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Distance Optimization and the Extremal Variety of the Grassmann variety $G_2(\mathbb{R}^n)$ *

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Abstract

The approximation of a multivector by a decomposable one is a distance-optimization problem between the multivector and the Grassmann variety of a projective space. When the multivector diverges from the Grassmann variety the approximate solution sought is the worst possible. In this paper it is showed that the worst solution of this problem is achieved when the eigenvalues of the matrix-form of a 2-vector are all equal. Then these solutions form a variety whose equation that describe it as well as its maximum distance from the related Grassmann variety are calculated. Several geometric and algebraic properties of this extremal variety are examined, providing a new aspect for the Grassmann varieties and the respective projective spaces.

Key words. Distance geometry problems, Optimization, Approximations, Projective varieties, Sums of squares and representations.

1 Introduction

The problem of approximating a multivector by a decomposable one is a key problem for many mathematical-based sciences; One of the most important and well-known implementations is met in control theory problems, where instead of deriving the actual controller for a system, its best approximation is calculated with the use of multivectors in a suitable projective space which are as closest to

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the corresponding Grassmann variety as possible, [Kar. 1], [Lev.1]. Approximate decomposability methods are also found in statistical analysis when dealing with least square-like problems, [Kan. 1], game theory and the so called *badly approximate n-tuples*, [Sch.1] and other fields. In general, the problem is viewed as the minimization problem

$$\min_{\underline{z}_1,\underline{z}_2\in\mathbb{R}^n} \|\underline{z}-\underline{z}_1\wedge\underline{z}_2\| \tag{1.1}$$

when $\underline{z}\in\wedge^2(\mathbb{R}^n)$. If the 2-vector $\underline{z}$ is as closest to the Grassmann variety $G_2(\mathbb{R}^n)$, [Hod. 1] as possible then a more satisfying solution for problem (1.1) may be achieved. This minimum distance case has been examined in [Lev.1] with the use of multivector decompositions, providing a closed-form solution of the problem. The worst case however, would be when $\underline{z}$ has the maximum distance from $G_2(\mathbb{R}^n)$.

In this paper we examine the case of those 2-vectors $\underline{z}\in\wedge^2(\mathbb{R}^n)$ for which the solution of (1.1) is the worst possible by considering them as elements of a new set which has the maximum distance from $G_2(\mathbb{R}^n)$. We prove that this set contains those 2-vectors $\underline{z}$ whose corresponding skew-symmetric matrix representations have equal eigenvalues and that this set is actually a variety, since it may be described by a polynomial equation. Moreover we provide a number of properties for its geometric and algebraic structure as well as its connection with the related Grassmann varieties, [Hod. 1] in the projective space.

Our extremal-variety approach may also constitute a very helpful tool for general approximation problems of the form

$$\min\|A-B\| \tag{1.2}$$

where one is looking to approximate a lower-rank matrix/tensor B from a nominal full-rank matrix/tensor A (if A is skew-symmetric and B is rank-one then the optimization problem (1.2) is equivalent to (1.1), [Land. 1]). Our approach will clearly show that when the eigenvalues of a multivector-related matrix are equal, the solution of (1.1) is the worst and for some Grassmann varieties their representations may actually be obtained, completing in this way the SVD-least squares techniques for matrices, [Eck. 1], [Gol. 1], as well as the lower-order tensor decomposition techniques, [Kol. 1], [Land. 1], in the $\wedge^2(\mathbb{R}^n)$ case, that usually examine the distinct singular values case only without providing adequate information about the solutions of (1.2) or (1.1) in cases of degeneracy, i.e., repeated or equal eigenvalues.

Throughout this paper the following notation is adopted: Scalars are denoted by lower-case letters, e.g., a, b, etc. Vectors and q-vectors (multivectors) are written as lower underline case letters, e.g., $\underline{x},\underline{y}$, etc. All q-vectors in this article are considered elements of the set $\wedge^q(\mathbb{R}^n)$ or its Hodge-dual, [Mar. 1], $\wedge^{n-q}(\mathbb{R}^n)$ where $q\leq n$. The respective Hodge-star operator for an n-dimensional oriented vector space $\mathcal{V}$ will be denoted as $(\cdot)^*$. The *wedge* or *exterior* product is denoted as $\underline{a}\wedge\underline{b}$. Matrices will be denoted in upper case, italic shape letters, e.g., X, Y, etc. To denote the *compound matrix*, [Mar. 2], of a matrix A, i.e., all the $q \times q$ minors of A we use the notation $C_q(A)$. A q-vector in $\wedge^q(\mathbb{R}^n)$ written as $\underline{a}_1 \wedge \underline{a}_2 \wedge \cdots \wedge \underline{a}_q$, with $\underline{a}_i$ in $\mathbb{R}^n$, $i = 1, ..., n$, will be called decomposable [Mar. 1], and equivalently $\underline{a}_1 \wedge \underline{a}_2 \wedge \cdots \wedge \underline{a}_q = C_q(A)$, $A := [\underline{a}_1, \underline{a}_2, ..., \underline{a}_q]$. Vector spaces, varieties or manifolds are denoted by calligraphic capital letters, e.g., $\mathcal{V}$. The Grassmann variety, [Hod. 1] of a real projective space $\mathbb{P}^{\binom{n}{q}-1}(\mathbb{R})$, i.e., the variety of the decomposable vectors of $\mathbb{P}^{\binom{n}{q}-1}(\mathbb{R})$, is denoted as $G_q(\mathbb{R}^n)$.

1.1 Background Results

In [Lev.1], it has been proved that a decomposition of a 2-vector $\underline{z} \in \wedge^2(\mathbb{R}^n)$ into a sum of decomposable 2-vectors may take the form

$$\underline{z} = \sigma_k \underline{x}_k + \sigma_{k-1}\underline{x}_{k-1} + ... + \sigma_1 \underline{x}_1 \tag{1.3}$$

where $\sigma_k \geq \sigma_{k-1} \geq \cdots \geq \sigma_1 \geq 0$ are the imaginary parts of the $k := [n/2]$ imaginary eigenvalues $\pm i\sigma_1, ..., \pm i\sigma_k$ of the matrix form of $\underline{z}$, i.e.,

$$T_{\underline{z}} = \begin{pmatrix} 0 & z_{12} & z_{13} & \cdots & z_{1,n} \\ -z_{12} & 0 & z_{23} & \cdots & z_{2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -z_{1,n-1} & \cdots & -z_{n-2,n-1} & 0 & z_{n-1,n} \\ -z_{1,n} & -z_{2,n} & \cdots & -z_{n-1,n} & 0 \end{pmatrix} \tag{1.4}$$

and $\underline{x}_k := \underline{e}_{2k} \wedge \underline{e}_{2k-1}, ..., \underline{x}_1 := \underline{e}_2 \wedge \underline{e}_1$, which correspond to the right complex eigenvectors (in $\mathbb{C}^{n\times 1}$): $\underline{e}_{2k} \pm i\underline{e}_{2k-1}, ..., \underline{e}_2 \pm i\underline{e}_1$ when $n = 2k$ and $0, \pm i\sigma_1, ..., \pm i\sigma_k$, $\underline{e}_{2k+1}, \underline{e}_{2k} \pm i\underline{e}_{2k-1}, ..., \underline{e}_2 \pm i\underline{e}_1$ when n is odd, with $\{\underline{e}_j\}_{j=1}^{2k}$, $\{\underline{e}_j\}_{j=1}^{2k+1}$ being two orthonormal bases for $\mathbb{R}^n$ when $n = 2k$, $n = 2k+1$, respectively. Decomposition (1.3) is referred to as the *prime decomposition*. With the use of this decomposition it was proved in [Lev.1] that standard distance formulae between "points"(equivalence classes) and "points" and subspaces in the projective space, [Wey. 1] may be expressed via s_i, i.e., the gap metric

$$\mathrm{gap}(\underline{z}, \underline{x}) = |\sin(\widehat{\underline{z}, \underline{x}})| = \min_{\lambda} \left\| \frac{\underline{z}}{\|\underline{z}\|} - \frac{\underline{x}}{\|\underline{x}\|} \cdot \lambda \right\| \tag{1.5}$$

between two "points" $\underline{z}, \underline{x}$ in the projective space has implied that the gap g between $\underline{z}$ and $G_2(\mathbb{R}^n)$ is equal to

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} \tag{1.6}$$

which is achieved at $\hat{\underline{z}} = \sigma_k \underline{x}_k$. Our aim in the following sections would be to elaborate on the degeneracy case $\sigma_1 = \cdots = \sigma_k = \sigma$ and the significance of this case to problem (1.1).

1.2 New Results and Contribution of the Paper

The key result of this paper is that when a 2-vector $\underline{z}$ is written as

$$\underline{z} = \sigma \underline{x}_k + \dots + \sigma \underline{x}_1 \tag{1.7}$$

then $\underline{z}$ has the maximum distance from all the decomposable vectors of the projective space, i.e., the Grassmann variety. Hence, the decomposable approximation of z (and equivalently its matrix rank-1 approximation) is the worst possible. Furthermore, if we define the extremal set which contains those 2-vectors of the form (1.7) as $\mathcal{V}_1$, then we may be able to find a polynomial equation that describes $\mathcal{V}_1$ as a projective variety and proceed to the solution of other related optimization problems, such as

$$\min_{z} g(\underline{z}, \mathcal{V}_1) \tag{1.8}$$

Another important result of this approach is that it gives a new geometric overview for some Grassmann varieties; we prove that for the Grassmann variety $G_2(\mathbb{R}^n)$, this new extremal variety $\mathcal{V}_1$ is pathwise connected, [Ful. 1] if $n \neq 4k$ and it is not connected for $n = 4k$. In addition, for the $G_2(\mathbb{R}^4)$ case, $\mathcal{V}_1$ is written as a union of two disjoint linear components. For the Grassmann variety $G_2(\mathbb{R}^5)$ we prove that we can actually find all the representations of (1.7), since as we show their are parametrized by the projective plane $\mathbb{P}^2(\mathbb{R})$, which is a very important result for vector and matrix decompositions with degeneracy issues. For the $G_2(\mathbb{R}^5)$ case, we also examine the sum of squares properties for the equation defying $\mathcal{V}_1$. Furthermore, we show that the gap between $\mathcal{V}_1$ and $G_2(\mathbb{R}^5)$ corresponds to a $\pi/4$ angle and that the two gaps, in a related projective space, of a vector from $\mathcal{V}_1$ and $G_2(\mathbb{R}^5)$ are complementary.

The paper is organized as follows: In Section 2 a new variety $\mathcal{V}_1$, as well as the equation that describes it, are presented. It is proved that this variety, consists of 2-vectors whose imaginary parts σ_i are equal and its gap from $G_2(\mathbb{R}^n)$ is maximal. The optimization problem of approximating a 2-vector by one in $\mathcal{V}_1$ (minimum distance from the extremal variety) is also presented. Section 3 is separated into 3 parts examining the application of the previous results to specific Grassmann varieties. In 3.1, the path-wise connectivity of $\mathcal{V}_1$ of $G_2(\mathbb{R}^n)$ is studied and in 3.2 all the parameterizations of a 2-vector with equal eigenvalues are presented for the $G_2(\mathbb{R}^5)$ case. In 3.3 it is showed that the equation describing $G_2(\mathbb{R}^5) \cup \mathcal{V}_1$ may be written as a Polynomial Sum of Squares. Finally, in 3.4 the complementarity of $G_2(\mathbb{R}^5)$ and $\mathcal{V}_1$ is examined in the related projective space, proving that $\mathcal{V}_1$ and $G_2(\mathbb{R}^5)$ form a constant $\pi/4$ angle between them.

2 The Extremal variety $\mathcal{V}_1$ of $G_2(\mathbb{R}^n)$

In this section we show that when the eigenvalues of a skew-symmetric matrix are equal, then the corresponding 2-vector belongs to a specific variety whose distance from the Grassmann variety $G_2(\mathbb{R}^n)$ is maximum.

Theorem 2.1. *Let a multi-vector $\underline{z} \in \wedge^2(\mathbb{R}^n)$ and its decomposition $\underline{z} = \sigma_k \underline{a}_k \wedge \underline{b}_k + ... + \sigma_1 \underline{a}_1 \wedge \underline{b}_1$ where $\sigma_k \geq \sigma_{k-1} \geq \cdots \geq \sigma_1 \geq 0$ and $\{\underline{a}_i, \underline{b}_i\}_{i=1}^k$ an orthonormal set. Then z has the maximum distance from the Grassmann variety $G_2(\mathbb{R}^n)$, if and only if $\underline{z} = \sigma(\underline{x}_k + \underline{x}_{k-1} + ... + \underline{x}_1)$.*

Proof. As shown in [Lev.1]

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\sqrt{\sigma_k^2 + \sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}} \tag{2.1}$$

Therefore

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{1}{\sqrt{1 + \dfrac{\sigma_k^2}{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}} = \sqrt{\frac{1}{1 + \dfrac{1}{s_{k-1} + s_{k-2} + \cdots + s_1}}} \tag{2.2}$$

where $s_i = \sigma_i^2/\sigma_k^2$. Thus, in order to maximize $g\left(\underline{z}, G_2(\mathbb{R}^n)\right)$, we need to solve the maximization problem

$$\max(s_{k-1} + s_{k-2} + \cdots + s_1) \ s.t. \ 1 \geq s_{k-1} \geq s_{k-2} \geq \cdots \geq s_1 \tag{2.3}$$

Problem (2.3) has the obvious solution $1 = s_{k-1} = \cdots = s_1$ which implies $\sigma_k = \sigma_{k-1} = \sigma_{k-2} = ... = \sigma_1 = \sigma$. Hence, the maximum distance from the Grassmann variety is achieved by those vectors of the form $\underline{z} = \sigma(\underline{a}_k \wedge \underline{b}_k + ... + \underline{a}_1 \wedge \underline{b}_1)$. □

We may now connect the maximum distance result in [Lev.1] with the degeneracy case $s = s_i, \ i = 1, ...k$.

Corollary 2.1. *The maximum distance from the Grassmann variety $G_2(\mathbb{R}^n$ is achieved by those vectors in $\wedge^2(\mathbb{R}^n)$ whose distance is $\sqrt{1 - 1/k}$.*

Proof. We proved that the maximum distance from $G_2(\mathbb{R}^n$ is achieved by vectors of the form $\underline{z} = \sigma(\underline{a}_k \wedge \underline{b}_k + ... + \underline{a}_1 \wedge \underline{b}_1)$. Hence

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\sqrt{\sigma_k^2 + \sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}} = \sqrt{\frac{(k-1)\sigma^2}{k\sigma^2}} = \sqrt{1 - 1/k}$$

Conversely, if $g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \sqrt{1 - 1/k}$ then

$$\frac{\sigma_k^2}{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2} = \frac{1}{k-1}$$

which implies

$$\underbrace{\sigma_k^2 + \cdots + \sigma_k^2}_{k-1 \text{ times}} = \sigma_{k-1}^2 + \cdots + \sigma_1^2$$

which in turn implies $\sigma_k = \sigma_{k-1} = \cdots = \sigma_1$. $\square$

The above results imply that when the eigenvalues of a skew-symmetric matrix are all equal, then its corresponding 2-vector is an element of a set whose distance from the Grassmann variety is maximum. Therefore, the respective decomposable approximation of the form $\sigma \underline{a}_i \wedge \underline{b}_i$ is the worst possible. In the next theorem we show that $\mathcal{V}_1$ is a projective variety where we derive the equation that describes it.

Theorem 2.2. *The set $\mathcal{V}_1$ is a variety described by the equation*

$$\|\underline{z} \wedge \underline{z}\| = \sqrt{2\left(1 - \frac{1}{k}\right)} \cdot \|\underline{z}\|^2 \tag{2.4}$$

Proof. $(\Rightarrow)$ If $\underline{z} \in \mathcal{V}_1$ then, due to Theorem 2.1, $\underline{z}$ is written as $\underline{z} = \sigma \underline{x}_1 + \sigma \underline{x}_2 + \ldots + \sigma \underline{x}_k$. Therefore, $\|\underline{z}\|^2 = k\sigma^2$ and $\|\underline{z} \wedge \underline{z}\| = 2\sigma^2 \sqrt{k(k-1)/2}$ because

$$\begin{aligned}
\|\underline{z} \wedge \underline{z}\|^2 &= 4\sigma^4 \|\underline{x}_1 \wedge \underline{x}_2 + \cdots + \underline{x}_{k-1} \wedge \underline{x}_k\|^2 = \\
&= 4\sigma^4 < \underline{x}_1 \wedge \underline{x}_2 + \cdots + \underline{x}_{k-1} \wedge \underline{x}_k, \underline{x}_1 \wedge \underline{x}_2 + \cdots + \underline{x}_{k-1} \wedge \underline{x}_k >= \\
&= 4\sigma^4 \left((k-1) + (k-2) + \cdots + 1\right) = 4\sigma^4 \frac{k(k-1)}{2}
\end{aligned}$$

From these two equations we obtain

$$\|\underline{z} \wedge \underline{z}\| = \frac{\sqrt{2k(k-1)}}{k} \cdot \|\underline{z}\|^2 = \sqrt{2\left(1 - \frac{1}{k}\right)} \cdot \|\underline{z}\|^2$$

$(\Leftarrow)$ As shown in [Lev.1], if

$$\underbrace{\underline{z} \wedge \cdots \wedge \underline{z}}_{\mu - factors} \equiv \underline{z}^{\wedge \mu} \tag{2.5}$$

then

$$\underline{z}^{\wedge \mu} = \mu! \sum_{1 \le i_1 < \ldots < i_\mu \le k} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_\mu} \underline{x}_{i_1} \wedge \underline{x}_{i_2} \wedge \ldots \wedge \underline{x}_{i_\mu},\ 2 \le \mu \le k \tag{2.6}$$

If $\|\underline{z} \wedge \underline{z}\| = \sqrt{2\left(1 - \frac{1}{k}\right)} \cdot \|\underline{z}\|^2$ then with the use of formula (2.6), we have that

$$(k-1) \sum_{i=1}^{k} \sigma_i^4 - 2 \sum_{\substack{i=1 \\ k > j > i}}^{k} \sigma_i^2 \sigma_j^2 = 0 \tag{2.7}$$

But

$$\sum_{\substack{i=1\\k>j>i}}^{k}(\sigma_i^2-\sigma_j^2)^2=\underbrace{(\sigma_1^4+\sigma_2^4+...+\sigma_k^4)+...+(\sigma_1^4+\sigma_2^4+...+\sigma_k^4)}_{k-1\text{ times}}-2\sum_{\substack{i=1\\k>j>i}}^{k}\sigma_i^2\sigma_j^2 \tag{2.8}$$

Therefore, from equations (2.7), (2.8) we have that

$$\sigma_i=\sigma_j,\ \forall i=1,2,...,k,\ k>j>i.$$

□

Corollary 2.2. *Let $\underline{z}=\sigma(\underline{x}_k+\underline{x}_{k-1}+...+\underline{x}_1)$. Then $\mathcal{V}_1$ can be also described by the equation*

$$\left(\frac{\|\underline{z}^{\wedge\mu}\|}{\mu!}\right)^2=\left(\frac{\|\underline{z}^2\|}{k}\right)^\mu\cdot\binom{k}{\mu} \tag{2.9}$$

Proof. Again, with the use of (2.6), for $\mu\leq k$, we have that

$$\|\underline{z}^{\wedge\mu}\|^2=(\mu!)^2\sigma^{2\mu}\cdot\binom{k}{\mu} \tag{2.10}$$

and since $\underline{z}=\sigma(\underline{x}_k+...+\underline{x}_1)$ we have that $\|\underline{z}\|^2=k\sigma^2$ and the result readily follows. □

We may now obtain the following definition.

Definition 2.1. *The extremal variety $\mathcal{V}_1:=Extr\,(G_2(\mathbb{R}^n))$ is the variety containing all points of $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$ of the form $\underline{z}=\sigma(\underline{a}_k\wedge\underline{b}_k+...+\underline{a}_1\wedge\underline{b}_1)$ that achieve the maximum distance from the Grassmann variety $G_2(\mathbb{R}^n)$.*

The derivation of the equation that describes $\mathcal{V}_1$ is very helpful, since we may now solve other optimization problems related to (1.1). We present the least distance problem from $\mathcal{V}_1$. Note that, optimization problems subject to manifold constraints, are usually addressed algorithmically via appropriate Numeric Algebraic Geometry Toolbox, [Eis. 1]. But in this case, a closed form solution may be obtained, as we show next.

Theorem 2.3. *Let the prime decomposition of a 2-vector $\underline{z}$ be written as $\underline{z}:=\sum_{i=1}^k\sigma_i\underline{x}_i\wedge\underline{y}_i$. The distance between $\underline{z}$ and $\mathcal{V}_1$ is equal to*

$$g\left(\underline{z},\mathcal{V}_1\right)=\sum_{i=1}^{k}(\sigma_i-\bar{\sigma})^2 \tag{2.11}$$

and is realized for $\underline{v}_0=\bar{\sigma}\sum_{i=1}^k\underline{x}_i\wedge\underline{y}_i$, where $\bar{\sigma}=\sum_{i=1}^k\sigma_i/n$.

Proof. Let $\underline{v} \in \mathcal{V}_1$. Then $\underline{v} = \sigma \sum_{i=1}^{k} \underline{a}_i \wedge \underline{b}_i$. Therefore, due to the (generalized) Cauchy-Schwartz Type Inequality, [Lev.2]

$$< \underline{z}_1, \underline{z}_2 > \leq \sum_{i=1}^{k} \sigma_i s_i \tag{2.12}$$

for two prime decompositions $\underline{z}_1 = \sum_{i=1}^{k} \sigma_i \underline{x}_{2i-1} \wedge \underline{x}_{2i}$, $\underline{z}_2 = \sum_{i=1}^{k} s_i \underline{y}_{2i-1} \wedge \underline{y}_{2i}$ we have that

$$\begin{aligned}
\|\underline{z} - \underline{v}\|^2 &= \left\| \sum_{i=1}^{k} \sigma_i \underline{x}_i \wedge \underline{y}_i - \sigma \sum_{i=1}^{k} \underline{a}_i \wedge \underline{b}_i \right\|^2 = \\
&= \sum_{i=1}^{k} \sigma_i^2 + n\sigma^2 - 2 < \sum_{i=1}^{k} \sigma_i \underline{x}_i \wedge \underline{y}_i, \sigma \sum_{i=1}^{k} \underline{a}_i \wedge \underline{b}_i > \geq \\
&\geq \sum_{i=1}^{k} \sigma_i^2 + n\sigma^2 - 2\sigma \sum_{i=1}^{k} \sigma_i := f(\sigma)
\end{aligned}$$

But $f(\sigma)$ is a quadratic and is minimized when $f'(\sigma) = 0$, i.e. $\sigma = \sum_{i=1}^{k} \sigma_i / n := \bar{\sigma}$. In this case we have that

$$f(\sigma) = \sum_{i=1}^{k} (\sigma_i - \bar{\sigma})^2 \tag{2.13}$$

and its minimizer is $\underline{v}_0 = \bar{\sigma} \sum_{i=1}^{k} \underline{x}_i \wedge \underline{y}_i$. $\square$

3 Properties of the Extremal Variety $\mathcal{V}_1$

Our aim in this section is to provide a number of properties of the newly defined set $\mathcal{V}_1$ and to obtain a new aspect for the algebraic and geometric structure of the Grassmann varieties. The properties usually examined for projective varieties, involve connectedness, [Ful. 1], quadratic forms and sums of squares properties, [Mum. 1], intersections-unions, [Net. 1], [Hod. 1] and others of similar nature, [Cil. 1]. The properties of $\mathcal{V}_1$ may act complementary to the rest of these properties and be very helpful for the examination of the geometry of the Grassmann varieties, [Koz. 1]. We examine the path-wise connectivity of $\mathcal{V}_1$ for $G_2(\mathbb{R}^n)$ and we show that all the 2-vectors with equal eigenvalues are parameterized by the projective plane for the $G_2(\mathbb{R}^5)$ case. For the same Grassmann variety the sum of squares properties is examined and we elaborate on the complementary between $\mathcal{V}_1$ and $G_2(\mathbb{R}^5)$. The latter, will imply that the gap between $\mathcal{V}_1$ and $G_2(\mathbb{R}^5)$ in the projective space is equal to $1/\sqrt{2}$ which corresponds to a $\pi/4$ angle, which is a very important new result for the geometry of the Grassmann varieties.

3.1 The path-wise connectivity of $\mathcal{V}_1$

At first, we will need the following definition.

Definition 3.1. *Let $\underline{z} \in \wedge^2(\mathbb{R}^{2k+1})$. If $\underline{z}^{\wedge k} \neq \underline{0}$, we define the vector $\underline{r}_{\underline{z}}$ as*

$$\underline{r}_{\underline{z}} := \frac{\left(\underline{z}^{\wedge k}\right)^*}{\|\underline{z}^{\wedge k}\|} \tag{3.1}$$

Theorem 3.1. *(Pathwise Connectedness) Let the Grassmann variety $G_2(\mathbb{R}^n)$.*

i) If $n = 2k+1$ then the extremal variety $\mathcal{V}_1$ of $G_2(\mathbb{R}^n)$ is pathwise connected.

ii) If $n = 4k+2$ then $\mathcal{V}_1$ is path-wise connected.

iii) If $n = 4k$ then $\mathcal{V}_1$ is not pathwise connected.

Proof. i) Let $\mathrm{sp}\{\underline{z}_1\}, \mathrm{sp}\{\underline{z}_2\}$ two elements of $\mathcal{V}_1$. Then,

$$\underline{z}_1 = \frac{1}{\sqrt{k}} \cdot (\underline{a}_1 \wedge \underline{b}_1 + \cdots \underline{a}_k \wedge \underline{b}_k)\,,\ \underline{z}_2 = \frac{1}{\sqrt{k}} \cdot (\underline{a}'_1 \wedge \underline{b}'_1 + \cdots \underline{a}'_k \wedge \underline{b}'_k)$$

with $\{\underline{a}_i, \underline{b}_i\}_{i=1}^k$, $\{\underline{a}'_i, \underline{b}'_i\}_{i=1}^k$ being two orthonormal sets. We consider their corresponding spectral matrices:

$$U_{\mathrm{sp}\{\underline{z}_1\}} := \left(\underline{a}_1, \underline{b}_1, \underline{a}_2, \underline{b}_2, ..., \underline{r}_{\underline{z}_1}\right),\ U_{\mathrm{sp}\{\underline{z}_2\}} := \left(\underline{a}'_1, \underline{b}'_1, \underline{a}'_2, \underline{b}'_2, ..., \underline{r}_{\underline{z}_2}\right)$$

both in $\mathrm{SO}_n(\mathbb{R})$. Now, let $U := U^{-1}_{\mathrm{sp}\{\underline{z}_1\}} \cdot U_{\mathrm{sp}\{\underline{z}_2\}} \in \mathrm{SO}_n(\mathbb{R})$. We consider a skew-symmetric matrix A, such that $e^A = U$. Then the path $U(t) = U_{\mathrm{sp}\{\underline{z}_1\}} \cdot e^{At}$, connects $U_{\mathrm{sp}\{\underline{z}_1\}}, U_{\mathrm{sp}\{\underline{z}_2\}}$ in $\mathrm{SO}_n(\mathbb{R})$. Indeed, $e^A \in \mathrm{SO}_n(\mathbb{R})$ (hence $U(t) \in \mathrm{SO}_n(\mathbb{R})$) and $U(0) = U_{\mathrm{sp}\{\underline{z}_1\}}, U(1) = U_{\mathrm{sp}\{\underline{z}_1\}} \cdot U^{-1}_{\mathrm{sp}\{\underline{z}_1\}} \cdot U_{\mathrm{sp}\{\underline{z}_2\}} = U_{\mathrm{sp}\{\underline{z}_2\}}$. Therefore, if $U(t) := \left(\underline{a}_1(t), \underline{b}_1(t), ..., \underline{a}_k(t), \underline{b}_k(t), \underline{r}_{\underline{z}}(t)\right)$, then

$$\underline{z}(t) = 1/\sqrt{k} \cdot (\underline{a}_1(t) \wedge \underline{b}_1(t) + \cdots + \underline{a}_k(t) \wedge \underline{b}_k(t)) \tag{3.2}$$

connects $\mathrm{sp}\{\underline{z}_1\}, \mathrm{sp}\{\underline{z}_2\}$ in $\mathcal{V}_1$.

ii) If $n = 4k+2$ and $\mathrm{sp}\{\underline{z}\} \in \mathcal{V}_1$ then $\underline{z} = 1/\sqrt{2k+1} \cdot \left(\underline{a}_1 \wedge \underline{b}_1 + \cdots \underline{a}_{2k+1} \wedge \underline{b}_{2k+1}\right)$ or $-\underline{z} = 1/\sqrt{2k+1} \cdot \left(\underline{b}_1 \wedge \underline{a}_1 + \cdots \underline{b}_{2k+1} \wedge \underline{a}_{2k+1}\right)$, for $\underline{z}$, $-\underline{z} \in \mathrm{sp}\{\underline{z}\}$. Then, $U_{\underline{z}} := \left(\underline{a}_1, \underline{b}_1, \underline{a}_2, \underline{b}_2, ..., \underline{a}_{2k+1}, \underline{b}_{2k+1}\right)$ or $U_{-\underline{z}} := \left(\underline{b}_1, \underline{a}_1, \underline{b}_2, \underline{a}_2, ..., \underline{b}_{2k+1}, \underline{a}_{2k+1}\right)$ should belong to $\mathrm{SO}_n(\mathbb{R})$. We define $U_{\mathrm{sp}\{\underline{z}\}}$ to be the one of the two matrices $U_{\underline{z}}$, $U_{-\underline{z}}$ in $\mathrm{SO}_n(\mathbb{R})$. In this setting, if we take two elements $\mathrm{sp}\{\underline{z}_1\}, \mathrm{sp}\{\underline{z}_2\} \in \mathcal{V}_1$, we let $U := U^{-1}_{\mathrm{sp}\{\underline{z}_1\}} \cdot U_{\mathrm{sp}\{\underline{z}_2\}} \in \mathrm{SO}_n(\mathbb{R})$. We consider the skew-symmetric matrix A such that $e^A = U$. Then the path $U(t) = U_{\mathrm{sp}\{\underline{z}_1\}} \cdot e^{At}$, connects $U_{\mathrm{sp}\{\underline{z}_1\}}, U_{\mathrm{sp}\{\underline{z}_2\}}$ in $\mathrm{SO}_n(\mathbb{R})$ and therefore, if $U(t) := \left(\underline{a}_1(t), \underline{b}_1(t), ..., \underline{a}_{2k+1}(t), \underline{b}_{2k+1}(t)\right)$, then the path

$$\underline{z}(t) = 1/\sqrt{2k+1} \cdot \left(\underline{a}_1(t) \wedge \underline{b}_1(t) + \cdots + \underline{a}_{2k+1}(t) \wedge \underline{b}_{2k+1}(t)\right) \tag{3.3}$$

for $t \in [0, 1]$ connects $\mathrm{sp}\{\underline{z}_1\}, \mathrm{sp}\{\underline{z}_2\}$ in $\mathcal{V}_1$.

iii) If $n = 4k$ we define the map $T : \mathcal{V}_1 \to \{-1, 1\}$ such that

$$T(\underline{z}) = \frac{\sqrt{2k}^{2k}\left(\underline{z}^{\wedge 2k}\right)^{\star}}{(2k)!\|\underline{z}\|^{2k}} \tag{3.4}$$

where

$$\underbrace{\underline{z} \wedge \cdots \wedge \underline{z}}_{2k-factors} \equiv \underline{z}^{\wedge 2k} \tag{3.5}$$

This map is well defined as

$$T(\lambda\underline{z}) = \frac{\sqrt{2k}^{2k}}{(2k)!}\frac{\lambda^{2k}\left(\underline{z}^{\wedge 2k}\right)^{\star}}{\lambda^{2k}\|\underline{z}\|^{2k}} = T(\underline{z})$$

and if $\underline{z} = \sigma(\underline{a}_1 \wedge \underline{b}_1 + \cdots + \underline{a}_{2k} \wedge \underline{b}_{2k})$ then $\underline{z}^{\wedge 2k} = (2k)!\sigma^{2k}\underline{a}_1 \wedge \underline{b}_1 \wedge \cdots \wedge \underline{a}_{2k} \wedge \underline{b}_{2k}$, with $\|\underline{z}\|^{2k} = (2k)^k\sigma^{2k}$ (since $\|\underline{z}\|^2 = 2k\sigma^2$). Therefore,

$$|T(z)| = \frac{\sqrt{2k}^{2k}(2k)!\sigma^{2k}}{(2k)!(2k)^k\sigma^{2k}} = 1 \tag{3.6}$$

Hence, $\mathcal{V}_1$ can be written as a disjoint union of $T^{-1}(-1) \cup T^{-1}(1)$.

□

Moreover, for the $G_2(\mathbb{R}^4)$ case one may imply a more explicit result regarding the connectivity of $\mathcal{V}_1$.

Corollary 3.1. *The extremal variety $\mathcal{V}_1$ of $G_2(\mathbb{R}^4)$ is not connected and it is a union of two disjoint linear components.*

Proof. If $\underline{z} = (z_1, z_2, ..., z_6) \in \wedge^2(\mathbb{R}^4)$ then the equation describing $\mathcal{V}_1$ is

$$\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2 = 0 \Leftrightarrow \left(\sum_{i=1}^{6} z_i^2\right)^2 - 4(z_1z_6 - z_2z_5 + z_3z_4)^2 = 0$$

or equivalently

$$\left[(z_1 - z_6)^2 + (z_2 + z_5)^2 + (z_3 - z_4)^2\right]\left[(z_1 + z_6)^2 + (z_2 - z_5)^2 + (z_3 + z_4)^2\right] = 0$$

Hence, $\mathcal{V}_1$ is a union of two linear sets in $\mathbb{P}^5(\mathbb{R})$, i.e., $\mathcal{V}_1^1 = \{\underline{z} : \underline{z} = \underline{z}^{\star}\}$, $\mathcal{V}_1^2 = \{\underline{z} : \underline{z} = -\underline{z}^{\star}\}$ which are obviously disjoint in $\mathbb{P}^5(\mathbb{R})$. □

Example 3.1. *We will construct a path between two elements of*

$$\underline{z}_1 = \frac{1}{\sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2)\,,\ \underline{z}_2 = \frac{1}{\sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 - \underline{b}_1 \wedge \underline{b}_2)$$

of $\mathcal{V}_1$ for the Grassmann variety $G_2(\mathbb{R}^5)$. For these elements we have $U_{\underline{z}_1} := (\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2, \underline{r})$, $U_{\underline{z}_2} := (\underline{a}_3, \underline{a}_4, \underline{b}_3, \underline{b}_4, -\underline{r})$. Therefore

$$U = U_{\underline{z}_1}^{-1} \cdot U_{\underline{z}_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and since $e^A = U$, then

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -\frac{\pi}{\sqrt{2}} \\ 0 & 0 & -\frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}} & 0 \end{pmatrix}, \; e^{At} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1+\cos \pi t}{2} & \frac{1-\cos \pi t}{2} & \frac{\sin \pi t}{\sqrt{2}} \\ 0 & 0 & \frac{1-\cos \pi t}{2} & \frac{1+\cos \pi t}{2} & -\frac{\sin \pi t}{\sqrt{2}} \\ 0 & 0 & -\frac{\sin \pi t}{\sqrt{2}} & \frac{\sin \pi t}{\sqrt{2}} & \cos \pi t \end{pmatrix}$$

Hence, $U(t) = U_{\underline{z}_1} \cdot e^{At} = (\underline{a}_1(t), \underline{a}_2(t), \underline{b}_1(t), \underline{b}_2(t), \underline{r}(t))$, where $\underline{a}_1(t) \equiv \underline{a}_1$, $\underline{a}_2(t) \equiv \underline{a}_2$, $\underline{b}_1(t) \equiv \dfrac{1+\cos \pi t}{2} \cdot \underline{b}_1 + \dfrac{1-\cos \pi t}{2} \cdot \underline{b}_2 - \dfrac{\sin \pi t}{\sqrt{2}} \cdot \underline{r}$, $\underline{b}_2(t) \equiv \dfrac{1-\cos \pi t}{2} \cdot \underline{b}_1 + \dfrac{1+\cos \pi t}{2} \cdot \underline{b}_2 + \dfrac{\sin \pi t}{\sqrt{2}} \cdot \underline{r}$, $\underline{r}(t) \equiv \dfrac{\sin \pi t}{\sqrt{2}} \cdot \underline{b}_1 - \dfrac{\sin \pi t}{\sqrt{2}} \cdot \underline{b}_2 + \cos \pi t$. Hence, $\underline{z}(t)$ is given by

$$\underline{z}(t) = \frac{1}{\sqrt{2}} \cdot \left(\underline{a}_1(t) \wedge \underline{a}_2(t) + \underline{b}_1(t) \wedge \underline{b}_2(t)\right),$$

where $\underline{z}(t) \in \mathcal{V}_1,\ \underline{z}(0) = \underline{z}_1,\ \underline{z}(1) = \underline{z}_2,\ \forall t \in [0,1]$

In the next sections we investigate the special properties that $\mathcal{V}_1$ may have in relation with specific Grassmann varieties. We will present the results for the $G_2(\mathbb{R}^5)$ case.

3.2 $\mathcal{V}_1$ and the Uniqueness of Decompositions

In this section we prove a result that clearly shows the advantages of the extremal-variety approach for best-approximation problems of the form (1.1); it reveals the representation of the non-unique decompositions in the case of equal eigenvalues, something not feasible with the SVD-low rank techniques for matrices or even tensors, we mentioned in Section 1 that work for decompositions with distinct eigenvalues.

Proposition 3.1. *Let $\underline{z} \in \mathcal{V}_1$ written as*

$$\underline{z} = \sigma \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma \cdot \underline{b}_1 \wedge \underline{b}_2 \tag{3.7}$$

for $n = 5$. Then decomposition (3.7) is not unique and all representations are parameterized by $P^2(\mathbb{R})$.

Proof. If $\underline{z} = \sigma \cdot \underline{a}'_1 \wedge \underline{a}'_2 + \sigma \cdot \underline{b}'_1 \wedge \underline{b}'_2$ is a second representation of $\underline{z}$, then $\mathrm{colspan}[\underline{b}_1, \underline{b}_2, \underline{a}_1, \underline{a}_2] = \mathrm{colspan}[\underline{b}'_1, \underline{b}'_2, \underline{a}'_1, \underline{a}'_2]$. If we consider a matrix U such that $[B, A] \cdot U = [B', A']$, $U = [U_1, U_2]$, where $B = [\underline{b}_1, \underline{b}_2]$, $A = [\underline{a}_1, \underline{a}_2]$, $B' = [\underline{b}'_1, \underline{b}'_2]$, $A' = [\underline{a}'_1, \underline{a}'_2]$, then we get $\underline{b}'_1 \wedge \underline{b}'_2 = C_2[B, A] \cdot C_2[U_1]$ and $\underline{a}'_1 \wedge \underline{a}'_2 = C_2[B, A] \cdot C_2[U_2]$, where $C_2[U_1], C_2[U_2] \in \wedge(\mathbb{R}^4)$. Hence, if $\underline{x} := C_2[U_1]$ then $\underline{x}^* := C_2[U_2]$. Therefore,

$$\sigma \cdot \underline{a}'_1 \wedge \underline{a}'_2 + \sigma \cdot \underline{b}'_1 \wedge \underline{b}'_2 = C_2[B, A]\,(\sigma \underline{x} + \sigma \underline{x}^*) \tag{3.8}$$

and

$$\sigma \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma \cdot \underline{b}_1 \wedge \underline{b}_2 = C_2[B, A]\,(\sigma \underline{e}_1 + \sigma \underline{e}_6) \tag{3.9}$$

Thus if, $C_2[B, A]\,(\sigma \underline{x} + \sigma \underline{x}^*) = C_2[B, A]\,(\sigma \underline{e}_1 + \sigma \underline{e}_6)$, by taking the left inverse matrix of $C_2[B, A]$ we have that $\sigma \underline{x} + \sigma \underline{x}^* = \sigma \underline{e}_1 + \sigma \underline{e}_6$. Hence, by applying the Hodge star operator we obtain

$$\sigma \cdot (\underline{x} + \underline{x}^*) = \sigma \cdot (\underline{e}_1 + \underline{e}_6) \tag{3.10}$$

If $\underline{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$, then $\underline{x}^* = (x_6, -x_5, x_4, x_3, -x_2, x_1)$ and from eqn.(3.10),we have that

$$x_1 + x_6 = 1,\; x_2 - x_5 = 0,\; x_3 + x_4 = 0 \tag{3.11}$$

Also, since $\underline{x}$ is decomposable in $\wedge^2(\mathbb{R}^4)$, it satisfies the unique Quadratic Plücker Relation

$$x_1 x_6 - x_2 x_5 + x_3 x_4 = 0 \tag{3.12}$$

Therefore, eqn.(3.12) due to equations (3.11) and the fact that $\|\underline{x}\| = 1$, is equivalent to

$$\left(x_1 - \frac{1}{2}\right)^2 + x_2^2 + x_3^2 = \frac{1}{4} \tag{3.13}$$

Hence, the representation of $\underline{z}$ is not unique. Also, the pair $(\underline{x}^*, \underline{x})$ corresponds to $\sigma \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma \cdot \underline{b}_1 \wedge \underline{b}_2$, whereas $(\underline{x}, \underline{x}^*)$ to $\sigma \cdot \underline{b}_1 \wedge \underline{b}_2 + \sigma \cdot \underline{a}_1 \wedge \underline{a}_2$, which are the same representatives. Hence, we have identified $\underline{x}, \underline{x}^*$ i.e., the antipodal points of the sphere above, which gives rise to the projective space $P^2(\mathbb{R})$. □

Note that, while $G_2(\mathbb{R}^3)$ is isomorphic to the projective plane, [Hod. 1], [Mar. 1], the extremal variety $\mathcal{V}_1$ of $G_2(\mathbb{R}^5)$ is parameterized by it, as the above proposition implies. Hence, there is a strong algebro-geometric connection between $\mathcal{V}_1$ and $G_2(\mathbb{R}^5)$. The next properties will verify this allegation.

3.3 Polynomial Sum of Squares

In this section we continue the investigation for the algebrogeometric structure of $\mathcal{V}_1$ and specifically we examine to what extent the polynomial $\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2$ implied by the equation $\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\| = 0$ that describes $\mathcal{V}_1$ may be written as a *Sum of Squares(SOS)*. This is a common problem in algebraic and projective varieties, [Choi. 1], [Mum. 1]. It can be proved, using the Matlab SOSTOOLS toolbox, [Pra. 1], that $\|\underline{z}\|^4 - \|\underline{z}\wedge\underline{z}\|^2$ can not be written as a SOS. Instead, we will prove that this is feasible for the variety $G_2(\mathbb{R}^5)\cup\mathcal{V}_1$.

Definition 3.2. *Let the* 10×2 *matrix* $A = (\underline{z}\wedge(\underline{z}\wedge\underline{z})^*, \underline{z}^*)$ *for* $\underline{z}\in\wedge^2(\mathbb{R}^5)$. *The fourth degree homogeneous polynomials* $f_i(\underline{z}), i = 1, ..., 45$ *are defined as the* 2×2 *minors of* A, *i.e.,* $C_2(A) = (f_1(\underline{z}), f_2(\underline{z}, ..., f_{45}(\underline{z})))^t$

Theorem 3.2. *Let* $\underline{z}\in\wedge^2(\mathbb{R}^5)$. *Then* $\|\underline{z}\wedge\underline{z}\|^2\cdot(\|\underline{z}\|^4 - \|\underline{z}\wedge\underline{z}\|^2) = \sum_{i=1}^{45} f_i^2(\underline{z})$.

Proof.

$$\sum_{i=1}^{45} f_i^2(\underline{z}) = C_2(A^t)\cdot C_2(A) = det\left(A^t\cdot A\right) =$$

$$= \begin{vmatrix} \|\underline{z}\wedge(\underline{z}\wedge\underline{z})^*\|^2 & <\underline{z}\wedge(\underline{z}\wedge\underline{z})^*, \underline{z}^*> \\ <\underline{z}\wedge(\underline{z}\wedge\underline{z})^*, \underline{z}^*> & \|\underline{z}\|^2 \end{vmatrix} =$$

$$= \|\underline{z}\|^2\cdot\|\underline{z}\wedge(\underline{z}\wedge\underline{z})^*\|^2 - \|\underline{z}\wedge\underline{z}\wedge(\underline{z}\wedge\underline{z})^*\|^2 =$$

$$= \|\underline{z}\|^2\cdot\|\underline{z}\|^2\cdot\|\underline{z}\wedge\underline{z}\|^2 - \|\underline{z}\wedge\underline{z}\|^4 = \|\underline{z}\wedge\underline{z}\|^2\cdot\left(\|\underline{z}\|^4 - \|\underline{z}\wedge\underline{z}\|^2\right)$$

□

Corollary 3.2. *The polynomial* $\|\underline{z}\wedge\underline{z}\|^2(\|\underline{z}\|^4 - \|\underline{z}\wedge\underline{z}\|^2)$ *whose zero locus defines the variety* $G_2(\mathbb{R}^5)\cup\mathcal{V}_1$ *is a polynomial SOS.*

Proof. This is straight-forward by the fact that $\underline{z}\wedge\underline{z} = \underline{0}$ defines $G_2(\mathbb{R}^5)$ and $\|\underline{z}\|^4 - \|\underline{z}\wedge\underline{z}\|^2 = 0$ defines $\mathcal{V}_1$. □

Now, the next corollary is evident.

Corollary 3.3. *The equations* $f_i(\underline{z}) = 0$ *define the variety* $G_2(\mathbb{R}^5)\cup\mathcal{V}_1$.

3.4 The complementarity of $G_2(\mathbb{R}^5)$ and $\mathcal{V}_1$

In this section we prove that the gaps between a fixed 2-vector $\underline{z}$ and the varieties $G_2(\mathbb{R}^5), \mathcal{V}_1$ are complementary. First we will need the $\mathcal{V}_1$-representation of a 2-vector.

Definition 3.3. *The* $\mathcal{V}_1$*-decomposition of a 2-vector* $\underline{z}\in\wedge^2(\mathbb{R}^5)$ *whose prime decomposition is* $\underline{z} = \sigma_1\cdot\underline{a}_1\wedge\underline{a}_2 + \sigma_2\cdot\underline{b}_1\wedge\underline{b}_2$ *is defined as*

$$\underline{z} = \frac{\sigma_2+\sigma_1}{2}\cdot(\underline{a}_1\wedge\underline{a}_2 + \underline{b}_1\wedge\underline{b}_2) + \frac{\sigma_2-\sigma_1}{2}\cdot(\underline{b}_1\wedge\underline{b}_2 - \underline{a}_1\wedge\underline{a}_2) \qquad (3.14)$$

From Theorem (2.3) and for $n = 5$ we see that the gap $g(\underline{z}, \mathcal{V}_1)$ between a 2-vector $\underline{z}$ and $\mathcal{V}_1$ is given by

$$g(\underline{z}, \mathcal{V}_1) = \frac{\sigma_2 - \sigma_1}{\sqrt{2}\sqrt{\sigma_2^2 + \sigma_1^2}} = \frac{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{\sqrt{2}\|\underline{z}\|} = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\|\underline{z} \wedge \underline{z}\|}{\|\underline{z}\|^2}} \tag{3.15}$$

Next theorem is one of the most important results of this paper, linking geometrically $G_2(\mathbb{R}^5)$ with $\mathcal{V}_1$.

Theorem 3.3. *If θ_1, θ_2 are the gap angles of $\underline{z} \in \mathbb{P}^9(\mathbb{R})$ from the varieties $G_2(\mathbb{R}^5)$ and $\mathcal{V}_1$ respectively, then*

$$\theta_1 + \theta_2 = \frac{\pi}{4}$$

Proof. The prime decomposition of any $\underline{z} \in \wedge^2(\mathbb{R}^5)$ is

$$\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 = \|\underline{z}\| \cdot \left(\frac{\sigma_2}{\|\underline{z}\|}\underline{b}_1 \wedge \underline{b}_2 + \frac{\sigma_1}{\|\underline{z}\|}\underline{a}_1 \wedge \underline{a}_2 \right) =$$

$$= \|\underline{z}\| \left(\cos\theta_1 \underline{b}_1 \wedge \underline{b}_2 + \sin\theta_1 \underline{a}_1 \wedge \underline{a}_2 \right)$$

since

$$\sin^2(\widehat{\underline{z}, \underline{x}}) = \frac{\|\underline{z}\|^2 \cdot \|\underline{x}\|^2 - < \underline{z}, \underline{x} >^2}{\|\underline{z}\|^2 \cdot \|\underline{x}\|^2} = \frac{\|\underline{z}\|^2 \cdot \sigma_2^2 - \sigma_2^4}{\|\underline{z}\|^2 \cdot \sigma_2^2} = \frac{\sigma_1^2}{\|\underline{z}\|^2}$$

for $\underline{x} = \sigma_2 \underline{b}_1 \wedge \underline{b}_2$ and $\underline{z} = \sigma_1 \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \underline{b}_1 \wedge \underline{b}_2$. Also, from Definition 3.3 we have that

$$\underline{z} = \|\underline{z}\| \cdot \left(\frac{\sigma_2 + \sigma_1}{\sqrt{2} \cdot \|\underline{z}\|\sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2) + \frac{\sigma_2 - \sigma_1}{\sqrt{2} \cdot \|\underline{z}\|\sqrt{2}} \cdot (\underline{b}_1 \wedge \underline{b}_2 - \underline{a}_1 \wedge \underline{a}_2) \right) =$$

$$= \|\underline{z}\| \cdot \left(\frac{\cos\theta_2}{\sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2) + \frac{\sin\theta_2}{\sqrt{2}} \cdot (\underline{b}_1 \wedge \underline{b}_2 - \underline{a}_1 \wedge \underline{a}_2) \right),$$

where $0 \le \theta_1, \theta_2 \le \frac{\pi}{4}$. Therefore, $\sin\theta_1 = g\left(\underline{z}, G_2(\mathbb{R}^5)\right), \sin\theta_2 = g\left(\underline{z}, \mathcal{V}_1\right)$. Thus, by eqn.(3.15) we have

$$\sin\theta_2 = \frac{\sigma_2 - \sigma_1}{\sqrt{2}\|\underline{z}\|} = \frac{1}{\sqrt{2}}\left(\cos\theta_1 - \sin\theta_1\right) = \sin\left(\frac{\pi}{4} - \theta_1\right)$$

Hence, $\theta_1 + \theta_2 = \pi/4$, since $0 \le \theta_1, \theta_2 \le \frac{\pi}{4}$. $\square$

4 Conclusions

The case where the eigenvalues of a nominal matrix are distinct provide a unique eigenvalue decomposition and thus a unique representation of the approximation of the matrix. When the eigenvalues are repeated the eigenvalue decompositions are automatically non-unique but the possible approximations are of uncertain nature, not being able to quarantee how close they really are to the initial matrix. In this paper we have answered the question that when all the eigenvalues of a skew-symmetric matrix are equal then the approximation is the worst possible and the corresponding 2-vector of the matrix is an element of a variety that has a maximum distance from the related Grassmann variety in the projective space. Furthermore we have calculated the equation that describes this extremal variety, similarly to the QPR set for the Grassmann varieties. Furthermore, several properties of the new extremal variety have been examined such as the path-wise connectedness and as its relation in geometric terms with a Grassmann variety. Future research will deal with futher examination of the extremal variety $\mathcal{V}_1$ for optimization problems of the form

$$\min_{\underline{z}_1,\underline{z}_2,\ldots,\underline{z}_p\in\mathbb{R}^n} \|\underline{z} - \underline{z}_1 \wedge \underline{z}_2 \wedge \cdots \wedge \underline{z}_p\| \tag{4.1}$$

and study the geometry of the corresponding higher order Grassmann varieties.

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