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CAPITAL REQUIREMENTS AND OPTIMAL INVESTMENT WITH SOLVENCY PROBABILITY CONSTRAINTS

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Abstract

Quantifying the economic capital and optimally allocating it into portfolios of financial instruments are two key topics in the Asset/Liability Management (ALM) of an insurance company. In general these problems are studied in the literature by minimizing standard risk measures such as the Value at Risk (VaR) and the Conditional Value at Risk (CVaR). Motivated by Solvency II regulations, we introduce a novel optimization problem to solve for the optimal required capital and the portfolio structure simultaneously, when the ruin probability is used as an insurance solvency constraint.

Besides the generic optimal required capital and portfolio problem formulation, we propose a two-model hierarchy of optimization models, where both models admit the so-called second-order conic reformulation, in turn making them particularly well suited for numerics. The first model, albeit naively asserting the normality of the returns on assets and liabilities, under minor further simplifications admits a closed form solution – a set of formulas, which may be used as simple decision-making guidelines in the analysis of more complex scenarios. A potentially more realistic second model aims to represent the "heavy-tailed" nature of the insurer's liabilities more accurately, while also allowing arbitrary distributions of asset returns via a semi-parametric approach. Extensive numerical simulations illustrate the sensitivity and robustness of the proposed approach relative to model's parameters. In addition, we explore the potential of insurance risk diversification and discuss if combining several liabilities into a single insurance portfolio may always be beneficial for the insurer. Finally, we propose an extension of the model with an expected return on capital constraint added.

Keywords and phrases: Optimal investment; portfolio efficient frontier; risk capital; chance contrained programming; Solvency II; second-order cone programming.

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1 Introduction

Asset allocation is one of the central issues in banking, finance and insurance industries. Markowitz (1952) pioneered the development of a quantitative single period model for optimal asset allocation based on Gaussian returns. Using the variance, or the standard deviation, of a portfolio's return as a measure of risk, the practical asset allocation problem comes down to a mean-variance optimization problem. This simplifies the problem and conveys essential intuition for analyzing the relationship between risk and return. The asset allocation problem was further investigated by Samuelson (1969) and Merton (1969) in a multi-period set-up and a continuous-time model, respectively. Many empirical studies have illustrated that the normality assumption in modeling asset returns is often violated, motivating the proposal of other distributions for capturing the skewness and excess kurtosis exhibited by financial data. Thus, the use of variance or standard deviation in the context of portfolio optimization might not be appropriate.

Value at Risk (VaR) has emerged as a simple and powerful tool for risk measurement, and describes the risk of a trading position by a single number such that the actual loss from the position exceeds this number over a fixed time period with a certain (small) probability. In practice, the time period is usually fixed as one day or two weeks, and the probability level is typically set as either 1% or 5%, respectively. For example, in 1995 the Basel Committee on Banking Supervision proposed the use of a 10 day VaR at 1% level. Despite its popularity, VaR has a few pitfalls. The major limitations consist of the failure to provide any information regarding the risk beyond its targeted confidence and to illustrate the intuitive advantage of risk aggregation. Artzner et al. (1999) recommended the use of *Tail Conditional Expectation*. Various equivalent formulations appeared in the literature, such as *Conditional Value at Risk* (CVaR), *Expected Shortfall* or *Tail Value-at-Risk*, and all of them express the same exposure to risk as long as the underlying risk distribution is continuous. Acerbi and Tasche (2002) and Hürliman (2003) provided a thorough description of all these risk measures. Essentially, CVaR is the average loss from a trading position when the loss of the position exceeds its VaR level. Among the advantages provided by the CVaR, its coherency properties for measuring risk and computational tractability are listed.

Portfolio optimization under VaR and CVaR has been the objective of many research studies in the financial literature in the last decade. Rockafellar and Uryasev (2000) investigated a portfolio optimization problem using a risk-based criterion based on CVaR. Their proposal reformulates the problem into a linear programming optimization by introducing a large number of auxiliary variables. This approach has been widely used in the literature; for example, Rockafellar and Uryasev (2002) used it for optimal allocation with general loss functions and Krokhmal et al. (2002) used a CVaR constraint to improve the skewness of a mean-variance optimization under transaction costs. In addition, several algorithms for dealing with scenario-based VaR optimization problems have been proposed (e.g. Gaivoronski and Pflug, 2004). Alexander et al. (2006) analyzed VaR/CVaR optimization for portfolio of derivatives and investigated the well-posedness of these problems. More recently, the modeling of risk-reward optimization problems has been extended by including other risk measures such as stochastic dominance, downside risk measures, deviation measures etc. Krokhmal et al. (2011) provided a survey of the most recent contributions.

Optimal investment is a key problem in *Asset-Liability Management* (ALM) of an insurance company and not only. Instead of allocating wealth optimally in order to maximize the overall investment return, an insurance company is interested in assessing the risk exposure where both assets and liabilities are included, and minimizing the risk of mis-match between assets and liabilities. Ferrari (1967), Kahane and Nye (1975) and Cummins and Nye (1981) were among the first to investigate portfolio selection for an insurer. The standard approach has been to apply the Markowitz portfolio theory to the ALM setting, and it has been further extended to other risk measures. For example, Bogentoft et al. (2001) introduced an optimization problem for a pension fund based on CVaR constraints, while Tian et al. (2010) studied a mean-variance asset-liability portfolio problem by adding multiple CVaR constraints.

The computation of capital requirements imposed by financial or insurance regulators is typically based on some of the above-mentioned risk measures.[5] For example, the US *Risk-based capital* (RBC) is constructed using the *Expected Policyholder Deficit* (EPD) concept (e.g. see Bustic (1994)), while Swiss-based insurers adopted the *Swiss Solvency test*, according to which the targeted capital is evaluated via the *Expected Shortfall* risk measure (FOPI, 2004). The regulatory capital designed for the insurance companies that operates within the European Union is specified in the recent *Solvency II Directive*, and it should ensure solvability over a one year period with a 99.5% probability, which can be further linked to a VaR computation (e.g. Sandström, 2006). An overview of advantages/disadvantages of the Solvency II regulations can be found in Eling et al. (2007). Cummins and Phillips (2009) provide a detailed comparison of these three capital standards.

In this paper, we construct a new optimization problem such that both, the capital requirement and optimal portfolio allocation for a non-life insurance company, are addressed. More specifically, the proposed approach relies on simultaneously minimizing the capital required according to a solvency criteria and optimally allocating it into a well-diversified portfolio of financial assets. Following the Solvency II regulations, the proposed solvency principle is defined through a *chance constraint*, which ensures that ruin occurs (i.e. liabilities exceeds the market value of company's assets) with a very small probability over a fixed period of time In general, except for some specific assumptions on the model distribution, such chance constraints do not have a closed form representation, and therefore different approximations are needed.[6]

We investigate our optimization problem using a two-model hierarchy approach. First, we assume that assets and liabilities have a multivariate normal distribution. In this case, the chance constrained can be written as a certainty equivalent constraint, and this allows us to reformulate our problem as a

[5]A review of theoretical properties of the most important risk measures used for quantifying the solvency capital can be found in Dhaene et al. (2006).

[6]Chance constrained programming was first introduced by Charnes et al. (1958) and was applied to portfolio selection for a non-life insurance company by Agnew et al. (1969). For theoretical background and recent advances and applications of stochastic programming based on chance constrained we refer to Prekopa (1995) and Shapiro et al. (2009), and the references therein.

second-order conic (SOC) optimization, which can be efficiently handled by standard solvers. Moreover, when portfolio short-selling is permitted, we provide conditions for the existence of a feasible solution and we derive a closed form expression for the optimal capital and portfolio allocation. Since both financial and loss data typically exhibit skewness and leptokurtosis, we need to investigate our problem under more realistic assumptions on the model distributions. Different approximations techniques have been recently developed for general probabilistic constraints. For example, Caliafore and Campi (2005) and Nemirovski and Shapiro (2005) proposed a scenario based approximation, while Nemirovski and Shapiro (2006) explored convex approximations based on the Bernstein scheme. Luedke and Ahmed (2008) used the sample average approximation method, which consists of replacing the probability constraint by its Monte-Carlo estimator. Depending on the objective function, the implementation of their approach is based on a non-convex *mixed-integer programming* (MIP) formulation. However, when the number of Monte-Carlo paths is too large, the implementation of MIP may not be efficiently performed with appropriate solvers.

Our approach is different and it is based on a semiparametric approximation of the solvency probability, by combining a parametric specification for the aggregate loss survival function with the empirical distribution of asset returns. Assuming standard regularity conditions required for the uniform convergence of the proposed Monte-Carlo estimator are satisfied, the equivalent problem is convex, provided that the aggregate liability random variable exhibits a convex tail behaviour. We test the accuracy of the semiparametric approximation for Gaussian distributed assets and losses. Our numerical results suggest that the optimal solutions are stable; moreover, even for a relatively small number of simulations, their values are very close to the ones obtained based on the SOC Mosek solver, which also coincides with the closed-form formula.

The Pareto distribution is one of the most commonly used distributions for modeling extreme events, and it has been generally used for fitting large non-life insurance claim data for various business lines (e.g. see McNeil (1997)). Since accurate modeling of extreme losses is crucial in any solvency analysis, we further explore the semiparametric approach based on Pareto liabilities. First, we give a SOC reformulation for our optimization problem, and then we numerically compare the the performance of two solvers, the SOC Mosek solver and the Matlab' s standard `fmincon` routine. We did not find any significant differences between the two methods relative to the optimal capital and portfolio structure. However, when the number of assets included in the portfolio increases, the Mosek solver becomes faster, as measured by the CPU time. The Pareto setup is further explored by providing a sensitivity analysis of the optimal solutions with respect to the loss parameters.

Finally, further extensions of our model are discussed. First, we provide a detailed numerical experiment to analyze the effects of diversifying the insurance risk by combining two portfolios with Pareto liabilities, for different values of the tail index and scale of the distribution The particular choice of our model parameters results in a non-negative diversification gain in all cases considered. However, this is not always the case, a simple scenario-based counter example being provided in this sense. The second extension relies on adding a more practical flavour to our problem, by further assuming an expected return on capital constraint along with the solvency requirement. For a portfolio formed with one riskless and two risky assets we construct efficient frontiers based on different levels of shareholders' required return on capital. Furthermore, we analyze the behaviour of the optimal regulatory capital and portfolio allocations when the correlation coefficient between the two assets varies, and in the case of Pareto and log-normally distributed liabilities.

The rest of the paper is organized as follows. In the next section, we introduce the joint optimization problem and the standard SOC framework. The two-model hierarchy, namely the Gaussian assumptions and the semiparametric approach, is proposed in Section 3. We derive exact solutions for our problem in the Gaussian case and we give a further SOC reformulation when losses are Pareto distributed. In Section 4 we provide extensive numerical simulations to assess the accuracy of our proposed method and we discuss on the advantages of having the SOC representation. Further extensions on insurance risk diversifications and optimization with expected return on capital constraint are discussed in Section 5. The paper concludes in Section 6.

2 General approach and preliminaries

2.1 The approach

In this subsection, a one-period portfolio optimization problem is considered for an all equity financed insurance company with one line of business. The insurer liability random variable Y represents a future claim payable at the end of the period. With multiple business lines, this random variable can be interpreted as the aggregate claim amount from a portfolio of claim payoffs. In exchange for paying the future liabilities, the insurance company receives a lump sum at the beginning of the period which is denoted by p. The choice for the amount of premiums is not relevant for our work, and all our numerical investigations assume the total premium to be proportional to the liability expected amount.

Let c denote the regulatory capital imposed to the insurer, which will be set to maintain a target solvency requirement at the end of the period. This capital is generally provided by the insurance company's shareholders and it is invested in the financial markets, so that, the total assets available for investment at the beginning of the period is $p+c$, and the insurance company can be viewed as a levered entity.

Investment decisions are assumed to be made initially over a one period with maturity t, when the liabilities are due to be paid. For example, the standard time frame proposed by Solvency II Directive is a one year period. A well diversified portfolio consisting of n assets is available for investment for which $\mathbf{R} = (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ represents the random vector describing the asset *gross return* processes; note $\mathbf{R}$ is nonnegative by definition. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be the vector of portfolio weights whose components satisfy both the standard budget constraint, $\sum_{i=1}^{n} \mathbf{x}_i = 1$ and the no short sales constraint, $\mathbf{x}_i \geq 0$, $i =$

$1, \ldots, n$.

The insurance company net loss at time t is defined as the excess of liabilities over assets:

$$L(c, \mathbf{x}) = Y - (p + c)\mathbf{R}^T\mathbf{x} \,. \tag{2.1}$$

In the absence of liabilities, the traditional approach in portfolio optimization is to minimize a given risk measure of the portfolio loss function subject to performance measure constraints. Similar ALM approaches have been followed by Cummins and Nye (1981), who investigated the optimal allocation problem for Property-Liability insurance companies using a mean-variance technique, and by Li (1995), who constructed quadratic programming problems with chance constraints. The vast majority of the current literature relies on only solving for the optimal portfolio allocation decision variable.

Our main objective is to construct an optimization problem that jointly solves for c and $\mathbf{x}$ and is consistent with the current regulatory regimes. Depending on how the solvency requirement is defined, one may construct appropriate optimization problems based on finding the “best” choice of the risk capital. For example, Bustic (1994) uses a target level for the *Expected Policyholder Deficit* (EPD) in defining the insurance insolvency, while Barth (2000) compares the risk capital based on EPD and ruin probability. Djehiche and Horfelt (2004) compute optimal capital when solvency is maintained with a negative value of a Standard Deviation Principle and a VaR risk measure for the portfolio loss random variable. More recently, Mankai and Bruneau (2009) propose a joint optimization problem for a non-life insurance company by maximizing the expected *Return on Risk Adjusted Capital* using a CVaR constraint.

Our approach is based on the recent regulatory capital requirements defined for the European based insurance companies, which uses the ruin probability as a measure of risk. More specifically, we identify the minimum risk capital c which corresponds to a ruin probability of 0.5% over a one year period. Essentially, this is given by the capital that allows insolvency for at most one in 200 possible scenarios within the considered period. Shareholders are typically looking to minimize the level of capital subject to the solvency constraint and a target level of expected return on the invested capital. However, there is a tradeoff theory regarding the optimal size of the capital provided by the shareholders. On the one hand, a high value of regulatory capital reduces the cost of financial distress, while a smaller amount would not benefit the owners due to increased frictional costs. As a simplification to the approach, it is assumed that there are no transactions or financial distress costs.

We introduce the general one-period optimization problem for finding the optimal capital requirement and portfolio allocation problem as follows:

$$\begin{aligned} & \min c \\ \text{s.t.} \quad & \Pr(L(c, \mathbf{x}) \ > 0) \leq \beta, \\ & \sum_{i=1}^{n} \mathbf{x}_i = 1, \\ & \mathbf{x} \geq 0, c \geq 0. \end{aligned} \tag{2.2}$$

Here, β represents the specified solvency level which is set to be 0.5% for all our numerical simulations. Obviously, the existence of a global solution for (2.2) is dictated by the convexity of the solvency probability constraint. We further notice that the chance constraint from (2.2) is equivalent to a VaR constraint. Although a few heuristic methods for solving optimization problems under VaR constraints based on discrete distributions for asset returns have been proposed in the literature[7], and their implementation into our ALM setting might be of potentially future interest, the objective of this paper is to provide algorithms for dealing exclusively with chance constrained programming.

2.2 Second-order conic programming

To a large extent throughout this study we rely on the second-order conic programming framework to solve problem (2.2) under various modeling assumptions. In turn, recasting (2.2) as SOC optimization problem allows for the use of highly efficient numerical software, such as Mosek, SeDuMi, CPLEX, Gurobi, etc., with many of these numerical packages already being very well recommended and recognized in both academic and industrial settings. In addition to the above, the SOC recasting of the most stylized model even permits a derivation of the closed-form solution.

Let us recall the SOC optimization framework next. The SOC optimization problem in the primal form is typically formulated as

$$\begin{aligned} & \inf \mathbf{f}^T \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in \mathcal{K}_{n_1} \times \mathcal{K}_{n_2} \times \cdots \mathcal{K}_{n_k}, \end{aligned} \tag{2.3}$$

where $A \in \Re^{m \times n}, m < n$, $\mathbf{f} \in \Re^n$, $\mathbf{b} \in \Re^m$ are fixed, and each $\mathcal{K}_{n_i} \subset \Re^{n_i}$ is the second-order cone defined as

$$\mathcal{K}_{n_i} = \left\{ (\mathbf{y}, \gamma) \in \Re^{n_i - 1} \times \Re : \ \|\mathbf{y}\| \leq \gamma \right\}, \text{ for } n_i > 1,$$
$$\mathcal{K}_{n_i} = \{ \gamma \in \Re : \ 0 \leq \gamma \}, \text{ for } n_i = 1,$$

with $n_1 + n_2 + \cdots + n_k = n$, $n_i \geq 1$ for all i. Note that $\gamma \in \mathcal{K}_1$ is simply a non-negativity constraint, and so linear optimization is a particular case of (2.3).

Some convex optimization problems admit reformulations into the so-called linear optimization problems over symmetric cones, in particular, linear optimization, second-order conic optimization, and positive semi-definite optimization. The above reformulations typically require introducing extra auxiliary variables, yet the advantages of such a reformulation may still outweigh the obvious burden of increasing problem's dimensions. Besides their extensive modeling capabilities, e.g., see Ben-Tal and Nemirovski (2001), SOC problems are particularly well suited to be handled by the interior-point methods – a family of efficient numerical algorithms that revolutionized the landscape of modern convex optimization in the

[7]For example, Larsen et al. (2002) proposed two VaR optimization methods by iteratively solving CVaR associated problems, while Wozabal et al. (2008) explored a difference of convex functions approach for dealing with risk-return portfolio selection problems.

past two decades. The extensive duality theory behind the second-order conic programming gives easy access to optimal Lagrange multiplies which may be used to streamline the sensitivity analysis and the optimal frontier navigation. Powerful SOC duality machinery permits computation of optimality and infeasibility certificates, and makes the optimization problems amenable to very powerful numerical software, capable of handling optimization problems with hundred thousands and even millions of variables and constraints.

In addition to the attractive features listed above, it may be noted that under some suitable assumptions on the nature of underlying uncertainties of the model's parameters, the so-called *robust counterpart* to a given SOC problem remains within the same SOC class, and thus is still computationally tractable. Note that robustness of the solution to at least some level of uncertainty in model's parameters is almost always a desirable characteristic of any practical model, which in turn makes the robust counterpart to our problem (2.2) into potentially valuable avenue of research we did not pursue in this manuscript. Instead, for the sake of conciseness, we carried out an extensive numerical stability and sensitivity study of the models proposed.

3 Two-model hierarchy

In general there is no closed form expression for the solvency probability unless some rather restrictive distributional assumptions for both liability and asset returns are made. The standard approach in the ALM literature is to consider that the asset gross return process $\mathbf{R}$ is governed by a multivariate normal distribution and each component is independent from Y, which is also assumed to be sampled from a Gaussian distribution. Under these assumptions, the chance constraints from the above optimization problems can be re-written in terms of the inverse of a standard normal random variable making its implementation tractable, e.g., Li and Huang (1996).

We begin by following this standard ALM literature assumption of normality in our first model. This not only allows us to recast our problem (2.2) into an elementary SOC formulation which can be handled very efficiently numerically, but, under a few further simplifying assumptions, to derive a closed-form solution to the problem. This later set of formulas, despite being based on a stylized model with implicit light-tailed liabilities assumptions, may in principle be used as a simple initial guideline towards the optimal decision-making.

The second model may be viewed as a more realistic set-up that takes into account the empirically observed heavy-tailed nature of insurer's liabilities, and permits an arbitrary distribution for the returns on company's assets. The liabilities are modeled via a parametric Pareto distribution. We adopt a non-parametric Monte-Carlo based approach to model the returns on company's assets. Again, the SOC reformulation of the corresponding problem (2.2) variant is provided with the supporting numerics to advocate the validity of our approach.

Throughout the manuscript we assume the liabilities and returns to be independent from one another.

The latter is a relatively strong yet unifying assumption, which in principle may be removed from our first model at a cost of making the closed form expression significantly more involved. As far as the second and more realistic model is concerned, relaxing the independence assumption while keeping the model computationally tractable is not so straightforward and is the subject of our future work.

3.1 First model: Gaussian losses and returns

Assume that the gross return process has a multivariate normal distribution $\mathbf{R} \sim \mathbf{N}(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu}$ is the $n-$dimensional mean return vector and Σ is the $n \times n$ positive definite covariance matrix. Likewise, assume $Y \sim \mathbf{N}(\mu_y, \sigma_y^2)$; Y is independent from $\mathbf{R}$.

Denoting $\mathbf{z} = (p+c)\mathbf{x}$, it follows that $Y - \mathbf{z}^T\mathbf{R} \sim \mathbf{N}(\mu_y - \mathbf{z}^T\boldsymbol{\mu}, \sigma_y^2 + \mathbf{z}^T\Sigma\mathbf{z})$. The probability constraint from (2.2) can now be reformulated as a certainty equivalent constraint

$$\mu_y - \boldsymbol{\mu}^T\mathbf{z} + \Phi^{-1}(1-\beta)\sqrt{\sigma_y^2 + \mathbf{z}^T\Sigma\mathbf{z}} \leq 0,$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of a standard normal random variable. Note that if $\beta \leq 0.5$, then $\Phi^{-1}(1-\beta)$ is positive, and thus, the constraint is convex. In addition, the constraint may be re-cast as

$$\left\| \begin{pmatrix} \phi\,\sigma_y \\ \phi\,\Sigma^{1/2}\mathbf{z} \end{pmatrix} \right\| \leq \boldsymbol{\mu}^T\mathbf{z} - \mu_y$$

with $\phi = \Phi^{-1}(1-\beta)$. Now, using the latter and noting that with p being fixed, minimizing c is equivalent to minimizing $p + c$, we can write the following SOC equivalent re-formulation of our problem (2.2):

$$\begin{aligned} &\min \mathbf{1}^T\mathbf{z} \\ \text{s.t.} \quad &\begin{pmatrix} (1\ \mathbf{0}^T) & 0 & \mathbf{0}^T & 0 \\ (\mathbf{0}\ I) & \mathbf{0} & -\phi\,\Sigma^{1/2} & \mathbf{0} \\ \mathbf{0}^T & -1 & \boldsymbol{\mu}^T & 0 \\ \mathbf{0}^T & 0 & \mathbf{1}^T & -1 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \gamma \\ \mathbf{z} \\ c \end{pmatrix} = \begin{pmatrix} \phi\,\sigma_y \\ \mathbf{0} \\ \mu_y \\ p \end{pmatrix}, \\ &(\mathbf{y}, \gamma) \in \mathcal{K}_{n+2}, \\ &\mathbf{z}_j \geq 0,\ j = 1, \ldots, n, \\ &c \geq 0, \end{aligned} \qquad (3.1)$$

where $\mathbf{y} \in \Re^{n+1}$, $\gamma \in \Re$, $\mathbf{z} \in \Re^n$, and $\mathbf{1}$, $\mathbf{0}$, and I are the vectors of all ones, all zeros, and the identity matrix of the corresponding dimensions.

In general, problem (3.1) may be efficiently solved numerically with an appropriate second-order conic solver, such as Mosek, SeDuMi, etc. However, since the problem has a single "true" second-order constraint, namely, norm restriction on $(\mathbf{y}, \gamma)$, we observe that under further simplifications of the model, namely, relaxation of the non-negativity constraints on $\mathbf{z}_j, c$, it is possible to derive its closed form solution. One may even argue that relaxing both $\mathbf{z}_j \geq 0$ and $c \geq 0$ allows for a realistic practical interpretation, as sign-unrestricted $\mathbf{z}_j$ amount to allowing short-selling in the portfolio, and negative c would indicate that the current level p already suffices to insure adequate level of solvency, and thus no extra capital c is required. Next, we derive the solution to a relaxed optimal required capital and portfolio problem

$$\begin{aligned}
&\min \mathbf{1}^T \mathbf{z} \\
\text{s.t.} \quad &\begin{pmatrix} (1\ \mathbf{0}^T) & 0 & \mathbf{0}^T \\ (\mathbf{0}\ I\,) & \mathbf{0} & -\phi\,\Sigma^{1/2} \\ \mathbf{0}^T & -1 & \boldsymbol{\mu}^T \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \gamma \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \phi\,\sigma_y \\ \mathbf{0} \\ \mu_y \end{pmatrix}, \\
&(\mathbf{y},\gamma) \in \mathcal{K}_{n+2}, \\
&\mathbf{z} - \text{unrestricted.}
\end{aligned}$$

Note that if $\phi = 0$, i.e., $\beta = 0.5$, the problem above reduces to an elementary linear optimization problem with a single inequality constraint stating that *the expected return of the portfolio exceeds or equals to the expected loss*, $\boldsymbol{\mu}^T \mathbf{z} \geq \mu_y$. Since the case of $\beta = 0.5$ presents little practical, as well as theoretical interest, from now one we focus on $\beta < 0.5$, implying positive ϕ. After eliminating $\mathbf{z}$, the last optimization problem becomes

$$\begin{aligned}
&\min \left(\Sigma^{-1/2}\mathbf{1}\right)^T \mathbf{y} \\
\text{s.t.} \quad &\begin{pmatrix} (1\ \mathbf{0}^T) & \mathbf{0}^T \\ (0\ \ \boldsymbol{\mu}^T\Sigma^{-1/2}) & -\phi \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \gamma \end{pmatrix} = \begin{pmatrix} \sigma_y \\ \mu_y \end{pmatrix}, \\
&(\mathbf{y},\gamma) \in \mathcal{K}_{n+2}.
\end{aligned} \tag{3.2}$$

It is well-known that SOC with a single conic constraint has a closed-form characterization, yet for self-containment we briefly sketch the derivation specialized to our setting, as it also allows us to shed some light onto feasibility of (3.2), and consequently (3.1).

Let us first discuss the feasibility of (3.2). Recall that the feasible region of the problem effectively corresponds to $\mathbf{z}$ such that

$$\sqrt{\sigma_y^2 + \mathbf{z}^T \Sigma \mathbf{z}} \leq \frac{1}{\phi} \boldsymbol{\mu}^T \mathbf{z} - \frac{\mu_y}{\phi}, \tag{3.3}$$

where $\phi > 0$. Using $\mathbf{y} = \Sigma^{1/2}\mathbf{z}$, the above may be re-written as

$$\sqrt{\sigma_y^2 + \mathbf{y}^T \mathbf{y}} \leq \frac{1}{\phi} \left(\Sigma^{-1/2}\boldsymbol{\mu}\right)^T \mathbf{y} - \frac{\mu_y}{\phi}. \tag{3.4}$$

Now, considering the left-hand side of the last expression separately, we observe that $\sqrt{\sigma_y^2 + \|\mathbf{y}\|^2}$ is convex and increasing in $\|\mathbf{y}\| \geq 0$, has horizontal tangent as $\|\mathbf{y}\| \to 0+$, and $\sqrt{\sigma_y^2 + \|\mathbf{y}\|^2}/\|\mathbf{y}\| \to 1$ as $\|\mathbf{y}\| \to \infty$. In turn, the right-hand side is a linear function with respect to $\|\mathbf{y}\|$ with values growing at the rate of at most $\|\Sigma^{-1/2}\boldsymbol{\mu}\|/\phi$, due to Cauchy-Swartz inequality, with the highest rate achieved with equality whenever $\mathbf{y}$ is a positive multiple of $\Sigma^{-1/2}\boldsymbol{\mu}$. Consequently, combined with $\mu_y, \phi > 0$, we may conclude that the constraint (3.4) is feasible (for sufficiently large $\mathbf{y}$) if and only if

$$\|\Sigma^{-1/2}\boldsymbol{\mu}\| > \phi, \tag{3.5}$$

where the last condition may be interpreted as a multivariate version of *relative standard deviation of the portfolio assets* not exceeding $1/\phi$ threshold.

Condition (3.5) not only is necessary and sufficient for feasibility of constraint (3.3) and thus problem (3.2), but also is necessary for feasibility of the original problem (3.1).

With this is mind and assuming (3.2) is feasible, that is, (3.5) is true, let us equivalently re-write problem (3.2) as

$$\begin{aligned} &\min \mathbf{1}^T\mathbf{z} \\ \text{s.t.} \quad &\sigma_y^2 + \mathbf{z}^T\Sigma\mathbf{z} \le \left(\tfrac{1}{\phi}\boldsymbol{\mu}^T\mathbf{z} - \tfrac{\mu_y}{\phi}\right)^2, \\ &\boldsymbol{\mu}^T\mathbf{z} - \mu_y \ge 0 \end{aligned}$$

with now smooth non-linear constraint. Since the problem is feasible, there are two option remaining: the optimal objective value is either finite or the problem is unbounded. To find the solution to problem (3.2) or establish its unboundedness, it suffices to examine the solutions to KKT equations corresponding to the problem above, ignoring the last linear inequality constraint. From the geometry of SOC formulation (3.2) we know that the feasible region of

$$\begin{aligned} &\min \mathbf{1}^T\mathbf{z} \\ \text{s.t.} \quad &\sigma_y^2 + \mathbf{z}^T\Sigma\mathbf{z} - \left(\tfrac{1}{\phi}\boldsymbol{\mu}^T\mathbf{z} - \tfrac{\mu_y}{\phi}\right)^2 \le 0, \end{aligned}$$

consists of the affine subspace intersected with a union of the (convex) SOC and its mirror reflection through the origin. Writing down KKT conditions we have

$$\Sigma\mathbf{w} + \frac{1}{\phi^2}\mu_y\boldsymbol{\mu} - \frac{\boldsymbol{\mu}^T\mathbf{z}}{\phi^2}\boldsymbol{\mu} = \tau\mathbf{1}, \;\; \tau \le 0,$$

which may be re-written as

$$\left(\Sigma - \frac{1}{\phi^2}\boldsymbol{\mu}\boldsymbol{\mu}^T\right)\mathbf{z} = \tau\mathbf{1} - \frac{\mu_y}{\phi^2}\,\boldsymbol{\mu}, \;\; \tau \le 0.$$

Applying Sherman-Morrison rank-one update formula for the inverse, we obtain that the optimal solution $\mathbf{z}^*$ must satisfy

$$\mathbf{z}^* = \left(\Sigma^{-1} + \frac{\Sigma^{-1}\boldsymbol{\mu}\boldsymbol{\mu}^T\Sigma^{-1}}{\phi^2 - \boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}}\right)\left(\tau^*\mathbf{1} - \frac{\mu_y}{\phi^2}\boldsymbol{\mu}\right) \tag{3.6}$$

for some $\tau^* \le 0$. Note that we by feasibility condition (3.5) we are guaranteed to have non-zero denominator $\phi^2 - \boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}$.

In turn, to determine τ^* we substitute $\mathbf{z}^*$ into our non-linear constraint

$$\sigma_y^2 + \mathbf{z}^T\Sigma\mathbf{z} \le \left(\frac{1}{\phi}\boldsymbol{\mu}^T\mathbf{z} - \frac{\mu_y}{\phi}\right)^2 \tag{3.7}$$

and solve the resulting quadratic equation. Namely, writing

$$\mathbf{z}^* = \underline{\mathbf{z}} + \overline{\mathbf{z}}\,\tau^*,$$

where

$$\underline{\mathbf{z}} = \frac{-\mu_y}{\phi^2}\left(\Sigma^{-1} + \frac{\Sigma^{-1}\boldsymbol{\mu}\boldsymbol{\mu}^T\Sigma^{-1}}{\phi^2 - \boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}}\right)\boldsymbol{\mu}$$

and

$$\overline{\mathbf{z}} = \left(\Sigma^{-1} + \frac{\Sigma^{-1}\boldsymbol{\mu}\boldsymbol{\mu}^T\Sigma^{-1}}{\phi^2 - \boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}}\right)\mathbf{1},$$

and denoting

$$u = \overline{\mathbf{z}}^T \Sigma \overline{\mathbf{z}} - \left(\frac{\boldsymbol{\mu}^T \overline{\mathbf{z}}}{\phi} \right)^2,$$

$$v = 2\underline{\mathbf{z}}^T \Sigma \overline{\mathbf{z}} + \frac{2\mu_y \, \boldsymbol{\mu}^T \overline{\mathbf{z}}}{\phi^2} - \frac{2\boldsymbol{\mu}^T \underline{\mathbf{z}} \, \boldsymbol{\mu}^T \overline{\mathbf{z}}}{\phi^2},$$

$$w = \underline{\mathbf{z}}^T \Sigma \underline{\mathbf{z}} + \sigma_y^2 - \left(\frac{\mu_y}{\phi} \right)^2 + \frac{2\mu_y \, \boldsymbol{\mu}^T \underline{\mathbf{z}}}{\phi^2} - \left(\frac{\boldsymbol{\mu}^T \underline{\mathbf{z}}}{\phi} \right)^2,$$

τ^* is obtained from solving the quadratic equation

$$u \, \tau^2 + v \, \tau + w = 0.$$

Hence, using quadratic formula we may potentially get two real solutions

$$\tau_{1,2} = \frac{-v \pm \sqrt{v^2 - 4uw}}{u^2}. \tag{3.8}$$

Now, comparing the original feasibility condition (3.3) with its squared version (3.7) we note that indeed, if the problem (3.2) has an optimum, two real roots $\tau_{1,2}$ are expected, with one being non-positive and corresponding to $\boldsymbol{\mu}^T \mathbf{z} - \mu_y \geq 0$. On the contrary, if neither τ_1 nor τ_2 is non-positive and satisfies $\boldsymbol{\mu}^T \mathbf{z} - \mu_y \geq 0$, then the problem (3.2) is unbounded. In turn, the unboundedness may be interpreted as *if portfolio short-sales and negative capital requirement are permitted, there is an opportunity for unlimited cash-flow to be generated with prescribed solvency requirements.*

To summarize, to explicitly solve relaxed version of optimal required capital and portfolio problem with portfolio short-sales and negative capital requirement allowed, one needs to

- verify the feasibility condition (3.5), and in case if the condition is violated, conclude that neither problem (3.2) nor problem (3.1) are feasible,
- compute $\tau_{1,2}$ according to (3.8) and out the two values of τ pick τ^* such that $\tau^* \leq 0$ and the corresponding $\mathbf{z}^*$ computed according to (3.6) satisfies $\boldsymbol{\mu}^T \mathbf{z}^* - \mu_y \geq 0$,
- if neither $\tau_{1,2}$ meet these requirements, conclude that problem (3.2) is unbounded, otherwise report optimal required capital $c^* = \mathbf{1}^T \mathbf{z}^*$ and portfolio allocation $\mathbf{z}^*$.

We want to stress that although (3.2) is a relaxed version of our main problem of interest, its solution may still contain useful information about the status of (3.1) that may further be given a narrative interpretation. For example, observe that if $\mathbf{z}^* \geq 0$ and yet $c^* < 0$, this indicates that no extra capital is needed no guarantee the prescribed level of solvency.

3.2 Second model: semi-parametric with heavy tailed losses

Since losses associated to certain business lines of non-life insurance companies generally exhibit heavier-tails, we propose a semiparametric method for dealing with such more realistic situations. One of the most popular distributions used for modelling extreme events is the Pareto law (e.g. see McNeil (1997)), and thus we derive further results under such an assumption for Y. Another special feature here is that the Pareto optimization setting also admits a SOC reformulation, which in turn is well-suited for subsequent numerics.

3.2.1 Solvency probability approximation

We now propose a semi-parametric approach for approximating the solvency probability in the constraint of Problem (2.2). The main idea is to use a Monte-Carlo type estimator for the expected survival function of the net liability distribution. The aggregate liability Y is assumed to be non-negative and independent of R, with the following survival function:

$$H(y) = \Pr(Y > y).$$

Using the above notations and $\mathbf{z} = (p + c)\mathbf{x}$, we approximate the ruin probability as follows:

$$\Pr(L(c, \mathbf{x}) > 0) \quad \approx \quad \frac{1}{N} \sum_{k=1}^{N} \Pr(Y > \mathbf{R}_{(k)}^T \mathbf{z}), \tag{3.9}$$

where each $\mathbf{R}_{(k)} \in \Re^n$ is sampled from $\mathbf{R}$. We further assume that standard regularity conditions, which ensure the uniform convergence with probability one over a compact set, of the Monte-Carlo estimator from (3.9) are satisfied.

Following the approximation from (3.9), the optimization problem (2.2) becomes

$$\begin{aligned} & \min c \\ \text{s.t.} \quad & \tfrac{1}{N} \sum_{k=1}^{N} \Pr(Y > \mathbf{R}_{(k)}^T \mathbf{z}) \le \beta, \\ & \sum_{i=1}^{n} \mathbf{z}_i = p + c, \\ & \mathbf{z} \ge \mathbf{0}, c \ge 0. \end{aligned} \tag{3.10}$$

A sufficient condition for the convexity of our optimization problem (3.10) is that the survival probability function $H(y)$ is a convex function in y on the support of Y, when short sales are prohibited.

The majority of the widely-used liability distributions have convex survival probability functions after some threshold point. The Pareto and exponential survival functions are convex on the entire domain for any parametrization. Some parametric forms of the Burr, Gamma and Weibull distributions have the same property on the whole domain. The Inverse Gamma, Inverse Gaussian and Log-Normal distributions have convex survival functions only on the right side of their domains. Therefore, in principle all these parametric models, further equipped with proper assumptions, may be suited for Y in our model (3.10) and still produce computationally tractable, i.e., convex, formulation.

3.2.2 Pareto-distributed losses

The Pareto law has been widely used for modeling extreme events and also represents one of the most appropriate ways of estimating the tail distribution of a heavy tailed events. The advantage of this particular parametrization is that once reformulated as SOC optimization, the problem can be solved with a more efficient numerical methods and consequently solvers, such as Mosek.

Beside the computational advantage created by this setting, it is further argued that the choice of the Pareto distribution represents a plausible assumption. One of the quantitative requirements designed within the Solvency II framework is that the regulatory capital should overcome extreme scenarios that occur with a specified, but small chance. The insurer claimss tend to have a heavy tail, for which the Extreme Value Theory suggests the use of the Pareto distribution when estimating these extreme events. Therefore, it is worth elaborating our model under the Pareto setting.

The survival function of a second order Pareto random variable with scale parameter $\lambda > 0$ and tail index $\alpha > 0$ (also known as Lomax distribution) is given by $H(y) = 1$ for $y < 0$ and

$$H(y) = \left(\frac{\lambda}{\lambda + y}\right)^{\alpha}, \quad y \geq 0. \tag{3.11}$$

Statistical inferences for the tail index have been a major interest in the mathematical statistics literature over the last several decades, and represents a standard procedure in estimating the extreme events associated with a univariate random outcome (e.g. Embrechts et al. (1997), Coles (2004), Resnick (2007). For this reason, model specification corresponding to any given data can be implemented with relative ease.

Note that when the tail index $\alpha \geq 1$, the survival function $H(y)$ is convex, and so is our main problem (3.10). Recall that in turn $\alpha \geq 1$ is a necessary and sufficient condition for the expected value of Y to be finite; this represents a quite reasonable modeling assumption of the total liabilities of having a finite mean value.

Assuming $\alpha \geq 1$, a SOC reformulation of the optimization problem (3.10) with the Pareto-distributed cumulative loss Y is further discussed. Using the survival function (3.11), we re-write (3.10) as:

$$\begin{aligned} & \min c \\ \text{s.t.} \quad & \tfrac{1}{N} \sum_{k=1}^{N} \left(\tfrac{\lambda}{\lambda + y_k}\right)^{\alpha} \leq \beta, \\ & y_k = \mathbf{R}_{(k)}^T \mathbf{z},\ k = 1, \ldots, N, \\ & \sum_{i=1}^{n} \mathbf{z}_i = p + c, \\ & \mathbf{z} \geq 0, c \geq 0, \end{aligned}$$

which, in turn, is equivalent to

$$\begin{aligned} & \min c \\ \text{s.t.} \quad & \tfrac{1}{N} \sum_{k=1}^{N} v_k^{\alpha} \leq \beta, && \text{(a)} \\ & \tfrac{\lambda}{\lambda + y_k} \leq v_k,\ k = 1, \ldots, N, && \text{(b)} \\ & y_k = \mathbf{R}_{(k)}^T \mathbf{z},\ k = 1, \ldots, N, \\ & \sum_{i=1}^{n} \mathbf{z}_i = p + c. \\ & \mathbf{z} \geq 0, c \geq 0. \end{aligned} \tag{3.12}$$

Recall that since $\mathbf{R}$ represents gross returns, $\mathbf{R}$ is nonnegative, and so is y_k. The remaining details of formulating (3.12), namely constraints (a,b), as SOC constraints, may be found in the Appendix.

Next we report on a numerical study focused primarily on the second model, as solving our first model, even numerically, is a fairly straightforward task.

4 Numerical illustrations and model's performance

Besides illustrating computational advantages of SOC reformulation of problem (3.12), we present numerical sensitivity analysis of the optimal solutions relative to the number of scenarios used, model parameters, and number of instruments in the portfolio.

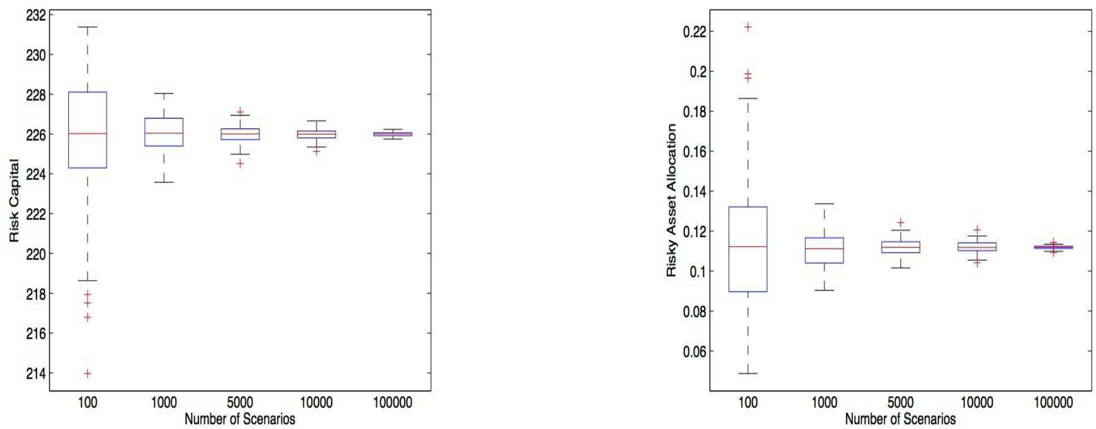
4.1 Accuracy of solvency probability approximation

We illustrate the accuracy of the proposed approximation using the following numerical example. Consider a portfolio formed by one risk-free asset and one risky asset. The gross return of the nearly risk-free asset is assumed to be distributed according to $\mathbf{N}(1.04, 10^{-12})$, while the gross return on the risky asset is assumed to be $\mathbf{N}(1.14, 0.04)$. The liability is Gaussian distributed as well such that $Y \sim \mathbf{N}(1000, 22500)$. Finally, the premium is set at $p = 1100$; the number of scenarios in Monte-Carlo estimator of the survival probability is fixed to 5000.

To evaluate the accuracy of our semi-parametric solvency probability approximation, we first numerically solve optimization problem (3.1) with Mosek. The resulting optimal capital requirement is $c^* = 225.99$ and the optimal portfolio is formed by investing 11.19% of its initial capital and insurance premiums into the risky asset. To confirm the accuracy of the solution we also apply the closed-form derivation procedure described in Subsection 3.1, and obtain the same optimal values that, due to non-negativity of the resulting $\mathbf{z}^*, c^*$, coincide with the optimal solution to (3.1).

Next, problem (3.10) is solved via the semi-parametric approach (3.10) using the Monte-Carlo based ruin probability estimator (3.9) for different number of scenarios N. We repeat each experiment 100 times and the average of the optimal capital requirement and the risky asset portion of the optimal portfolio, together with the corresponding standard deviations, are reported in the box-plots in Figure 4.1. As expected, the results from the solvency probability approximation are quite similar to those obtained by using the certainty equivalent constraint. The standard deviations decrease as the number of scenarios used in the Monte Carlo approximation increases. More specifically, the first and third quartile converge to the optimal required capital and risky asset allocation respectively. There is very little variation in the optimal solution whenever the number of scenarios is large.

Altogether, we believe that these observations further support the use of the proposed ruin probability approximation.



(a) optimal capital

(b) optimal portfolio

Figure 4.1: Optimal capital and risky asset allocation approximation.

4.2 Some advantages of SOC reformulation

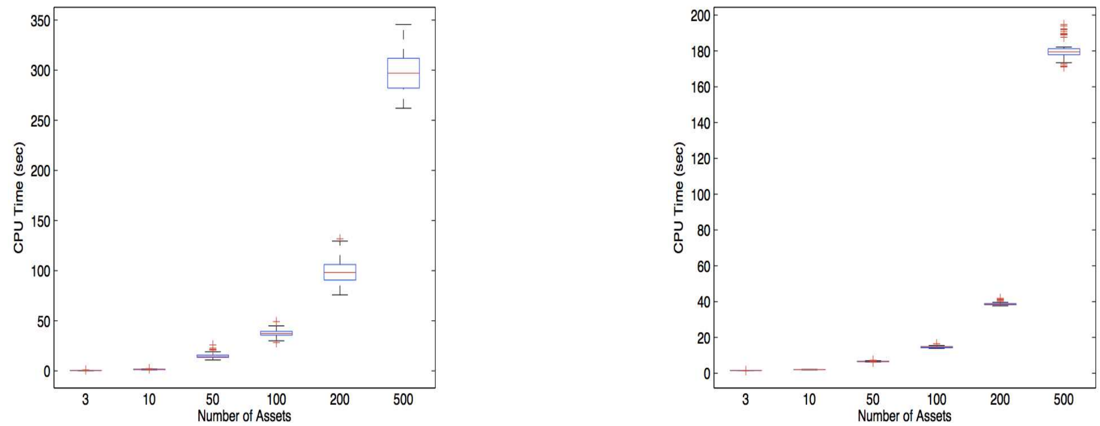
Next we explore the computational advantages of the SOC reformulation of problem (3.12). In particular, we compare the performance of SOC Mosek solver to that of Matlab's standard non-linear optimization routine `fmincon` applied to more standard convex formulation (3.10). The optimization experiments are carried out for portfolios ranging from small, $n = 3$ assets, to large, $n = 500$ assets, sizes. To solve (3.10), the interior-point option in `fmincon` was used.

We assume that asset returns are independent log-normally distributed, that is, $\mathbf{R}_i \sim \mathbf{LGN}(\mu_i, \sigma_i^2)$, with $\mu_i \in [0.004, 0.007]$ and $\sigma_i \in [0.4, 0.7]$ chosen uniformly for all $1 \leq i \leq n$. In addition, the liabilities are assumed to have the Pareto distribution with $\alpha = 4, \lambda = 3000$. Note that the parameters for the asset returns are chosen so that the annual gross returns are approximately between 8.76% and 28.67%. The Pareto parameters ensure the expected annual liability of 1000, and so the insurance premium is set to be 10% higher than the expected liabilities, $p = 1100$. For every distinct number of assets considered, the optimization runs were replicated 100 times, varying the distribution parameters.

First, we report the CPU time of the two methods relative to the number of assets included in the portfolio. The running times, in seconds, are reported via box-plots in Figure 4.2.

Note that optimization over SOC reformulation of (3.12) becomes faster as compared to its convex nonlinear counterpart (3.10) as the number of assets increases. For example, for portfolios with 500 assets, the median time for SOC is around 180 seconds compared to 300 seconds for the Matlab solver. The CPU time is also less variable with Mosek than with Matlab's `fmincon`. When the number of assets is relatively small (less than 20 assets), the differences are not significant.

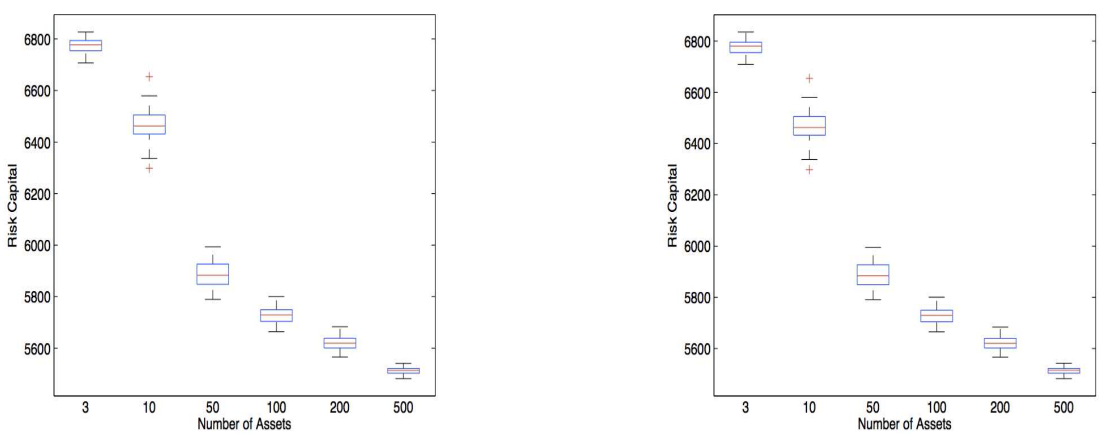
For an indirect evidence to the comparative accuracy of the two approaches, one may examine the box-plots in Figure 4.3 constructed for the optimal risk capitals with fixed number of assets. The box plots



(a) CPU time for `fmincon` (b) CPU time for Mosek

Figure 4.2: CPU times for `fmincon` and Mosek.

(a) and (b) being virtually identical regardless of the modeling and numerical approaches used suggests that there is no significant difference between `fmincon` and Mosek with regards to the optimal capital allocation. Interestingly, when the number of assets is large, namely $n = 200$ and 500, the distribution of the optimal solution exhibits a smaller variance as compared to the case of smaller size portfolios. In addition, the optimal capital decreases as more assets are added into the portfolio. Therefore, one may speculate that indeed in general portfolio diversification is beneficial and should be encouraged.



(a) optimal capital with `fmincon` (b) optimal capital with Mosek

Figure 4.3: Optimal capital allocation using `fmincon` and Mosek.

In order to assess if there is a significant difference between the optimal portfolio structures computed with `fmincon` and Mosek respectively, we apply ℓ_1-norm to the difference portfolio allocation as follows

$$\|\mathbf{x}^*_{\text{Mat}} - \mathbf{x}^*_{\text{Mos}}\|_1 = \sum_{i=1}^{n} |\mathbf{x}^*_{\text{Mat}_i} - \mathbf{x}^*_{\text{Mos}_i}|.$$

Here, $\mathbf{x}^*_{\text{Mat}}$ and $\mathbf{x}^*_{\text{Mos}}$ represent the optimal allocation for the asset portfolio obtained from the Matlab's `fminconv` and Mosek solver, respectively. The differences between the optimal portfolios are quite small, with the largest ℓ_1-norm portfolio difference being 1.21% for one particular simulation run corresponding to 500 assets, see Figure 4.4. Both solvers were run with default parameter settings.

It should be noted that in general we expect the approximate optimal solution computed with Mosek to be more precise than the ones computed with `fmincon`. The reasons for the latter is that, unlike the generic minimizer `fmincon`, Mosek is highly-specialized to solving SOC problem. For example, even when examining the solver's termination criteria, we know that by default Mosek stops with a (near) feasible solution whenever the so-called duality gap falls below the default threshold of 10^{-8}, which in turn guarantees high accuracy of the optimal required capital allocation. `fmincon`, on the other hand, typically terminates based on the first-order-type optimality conditions typically with a higher default precision threshold.

Overall, our numerical experiments indicate that reformulating our problem (3.10) as SOC optimization has an advantage of shorter optimization runs with respect to the CPU time, especially as the number of the assets gets large. However, it should be noted that as the number of scenarios for the ruin probability estimator gets large, the computational cost associated with SOC reformulation may become prohibitively large, in which case one may resort back to a generic optimization solver such as Matlab's `fmincon`.

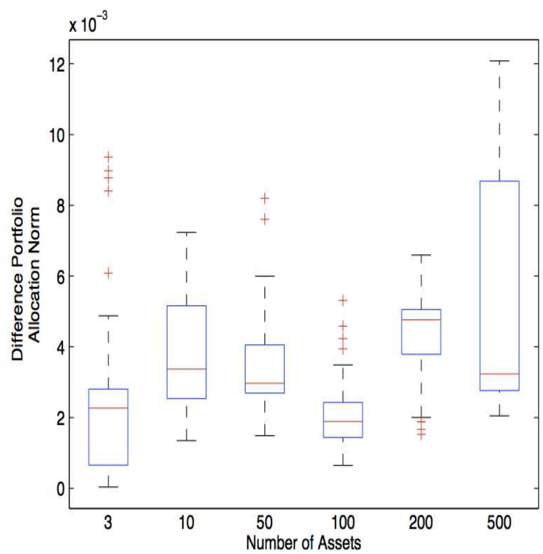


Figure 4.4: `fmincon` and Mosek-computed portfolios difference in ℓ_1-norm.

4.3 Sensitivity to model's parameters

The parameter uncertainty embedded in any statistical estimation indicates the need to perform at least an elementary sensitivity analysis of our approach. Under various settings of the liabilities parameters, the behavior of the optimal risk capital and asset allocation are now investigated for a two asset portfolio with only one risky asset. The risky asset is assumed to have a log-normal distributed gross return, $\mathbf{R}_1 \sim \mathbf{LGN}(0.005, 0.25)$, while the risk-free asset has its gross return set to 1.04. The Pareto parameters for the liability are chosen as in the previous section, with the same premium $p = 1100$. Four scenarios are examined in order to evaluate the impact of the potential estimation error in Y parameters on the optimal solution,

1. fixed $E[Y]$, $Var[Y]$ increases by 5%, i.e., $\alpha = 3.82, \lambda = 2818.18$,
2. fixed $E[Y]$, $Var[Y]$ decreases by 5%, i.e., $\alpha = 4.22, \lambda = 3222.22$,
3. fixed $Var[Y]$, $E[Y]$ increases by 5%, i.e., $\alpha = 4.46, \lambda = 3629.67$,
4. fixed $Var[Y]$, $E[Y]$ decreases by 5%, i.e., $\alpha = 3.64, \lambda = 2512.41$.

The Monte-Carlo solvency probability approximation is based on $N = 10000$ scenarios and the average optimal capital and portfolio weights are computed from 10000 replications. The averages for the optimal capital c^* and the portfolio proportion allocated to risk-free asset $1 - \mathbf{x}_1^*$ together with their standard deviations (in parenthesis) are reported in Table 4.3, and suggest that the relative changes in the optimal values of c and $\mathbf{x}_1$ are quite small. For example, a decrease of 5% in the expected loss, while the variance

Optimal values	Initial	Scenario 1	Scenario 2	Scenario 3	Scenario 4
c^*	6,831.00	7,010.56	6,637.56	6,837.18	6,788.23
	(3.6656)	(3.8128)	(3.4979)	(3.5783)	(3.8020)
$1 - \mathbf{x}_1^*$	0.9097	0.9082	0.9112	0.9128	0.9067
	(0.0052)	(0.0053)	(0.0051)	(0.0051)	(0.0054)

Table 4.1: Sensitivity analysis with respect to liability distribution parameters.

remains constant, would result in relative decreases of 0.66% in the optimal capital and 0.33% in the risky asset allocation.

On the qualitative side, one may note that more capital needs to be held if the liability mean or variance increases, scenarios 1 an 3. Furthermore, we note that the insurer invests more into the risk-free asset whenever the variance of the losses increases with the expected value of losses kept constant, scenario 1. In other words, higher variability of liabilities generally leads to the insurer investing more in risk-free assets. The latter concurs well with our initial intuition.

Analogous numerical experiments were carried out for portfolios with three assets. Since the similar changes in the optimal solutions were observed, the results are omitted.

5 Further analysis and extensions of the framework

In this section we discuss how our framework may be employed to analyze the effects of insurance risk diversification and how the model may be extended further to accommodate other practically relevant constraints.

5.1 Effects of insurance risk diversification

Using the proposed framework we can explore the risk diversification effect of combining two or more insurance portfolios with respect to its effect on the optimal capital requirement.

For simplicity we consider the following basic setup. Two lines of business with random liabilities, Y_1 and Y_2, are considered. The received premiums are denoted by p_1 and p_2, and the optimal required capitals are c_1 and c_2. The combined portfolio has a random liability $Y_1 + Y_2$, and the corresponding optimal capital required is denoted by c_{12}. Since each premium is proportional to its expected liability, the portfolio aggregate premium is assumed to be $p_1 + p_2$. The impact of liabilities aggregation may be measured by the *diversification gain* defined as

$$DG = 1 - \frac{c_{12}}{c_1 + c_2},$$

where positive values of DG correspond to aggregation being beneficial to the insurer with respect to reducing the required capital; note $DG \leq 1$.

In general, intuitively one may expect to witness a positive impact of combining multiple liabilities into a single insurance portfolio as one would expect the overall risk to be reduced. However, here we want to emphasize the main intrinsic difference between our problem of simultaneously optimizing the required capital and the investment portfolio allocation (2.2) and the well-known mean-variance investment portfolio optimization problem where no liabilities are present. Unlike the latter, the optimal required capital determined by solving (2.2) is not sub-additive with respect to aggregating the liabilities into a single insurance product. That is, in general aggregating several liabilities may lead to much higher required capital amount as compared to maintaining several distinct lines of business. To illustrate this fact, consider the following artificial elementary example.

As before, the merger of two lines of business are considered. For simplicity we consider a single investment asset $\mathbf{R}_1$, and suppose that the underlying probability distributions for $Y_1, Y_2, \mathbf{R}_1$ are atomic corresponding to three possible outcomes, see Table 5.1. For convenience, we assume that both premiums $p_1 = p_2$ are 0 (the numerical values in this example can be easily changed to accommodate for non-zero premiums). With solvency probability requirement set to β, we note that the first line of business has the option of not covering either of the outcomes 2 or 3, and thus the optimal strategy corresponds to investing a single capital unit into $\mathbf{R}_1$ leaving out the outcome 3, resulting the minimal required capital $c_1 = 1$; analogously, $c_2 = 1$. On the other hand, to achieve solvency level β in case of the merger of two liabilities corresponding to $Y_1 + Y_2$, we note that at least one of the outcomes 2 or 3 has to be covered,

Random variable	Outcome 1 w.p. $1-2\beta$	Outcome 2 w.p. β	Outcome 3 w.p. β
Y_1	1	2	100
Y_2	1	100	2
Y_1+Y_2	2	102	102
$\mathbf{R}_1$	1	2	2

Table 5.1: Distributions of Y_1, Y_2, Y_1+Y_2 and $\mathbf{R}_1$.

leading to the capital requirement $c_{12} = 102/2 = 51$ which is much larger than $c_1 + c_2$. Note that the downside potential of the merger measured by DG is bounded only by the loss amounts in the outcomes 2 and 3, and so in principle may grow arbitrarily large, that is, an example with arbitrarily large negative DG may be easily constructed.

In summary, the above discussion emphasizes the importance of a thorough analysis to be carried out prior to merging several liabilities into a single insurance product, as in simple words, one may get particularly unlucky and/or the findings may be quite counter-intuitive. Our framework may be employed to do exactly that. The numerical example below illustrates the risk diversification effects when two risks are combined. The sensitivity to liability parameters is also analyzed numerically.

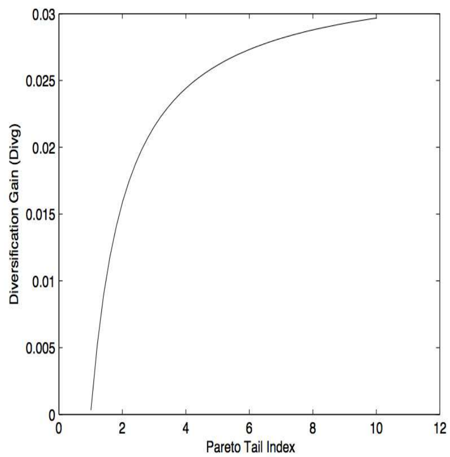
The liabilities Y_1, Y_2 are assumed to be jointly Pareto-distributed with

- the joint survival function $H_{Y_1,Y_2}(y_1, y_2) = \left(1 + \frac{y_1}{\lambda_1} + \frac{y_2}{\lambda_2}\right)^{-\alpha}$ with $\alpha > 1$,
- the marginal survival probabilities for Y_1, Y_2 being Pareto with parameters (α, λ_1) and (α, λ_2) respectively,
- the correlation coefficient between Y_1 and Y_2 being equal to $\frac{1}{\alpha}$, and
- the survival function of $Y_1 + Y_2$ being given by

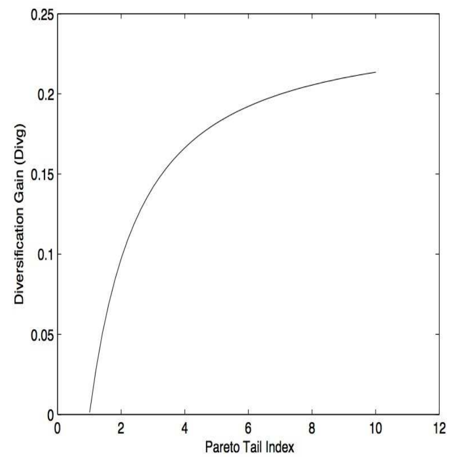
$$H_{Y_1+Y_2}(y) = \begin{cases} \lambda_1^{\alpha}(y+\lambda_1)^{-\alpha-1}(\alpha y + y + \lambda_1) & \text{if } \lambda_1 = \lambda_2, \\ \frac{\lambda_1\left(\frac{\lambda_1}{y+\lambda_1}\right)^{\alpha} - \lambda_2\left(\frac{\lambda_2}{y+\lambda_2}\right)^{\alpha}}{\lambda_1 - \lambda_2} & \text{if } \lambda_1 > \lambda_2. \end{cases}$$

For our numerical experiments the first scale parameter $\lambda_1 = 3000$ is kept constant, while the second scale parameter assumes four distinct values, $\lambda_2 = 100, 1000, 2000, 3000$. The tail index α varies from 1.01 to 10. Premium and capital corresponding to each risk are invested in the same two assets portfolio used in Subsection 4.3.

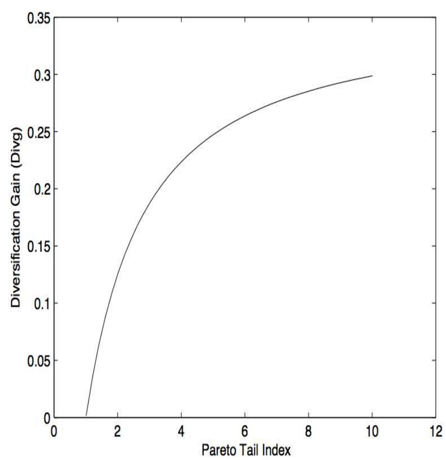
Examining the numerical results in Figure 5.1 one may observe that indeed under our scenarios combining two liabilities into one has a positive impact on the required capital, that is, $DG > 0$. Interestingly, Figure 5.1 shows that the diversification gain increases when the two liabilities become less correlated, i.e., as α increases. The diversification effect is more pronounced for "heavier-tailed" liabilities, namely,



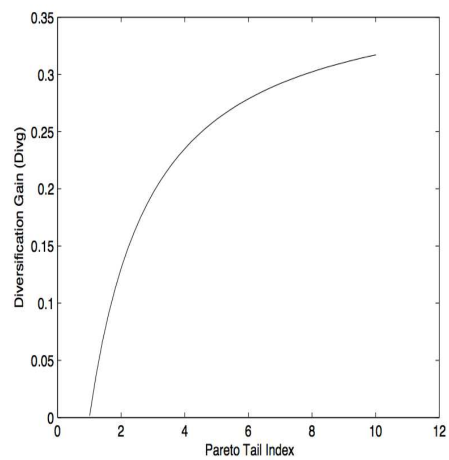
(a) $\lambda_2 = 100$



(b) $\lambda_2 = 1{,}000$



(c) $\lambda_2 = 2{,}000$

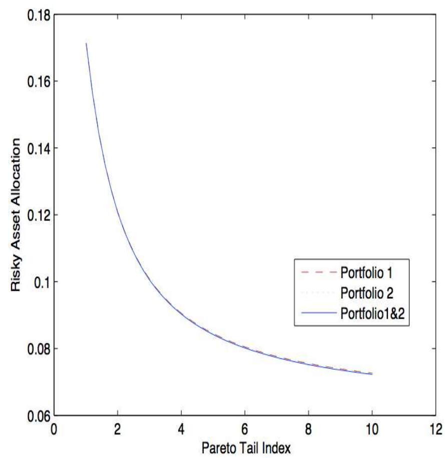


(d) $\lambda_2 = 3{,}000$

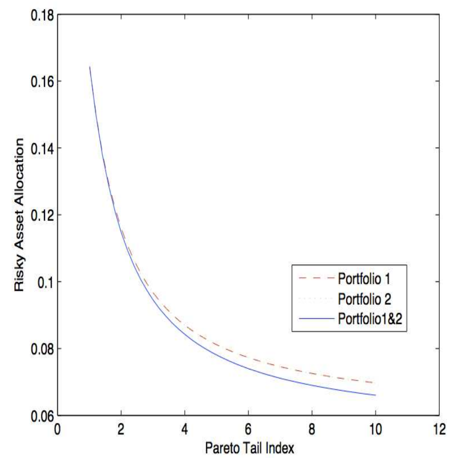
Figure 5.1: Diversification gain for different values of λ_2.

when $\alpha \in (1, 4]$ –recall that $\alpha \geq 2$ is necessary and sufficient for existence of the second moment– and slowly diminishes for larger values of α. The maximum diversification gain also depends on the choice of the second scale parameter. For example, when $\lambda_2 = 100$, a diversification gain higher then 3% can not be obtained. This result is not surprising, since the $\alpha = 10$ gives $E[Y_2] = 11.1$, which is much smaller when compared to the expected liability from the first business line. Therefore, diversifying with a risk in a disproportionate size does not have a significant impact on the overall optimal capital requirement. It can be noticed that large values of λ_2 increases the effect of diversification.

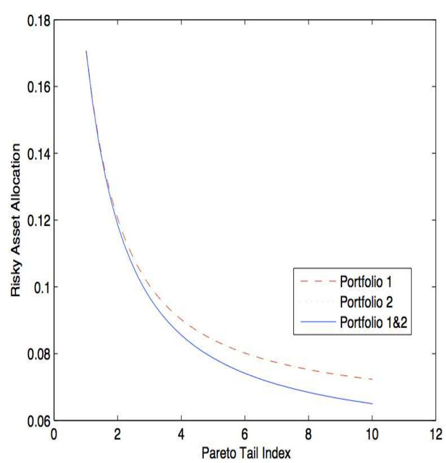
Now we turn our attention to the changes in the structure of the optimal portfolios. Figure 5.2 displays the behavior of the optimal risky asset portfolio allocation for Y_1, Y_2 and $Y_1 + Y_2$ liabilities and for different values of the second scale parameter. There is almost no difference in the optimal portfolio



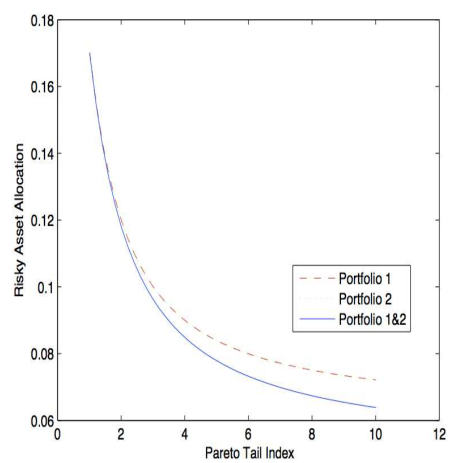
(a) $\lambda_2 = 100$



(b) $\lambda_2 =$ 1,000



(c) $\lambda_2 =$ 2,000



(d) $\lambda_2 =$ 3,000

Figure 5.2: Optimal risky asset allocations for different values of λ_2; Portfolio 1 corresponds to Y_1 only, Portfolio 2 to Y_2, and Portfolio 1&2 to $Y_1 + Y_2$.

allocations for the smallest value $\lambda_2 = 100$, while shareholders would allocate around 1% more into the risk-free asset of the diversified portfolio; the difference corresponds only to high values of α, so when diversification effect is at its maximum when $\lambda_2 = 3000$. It may also be concluded that the optimal portfolio structure is robust with respect to the scale parameter. However, the tail index parameter plays more significant role in determining the optimal portfolio allocations.

5.2 Optimization with expected return on capital constraint

So far optimization problem (2.2) has been studied without imposing any additional constraint involving a performance measure. As mentioned earlier, shareholders seek to gain some return for providing the regulatory capital. A natural candidate for the performance measure is the *expected return on capital* (ROC), which is defined as the ratio between the company's expected payoff and the capital invested. Generally speaking, shareholders may require a higher return on their capital than they would receive if they were to invest by themselves on the financial markets, since the shareholders would now face an additional risk induced by the insurance risk. We now add an expected return constraint to our optimization problem. A detailed simulation study is performed for a three-asset portfolio with only two risky assets, in order to investigate the efficient frontier structure and its dependence on the correlation coefficient between the risky assets. Our results are displayed for both Pareto and log-normally distributed liabilities.

The expected return on capital is given by

$$E[ROC] = \frac{(p+c)E[\mathbf{R}]^T\mathbf{x} - E[Y]}{c},$$

and the new constrained optimization problem may be formulated as

$$\begin{aligned} & \min c \\ \text{s.t.} \quad & \tfrac{1}{N}\sum_{k=1}^{N} \Pr(Y > \mathbf{R}_{(k)}^T\mathbf{z}) \le \beta, \\ & E[ROC] \ge ROC_\beta, \\ & \sum_{i=1}^{n} \mathbf{z}_i = p + c, \\ & \mathbf{z} \ge \mathbf{0}, c \ge 0. \end{aligned} \tag{5.1}$$

Here, ROC_β represents the minimum level of return on capital used for constructing the efficient frontiers. Problem (5.1) can be reformulated as a standard expected return maximization problem with the same efficient frontier

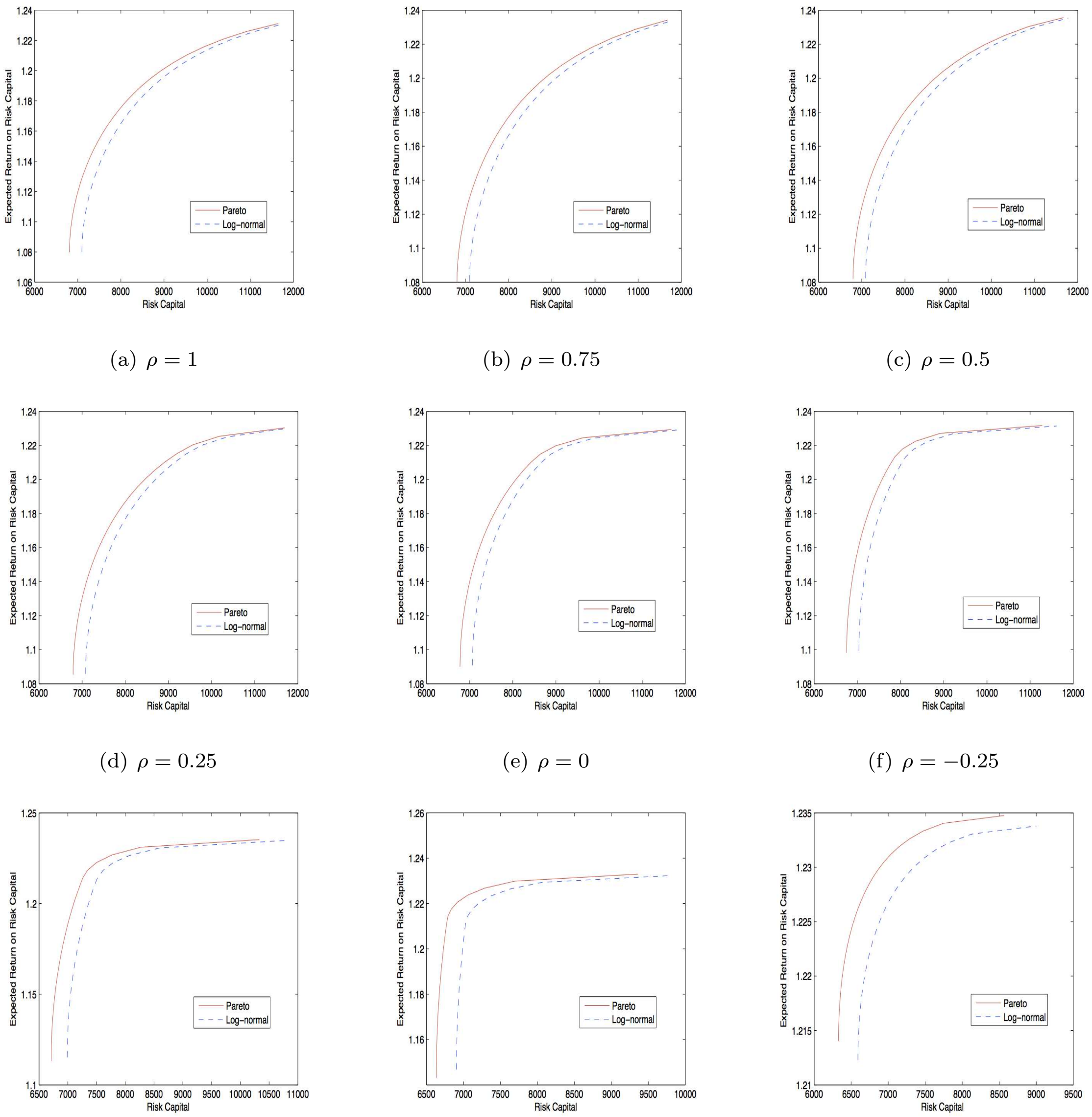
$$\begin{aligned} & \max E[ROC] \\ \text{s.t.} \quad & \tfrac{1}{N}\sum_{k=1}^{N} \Pr(Y > \mathbf{R}_{(k)}^T\mathbf{z}) \le \beta, \\ & c \le c_\beta, \\ & \sum_{i=1}^{n} \mathbf{z}_i = p + c, \\ & \mathbf{z} \ge \mathbf{0}, c \ge 0. \end{aligned} \tag{5.2}$$

The lower and upper bounds for c_β and ROC_β are chosen such that the constraints of the two optimization problems are binding. Note that both problems above still admit equivalent convex reformulations provided the survival probability $H(y)$ is convex in the respective domain.

As an illustration, consider a portfolio formed by one risk-free asset with gross return 4% and two risky assets $\mathbf{R}_1 \sim \mathbf{LGN}(0.005, 0.25)$, $\mathbf{R}_2 \sim \mathbf{LGN}(0.006, 0.36)$ with correlation coefficient ρ; $\mathbf{R}_2$ has an expected gross return of 1.20 and a variance of 0.63, both being larger than the respective mean and variance of $\mathbf{R}_1$. The liability is assumed to have either the Pareto distribution with parameters $\alpha = 4, \lambda = 3000$ or $\mathbf{LGN}(6.358445, 1.048147^2)$, chosen such that the first two moments match of Y match. Efficient frontiers and optimal portfolio allocations for the Pareto and log-normally distributed liabilities are illustrated in Figures 5.3 and 5.4. Since both problems provide the same efficient frontiers, only the solutions obtained from solving (5.2) are presented. The plots correspond to various values of the risky asset correlation coefficient ρ.

Figure 5.3 suggests that under most scenarios shareholders benefit more when losses have log-normal as opposed to the Pareto distribution, since they receive a higher expected return on capital for the same level of capital invested in the former case. This can be explained by a higher and more conservative VaR at 99.5% of the log-normal distribution as compared to that of the Pareto counterpart under our parametric assumptions. The shape of the efficient frontiers also depends on the correlation coefficient between the two risky assets. For example, for negative values of ρ, the efficient frontiers for both liabilities flatten out, suggesting that the increase in the expected return on shareholders' investment is not significant after a specific level of risk capital. Moreover, generally the difference between the two efficient frontiers is very small. The diversification effect can also be observed by monitoring the changes in value of the optimal capital required relative to the correlation coefficient. From Figure 5.3 a), an expected gross return of 1.2 corresponds to an optimal risk capital of around 9000 for Pareto losses, while when $\rho = -0.75$, the same return corresponds to an optimal regulatory capital of approximately 7500. As expected, the optimal risk capital attain its lowest level in the case of a perfect hedge.

Some interesting patterns regarding the structure of the optimal portfolios and their dependence on ρ can be observed in Figure 5.4. The optimal allocation into the risk-free asset decreases with respect to the level of risk capital, while the allocation into the most risky asset increases with the regulatory capital. Thus, the higher the return that is expected by the shareholders, the more should be allocated to the most risky asset with higher expected return. The parameter ρ has a great impact on the optimal portfolio allocation. For example, in the case of very little or no diversification, the optimal portfolio is formed by different combinations of the risk-free and most risky asset. A portfolio constructed based only on the two risky entities is obtained in the perfect hedge case. Indeed, panel *i*) of Figure 5.4 indicates there should be no investment into the risk-free asset in the case of a perfect hedge. The optimal allocation into a more risky asset starts from 50% and increases to almost 90%. When the capital level is low and assets are positively correlated, shareholders should invest approximately $80\% - 90\%$ of their wealth into the risk-free asset, and this allocation converges to zero when the expected gross return reaches its

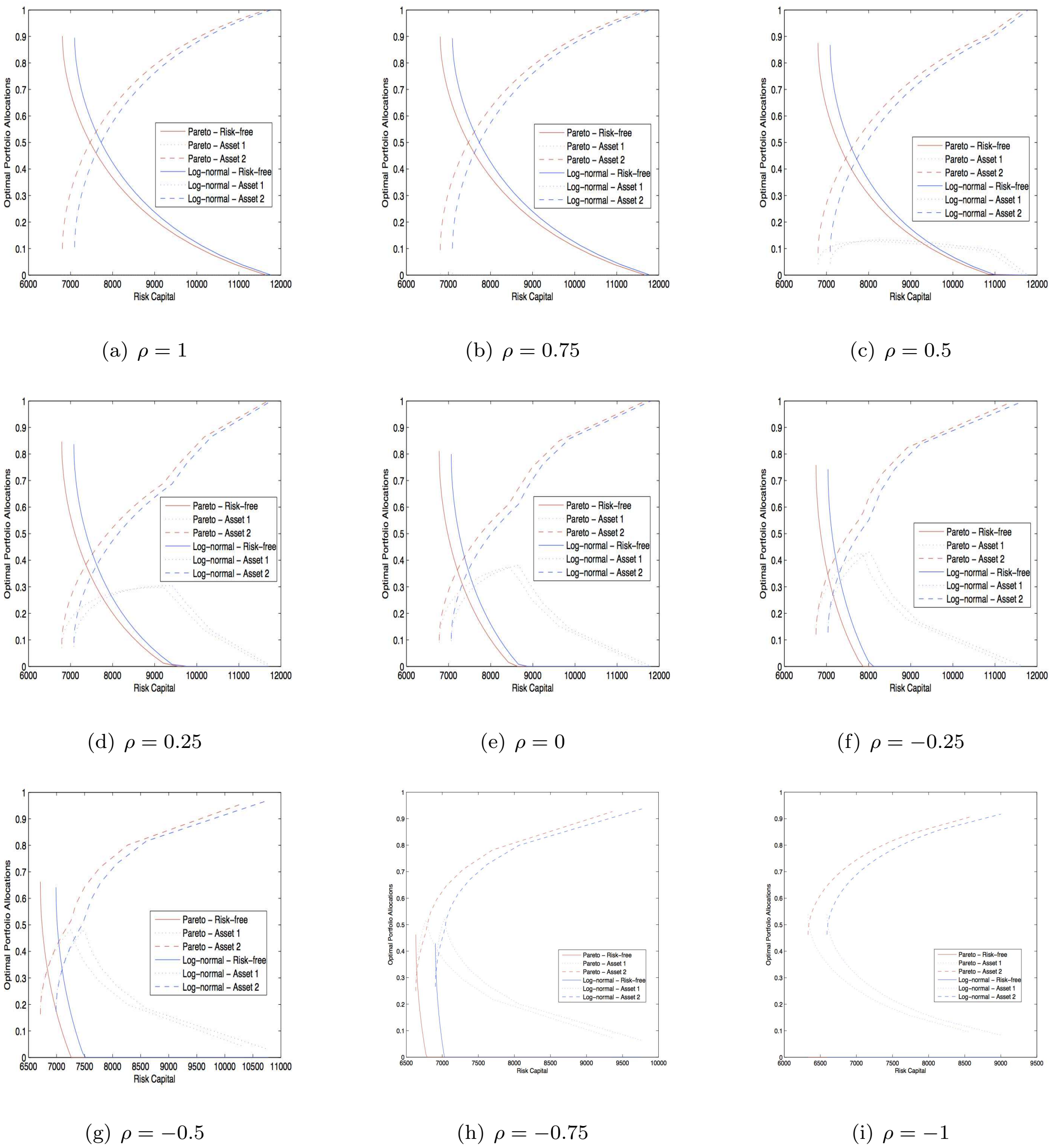


(a) $\rho = 1$ (b) $\rho = 0.75$ (c) $\rho = 0.5$

(d) $\rho = 0.25$ (e) $\rho = 0$ (f) $\rho = -0.25$

(g) $\rho = -0.5$ (h) $\rho = -0.75$ (i) $\rho = -1$

Figure 5.3: Efficient frontiers for the Pareto and log-normally distributed liabilities.



(a) $\rho = 1$ (b) $\rho = 0.75$ (c) $\rho = 0.5$

(d) $\rho = 0.25$ (e) $\rho = 0$ (f) $\rho = -0.25$

(g) $\rho = -0.5$ (h) $\rho = -0.75$ (i) $\rho = -1$

Figure 5.4: Optimal portfolio allocations for the Pareto and log-normally distributed liabilities.

maximum. In addition, the optimal allocation into the most risky asset converges to 100% as c increases, except for the case when $\rho = -0.75$ and $\rho = -1$. Thus, the diversification effect has an important role in determining the choice of the optimal portfolio for our proposed optimization framework.

6 Conclusions

We propose a novel optimization problem with a Solvency II inspired chance constraint, which simultaneously determines the minimum regulatory capital for an insurance company, as well as its optimal allocation into a portfolio of financial securities. The solvency requirement prescribes a certain threshold for the ruin probability over a predetermined period of time. This optimization problem is investigated based on a two-model hierarchy analysis. First, when both assets and losses are multivariate Gaussian distributed, the problem can be easily reformulated as a SOC program. In the absence of short-sales, we derive a closed form expression for the optimal solutions and provide necessary and sufficient conditions for the feasibility of the problem. In the second stage, we propose a semiparametric approach for the solvency probability, based on a parametric distributional assumption for the aggregate liability and the empirical distribution of asset returns. When losses follow a Pareto distribution of the second order, we derive a SOC representation. The accuracy of the is numerically tested for the Gaussian case by comparing the semiparametric method with the exact solution. Our results do not indicate a significant difference, even for a small number of scenarios (e.g. N = 100). The variability of the optimal solutions decreases as the number of Monte Carlo simulations increases. For Pareto losses we compare the SOC representation based on Mosek to the `fmincon` routine from Matlab. The optimal solutions are almost identical, the main difference coming from the CPU time (less for Mosek) when the portfolio size increases. The insurance risk diversification is also analyzed. Using a simple counterexample, we show that it is not always beneficial for a company to combine two or more liabilities. However, based on a two Pareto losses with the same tail index parameter example, we provide an analysis on the diversification gain relative to changes in the distribution parameters. More specifically, the diversification gain increases when the two liabilities become less correlated and this effect is more pronounced for heavier-tailed losses. Finally, we investigate our proposed problem when an expected return on capital constraint is required. Efficient frontiers are constructed for three-asset portfolios for both Pareto and log-normal liabilities. We analyze the impact of the correlation between the portfolio assets to the optimal solutions and illustrate how this affects the shape of the efficient frontiers. Although this proposed optimization has been extensively analyzed in this paper, there are several extensions which can be further explored. For example, one can consider a similar problem by relaxing the independence assumption between assets and liabilities. It may be also interesting to construct similar problems based on other solvency regulations, such as those constructed based on the CVaR or EPD.

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Appendix

Observe that for a three dimensional second-order cone $\text{SOC}_3 = \{(w_1, w_2, t) \in \Re^2 \times \Re : \ \|(w_1, w_2)\| \leq t\}$ we have

$$(w_1, w_2, t) \in \text{SOC}_3 \Leftrightarrow w_1^2 + w_2^2 \leq t^2, \ t \geq 0 \Leftrightarrow w_1^2 \leq (t - w_2)(t + w_2), \ t \geq 0. \tag{6.1}$$

The latter provides the basis for our SOC reformulation.

We will demonstrate the procedure for a single fixed k, which obviously may be repeated as many times as necessary to accommodate all of the N variables.

Considering (3.12) part (b), note that with $\lambda, y_k, v_k \geq 0$,

$$\frac{\lambda}{\lambda + y_k} \leq v_k \Leftrightarrow \lambda \leq v_k(\lambda + y_k),$$

which by (6.1) are equivalent to

$$\begin{cases} (w_1, w_2, t) \in \text{SOC}_3, \\ w_1 = \sqrt{\lambda}, \\ t - w_2 = v_k, \\ t + w_2 = \lambda + y_k. \end{cases} \tag{6.2}$$

Considering (3.12) part (a),

$$\frac{1}{N} \sum_{k=1}^{N} v_k^\alpha \leq \beta,$$

noting also that $v_k \geq 0$, we restrict ourselves to only rational $\alpha \geq 1$. Clearly, if $\alpha = 1$ we simply have one linear inequality. We consider two cases: (1) $\alpha = 2^\ell$ where ℓ is a positive integer, and (2) $\alpha = p/q > 1$ where p, q are positive integers. Without loss of generality $\alpha > 1$ is further assumed, since otherwise the expected value would not be finite. Case (1) is simpler in a sense that it requires fewer auxiliary variables and cones, while case (2) is generic and certainly accommodates case (1), but requires more work. We show how to represent each of the summands of the form v_k^α, so that (3.12) part (a) can be replaced by t_k satisfying

$$\begin{cases} v_k^\alpha \leq s_k, \ k = 1, \ldots, N, \\ \sum_{k=1}^{N} s_k = N\beta. \end{cases} \tag{6.3}$$

Case (1): $\alpha = 2^\ell$ where ℓ is a positive integer. Note that $v_k^\alpha \leq s_k$ is equivalent to

$$\begin{cases} v_k^2 \leq u_1, \\ u_1^2 \leq u_2, \\ u_2^2 \leq u_3, \\ \vdots \\ u_{\ell-1}^2 \leq s_k, \end{cases}$$

which, in turn, by (6.1) is equivalent to

$$\begin{cases} (w_1, w_2, t)^{(i)} \in \mathrm{SOC}_3^{(i)}, \ i = 1, \dots, \ell, \\ w_1^{(1)} = v_k, \ t^{(i)} - w_2^{(1)} = 1, \ t^{(i)} + w_2^{(1)} = u_1, \\ w_1^{(i)} = u_{i-1}, \ t^{(i)} - w_2^{(i)} = 1, \ t^{(i)} + w_2^{(1)} = u_i, \ \ i = 2, \dots, \ell - 1, \\ w_1^{(\ell)} = u_{\ell-1}, \ t^{(\ell)} - w_2^{(\ell)} = 1, \ t^{(\ell)} + w_2^{(\ell)} = s_k, \end{cases} \tag{6.4}$$

where $(w_1, w_2, t)^{(i)} \in \mathrm{SOC}_3^{(i)}$ denotes the i^{th} three-dimensional vector with coordinates $w_1^{(i)}, w_2^{(i)}, t^{(i)}$ and $\mathrm{SOC}_3^{(i)}$ – the corresponding three-dimensional cone.

Case (2): $\alpha = p/q > 1$ where p, q are positive integers. First we demonstrate how to represent the (epigraph of the negated) *geometric mean* function

$$\sqrt[2^\ell]{u_1 u_2 u_3 \cdots u_{2^\ell}},$$

where $u_i \geq 0, i = 1, 2, \dots, 2^\ell$ for some fixed positive integer ℓ. Here, cascading construction is applied in the sense that at every step we aggregate our variables u_i into groups of two and recursively repeat the procedure until only one variable remains. Specifically, for $u_i \geq 0$ with $i \geq 1$,

$$u_1 u_2 u_3 \cdots u_{2^\ell} \geq s^{2^\ell}$$

is equivalent to

$$\begin{cases} u_1 u_2 \geq u_{S_1+1}^2, \ u_3 u_4 \geq u_{S_1+2}^2, \dots, u_{S_1-1} u_{S_1} \geq u_{S_1+K_1/2}^2, & K_1 = 2^\ell, S_1 = K_1, \\ u_{S_1+1} u_{S_1+2} \geq u_{S_2+1}^2, \dots, u_{S_2-1} u_{S_2} \geq u_{S_2+K_2/2}^2, & K_2 = K_1/2, S_2 = S_1 + K_2, \\ \vdots & \\ u_{S_{\ell-2}+1} u_{S_{\ell-2}+2} \geq u_{S_{\ell-1}+1}^2, \ u_{S_{\ell-1}-1} u_{S_{\ell-1}} \geq u_{S_{\ell-1}+2}^2, & K_{\ell-1} = 4, S_{\ell-1} = S_{\ell-2} + 2, \\ u_{S_{\ell-1}+1} u_{S_{\ell-1}+2} \geq s^2, & \end{cases}$$

which, in turn, by (6.1), is equivalent to

$$\begin{cases} i = 1, \dots, \ell, \\ S_1 = K_1 = 2^\ell, \\ K_i = K_{i-1}/2, S_i = S_{i-1} + K_i, i = 2, \dots, \ell - 1, \\ \\ (w_1, w_2, t)^{(i,j)} \in \mathrm{SOC}_3^{(i,j)}, \ \text{where for a fixed } i, & j = 1, \dots, 2^{\ell-i}, \\ w_1^{(1,j)} = u_{S_1+j}, \ t^{(1,j)} - w_2^{(1,j)} = u_{2j-1}, \ t^{(1,j)} + w_2^{(1,j)} = u_{2j}, & \\ w_1^{(i,j)} = u_{S_i+j}, \ t^{(i,j)} - w_2^{(i,j)} = u_{S_{i-1}+2j-1}, \ t^{(i,j)} + w_2^{(i,j)} = u_{S_{i-1}+2j}, & i = 2, \dots, \ell - 1, \\ w_1^{(\ell,1)} = s, \ t^{(\ell,1)} - w_2^{(\ell,1)} = u_{S_{\ell-1}+1}, \ t^{(\ell,1)} + w_2^{(\ell,1)} = u_{S_{\ell-1}+2}, & \end{cases} \tag{6.5}$$

where $(w_1, w_2, t)^{(i,j)} \in \mathrm{SOC}_3^{(i,j)}$ denotes the $(i,j)^{\text{th}}$ three-dimensional vector with coordinates $w_1^{(i,j)}, w_2^{(i,j)}, t^{(i,j)}$ and $\mathrm{SOC}_3^{(i,j)}$ – the corresponding three-dimensional cone.

Now, observe that if an integer ℓ is chosen such that $2^\ell \geq p + q$, and consider

$$u_1 u_2 u_3 \cdots u_{2^\ell} \geq s^{2^\ell}$$

combined with

$$\begin{cases} s = u_1 = u_2 = \cdots = u_{2^\ell - p} = v_k, \\ u_{2^\ell - p + 1} = \cdots = u_{2^\ell - p + q} = s, \\ u_i = 1, \;\; i = 2^\ell - p + q, \ldots, 2^\ell, \end{cases} \tag{6.6}$$

the desired representation $v_k^\alpha \leq s$ arises, since the above is equivalent to

$$\underbrace{v_k v_k \cdots v_k}_{2^\ell - p} \cdot \underbrace{s \cdots s}_{q} \geq v_k^{2^\ell} \Leftrightarrow s^q \geq v_k^p \Leftrightarrow s \geq v_k^{p/q},$$

with $p > q$ as a result of $\alpha = p/q > 1$. Note that for brevity of notation in the above construction we omit subindex k for the variable s, while it should be understood that for every v_k one must use an individual copy of s_k to generate N constraints $v_k^\alpha \leq s_k$.

In summary, the SOC reformulation of (3.12) may be obtained by taking (3.12), replacing part (b) with (6.2), and replacing part (a) with (6.3) combined with either (6.4) or (6.5) and (6.6), depending on α being either an integer power of 2 or not, applied to each $k = 1, \ldots, N$.