Eisele, F. (2012). P-Adic lifting problems and derived equivalences. Journal of Algebra, 356(1), pp. 90-114. doi: 10.1016/j.jalgebra.2012.01.015

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p-Adic Lifting Problems and Derived Equivalences

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Abstract

For two derived equivalent k-algebras $\overline{\Lambda}$ and $\overline{\Gamma}$, we introduce a correspondence between $\mathcal{O}$-orders reducing to $\overline{\Lambda}$ and $\mathcal{O}$-orders reducing to $\overline{\Gamma}$. We outline how this may be used to transfer properties like uniqueness (or non-existence) of a lift between $\overline{\Lambda}$ and $\overline{\Gamma}$. As an application, we look at tame algebras of dihedral type with two simple modules, where, most notably, we are able to show that among those algebras only the algebras $\mathcal{D}^{\kappa,0}(2A)$ and $\mathcal{D}^{\kappa,0}(2B)$ can actually occur as basic algebras of blocks of group rings of finite groups.

Keywords: Orders, Integral Representations, Derived Equivalences, Dihedral Defect

1. Introduction

In this article we consider, roughly speaking, the following problem: Given a finite-dimensional k-algebra $\overline{\Lambda}$, how many $\mathcal{O}$-orders Λ are there with $k \otimes \Lambda \cong \overline{\Lambda}$? Here k is a field of characteristic $p > 0$ and $\mathcal{O}$ is a complete discrete valuation ring with residue field k. Moreover, the field of fractions K of $\mathcal{O}$ is assumed to be of characteristic zero. Of course, there are usually infinitely many such lifts Λ of $\overline{\Lambda}$, and we may want to impose further restrictions on Λ to get a meaningful answer. Our main focus lies on the case where $\overline{\Lambda}$ is a block of kG for some finite group G, and the restrictions imposed on Λ should therefore be known properties of the corresponding block of $\mathcal{O}G$. The semisimplicity of $K \otimes \Lambda$ and the symmetry of Λ are certainly the simplest of those properties, but further properties may come from knowledge of the decomposition matrix and character values of G. The ideal outcome, for any given block of a group algebra kG, would be that any $\mathcal{O}$-order Λ subject to a certain set of conditions is isomorphic to the corresponding block of $\mathcal{O}G$.

For some algebras we can tackle these sorts of questions directly, essentially using linear algebra. The method we devise in this paper is the transfer, at least to some extent, of answers to the above questions via derived equivalences of k-algebras. The key point is that although, for two given derived equivalent finite-dimensional k-algebras $\overline{\Lambda}$ and $\overline{\Gamma}$, the respective problems of lifting $\overline{\Lambda}$ to an $\mathcal{O}$-order and lifting $\overline{\Gamma}$ to an $\mathcal{O}$-order may appear to be of a different degree of difficulty from the point of view of elementary linear algebra, we will show that they are in fact essentially equivalent. For this we associate to a two-sided tilting complex $X \in \mathcal{D}^b(\overline{\Lambda}^{\mathrm{op}} \otimes_k \overline{\Gamma})$ a bijection

$$\Phi_X : \widehat{\mathfrak{L}}(\overline{\Lambda}) \longrightarrow \widehat{\mathfrak{L}}(\overline{\Gamma}) \tag{1}$$

where $\widehat{\mathfrak{L}}(\overline{\Lambda})$ denotes the set of equivalence classes of pairs (Λ, φ), Λ being an $\mathcal{O}$-order and $\varphi : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda}$ being an isomorphism (see Definition 3.1 for a precise definition). $\widehat{\mathfrak{L}}(\overline{\Gamma})$ is defined in the same way. Of course some more work is needed to make this map useful, as, for instance, the sets $\widehat{\mathfrak{L}}(\overline{\Lambda})$ and $\widehat{\mathfrak{L}}(\overline{\Gamma})$ usually contain many elements representing one and the same order.

The idea behind this map Φ_X can be explained fairly easily in the case of a Morita-equivalence: If $\overline{\Lambda}$ and $\overline{\Gamma}$ are two Morita-equivalent k-algebras, then there is some invertible $\overline{\Lambda}$-$\overline{\Gamma}$-bimodule X with inverse X^{-1}. We can restrict X^{-1} to a projective right $\overline{\Lambda}$-module $\overline{P}$. The endomorphism ring of $\overline{P}$ can be identified with $\overline{\Gamma}$. Now for any pair $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ we can use φ to turn $\overline{P}$ into a projective $k \otimes \Lambda$-module. This

projective $k \otimes \Lambda$-module will lift uniquely to a projective Λ-module P. The endomorphism ring (let us call it Γ) of P will then be a lift of $\overline{\Gamma}$. What is still missing at this point is an isomorphism $k \otimes \Gamma \xrightarrow{\sim} \overline{\Gamma}$. Since P may be construed as a Γ-Λ-bimodule, we just choose $\psi : k \otimes \Gamma \xrightarrow{\sim} \overline{\Gamma}$ so that when we turn X^{-1} into a $k \otimes \Gamma$-$k \otimes \Lambda$-bimodule using φ and ψ, it becomes isomorphic to $k \otimes P$. Note that this does in fact determine ψ uniquely. (Γ, ψ) will then be the lift of $\overline{\Gamma}$ we associated to (Λ, φ).

A field of application are tame blocks of group algebras over k. Here the appendix of [3] gives a list of k-algebras which may occur as basic algebras. The first question is what the corresponding blocks of group algebras over $\mathcal{O}$ look like, and under which conditions two tame blocks of group algebras over $\mathcal{O}$ are Morita-equivalent given that the corresponding blocks of the group algebras over k are Morita-equivalent. Concretely, we look at blocks with dihedral defect group and two simple modules. The upshot here is that two such blocks are Morita-equivalent over $\mathcal{O}$ if and only if their corresponding blocks defined over k are Morita-equivalent and their centers are equal. Lifts for these blocks are then determined explicitly in Theorem 6.13.

We also narrow down which algebras given in Erdmann's list may actually occur as blocks of group rings. The key point here is that blocks of group rings defined over k possess a lift, namely the corresponding block defined over $\mathcal{O}$, which has all the well-known properties that blocks of group rings share. We show that such a lift does not exist for algebras of dihedral type with two simple modules and parameter $c = 1$, implying that only those algebras with parameter $c = 0$ occur as basic algebras of blocks of group rings (Corollary 6.9). The last assertion was recently proved for principal blocks in [1], but the general case was still open.

2. Foundations and Notation

In this section we recall some definitions and theorems, and state some corollaries for later use. Throughout this article, $(K, \mathcal{O}, k)$ will denote a p-modular system for some $p > 0$, and we assume that $\mathcal{O}$ is complete. We let π be a uniformizer of $\mathcal{O}$.

Notation 2.1. *If A is a ring, we denote by $\mathbf{mod}_A$ the category of finitely generated right A-modules, and by $\mathbf{proj}_A$ the category of finitely generated projective right A-modules. By a "module" we will always mean a right module (unless we explicitly say otherwise). By $\mathcal{C}^b(\mathbf{proj}_A)$ we denote the category of bounded complexes over $\mathbf{proj}_A$, and by $\mathcal{K}^b(\mathbf{proj}_A)$ we denote the corresponding homotopy category. By $\mathcal{D}^b(A) = \mathcal{D}^b(\mathbf{mod}_A)$ (respectively $\mathcal{D}^-(A) = \mathcal{D}^-(\mathbf{mod}_A)$) we denote the bounded derived category of A (respectively the right-bounded derived category of A). By "$-\otimes_A^{\mathbb{L}} =$" we will always denote a left-derived tensor product.*

Notation 2.2. *Let $R \in \{\mathcal{O}, k\}$, and let A, B and B' be R-algebras. Let $\alpha : B \to A, \beta : B' \to A$ be R-algebra homomorphisms. Then define $_\alpha A_\beta$ to be the B-B'-bimodule which is (as a set) equal to A with the action*

$$B \times A \times B' \longrightarrow A : (b, x, b') \mapsto \alpha(b) \cdot x \cdot \beta(b') \tag{2}$$

Definition 2.3. *Let $R \in \{\mathcal{O}, k\}$, and let A be any R-algebra that is free and finitely generated as an R-module (i. e. an order if $R = \mathcal{O}$). Then define the Picard group of A as follows:*

$$\mathrm{Pic}_R(A) := \{ \text{Isomorphism classes of invertible } A^{\mathrm{op}} \otimes_R A\text{-modules} \} \tag{3}$$

Note that we will always identify $A^{\mathrm{op}} \otimes_R B$-modules with the corresponding A-B-bimodules. An $A^{\mathrm{op}} \otimes_R A$-module X is invertible (by definition) if it is projective as a left and as a right A-module and there is an $A^{\mathrm{op}} \otimes_R A$-module Y (also projective as a left and as a right A-module) such that $X \otimes_A Y \cong Y \otimes_A X \cong {}_A A_A$. Now $\mathrm{Pic}_R(A)$ becomes a group with "$-\otimes_A =$" as its product.

Similarly define the derived Picard group of A as follows:

$$\mathrm{TrPic}_R(A) := \{ \text{Isomorphism classes of invertible objects in } \mathcal{D}^b(A^{\mathrm{op}} \otimes_R A) \} \tag{4}$$

We say a complex X in $\mathcal{D}^b(A^{\mathrm{op}} \otimes_R A)$ is invertible if there is a Y in $\mathcal{D}^b(A^{\mathrm{op}} \otimes_R A)$ such that $X \otimes_A^{\mathbb{L}} Y \cong Y \otimes_A^{\mathbb{L}} X \cong 0 \to {}_A A_A \to 0$. $\mathrm{TrPic}_R(A)$ is a group with "$-\otimes_A^{\mathbb{L}} =$" as its product.

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Remark 2.4. *Situation as above. There is a group homomorphism*

$$(\mathrm{Aut}_R(A), \circ) \to (\mathrm{Pic}_R(A), \otimes_A): \ \alpha \mapsto {}_{\mathrm{id}}A_\alpha \cong {}_{\alpha^{-1}}A_{\mathrm{id}} \tag{5}$$

The kernel of this homomorphism consists of all inner automorphisms of A, and we denote its image by $\mathrm{Out}_R(A)$. In case $R = k$ is an algebraically closed field, $\mathrm{Out}_k(A)$ is a linear algebraic group defined over k, and we denote its connected component by $\mathrm{Out}_k^0(A)$.

Remark 2.5. *Situation as above. If $X \in \mathrm{Pic}_R(A)$, then X is projective and finitely generated as a left A-module and as a right A-module. Hence if $P \in \mathbf{proj}_A$, then $P \otimes_A X$ is again in $\mathbf{proj}_A$. If P is indecomposable, then so is $P \otimes_A X$, since X is invertible. This implies that there is a group homomorphism from the Picard group of A into the symmetric group $\mathrm{Sym}(\mathcal{P})$ on $\mathcal{P}$*

$$\mathrm{Pic}_R(A) \longrightarrow \mathrm{Sym}(\mathcal{P}): \ X \mapsto [P \mapsto P \otimes_A X] \tag{6}$$

where $\mathcal{P}$ is the set of all isomorphism classes of finitely generated projective indecomposable A-modules. Define $\mathrm{Pic}_R^s(A)$ to be the kernel of this group homomorphism, and $\mathrm{Out}_R^s(A)$ to be the intersection of $\mathrm{Pic}_R^s(A)$ with $\mathrm{Out}_R(A)$. Define $\mathrm{Aut}_R^s(A)$ to be the preimage of $\mathrm{Out}_R^s(A)$ under the canonical epimorphism $\mathrm{Aut}_R(A) \twoheadrightarrow \mathrm{Out}_R(A)$.

Remark 2.6. *Situation as above. Then we get a series of embeddings*

$$\mathrm{Out}_R(A) \hookrightarrow \mathrm{Pic}_R(A) \hookrightarrow \mathrm{TrPic}_R(A) \tag{7}$$

Notation 2.7. *If A is a ring and $T \in \mathcal{K}^b(\mathbf{proj}_A)$ is a tilting complex with endomorphism ring B, then we denote by*

$$\mathcal{G}_T: \ \mathcal{D}^b(A) \xrightarrow{\sim} D^b(B) \tag{8}$$

an equivalence which agrees on objects with taking T-resolutions. For proper definitions and proof of the existence of such an equivalence we refer the reader to [11]. For our purposes, the most important property of $\mathcal{G}_T$ is that $\mathcal{G}_T(T)$ is isomorphic to the stalk complex $0 \to B \to 0$.

Remark 2.8. *Let Λ be an $\mathcal{O}$-order. The functor $k \otimes_{\mathcal{O}} -: \mathbf{mod}_\Lambda \longrightarrow \mathbf{mod}_{k \otimes \Lambda}$ has a (unique) left-derived functor $k \otimes_{\mathcal{O}}^{\mathbb{L}} -: \mathcal{D}^-(\Lambda) \longrightarrow \mathcal{D}^-(k \otimes \Lambda)$, which restricts to a functor from $\mathcal{K}^b(\mathbf{proj}_\Lambda)$ to $\mathcal{K}^b(\mathbf{proj}_{k \otimes \Lambda})$. For a complex $C \in \mathcal{K}^b(\mathbf{proj}_\Lambda)$, $k \otimes_{\mathcal{O}}^{\mathbb{L}} C$ is obtained by simply applying $k \otimes_{\mathcal{O}} -$ to this complex viewed as a sequence of modules. Hence, for objects $C \in \mathcal{K}^b(\mathbf{proj}_\Lambda)$, there is no harm in writing $k \otimes_{\mathcal{O}} C$ instead of $k \otimes_{\mathcal{O}}^{\mathbb{L}} C$.*

Remark 2.9. *Let A and B be R-algebras and let $\mathcal{F}: \ \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(B)$ be an equivalence that sends the stalk complex $0 \to A \to 0$ to $0 \to B \to 0$. Then there is an $\alpha: \ A \xrightarrow{\sim} B$ such that $\mathcal{F}(X) \cong X \otimes_A^{\mathbb{L}} {}_\alpha B_{\mathrm{id}}$ for all objects $X \in \mathcal{D}^b(A)$. This follows from [11, Proposition 7.1].*

Lemma 2.10. *Let A be a finite-dimensional k-algebra and $T \in \mathcal{K}^b(A)$ a tilting complex with endomorphism ring B. Then there exists a two-sided tilting complex ${}_BX_A \in \mathcal{D}^b(B^{\mathrm{op}} \otimes_k A)$ with restriction to $\mathcal{D}^b(A)$ isomorphic to T.*

Proof. By [11], there exists a functor $\mathcal{F}: \ \mathcal{D}^b(B) \longrightarrow \mathcal{D}^b(A)$ sending $0 \to B \to 0$ to T. By [12, Corollary 3.5] this equivalence is afforded by $\mathrm{RHom}_B(Y, -)$ for some $Y \in \mathcal{D}^b(A^{\mathrm{op}} \otimes B)$. This Y has an inverse $X \in \mathcal{D}^b(B^{\mathrm{op}} \otimes A)$ such that $\mathrm{RHom}_B(Y, -) \cong - \otimes_B^{\mathbb{L}} X$ (see [12, Definition 4.2] and the remarks following it). Since $B \otimes_B^{\mathbb{L}} X \cong \mathcal{F}(B) \cong T$, X has the desired properties. $\square$

Lemma 2.11. *Let A be a finite-dimensional symmetric k-algebra and let*

$$T = 0 \to P_1 \to P_0 \to 0 \tag{9}$$

be a two-term tilting complex. Then $\mathcal{G}_T(0 \to A \to 0)$ is again a two-term tilting complex.

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Proof. Set $B := \operatorname{End}_{\mathcal{D}^b(A)}(T)$. Let $X \in \mathcal{D}^b(B^{\mathrm{op}} \otimes A)$ be a two-sided tilting complex with restriction to $\mathcal{D}^b(A)$ isomorphic to T. Let $Y \in \mathcal{D}^b(A^{\mathrm{op}} \otimes B)$ be the inverse of X. By [7, Lemma 9.2.6] we may assume that X is a bounded complex of A-B-bimodules that become projective upon restriction to A and restriction to B. We may then furthermore assume that $Y = \operatorname{Hom}_k(X, k)$ (see [7, Corollary 9.2.5]; Note that both Lemma 9.2.6 and Corollary 9.2.5 in [7] use that A is symmetric). Hence Y has non-vanishing homology in precisely two adjacent degrees, since the same can be said about X and $\operatorname{Hom}_k(-, k)$ is exact on vector spaces. Now $- \otimes_A^{\mathbb{L}} Y$ sends T to $0 \to B \to 0$, which implies that for some automorphism $\gamma : B \to B$ the functor $- \otimes_A^{\mathbb{L}} Y \otimes_B^{\mathbb{L}} {}_{\mathrm{id}}B_\gamma$ agrees with $\mathcal{G}_T(-)$ on objects. Hence the image of $0 \to A \to 0$ under $\mathcal{G}_T(-)$ is equal to the restriction $Y \otimes_B^{\mathbb{L}} {}_{\mathrm{id}}B_\gamma$ to $\mathcal{D}^b(B)$. Therefore it is a bounded complex of projective B-modules that has non-zero homology (at most) in two (adjacent) degrees. Since A and B are symmetric (so in particular self-injective), any injection of a projective module and any epimorphism onto a projective module splits. Hence $Y \otimes_B^{\mathbb{L}} {}_{\mathrm{id}}B_\gamma$ is isomorphic in $\mathcal{K}^b(\mathbf{proj}_B)$ to a two-term complex. $\qquad\square$

For the rest of the section, let k be algebraically closed.

Theorem 2.12 (Rouquier, Huisgen-Zimmermann, Saorín). *Let A, B be finite-dimensional k-algebras and X a bounded complex of A-B-bimodules inducing an equivalence between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ (i. e., a two-sided tilting complex). Then there exists a (unique) isomorphism of algebraic groups*

$$\sigma : \operatorname{Out}_k^0(A) \xrightarrow{\sim} \operatorname{Out}_k^0(B) \tag{10}$$

such that

$${}_{\mathrm{id}}A_\alpha \otimes_A^{\mathbb{L}} X \cong X \otimes_B^{\mathbb{L}} {}_{\mathrm{id}}B_{\sigma(\alpha)} \tag{11}$$

for all $\alpha \in \operatorname{Out}_k^0(A)$.

Proof. The Theorem was stated in this form in [15, Theorem 3.4]. A proof can be found in [5] or in [14]. $\qquad\square$

Theorem 2.13 (Jensen, Su, Zimmermann). *Let A be a finite-dimensional k-algebra. Then up to isomorphism in $\mathcal{K}^b(\mathbf{proj}_A)$ there exists at most one two-term (partial) tilting complex*

$$0 \to P_1 \to P_0 \to 0 \tag{12}$$

with fixed homogeneous components P_0 and P_1.

Proof. See [6, Corollary 8]. $\qquad\square$

Corollary 2.14. *Let A be a finite-dimensional k-algebra and T a tilting complex over A. Then*

1. *$T \otimes_A {}_{\mathrm{id}}A_\gamma \cong T$ for all $\gamma \in \operatorname{Out}_k^0(A)$.*

2. *If T is a two-term complex, then $T \otimes_A {}_{\mathrm{id}}A_\gamma \cong T$ for all $\gamma \in \operatorname{Out}_k^s(A)$.*

Proof. The first point follows from Theorem 2.12 and Lemma 2.10. The second point follows from Theorem 2.13 and the definition of $\operatorname{Out}_k^s(A)$. $\qquad\square$

3. A Correspondence of Lifts

In this section we introduce a bijection between "lifts" of derived equivalent finite-dimensional k-algebras.

Definition 3.1. *For a finite-dimensional k-algebra $\overline{\Lambda}$ define its set of lifts as follows:*

$$\widehat{\mathfrak{L}}(\overline{\Lambda}) := \left\{ (\Lambda, \varphi) \mid \Lambda \text{ is an } \mathcal{O}\text{-order and } \varphi : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda} \text{ is an isomorphism} \right\} \Big/ \sim \tag{13}$$

where we say $(\Lambda, \varphi) \sim (\Lambda', \varphi')$ if and only if

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1. *There is an isomorphism $\alpha : \Lambda \xrightarrow{\sim} \Lambda'$*

2. *There is a $\beta \in \mathrm{Aut}_k(\overline{\Lambda})$ such that the functor $- \otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\beta}\overline{\Lambda}_{\mathrm{id}}$ fixes all isomorphism classes of tilting complexes in $\mathcal{K}^b(\mathbf{proj}_{\overline{\Lambda}})$*

such that $\varphi = \beta \circ \varphi' \circ (\mathrm{id}_k \otimes \alpha)$.

Our bijection will be based on the following theorem of Rickard:

Theorem 3.2 ([13, Theorem 3.3.]). *Let Λ be an $\mathcal{O}$-order and let $\overline{T} \in \mathcal{K}^b(\mathbf{proj}_{k \otimes \Lambda})$ be a tilting complex for $k \otimes \Lambda$. Then there exists a unique (up to isomorphism in $\mathcal{D}^b(\Lambda)$) tilting complex $T \in \mathcal{K}^b(\mathbf{proj}_{\Lambda})$ with $k \otimes T \cong \overline{T}$. $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is torsion-free and*

$$k \otimes \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T) \cong \mathrm{End}_{\mathcal{D}^b(k \otimes \Lambda)}(\overline{T}) \tag{14}$$

Remark 3.3. *By [13, Proposition 3.1.] it is immediately clear that we can replace the word "tilting complex" by "partial tilting complex" in the above theorem (where we understand "partial tilting complex" as defined in [7, Definition 3.2.1.]).*

Lemma 3.4. *Let Λ be an $\mathcal{O}$-order and $T \in \mathcal{K}^b(\mathbf{proj}_{\Lambda})$ a tilting complex. Define $\Gamma := \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$, and assume that Γ is also an $\mathcal{O}$-order. Then $k \otimes T$ is a tilting complex and the k-algebras $k \otimes \Gamma$ and $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(k \otimes T)$ are (canonically) isomorphic. Moreover, the diagram*

$$
\begin{array}{ccc}
\mathcal{D}^-(\Lambda) & \xrightarrow{\;\mathcal{G}_T\;} & \mathcal{D}^-(\Gamma) \\
{\scriptstyle k\otimes^{\mathbb{L}}-}\Big\downarrow & & \Big\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^-(k \otimes \Lambda) & \xrightarrow{\;\mathcal{G}_{k\otimes T}\;} & \mathcal{D}^-(k \otimes \Gamma)
\end{array}
\tag{15}
$$

commutes on objects.

Proof. This follows from [12, Proposition 2.4.]. $\qquad\square$

For the rest of the section let $\overline{\Lambda}$ and $\overline{\Gamma}$ be two derived equivalent finite-dimensional k-algebras. Furthermore let $X \in \mathcal{D}^b(\overline{\Lambda}^{\mathrm{op}} \otimes_k \overline{\Gamma})$ be a two-sided tilting complex, and let X^{-1} be its inverse. Let $\overline{T}$ be the restriction of X^{-1} to $\mathcal{D}^b(\mathbf{proj}_{\overline{\Lambda}})$ and likewise let $\overline{S}$ be the restriction of X to $\mathcal{D}^b(\mathbf{proj}_{\overline{\Gamma}})$.

Definition 3.5. *Define a map*

$$\Phi_X : \; \widehat{\mathfrak{L}}(\overline{\Lambda}) \longrightarrow \widehat{\mathfrak{L}}(\overline{\Gamma}) \tag{16}$$

as follows: Assume $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$. Let T be the lift of $\overline{T} \otimes_{\overline{\Lambda} \mathrm{id}} \overline{\Lambda}_{\varphi}$ (which exists and is unique by Theorem 3.2). We put $\Phi_X(\Lambda, \varphi) = (\Gamma, \psi)$, where $\Gamma = \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ and $\psi : k \otimes \Gamma \xrightarrow{\sim} \overline{\Gamma}$ is an isomorphism such that the following diagram commutes on objects:

$$
\begin{array}{ccccc}
\mathcal{D}^-(\Lambda) & \xrightarrow{\;\mathcal{G}_T(-)\;} & \mathcal{D}^-(\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)) & =\!=\!=\!=\!= & \mathcal{D}^-(\Gamma) \\
{\scriptstyle k\otimes^{\mathbb{L}}-}\Big\downarrow & & {\scriptstyle k\otimes^{\mathbb{L}}-}\Big\downarrow & & \Big\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^-(k \otimes \Lambda) & \xrightarrow{\;\mathcal{G}_{k\otimes T}(-)\;} & \mathcal{D}^-(\mathrm{End}_{\mathcal{D}^b(k\otimes\Lambda)}(k \otimes T)) & =\!=\!=\!= & \mathcal{D}^-(k \otimes \Gamma) \\
{\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Lambda\,\varphi}\overline{\Lambda}_{\mathrm{id}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{E}} & & \Big\downarrow{\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Gamma\,\psi}\overline{\Gamma}_{\mathrm{id}}} \\
\mathcal{D}^-(\overline{\Lambda}) & \xrightarrow[{\scriptstyle -\otimes^{\mathbb{L}}_{\overline{\Lambda}}X}]{} & \mathcal{D}^-(\overline{\Gamma}) & =\!=\!=\!= & \mathcal{D}^-(\overline{\Gamma})
\end{array}
\tag{17}
$$

Here, $\mathcal{E}$ is defined so that the bottom left square commutes.

Proof of well-definedness. First note that the top left square commutes on objects by Lemma 3.4. Thus the left half of the diagram will commute on objects. Note furthermore that $\mathcal{E}$ sends $0 \to k \otimes \Gamma \to 0$ to $0 \to \overline{\Gamma} \to 0$, and hence a ψ making the diagram commutative on objects can be chosen due to Remark 2.9. This ψ is unique up to an automorphism β of $\overline{\Gamma}$ such that $- \otimes_{\overline{\Gamma}}^{\mathbb{L}} {}_{\mathrm{id}}\overline{\Gamma}_{\beta}$ fixes all objects in $\mathcal{D}^-(\overline{\Gamma})$, and hence in particular fixes all tilting complexes. Therefore the equivalence class of (Γ, ψ) is certainly independent of the particular choice of ψ.

Now assume $(\Lambda, \varphi) \sim (\Lambda', \varphi')$, that is, there are α and β as in Definition 3.1 such that $\varphi = \beta \circ \varphi' \circ (\mathrm{id}_k \otimes \alpha)$. We need to show that $(\Gamma, \psi) := \Phi_X(\Lambda, \varphi) \sim \Phi_X(\Lambda', \varphi') =: (\Gamma', \psi')$, Φ_X being given by the construction above. We get the following diagram (where we define T' analogous to T):

$$(18)$$

$$
\begin{array}{ccccccc}
\mathcal{D}^-(\Gamma') & \xleftarrow{\mathcal{G}_{T'}(-)} & \mathcal{D}^-(\Lambda') & \xrightarrow{-\otimes_{\Lambda'}^{\mathbb{L}}{}_{\mathrm{id}}\Lambda'_\alpha} & \mathcal{D}^-(\Lambda) & \xrightarrow{\mathcal{G}_T(-)} & \mathcal{D}^-(\Gamma) \\
\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^-(k\otimes\Gamma') & \xleftarrow{\mathcal{G}_{k\otimes T'}(-)} & \mathcal{D}^-(k\otimes\Lambda') & \xrightarrow[-\otimes_{k\otimes\Lambda'}^{\mathbb{L}}{}_{\mathrm{id}}k\otimes\Lambda'_{\mathrm{id}_k\otimes\alpha}]{} & \mathcal{D}^-(k\otimes\Lambda) & \xrightarrow{\mathcal{G}_{k\otimes T}(-)} & \mathcal{D}^-(k\otimes\Gamma) \\
\downarrow{\scriptstyle -\otimes_{k\otimes\Gamma'}^{\mathbb{L}}{}_{\psi'}\overline{\Gamma}_{\mathrm{id}}} & & \downarrow{\scriptstyle -\otimes_{k\otimes\Lambda'}^{\mathbb{L}}{}_{\varphi'}\overline{\Lambda}_{\mathrm{id}}} & & \downarrow{\scriptstyle -\otimes_{k\otimes\Lambda}^{\mathbb{L}}{}_{\varphi}\overline{\Lambda}_{\mathrm{id}}} & & \downarrow{\scriptstyle -\otimes_{k\otimes\Gamma}^{\mathbb{L}}{}_{\psi}\overline{\Gamma}_{\mathrm{id}}} \\
\mathcal{D}^-(\overline{\Gamma}) & \xleftarrow{-\otimes_{\overline{\Lambda}}^{\mathbb{L}}X} & \mathcal{D}^-(\overline{\Lambda}) & = & \mathcal{D}^-(\overline{\Lambda}) & \xrightarrow{-\otimes_{\overline{\Lambda}}^{\mathbb{L}}X} & \mathcal{D}^-(\overline{\Gamma})
\end{array}
$$

This diagram will commute at the very least on tilting complexes (that is, if we take a tilting complex in any of those categories and take its image under a series of arrows in the above diagram, the isomorphism class of the outcome will not depend on the path we have chosen). Note that all horizontal arrows are equivalences, and so we get a diagram (again commutative on tilting complexes)

$$(19)$$

$$
\begin{array}{ccc}
\mathcal{D}^-(\Gamma') & \xrightarrow{\mathcal{F}_1(-)} & \mathcal{D}^-(\Gamma) \\
\downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} & & \downarrow{\scriptstyle k\otimes^{\mathbb{L}}-} \\
\mathcal{D}^-(k\otimes\Gamma') & \xrightarrow{\mathcal{F}_2(-)} & \mathcal{D}^-(k\otimes\Gamma) \\
\downarrow{\scriptstyle -\otimes_{k\otimes\Gamma'}^{\mathbb{L}}{}_{\psi'}\overline{\Gamma}_{\mathrm{id}}} & & \downarrow{\scriptstyle -\otimes_{k\otimes\Gamma}^{\mathbb{L}}{}_{\psi}\overline{\Gamma}_{\mathrm{id}}} \\
\mathcal{D}^-(\overline{\Gamma}) & = & \mathcal{D}^-(\overline{\Gamma})
\end{array}
$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are two equivalences. Due to commutativity on tilting complexes, $\mathcal{F}_2$ needs to send $0 \to k \otimes \Gamma' \to 0$ to $0 \to k \otimes \Gamma \to 0$. Due to unique lifting (and again commutativity), $\mathcal{F}_1$ needs to send $0 \to \Gamma' \to 0$ to $0 \to \Gamma \to 0$. Hence there is an isomorphism $\alpha : \Gamma' \to \Gamma$ such that $\mathcal{F}_1(-)$ agrees on objects with $- \otimes_{\Gamma'}^{\mathbb{L}} {}_\alpha\Gamma_{\mathrm{id}}$. Due to commutativity, $\mathcal{F}_2(-)$ then needs to agree on tilting complexes with $- \otimes_{k\otimes\Gamma'}^{\mathbb{L}} {}_{\mathrm{id}_k\otimes\alpha}k \otimes \Gamma_{\mathrm{id}}$ (this is owed to the fact that every tilting complex lies in the image of $k \otimes^{\mathbb{L}} -$ due to Theorem 3.2). Commutativity on tilting complexes of the lower square then implies that $\psi' = \beta \circ \psi \circ (\mathrm{id}_k \otimes \alpha)$ for some $\beta \in \mathrm{Aut}_k(\overline{\Gamma})$ so that $- \otimes_{\overline{\Gamma}}^{\mathbb{L}} {}_\beta\overline{\Gamma}_{\mathrm{id}}$ fixes all tilting complexes in $\mathcal{K}^b(\mathbf{proj}_{\overline{\Gamma}})$. By definition this means $(\Gamma, \psi) \sim (\Gamma', \psi')$. $\qquad\square$

Proposition 3.6. *The maps Φ_X and $\Phi_{X^{-1}}$ are mutually inverse. In particular, they induce a bijection*

$$\widehat{\mathfrak{L}}(\overline{\Lambda}) \longleftrightarrow \widehat{\mathfrak{L}}(\overline{\Gamma}) \tag{20}$$

Proof. We keep the notation of Definition 3.5. Set $(\Gamma, \psi) := \Phi_X(\Lambda, \varphi)$ and $(\Lambda', \tilde{\varphi}) := \Phi_{X^{-1}}(\Gamma, \psi)$. Furthermore, let S be the lift of $\overline{S} \otimes_{\overline{\Gamma}} {}_{\mathrm{id}}\overline{\Gamma}_\psi$. Consider the following diagram (obtained by composing (17) with itself)

$$\begin{array}{ccccc}
\mathcal{D}^-(\Lambda) & \xrightarrow{\;\mathcal{G}_T(-)\;} & \mathcal{D}^-(\Gamma) & \xrightarrow{\;\mathcal{G}_S(-)\;} & \mathcal{D}^-(\Lambda') \\
{\scriptstyle k\otimes^{\mathbb{L}}-}\downarrow & & {\scriptstyle k\otimes^{\mathbb{L}}-}\downarrow & & {\scriptstyle k\otimes^{\mathbb{L}}-}\downarrow \\
\mathcal{D}^-(k\otimes\Lambda) & \xrightarrow{\;\mathcal{G}_{k\otimes T}(-)\;} & \mathcal{D}^-(k\otimes\Gamma) & \xrightarrow{\;\mathcal{G}_{k\otimes S}(-)\;} & \mathcal{D}^-(k\otimes\Lambda') \\
{\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Lambda\,\varphi}\overline{\Lambda}_{\mathrm{id}}}\downarrow & & {\scriptstyle \otimes^{\mathbb{L}}_{k\otimes\Gamma\,\psi}\overline{\Gamma}_{\mathrm{id}}}\downarrow & & {\scriptstyle -\otimes^{\mathbb{L}}_{k\otimes\Lambda'\,\tilde{\varphi}}\overline{\Lambda}_{\mathrm{id}}}\downarrow \\
\mathcal{D}^-(\overline{\Lambda}) & \xrightarrow[{-\otimes^{\mathbb{L}}_{\overline{\Lambda}}X}]{} & \mathcal{D}^-(\overline{\Gamma}) & \xrightarrow[{-\otimes^{\mathbb{L}}_{\overline{\Gamma}}X^{-1}}]{} & \mathcal{D}^-(\overline{\Gamma})
\end{array} \tag{21}$$

Commutativity on objects implies that $\mathcal{G}_{k\otimes S}\circ\mathcal{G}_{k\otimes T}$ sends the stalk complex $0\to k\otimes\Lambda\to 0$ to the stalk complex $0\to k\otimes\Lambda'\to 0$. Hence, due to unique lifting, $\mathcal{G}_S\circ\mathcal{G}_T$ sends $0\to\Lambda\to 0$ to $0\to\Lambda'\to 0$. $\mathcal{G}_S\circ\mathcal{G}_T$ hence agrees on objects with $-\otimes^{\mathbb{L}}_{\Lambda\,\mathrm{id}}\Lambda_\alpha$ for some isomorphism $\alpha:\Lambda'\xrightarrow{\sim}\Lambda$. Thus, $\mathcal{G}_{k\otimes S}\circ\mathcal{G}_{k\otimes T}$ will agree on tilting complexes with $-\otimes^{\mathbb{L}}_{k\otimes\Lambda\,\mathrm{id}}k\otimes\Lambda_{\mathrm{id}_k\otimes\alpha}$. Hence, due to commutativity on objects, we must have that $-\otimes^{\mathbb{L}}_{k\otimes\Lambda\,\tilde{\varphi}}\overline{\Lambda}_{\mathrm{id}}$ and $-\otimes^{\mathbb{L}}_{k\otimes\Lambda\,\mathrm{id}_k\otimes\alpha}k\otimes\Lambda_{\mathrm{id}}\otimes^{\mathbb{L}}_{k\otimes\Lambda\,\varphi}\overline{\Lambda}_{\mathrm{id}}=-\otimes^{\mathbb{L}}_{k\otimes\Lambda\,\varphi\circ(\mathrm{id}_k\otimes\alpha)}\overline{\Lambda}_{\mathrm{id}}$ agree on tilting complexes. This however is the same as saying that $\tilde{\varphi}=\beta\circ\varphi\circ(\mathrm{id}_k\otimes\alpha)$, where $\beta\in\mathrm{Aut}_k(\overline{\Lambda})$ is an automorphism such that $-\otimes^{\mathbb{L}}_{\overline{\Lambda}\,\beta}\overline{\Lambda}_{\mathrm{id}}$ fixes all tilting complexes. This means, by definition, that $(\Lambda,\varphi)\sim(\Lambda',\tilde{\varphi})$. So we proved that $\Phi_{X^{-1}}\circ\Phi_X=\mathrm{id}$, and $\Phi_X\circ\Phi_{X^{-1}}=\mathrm{id}$ follows by swapping the roles of X and X^{-1}. $\qquad\square$

Proposition 3.7. $\mathrm{Out}_k(\overline{\Lambda})$ *acts on* $\widehat{\mathfrak{L}}(\overline{\Lambda})$ *from the left via*

$$\alpha\cdot(\Lambda,\varphi):=(\Lambda,\alpha\circ\varphi) \tag{22}$$

Proof. The above formula clearly defines an action of $\mathrm{Aut}_k(\overline{\Lambda})$. In order to verify that it defines an action of $\mathrm{Out}_k(\overline{\Lambda})$, we just need to check that for any inner automorphism α of $\overline{\Lambda}$ we have $(\Lambda,\varphi)\sim(\Lambda,\alpha\circ\varphi)$. But an inner automorphism α of $\overline{\Lambda}$ gives us an inner automorphism $\varphi^{-1}\circ\alpha\circ\varphi$ of $k\otimes\Lambda$, which lifts to an inner automorphism $\hat{\alpha}$ of Λ (since the natural map of unit groups $\Lambda^\times\to(k\otimes\Lambda)^\times$ is surjective). (Λ,φ) and $(\Lambda,\alpha\circ\varphi)=(\Lambda,\varphi\circ(\mathrm{id}_k\otimes\hat{\alpha}))$ are then clearly equivalent in the sense of Definition 3.1. $\qquad\square$

Proposition 3.8. *If k is algebraically closed, then $\mathrm{Out}_k^0(\overline{\Lambda})$ lies in the kernel of the action of $\mathrm{Out}_k(\overline{\Lambda})$ on* $\widehat{\mathfrak{L}}(\overline{\Lambda})$.

Proof. This follows directly from Theorem 2.12. $\qquad\square$

Proposition 3.9. *Let $\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}$ respectively $\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}}$ denote the stabilizers of the isomorphism classes of $\overline{T}$ respectively $\overline{S}$. There is an isomorphism*

$$-^X:\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}\xrightarrow{\sim}\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}} \tag{23}$$

such that for all $\alpha\in\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}$ we have

$$\Phi_X(\alpha\cdot(\Lambda,\varphi))=\alpha^X\cdot\Phi_X(\Lambda,\varphi) \tag{24}$$

In particular, Φ_X induces a bijection

$$\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}\backslash\widehat{\mathfrak{L}}(\overline{\Lambda})\longleftrightarrow\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}}\backslash\widehat{\mathfrak{L}}(\overline{\Gamma}) \tag{25}$$

Proof. Set

$$-^X:\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}}\longrightarrow\mathrm{TrPic}(\overline{\Gamma}):\alpha\mapsto X^{-1}\otimes^{\mathbb{L}}_{\overline{\Lambda}\,\mathrm{id}}\overline{\Lambda}_\alpha\otimes^{\mathbb{L}}_{\overline{\Lambda}}X \tag{26}$$

First note that the restriction of X^{-1} to $\mathcal{D}^b(\Lambda)$ is isomorphic to $\overline{T}$ by definition of $\overline{T}$. Since α stabilizes the isomorphism class of $\overline{T}$, the restriction of $X^{-1}\otimes^{\mathbb{L}}_{\overline{\Lambda}\,\mathrm{id}}\overline{\Lambda}_\alpha\otimes^{\mathbb{L}}_{\overline{\Lambda}}X$ to $\mathcal{D}^b(\overline{\Gamma})$ is isomorphic to $0\to\overline{\Gamma}\to 0$. Thus $X^{-1}\otimes^{\mathbb{L}}_{\overline{\Lambda}\,\mathrm{id}}\overline{\Lambda}_\alpha\otimes^{\mathbb{L}}_{\overline{\Lambda}}X$ is isomorphic to $0\to{}_{\mathrm{id}}\overline{\Gamma}_\beta\to 0$ for some $\beta\in\mathrm{Aut}_k(\overline{\Gamma})$. That is, the image of $-^X$ as

defined above is indeed contained in $\mathrm{Out}_k(\overline{\Gamma}) \leqslant \mathrm{TrPic}(\overline{\Gamma})$. Now $\overline{S}$ is by definition just the restriction of X to $\mathcal{D}^b(\overline{\Gamma})$, and hence $\overline{S} \otimes^{\mathbb{L}}_{\overline{\Gamma}} X^{-1} \otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\mathrm{id}}\overline{\Lambda}_\alpha \otimes^{\mathbb{L}}_{\overline{\Lambda}} X$ is isomorphic to the restriction of ${}_{\mathrm{id}}\overline{\Lambda}_\alpha \otimes^{\mathbb{L}}_{\overline{\Lambda}} X$ to $\mathcal{D}^b(\overline{\Gamma})$ which is again isomorphic to $\overline{S}$. So $-^X$ does indeed define a map with image contained in $\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}}$. It is also easy to see that $-^X$ is a group homomorphism, and that $-^{X^{-1}}$ is a two-sided inverse for $-^X$.

Now the claim of (24) follows from the commutativity of the following diagram

$$
\begin{array}{ccc}
\mathcal{D}^-(\overline{\Lambda}) & \xrightarrow{\;-\otimes^{\mathbb{L}}_{\overline{\Lambda}} X\;} & \mathcal{D}^-(\overline{\Gamma}) \\[2pt]
{\scriptstyle -\otimes^{\mathbb{L}}_{\overline{\Lambda}}{}_{\mathrm{id}}\overline{\Lambda}_\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle -\otimes^{\mathbb{L}}_{\overline{\Gamma}}{}_{\mathrm{id}}\overline{\Gamma}_\alpha X} \\[2pt]
\mathcal{D}^-(\overline{\Lambda}) & \xrightarrow[\;-\otimes^{\mathbb{L}}_{\overline{\Lambda}} X\;]{} & \mathcal{D}^-(\overline{\Gamma})
\end{array}
\tag{27}
$$

by gluing it below diagram (17). $\qquad\square$

Definition 3.10. *Define the set $\mathfrak{L}(\overline{\Lambda})$ to be the set of all isomorphism classes of $\mathcal{O}$-orders Λ such that $k \otimes \Lambda \cong \overline{\Lambda}$. Clearly, $\mathfrak{L}(\overline{\Lambda})$ is in bijection with $\mathrm{Out}_k(\overline{\Lambda}) \setminus \widehat{\mathfrak{L}}(\overline{\Lambda})$. Furthermore, we define the projection map*

$$
\Pi : \widehat{\mathfrak{L}}(\overline{\Lambda}) \longrightarrow \mathfrak{L}(\overline{\Lambda}) : (\Lambda, \varphi) \mapsto \Lambda
\tag{28}
$$

Corollary 3.11. *Assume k is algebraically closed, $\overline{\Lambda}$ is symmetric, and $\overline{T}$ is a two-term complex. Assume furthermore that $\mathrm{Out}_k^s(\overline{\Lambda}) = \mathrm{Out}_k(\overline{\Lambda})$ and $\mathrm{Out}_k^s(\overline{\Gamma}) = \mathrm{Out}_k(\overline{\Gamma})$ (a sufficient criterion for this is for instance that the Cartan matrices of $\overline{\Lambda}$ and $\overline{\Gamma}$ have no non-trivial permutation symmetries). Then there is a bijection*

$$
\mathfrak{L}(\overline{\Lambda}) \longleftrightarrow \mathfrak{L}(\overline{\Gamma})
\tag{29}
$$

Proof. Lemma 2.11 implies that $\overline{S}$ may be assumed to be a two-term complex as well. The assertion now follows from (25) together with Theorem 2.13, since the latter implies that $\mathrm{Out}_k(\overline{\Lambda})_{\overline{T}} = \mathrm{Out}_k(\overline{\Lambda})$ and $\mathrm{Out}_k(\overline{\Gamma})_{\overline{S}} = \mathrm{Out}_k(\overline{\Gamma})$. $\qquad\square$

The following proposition is useful to prove a "unique lifting property" for the group ring of $\mathrm{SL}_2(p^f)$ in defining characteristic, which we will do in a later paper.

Proposition 3.12. *Assume k is algebraically closed. Let $\Lambda \in \mathfrak{L}(\overline{\Lambda})$, and let $\gamma : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda}$ be an isomorphism. Now assume*

$$
\overline{\mathrm{Aut}_{\mathcal{O}}(\Lambda)} \cdot \mathrm{Out}_k^0(\overline{\Lambda}) = \mathrm{Out}_k(\overline{\Lambda})
\tag{30}
$$

where $\overline{\mathrm{Aut}_{\mathcal{O}}(\Lambda)}$ is the image of $\mathrm{Aut}_{\mathcal{O}}(\Lambda)$ in $\mathrm{Out}_k(\overline{\Lambda})$ (here we identify $k \otimes \Lambda$ with $\overline{\Lambda}$ via γ). Then the fiber $\Pi^{-1}(\{\Lambda\})$ has cardinality one.

Proof. Let $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ for some $\varphi : k \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda}$. Now if (30) holds, we can write $\gamma \circ \varphi^{-1} = \gamma \circ (\mathrm{id}_k \otimes \hat\alpha) \circ \gamma^{-1} \circ \beta$ for some $\hat\alpha \in \mathrm{Aut}_{\mathcal{O}}(\Lambda)$ and $\beta \in \mathrm{Aut}_k(\overline{\Lambda})$ such that the image of β in $\mathrm{Out}_k(\overline{\Lambda})$ lies in $\mathrm{Out}_k^0(\overline{\Lambda})$. Hence $\gamma \circ (\mathrm{id}_k \otimes \hat\alpha^{-1}) = \beta \circ \varphi$. Proposition 3.8 (together with the definition of "$\sim$") implies $(\Lambda, \gamma) \sim (\Lambda, \beta^{-1} \circ \gamma \circ (\mathrm{id}_k \otimes \hat\alpha^{-1})) = (\Lambda, \varphi)$. $\qquad\square$

4. Tilting Orders in Semisimple Algebras

What we would want to do now is to partition the set $\widehat{\mathfrak{L}}(\overline{\Lambda})$ into manageable pieces, so that the map Φ_X defined in the previous section restricts to a bijection between corresponding pieces of $\widehat{\mathfrak{L}}(\overline{\Lambda})$ and $\widehat{\mathfrak{L}}(\overline{\Gamma})$. In order to do that in the next section, we first need to study how those properties of orders that we are interested in behave under derived equivalences. The results in this section are elementary and therefore certainly known, but we were unable to find explicit references for most of them, which is why we include proofs. We assume throughout this section that Λ is an $\mathcal{O}$-order in a finite-dimensional semisimple K-algebra A.

Lemma 4.1. *If $T \in \mathcal{K}^b(\mathbf{proj}_\Lambda)$ is a tilting complex for Λ and $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is torsion-free as an $\mathcal{O}$-module, then $K \otimes T$ is a tilting complex for A. Furthermore, $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is a full $\mathcal{O}$-order in $\mathrm{End}_{\mathcal{D}^b(A)}(K \otimes T)$.*

Proof. First we show that $\mathrm{Hom}_{\mathcal{D}^b(A)}(K \otimes T, K \otimes T[i]) = 0$ for $i \neq 0$. Assume that $\varphi \in \mathrm{Hom}_{\mathcal{D}^b(A)}(K \otimes T, K \otimes T[i])$ for some i. Then we may view φ (or rather a representative of it) as a morphism of graded modules $K \otimes T \to K \otimes T[i]$ commuting with the differential. As such we may restrict it to T, and for a large enough $n \in \mathbb{N}$, we will have $\mathrm{Im}(\pi^n \cdot \varphi) \subseteq T[i]$. Hence $\pi^n \cdot \varphi$ defines an element in $\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(T, T[i])$. For $i = 0$ this implies that $\mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ is a full $\mathcal{O}$-lattice in $\mathrm{End}_{\mathcal{D}^b(A)}(K \otimes T)$. For $i \neq 0$ this implies that $\pi^n \cdot \varphi$ is homotopic to zero, and hence so is φ (by dividing the homotopy by π^n).

Since $K \otimes^{\mathbb{L}} - : \mathcal{D}^-(\Lambda) \to \mathcal{D}^-(A)$ is an exact functor between triangulated categories that maps T to $K \otimes T$, it is clear that $\mathrm{add}(K \otimes T)$ contains the image of $\mathrm{add}(T)$. But $\mathrm{add}(T)$ is equal to $\mathcal{K}^b(\mathbf{proj}_\Lambda)$ by definition, and so in particular contains $0 \to \Lambda \to 0$, which maps to $0 \to A \to 0$, which in turn clearly generates $\mathcal{K}^b(\mathbf{proj}_A)$. Hence $\mathrm{add}(K \otimes T) = \mathcal{K}^b(\mathbf{proj}_A)$. $\square$

Lemma 4.2. *Situation as above. Let $V_1, \ldots, V_n$ be representatives for the isomorphism classes of simple A-modules. Then there are sets Ω_i for $i \in \mathbb{Z}$ with $\biguplus_i \Omega_i = \{1, \ldots, n\}$ and numbers $\delta_j \in \mathbb{Z}_{>0}$ for $j \in \{1, \ldots, n\}$ such that*

$$K \otimes T \cong_{\mathcal{D}^b(A)} \ldots \xrightarrow{0} \underbrace{\bigoplus_{j \in \Omega_i} V_j^{\delta_j}}_{degree\ i} \xrightarrow{0} \bigoplus_{j \in \Omega_{i+1}} V_j^{\delta_j} \xrightarrow{0} \ldots \tag{31}$$

In particular, each V_j occurs as a direct summand of precisely one of the $H^i(K \otimes T)$. Also, it follows that $H^i(K \otimes T) \cong \bigoplus_{j \in \Omega_i} V_j^{\delta_j}$, and the map below is an isomorphism:

$$\bigoplus_i H^i : \ \mathrm{End}_{\mathcal{D}^b(A)}(K \otimes T) \xrightarrow{\sim} \bigoplus_i \mathrm{End}_A(H^i(K \otimes T)) \cong \bigoplus_i \bigoplus_{j \in \Omega_i} \mathrm{End}_A(V_j)^{\delta_j \times \delta_j} \tag{32}$$

Proof. $K \otimes T$ is, as a complex over A, certainly split, and hence isomorphic in the homotopy category to a complex C with differential equal to zero. Clearly $H^i(C) = C^i$ and $\mathrm{End}_{\mathcal{D}^b(A)}(C) = \bigoplus_i \mathrm{End}_A(C^i)$. So all that remains to show is that any V_j occurs in precisely one C^i. But $\mathrm{Hom}_{\mathcal{D}^b(A)}(C, C[l]) = 0$ for $l \neq 0$ implies that $\mathrm{Hom}_A(C^i, C^{i+l}) = 0$ for all $l \neq 0$ and hence that V_j occurs in at most one C^i. The fact that $\mathrm{add}(C) = \mathcal{D}^b(A)$ implies that each V_j has to occur in some C^i. $\square$

Definition 4.3. *The above lemma contains a definition of sets Ω_i and numbers δ_j associated to the tilting complex T. We keep this notation. In addition to those, define $\varepsilon : \{1, \ldots, n\} \to \mathbb{Z}$ to map j to the unique i such that $j \in \Omega_i$.*

Note that in the context of perfect isometries, the numbers $(-1)^{\varepsilon(i)}$ are known as the "signs" in the "bijection with signs" induced by a perfect isometry.

Theorem 4.4 ([16, Theorem 1])**.** *Assume Λ is a symmetric order. Then any $\mathcal{O}$-algebra Γ which is derived equivalent to Λ is again an $\mathcal{O}$-order, and symmetric.*

We close this section by making Theorem 4.4 constructive. We wish to give an explicit symmetrizing form (as defined below) for Γ, provided we know one for Λ (which we usually do, for instance in the case when Λ is a block of a group ring).

Definition 4.5 (Symmetrizing Form)**.** *Assume in this definition that the Wedderburn-decomposition of A is given by*

$$A \cong \bigoplus_{i=1}^n D_i^{d_i \times d_i} \tag{33}$$

for certain skew-fields D_i (finite-dimensional over K) and certain numbers d_i. Let $\varepsilon_i \in Z(A)$ (for $i \in \{1, \ldots, n\}$) be the central primitive idempotent belonging to the Wedderburn-component $D_i^{d_i \times d_i}$. For any

element $u \in Z(A)^{\times} \cong \bigoplus_i Z(D_i)^{\times}$ define the non-degenerate associative symmetric bilinear form

$$T_u: \ A \times A \to K: \ (a,b) \mapsto \sum_{i=1}^{n} \mathrm{tr.}_{Z(D_i)/K} \, \mathrm{tr.\,red.}_{D_i^{d_i \times d_i}/Z(D_i)}(\varepsilon_i \cdot u \cdot a \cdot b) \tag{34}$$

Here $\mathrm{tr.}_{Z(D_i)/K}: \ Z(D_i) \to K$ denotes the usual trace for field-extensions.

We call a full $\mathcal{O}$-lattice $L \subset A$ self-dual *with respect to* T_u if it is equal to its dual lattice $L^{\sharp} := \{a \in A \mid T_u(a,L) \subseteq \mathcal{O}\}$. If Λ is self-dual with respect to T_u, then we call T_u a symmetrizing form *for* Λ, and u a symmetrizing element.

Remark 4.6. *1. Any non-degenerate symmetric and associative K-bilinear form on A is equal to $T_u(-, =)$ for some $u \in Z(A)^{\times}$. This follows fairly easily from the structure theory of finite-dimensional semisimple algebras.*

2. We sometimes write $T_u(a)$ (where $a \in A$) instead of $T_u(a, 1)$.

Theorem 4.7 (Transfer of the Symmetrizing Form)**.** *Let Λ be symmetric, and let $T \in \mathcal{K}^b(\mathbf{proj}_{\Lambda})$ be a tilting complex. Set $\Gamma = \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$, and $B = \mathrm{End}_{\mathcal{D}^b(A)}(K \otimes T)$. Identify*

$$Z(A) = \bigoplus_{j=1}^{n} Z(\mathrm{End}_A(V_j)) = Z(B) \tag{35}$$

Let $u = (u_1, \ldots, u_n) \in Z(A)^{\times}$ such that Λ is self-dual with respect to the trace bilinear form $T_u : A \times A \to K$ induced by u. Then Γ is self-dual with respect to the trace bilinear form $T_{\tilde{u}} : B \times B \to K$, where

$$\tilde{u} = ((-1)^{\varepsilon(1)} \cdot u_1, \ldots, (-1)^{\varepsilon(n)} \cdot u_n) \in Z(B)^{\times} \tag{36}$$

where ε is as defined in Definition 4.3.

Proof. Let $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n) \in Z(B)$ be an element such that Γ is actually self-dual with respect to $T_{\hat{u}}$. Then the p-valuations of the $\hat{u}_i$ are in fact independent of the particular choice of $\hat{u}$, since the coset $\hat{u} \cdot Z(\Gamma)^{\times} \in Z(B)^{\times}/Z(\Gamma)^{\times}$ is. Furthermore, the $u' \in Z(B)^{\times}$ such that Γ is integral with respect to $T_{u'}$ are precisely the elements of $\hat{u} \cdot (Z(\Gamma) \cap Z(B)^{\times})$. An element of $\hat{u} \cdot Z(\Gamma) \cap Z(B)^{\times}$ lies in $\hat{u} \cdot Z(\Gamma)^{\times}$ if and only in $\nu_p(u'_i) = \nu_p(\hat{u}_i)$ for all i (all of those assertions are elementary). Now assume we had shown that Γ is integral with respect to $T_{\tilde{u}}$. Then we have $\tilde{u} \in \hat{u} \cdot (Z(\Gamma) \cap Z(B)^{\times})$. Thus $\nu_p(\tilde{u}_i) \geqslant \nu_p(\hat{u}_i)$ for all i, and equality for all i holds if and only if $\tilde{u} \in \hat{u} \cdot Z(\Gamma)^{\times}$, that is, if Γ is self-dual with respect to $T_{\tilde{u}}$. So we have seen (up to the assumption above that we have yet to prove) that if Λ is self-dual with respect T_u and Γ is self-dual with respect to $T_{\hat{u}}$, then $\nu_p(u_i) \geqslant \nu_p(\hat{u}_i)$, and, by swapping the roles of Λ and Γ, also $\nu_p(\hat{u}_i) \geqslant \nu_p(u_i)$. In conclusion, we have $\nu_p(\tilde{u}_i) = \nu_p(u_i) = \nu_p(\hat{u}_i)$ for all i, which, by the above considerations, implies that Γ is self-dual with respect to $T_{\tilde{u}}$.

So far we have reduced the problem to showing that Γ is integral with respect to $T_{\tilde{u}}$, which we will do now. So let $\varphi \in \mathrm{End}_{\mathcal{D}^b(\Lambda)}(T)$ (and fix a representative in $\mathrm{End}_{\mathcal{C}^b(\mathbf{proj}_{\Lambda})}(T)$). Then φ^i induces an endomorphism of T^i, and we can decompose the A-module $K \otimes T^i$ as follows

$$K \otimes T^i = \underbrace{H^i(K \otimes T)}_{=:H^i} \oplus \underbrace{\mathrm{Im}(\mathrm{id}_K \otimes d^{i-1})}_{=:Z_-^i} \oplus \underbrace{K \otimes T^i / \mathrm{Ker}(\mathrm{id}_K \otimes d^i)}_{=:Z_+^i} \tag{37}$$

Define π_{H^i}, $\pi_{Z_-^i}$ and $\pi_{Z_+^i}$ to be the corresponding projections. Define $B^i := \mathrm{End}_A(K \otimes T^i)$, $B_H^i :=$

$\pi_{H^i} B^i \pi_{H^i} = \operatorname{End}_A(H^i)$, $B_+^i := \pi_{Z_+^i} B^i \pi_{Z_+^i} = \operatorname{End}_A(Z_+^i)$ and $B_-^i := \pi_{Z_-^i} B^i \pi_{Z_-^i} = \operatorname{End}_A(Z_-^i)$. Now we have

$$
\begin{aligned}
&\sum_i (-1)^i \cdot T_{1_{B^i} \cdot u}(\varphi^i) \\
=\ & \sum_i T_{\pi_{H^i} \cdot \tilde{u}}(\pi_{H^i} \varphi^i \pi_{H^i}) + (-1)^i \cdot T_{\pi_{Z_+^i} \cdot u}(\pi_{Z_+^i} \varphi^i \pi_{Z_+^i}) + (-1)^i \cdot T_{\pi_{Z_-^i} \cdot u}(\pi_{Z_-^i} \varphi^i \pi_{Z_-^i}) \\
\overset{(*)}{=}\ & \sum_i T_{\pi_{H^i} \cdot \tilde{u}}(\pi_{H^i} \varphi^i \pi_{H^i}) \overset{(**)}{=} T_{\tilde{u}}(\varphi)
\end{aligned}
\tag{38}
$$

Here $(*)$ holds because

$$
T_{\pi_{Z_+^i} \cdot u}(\pi_{Z_+^i} \varphi^i \pi_{Z_+^i}) = T_{\pi_{Z_-^{i+1}} \cdot u}(\pi_{Z_-^{i+1}} \varphi^{i+1} \pi_{Z_-^{i+1}})
\tag{39}
$$

as φ is a map of chain complexes. The equality $(**)$ holds in fact just by definition, as we have identified $\bigoplus \operatorname{End}_A(H^i) = B$. The left side is trivially integral, as $\varphi^i \in \operatorname{End}_\Lambda(T^i)$, and $\operatorname{End}_\Lambda(T^i)$ is a self-dual (and so in particular integral) lattice in B^i with respect to $T_{1_{B^i} \cdot u}$. Hence the right side is also integral. So Γ is indeed integral with respect to $T_{\tilde{u}}$. This concludes the proof. $\qquad\square$

5. Partitioning $\widehat{\mathfrak{L}}(\overline{\Lambda})$ by Rational Conditions

Now we continue with what we started in Section 3. We want to define "rational conditions" on lifts that behave well under the map Φ_X, that is, conditions such that Φ_X restricts to a bijective map between the lifts of $\overline{\Lambda}$ that fulfill the given conditions and the lifts of $\overline{\Gamma}$ that fulfill certain corresponding conditions. Probably the simplest of those conditions is to demand that the K-span of Λ shall be Morita-equivalent to a certain semisimple K-algebra A. It follows from the previous section that Φ_X sends lifts of $\overline{\Lambda}$ with K-span Morita-equivalent to A to lifts of $\overline{\Gamma}$ with the same property (and Φ_X^{-1} does it the other way round).

Since it will make things easier for us, we first give a slightly non-standard definition of decomposition matrices (which is linked to the usual definition via Brauer reciprocity, and coincides with the usual definition in the split case).

Definition 5.1 (Decomposition matrix). *We define the decomposition matrix D^Λ of an $\mathcal{O}$-order Λ with $K \otimes \Lambda$ semisimple to be the transposed of the matrix of the canonical map of Grothendieck groups*

$$
K_0(\mathbf{proj}_{k \otimes \Lambda}) \cong K_0(\mathbf{proj}_\Lambda) \to K_0(\mathbf{mod}_{K \otimes \Lambda})
\tag{40}
$$

sending $[P]$ to $[K \otimes P]$ with respect to the bases consisting of projective indecomposable modules on the left and simple $K \otimes \Lambda$-modules on the right. We call this map the "decomposition map". Note that the rows of D^Λ may be thought of as being labeled by the central primitive idempotents in $Z(K \otimes \Lambda)$ (resp. Wedderburn-components of $K \otimes \Lambda$).

Theorem 5.2. *Let $\overline{\Lambda}$ and $\overline{\Gamma}$ be finite-dimensional k-algebras that are derived equivalent. Let the derived equivalence be afforded by the (one-sided) tilting complex $\overline{T}$, and let X be a two-sided tilting complex such that its inverse has restriction to $\mathcal{D}^b(\mathbf{proj}_{\overline{\Lambda}})$ isomorphic to $\overline{T}$. Set $\Phi := \Pi \circ \Phi_X$. Define*

$$
\widehat{\mathfrak{L}}_s(\overline{\Lambda}) := \{(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda}) \mid K \otimes \Lambda \text{ is semisimple }\}
\tag{41}
$$

Then Φ_X induces a bijection

$$
\widehat{\mathfrak{L}}_s(\overline{\Lambda}) \longleftrightarrow \widehat{\mathfrak{L}}_s(\overline{\Gamma})
\tag{42}
$$

The following holds:

(i) If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ are two lifts with $Z(K \otimes \Lambda) \cong Z(K \otimes \Lambda')$, then

$$
Z(K \otimes \Phi(\Lambda, \varphi)) \cong Z(K \otimes \Phi(\Lambda', \varphi'))
\tag{43}
$$

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and every choice of an isomorphism $\gamma : Z(K \otimes \Lambda) \to Z(K \otimes \Lambda')$ gives rise to a (canonically defined) isomorphism $\Phi(\gamma) : Z(K \otimes \Phi(\Lambda, \varphi)) \to Z(K \otimes \Phi(\Lambda', \varphi'))$.

(ii) *If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ are two lifts and $\gamma : Z(\Lambda) \xrightarrow{\sim} Z(\Lambda')$ is an isomorphism, then $\Phi(\gamma) : Z(\Phi(\Lambda, \varphi)) \to Z(\Phi(\Lambda', \varphi'))$ is well defined and an isomorphism as well.*

(iii) *If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}_s(\overline{\Lambda})$ are two lifts, and $\gamma : Z(K \otimes \Lambda) \xrightarrow{\sim} Z(K \otimes \Lambda')$ is an isomorphism such that $D^\Lambda = D^{\Lambda'}$ up to permutation of columns (where rows are identified via γ), then $D^{\Phi(\Lambda, \varphi)} = D^{\Phi(\Lambda', \varphi')}$ up to permutation of columns (where rows are identified via $\Phi(\gamma)$).*

(iv) *If $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}_s(\overline{\Lambda})$ are two lifts with $D^\Lambda = D^{\Lambda'}$ up to permutation of rows and columns then $D^{\Phi(\Lambda, \varphi)} = D^{\Phi(\Lambda', \varphi')}$ up to permutation of rows and columns.*

Proof. The fact that Φ_X induces a bijection between $\widehat{\mathfrak{L}}_s(\overline{\Lambda})$ and $\widehat{\mathfrak{L}}_s(\overline{\Gamma})$ follows from the last section.

Let $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$. Then $Z(K \otimes \Phi(\Lambda))$ is naturally isomorphic to $K \otimes Z(\Phi(\Lambda))$. But there is an isomorphism between $Z(\Lambda)$ and $Z(\Phi(\Lambda, \varphi))$ (letting $c \in Z(\Lambda)$ correspond to the endomorphism of the tilting complex that is given by multiplication with c in every degree). That proves (i), and shows how $\Phi(\gamma)$ should be defined. The claim of (ii) also follows.

To the proof of (iii): Let $T \in \mathcal{C}^b(\mathbf{proj}_\Lambda)$ be the lift of $\overline{T}$ (we identify $k \otimes \Lambda$ and $\overline{\Lambda}$ via φ). Write $\overline{T} = \overline{T}_0 \oplus \overline{T}_1$ such that $\mathcal{G}_{\overline{T}}(\overline{T}_0) \cong 0 \to \overline{P} \to 0$ for a projective indecomposable $\overline{\Gamma}$-module $\overline{P}$. By Remark 3.3 there is a corresponding direct sum decomposition $T = T_0 \oplus T_1$ and we will have $\mathcal{G}_T(T_0) \cong 0 \to P \to 0$, where P is the unique projective indecomposable $\Phi(\Lambda, \varphi)$-module with $k \otimes P \cong \overline{P}$. Then take e_P to be the endomorphism of T inducing the identity on T_0 and the zero map on T_1. Clearly this is a primitive idempotent in $\Phi(\Lambda, \varphi)$ (which is just $\operatorname{End}_{\mathcal{D}^b(\Lambda)}(T)$, so this statement makes sense) with $e_P \Phi(\Lambda, \varphi) \cong P$. So the decomposition number associated to P and the simple $K \otimes \Phi(\Lambda, \varphi)$-module corresponding to the simple $K \otimes \Lambda$-module V_j (under the isomorphism of the centers) is just the $\operatorname{End}_{K \otimes \Lambda}(V_j)$-rank of the image of e_P in $\operatorname{End}_{K \otimes \Lambda}(V_j)^{\delta_j \times \delta_j}$ under the map given in Lemma 4.2. On the other hand (due the way Lemma 4.2 was obtained) this is just the absolute value of the coefficient of $[V_j] \in K_0(\mathbf{mod}_{K \otimes \Lambda})$ in the image under the decomposition map of $\sum_i (-1)^i \cdot [T_0^i] \in K_0(\mathbf{proj}_\Lambda)$. But due to the isomorphism $K_0(\mathbf{proj}_\Lambda) \cong K_0(\mathbf{proj}_{\overline{\Lambda}})$ we can compute this coefficient, and hence the decomposition matrix of $\Phi(\Lambda, \varphi)$, from the knowledge of a direct sum decomposition of $\overline{T}$ and the knowledge of the decomposition matrix of Λ (since the latter determines the map $K_0(\mathbf{proj}_{\overline{\Lambda}}) \longrightarrow K_0(\mathbf{mod}_{K \otimes \Lambda})$). Therefore, if the decomposition matrices of Λ and Λ' coincide, then so do the decomposition matrices of $\Phi(\Lambda, \varphi)$ and $\Phi(\Lambda', \varphi')$. This concludes the proof of (iii). The explicit formula for the decomposition matrix of $\Phi(\Lambda, \varphi)$ we obtained above is in fact independent of the knowledge of $Z(K \otimes \Lambda)$. This implies (iv). $\square$

Remark 5.3. *The last theorem shows that the lifts $(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda})$ that satisfy certain conditions (as listed in the theorem) correspond via Φ_X to lifts $(\Gamma, \psi) \in \widehat{\mathfrak{L}}(\overline{\Gamma})$ that satisfy a corresponding set of conditions. We shall call these kinds of conditions on Λ "rational conditions".*

6. 2-Blocks with Dihedral Defect Group

In this section we specialize K to be the 2-adic completion of the maximal unramified extension of $\mathbb{Q}_2$ (so, in particular, k will be algebraically closed). We fix a finite group G and a block Λ of $\mathcal{O}G$ with dihedral defect group D_{2^n} for some fixed $n \geqslant 3$ (we use the convention where $|D_{2^n}| = 2^n$). Set $A := K \otimes \Lambda$ and $\overline{\Lambda} := k \otimes \Lambda$. For any $i \geqslant 2$ we denote by ζ_i a primitive 2^i-th root of unity in $\overline{K}$ (that is, we fix a choice for each i). In what follows, by a "character" we always mean an absolutely irreducible ordinary character with values in $\overline{K}$.

6.1. Generalities

Lemma 6.1 (Facts from Number Theory). *(i) Define $K_i := K(\zeta_i + \zeta_i^{-1})$. K_i/K is a field extension of degree 2^{i-2}. Its Galois group is cyclic, and we denote by γ_i one of its generators. Hence the subfield lattice of K_i is just a chain, and in fact equal to*

$$K = K_2 \subset K_3 \subset \ldots \subset K_i \tag{44}$$

We denote by $\mathcal{O}_i$ the integral closure of $\mathcal{O}$ in K_i.

(ii) The field extension K_i/K is totally ramified and the 2-valuation of its discriminant is equal to $(i-1) \cdot 2^{i-2} - 1$.

(iii) If G is any finite group, then KG is isomorphic to a direct sum of matrix rings over fields (i. e., no non-commutative division algebras occur in the Wedderburn decomposition of KG).

Proof. (i) This is elementary Galois theory.

(ii) This is [8, Theorem 1]. The result from that paper carries over to our situation without change, as the 2-valuation of the discriminant of $K(\zeta_i + \zeta_i^{-1})/K$ equals the 2-valuation of the discriminant of $\mathbb{Q}(\zeta_i + \zeta_i^{-1})/\mathbb{Q}$ due to both extensions having the same degree.

(iii) To see this let D be a skew-field that occurs in the Wedderburn decomposition of $\mathbb{Q}_2 G$, and denote the center of D by E. Then by [10, Corollary 31.10] the unique unramified extension E' of E of degree equal to the index of D will split D. We may write $E' = E \cdot F$ for some unramified extension F of $\mathbb{Q}_2$. Then $F \otimes_{\mathbb{Q}_2} D$ will be isomorphic to a direct sum of matrix rings over E'. Since F is contained in K, this proves the assertion. $\qquad\square$

Theorem 6.2 (Brauer). *(i) There are precisely $2^{n-2} + 3$ characters in Λ. Four of these characters have height zero, the rest has height one. See [2, Theorem 1].*

(ii) All characters of Λ take values in K_{n-1} (see [2, Proposition (5A)]). There are exactly 5 characters in Λ with values in K. The remaining characters lie in families F_r for $r = 1, \ldots, n-3$, where each F_r is a single $\mathrm{Gal}(K_{n-1}/K)$-conjugacy class of characters. Each F_r consists of 2^r elements (see [2, Theorem 3]). Together with Lemma 6.1 (i) and elementary Galois theory the latter implies that a character in F_r takes values in K_{r+2}.

(iii) The four characters of height zero in Λ take values in K. See [2, Theorem 4].

Note that we may as well denote the one-element set containing the unique K-rational character of height one by F_0, and use indices $r = 0, \ldots, n-3$. The grouping of the characters into four height zero characters and $n-2$ families F_r of height one characters seems more natural in what follows.

Corollary 6.3. *From the above it follows immediately that Λ is an $\mathcal{O}$-order in*

$$A = \bigoplus_{i=1}^{4} K^{\delta_i \times \delta_i} \oplus \bigoplus_{r=0}^{n-3} K_{r+2}^{\delta_r' \times \delta_r'} \quad \text{for certain } \delta_i, \delta_i' \in \mathbb{Z}_{>0} \tag{45}$$

that is self-dual with respect to T_u, where $u = (u_1, u_2, u_3, u_4, \ldots, u_{n+2}) \in Z(A)$ with $\nu_2(u_i) = -n$ for $i = 1, \ldots, 4$ and $\nu_2(u_i) = -n + 1$ for $i > 4$. Of course the analogous statement will hold for a basic order of Λ.

Proof. Lemma 6.1 (iii) implies that $K \otimes \Lambda$ is a direct sum of matrix rings over fields (and not merely skew-fields). Therefore $K \otimes \Lambda$ is Morita-equivalent to its center. From ordinary representation theory we know that given a finite group G we have

$$Z(KG) \cong \bigoplus_{\chi} K(\chi) \tag{46}$$

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where χ runs over representatives for all Galois conjugacy classes of absolutely irreducible characters of G with values in the algebraic closure of K. Theorem 6.2 says that in a block of defect D_{2^n} there are $n + 2$ Galois conjugacy classes of characters (four K-valued characters of height zero and one conjugacy class of K_{r+2}-valued characters for each $r = 0, \ldots, n - 3$). This shows that $K \otimes \Lambda \cong A$ (with A as given in (45) for some choice of numbers δ_i and δ_i'). As for the choice of u, note that the group ring $\mathcal{O}G$ of a finite group G is self-dual with respect to T_u where u is defined as follows:

$$u = \sum_{\chi \in \mathrm{Irr}_{\bar{K}}(G)} \frac{\chi(1)}{|G|} \varepsilon_\chi \in Z(\mathbb{Q}G) \subset Z(KG) \tag{47}$$

Here, ε_χ denotes the central primitive idempotent in $Z(\bar{K}G)$ associated to χ. The entry of this element u in the Wedderburn component of the right hand side of (46) associated to the absolutely irreducible character χ is just $\chi(1)/|G|$. Now the assertions on the heights of the characters in a dihedral block made in Theorem 6.2 imply our assertion on the p-valuations of the u_i. The fact that this symmetrizing element u carries over to a basic order may be seen as a consequence of Theorem 4.7 (since a Morita equivalence is a special case of a derived equivalence). $\qquad\square$

Theorem 6.4 (Erdmann). *The basic algebra of $\overline{\Lambda}$ is isomorphic to one of the algebras of dihedral type in the list given in the appendix of [3].* (Technically, this follows from [3, Lemma IX.2.2] together with the fact that $\overline{\Lambda}$ is known to be of tame representation type and thus has to occur in the list.)

Theorem 6.5 (Holm and Linckelmann). *(i) In Erdmann's classification, the algebras $\mathcal{D}(2A)^{\kappa,c}$ and $\mathcal{D}(2B)^{\kappa,c}$, for any combination $\kappa = 2^{n-2} \geqslant 1$ and $c \in \{0, 1\}$, are derived equivalent. In particular, for fixed n, there are at most two derived equivalence classes of 2-blocks over k with defect group D_{2^n} and two simple modules. See [4].*

(ii) There is precisely one derived equivalence class of 2-blocks over k with defect group D_{2^n} and three simple modules. See [9, Theorem 1].

6.2. Blocks with Two Simple Modules

Assume in this subsection that Λ has precisely two isomorphism classes of simple modules. We first assume that $\overline{\Lambda}$ is Morita-equivalent to $\mathcal{D}(2B)^{\kappa,c}$ for some $c \in \{0, 1\}$ and $\kappa = 2^{n-2}$ (the latter is implied by $\kappa + 3 = \dim_k Z(\mathcal{D}(2B)^{\kappa,c}) = \dim_K Z(A) = 2^{n-2} + 3$). Now let Λ_0 be a basic algebra of Λ. From [3] we know that $k \otimes \Lambda_0 \cong kQ/I$, where

$$Q = \alpha \,\circlearrowright\, \bullet_0 \underset{\beta}{\overset{\gamma}{\rightleftarrows}} \bullet_1 \,\circlearrowleft\, \eta \tag{48}$$

and

$$I = \left\langle \beta\eta, \ \eta\gamma, \ \gamma\beta, \ \alpha^2 - c \cdot \alpha\beta\gamma, \ \alpha\beta\gamma - \beta\gamma\alpha, \ \gamma\alpha\beta - \eta^\kappa \right\rangle \tag{49}$$

We may assume the following rational structure on Λ_0

$\mathbf{Z(A)}$	$\mathbf{u}$	$\mathbf{0}$	$\mathbf{1}$	
K	u_1	1	0	
K	u_1	1	0	
K	u_2	1	1	
K	u_2	1	1	
K_{r+2}	u_3	0	1	[exactly once for each $r = 0, \ldots, n - 3$]

$$\tag{50}$$

where $u_1, u_2 \in K$ have 2-valuation $-n$ and $u_3 \in K$ has 2-valuation $-n + 1$.

Remark 6.6. *We say that a lift Γ of $k \otimes \Lambda_0$ satisfies the rational conditions given above if all of the following conditions hold:*

(i) $K \otimes \Gamma$ is Morita equivalent to $K \oplus K \oplus K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2}$ (so in particular $K \otimes \Gamma$ will be semisimple).

(ii) The decomposition matrix of Γ is as in (50), where the individual rows pertain to the summand of the center that is given on the left of the table.

(iii) There exists some $u = (u_1, u_1, u_2, u_2, u_3, \ldots, u_3) \in K^{n+2} \subseteq K \oplus K \oplus K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2}$ with $\nu_2(u_1) = \nu_2(u_2) = -n$ and $\nu_2(u_3) = -n+1$ such that Γ is self-dual with respect to T_u.

We should probably also explain what we mean when we say that two lifts Γ and Γ' of $k \otimes \Lambda_0$ subject to the above rational conditions have equal *center. The point is that the rows of the decomposition matrix of Γ are canonically in bijection with the Wedderburn components of $Z(K \otimes \Gamma)$ (or, equivalently, central primitive idempotents in $K \otimes \Gamma$). Naturally we demand that there should be an isomorphism $\gamma : Z(\Gamma) \xrightarrow{\sim} Z(\Gamma')$ such that if $\varepsilon \in Z(K \otimes \Gamma)$ is a central primitive idempotent, then the rows in the respective decomposition matrices pertaining to ε respectively $(\mathrm{id}_K \otimes \gamma)(\varepsilon)$ are equal (up to some fixed permutation of the columns).*

Lemma 6.7. *Let*

$$\Gamma \subseteq \mathcal{O} \oplus \mathcal{O} \oplus \bigoplus_{r=0}^{n-3} \mathcal{O}_{r+2} \tag{51}$$

be a local $\mathcal{O}$-order such that $k \otimes \Gamma$ is generated by a single nilpotent element η (so, in particular, $k \otimes \Gamma = k[\eta]$). Furthermore assume that Γ is symmetric with respect to T_u, where $u = (u_1, u_2, u_3 \ldots, u_n) \in K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2}$ with $\nu_2(u_1) = \nu_2(u_2) = -n$ and $\nu_2(u_i) = -n+1$ for all $i > 2$. Then for some $x \in k^\times$ there exists a preimage $\hat{\eta}$ of $x \cdot \eta$ in Γ of the form

$$(0, 4, \pi_0, \ldots, \pi_{n-3}) \tag{52}$$

where the π_r are prime elements in the ring $\mathcal{O}_{r+2}$.

Proof. If $\hat{\eta} = (a, b, d_0 \ldots, d_{n-3})$ is a preimage of η, then $a \in (2)_\mathcal{O}$, and hence $\hat{\eta} - a \cdot (1, \ldots, 1)$ is a preimage of η as well. So we may assume without loss that $a = 0$. Hence some non-zero scalar multiple of η will have a preimage in Γ of the following shape:

$$\hat{\eta} = (0, 2^l, \pi_0, \ldots, \pi_{n-3}) \quad \text{with } \pi_r \in \mathrm{Jac}(\mathcal{O}_{r+2}) \tag{53}$$

Note that we do not know yet that the π_r are prime elements in $\mathcal{O}_{r+2}$. All we can say at this point is $\nu_2(\pi_r) \geqslant 2^{-r}$ (because we know the ramification indices of the extensions K_{r+2}/K to be 2^r). The fact that Γ is self-dual with respect to u implies that

$$\nu_2 \left(\left[\mathcal{O}_{n-1}^{2^{n-2}+1} : \mathcal{O}_{n-1} \otimes_\mathcal{O} \Gamma \right] \right) = \frac{1}{2} \left(2n + (2^{n-2} - 1)(n - 1) \right) \tag{54}$$

Here, for two $\mathcal{O}_{n-1}$-lattices $N \subseteq M$ such that M/N is a torsion module, we denote by $[M : N]$ the product of all elementary divisors of M/N (of course, this is only well-defied up to units). The left-hand side of the above equation is equal to the 2-valuation of the determinant of the $(2^{n-2} + 1) \times (2^{n-2} + 1)$ Vandermonde matrix $M(S)$ associated to the values

$$S = \{s_1, \ldots, s_{2^{n-2}+1}\} := \{0, 2^l, \pi_r^{\alpha_r} \mid r = 0, \ldots, n-3, \; \alpha_r \in \mathrm{Gal}(K_{r+2}/K)\} \tag{55}$$

But the factorization (note that we fix an arbitrary total ordering on the Galois groups $\mathrm{Gal}(K_i/K)$)

$$
\begin{aligned}
\prod_{i>j}(s_i - s_j) = \;\; &\pm 2^l \cdot \left(\prod_{r=0}^{n-3}\prod_{\alpha\in\mathrm{Gal}(K_{r+2}/K)} \pi_r^\alpha\right) \cdot \left(\prod_{r=0}^{n-3}\prod_{\alpha\in\mathrm{Gal}(K_{r+2}/K)}(2^l - \pi_r^\alpha)\right) \\
&\cdot \prod_{r=0}^{n-3}\left(\left(\prod_{q=r+1}^{n-3}\prod_{\alpha\in\mathrm{Gal}(K_{r+2}/K)}\prod_{\beta\in\mathrm{Gal}(K_{q+2}/K)}(\pi_r^\alpha - \pi_q^\beta)\right)\right. \\
&\qquad\qquad \left. \cdot \prod_{\alpha>\beta\in\mathrm{Gal}(K_{r+2}/K)}(\pi_r^\alpha - \pi_r^\beta)\right)
\end{aligned}
\tag{56}
$$

of $\det M(S)$ yields the following estimate of its 2-valuation:

$$
\begin{aligned}
&\nu_2(\det M(S)) \\
\geqslant \;\; & l + \sum_{r=0}^{n-3}\frac{1}{2^r}\cdot 2^r + \sum_{r=0}^{n-3}\frac{1}{2^r}\cdot 2^r + \sum_{r=0}^{n-3}\left(\sum_{q=r+1}^{n-3}(2^{r+q}\cdot\frac{1}{2^q}) + \frac{1}{2}\nu_2\,\mathrm{discrim}_K(K_{r+2})\right) \\
= \;\; & l + 2(n-2) + \sum_{r=0}^{n-3}\left((n-3-r)\cdot 2^r + \frac{1}{2}\left((r+1)\cdot 2^r - 1\right)\right) \\
= \;\; & \frac{1}{2}n + \frac{1}{8}2^n n + l - \frac{1}{8}2^n - \frac{3}{2}
\end{aligned}
\tag{57}
$$

Here we used that for any $x \in \mathrm{Jac}(\mathcal{O}_i)$ we have

$$
\begin{aligned}
\nu_2 \prod_{\alpha>\beta\in\mathrm{Gal}(K_i/K)}(x^\alpha - x^\beta) \;\; = \;\; & \nu_2\left(\left[\mathcal{O}_{n-1}^{2^{i-2}} : \mathcal{O}_{n-1}\otimes_{\mathcal{O}}\mathcal{O}[x]\right]\right) \\
\geqslant \;\; & \nu_2\left(\left[\mathcal{O}_{n-1}^{2^{i-2}} : \mathcal{O}_{n-1}\otimes_{\mathcal{O}}\mathcal{O}_i\right]\right) = \frac{1}{2}\nu_2(\mathrm{discrim}_K(K_i))
\end{aligned}
\tag{58}
$$

Now the right hand side of (54) has to be greater than or equal to the right hand side of (57). This implies $l \leqslant 2$. On the other hand, the assumptions on u would imply that $\nu_2(T_u(\hat\eta)) < 0$ if $l \leqslant 1$, which is of course impossible. Hence $l = 2$, and in particular the "$\geqslant$" in (57) is really an equality, which is easily seen to be equivalent to $\nu_2(\pi_r) = 2^{-r}$ for all $r = 0, \ldots, n-3$. $\qquad\square$

Theorem 6.8. *If* $\Gamma, \Gamma' \in \mathfrak{L}(\mathcal{D}(2B)^{\kappa,c})$ *(where* $\kappa = 2^{n-2}$*) satisfy the rational conditions stated in (50) and* $Z(\Gamma) = Z(\Gamma')$*, then* $\Gamma \cong \Gamma'$*. Furthermore, the existence of such a lift implies* $c = 0$*.*

Proof. Our general approach is to determine the structure of Γ up to some parameters, and then conclude that these parameters are determined by the knowledge of $Z(\Gamma)$. We assume (without loss) that

$$
\Gamma \subseteq \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}^{2\times 2} \oplus \mathcal{O}^{2\times 2} \oplus \bigoplus_{r=0}^{n-3}\mathcal{O}_{r+2}
\tag{59}
$$

Choose lifts $\hat e_0$ and $\hat e_1$ in Γ of the idempotents e_0 and e_1 in $\mathcal{D}(2B)^{\kappa,c}$. Assume without loss that these idempotents $\hat e_0$ and $\hat e_1$ are diagonal in each direct summand on the right hand side of (59) (this is of course only a non-trivial condition in the two summands which 2×2-matrix rings), and identify in the obvious way

$$
\Gamma_{00} := \hat e_0\Gamma\hat e_0 \subseteq \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \quad \Gamma_{11} := \hat e_1\Gamma\hat e_1 \subseteq \mathcal{O}\oplus\mathcal{O}\oplus\bigoplus_{r=0}^{n-3}\mathcal{O}_{r+2}
\tag{60}
$$

$$
\Gamma_{10} := \hat e_1\Gamma\hat e_0 \subseteq \mathcal{O}\oplus\mathcal{O} \qquad\qquad \Gamma_{01} := \hat e_0\Gamma\hat e_1 \subseteq \mathcal{O}\oplus\mathcal{O}
$$

We first look at Γ_{11}. Note that $e_1 \mathcal{D}(2B)^{\kappa,c} e_1 \cong k[\eta]$, and therefore Lemma 6.7 tells us that there is a lift $\hat\eta \in \Gamma_{11}$ of some non-zero scalar multiple of η of the form $(0, 4, \pi_0, \dots, \pi_{n-3})$.

Now we consider Γ_{00}. We may assume without loss that Γ_{00} is equal to the row space of

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 2^a & x & y \\
0 & 0 & 2^b & z \\
0 & 0 & 0 & 2^n
\end{bmatrix}
\quad \text{for certain } a, b \in \mathbb{Z}_{>0} \text{ and } x, y, z \in (2)_{\mathcal{O}}
\tag{61}
$$

We may furthermore assume without loss that $\hat\alpha := [0, 2^a, x, y]$ is a lift of a (non-zero) scalar multiple of α. To see this first note that $\hat\alpha \notin \Gamma_{01} \cdot \Gamma_{10} + 2 \cdot \Gamma_{00}$, and therefore the image of $\hat\alpha$ in $\mathcal{D}(2B)^{\kappa,c}$ will be of the form $c_1 \cdot \alpha + c_2 \cdot \beta\gamma + c_3 \cdot \alpha\beta\gamma$ with $c_1, c_2, c_3 \in k$ and $c_1 \neq 0$. For all $c_1, c_2 \in k$ there is an automorphism of $\mathcal{D}(2B)^{\kappa,c}$ with $\alpha \mapsto \alpha + c_1 \cdot \beta\gamma + c_2 \cdot \alpha\beta\gamma$, $\beta \mapsto \beta$, $\gamma \mapsto \gamma$ and $\eta \mapsto \eta$ (to verify this just plug the right hand sides into the defining relations of $\mathcal{D}(2B)^{\kappa,c}$). Thus we may replace α by an appropriate multiple of the image of $\hat\alpha$ in $\mathcal{D}(2B)^{\kappa,c}$.

Next we look at the trace form T_u to get some restrictions on the parameters (by "$\sim$" we mean "equal up to units in $\mathcal{O}$"):

$$
\begin{aligned}
T_u([1,1,1,1]) &\sim 2^{-n} \cdot \left(2 + 2 \cdot \tfrac{u_2}{u_1}\right) \overset{!}{\in} \mathcal{O} &\implies&\quad \tfrac{u_1}{u_2} \equiv -1 \mod (2^{n-1}) \\
T_u([0,0,2^b,z]) &\sim 2^{-n} \cdot (2^b + z) \overset{!}{\in} \mathcal{O} &\implies&\quad z \equiv -2^b \mod (2^n) \\
&&\overset{\text{w.l.o.g.}}{\implies}&\quad z = -2^b \\
T_u(\hat\alpha) &= 2^{-n} \cdot \left(2^a + (x+y) \cdot \tfrac{u_2}{u_1}\right) \overset{!}{\in} \mathcal{O} &\implies&\quad x + y \equiv -\tfrac{u_1}{u_2} \cdot 2^a \mod (2^n) \\
&&\overset{\text{w.l.o.g.}}{\implies}&\quad x = 2^a - y
\end{aligned}
\tag{62}
$$

Now let $\hat\gamma \in \Gamma_{10}$ and $\hat\beta \in \Gamma_{01}$ be lifts of non-zero scalar multiples of γ and β such that

$$
\hat\beta \cdot \hat\gamma = [0, 0, 2^b, -2^b] + \xi \cdot [0, 0, 0, 2^n] \quad \text{for some } \xi \in \mathcal{O}
\tag{63}
$$

Then we have

$$
\hat\gamma \cdot \hat\beta = [2^b, -2^b + \xi \cdot 2^n, 0, \dots, 0] \in \Gamma_{11}
\tag{64}
$$

Since $\beta\eta = 0$ we have $\frac{1}{2} \cdot \hat\gamma \cdot \hat\beta \cdot \hat\eta \in \Gamma$, and thus

$$
T_u\left(\frac{1}{2} \cdot \hat\gamma \cdot \hat\beta \cdot \hat\eta\right) = u_2 \cdot (-2^{b+1} + \xi 2^{n+1}) \sim 2^{b-n+1} \overset{!}{\in} \mathcal{O} \implies b \geqslant n - 1
\tag{65}
$$

But $a + b = n$ and a, b are both strictly greater than zero. This implies $b = n - 1$ and $a = 1$. To summarize: At this point we know that Γ_{00} is equal to the row space of

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 2 & x & 2 - x \\
0 & 0 & 2^{n-1} & 2^{n-1} \\
0 & 0 & 0 & 2^n
\end{bmatrix}
\quad \text{for some } x \in (2)_{\mathcal{O}}
\tag{66}
$$

Note that (for large n) this row space will not be multiplicatively closed for all values of x. So this gives us a condition on x:

$$
\hat\alpha^2 - 2\hat\alpha = [0, 0, x^2 - 2x, x^2 - 2x] \overset{!}{\in} \left\langle [0, 0, 2^{n-1}, 2^{n-1}], [0, 0, 0, 2^n] \right\rangle_{\mathcal{O}}
\tag{67}
$$

This is equivalent to $x^2 \equiv 2x \mod (2^{n-1})$, which in turn is equivalent to

$$
x \equiv 0 \mod (2^{n-2}) \quad \text{or} \quad x \equiv 2 \mod (2^{n-2})
\tag{68}
$$

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For now let us assume $x \equiv 0 \mod (2^{n-2})$. Then $x = 2^{n-2} \cdot \xi$ for some $\xi \in \mathcal{O}$. But then

$$\hat{\alpha}^2 - 2\hat{\alpha} = \xi(2^{n-3}\xi - 1) \cdot [0, 0, 2^{n-1}, 2^{n-1}] \tag{69}$$

and

$$\hat{\alpha} \cdot \Gamma_{01} \cdot \Gamma_{10} + 2 \cdot \Gamma_{00} \subseteq \langle [0, 0, 0, 2^n] \rangle_{\mathcal{O}} + 2 \cdot \Gamma_{00} \tag{70}$$

Hence α^2 and $\alpha\beta\gamma$ would be linearly independent over k if $\xi(2^{n-3}\xi - 1) \in \mathcal{O}^{\times}$. The relation $\alpha^2 - c \cdot \alpha\beta\gamma$ prohibits this though. Therefore we must have $\xi(2^{n-3}\xi - 1) \in (2)_{\mathcal{O}}$, and thus $\alpha^2 = 0$. This implies the assertion that the existence of a lift implies $c = 0$. Furthermore, if $n > 3$, the fact that $\xi(2^{n-3}\xi - 1) \in (2)_{\mathcal{O}}$ implies $x \equiv 0 \mod (2^{n-1})$. If $n = 3$, the fact that $\xi(2^{n-3}\xi - 1) \in (2)_{\mathcal{O}}$ implies that either $x \equiv 0 \mod (2^{n-1})$ or $x \equiv 2 \mod (2^{n-1})$. Had we started with the assumption $x \equiv 2 \mod (2^{n-2})$, we would in the same fashion have arrived at $x \equiv 2 \mod (2^{n-1})$ (again with the exception of $n = 3$ where $x \equiv 0 \mod (2^{n-1})$ is also possible). Hence independent of our assumptions on x it follows that either $x \equiv 0 \mod (2^{n-1})$ or $x \equiv 2 \mod (2^{n-1})$, which means that Γ_{00} is equal to the row space of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 0 & 2^n \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 0 & 2^n \end{bmatrix} \tag{71}$$

The row space of the second matrix is obtained from the row space of the first matrix by swapping the first two columns. This swapping of columns is induced by an automorphism of $K \otimes \Gamma$. Hence we may assume that we are in the case where Γ_{00} is equal to the row space of the leftmost matrix in (71). Note that the aforementioned automorphism which swaps the first two Wedderburn components of $Z(K \otimes \Gamma)$ might not fix $Z(\Gamma)$. This will however not matter to us since we only use that the projection of $Z(\Gamma)$ to all but the first two Wedderburn components is equal to the projection of $Z(\Gamma')$ to all but the first two Wedderburn components (instead of $Z(\Gamma) = Z(\Gamma')$; in particular, we could have made a slightly stronger assertion in the statement of the theorem).

Now if we project Γ_{00} onto its last two Wedderburn components we get an order $\Gamma'_{00} := \langle [1, 1], [0, 2] \rangle_{\mathcal{O}}$. Clearly Γ_{01} and Γ_{10} are both Γ'_{00}-lattices with the natural action. However, Γ'_{00} has only two non-isomorphic lattices L with $K \otimes L \cong K \otimes \Gamma'_{00}$, namely $L_1 = \mathcal{O} \oplus \mathcal{O}$ and $L_2 = \Gamma'_{00}$. Both of them are self-dual lattices in $K \otimes \Gamma'_{00}$. Assume $\Gamma_{01} = L_1$ (if we assume $\Gamma_{01} \cong L_1$, we may as well assume equality, by means of conjugation). By self-duality of Γ, we would then have $\Gamma_{10} = 2^n \cdot L_1$, and hence $\Gamma_{01}\Gamma_{10} \subset \mathrm{Jac}^2(\Gamma_{00})$. But $\beta\gamma$ certainly is not contained in $\mathrm{Jac}^2(e_0 \mathcal{D}(2B)^{\kappa,c} e_0)$. Hence we have a contradiction. This implies (without loss) $\Gamma_{01} = L_2$ and $\Gamma_{10} = [2^{n-1}, -2^{n-1}] \cdot L_2$.

All that is left to verify is that the choice of the π_i in Γ_{11} can be reconstructed from $Z(\Gamma)$. But from our knowledge of Γ_{00} and Γ_{11} we know that the following element is in $Z(\Gamma)$:

$$[0, 4, 0, 4, \pi_0, \ldots, \pi_{n-3}] \in Z(\Gamma) \subset K \oplus K \oplus K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2} \tag{72}$$

Hence the natural homomorphism $Z(\Gamma) \to \Gamma_{11}$ is surjective. This concludes the proof. $\qquad\square$

Now assume that $\overline{\Lambda}$ is Morita-equivalent to $\mathcal{D}(2A)^{\kappa,c}$. Then we may assume the following rational structure of Λ_0

$\mathbf{Z(A)}$	$\mathbf{u}$	$\mathbf{0}$	$\mathbf{1}$	
K	u_1	1	0	
K	u_1	1	0	
K	u_2	1	1	
K	u_2	1	1	
K_{r+2}	u_3	2	1	[exactly once for each $r = 0, \ldots, n-3$]

(73)

where $u_1, u_2 \in K$ have 2-valuation $-n$ and $u_3 \in K$ has 2-valuation $-n + 1$. We also know from [4] that there is a tilting complex $\overline{T} \in \mathcal{K}^b(\mathbf{proj}_{\mathcal{D}(2A)^{\kappa,c}})$ with $\mathrm{End}_{\mathcal{D}^b(\mathcal{D}(2A)^{\kappa,c})}(\overline{T}) \cong \mathcal{D}(2B)^{\kappa,c}$ looking as follows:

$$\overline{T} = [0 \to P_1 \oplus P_1 \to P_0 \to 0] \oplus [0 \to P_1 \to 0 \to 0] \tag{74}$$

Let X be a two-sided tilting complex the inverse of which restricts to $\overline{T}$. Then clearly Φ_X maps a lift of $\mathcal{D}(2A)^{\kappa,c}$ satisfying the rational conditions (73) to a lift of $\mathcal{D}(2B)^{\kappa,c}$ satisfying the rational conditions (50). Hence we get the following Corollary directly:

Corollary 6.9. *If there is a $\Gamma \in \mathfrak{L}(\mathcal{D}(2A)^{\kappa,c})$ subject to the rational conditions stated in (73), then $c = 0$. In particular, if B is a 2-block of kG with defect group D_{2^n} (where $n \geqslant 3$), and B has exactly two simple modules, then B is Morita-equivalent to either $\mathcal{D}(2A)^{\kappa,0}$ or $\mathcal{D}(2B)^{\kappa,0}$ with $\kappa = 2^{n-2}$.*

Corollary 6.10. *If $\Gamma, \Gamma' \in \mathfrak{L}(\mathcal{D}(2A)^{\kappa,c})$ (where $\kappa = 2^{n-2}$) satisfy the rational conditions stated in (73) and $Z(\Gamma) = Z(\Gamma')$, then $\Gamma \cong \Gamma'$.*

Proof. By Corollary 3.11 Φ_X induces a bijection between $\mathfrak{L}(\mathcal{D}(2A)^{\kappa,c})$ and $\mathfrak{L}(\mathcal{D}(2B)^{\kappa,c})$. Note that Φ_X maps the lifts of $\mathcal{D}(2A)^{\kappa,c}$ satisfying rational conditions as in (50) to lifts of $\mathcal{D}(2B)^{\kappa,c}$ satisfying rational conditions as in (73). Hence our assertion follows from Theorem 6.8. $\qquad\square$

6.3. Explicit Computation of the Lifts

In this section we will compute the unique lift of $\mathcal{D}(2A)^{\kappa,c}$ explicitly (depending, of course, on a prescribed center). We know already that we may assume $c = 0$. Define a complex of $\mathcal{D}(2B)^{\kappa,0}$-modules

$$\overline{T} := \underbrace{0 \to P_1 \oplus P_1 \xrightarrow{\left[\begin{smallmatrix} \gamma \\ \gamma\alpha \end{smallmatrix}\right]} P_0 \to 0}_{=:\overline{T}_0} \oplus \underbrace{0 \to P_1 \to 0 \to 0}_{=:\overline{T}_1} \tag{75}$$

Here, for the sake of simplicity, we identify the generators of $\mathcal{D}(2B)^{\kappa,0}$ with homomorphisms between projective indecomposables satisfying the same relations as the original generators (as opposed to the opposite relations). We can do this since the algebra $\mathcal{D}(2B)^{\kappa,0}$ is isomorphic to its opposite algebra (it even carries an involution).

Remark 6.11. *The algebra $\mathcal{D}(2A)^{\kappa,0}$ has* Ext-*quiver*

$$Q' = \ \alpha' \ \overset{\curvearrowright}{\bullet_0} \underset{\beta'}{\overset{\gamma'}{\rightleftarrows}} \bullet_1 \tag{76}$$

with ideal of relations

$$I' = \left\langle \gamma'\beta', \ \alpha'^2, \ (\alpha'\beta'\gamma')^\kappa - (\beta'\gamma'\alpha')^\kappa \right\rangle_{kQ'} \tag{77}$$

where $\kappa = 2^{n-2}$. Its Cartan matrix is

$$\begin{bmatrix} 4\kappa & 2\kappa \\ 2\kappa & \kappa + 1 \end{bmatrix} \tag{78}$$

Lemma 6.12. *$\overline{T}$ as defined in (75) is a tilting complex with endomorphism ring $\mathcal{D}(2A)^{\kappa,0}$.*

Proof. First note that γ and $\gamma\alpha$ form a k-basis of $\mathrm{Hom}(P_1, P_0)$. and $\beta, \alpha\beta$ form a k-basis of $\mathrm{Hom}(P_0, P_1)$. Now let $\varphi = c_1 \cdot \beta + c_2 \cdot \alpha\beta \in \mathrm{Hom}(P_0, P_1)$. Then

$$\begin{bmatrix} \gamma \\ \gamma\alpha \end{bmatrix} \cdot \varphi = 0 \iff \begin{bmatrix} c_2 \cdot \gamma\alpha\beta \\ c_1 \cdot \gamma\alpha\beta \end{bmatrix} = 0 \iff \varphi = 0 \tag{79}$$

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This implies $\mathrm{Hom}(\overline{T}_0, \overline{T}[-1]) = 0$ (already in $\mathcal{C}^b(\mathbf{proj}_{\mathcal{D}(2A)^{\kappa,0}})$). $\mathrm{Hom}(\overline{T}_1, \overline{T}[-1]) = 0$ is clear since in any degree at least one of these complexes is the zero module. Now assume $\varphi = c_1 \cdot \gamma + c_2 \cdot \gamma\alpha \in \mathrm{Hom}(P_1, P_0)$. Then clearly

$$\varphi = \left[\begin{array}{cc} c_1 & c_2 \end{array}\right] \cdot \left[\begin{array}{c} \gamma \\ \gamma\alpha \end{array}\right] \tag{80}$$

which implies that every chain map from $\overline{T}$ to $\overline{T}[1]$ is homotopic to zero. Furthermore $\overline{T}$ generates $\mathcal{D}^b(\mathcal{D}(2B)^{\kappa,0})$, since $P_1[1]$ is a summand of $\overline{T}$, and the mapping cone of the projection map $\overline{T}_0 \to \overline{T}_1 \oplus \overline{T}_1$ is isomorphic to $P_0[0]$. So we have seen that $\overline{T}$ is a tilting complex.

Now we claim that the endomorphisms

$$
\begin{array}{ccc}
P_1 \oplus P_1 & \xrightarrow{\left[\begin{array}{c} \gamma \\ \gamma\alpha \end{array}\right]} & P_0 \\
{\scriptstyle \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} \\
P_1 \oplus P_1 & \longrightarrow & P_0
\end{array}
\tag{81}
$$

(which we denote by α') and

$$
\begin{array}{ccccccc}
P_1 \oplus P_1 & \longrightarrow & P_0 & & P_1 & \longrightarrow & 0 \\
{\scriptstyle \left[\begin{array}{c} 0 \\ 1 \end{array}\right]}\Big\downarrow & & \Big\downarrow & & {\scriptstyle \left[\begin{array}{cc} \eta & 0 \end{array}\right]}\Big\downarrow & & \Big\downarrow \\
P_1 & \longrightarrow & 0 & & P_1 \oplus P_1 & \longrightarrow & P_0
\end{array}
\tag{82}
$$

(which we denote by β' and γ') together with the idempotent endomorphisms coming from the decomposition $\overline{T} = \overline{T}_0 \oplus \overline{T}_1$ (which we denote by e_0' and e_1') generate the endomorphism ring of $\overline{T}$. To prove this, we determine the dimension of the subalgebra of $\mathrm{End}(\overline{T})$ they generate. It should be noted that one can deduce from the shape of $\overline{T}$ and the Cartan matrix of $\mathcal{D}(2B)^{\kappa,0}$ that the Cartan matrix of $\mathrm{End}(\overline{T})$ is equal to that of $\mathcal{D}(2A)^{\kappa,0}$. First look at the endomorphism ring of $\overline{T}_0$ in the category $\mathcal{C}^b(\mathcal{D}(2B)^{\kappa,0})$ (which we identify as a subring of $\mathrm{End}(P_1 \oplus P_1) \oplus \mathrm{End}(P_0)$). Here α' and $\beta' \cdot \gamma'$ generate the subalgebra

$$\left\langle \left(\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], 1 \right) \right\rangle_k \oplus \left\langle \left(\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], \alpha \right) \right\rangle_k \oplus \left(\left[\begin{array}{cc} \eta k[\eta] & \eta k[\eta] \\ \eta k[\eta] & \eta k[\eta] \end{array}\right], 0 \right) \tag{83}$$

which has dimension $2 + 4 \cdot 2^{n-2}$. The zero-homotopic endomorphisms generate the subspace

$$\left\langle \left(\left[\begin{array}{cc} \gamma\alpha\beta & 0 \\ 0 & 0 \end{array}\right], \alpha\beta\gamma \right), \left(\left[\begin{array}{cc} 0 & \gamma\alpha\beta \\ 0 & 0 \end{array}\right], 0 \right), \left(\left[\begin{array}{cc} 0 & 0 \\ \gamma\alpha\beta & 0 \end{array}\right], \beta\gamma \right), \left(\left[\begin{array}{cc} 0 & 0 \\ 0 & \gamma\alpha\beta \end{array}\right], \alpha\beta\gamma \right) \right\rangle_k \tag{84}$$

which has two-dimensional intersection with the vector space in (83). Hence the subalgebra of the endomorphism ring (in $\mathcal{D}^b(\mathcal{D}(2B)^{\kappa,0})$) of $\overline{T}_0$ generated by α' and $\beta' \cdot \gamma'$ is 2^n-dimensional. Since we know the dimension of $\mathrm{End}(\overline{T}_0)$ to be 2^n, it follows that α' and $\beta' \cdot \gamma'$ generate $\mathrm{End}(\overline{T}_0)$.

With much less effort one can see that (in the category $\mathcal{C}^b(\mathcal{D}(2B)^{\kappa,0})$) we have $\mathrm{Hom}(\overline{T}_0, \overline{T}_1) \cong k[\eta] \oplus k[\eta]$, and β' generates this space as an $\mathrm{End}(\overline{T}_0)$-module. Similarly $\mathrm{Hom}(\overline{T}_1, \overline{T}_0) \cong \eta k[\eta] \oplus \eta k[\eta]$ and γ' generates this space as an $\mathrm{End}(\overline{T}_0)$-module. Furthermore $\gamma' \cdot \alpha' \cdot \beta' = \eta$ generates $\mathrm{End}(\overline{T}_1) = \mathrm{End}(P_1)$ as a k-algebra. The above considerations imply that $e_0', e_1', \alpha', \beta'$ and γ' generate the endomorphism ring (in $\mathcal{D}^b(\mathcal{D}(2B)^{\kappa,0})$) of $\overline{T}$ as a k-algebra.

Now one can easily verify that α', β' and γ' satisfy the relations given in (76), and this is all we have to check, since we know that the endomorphism ring of $\overline{T}$ has the same dimension as $\mathcal{D}(2A)^{\kappa,0}$. $\qquad\square$

Theorem 6.13. *Define K-algebras A and B as follows:*

$$A := K \oplus K \oplus K^{2\times2} \oplus K^{2\times2} \oplus \bigoplus_{r=0}^{n-3} K_{r+2} \quad B := K \oplus K \oplus K^{2\times2} \oplus K^{2\times2} \oplus \bigoplus_{r=0}^{n-3} K_{r+2}^{3\times3} \qquad (85)$$

Define idempotents $\hat{e}_0, \hat{e}_1 \in A$:

$$\hat{e}_0 := \left(1, 1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0, \dots, 0\right) \quad \hat{e}_1 := 1_A - \hat{e}_0 \qquad (86)$$

and define idempotents $\hat{e}'_0, \hat{e}'_1 \in B$:

$$\hat{e}'_0 := \left(1, 1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \quad \hat{e}'_1 := 1_B - \hat{e}'_0 \qquad (87)$$

Any lift Λ of $\mathcal{D}(2B)^{\kappa,0}$ subject to the rational conditions in (50) is isomorphic to the $\mathcal{O}$-order in A generated by the idempotents $\hat{e}_0$, $\hat{e}_1$ and

$$\begin{array}{rclcccc}
\hat{e}_0 A \hat{e}_0 \ni \hat{\alpha} & = & (0, & 2, & 0, & 2) & \\
\hat{e}_1 A \hat{e}_1 \ni \hat{\eta} & = & & (0, & 4, & \pi_0, & \dots, & \pi_{n-3}) \\
\hat{e}_0 A \hat{e}_1 \ni \hat{\beta} & = & & (1, & 1) & \\
\hat{e}_1 A \hat{e}_0 \ni \hat{\gamma} & = & & (2^{n-1}, & 2^{n-1}) &
\end{array} \qquad (88)$$

for certain prime elements $\pi_i \in K_{i+2}$. Any lift Γ of $\mathcal{D}(2A)^{\kappa,0}$ subject to the rational conditions in (73) is isomorphic to the $\mathcal{O}$-order in B generated by the idempotents $\hat{e}'_0$, $\hat{e}'_1$ and

$$\hat{e}'_0 B \hat{e}'_0 \ni \hat{\alpha}' = \left(0, \ 2, \ 2, \ 0, \ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \ \dots, \ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right)$$

$$\hat{e}'_0 B \hat{e}'_1 \ni \hat{\beta}' = \left(1, \ 1, \ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \dots, \ \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \qquad (89)$$

$$\hat{e}'_1 B \hat{e}'_0 \ni \hat{\gamma}' = \left(-2, \ -2, \ \begin{bmatrix} \pi_0 & -2 \end{bmatrix}, \ \dots, \ \begin{bmatrix} \pi_{n-3} & -2 \end{bmatrix}\right)$$

for certain prime elements $\pi_i \in K_{i+2}$. In particular, any block with dihedral defect group D_{2^n} and two simple modules is isomorphic to an order of one of the above shapes.

Furthermore, if X is a two-sided tilting complex the inverse of which restricts to $\overline{T}$, the lifts of (88) and (89) with equal π_i correspond to each other under the bijection Φ_X.

Proof. We have already seen in the proof of Theorem 6.8 that Λ has to be as in (88). We did however not see (and in general it is not true) that $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\eta}$ may be assumed to be lifts of the elements α, β, γ and η. What we did see is that $\hat{\alpha}$ and $\hat{\eta}$ may be assumed to reduce to scalar multiples of α and η. Since we will need it below we now show that we may in fact assume that $\hat{\alpha}, \hat{\gamma}$ and $\hat{\eta}$ reduce to α, γ and η. To see that one simply verifies that for all $c_1, c_2, c_3, c_4 \in k$ with $c_1, c_2, c_4 \neq 0$ the following

$$\mathcal{D}(2B)^{\kappa,0} \longrightarrow \mathcal{D}(2B)^{\kappa,0} : \ \alpha \mapsto c_1 \alpha, \ \beta \mapsto \frac{c_4^\kappa}{c_1 c_2} \beta + \frac{c_3 c_4^\kappa}{c_1 c_2^2} \alpha\beta, \ \gamma \mapsto c_2 \gamma + c_3 \gamma\alpha, \ \eta \mapsto c_4 \eta \qquad (90)$$

defines an automorphism of $\mathcal{D}(2B)^{\kappa,0}$.

Now we show that Γ as given in (89) equals $\Phi_X(\Lambda)$. We choose

$$T := \underbrace{0 \to \hat{P}_1 \oplus \hat{P}_1 \xrightarrow{\left[\begin{smallmatrix} \hat{\gamma} \\ \hat{\gamma}\hat{\alpha} \end{smallmatrix}\right]} \hat{P}_0 \to 0}_{=:T_0} \ \oplus \ \underbrace{0 \to \hat{P}_1 \to 0 \to 0}_{=:T_1} \tag{91}$$

as a lift of $\overline{T}$ (where the $\hat{P}_i$ are the projective indecomposable Λ-modules). Now

$$
\begin{array}{ccc}
\hat{P}_1 \oplus \hat{P}_1 & \longrightarrow & \hat{P}_0 \\
{\scriptstyle \left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right]} \downarrow & & \downarrow {\scriptstyle \hat{\alpha}} \\
\hat{P}_1 \oplus \hat{P}_1 & \longrightarrow & \hat{P}_0
\end{array}
\tag{92}
$$

is a lift of α' (which we denote by $\hat{\alpha}'$), and

$$
\begin{array}{ccc}
\hat{P}_1 \oplus \hat{P}_1 & \longrightarrow & \hat{P}_0 \\
{\scriptstyle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \downarrow & & \downarrow \\
\hat{P}_1 & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccc}
\hat{P}_1 & \longrightarrow & 0 \\
{\scriptstyle \left[\begin{smallmatrix} \hat{\eta} & -2 \end{smallmatrix}\right]} \downarrow & & \downarrow \\
\hat{P}_1 \oplus \hat{P}_1 & \longrightarrow & \hat{P}_0
\end{array}
\tag{93}
$$

are lifts of β' and γ' (which we denote by $\hat{\beta}'$ and $\hat{\gamma}'$). We now have to calculate the action of those endomorphisms on homology. For that identify $K \otimes P_0 \cong K \oplus K \oplus K \oplus K$ and $K \otimes P_1 \cong K \oplus K \oplus \bigoplus_{r=0}^{n-3} K_{r+2}$. Only for the third and fourth Wedderburn-component we need to do any actual work. Choose $\begin{bmatrix} 0 & 1 \end{bmatrix}$ as a basis for the projection of the kernel of the differential to the third Wedderburn-component, and $\begin{bmatrix} -2 & 1 \end{bmatrix}$ as a basis of the projection to the fourth Wedderburn-component. Now, for instance,

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = 0 \cdot \begin{bmatrix} -2 & 1 \end{bmatrix} \tag{94}$$

which leads to the corresponding entries of $\hat{\alpha}'$ in the third and fourth Wedderburn-component. $\qquad\square$

Acknowledgments

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