivation for studying systems with no state overshoots comes from applications, where stability properties may only be inferred by finite time observation of the state trajectory. For such cases it may be difficult to distinguish between a stable overshooting trajectory and an unstable behaviour and hence no-overshoot conditions are sufficient for predicting stability on the basis of finite time observation.

This research is driven by the following question: Assuming the system is stable (asymptotically, or in the sense of Lyapunov), is it possible to have overshoots in the state-free response even for a single initial condition? If yes, then characterize the type of state matrices A for which such property holds true and relate this non-overshooting property to other system properties. This study requires a proper definition of state-space overshoots as shown below:

Definition 2.1: The system $\mathcal{S}(A)$ exhibits state-space overshoots, if for at least one initial condition in the sphere $Sp(0, r)$ (centred at the origin and with radius r), the resulting trajectory $x(t)$ satisfies

$$\|x(t)\| > r$$

for some interval $[t_0, t_1]$ where $\|\cdot\|$ denotes the Euclidean norm. $\square$

The property of avoiding overshoots for all possible initial conditions introduces stronger notions of stability, the *strong asymptotic* and *strong Lyapunov stability* properties defined below. We start by quoting the classical notions of stability (e.g. see [15]):

Definition 2.2: For a linear system $\mathcal{S}(A)$ we define:

1. $\mathcal{S}(A)$ is *Lyapunov stable* iff for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|x(t_0)\| < \delta(\epsilon)$ implies that $\|x(t)\| < \epsilon$ for all $t \geq t_0$.
2. $\mathcal{S}(A)$ is *asymptotically stable* iff it is Lyapunov stable and $\delta(\epsilon)$ in part (1) of the definition can be selected so that $\|x(t)\| \to 0$ as $t \to \infty$. $\square$

Remark 2.1: For linear time-invariant systems $\mathcal{S}(A)$, a necessary and sufficient condition for asymptotic stability is that the spectrum of A is contained in the open left-half plane (all eigenvalues have negative real parts); a necessary and sufficient condition for Lyapunov stability is that the spectrum of A lies in the closed left-half plane ($\mathrm{Re}(s) \leq 0$) and, in addition, any eigenvalue on the imaginary axis has simple structure (i.e. equal algebraic and geometric multiplicity) [15]. Note that asymptotic stability is here taken to mean that the origin is the unique equilibrium point and that it is asymptotically stable (in the sense of Definition 2.2 part 2).

Three definitions of strong stability related to the absence of overshoots are stated below:

Definition 2.3: For the LTI system $S(A)$ we define:

1. The system $\mathcal{S}(A)$ is *strongly Lyapunov stable* iff $\|x(t)\| \leq \|x(t_0)\|$, $\forall t > t_0$ and $\forall\, x(t_0) \in \mathcal{R}^n$.
2. The system $\mathcal{S}(A)$ is *strongly asymptotically stable w.s.* (in the wide sense), iff $\|x(t)\| < \|x(t_0)\|$, $\forall t > t_0$ and $\forall\, x(t_0) \neq 0$.
3. The system $\mathcal{S}(A)$ is *strongly asymptotically stable s.s.* (in the strict sense, or simply strongly asymptotically stable) iff $\frac{d\|x(t)\|}{dt} < 0$, $\forall t \geq t_0$ and $\forall\, x(t_0) \neq 0$. $\square$

Remark 2.2: Strong Lyapunov stability implies Lyapunov stability. To see this note that if $\mathcal{S}(A)$ is strongly Lyapunov stable we can choose $\delta(\epsilon) = \epsilon$ in Definition 2.2 part 1. Strong asymptotic stability (in either the wide or strict sense) implies asymptotic stability: Since (from time invariance) $\|x(t)\|$ is a strictly decreasing function when $S(A)$ is strongly asymptotically stable, $\|x(t)\|$ must converge as $t \to \infty$ (since the norm is bounded from below by zero). Thus, if the limit is not zero, the trajectory $x(t)$ must converge to an equilibrium point $x_e(t) \neq 0$ or to a limit cycle (which for LTI systems is always an oscillatory trajectory associated with a pair of conjugate imaginary axis eigenvalues of A). This, however, violates the definition of strong asymptotic stability, since choosing as initial condition $x(t_0) = x_e$ or a point on the limit cycle implies that the norm of the ensuing trajectory is not strictly decreasing (i.e. stays constant). Note also that for LTI systems asymptotic stability (of the origin) is always global. $\square$

Example 2.2: The requirement in Definition 2.3 part 2, that $\|x(t)\| \leq \|x(t_0)\|$ *for every* initial condition $x(t_0) \neq 0$ is crucial. Consider the system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

and an initial state:

$$x(0) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}'$$

The trajectory of the system if given as $x_1(t) = \cos t$, $x_2(t) = \sin t$ and $x_3(t) = e^{-t}$. Thus $\|x(t)\| = \sqrt{1+e^{-2t}}$ is monotonically decreasing and converges to $\lim_{t\to\infty} \|x(t)\| = 1$. However the system is not strongly asymptotically stable (in fact not even asymptotically stable) since releasing it from any initial condition $x(t_0)$ on the (x_1, x_2) plane results in a constant-norm trajectory $\|x(t)\| = \|x(t_0)\|$, $t \geq t_0$. □

Remark 2.3: The three definitions of strong stability introduced above make precise the notion of non-overshooting responses. Thus, strong Lyapunov stability does not allow state trajectories to exit (at any time) the (closed) hyper-sphere with centre the origin and radius the norm of the initial state vector $r_0 = \|x(t_0)\|$ (although motion on the boundary of the sphere $\|x(t)\| = r_0$ is allowed, e.g. an oscillator's trajectory). Strong asymptotic stability (strict sense) requires that all state trajectories enter each hyper-sphere $\|x(t)\| = r \leq r_0$ from a non-tangential direction, whereas for systems which are strongly asymptotically stable (wide-sense), tangential entry is allowed. Clearly strong asymptotic stability in the strict sense implies strong asymptotic stability in the wide sense which in turn implies strong Lyapunov stability. □

Example 2.3: A simple example of strong asymptotic stability (s.s.) is provided by the system $\dot{x}(t) = -x(t)$. Next we present an example of a system which is strongly asymptotically stable (w.s.) but not strongly asymptotically stable (s.s.). Let

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 2\sqrt{2} \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Clearly, the system is asymptotically stable and its trajectory is described by:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} x_1(0) + \begin{pmatrix} 2\sqrt{2}(e^{-t} - e^{-2t}) \\ e^{-2t} \end{pmatrix} x_2(0)$$

Further,

$$V(x(t)) := \|x(t)\|^2 = x'(0)\Phi(t)x_0$$

where,

$$\Phi(t) = \begin{pmatrix} e^{-2t} & 2\sqrt{2}(e^{-2t} - e^{-3t}) \\ 2\sqrt{2}(e^{-2t} - e^{-3t}) & 8e^{-2t} - 16e^{-3t} + 9e^{-4t} \end{pmatrix}$$

Thus,

$$\|x(0)\|^2 - \|x(t)\|^2 = x'(0)\Theta(t)x(0)$$

where $\Theta(t) = I_2 - \Phi'(t)\Phi(t)$. It can be easily seen (after some algebra) that $\Theta_{11}(t) = 1 - e^{-2t} > 0$ for all $t > 0$ and

$$|\Theta(t)| = (e^{-t} - 1)^4(e^{-2t} + 4e^{-t} + 1) > 0$$

for all $t \neq 0$. Thus $\Theta(t) > 0$ for all $t > 0$ and hence $\|x(t)\| \leq \|x(0)\|$ for all $t > 0$, $x(0) \neq 0$, which implies strong asymptotic stability (w.s.). However the system is not strongly asymptotically stable (s.s.): Evaluating

$$\dot{V}(x(t)) = \frac{d\|x(t)\|^2}{dt} = x'(0)\dot{\Phi}(t)x(0)$$

at $t = 0$ shows that

$$\dot{V}(x(0)) = 2x'(0)\begin{pmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}x(0)$$

which is equal to zero if the initial state is selected in the direction $x(0) = (\sqrt{2}\ \ 1)'$. Note that this example was constructed to violate the condition given in Definition 2.3 at $t = t_0 = 0$. Due to time-invariance, the example can easily be modified to violate this condition at an arbitrary $t_1 > t_0 = 0$, e.g. by propagating the dynamics backwards in time up to $t = -t_1$ and then shifting the time axis by t_1. $\square$

The characterization of the properties of LTI systems for which we may have, or can avoid, overshoots is a property dependent entirely on the matrix A and it is the subject considered next. The definitions given for the system are also used for the corresponding matrices. We first note:

Remark 2.4: A system that exhibits no overshoots in the sense of Definition 2.1 is (at least) Lyapunov stable, but not vice-versa. Instability also implies the existence of overshoots is the sense of Definition 2.1. Furthermore, for linear systems the radius of the sphere $Sp(0, r)$ does not affect the overshooting property and we can always assume $r = 1$. $\square$

Necessary and sufficient conditions for strong asymptotic stability (in the three senses of Definition 2.3) are derived in Theorem 2.1 below. Before stating and proving this theorem we note the following standard result:

Lemma 2.1: The quadratic $x'Ax$ is generated by the symmetric part of A, where $\bar{A} = \frac{1}{2}(A + A')$ i.e. $Q(x, A) = x'Ax$. Furthermore the quadratic $Q(x, A) = x'Ax$ is negative definite (semi-definite), if and only if $\bar{A} = \frac{1}{2}(A + A')$ satisfies either of the equivalent conditions: (i) $\bar{A}$ is negative definite (semi-definite), (ii) $\bar{A}$ has all its eigenvalues negative (non-positive). $\square$

Theorem 2.1: For the system $\mathcal{S}(A)$, the following properties hold true:

(i) $\mathcal{S}(A)$ is strongly asymptotically stable (s.s.) if and only if $A + A' < 0$. $\square$

(ii) $\mathcal{S}(A)$ is strongly asymptotically stable (w.s.) if and only if A is asymptotically stable and $A + A' \leq 0$.

(iii) $\mathcal{S}(A)$ is strongly Lyapunov stable, if and only if $A + A' \leq 0$.

Proof:

(i) Direct differentiation of $\|x(t)\|^2 = x'(t)x(t)$ gives

$$\frac{d\|x(t)\|^2}{dt} = \dot{x}'(t)x(t) + x'(t)\dot{x}(t) = x'(t)(A + A')x(t)$$

or

$$\frac{d\|x(t)\|^2}{dt} = x(t_0)'(t)e^{A'(t-t_0)}(A + A')e^{A(t-t_0)}x(t_0)$$

Since $e^{A(t-t_0)}$ is non-singular, $\frac{d\|x(t)\|^2}{dt} < 0$ for all $t \geq t_0$, $x(t_0) \neq 0$ if and only if $A + A' < 0$.

(ii) Suppose $\mathcal{S}(A)$ is strongly asymptotically stable (w.s.). Asymptotic stability of A then follows immediately (see Remark 2.2). Further, $\frac{d\|x(t)\|}{dt} \leq 0$ for all $t \geq t_0$ and all $x(t_0) \in \mathcal{R}^n$. Since

$$\frac{d\|x(t)\|^2}{dt} = x'(t_0)e^{A'(t-t_0)}(A + A')e^{A(t-t_0)}x(t_0)$$

this implies that $A + A' \leq 0$. Conversely, suppose that A is asymptotically stable and $A + A' \leq 0$. Suppose for contradiction that for some $t_1 > t_0$ we have $\|x(t_1)\| \geq \|x(t_0)\|$. Since the condition $A + A' \leq 0$ implies that $\|x(t)\|$ is non-increasing for all $t > t_0$, we conclude that $\|x(t)\| = \|x(t_0)\|$ for all $t \in [t_0,\ t_1]$. An analytic continuation argument may now be used to show that this implies that $\|x(t)\| = \|x(t_0)\|$ *for every* $t \geq t_0$. This, however, contradicts asymptotic stability.

(iii) The equivalence of the two conditions follows immediately on noting that $\|x(t)\|$ is non-increasing along any trajectory and initial condition $x(t_0)$ if and only if $A + A' \leq 0$. $\square$

Remark 2.5: It follows from Theorem 2.1 part (iii) that if $A + A' \leq 0$ then A is Lyapunov stable, i.e. all its eigenvalues are contained in the closed left-half plane and any eigenvalue on the imaginary axis has simple structure (equal algebraic and geometric multiplicity). We next establish this fact via a direct linear-algebraic argument which is independent of Lyapunov stability theory. We first need the following two Lemmas:

Lemma 2.2: Suppose $A = A' < 0$. Then for every $B = -B'$, $\lambda(A + B) \subseteq \mathcal{C}_-$.

Proof: It is first shown that $C = A + B$ cannot have an eigenvalue on the imaginary axis. For suppose that $j\omega$ was such an eigenvalue with $x \neq 0$ the corresponding eigenvector. Then $Cx = j\omega x$ which implies that $x^*C' = -j\omega x^*$. Pre-multiplying the first equation by x^* and post-multiplying the second equation by x gives $x^*Cx = j\omega\|x\|^2$ and $x^*C'x = -j\omega\|x\|^2$, respectively; adding these two equations gives $x^*Ax = 0$ which implies that $x = 0$, a contradiction. To show that C cannot have eigenvalues in the open right half plane either, consider the locus of eigenvalues of $A + \epsilon B$ where $\epsilon \geq 0$. For $\epsilon = 0$ all eigenvalues lie on the negative real line of the complex plane. Now since the eigen-loci of $A + \epsilon B$ vary continuously with ϵ, if $C = A + B$ had an eigenvalue in the open right half plane there would exist an ϵ_0, $0 < \epsilon_0 \leq 1$, such that $A + \epsilon_0 B$ has an eigenvalue on the imaginary axis, which is impossible by the previous argument (with B replaced by $\epsilon_0 B$). $\square$

Lemma 2.3: Suppose $A = A' \leq 0$. Then for every $B = -B'$, $\lambda(A + B) \subseteq \overline{\mathcal{C}}_-$. Further, if x is an eigenvector of $A + B$ corresponding to an imaginary axis eigenvalue, then $x \in \text{Ker}(A)$.

Proof: Suppose that $C = A + B$ has an eigenvalue $\alpha + j\omega$ with $\alpha \geq 0$ and corresponding eigenvector $x \neq 0$. Then $Cx = (\alpha + j\omega)x$ which implies that $x^*C' = (\alpha - j\omega)x^*$. By a similar argument as in the proof of Lemma 2.2 it follows that $x^*Ax = \alpha\|x\|^2$. The left hand side of this equality is non-positive while the right hand side is non-negative. Thus both terms are zero, so that $\alpha = 0$ and $x^*Ax = 0 \Rightarrow Ax = 0$ since $A = A' \leq 0$. $\square$

Corollary 2.1: Suppose that $A = A' \leq 0$. Then for any skew-symmetric matrix B, $\lambda(A + B) \subseteq \overline{\mathcal{C}}_-$ and $A + B$ has simple structure on the imaginary axis, so that $A + B$ is Lyapunov-stable.

Proof: The fact that $\lambda(A + B) \subseteq \overline{\mathcal{C}}_-$ follows from Lemma 2.3. To show that $A + B$ has simple structure on the imaginary axis (and is hence Lyapunov-stable), assume that $(A + B)x = j\omega x$, $\omega \in \mathcal{R}$, $x \neq 0$. Then from Lemma 2.3 we have that $Ax = 0$. Thus $Bx = j\omega x$. Since B is skew-symmetric (and therefore normal), $j\omega$ has equal algebraic and geometric multiplicity. Repeating the argument for every imaginary eigenvalue of $A+B$ axis shows that $A+B$ has a simple structure on the imaginary axis. $\square$

The following Proposition shows that an alternative equivalent condition of strong asymptotic stability w.s. (see Theorem 2.1 part (ii)) is that $A + A' \leq 0$ and the pair $(A, A + A')$ is observable.

Proposition 2.1: Suppose that $A + A' \leq 0$. Then A is asymptotically stable if and only if the pair $(A, A + A')$ is observable.

Proof: (i) *Sufficiency:* Suppose for contradiction that the pair $(A, A + A')$ is unobservable. Then there exists $\lambda \in \mathcal{C}$ and $x \neq 0$ such that:

$$\begin{pmatrix} \lambda I_n - A \\ A + A' \end{pmatrix} x = 0 \tag{2}$$

This implies that $Ax = \lambda x$ and $(A + A')x = 0$. Thus,

$$(A + A')x = 0 \Rightarrow Ax = -A'x \Rightarrow (\lambda I + A')x = 0$$

which is a contradiction since λ and $-\lambda$ cannon be simultaneously eigenvalues of A if A is asymptotically stable. (ii) *Necessity:* Here we suppose that the pair $(A, A + A')$ is observable and need to show that A is asymptotically stable. Since $A + A' \leq 0$, we know from Lemma 2.3 that A has all its eigenvalues in the closed left-half plane; further if $Ax = j\omega x$ for some $\omega \in \mathcal{R}$, $x \neq 0$, then $(A + A')x = 0$. But in such a case equation (2) holds (with $\lambda = j\omega$) contradicting the observability of $(A, A + A')$. Hence A is free from imaginary axis eigenvalues and thus it is asymptotically stable. $\square$

Corollary 2.2: A is strongly asymptotically stable (w.s.) if and only if $A + A' \leq 0$ and the pair $(A, A + A')$ is observable.

Proof: Follows immediately from Theorem 2.1 (ii) and Proposition 2.1. $\square$

Remark 2.6: Note that if $A + A' < 0$ then $(A, A + A')$ is necessarily observable, although the reverse implication does not necessarily hold. Thus, as expected, strong asymptotic stability (s.s.) is a stronger condition than strong asymptotic stability (w.s.). $\square$

Example 2.4: Consider the system $\dot{x} = Ax$ with

$$A = \begin{pmatrix} -1 & 2\sqrt{2} \\ 0 & -2 \end{pmatrix} \;\Rightarrow\; A + A' = \begin{pmatrix} -2 & 2\sqrt{2} \\ 2\sqrt{2} & -4 \end{pmatrix}$$

introduced in Example 2.3 above. Since $A + A' \leq 0$ and singular, A is not strongly asymptotically stable (s.s.). However, since A is asymptotically stable it is also strongly asymptotically stable (w.s.). Equivalently, the pair $(A, A + A')$ is observable, since

$$\begin{pmatrix} -2 & 2\sqrt{2} \\ 2\sqrt{2} & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mu \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \quad \mu \in \mathcal{R}$$

which is not an eigenvector of A. $\square$

In the last part of this section we explore the observability condition of Corollary 2.2 by stating and proving the following result.

Proposition 2.2: Consider the system $\mathcal{S}: \ \dot{x} = Ax, \ x(0) = x_0$ with $A \in \mathcal{R}^{n \times n}$ and $A + A' \leq 0$. Further, define $V(x) = \|x\|^2$ and let $V^{(k)}(x(t))$ denote the k-th derivative of $V(x)$ with respect to the time-variable t, evaluated along the trajectory $x(t) = e^{At}x_0$ of $\mathcal{S}$. Then:

(i) Suppose that $A^{i-1}x \in \mathrm{Ker}(A + A')$ for $i = 1, 2, \ldots, k$. Then,

$$V^{(2k)}(x) = x' \left(\sum_{i=0}^{2k-1} \binom{2k-1}{i} (A')^{2k-1-i}(A + A')A^i \right) x = 0$$

(ii) Suppose that $A^{i-1}x \in \mathrm{Ker}(A + A')$ for $i = 1, 2, \ldots, k$. Then,

$$V^{(2k+1)}(x) = x' \left(\sum_{i=0}^{2k} \binom{2k}{i} (A')^{2k-i}(A + A')A^i \right) x = 0 \quad \text{if } A^k x \in \mathrm{Ker}(A + A')$$

and

$$V^{(2k+1)}(x) = \binom{2k}{k} x'(A')^k(A + A')A^k x < 0 \quad \text{if } A^k x \notin \mathrm{Ker}(A + A')$$

(iii) If there exists an $x \neq 0$ for which $V^{(k)}(x) = 0$ for all $k \in \mathbb{N}$, then $(A, A + A')$ is unobservable and $x \in \mathrm{Ker}(\Gamma_o)$ where Γ_o denotes the observability matrix

$$\Gamma_o = \begin{pmatrix} A + A' \\ (A + A')A \\ \vdots \\ (A + A')A^{n-1} \end{pmatrix}$$

Proof: Differentiation of $V(x) = \|x\|^2 = x'x$ gives:

$$V^{(1)}(x) = \dot{x}'x + x'\dot{x} = x'(A + A')x$$

Thus, $V^{(1)}(x) < 0$ iff $x \notin \mathrm{Ker}(A + A')$ and $V^{(1)}(x) = 0$ otherwise. The second derivative is:

$$V^{(2)}(x) = \dot{x}'(A + A')x + x'(A + A')\dot{x} = x' \left[A'(A + A') + (A + A')A \right] x$$

and hence $V^{(2)}(x) = 0$ if $x \in \mathrm{Ker}(A + A')$. Similarly,

$$V^{(3)}(x) = x'A' \left[A'(A + A') + (A + A')A \right] x + x' \left[A'(A + A') + (A + A')A \right] Ax$$

or

$$V^{(3)}(x) = x' \left[(A')^2(A + A') + 2A'(A + A')A + (A + A')A^2 \right] x = 2x'A'(A + A')Ax$$

if $x \in \mathrm{Ker}(A + A')$. Thus,

$$\begin{aligned} V^{(3)}(x) < 0 \ \text{ if } \ x \in \mathrm{Ker}(A + A') \ \text{ and } \ Ax \notin \mathrm{Ker}(A + A') \\ V^{(3)}(x) = 0 \ \text{ if } \ x \in \mathrm{Ker}(A + A') \ \text{ and } \ Ax \in \mathrm{Ker}(A + A') \end{aligned}$$

A formal inductive argument on k many now be used to establish parts (i) and (ii) (details are omitted). To show (iii) note that there exists an $x \in \mathcal{R}^n$, $x \neq 0$, such that $V^{(k)}(x) = 0$ for all $k \in \mathbb{N}$, if and only if $A^k x \in \mathrm{Ker}(A + A')$ for all $k \in \mathbb{N}$. This implies in particular that $\Gamma_o x = 0$ and hence the pair $(A, A + A')$ is unobservable. $\square$

Remark 2.7: (i) The above proposition shows that if $A+A' \leq 0$, the norm of $x(t)$ along any trajectory is non-increasing. According to Definition 2.3 (i) this says that the system is strongly Lyapunov stable and according to Theorem 2.1 (i). (ii) If $A + A' \leq 0$ and the pair $(A, A + A')$ is observable, then (according to Proposition 2.1) A is asymptotically stable and (from Theorem 2.1) the system is strongly asymptotically stable (w.s.), i.e. the norm of x is strictly decreasing along any trajectory (unless the initial condition is the origin). Now, for any given $x(t)$ the function $V(x) = \|x(t)\|^2$ is locally strictly decreasing in some interval $(t - \epsilon,\ t + \epsilon)$ if there exists an integer $2k + 1$ (arbitrarily large) such that $V^{(2k+1)}(x) < 0$, while $V^{(i)}(x) = 0$ for $i = 1, 2, \ldots, 2k$. According to the proposition, this condition occurs if:

$$x \notin A^{-k}\mathrm{Ker}(A + A') \ \text{ and } \ x \in \bigcap_{i=0}^{k-1} A^{-i}\mathrm{Ker}(A + A')$$

Thus the function is locally strictly decreasing unless $A^k x \in \mathrm{Ker}(A + A')$, $k \geq 0$, which implies the unobesvability of the pair $(A, A + A')$. Thus strong asymptotic stability is lost when $(A, A + A')$ is unobservable. □

In the remaining part of the paper we refer to "strong asymptotic stability in the strict sense (s.s.)" as simply "strong stability".

3. Strong Stability: Basic Results

A question that naturally arises is the characterization of the matrix A which guarantees the statements of Theorem 2.1, in particular the properties of A which guarantee or violate the negative-definiteness of the symmetric part of A, $\bar{A}$. We first state the following result.

Proposition 3.1: For the matrix A the following properties hold true:

(i) If A is unstable then $\bar{A}$ is either sign indefinite or positive definite.

(ii) Necessary conditions for $\bar{A}$ to be negative definite, is that A is stable.

Proof: If A is unstable, then there exist initial conditions for which $x(t) = e^{At}x(0)$ leaves the sphere $Sp(0, r)$, i.e. the cosine of the angle of $\langle \dot{x}, \nabla V(x) \rangle$ is positive for some $x(0)$ on the sphere and thus the quadratic $x'Ax$ is positive in some regions at least, which proves the result. Part (ii) follows immediately from Lemma 2.2). □

Next, we consider the family of (asymptotically) stable matrices A and investigate the special conditions which guarantee negative definiteness of $\bar{A}$, or lead to violation of this property. The following example demonstrates the simple fact that not every stable matrix $A \in \mathcal{R}^{n\times n}$ has a symmetric part that is negative definite.

Example 3.1: Consider the stable matrix:

$$A = \begin{pmatrix} -1 & 4 \\ 0 & -3 \end{pmatrix} \ \text{ so that } \ \bar{A} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = -3$ and its symmetric part is sign-indefinite. Similarly for the matrices of Example 2.1 we have:

- In the first case:
$$A = \begin{pmatrix} -1 & 6 \\ 0 & -3 \end{pmatrix} \text{ with } \bar{A} = \begin{pmatrix} -1 & 3 \\ 3 & -3 \end{pmatrix}$$
and spectrum $\lambda(\bar{A}) = \{-2-\sqrt{10}, -2+\sqrt{10}\}$; then A is not strongly stable.

- In the second case:
$$A = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}, \text{ with } \bar{A} = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}$$
and spectrum $\lambda(\bar{A}) = \{-2-\sqrt{2}, -2+\sqrt{2}\}$; then A is strongly stable. □

Conditions which guarantee the negative-definiteness of a matrix may be derived by using Sylvester's theorem [9] and this will be illustrated by the following example:

Example 3.2: Consider the 2×2 case first:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \bar{A} = \begin{pmatrix} a_{11} & \frac{1}{2}(a_{12}+a_{21}) \\ \frac{1}{2}(a_{12}+a_{21}) & a_{22} \end{pmatrix} := \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} \tag{3}$$

For the matrix $\bar{A}$ the Sylvester theorem conditions lead to a set of nonlinear inequalities, i.e.

$$\alpha_{11} < 0, \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{vmatrix} > 0 \qquad \text{or} \qquad a_{11} < 0, (a_{12}+a_{21})^2 > 4a_{11}a_{22} \tag{4}$$

□

The above example demonstrates that a natural way to parametrise the family of strongly stable matrices is to use Sylvester conditions on $\bar{A}$, which however become complicated for dimensions higher than two.

An interesting question that arises is whether there exist special forms of stable matrices which cannot satisfy strong stability, or satisfy strong stability under simple conditions. We consider first the case of companion type matrices and then the case of Jordan canonical descriptions, as two representatives.

Example 3.3: Consider a matrix A in companion form, i.e. say

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} \in \mathcal{R}^{3\times 3} \text{ with } a_0, a_1, a_2 > 0, \text{ where } 2\bar{A} = \begin{pmatrix} 0 & 1 & -a_0 \\ 1 & 0 & 1-a_1 \\ -a_0 & 1-a_1 & -2a_2 \end{pmatrix}$$

The Sylvester conditions give the leading minors as $\Delta_1 = 0$, $\Delta_2 = -1$ and $\Delta_3 = 2a_2$. Clearly, since $\Delta_1 = 0$ and $\Delta_2 < 0$, $x'Ax$ is sign indefinite. □

The example demonstrates that for certain types of matrices strong stability is not possible. In fact, we can state the following proposition.

Proposition 3.2: If $A \in \mathcal{R}^{n\times n}$ is in companion form, then it cannot be strongly stable.

Proof: Consider for the sake of simplicity the case where $n = 4$. Then, if

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix}$$

where $a_0, a_1, a_2, a_3 > 0$,

$$2\bar{A} = \begin{pmatrix} 0 & 1 & 0 & -a_0 \\ 1 & 0 & 1 & -a_1 \\ 0 & 1 & 0 & 1-a_2 \\ -a_0 & -a_1 & 1-a_2 & -2a_3 \end{pmatrix}$$

and thus $\Delta_1 = 0$, $\Delta_2 = -1$ and $\Delta_3 = 0$ and no matter what the value of Δ_4 is, the negative definiteness (or semi-definiteness) conditions are violated. In the general $n \times n$ case, it is easy to see that any matrix B generated by selecting the $\{i_1, i_2, \ldots, i_m\}$ rows and columns of a strongly stable matrix A must also be strongly stable (for if $B + B'$ is not negative definite, neither is $A + A'$). Hence, by noting that the first $n-1$ minors of $2\bar{A}$ oscillate between the values $\Delta_{2i-1} = 0$ and $\Delta_{2i} = -1$, it follows that no matrix in companion form can be strongly stable. $\square$

In the sequel, we investigate the strong stability property when A is in Jordan canonical description. Note that, any square matrix A can be transformed via similarity transformations to a matrix in Jordan form:

$$J(A) = \text{block-diag}(J_1(\lambda_1), J_2(\lambda_2), \ldots, J_k(\lambda_k)) \tag{5}$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ denote the non-repeated eigenvalues of A. The size of each block $J_i(\lambda_i)$ is equal to the algebraic multiplicity of λ_i and, in general, has the form:

$$J_i(\lambda_i) = \text{block-diag}(J_{i1}(\lambda_i), J_{i2}(\lambda_i), \ldots, J_{ir_i}(\lambda_i))$$

where $J_{ij}(\lambda_i)$ has the general form:

$$J_{ij}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{pmatrix} \tag{6}$$

and r_i is the geometric multiplicity of λ_i. In the following we examine the necessary and sufficient conditions for a matrix in Jordan form to be strongly stable. To establish the main result we need the following Lemma.

Lemma 3.1: Consider the $n \times n$ tridiagonal matrix of special form:

$$R_n = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix} \tag{7}$$

i.e. $R_n(i, i+1) = R_n(i+1, i) = 1$ for all i, $R_n(i,j) = 0$ otherwise. Let $D_n(\lambda) = |\lambda I_n - R_n|$ denote the characteristic polynomial of R_n. Then:

(i) $D_n(\lambda)$ is generated recursively as: $D_{n+2}(\lambda) = \lambda D_{n+1}(\lambda) - D_n(\lambda)$ with $D_1(\lambda) = \lambda$ and $D_2(\lambda) = \lambda^2 - 1$.

(ii) The eigenvalues of R_n are given as $\lambda_k(R_n) = 2\cos\frac{k\pi}{n+1}$, $k = 1, 2, \ldots, n$.

Proof: (i) Expanding D_n along its first row (or column) gives:

$$D_n(\lambda) = \left| \begin{pmatrix} \lambda & -1 & 0 & \ldots & 0 \\ -1 & \lambda & -1 & \ddots & 0 \\ 0 & -1 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \ldots & 0 & -1 & \lambda \end{pmatrix} \right| = \lambda D_{n-1}(\lambda) + \left| \begin{pmatrix} -1 & -1 & 0 & \ldots & 0 \\ 0 & \lambda & -1 & \ddots & 0 \\ 0 & -1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \ldots & 0 & -1 & \lambda \end{pmatrix} \right|$$

Note that in the above expression the first matrix inside the determinant has dimension n and the second has dimension $n-1$. Thus

$$D_n(\lambda) = \lambda D_{n-1}(\lambda) - D_{n-2}(\lambda)$$

resulting in the λ-dependent difference equation:

$$D_n(\lambda) - \lambda D_{n-1}(\lambda) + D_{n-2}(\lambda) = 0$$

with initial conditions:

$$D_1(\lambda) = \lambda \quad \text{and} \quad D_2(\lambda) = \left| \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} \right| = \lambda^2 - 1$$

(ii) First note that R_n can be expressed as the sum of two matrices J_n and J_n' with J_n being the zero matrix except for entries equal to one above the main diagonal. Thus, denoting by $\rho(\cdot)$ the spectral radius of a matrix,

$$\rho(R_n) = \rho(J_n + J_n') \leq \|J_n\| + \|J_n'\| = 2$$

and hence $-2 \leq \lambda_k(R_n) \leq 2$ for all $k = 1, 2, \ldots, n$. Moreover it can be easily be shown that $-2, 2 \notin \lambda(R_n)$ and hence $-2 < \lambda_k(R_n) < 2$ for all $k = 1, 2, \ldots, n$. We will define a parametric expansion of $D_n(\lambda)$ for all real λ in the interval $-2 < \lambda < 2$. Calculating the roots of this expression will then produce *all* eigenvalues of R_n, since R_n is symmetric and all its eigenvalues lie in this interval. Consider the parametric quadratic equation:

$$z^2(\lambda) - \lambda z(\lambda) + 1 = 0$$

with $\lambda \in (-2,\ 2)$. This has the solutions:

$$z_1(\lambda) = \frac{\lambda}{2} + j\sqrt{1 - \left(\frac{\lambda}{2}\right)^2} \quad \text{and} \quad z_2(\lambda) = \frac{\lambda}{2} - j\sqrt{1 - \left(\frac{\lambda}{2}\right)^2}$$

Since $z_1(\lambda) \neq z_2(\lambda)$ for all $\lambda \in (-2, 2)$, the general solution to the parametric difference equation (7) is of the form:

$$D_n(\lambda) = A(\lambda) z_1^n(\lambda) + B(\lambda) z_2^n(\lambda)$$

or equivalently as:

$$D_n(\lambda) = A(\lambda) e^{jn\cos^{-1}(\lambda/2)} + B(\lambda) e^{-jn\cos^{-1}(\lambda/2)}$$

For a real valued solution we must have $A(\lambda) = \overline{B}(\lambda)$ and hence $D_n(\lambda)$ can be written in real form as:

$$D_n(\lambda) = \hat{A}(\lambda)\cos\left(n\cos^{-1}(\lambda/2)\right) + \hat{B}(\lambda)\sin\left(n\cos^{-1}(\lambda/2)\right)$$

where $\hat{A}(\lambda)$ and $\hat{B}(\lambda)$ are now real valued functions. To determine $\hat{A}(\lambda)$ and $\hat{B}(\lambda)$, we use the following two initial conditions:

$$D_1(\lambda) = \frac{\lambda \hat{A}(\lambda)}{2} + \hat{B}(\lambda)\sqrt{1 - \frac{\lambda^2}{4}} = \lambda$$
$$D_2(\lambda) = \hat{A}(\lambda)\left(\frac{\lambda^2}{2} - 1\right) + \lambda \hat{B}(\lambda)\sqrt{1 - \frac{\lambda^2}{4}} = \lambda^2 - 1$$

Solving simultaneously gives:

$$\hat{A}(\lambda) = 1 \quad \text{and} \quad \hat{B}(\lambda) = \frac{\lambda/2}{\sqrt{1 - (\lambda/2)^2}}$$

Setting $\psi = \cos^{-1}(\lambda/2)$ then gives:

$$\begin{aligned} D_n(\lambda) &= \cos[n\cos^{-1}(\lambda/2)] + \frac{\lambda/2}{\sqrt{1-(\lambda/2)^2}} \sin[n\cos^{-1}(\lambda/2)] \\ &= \frac{1}{\sin\psi}(\sin\psi\cos n\psi + \cos\psi\sin n\psi) = \frac{\sin[(n+1)\psi]}{\sin\psi} \\ &= \frac{\sin[(n+1)\cos^{-1}(\lambda/2)]}{\sin[\cos^{-1}(\lambda/2)]} \end{aligned}$$

Note that $\sin[\cos^{-1}(\lambda/2)] \neq 0$ for $\lambda \in (-2,\ 2)$. Thus the eigenvalues of R_n are given by the roots of $D_n(\lambda) = 0$, i.e. $(n+1)\cos^{-1}(\lambda/2) = k\pi$, $k \neq 0$, so that,

$$\lambda_k(R_n) = 2\cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, 2, \ldots, n$$

as required. □

Any square matrix A can be transformed via similarity transformations to its Jordan form:

$$J(A) = \text{block-diag}(J_1(\lambda_1), J_2(\lambda_2), \ldots, J_k(\lambda_k))$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ denote the non-repeated eigenvalues of A. The size of each block $J_i(\lambda_i)$ is equal to the algebraic multiplicity of λ_i and, in general, has the form:

$$J_i(\lambda_i) = \text{block-diag}(J_{i1}(\lambda_i), J_{i2}(\lambda_i), \ldots, J_{ir_i}(\lambda_i))$$

where $J_{ij}(\lambda_i)$ has the general form:

$$J_{ij}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{pmatrix}$$

and r_i is the geometric multiplicity of λ_i. The following proposition gives necessary and sufficient conditions for a matrix in Jordan form to be strongly stable:

Proposition 3.3: Let $J(A)$ be the Jordan form of A. Then in the notation above $J(A)$ is strongly stable if and only if:

$$\max_{i=1,2,\ldots,k}\left(\text{Re}(\lambda_i) + \cos\left(\frac{\pi}{m_i+1}\right)\right) < 0$$

where m_i denotes the size of the largest block J_{ij} $(j = 1, 2, \ldots, r_i)$ in J_i.

Proof: $J(A)$ is strongly stable if and only if $J(A) + J'(A) < 0$. This condition is equivalent to $J_i(\lambda_i) + J_i'(\lambda_i) < 0$ for all $i \in \{1, 2, \ldots, k\}$. For a specific i, $J_i(\lambda_i) + J_i'(\lambda_i) < 0$ if and only if $J_{ij}(\lambda_i) + J_{ij}'(\lambda_i) < 0$ for all $j = 1, 2, \ldots, r_i$. Let the size of $J_{ij}(\lambda_i)$ be m_{ij}. Then, using the notation of Lemma 3.1,

$$J_{ij}(\lambda_i) + J_{ij}'(\lambda_i) = 2\text{Re}(\lambda_i)I_{m_{ij}} + R_{m_{ij}}$$

and thus its eigenvalues are: $2\text{Re}(\lambda_i) + 2\cos\left(\frac{m\pi}{m_{ij}+1}\right)$, $m = 1, 2, \ldots m_{ij}$. Thus $J_{ij}(\lambda_i) + J_{ij}'(\lambda_i) < 0$ if and only if $\text{Re}(\lambda_i) + \cos\left(\frac{\pi}{m_{ij}+1}\right) < 0$. Further, since for any two positive integers m and n with $m > n$ we have that:

$$\cos\left(\frac{\pi}{m+1}\right) > \cos\left(\frac{\pi}{n+1}\right)$$

we conclude that $J_i(\lambda_i) + J_i'(\lambda_i) < 0$ if and only if $\text{Re}(\lambda_i) + \cos\left(\frac{\pi}{k_i+1}\right) < 0$, where $k_i = \max_{j\in\{1,2,\ldots,r_i\}} m_{ij}$. Repeating the argument for all $i = 1, 2, \ldots, k$ and requiring that $J_i(\lambda_i) + J_i'(\lambda_i) < 0$ establishes the condition stated in the Proposition. □

Remark 3.1: If $m_i = 1$ for all $i = 1, 2, \ldots, k$ and hence $J(A)$ is diagonal, we recover the condition that $J(A)$ is strongly stable if and only if it is asymptotically stable.

The above result on Jordan forms will be used later on to investigate the role of eigenframes on strong stability.

4. Invariance and Properties of Strong stability under Orthogonal Transformations and the Schur Form

The problem of overshoots which is considered here makes sense only when the state variables which are considered are natural and thus it makes sense to impose constraints on their behaviour. Thus, carrying out arbitrary coordinate transformations and then studying strong stability is a problem that does not make sense. It is thus clear, that strong stability is a property of the original coordinate frame and thus the specific description of matrix A. Here we investigate the existence of special coordinate transformations, such that the strong stability property is invariant. If such transformations exist, we aim to define a canonical form that may simplify the parametrisation of matrices with the strong stability property.

Consider $A \in \mathcal{R}^{n\times n}$ and the quadratic form $x'Ax$. If $Q \in \mathcal{R}^{n\times n}$ and Q is orthogonal, i.e. $Q'Q = I_n$, we can define the coordinate transformation $x = Q\hat{x}$. We note first the following property:

Lemma 4.1: Let $A \in \mathcal{R}^{n\times n}$ and $Q \in \mathcal{R}^{n\times n}$, $Q'Q = I_n$ be a coordinate transformation such that $A \to \hat{A} = Q'AQ$. If $\bar{A}$, $\bar{\hat{A}}$ are the symmetric parts of A, $\hat{A}$, respectively, then $\bar{\hat{A}} = Q'\bar{A}Q$ and $\bar{A} = Q\bar{\hat{A}}Q'$.

Proof: Under the coordinate transformation the quadratic $V(x)$ becomes:

$$V(\hat{x}) = (Q\hat{x})'AQ\hat{x} = \hat{x}'Q'AQ\hat{x} = \hat{x}'\hat{A}\hat{x}$$

where $\hat{A} = Q'AQ$. Since $\bar{A} = \frac{1}{2}(A + A')$, then

$$Q'\bar{A}Q = \frac{1}{2}(Q'AQ + Q'A'Q) = \frac{1}{2}(\hat{A} + \hat{A}')$$

and thus

$$Q'\bar{A}Q = \frac{1}{2}(\hat{A} + \hat{A}') = \bar{\hat{A}}$$

where $Q'Q = QQ' = I_n$. □

This Lemma together with the properties of congruence provide the means in establishing one of the central results here, that is the invariance of strong stability under orthogonal transformations. We first define some basic results on congruence: If $A, B \in \mathcal{R}^{n\times n}$ and $P \in \mathcal{R}^{n\times n}$, $|P| \neq 0$ such that $B = P'AP$ then A, B are called congruent over $\mathcal{R}$. In general P is arbitrary and not necessarily orthogonal. Note that if A is symmetric, i.e. $A = A'$, then

$$B' = (P'AP)' = P'A'P = P'AP = B$$

and thus symmetry is preserved under congruence.

Definition 4.1: Let $A \in \mathcal{C}^{n\times n}$ and Hermitian, then we define as the *inertia* of the matrix A the tripple of non-negative integers

$$\mathcal{I}_n(A) = \{i_+(A), i_-(A), i_0(A)\}$$

where $i_+(A)$, $i_-(A)$, $i_0(A)$ are respectively the number of eigenvalues of A, counted with their algebraic multiplicities, which have real part positive, negative and zero correspondingly. Notice that $\text{rank}(A) = i_+(A) + i_-(A)$, while $s_n(A) = i_+(A) - i_-(A)$ is defined as the *signature* of A. □

Inertia is an important concept because the behaviour of systems of linear differential equations (instability, periodicity, etc) depends on the distribution in the complex plane of the eigenvalues of the coefficient matrix. In the subsequent analysis we will use the following standard result:

Lemma 4.2: [9] Let $A, B \in \mathcal{C}^{n\times n}$ be Hermitian matrices. There is a nonsingular matrix $Q \in \mathcal{C}^{n\times n}$ such that $A = Q^*BQ$ if and only if A and B have the same inertia. □

The above result establishes the important property that for Hermitian matrices the inertia defines a complete invariance under congruence. Obviously this result also applies to real symmetric matrices under real congruence and can be expressed as:

Lemma 4.3: [9], [4] Let $A \in \mathcal{R}^{n\times n}$ be symmetric and let $\mathcal{I}_n(A)$ be the inertia of A. Then, $\mathcal{I}_n(A)$ is a complete invariant under congruence. □

With these preliminary results we may return to the problem of strong stability and will examine this property under orthogonal equivalence. First, if $A \in \mathcal{R}^{n\times n}$, then we define the orthogonal equivalence on A by

$$\hat{A} = Q'AQ, \quad Q \in \mathcal{R}^{n\times n}, Q'Q = I_n$$

and the set of all such matrices $\hat{A}$ will be denotes by $\mathcal{E}_{or}(A)$ and referred to as the *orthogonal orbit* of A. We examine next the property of strong stability under $\mathcal{E}_{or}$ equivalence. We fisrt note:

Remark 4.1: The matrix A is strongly stable if the inertia $\mathcal{I}_n(\bar{A}) = \{0, i_-(\bar{A}), 0\}$, i.e. if all eigenvalues of $\bar{A}$ are negative and thus $\bar{A}$ is negative definite. □

The set $\mathcal{I}_n(\bar{A}) = \{i_+(\bar{A}), i_-(\bar{A}), i_0(\bar{A})\}$ may be also referred as the *inertia characteristic* of A and denoted by $\mathcal{I}_c(A)$ (i.e. $\mathcal{I}_c(A) = \mathcal{I}_n(\bar{A})$). We examine next the property of strong stability under $\mathcal{E}_{or}$ equivalence.

Theorem 4.1: Let $A \in \mathcal{R}^{n\times n}$ and $\mathcal{E}_{or}(A)$ denote the orthogonal equivalence orbit of A. Then:

(i) The inertia characteristics of A, $\mathcal{I}_c(A)$ is an invariant of $\mathcal{E}_{or}(A)$.

(ii) The strong stability of A, $\mathcal{I}_c(A)$ is an invariant of $\mathcal{E}_{or}(A)$.

Proof: By Lemma 4.1 it follows that if $A\mathcal{E}_{or}\hat{A}$, i.e. $\hat{A} = Q'AQ$, $Q \in \mathcal{R}^{n\times n}$, $Q'Q = I_n$, then also for the symmetric parts we have

$$\bar{\hat{A}} = Q'\bar{A}Q \text{ and } \bar{A} = Q\bar{\hat{A}}Q'$$

By Lemma 4.3 it follows that $\bar{A}\mathcal{E}_{or}\bar{\hat{A}}$ (they are also congruent) and thus $\mathcal{I}_n(\bar{A}) = \mathcal{I}_n(\bar{\hat{A}})$; however by definition $\mathcal{I}_c(A) = \mathcal{I}_n(\bar{A})$ and $\mathcal{I}_c(\hat{A}) = \mathcal{I}_n(\bar{\hat{A}})$ and the latter implies $\mathcal{I}_c(A) = I_c(\hat{A})$ which proves part (i). (ii) Clearly if A is strongly stable, then $\mathcal{I}_c(A) = \{0, n, 0\}$. By part (i) $\mathcal{I}_c(\hat{A}) = \mathcal{I}_c(A) = \{0, n, 0\}$ and this establishes the invariance of strong stability. □

The above result suggests that the property of strong stability, or the lack of strong stability, may be studied using elements of $\mathcal{E}_{or}(A)$ orbit. We first note the following standard result:

Lemma 4.4: [9] Let $A \in \mathcal{R}^{n\times n}$. There always exists a real orthogonal matrix $Q \in \mathcal{R}^{n\times n}$, $Q'Q = I_n$, such that

$$T = Q'AQ = \begin{pmatrix} A_1 & x & x & x \\ 0 & A_2 & x & x \\ 0 & 0 & \ddots & x \\ 0 & 0 & 0 & A_k \end{pmatrix} \in \mathcal{R}^{n\times n}, 1 \leq k \leq n$$

where each A_i is a real 1×1 matrix or a real 2×2 matrix with a non-real pair of complex conjugate eigenvalues. □

The above result is the version of Schur's theorem for real matrices. If the complex eigenvalues of A are $\sigma_i \pm j\omega_i$, then the corresponding 2×2 block in the above decomposition is of the the type

$$A(\sigma_i, \omega_i) = \begin{pmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{pmatrix}$$

The above decomposition is not necessarily unique but in the special case of distinct real eigenvalues we have [13]:

Lemma 4.5: If $A \in \mathcal{C}^{n\times n}$ has distinct eigenvalues and their order along their main diagonal is prescribed, i.e.

$$T = U^*AU = \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

where U is the unitary matrix that reduces A to Schur form, then U is determined uniquely to within post-multiplication by a diagonal unitary matrix and the triangular matrix T is determined uniquely to within unitary similarity by a diagonal unitary matrix. □

Remark 4.2: If $A \in \mathcal{R}^{n\times n}$ has distinct real eigenvalues there exists a unique orthogonal matrix $Q \in \mathcal{R}^{n\times n}$, $Q'Q = I_n$ such that the matrix A is reduced to a unique upper-triangular form T, i.e.

$$T = Q'AQ = \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

which is referred as the real Schur form. $\square$

The advantage of triangulation is that it permits the study of strong stability on the set of fewer parameters of T, as well as it allows the establishment of links between the eigenframe properties of T and the property of strong stability. This is demonstrated by an example.

Example 4.1: Consider $A \in \mathcal{R}^{3\times 3}$ as an upper triangular matrix, i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

The eigenvalues appear on the diagonal while the eigenvector frame may be explicitly defined by the parameters of the matrix

$$U = \begin{pmatrix} 1 & -\frac{a_{12}}{a_{11}-a_{22}} & \frac{a_{12}a_{23}-a_{13}a_{22}+a_{13}a_{33}}{(a_{11}-a_{33})(a_{22}-a_{33})} \\ 0 & 1 & -\frac{a_{23}}{a_{22}-a_{33}} \\ 0 & 0 & 1 \end{pmatrix}$$

after some algebra. The symmetric part of A is defined by:

$$2\bar{A} = \begin{pmatrix} 2a_{11} & a_{12} & a_{13} \\ a_{12} & 2a_{22} & a_{23} \\ a_{13} & a_{23} & 2a_{33} \end{pmatrix}$$

and the conditions for negative definiteness are:

$$\Delta_1 = a_{11} < 0, \;\; \Delta_2 = 4a_{11}a_{22} - a_{12}^2 > 0$$

and

$$\Delta_3 = 4a_{11}a_{22}a_{33} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 + a_{12}a_{23}a_{13} < 0$$

$\square$

The above suggests that the Schur form provides the means to connect strong stability conditions with the properties of the eigenframe in an explicit way. Thus, derivation of a general formula for the eigenframe of a Schur matrix in parametric form as indicated above is crucial for linking the properties of the eigenframe of the Schur triangular matrix and for establishing conditions that lead to lack of strong stability. The development of the structure of eigenframe of upper triangular matrices requires some appropriate definitions:

Definition 4.2: Consider the $n \times n$ Schur canonical matrix described as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & a_{22} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a_{n,n} \end{pmatrix}$$

(a) Let $i \in \underline{n}$ and consider the set of integers $\mathcal{I}_i^{\mu} = \{i, i+1, \ldots, \mu; \mu \leq n\}$. Any ordered pair of indices from $\mathcal{I}_i^{\mu}(j_1, j_2) : j_1 < j_2$ will be called an *arc* and any set of arcs with ρ elements such that

$$\omega_{i\mu}^{\rho} := \{(i, j_1), (j_1, j_2), \ldots, (j_\rho, \mu) : i < j_1 < \ldots < j_\rho < \mu\}$$

will be called a *ρ-length path* and will be denoted in short as $\omega_{i,\mu}^{\rho} := [i, j_1, j_2, \ldots, j_\rho, \mu]$. The set of all ρ-length paths of $\mathcal{I}_i^{\mu}$ will be denoted by $\langle \omega_{i,\mu}^{\rho} \rangle$ and the set of all paths of all possible length will be denoted by $\Omega_{i,\mu} = \langle \omega_{i,\mu}^{1} \rangle \cup \ldots \cup \langle \omega_{i,\mu}^{\mu-i} \rangle$ or simply as:

$$\Omega_{i,\mu} := \left\{ \langle \omega_{i,\mu}^{1} \rangle ; \langle \omega_{i,\mu}^{2} \rangle ; \ldots ; \langle \omega_{i,\mu}^{\mu-i} \rangle \right\}$$

(b) For every arc (j_1, j_2) of $\mathcal{I}_i^{\mu}$ we define as its *trace in A* the number $t_{j_1,j_2} = a_{j_1,j_2}/(a_{j_1,j_1} - a_{\mu,\mu})$ and for the $\omega_{i,\mu}^{\rho}$ ρ-length path we define its *trace in A* by

$$t[\omega_{i,\mu}^{\rho}] = (-1)^{\rho} t_{i,j_1} t_{j_1,j_2} \ldots t_{j_\rho,\mu}$$

If $\sum t(\omega_{i,\mu}^{\rho})$ denotes the sum corresponding to all terms in the set $\langle \omega_{i,\mu}^{\rho} \rangle$, then we define as the $\mathcal{I}_i^{\mu}$-value of A:

$$u_{i,\mu} = \sum t[\omega_{i,\mu}^{1}] + \sum t[\omega_{i,\mu}^{2}] + \ldots + \sum t[\omega_{i,\mu}^{\mu-i}]$$

□

Example 4.2: Consider the case of a Schur matrix of 5×5 dimension. In this case the fundamental set of indices is $\{1, 2, 3, 4\}$. Consider the largest set $\mathcal{I}_1^{5} = \{1, 2, 3, 4, 5\}$. We shall describe all length paths for this set and the corresponding traces in A.

(a) Different length paths:

Paths of length (1): $\langle \omega_{1,5}^{1} \rangle = \{[(1, 5)]\}$.

Paths of length (2): $\langle \omega_{1,5}^{2} \rangle = \{[(1, 2), (2, 5)]; [(1, 3); (3, 5)]; [(1, 4); (4, 5)]\} = \{[1, 2, 5], [1, 3, 5], [1, 4, 5]\}$.

Paths of length (3): $\langle \omega_{1,5}^{3} \rangle = \{[(1, 2), (2, 3), (3, 5)]; [(1, 2), (2, 4), (4, 5)]; [(1, 3), (3, 4), (4, 5)]\}$
$\{[1, 2, 3, 5]; [1, 2, 4, 5]; [1, 3, 4, 5]\}$.

Paths of length (3): $\langle \omega_{1,5}^{4} \rangle = \{[(1, 2), (2, 3), (3, 4), (4, 5)]\} = \{[1, 2, 3, 4, 5]\}$.

Using the above definitions we now have: (b) Compute traces and $\mathcal{I}_1^{5}$ values:

$[(1, 5)]: \quad t(1, 5) = (-1)^1 \frac{a_{15}}{a_{11} - a_{55}}$.

$[(1, 2), (2, 5)]: \quad t(1, 2, 5) = (-1)^2 \frac{a_{12} a_{25}}{(a_{11} - a_{55})(a_{22} - a_{55})}$.

$[(1, 3), (3, 5)]: \quad t(1, 3, 5) = (-1)^2 \frac{a_{13} a_{35}}{(a_{11} - a_{55})(a_{33} - a_{55})}$.

$[(1, 4), (4, 5)]: \quad t(1, 4, 5) = (-1)^2 \frac{a_{14} a_{45}}{(a_{11} - a_{55})(a_{44} - a_{55})}$.

$[(1, 2), (2, 3), (3, 5)]: \quad t(1, 2, 3, 5) = (-1)^3 \frac{a_{12} a_{23} a_{35}}{(a_{11} - a_{55})(a_{22} - a_{55})(a_{33} - a_{55})}$.

$[(1, 2), (2, 4), (4, 5)]: \quad t(1, 2, 4, 5) = (-1)^3 \frac{a_{12} a_{24} a_{45}}{(a_{11} - a_{55})(a_{22} - a_{55})(a_{44} - a_{55})}$.

$[(1, 2), (3, 4), (4, 5)]: \quad t(1, 3, 4, 5) = (-1)^3 \frac{a_{13} a_{34} a_{45}}{(a_{11} - a_{55})(a_{33} - a_{55})(a_{44} - a_{55})}$.

$[(1, 2), (2, 3), (3, 4), (4, 5)]: \quad t(1, 2, 3, 4, 5) = (-1)^4 \frac{a_{12} a_{23} a_{34} a_{45}}{(a_{11} - a_{55})(a_{11} - a_{55})(a_{33} - a_{55})(a_{44} - a_{55})}$.

and thus the value of $\mathcal{I}_1^5$ is defined as:

$$u_{1,5} = t(1,5) + t(1,2,5) + t(1,3,5) + t(1,4,5) + t(1,2,3,5) + t(1,2,4,5) + t(1,3,4,5) + t(1,2,3,4,5)$$

$\square$

An issue that emerges is the generation of the set $\Omega_{i,\mu}$ for any set $\mathcal{I}_i^\mu$. This is done following the procedure described below:

Algorithm for generating $\Omega_{i,\mu}$ ***set:*** Given the set $\mathcal{I}_i^\mu = \{i, i+1, \ldots, \mu-1, \mu\}$ we define by $\hat{\mathcal{I}}_i^\mu$ the subset $\hat{\mathcal{I}}_i^\mu = \{i+1, \ldots, \mu-1\}$ with $\mu - i - 1$ elements. The set $\Omega_{i,\mu}$ is defined as:

(a) $\langle \omega_{i,\mu}^1 \rangle := \{[i, \mu]\}$.

(b) From the set $\hat{\mathcal{I}}_i^\mu = \{i+1, \ldots, \mu-1\}$ with $\mu - i - 1$ elements define the set of all lexicographically ordered sequences of σ elements, $\sigma = 1, 2, \ldots, \mu - i - 1$ and denote this set as

$$\hat{\mathcal{I}}_i^\mu(\sigma) := \{\rho(i, \mu; \sigma)\}$$

The set $\langle \omega_{i,\mu}^{\sigma+1} \rangle$ is then defined as:

$$\langle \omega_{i,\mu}^{\sigma+1} \rangle := \{[i, \rho(i, \mu; \sigma), \mu], \ \forall \rho(i, \mu; \sigma) \in \hat{\mathcal{I}}_i^\mu(\sigma)\}$$

$\square$

Example 4.3: Consider the set $\mathcal{I}_3^8 = \{3, 4, 5, 6, 7, 8\}$. Then $\hat{\mathcal{I}}_3^8 = \{4, 5, 6, 7\}$ and:

$$\begin{aligned}
\hat{\mathcal{I}}_3^8(1) &= \{(4), (5), (5), (7)\} \\
\hat{\mathcal{I}}_3^8(2) &= \{(4,5), (4,6), (4,7), (5,6), (5,7), (6,7)\} \\
\hat{\mathcal{I}}_3^8(3) &= \{(4,5,6), (4,5,7), (4,6,7), (5,6,7)\} \\
\hat{\mathcal{I}}_3^8(4) &= \{(4,5,6,7)\}
\end{aligned}$$

and thus the corresponding sets $\langle \omega_{3,8}^\sigma \rangle$, $\sigma = 1, 2, 3, 4, 5$ are:

$$\begin{aligned}
\langle \omega_{3,8}^1 \rangle &= \{[3,8]\} \\
\langle \omega_{3,8}^2 \rangle &= \{[3,4,8], [3,5,8], [3,6,8], [3,7,8]\} \\
\langle \omega_{3,8}^3 \rangle &= \{[3,4,5,8], [3,4,6,8], [3,4,7,8], [3,5,6,8], [3,5,7,8], [3,6,7,8]\} \\
\langle \omega_{3,8}^4 \rangle &= \{[3,4,5,6,8], [3,4,5,7,8], [3,4,6,7,8], [3,5,6,7,8]\} \\
\langle \omega_{3,8}^5 \rangle &= \{[3,4,5,6,7,8]\}
\end{aligned}$$

$\square$

The notation and examples introduced above allows us to express the eigenstructure of the Schur matrix and this is established below:

Theorem 4.2: Consider the n-order Schur canonical matrix described above. The eigenvalues of A are $\{a_{11}, a_{22}, \ldots, a_{nn}\}$ and the eigenvector matrix may be expressed as

$$U_A = [u_1, u_2, \ldots, u_n] = \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where the elements u_{ij} are $\mathcal{I}_i^j$-values of A as given in Definition 4.2.

Proof: The proof of the above result follows by developing a simple example. A straightforward inductive argument can then be used to generalise the result. Consider the case of the 4×4 Schur matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$$

The eigenvalues appear on the diagonal and the eigenvector matrix has the form

$$U = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix} = \begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The elements of U can be derived as follows:

$$u_{12} = -\frac{a_{12}}{a_{11} - a_{22}}, \; u_{23} = -\frac{a_{23}}{a_{22} - a_{33}}, \; u_{34} = -\frac{a_{34}}{a_{33} - a_{44}},$$
$$u_{13} = -\frac{a_{13}}{a_{11} - a_{33}} + \frac{a_{12}a_{23}}{(a_{11} - a_{33})(a_{22} - a_{33})}, \; u_{24} = -\frac{a_{24}}{a_{22} - a_{44}} + \frac{a_{23}a_{34}}{(a_{22} - a_{44})(a_{33} - a_{44})},$$
$$u_{14} = -\frac{a_{14}}{a_{11} - a_{44}} + \frac{a_{13}a_{34}}{(a_{11} - a_{44})(a_{33} - a_{44})} + \frac{a_{12}a_{24}}{(a_{11} - a_{44})(a_{22} - a_{44})} - \frac{a_{12}a_{23}a_{34}}{(a_{11} - a_{44})(a_{22} - a_{44})(a_{33} - a_{44})}$$

□

The Schur form is significant because it provides an explicit description of the eigenframe in terms of the Schur invariant parameters. Given that properties of skewness of eigenframe of a given matrix are invariant under orthogonal transformations, this makes this investigation relevant also for the eigenframe of the original physical system description. The study of properties of eigenframes is considered next. In the following we shall denote by

$$\mathcal{GL}_{or}(n, \mathcal{R}) := \{Q \in \mathcal{R} : Q'Q = I_n\}$$

the subgroup of the general linear group made up of orthogonal matrices.

Proposition 4.1: Let $A \in \mathcal{R}^{n \times n}$, $Q \in \mathcal{GL}_{or}(n, \mathcal{R})$ and let $A = UJV$ be the Jordan decomposition of A. If $\tilde{A} = QAQ'$, then the Jordan decomposition of $\tilde{A}$ is $\tilde{A} = \tilde{U}J\tilde{V}$ where $\tilde{U} = QY$, $\tilde{V} = VQ'$.

Proof: If $AU = UJ$ and $Q'Q = I_n$, then $QAU = QUJ$ and $QAQ'QU = QUJ = \tilde{A}QU$. By defining $QU = \tilde{U}$ this clearly establishes the result. □

Remark 4.3: Orthogonal transformations on A imply also orthogonal transformations on the corresponding eigenframe and thus skewness of the eigenframe, as this is measured either by the Grammian or condition number of the eigenframe is invariant under $\mathcal{GL}_{or}(n, \mathcal{R})$.

A key problem we have to consider here is the derivation of the conditions under which asymptotic stability implies strong stability. This problem is considered first for a special type of matrices, the set of normal matrices.

Lemma 4.6: [13] Let $A \in \mathcal{R}^{n\times n}$ be normal $AA' = A'A$. If $r_1, \ldots, r_k$ are the real eigenvalues and $\sigma \pm j\omega_i$ are the complex eigenvalues of A, then there exists a real orthogonal matrix U such that

$$U'AU = \text{block-diag}\left\{ r_1, \ldots r_k, \ldots, \begin{pmatrix} \sigma & -\omega_i \\ \omega_i & \sigma_i \end{pmatrix}, \ldots \right\}$$

□

Using the above result we may state:

Theorem 4.3: If $A \in \mathcal{R}^{n\times n}$ is asymptotically stable and normal, then it is also strongly stable.

Proof: Note that:

$$U'A'U = \text{block-diag}\left\{ r_1, \ldots r_k; \ldots, \begin{pmatrix} \sigma & -\omega_i \\ \omega_i & \sigma_i \end{pmatrix}, \ldots \right\}$$

and thus

$$U'(A + A')U = 2U'\bar{A}U = \text{block-diag}\left\{ 2r_1, \ldots, 2r_k; \ldots, \begin{pmatrix} 2\sigma_i & 0 \\ 0 & 2\sigma_i \end{pmatrix}, \ldots \right\}$$

or

$$U'\bar{A}U = \text{block-diag}\{r_1, \ldots, r_k; \ldots, \sigma_i, \sigma_i, \ldots\}$$

and from asymptotic stability $r_i < 0$, $\sigma_i < 0$ and this establishes the negative-definiteness of $\bar{A}$. □

The above result establishes a sufficient condition for strong stability in terms of the property of normality.

Remark 4.4: If A is symmetric ($A = A'$) or orthogonal ($A' = A^{-1}$) then it is also normal; thus if A is asymptotically stable it is also strongly stable. □

The condition of normality for A implies that asymptotic stability yields also strong stability. Using the Schur canonical form we may also establish conditions under which matrices which are not normal also have the property that asymptotic stability implies strong stability. We first note the following useful result:

Proposition 4.2: Let $A \in \mathcal{R}^{n\times n}$, $Q \in \mathcal{R}^{n\times n}$, $Q'Q = I_n$, be such that A is reduced to real Schur form, i.e.

$$T = Q'AQ = \begin{pmatrix} A_1 & x & x & x \\ 0 & A_2 & x & x \\ 0 & 0 & \ddots & x \\ 0 & 0 & 0 & A_k \end{pmatrix}, 1 \leq k \leq n$$

where A_i is a real 1×1 matrix or a real 2×2 matrix with $\sigma_i \pm j\omega_i$ pair of complex conjugate eigenvalues, where:

$$A_i = \begin{pmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{pmatrix}$$

The strong stability properties of A may now be studied using the matrix $\hat{T}$ which is obtained from T by substituting every block A_i corresponding the $\sigma \pm j\omega_i$ pair the block of the form:

$$\hat{A}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

while retaining all other parameters as in T.

Proof: Strong stability is invariant under orthogonal transformations and thus its study can be based on T instead of A. Consider the symmetric part of T,

$$\bar{T} = \frac{1}{2}(T + T') = \begin{pmatrix} A_1 + A_1' & x & x & x \\ x & A_2 + A_2' & x & x \\ x & x & \ddots & x \\ x & x & x & A_k + A_k' \end{pmatrix}$$

If, say

$$A_k = \begin{pmatrix} \sigma_k & \omega_k \\ -\omega_k & \sigma_k \end{pmatrix}$$

then:

$$A_k + A_k' = \begin{pmatrix} 2\sigma_k & 0 \\ 0 & 2\sigma_k \end{pmatrix}$$

It is clear from the above analysis that the conclusions about strong stability are not affected if the off-diagonal elements the A_k blocks are set to zero. The matrix $\hat{T}$ defined by the above procedure is referred to as the *extended Schur form* of A. □

Example 4.4: Let the real Schur form of a matrix A with complex eigenvalues be

$$T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & \sigma_1 & \omega_1 & a_{35} & a_{36} \\ 0 & 0 & -\omega_1 & \sigma_1 & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & \sigma_2 & \omega_2 \\ 0 & 0 & 0 & 0 & -\omega_2 & \sigma_2 \end{pmatrix}$$

The extended Schur form is obtained from T by zeroing the elements corresponding to the ω_1, ω_2 elements, i.e.

$$\hat{T} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & \sigma_1 & 0 & a_{35} & a_{36} \\ 0 & 0 & 0 & \sigma_1 & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2 \end{pmatrix}$$

It is clear that $\hat{T}$ is obtained from the real Schur form T by setting all elements ω_i appearing in the 2×2 blocks to zero. This procedure leads to a matrix $\hat{T}$ which is upper-triangular. □

Remark 4.5: For a matrix with distinct eigenvalues the extended Schur form is uniquely defined. The strong stability properties of A are independent of the imaginary parts of the eigenvalues and depend only on the real parts and the remaining elements of the real Schur form. □

In the following we shall investigate sufficient for strong stability under the assumption of asymptotic stability for the original matrix. We shall consider the matrix A expressed in extended Schur form and denoted as:

$$\hat{T} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,i} & a_{1,i+1} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2,i} & a_{2,i+1} & \dots & a_{2,n} \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & a_{ii} & a_{i,i+1} & \dots & a_{in} \\ \vdots & & & 0 & a_{i+1,i+1} & \dots & a_{i+1,n} \\ \vdots & & & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & a_{nn} \end{pmatrix}$$

where if $a_{ii} = a_{i+1,i+1} = \sigma_i$ is the real part of a complex conjugate eigenvalue of A, then $a_{i,i+1} = 0$. Clearly if A has real distinct eigenvalues, then $\hat{T} = T$. A sufficient condition for strong stability using the Gershgorin Theorem and the extended Schur matrix is defined by the following result:

Theorem 4.4: Let $A \in \mathcal{R}^{n \times n}$ be asymptotically stable and with an extended Schur as shown in Remark 4.5. Then A is strongly stable if the following conditions are satisfied:

$$2|a_{ii}| - |a_{i1}| - \dots - |a_{i,i-1}| - |a_{i,i+1}| - \dots - |a_{i,n}| > 0$$

for all $i = 1, 2, \dots, n$.

Proof: By proposition 4.2, the study of strong stability of A may be reduced to the equivalent problem on the extended Schur matrix $\hat{T}$, which has as elements on the diagonal a_{ii}, $i \in \underline{n}$, the real parts of the eigenvalues of A which are all on the negative real axis, due to the asymptotic stability assumption. The symmetric part of T and $\hat{T}$ is the same (from last proposition) and can be expressed as:

$$2\bar{T} = \begin{pmatrix} 2a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{12} & 2a_{22} & a_{23} & \dots & a_{2n} \\ a_{13} & a_{23} & 2a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & 2a_{nn} \end{pmatrix}$$

Clearly A is strongly stable if $2\bar{T}$ is asymptotically stable. If $2\bar{T} = [\alpha_{ij}]$, then applying Geshgorin's Theorem [9] shows that the eigenvalues lie in the union of the discs defined by

$$|z - \alpha_{ii}| \leq r_i,\ r_i = \sum_{j=1, j \neq i}^{n} |\alpha_{ij}|,\ j = 1, 2, \dots, n \tag{8}$$

where $\alpha_{ii} = 2a_{ii}$ and α_{ij} are defined by the structure of $2\bar{T}$. Given that all these discs have their centres on the negative real axis, then if the Geshgorin conditions (8) are satisfied, each disc lies entirely in the open left half of the complex plane, since for every i, $2|a_{ii}| - r_i > 0$, and thus also their union. Now $2\bar{T}$ has real eigenvalues (being symmetric) which thus lie entirely on the negative real axis and $\bar{T}$, as well as $\bar{A}$, is asymptotically stable, and thus $\bar{A}$ is negative-definite. This establishes the result. □

Note that the above conditions express an important property of matrices known as strict diagonal dominance which is formally defined below [9].

Definition 4.3: Let $A = [a_{ij}] \in \mathcal{R}^{n \times n}$. The matrix A is said to be *strictly diagonally dominant*, if

$$|a_{ij}| > \sum_{j=1, j \neq i}^{n} |a_{ij}| = R_i \text{ for all } i = 1, 2, \dots, n$$

□

The above property and Theorem 4.4 lead to:

Corollary 4.1: Let $A \in \mathcal{R}^{n\times n}$ be asymptotically stable and with an extended Schur form $\hat{T}$. Then A is strongly stable if $\hat{T}$ is strictly diagonally dominant. □

Note that due to the symmetry property of $\bar{T}$, the result stated above cannot be improved by using the column version of the theorem, neither the more general form provided by the Ostrowski theorem [13]. We conclude the section by stating further general properties related to the notion of strong stability.

Proposition 4.3: If A is strongly stable then AP is asymptotically stable for every symmetric positive-definite matrix P.

Proof: We invoke the well-known Lyapunov stability theorem which states that a matrix A is asymptotically stable if and only if there exists a $P = P' > 0$ such that $PA + A'P < 0$ [9]. Now suppose that A is strongly stable, so that $A + A' < 0$. For every symmetric positive definite P we can then write $P(A + A')P < 0$, or equivalently $P(AP) + (AP)'P < 0$ and hence AP is asymptotically stable. □

The above result provides a direct proof of the fact that strong stability implies asymptotic stability which does not rely on the use of Lyapunov functions. This is stated in Corollary 4.2 below, whose proof follows immediately from Proposition 4.3 by setting $P = I$:

Corollary 4.2: If A is strongly stable then A is asymptotically stable.

We conclude this section by establishing some general properties of the set of all strongly-stable matrices (of a fixed dimension) by showing that they form a *convex invertible cone (cic)* [3], [6], [11].

Proposition 4.4: Let $A = [a_{ij}] \in \mathcal{R}^{n\times n}$. The following properties hold true:

(i) If A is strongly stable then A^{-1} exists and is also strongly stable.

(ii) If A_1 and A_2 are strongly stable, then $\lambda A_1 + (1-\lambda)A_2$ is also strongly stable for any $0 \leq \lambda \leq 1$.

(iii) If A is strongly stable then so is λA for any $\lambda > 0$.

(iv) If A is strongly stable then so is A'.

Proof: (i) If A is strongly stable it is also asymptotically stable and hence invertible. Now $A + A' < 0$ implies that $A^{-1}(A + A')A^{-\prime} < 0$ since the inertia of a matrix is invariant under congruent transformations (Sylvester's law of inertia) and hence $A^{-1} + A^{-\prime} < 0$, or A^{-1} is strongly stable. (ii) Since A_1 and A_2 are strongly stable $A_1 + A_1' < 0$ and $A_2 + A_2' < 0$ form which it follows that $[\lambda A_1 + (1-\lambda)A_2)] + [\lambda A_1 + (1-\lambda)A_2)]' < 0$ for every $\lambda \in [0,\ 1]$ and hence $\lambda A_1 + (1-\lambda)A_2)$ is strongly stable. Part (iii) follows similarly to (ii) while (iv) follows directly from the definition. □

5. Deviation from normality and Skewness of Eigenframe

An interesting question that arises is to investigate the degree of divergence from normality that allows preservation of strong stability, or the degree of skewness of the eigenframe that leads to violation of strong stability. These issues are addressed in this section of the paper. Conditions related to the violation of the strong stability property are also important since they may provide indicators for characteristics which need to be avoided. The violation of strong stability is demonstrated first by examples which reveal the role of skewness of the eigenvectors of A on the structure of the matrix A. The results in the previous section suggest that for a given matrix A the study of the effect of skewness of the eigenframe can be reduced to the study of orthogonality of the eigenframe of the Schur matrix, which is parametrically explicitly expressed in terms of the Schur invariants.

Example 5.1: Continue with example 4.1 and consider conditions for skewness of the eigenframe of the matrix expressed in the Shcur form. Then the eigenframe is:

$$U_3 = \begin{pmatrix} 1 & -\frac{a_{12}}{a_{11}-a_{22}} & \frac{a_{12}a_{23}-a_{13}a_{22}+a_{13}a_{33}}{(a_{11}-a_{33})(a_{22}-a_{33})} \\ 0 & 1 & -\frac{a_{23}}{a_{22}-a_{33}} \\ 0 & 0 & 1 \end{pmatrix} = [u_1 \; u_2 \; u_3] \tag{9}$$

Assume that

$$\rho = \left| \frac{a_{12}}{a_{11}-a_{22}} \right| \gg 1 \Rightarrow a_{12}^2 \gg (a_{11}-a_{22})^2$$

and consider values:

(i) $a_{12} = 3$, $a_{11} = -1.011$, $a_{22} = -1.010$. Then $\rho = 3 \cdot 10^3$. Clearly the first two eigenvectors:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0.33 \cdot 10^{-3} \\ 0 \end{pmatrix}$$

are nearly dependent. Then condition $\Delta_2 > 0$ is violated since

$$\Delta_2 = 4a_{11}a_{22} - a_{12}^2 = 4 \times 1.01 \times 1.010 - 9 = 4.084 - 9 < 0$$

and thus we must have overshoots for certain initial conditions.

(ii) Take $a_{12} = 8$, $a_{11} = -1$, $a_{22} = -2$. Then:

$$-\frac{a_{12}}{a_{11}-a_{22}} = \frac{-8}{-1+2} = -8 \;\text{ and }\; \rho = 8$$

Clearly the angle between the first two eigenvectors is small, since

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -\frac{1}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -0.125 \\ 0 \end{pmatrix}$$

With this set of values we have

$$\Delta_2 = 4a_{11}a_{22} - a_{12}^2 = 4 \times (-1)(-2) - 8^2 = 8 - 64 = -56 < 0$$

and the strong stability condition is violated, i.e. we have overshoots. $\square$

The natural question that arises is the investigation of the effect of skewness of the eigenframe, as this may be estimated by a measure of the angle of the frame, on the preservation or violation of the strong stability property.

Remark 5.1: Expression (9) is valid for the 3×3 Schur matrix; however for the general $n \times n$ case the matrix U_n always has the form

$$U_n = \left(\begin{array}{c|ccc} U_3 & x & x & x \\ \hline 0 & 1 & x & x \\ \vdots & \ddots & \ddots & x \\ 0 & \dots & 0 & 1 \end{array} \right)$$

i.e. it can be extended by augmentation and it is also upper triangular. $\square$

Example 5.1 suggests an interesting case of skewness where two eigenvectors are very close to each other. Consider the vectors u_1, u_2 of U_n which are expressed as

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -\frac{a_{12}}{a_{11}-a_{22}} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

If we denote by

$$\rho_{12} = \left| \frac{a_{12}}{a_{11} - a_{22}} \right|$$

and $\rho_{12} \gg 1$, then clearly the overall frame becomes skewed since the angle of the two vectors is

$$\cos(u_1, u_2) = \frac{|u_1' u_2|}{\|u_1\| \, \|u_2\|} = \frac{\rho_{12}}{\sqrt{\rho_{12}^2 + 1}}$$

The value of $\cos(u_1, u_2)$ defines the *degree of skewness* of the $\{u_1, u_2\}$ vectors and if $\cos(u_1, u_2) = 1$ the vectors are linearly dependent. If $\rho_{12} \gg 1$ then $\cos(u_1, u_2) \approx 1$ and the angle $\sphericalangle(u_1, u_2)$ is very small. The question that now arises is to determine the minimum value of ρ_{12}, or degree of skewness, for which $\Delta_2 < 0$ and thus the strong stability property is violated. This is defined by the following result:

Proposition 5.1: For the eigenframe in the standard form (9), for all values of ρ_{12} for which

$$\rho_{12} > \frac{2\sqrt{a_{11} a_{22}}}{|a_{11} - a_{22}|}$$

the eigenframe is skewed and the skewness of $\{u_1, u_2\}$ implies violation of the strong stability property.

Proof: Note that $\Delta_2 = 4a_{11}a_{22} - a_{12}^2$. By the definition of ρ_{12}

$$\rho_{12}^2 = \frac{a_{12}^2}{(a_{11} - a_{22})^2} \Rightarrow a_{12}^2 = \rho_{12}^2 (a_{11} - a_{22})^2$$

and thus

$$-a_{12}^2 = -\rho_{12}^2 (a_{11} - a_{22})^2 \Leftrightarrow \Delta_2 = 4a_{11}a_{22} - \rho_{12}^2 (a_{11} - a_{22})^2$$

For $\Delta_2 < 0$ we must have:

$$4a_{11}a_{22} - \rho_{12}^2(a_{11} - a_{22})^2 < 0, \text{ or } \rho_{12}^2 > \frac{4a_{11}a_{22}}{(a_{11} - a_{22})^2}$$

Clearly, for all ρ_{12} satisfying the latter condition, strong stability is violated and this defines the minimum value of required degree of skewness. $\square$

A convenient measure of skewness of the eigenframe is provided by the Grammian of U_n in terms of the elements of u_{ij} and thus eventually in terms of the elements of the Schur matrix. We first note:

Proposition 5.2: Let U_n be the eigenframe of the n-dimensional Schur matrix of A and let $D_n = U_n' U_n$. Then $|D_n| = 1$.

Proof:

$$|D_n| = |U_n' U_n| = |U_n'|\ |U_n| = |U_n|^2 = 1$$

since U_n is upper triangular with diagonal elements equal to 1. $\square$

The eigenframe U_n of the Schur matrix may be normalised (unit length vectors) and the normalised eigenframe is denoted by

$$\tilde{U}_n = [u_1\ u_2\ \ldots\ u_n]\mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n) = U_n\Delta$$

where $\delta_i = 1/\|u_i\|$, $i \in \underline{n}$. $\tilde{U}_n$ will be called the *canonical Schur eigenframe*. For this frame we have the following result.

Theorem 5.1: If U_n is the Schur eigenframe, $\delta_i = 1/\|u_i\|$, $i \in \underline{n}$, and $\tilde{U}_n$ is the canonical Schur eigenframe, then its Grammian has the value

$$G(\tilde{U}_n) = |\tilde{U}_n' \tilde{U}_n| = \delta_1^2\delta_2^2 \ldots \delta_n^2 = \rho$$

and satisfies the Haddamard inequality $0 \leq G(\tilde{U}_n) \leq 1$.

Proof: Note that since $\tilde{U}_n = U_n\Delta$, then $\tilde{U}_n'\tilde{U}_n = \Delta U_n' U_n \Delta$ and thus

$$G(\tilde{U}_n) = |\tilde{U}_n'\tilde{U}_n| = |\Delta U_n' U_n \Delta| = |\Delta|^2 |U_n' U_n| = |\Delta|^2$$

since by Proposition 5.1 $|D_n| = |U_n' U_n| = 1$. $\square$

Example 5.2: For the case $n = 3$ we have:

$$\delta_1^2 = 1,\ \delta_2^2 = \frac{1}{1 + \frac{a_{23}^2}{(a_{22}-a_{33})^2}}$$

and

$$\delta_3^2 = \frac{1}{1 + \frac{a_{23}^2}{(a_{22}-a_{33})^2} + \left[-\frac{a_{13}}{a_{11}-a_{33}} + \frac{a_{12}a_{13}}{(a_{11}-a_{33})(a_{22}-a_{33})}\right]^2}$$

and thus

$$\rho = G(\tilde{U}_3) = \delta_1^2\delta_2^2\delta_3^2 = \frac{1}{\left\{1 + \frac{a_{23}^2}{(a_{22}-a_{33})^2}\right\}\left\{1 + \frac{a_{23}^2}{(a_{22}-a_{33})^2} + \left[-\frac{a_{13}}{a_{11}-a_{33}} + \frac{a_{12}a_{13}}{(a_{11}-a_{33})(a_{22}-a_{33})}\right]^2\right\}}$$

$\square$

A "co-ordinate free" necessary condition for strong stability for the special class of matrices with distinct eigenvalues is provided by the following Theorem:

Theorem 5.2: Let $A \in \mathcal{R}^{n\times n}$ be strongly stable and have a distinct set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and a corresponding set of eigenvectors $\{w_1, w_2, \ldots, w_n\}$, assumed normalised so that $\|w_i\| = 1$ for all $i = 1, 2, \ldots, n$. Then for every pair (i, j) with $1 \le i, j \le n$, $i \ne j$ we have:

$$\cos\theta_{ij} < \frac{2\sqrt{\mathrm{Re}(\lambda_i)\mathrm{Re}(\lambda_j)}}{|\lambda_i + \bar{\lambda}_j|} < 1 \tag{10}$$

where $\cos\theta_{ij} = |\langle w_i, w_j\rangle| = |w_i^* w_j|$.

Proof: Since A is strongly stable, it is also asymptotically stable and hence $\mathrm{Re}(\lambda_i) < 0$ for all $i = 1, 2, \ldots, n$. Since by assumption A has distinct eigenvalues, it is diagonalisable and hence $A = W\Lambda W^{-1}$ where $W = [w_1, w_2, \ldots w_n]$ and $\Lambda = \mathrm{diag}(\lambda_i)$. Thus $A' = A^* = W^{-*}\bar{\Lambda}W^*$ and hence

$$A + A' = W\Lambda W^{-1} + W^{-*}\bar{\Lambda}W^* < 0$$

since A is strongly stable. Since congruent transformations do not affect the inertia of a matrix we have that

$$W^*(W\Lambda W^{-1} + W^{-*}\bar{\Lambda}W^*)W < 0 \Rightarrow (W^*W)\Lambda + \bar{\Lambda}(W^*W) < 0$$

Now

$$W^*W = \begin{pmatrix} w_1^* \\ w_2^* \\ \vdots \\ w_n^* \end{pmatrix} \begin{pmatrix} w_1 & w_2 & \ldots & w_n \end{pmatrix} = \begin{pmatrix} 1 & \langle w_1, w_2\rangle & \ldots & \langle w_1, w_n\rangle \\ \langle w_2, w_1\rangle & 1 & \ldots & \langle w_2, w_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle w_n, w_1 & \langle w_n, w_2 & \ldots & 1 \end{pmatrix}$$

and hence

$$B = \begin{pmatrix} \lambda_1 & \lambda_2\langle w_1, w_2\rangle & \ldots & \lambda_n\langle w_1, w_n\rangle \\ \lambda_1\langle w_2, w_1\rangle & \lambda_2 & \ldots & \lambda_n\langle w_2, w_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1\langle w_n, w_1 & \lambda_2\langle w_n, w_2 & \ldots & \lambda_n \end{pmatrix} + \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_1\langle w_1, w_2\rangle & \ldots & \bar{\lambda}_1\langle w_1, w_n\rangle \\ \bar{\lambda}_2\langle w_2, w_1\rangle & \bar{\lambda}_2 & \ldots & \bar{\lambda}_2\langle w_2, w_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\lambda}_n\langle w_n, w_1 & \bar{\lambda}_n\langle w_n, w_2 & \ldots & \bar{\lambda}_n \end{pmatrix} < 0$$

Thus $B = [B_{ij}] < 0$ where $B_{ij} = (\bar{\lambda}_i + \lambda_j)\langle w_i, w_j\rangle$ if $i \ne j$ and $B_{ij} = 2\mathrm{Re}(\lambda_i)$ if $i = j$. Let $B_{(i,j)}$ denote the sub-matrix of B formed by selecting its (i, j) rows and columns. Then

$$B_{(i,j)} = \begin{pmatrix} 2\mathrm{Re}(\lambda_i) & (\bar{\lambda}_i + \lambda_j)\langle w_i, w_j\rangle \\ (\lambda_i + \bar{\lambda}_j)\langle w_j, w_i\rangle & 2\mathrm{Re}(\lambda_j) \end{pmatrix} < 0$$

for every $i \ne j$ and hence

$$4\mathrm{Re}(\lambda_i)\mathrm{Re}(\lambda_j) > |\bar{\lambda}_i + \lambda_j|^2 |\langle w_i, w_j\rangle|^2$$

since $\langle w_j, w_i\rangle = \overline{\langle w_j, w_i\rangle}$. Thus

$$|\langle w_i, w_j\rangle|^2 = \cos^2\theta_{ij} < \frac{4\mathrm{Re}(\lambda_i)\mathrm{Re}(\lambda_j)}{|\lambda_i + \bar{\lambda}_j|^2}$$

To prove the last inequality set $\lambda_i = \sigma_i + j\omega_i$ and $\lambda_j = \sigma_j + j\omega_j$ and note that

$$\frac{4\sigma_i\sigma_j}{|\sigma_i + j\omega_i + \sigma_j - j\omega_j|^2} < 1 \Leftrightarrow 4\sigma_i\sigma_j < (\sigma_i + \sigma_j)^2 + (\omega_i - \omega_j)^2 \Leftrightarrow (\sigma_i - \sigma_j)^2 + (\omega_i - \omega_j)^2 > 0$$

which is true since $\lambda_i \neq \lambda_j$ by assumption. $\square$

Remark 5.2: The inequalities (10) define necessary conditions for strong stability and they may interpreted as follows: These conditions may be used to show that an asymptotically stable matrix A with distinct eigenvalues cannot be strongly stable if any eigenvector pair violates the inequality stated in the Theorem. Clearly if the eigenvectors of A are mutually orthogonal we have that $\cos\theta_{ij} = 0$ for every pair (i, j) with $i \neq j$, and so this inequality is trivially satisfied. **In general, the strong stability property guarantees that the"skewness" of the eigenframe cannot exceed a certain maximum level, which depends on the distribution of the eigenvalues.** $\square$

Remark 5.3: Conditions 10 provide necessary conditions based on the eigenvalue pattern in the form of inequalities, which the angles of the eigenvectors of an asymptotically stable matrix with distinct eigenvalues must satisfy to guarantee strong stability. For the asymptotically stable matrix $A \in \mathcal{R}^{n\times n}$ with distinct set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ the cosines $\cos\theta_{ij}$, $\forall(i, j)\, 1 \leq i,\, j \leq n,\, i \neq j$, have as upper bounds the numbers

$$\psi(\lambda_i, \bar{\lambda}_j) = 2\sqrt{\mathrm{Re}(\lambda_i)\mathrm{Re}(\lambda_j)}/|\lambda_i + \bar{\lambda}_j| \tag{11}$$

Violation of at least one of such upper bounds by a pair of eigenvectors clearly implies violation of strong stability. $\square$

Remark 5.4: Consider an asymptotically stable 2×2 matrix A with distinct eigenvalues and normalised eigenvectors (w_1, w_2). Then, in the notation of Theorem 5.2, strong stability of A is equivalent to:

$$B = (W^*W)\Lambda + \bar{\Lambda}(W^*W) < 0$$

where

$$B = \begin{pmatrix} 2\mathrm{Re}(\lambda_1) & (\bar{\lambda}_1 + \lambda_2)\langle w_1, w_2\rangle \\ (\lambda_1 + \bar{\lambda}_2)\langle w_2, w_1\rangle & 2\mathrm{Re}(\lambda_2) \end{pmatrix}$$

Now, since A is assumed asymptotically stable, the diagonal elements of B are negative and thus negative-definiteness of B is equivalent to the condition:

$$|B| > 0 \Leftrightarrow |\langle w_1, w_2\rangle| = \cos\theta_{12} < \frac{2\sqrt{\mathrm{Re}(\lambda_1)\mathrm{Re}(\lambda_2)}}{|\lambda_1 + \bar{\lambda}_2|} = \psi(\lambda_1, \bar{\lambda}_2)$$

Thus in the 2×2 case the conditions of Theorem 5.2 are both necessary and sufficient. $\square$

The following example illustrates this result for a simple 2×2 example.

Example 5.3: Consider the 2×2 matrix A_ϵ with eigenvalue-eigenvector decomposition:

$$A_\epsilon = \begin{pmatrix} -1 & -\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ 0 & \sqrt{1-\varepsilon^2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \\ 0 & \frac{1}{\sqrt{1-\varepsilon^2}} \end{pmatrix}$$

in which ε is a real parameter in the interval $-1 < \varepsilon < 1$. This parameter controls the skewness of the eigenframe of A_ε; in fact in can be easily seen that $|\varepsilon| = \cos(\theta_{12})$, the cosine of the angle of the two eigenvectors of A_ε. Now, A_ε is strongly stable if and only if:

$$A_\varepsilon + A_\varepsilon' = \begin{pmatrix} -2 & \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \\ \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} & -4 \end{pmatrix} < 0$$

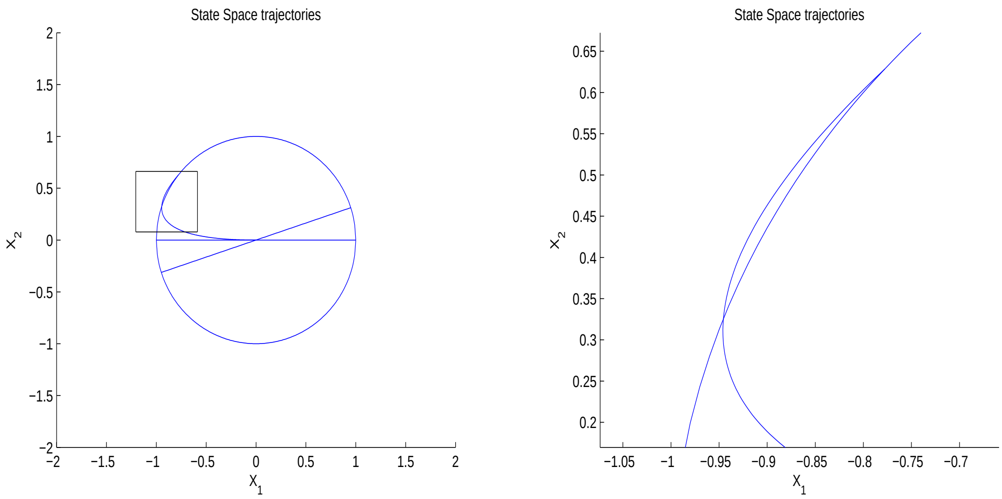


Figure 2: Example 5.3 ($\varepsilon = 0.95 > \varepsilon^*$)

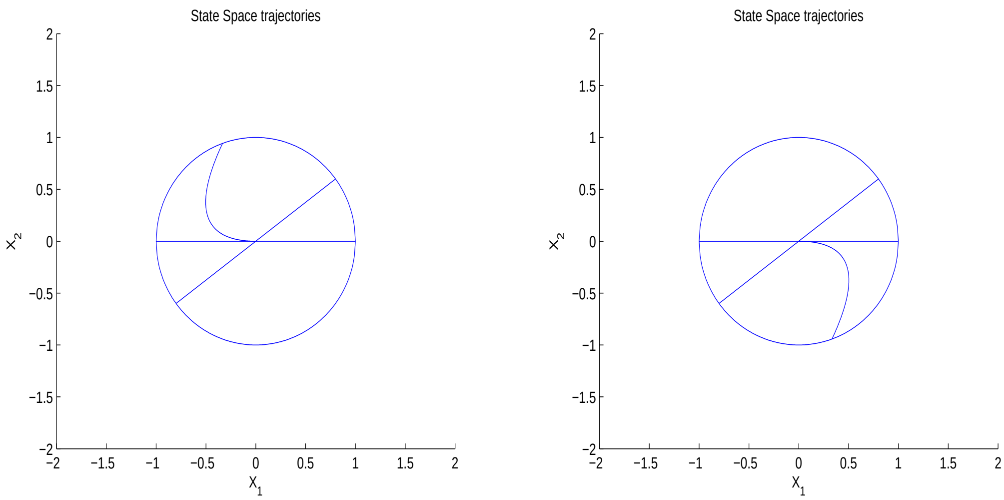


Figure 3: Example 5.3 ($\varepsilon = 0.8 < \varepsilon^*$)

or, equivalently iff:

$$\cos(\theta_{12}) = |\epsilon| < \epsilon^* = \frac{2\sqrt{2}}{3} \approx 0.9428 = \psi(\lambda_1, \bar{\lambda}_2)$$

in agreement with Theorem 5.2 and Remark 5.4. Figure 2 illustrates one overshooting trajectory of the system with state matrix A_ε when $\varepsilon = 0.95 > \varepsilon^*$. Two non-overshooting trajectories of the (strongly stable) system corresponding to $\varepsilon = 0.80 < \varepsilon^*$ are also shown for comparison in Figure 3. Note that the straight lines in the two figures correspond to the eigenvector directions. $\square$

Remark 5.5: Consider a 2×2 asymptotically stable matrix A with a complex conjugate eigenvalue pair $\lambda = \sigma \pm j\omega$, where $\sigma < 0$ and $\omega > 0$. In this case, the condition of Theorem 5.2 (which is actually necessary and sufficient - see Remark 5.4) can be written as:

$$\cos\theta_{12} < \frac{2|\sigma|}{|2\sigma + 2j\omega|} = \frac{1}{\sqrt{1 + (\omega/\sigma)^2}} < 1$$

Thus the system is strongly stable if and only if:

$$1 + \left(\frac{\omega}{\sigma}\right)^2 < \sec^2(\theta_{12}) \Leftrightarrow \left|\frac{\omega}{\sigma}\right| < |\tan(\theta_{12})|$$

Consider the (constant damping) line in the s-plane passing through the origin and the eigenvalue $\sigma + j\omega$. Let $\varphi(\omega, \sigma)$ be the angle formed by this line and the negative real axis ($\mathrm{Re}(s) \leq 0$). Then for strong stability we require $\varphi(\omega, \sigma) < \theta_{12}$. Equivalently, A is strongly stable if and only if the eigenvalues of A have a damping factor which exceeds $\cos\theta_{12}$. $\square$

The following example considers a concrete example with complex eigenvalues.

Example 5.4: Consider the 2×2 matrix

$$A_\epsilon = \begin{pmatrix} \sigma - \varepsilon\omega & (\varepsilon^2 + 1)\omega \\ -\omega & \sigma + \varepsilon\omega \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon \\ 0 & 1 \end{pmatrix}$$

with complex-conjugate eigenvalues $\{\sigma \pm j\omega\}$ in which ε is an arbitrary real parameter. We assume that $\sigma < 0$ and $\omega > 0$ (so that A_ε is asymptotically stable) and examine under what conditions A_ε is strongly stable. The strong stability property is satisfied if and only if:

$$A_\epsilon + A'_\epsilon = \begin{pmatrix} 2(\sigma - \varepsilon\omega) & \omega\varepsilon^2 \\ \omega\varepsilon^2 & 2(\sigma + \varepsilon\omega) \end{pmatrix} < 0$$

or, equivalently, if and only if:

$$\left\{ \begin{array}{l} \sigma - \varepsilon\omega < 0 \Rightarrow \varepsilon > \frac{\sigma}{\omega} \\ \sigma + \varepsilon\omega < 0 \Rightarrow \varepsilon < -\frac{\sigma}{\omega} \end{array} \right\} \Rightarrow |\varepsilon| < -\frac{\sigma}{\omega} \tag{12}$$

and

$$4(\sigma^2 - \varepsilon^2\omega^2) - \omega^2\varepsilon^4 > 0 \Rightarrow \varepsilon^4 + 4\varepsilon^2 - 4\frac{\sigma^2}{\omega^2} < 0 \Rightarrow 0 \leq \varepsilon^2 < 2\sqrt{1 + \frac{\sigma^2}{\omega^2}} - 2$$

The last condition may be alternatively written as $\varepsilon^2 < \sigma^2/\omega^2 - \varepsilon^4/4$ and hence is stronger than condition (12). In conclusion, A_ε is strongly stable if and only if

$$|\varepsilon| < \varepsilon^* = \sqrt{2\sqrt{1 + \frac{\sigma^2}{\omega^2}} - 2} \tag{13}$$

For example, if $|\sigma|/\omega = 1$, the condition for strong stability is $\varepsilon < \sqrt{2\sqrt{2} - 2} \approx 0.9102$. Next, consider the spectral decomposition of $A_\varepsilon = W\Lambda W^{-1}$, which can be written out in full as:

$$A_\epsilon = \begin{pmatrix} \frac{1+\varepsilon j}{\sqrt{2+\varepsilon^2}} & \frac{1-\varepsilon j}{\sqrt{2+\varepsilon^2}} \\ \frac{j}{\sqrt{2+\varepsilon^2}} & \frac{-j}{\sqrt{2+\varepsilon^2}} \end{pmatrix} \begin{pmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2+\varepsilon^2}}{2} & -\frac{\sqrt{2+\varepsilon^2}(\varepsilon + j)}{2} \\ \frac{\sqrt{2+\varepsilon^2}}{2} & -\frac{\sqrt{2+\varepsilon^2}(\varepsilon - j)}{2} \end{pmatrix}$$

Note that the columns of W (w_1 and w_2 say) are the right eigenvectors of A_ε and have been normalised as $\|w_1\| = \|w_2\| = 1$. The condition for strong stability given by Theorem 5.2 is:

$$\cos\theta_{12} = |w_1^* w_2| = \left| \begin{pmatrix} \frac{1-\varepsilon j}{\sqrt{2+\varepsilon^2}} & \frac{-j}{\sqrt{2+\varepsilon^2}} \end{pmatrix} \begin{pmatrix} \frac{1-\varepsilon j}{\sqrt{2+\varepsilon^2}} \\ \frac{-j}{\sqrt{2+\varepsilon^2}} \end{pmatrix} \right| < 2\sqrt{\mathrm{Re}(\lambda_1)\mathrm{Re}(\lambda_2)}/|\lambda_1 + \bar{\lambda}_2| = \frac{1}{\sqrt{1 + (\frac{\omega}{\sigma})^2}}$$

This gives:

$$\frac{|\varepsilon|\sqrt{\varepsilon^2 + 4}}{2 + \varepsilon^2} < \frac{1}{\sqrt{1 + (\frac{\omega}{\sigma})^2}} \Leftrightarrow \left(\frac{\omega}{\sigma}\right)^2 > \frac{4}{\varepsilon^2(\varepsilon^2 + 4)}$$

which can be rearranged to give an identical condition with equation (13). $\square$

Theorem 5.2 provides necessary conditions for strong stability on the family of asymptotically stable matrices. It is not clear whether the above established bounds are the best that can be defined.

The effect of the shape of the eigenvalue pattern on these constraints is also a challenging issue. We conclude this section by revisiting Proposition 3.3 on the link between Jordan forms and strong stability by examining the effect of the eigenframe on the conditions for strong stability for arbitrary matrices with repeated eigenvalues. We use Proposition 3.3 on Jordan forms and strong stability and we establish the following result:

Theorem 5.3: Let $A = WJW^{-1}$ be a Jordan decomposition of A with J strongly stable. Assume that:

$$\|W^*W - I\| \leq \epsilon := \frac{\lambda_{max}(-J - J^*)}{2\|J\|} > 0$$

Then A is strongly stable. In particular in the notation of Proposition 3.3, A is strongly stable provided that

$$\|W^*W - I\| \leq \epsilon_1 := -\frac{\min_i \left\{ \mathrm{Re}(\lambda_i) + \cos\left(\frac{m_i \pi}{m_i + 1}\right) \right\}}{\max\{\max_i\{|\lambda_i| : m_i = 1\}, \max_i\{|\lambda_i| + 1 : m_i > 1\}\}}$$

and J is strongly stable.

Proof: Set $\Delta = W^*W - I$. Then,

$$\begin{aligned}
\|\Delta\| < \frac{\lambda_{max}(-J - J^*)}{2\|J\|} &\Leftrightarrow \lambda_{max}(J + J^*) + 2\|\Delta\|\|J\| < 0 \\
&\Rightarrow \lambda_{max}(J + J^*) + \|\Delta J\| + \|J^*\Delta\| < 0 \\
&\Rightarrow \lambda_{max}(J + J^*) + \|\Delta J + J^*\Delta\| < 0 \\
&\Rightarrow \ \lambda_{max}(J + J^*) + \lambda_{max}(\Delta J + J^*\Delta) < 0 \\
&\Rightarrow \lambda_{max}(J + J^* + \Delta J + J^*\Delta) < 0 \\
&\Leftrightarrow \ J + J^* + \Delta J + J^*\Delta < 0
\end{aligned}$$

using Weyl's inequality [9]. Hence:

$$(I + \Delta)J + J^*(I + \Delta) < 0 \ \Leftrightarrow \ W^*WJ + J^*W^*W < 0 \ \Leftrightarrow \ W^{-*}[W^*WJ + J^*W^*W]W^{-1} < 0$$

and hence

$$WJW^{-1} + W^{-*}J^*W^* < 0 \ \Leftrightarrow \ A + A^* < 0$$

and thus A is strongly stable. A direct calculation similar to the proof of Proposition 3.3 shows that

$$\lambda_{max}(-J - J^*) = -2 \min_i \left\{ \mathrm{Re}(\lambda_i) + \cos\left(\frac{m_i \pi}{m_i + 1}\right) \right\}$$

Further,

$$\begin{aligned}
\|J\| &= \max_{i=1,2,\ldots,k} \ \max_{j=1,2,\ldots,r_i} \|J_{ij}\| = \max\{\max_i\{\|\lambda_i I_{m_i}\| : m_i = 1\}, \max_i\{\|\lambda_i I_{m_i} + E_{m_i}\| : m_i > 1\}\} \\
&\leq \max\{\max_i\{\|\lambda_i I_{m_i}\| : m_i = 1\}, \max_i\{\|\lambda_i I_{m_i}\| + \|E_{m_i}\| : m_i > 1\}\} \\
&= \max\{\max_i\{|\lambda_i| : m_i = 1\}, \max_i\{|\lambda_i| + 1 : m_i > 1\}\}
\end{aligned}$$

Here E_{m_i} denotes the $m_i \times m_i$ zero matrix except from elements of one above the main diagonal. Using the two expressions for $\lambda_{max}(-J - J^*)$ and $\|J\|$ gives the estimate ϵ_1 for strong stability stated above. $\square$

Clearly, $\psi = \|W^*W - I\|$ provides a measure of departure of the eigenframe from normality. The above result provides criteria based on the eigenvalues and Jordan pattern of the matrix that indicate acceptable departures from normality for which we can retain the strong asymptotic stability property of the matrix.

Remark 5.6: Theorem 5.3 shows that "small" perturbations of the (generalised) eigenvector matrix from normality do not destroy the strong stability property of the matrix, provided that its Jordan form is strongly stable.

Remark 5.7: **Note that all the techniques of the paper rely on standard linear algebraic techniques and are therefore easily implementable. A full numerical analysis of the strong stability notion will be undertaken in future work, together with explicit results on distance measures of arbitrary matrices to the (convex invertible) cone of strongly stable matrices.**

6. Conclusions

The paper has introduced a new notion of internal stability, strong stability, which characterises the absence of overshoots in the free system response. This problem makes sense for state space descriptions expressed in terms of physical variables and strong stability is a property associated with the given coordinate frame and in general changes under general coordinate transformations. We have focused here on the Linear Systems case, but the problem has a more general character and can also be considered for the nonlinear case. Three precise notions of strong stability have been introduced and necessary and sufficient conditions have been derived for each one terms of the negative definiteness or semi-definiteness of the symmetric part of the state matrix A and a two additional system properties(asymptotic stability, observability). Although the strong stability property changes under general coordinate transformations, it remains invariant under orthogonal transformations and this allows the use of the Schur form as a natural vehicle for studying strong stability.

The association of strong stability to specific types of coordinate frames motivates the study of this property for two distinct classes of asymptotically stable matrices, the companion type matrices and the general Jordan form. It has been shown first, that no companion matrix can be strongly stable, whereas for the Jordan case, necessary and sufficient conditions for strong stability have been obtained in terms of the spread of the eigenvalues and the Segre characteristics of its Jordan structure. The latter results provide new tests for characterising the quality of Jordan forms in terms of strong stability properties.

The use of Schur's form analysis simplifies the initial structure of the problem. It has been shown that the strong stability properties are independent from the properties of the imaginary parts of the eigenvalues and depend only on the real parts and the remaining elements of the real Schur form. A simple sufficient condition of strong stability (for an asymptotically stable matrix A) is that the extended Schur form of A is strictly diagonally dominant. The structure of the Schur matrix allows the derivation of explicit formulae for the eigenvectors and this allows us to establish links between the degree of skewness of the eigenframe and conditions indicating the violation of the strong stability property. For the case of asymptotically stable matrices with distinct eigenvalues, necessary conditions for strong stability have been derived in terms of properties of the eigenframe. These conditions provide new tests for characterising the spread of eigenvalues that can precondition strong stability. Finally, for a general matrix having a strongly stable Jordan form, measures of departure of an eigenframe from normality have been defined which guarantee that the strong stability property is retained. **Note that the derived conditions are only necessary and sufficient for the 2×2 case. Tighter bounds for general $n \times n$ matrices (or for special classes of matrices) may well be possible, as issue which we will explore in future work, along with the effects of different eigenvalue distributions in the complex plane.**

The notion of strong stability presented in this work has numerous applications in Systems and Control. In many practical applications, classical notions of stability (such as asymptotic or Lyapunov stability) may be too weak for characterizing satisfactorily operation of systems under feedback control. This is especially true for many real-time process-control and economic applications, where a non-overshooting response is often desirable. The notion of strong stability introduced here is directly related to the absence of state-space overshoots for systems described by physical variables and can be applied, if required, to limit the exponential growth of the system's response [7], [8], [16] or its transient energy and to address objectives related to energy dissipation [18]. A second area of applications involves the solution of stabilization problems under state or output feedback. Our recent work in the area of strong stabilization (which we plan to report in a future publication) suggests that problems of this type are easily solvable via convex programming techniques and, further, that closed-form parametrisations of the optimal solution sets can be obtained. Note that this is contrast to the standard output feedback stabilization problem which to this day remains essentially unsolved. Further work in this area may prove useful for the solution of this problem, and could also help to establish a conceptual framework for tackling other related problems in this area, such as partial eigenvalue/eigenvector assignment. One possible approach in this case is to employ the link between the notion of "strong stability" and eigenframe skewness of the state matrix (along the lines developed in this work), combined with a distance-to-strong stability optimization framework; solution techniques for the latter class of problems are already in place and will be reported elsewhere. Finally, other areas of application of the notion of strong stability include the control of switched linear systems, common Lyapunov function methods and related techniques.

Further research in this area aims to characterise the strong stability properties of special types of matrices, which may provide qualitative results for different types of coordinate systems. The link of strong stability to measures of skewness of eigenframes and the link of eigenframe skewness to robustness [17], [4] suggests that there may be some links between strong stability and robustness of stability. **In future work, this link will be investigated in the context of partial eigenvalue/eigenvector assignment under state feedback**. **Finally, the results of this work provide establish the basis for addressing the problem of feedback design for strong stabilisation using state feedback, output injection and output feedback**. These, together with extensions **of the work to discrete-time, time-varying and nonlinear cases** are also issues for future work.

References

[1] S. Barnett, *Matrices in Control Theory*, Van Nostrad Reinhold, London, 1971.

[2] N.B. Bhatia and G.P. Szego, *Stability theory of Dynamic Systems*, Classics in Mathematics Series, Springer Verlag, New York, 1970.

[3] N. Cohen and I. Lewkowicz, *Convex Invertible Cones and the Lyapunov Equation*, Linear Algebra and its Applications, $250 : 105 - 131$, 1997.

[4] G.H. Golub and C.F. van Loan, *Matrix Computations*, The John Hopkins University Press, 1996.

[5] G. Halikias, N. Karcanias and A. Papageorgiou, *Strong Stabilisation under state and output feedback*, Internal Report, Control Engineering Research Centre (CERC), City University, May 2009.

[6] D. Hershkowitz, *On cones and stability*, Linear Algebra and its Applications, $275 - 276$, pp. $249 - 259$, 1998.

[7] D. Hinrichsen and A.J. Pritchard, *Mathematical Systems Theory I*, Texts in Applied Mathematics Vol. 41, Springer-Verlag, New York, 2005.

[8] D. Hinrichsen and A.J. Pritchard (2000). *On the transient behaviour of stable linear systems.* Proc. 14th Int. Symp. Math. Theory Networks and Systems, Perpignan, France, 2000.

[9] Horn and Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1980.

[10] N. Karcanias, G. Halikias and A. Papageorgiou, *Strong Stability of Internal System Descriptions*, ECC Conference, Kos, Greece, 2007.

[11] I. Lewkowicz, *Convex invertible cones of matrices - a unified framework for the equations of Sylvester, Lyapunov and Riccati equations*, Linear Algebra and its Applications, 286, pp. $107-133$, 1999.

[12] D.O. Logofet, *Stronger-than-Lyapunov notions of matrix stability, or how flowers help solve problems in mathematical ecology*, Linear Algebra and its Applications, 398, pp. 75100, 2005.

[13] M. Marcus and H. Minc, *A survey of Matrix Theory and Matrix Inequalities*, Dover Publications, New York, 1964.

[14] O. Mason and R. Shorten, *On the simultaneous diagonal stability of a pair of positive linear systems*, Linear Algebra and its Applications, 413, pp. 1323, 2006.

[15] H.K. Khalil, *Nonlinear Systems*, MacMillan Publ. Comp., New York, 1992.

[16] von Elmar Plischke, *Transient Effects of Linear Dynamical Systems*, PhD thesis, University of Bremen, July 2005.

[17] J.E. Wilkinson, *The algebraic eigenvalue problem*, Oxford University Press, 1965.

[18] J.F. Whidborne, J. McKernan and G. Papadakis, *Minimal transient energy growth for plane poiseuille flow*, Proc. Inst. Mech. Eng., Part I: Journal of Systems and Control Engineering, 222(5):323-331, 2008.

[19] W.C. Yueh, *Eigenvalues of several tridiagonal matrices*, Applied Mathematics E-notes, 5:66-74, 2005.