EXTREME BEHAVIOR OF BIVARIATE
ELLIPTICAL DISTRIBUTIONS

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Abstract

This paper exploits a stochastic representation of bivariate elliptical distributions in order to obtain asymptotic results which are determined by the tail behavior of the generator. Under certain specified assumptions, we present the limiting distribution of componentwise maxima, the limiting upper copula, and a bivariate version of the classical peaks over threshold result.

Keywords: Componentwise maxima; Elliptical distribution; Pickands' representation; Regular variation, Threshold exceedances

1 Introduction

During the past few decades there has been an extensive amount of work on the understanding of the elliptical class of distributions. The first comprehensive work was given by Fang *et al.* (1990). Primarily, these distributions allow an alternative and extension of the normal law. Elliptical distributions are easily implemented and simulated (see,

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for example, Breymann *et al.*, 2003; Hodgson *et al.*, 2002; Johnson, 1987; Li *et al.*, 1997; Manzotti *et al.*, 2002), and they are useful for actuarial and financial applications.

Modelling of extreme or rare events is an important and well-researched topic. When there are several random variables of interest, the dependence structure must be considered in investigating their extreme behavior. This is addressed in the growing literature on multivariate extreme value theory (see, for example, Beirlant, *et al.*, 2004).

The extreme behavior of elliptically distributed random vectors is closely related to the asymptotic property of its generator (see Berman, 1992 and Hashorva, 2005). Starting with the work of Sibuya (1960), recently many other papers have studied the extreme behavior of elliptical random vectors, see for example Hult and Lindskog (2002), Schmidt (2002), Abdous *et al.* (2005), Demarta and McNeil (2005), and Hashorva (2005).

In this paper, we present some results on the extreme behavior of bivariate elliptical distributions. These results hold under certain conditions on the tail behavior of the generator. Specifically, we give the limiting distribution of componentwise maxima of iid elliptical random vectors and find that it is exactly that obtained by Demarta and McNeil (2005) for the special case of the Student t distribution. We then present results concerning joint exceedances over a high threshold. We first provide a characterization of the limiting upper copula. We then give a bivariate version of the classical peaks over threshold result (see Balkema and de Haan, 1974, and Pickands, 1975). We close the paper with an illustration.

2   Definitions and examples

Let $\mathbf{Z}_i = (X_i, Y_i), i = 1, 2, \ldots$ be a sequence of independent random vectors with common distribution F, and let

$$\mathbf{M}_n = (\max_{i=1,\ldots,n} X_i, \max_{i=1,\ldots,n} Y_i).$$

That is, $\mathbf{M}_n$ is the vector of componentwise maxima of $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$. If there exist sequences of vectors of constants $\mathbf{a}_n, \mathbf{b}_n \in \Re^2$ and a random vector $\mathbf{Z}$ with distribution

G and nondegenerate marginals such that $\mathbf{a}_n\mathbf{M}_n + \mathbf{b}_n$ converges weakly to $\mathbf{Z}$, then G, the limit distribution of normalized componentwise maxima, is said to be a *bivariate extreme value distribution*. We then say that F is in the *maximum domain of attraction* of G with *normalizing vectors of constants* $\mathbf{a}_n$ and $\mathbf{b}_n$ and write $F \in \mathrm{MDA}(G)$. It is useful to note that

$$\lim_{n \to \infty} F^n(\mathbf{a}_n\mathbf{x} + \mathbf{b}_n) = G(\mathbf{x}) \Leftrightarrow \lim_{n \to \infty} n[1 - F(\mathbf{a}_n\mathbf{x} + \mathbf{b}_n)] = -\log G(\mathbf{x}), \qquad (1)$$

for all $\mathbf{x}$ such that $G(\mathbf{x}) > 0$.

A characterization of the maximum domain of attraction of multivariate extreme value distributions is given by Marshall and Olkin (1983). Necessary conditions for (1) are that each marginal F_i of F is in the (univariate) MDA of the corresponding component G_i of G. Classical results concerning univariate maxima are given by Gnedenko (1943). In particular, if $F_i \in \mathrm{MDA}(G_i)$ then, by the Fisher-Tippett theorem, G_i belongs to the type of the distribution

$$H_\xi(x) = \begin{cases} \exp\left\{-(1 + \xi x)^{-1/\xi}\right\}, & 1 + \xi x > 0, \quad \xi \neq 0 \\ \exp\{-e^{-x}\}, & -\infty < x < \infty, \quad \xi = 0 \end{cases}. \qquad (2)$$

H_ξ is known as the *generalized extreme value distribution*. For $\alpha > 0$, $\Phi_\alpha(x) := H_{1/\alpha}(\alpha(x - 1))$ is the standard Fréchet distribution, $\Psi_\alpha(x) := H_{-1/\alpha}(\alpha(x + 1))$ is the standard Weibull distribution, and $\Lambda(x) := H_0(x)$ is the standard Gumbel distribution.

It is well-known (see, for example, Embrechts *et al.*, 1997) that $F_i \in MDA(H_\xi)$ if and only if there exists a positive, measurable function $a(\cdot)$ such that

$$\lim_{t \uparrow x_{F_i}} \frac{\bar{F}_i(t + xa(t))}{\bar{F}_i(t)} = \begin{cases} (1 + \xi x)^{-1/\xi}, & 1 + \xi x > 0, \quad \text{if } \xi \neq 0 \\ e^{-x}, & -\infty < x < \infty, \quad \text{if } \xi = 0 \end{cases}, \qquad (3)$$

where x_{F_i} is the right endpoint of the support of F_i. The right-hand side of (3) is the survival function of the *generalized Pareto distribution*.

Returning to the bivariate setup, the bivariate extreme value distribution can be represented as follows

$$G(x, y) = \exp\left\{\log\{G_1(x)G_2(y)\} A\left(\frac{\log G_1(x)}{\log\{G_1(x)G_2(y)\}}\right)\right\}, \qquad (4)$$

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where A is the Pickands' representation function, which is a convex function on $[0, 1]$ such that $\max(t, 1 - t) \leq A(t) \leq 1$ (see Pickands, 1981).

The dependence structure associated with the distribution of a random vector can be characterized in terms of a *copula*. A two-dimensional copula is a bivariate distribution function defined on $[0, 1]^2$ with uniformly distributed marginals. Due to Sklar's Theorem (see Sklar, 1959), if F is a joint distribution function with continuous marginals F_1 and F_2 respectively, then there exists a unique copula, C, given by

$$C(u, v) = F(F_1^{\leftarrow}(u), F_2^{\leftarrow}(v)), \tag{5}$$

where $h^{\leftarrow}(u) = \inf\{x : h(x) \geq u\}$ is the generalized inverse function. Similarly, the *survival copula* is defined as the copula relative to the joint survival function and is given by

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v). \tag{6}$$

A more formal definition, properties and examples of copulas are given in Nelsen (1999). Let (U, V) be a random vector with copula C, and standard uniformly distributed marginals. The upper copula at level u is defined as follows:

$$C_u^{up}(x, y) = \Pr(U \leq F_{1,u}^{\leftarrow}(x), V \leq F_{2,u}^{\leftarrow}(y)|U > u, V > u), \tag{7}$$

where $F_{1,u}(x) = \Pr(U \leq x|U > u, V > u)$ and $F_{2,u}(y) = \Pr(V \leq y|U > u, V > u)$.

A fundamental concept in Extreme Value Theory is that of regular variation, which we now define.

Definition 1 *A positive measurable function h defined on $(0, \infty)$ and satisfying*

$$\lim_{t \to \infty} \frac{h(tx)}{h(t)} = x^{\alpha}, \; x > 0, \tag{8}$$

is said to be regularly varying at ∞ with index $\alpha \in \Re$, and we denote this by $h \in RV_{\alpha}^{\infty}$.

For a more thorough background on regular variation see Bingham *et al.* (1987).

We now introduce the bivariate elliptical family of distribution, using the approach of Abdous *et al.* (2005). For other properties see Fang *et al.* (1990).

Definition 2 *A bivariate elliptical random vector has the following stochastic representation:*

$$(X, Y) \overset{d}{=} (\mu_X, \mu_Y) + (\sigma_X R D U_1, \sigma_Y \rho R D U_1 + \sigma_Y \sqrt{1 - \rho^2} \, R \, \sqrt{1 - D^2} \, U_2), \quad (9)$$

where $U_1, U_2, R,$ and D are mutually independent random variables, $\mu_X, \mu_Y \in \Re$ are the respective means of X and Y, $\sigma_X, \sigma_Y > 0$ are the standard deviations, ρ is the Pearson correlation between X and Y, and $\Pr(U_i = -1) = \Pr(U_i = 1) = \frac{1}{2}$, $i = 1, 2$. Both D and R are positive random variables and D has probability density function

$$f_D(s) = \frac{2}{\pi \sqrt{1 - s^2}}, \quad 0 < s < 1. \quad (10)$$

The random variable R is called the generator of the elliptical distributed random vector.

Throughout this paper it is assumed that $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$. Therefore, the joint distribution of X and Y is symmetric, and X and Y are identically distributed. Our results can be extended to the more general setup.

The following examples give the generator pdfs for some well-known bivariate elliptical distributions. We refer to these examples later in the paper. For more examples, see Fang *et al.* (1990), who use a more classical representation. Abdous *et al.* (2005) explain the relationship between the two representations.

Example 1 *Pearson type VII*

$$f_R(x) = \frac{2(N-1)}{m} x \left(1 + \frac{x^2}{m}\right)^{-N}, \; x > 0, \; N > 1, m > 0.$$

When $m = 1$ and $N = 3/2$, we have the Cauchy distribution, and when $N = (m+2)/2$ we have the Student t distribution with m degrees of freedom.

Example 2 *Logistic*

$$f_R(x) = 4 \, x \, \frac{\exp\{-x^2\}}{(1 + \exp\{-x^2\})^2}, \; x > 0.$$

Example 3 *Kotz*

$$f_R(x) = \frac{2s}{r^{-N/s} \Gamma(N/s)} x^{2N-1} \exp\{-rx^{2s}\}, \; x > 0, \; N, r, s > 0.$$

When $N = 1$, $s = 1$, and $r = 1/2$, we have the normal distribution.

3 Main results

3.1 Componentwise maxima

The limiting distribution of componentwise maxima of iid elliptical random vectors is discussed in detail by Hashorva (2005). The following result shows that, in the bivariate case where the generator $R \in MDA(\Phi_\alpha)$, the limiting distribution of componentwise maxima of iid bivariate elliptical random vectors is exactly that obtained by Demarta and McNeil (2005) for the bivariate Student t distribution.

Proposition 1 *Let (X, Y) be a bivariate standardized elliptical random vector, and F its distribution function. If $R \in MDA(\Phi_\alpha)$, then $(X, Y) \in MDA(G)$, where G has Fréchet marginals, Φ_α, and the Pickands' representation is given by*

$$A(t) = t\bar{T}_{\alpha+1}\left\{\frac{[(\frac{1-t}{t})^{\frac{1}{\alpha}} - \rho]\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right\} + (1-t)\bar{T}_{\alpha+1}\left\{\frac{[(\frac{t}{1-t})^{\frac{1}{\alpha}} - \rho]\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right\}, \quad (11)$$

where $\bar{T}_\alpha$ is the survival function of a univariate Student t random variable with α degrees of freedom.

Proof. First, we show that $X \in MDA(\Phi_\alpha)$ whenever the generator $R \in MDA(\Phi_\alpha)$. The latter implies that $\bar{F}_R \in RV_{-\alpha}^\infty$ (see, for example, Embrechts *et al.*, 1997). Therefore, for $x > 0$, by conditioning on U_1 in (9) we get

$$\frac{\bar{F}_X(x)}{\bar{F}_R(x)} = \frac{\Pr(RDU_1 > x)}{\bar{F}_R(x)} = \frac{1}{2}\frac{\Pr(RD > x)}{\bar{F}_R(x)}$$

$$= \frac{1}{2}\int_0^1 \frac{\bar{F}_R(\frac{x}{u})}{\bar{F}_R(x)} f_D(u)\, du \to \frac{1}{2}\int_0^1 u^\alpha f_D(u)\, du \text{ as } x \to \infty, \quad (12)$$

where the Dominated Convergence Theorem is used in the last step, since for x sufficiently large, the integrand is bounded by $u^{\alpha-1/2}f_D(u)$. The result can also be obtained from Lemma 2.2 of Hashorva (2005). Thus, $X \in MDA(\Phi_\alpha)$, and the normalizing constants for the maxima are given by $a_n \sim F_X^\leftarrow(1 - n^{-1})$ and $b_n = 0$ (see page 131 of Embrechts *et al.* 1997). It is sufficient to verify convergence criterion (1):

$$n[1 - \Pr(X \le a_n x, Y \le a_n y)]$$

$$= n\Pr(X > a_n x) + n\Pr(Y > a_n y) - n\Pr(X > a_n x, Y > a_n y). \quad (13)$$

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Since X and $Y \in MDA(\Phi_\alpha)$, the first two terms on the right hand side of (13) have limits $x^{-\alpha}$ and $y^{-\alpha}$, respectively, and from Theorem 1 of Abdous *et al.* (2005) we have

$$
\begin{aligned}
&n \Pr(X > a_n x, Y > a_n y) \\
&= \quad \frac{\Pr(X > a_n x, Y > a_n y)}{\Pr(X > a_n x)} n \bar{F}_X(a_n x) \\
&\rightarrow \quad x^{-\alpha} \bar{T}_{\alpha+1} \left\{ \frac{\left(\frac{y}{x} - \rho \right) \sqrt{\alpha + 1}}{\sqrt{1 - \rho^2}} \right\} + y^{-\alpha} \bar{T}_{\alpha+1} \left\{ \frac{\left(\frac{x}{y} - \rho \right) \sqrt{\alpha + 1}}{\sqrt{1 - \rho^2}} \right\}, \qquad (14) \\
&\quad \text{as } n \rightarrow \infty.
\end{aligned}
$$

Combining (1), (4), (13) and (14) completes the proof. ∎

3.2 Joint threshold exceedances

In financial applications, the limiting distribution of joint threshold exceedances is important in assessing the impact of extreme events affecting two or more variables of interest. For example, the losses in value of several different assets that result from a stock market crash can be viewed as dependent random variables. In analyzing the overall effect of the crash on the value of a portfolio, the dependence structure of these losses must be considered. If we are primarily interested in extreme cases, it is useful to understand the behavior of joint exceedances over a high threshold.

When the threshold of interest for each asset is the Value at Risk (VaR), then we are interested in exceedances above high quantiles. The joint distribution of these exceedances is given by the upper copula.

The next result is motivated by the work of Breymann *et al.* (2003). There, an empirical approach was given to illustrate that the limiting upper copula of a bivariate elliptical random vector is well-fitted by the survival Clayton copula. If $R \in MDA(\Phi_\alpha)$, then under the assumption that the distribution function of the elliptical random vector is continuous with strictly increasing marginals, we can obtain an asymptotic result for the upper copula. This result is a direct implication of Theorem 2.3 of Juri and Wüthrich (2003).

Proposition 2 *Let (X, Y) be a standardized continuous elliptical random vector with strictly increasing margins. If $R \in MDA(\Phi_\alpha)$, then the limiting survival upper copula is given by*

$$\lim_{u \uparrow 1} \hat{C}_u^{up}(x, y) = g^{-1}(y) g\left(\frac{g^{-1}(x)}{g^{-1}(y)}\right), \tag{15}$$

where

$$g(x) = \frac{x \bar{T}_{\alpha+1}\left((x^{1/\alpha} - \rho)\frac{\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right) + \bar{T}_{\alpha+1}\left((x^{-1/\alpha} - \rho)\frac{\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right)}{2 \, \bar{T}_{\alpha+1}\left\{(1-\rho)\sqrt{\frac{\alpha+1}{1-\rho^2}}\right\}}. \tag{16}$$

Remarks:

1. Proposition 2 is useful because it expresses the limiting distribution in terms of the two parameters α and ρ, which can be estimated using standard methods.

2. A comparison of contour plots (not shown) indicate that the copula in (15) is indeed similar to the Clayton copula.

Proof. Letting $x > 0$, we only need to check the sufficient condition from Theorem 2.3 of Juri and Wüthrich (2003) as follows:

$$
\begin{aligned}
\frac{\hat{C}(xv, v)}{\hat{C}(v, v)} &= \frac{\Pr(X > \bar{F}_X^{\leftarrow}(xv), Y > \bar{F}_X^{\leftarrow}(v))}{\Pr(X > \bar{F}_X^{\leftarrow}(v), Y > \bar{F}_X^{\leftarrow}(v))} \\
&\sim \frac{\Pr(X > x^{-1/\alpha} \bar{F}_X^{\leftarrow}(v), Y > \bar{F}_X^{\leftarrow}(v))}{\Pr(X > \bar{F}_X^{\leftarrow}(v), Y > \bar{F}_X^{\leftarrow}(v))} \\
&\to g(x), \text{ as } v \downarrow 0,
\end{aligned}
$$

which gives the required result by applying Theorem 1 of Abdous *et al.* (2005) and the result of de Haan (1970, see page 22). ∎

The main result of this paper establishes the joint distribution of the exceedances over a high threshold when $R \in MDA(\Phi_\alpha)$ and when $R \in MDA(\Lambda)$. We first give some preliminary results.

If a distribution function $F \in MDA(\Lambda)$ with infinite support, then the auxiliary function $a(\cdot)$ that satisfies (3) is absolutely continuous with density $a'(\cdot)$ such that

$$\lim_{t \to \infty} \frac{a(t)}{t} = 0, \ \lim_{t \to \infty} a'(t) = 0, \text{ and } \lim_{t \to \infty} \frac{a(t + xa(t))}{a(t)} = 1, \tag{17}$$

locally uniformly in $x \in \Re$. For further details see Resnick (1987, p. 40).

The following lemma will be useful in proving the main result.

Lemma 1 *If $F \in MDA(\Lambda)$ with $x_F = \infty$ and auxiliary function $a(\cdot)$, then, provided that $h(t) = o(a(t))$, the following holds for any x:*

$$\lim_{t \to \infty} \frac{\bar{F}(t + xa(t) + h(t))}{\bar{F}(t)} = \exp\{-x\}. \tag{18}$$

Proof. Let $h(t) = o(a(t))$. Then it is sufficient to verify that $\bar{F}(t + h(t)) \sim \bar{F}(t)$. Using a representation of Von Mises functions (see Resnick, 1987, p. 40) we need only prove that

$$\lim_{t \to \infty} \int_{t}^{t+h(t)} \frac{1}{a(u)} \, du = 0. \tag{19}$$

Let $\varepsilon, \delta > 0$, then since $a(\cdot)$ is positive, for t sufficiently large we get

$$\int_{t}^{t+h(t)} \frac{1}{a(u)} \, du \le \int_{t}^{t+a(t)\varepsilon} \frac{1}{a(u)} \, du = \int_{0}^{\varepsilon} \frac{a(t)}{a(t + za(t))} \, dz < (1 + \delta)\varepsilon,$$

where the last inequality is implied by (17), which completes the proof. ∎

Theorem 1 *Let (X, Y) be a bivariate standard elliptical random vector with $-1 < \rho < 1$.*

(a) Let $R \in MDA(\Phi_\alpha)$. Then whenever $x, y > 0$,

$$\lim_{t \to \infty} \Pr(X > t + xa(t), Y > t + ya(t) | X > t, Y > t) \tag{20}$$

$$= \frac{\left(1 + \frac{x}{\alpha}\right)^{-\alpha} \bar{T}_{\alpha+1}\left\{\left(\frac{\alpha+y}{\alpha+x} - \rho\right)\sqrt{\frac{\alpha+1}{1-\rho^2}}\right\} + \left(1 + \frac{y}{\alpha}\right)^{-\alpha} \bar{T}_{\alpha+1}\left\{\left(\frac{\alpha+x}{\alpha+y} - \rho\right)\sqrt{\frac{\alpha+1}{1-\rho^2}}\right\}}{2\,\bar{T}_{\alpha+1}\left\{(1 - \rho)\sqrt{\frac{\alpha+1}{1-\rho^2}}\right\}},$$

where $a(\cdot)$ is defined by (3).

(b) Let $R \in MDA(\Lambda)$ with auxiliary function $a(\cdot)$ and infinite right endpoint. If $a \in RV_\alpha^\infty$, $\alpha \le 1$, then whenever $x, y > 0$,

$$\lim_{t \to \infty} \Pr(X > t + xa(t), Y > t + ya(t) | X > t, Y > t) = \exp\left\{-\frac{x + y}{2}K^{\alpha-1}(\rho)\right\}, \tag{21}$$

where $K(\rho) = \sqrt{(\rho + 1)/2}$.

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If $\rho = 1$, then

$$\lim_{t \to \infty} \Pr(X > t + xa(t), Y > t + xa(t)|X > t, Y > t) = \exp\{-\max(x, y)\}.$$

Remarks:

1. When $\rho = -1$, there exists a $t_0 > 0$ such that for all $t > t_0$, $\Pr(X > t, Y > t) = 0$. Since it does not make sense to condition on the event $\{X > t, Y > t\}$ in this case, an equivalent result cannot be obtained.

2. In the Gaussian case, $\alpha = -1$ and (21) coincides with the result of Juri and Wüthrich (2003).

Proof. (a) If $R \in MDA(\Phi_\alpha)$, then $a(t) \sim \frac{t}{\alpha}$ (see p. 159 Embrechts *et al.* 1997). Then the proof of (a) follows from Theorem 1 of Abdous *et al.* (2005).

(b) Let $x, y \geq 0$, and we assume that $\rho \in [0, 1)$ (the $\rho \in (-1, 0)$ case follows the same reasoning). We now prove that when $t \to \infty$ the following holds:

$$\frac{\Pr(X > t + xa(t), Y > t + xa(t))}{\bar{F}_R\left(\frac{t}{K(\rho)}\right)} \sim 2\frac{a(t)}{t}\frac{K^{2-\alpha}(\rho)}{\sqrt{1-K^2(\rho)}} \exp\left\{-K^{\alpha-1}(\rho)\frac{x+y}{2}\right\}. \quad (22)$$

By conditioning on U_1, U_2 and D, from Definition 2, for t sufficiently large, we obtain

$$\Pr(X > t + xa(t), Y > t + xa(t))$$
$$= \frac{1}{2\pi}\left[\int_0^1 \bar{F}_R\left(\max\left\{\frac{t + a(t)x}{u}, \frac{t + a(t)y}{f(u, \rho)}\right\}\right)\frac{1}{\sqrt{1 - u^2}} du \right.$$
$$\left. + \int_{\sqrt{1-\rho^2}}^1 \bar{F}_R\left(\frac{t + a(t)x}{g(u, \rho)}\right)\frac{1}{\sqrt{1 - u^2}} du\right], \quad (23)$$

where $f(u, \rho) = \rho u + \sqrt{1 - \rho^2}\sqrt{1 - u^2}$ and $g(u, \rho) = \rho u - \sqrt{1 - \rho^2}\sqrt{1 - u^2}$. Note that we have used the fact that $g(u, \rho) < 0$ when $u < \sqrt{1 - \rho^2}$. Some simple algebraic computations allow one to express (23) as

$$\Pr(X > t + xa(t), Y > t + xa(t)) = \frac{1}{2\pi}\{I_1(t, x, y, \rho) + I_2(t, x, y, \rho) + I_3(t, x, y, \rho)\}, \quad (24)$$

where the three integrals I_1, I_2, and I_3 are

$$I_1(t, x, y, \rho) = \int_0^{u(t,x,y,\rho)} \bar{F}_R \left(\frac{t + a(t)x}{u} \right) \frac{1}{\sqrt{1 - u^2}} \, du, \qquad (25)$$

$$I_2(t, x, y, \rho) = \int_{u(t,x,y,\rho)}^1 \bar{F}_R \left(\frac{t + a(t)y}{f(u, \rho)} \right) \frac{1}{\sqrt{1 - u^2}} \, du, \qquad (26)$$

$$I_3(t, x, y, \rho) = \int_{\sqrt{1-\rho^2}}^1 \bar{F}_R \left(\frac{t + a(t)y}{g(u, \rho)} \right) \frac{1}{\sqrt{1 - u^2}} \, du, \qquad (27)$$

and

$$u(t, x, y, \rho) = \left(\frac{\left(\frac{t+a(t)y}{t+a(t)x} \right)^2 - 2\rho \left(\frac{t+a(t)y}{t+a(t)x} \right) + 1}{1 - \rho^2} \right)^{-1/2}. \qquad (28)$$

We now have to determine the rates of convergence for each of the three integrals defined in (25), (26), and (27). First, we establish that

$$I_1(t, x, y, \rho) \sim \frac{K^{2-\alpha}(\rho)}{\sqrt{1 - K^2(\rho)}} \frac{a(t)}{t} \bar{F}_R \left(\frac{t}{K(\rho)} \right) \exp \left\{ -K^{\alpha-1}(\rho) \frac{x + y}{2} \right\}, \text{ as } t \to \infty. \quad (29)$$

The change of variable $u(t, x, y, \rho)/u = 1 + za(t)/t$ in (25) gives

$$I_1(t, x, y, \rho) = \frac{a(t)}{t} u(t, x, y, \rho) \times$$
$$\int_0^\infty \bar{F}_R \left(\frac{t + (x + z)a(t) + xza^2(t)/t}{u(t, x, y, \rho)} \right) \frac{(1 + za(t)/t)^{-2}}{\sqrt{1 - \left(\frac{u(t,x,y,\rho)}{1+za(t)/t} \right)^2}} \, dz. \qquad (30)$$

Using Lemma 1 and the fact that $a(\cdot) \in RV_\alpha^\infty$, straightforward computations yield that

$$\frac{\bar{F}_R \left(\frac{t+(x+z)a(t)+xza^2(t)/t}{u(t,x,y,\rho)} \right)}{\bar{F}_R(t/K(\rho))} \sim \exp \left\{ -K^{\alpha-1}(\rho) \left(z + \frac{x + y}{2} \right) \right\}, \text{ as } t \to \infty. \qquad (31)$$

Since $e^{-z} < 1/z(z+1)$ for $z \geq 2$ the integral in (30) is bounded, and the Dominated Convergence Theorem together with (17), (28), and (31) leads to (29).

In a similar manner asymptotic equivalences for I_2 and I_3 can be found. The one-to-one mapping $u \mapsto f(z, \rho)$ reduces (26) to

$$I_2(t, x, y, \rho) = \int_\rho^{z(t,x,y,\rho)} \bar{F}_R \left(\frac{t + a(t)y}{z} \right) \frac{1}{\sqrt{1 - u^2}} \, du, \qquad (32)$$

where

$$z(t, x, y, \rho) = f(u(t, x, y, \rho), \rho). \qquad (33)$$

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The change of variable $z - \rho = (z(t, x, y, \rho) - \rho) / (1 + \varsigma a(t)/t)$ in (32) yields

$$I_2(t, x, y, \rho) = (z(t, x, y, \rho) - \rho)\frac{a(t)}{t} \times$$
$$\int_0^\infty \bar{F}_R\left(\frac{t + (y + \varsigma)a(t) + y\varsigma a^2(t)/t}{z(t, x, y, \rho) + \rho\varsigma a(t)/t}\right) \frac{(1 + \varsigma a(t)/t)^{-2}}{\sqrt{1 - \left(\frac{z(t,x,y,\rho)+\rho\varsigma \frac{a(t)}{t}}{1+\varsigma a(t)/t}\right)^2}} \, d\varsigma, \qquad (34)$$

and straightforward computations together with Lemma 1 and the Dominated Convergence Theorem give

$$I_2(t, x, y, \rho) \sim \frac{K^{2-\alpha}(\rho)}{\sqrt{1 - K^2(\rho)}} \frac{a(t)}{t} \bar{F}_R\left(\frac{t}{K(\rho)}\right) \exp\left\{-K^{\alpha-1}(\rho)\frac{x+y}{2}\right\}, \text{ as } t \to \infty. \quad (35)$$

The change of variable $z = g(u, \rho)$ in (27) yields

$$I_3(t, x, y, \rho) = \int_0^\rho \bar{F}_R\left(\frac{t + a(t)y}{z}\right) \frac{1}{\sqrt{1 - z^2}} \, dz. \qquad (36)$$

In a similar way as for the previous two integrals, the rate of convergence for I_3 can be found when $\rho > 0$:

$$I_3(t, x, y, \rho) \sim \frac{\rho^{2-\alpha}}{\sqrt{1 - \rho^2}} \frac{a(t)}{t} \bar{F}_R\left(\frac{t}{\rho}\right) \exp\left\{-\rho^{\alpha-1}y\right\}, \text{ as } t \to \infty, \qquad (37)$$

and by (27) $I_3 \equiv 0$ when $\rho = 0$. Moreover, when $\rho \geq 0$ it follows that $\rho < K(\rho)$, and since $\bar{F}_R$ is rapidly varying (see, for example, Embrechts, *et al.* 1997, p. 140) and using (37) we get

$$I_3(t, x, y, \rho) = o\left(\bar{F}_R\left(\frac{t}{K(\rho)}\right)\frac{a(t)}{t}\right). \qquad (38)$$

Combining (24), (29), (35) and (38) gives (22) and (21), which completes the proof. ∎

The Pearson type VII generator given in Example 1 is in the maximum domain of attraction of Fréchet distribution with $\alpha = 2(N - 1)$, and the generators given in Examples 2 and 3 are in the maximum domain of attraction of the Gumbel distribution. The auxiliary functions $a(\cdot)$ are regularly varying with indices -1 and $1 - 2s$ for the Logistic and Kotz cases, respectively.

4   Illustration

In this section, we explore the sensitivity of the probabilities obtained from the limit distribution given by (20) in Theorem 1 to the values of α and ρ, and we illustrate how

the theorem can be used in analyzing the joint distribution of returns on two stocks in the presence of an extreme event such as a market crash.

Table 1: Probabilities from equation (20) with $x = 1$ and $y = 3$ for various α and ρ

$\alpha \backslash \rho$	-0.9	-0.7	-0.2	0	0.1	0.5	0.8
3	0.2155	0.2145	0.2110	0.2089	0.2076	0.1994	0.1835
4	0.1971	0.1962	0.1929	0.1910	0.1898	0.1820	0.1664
5	0.1856	0.1847	0.1817	0.1799	0.1788	0.1714	0.1563

Table 1 shows joint probabilities obtained from equation (20) with $x = 1$ and $y = 3$ for several values of α and ρ. We observe that these probabilities are sensitive to the value of α, while the value of ρ does not have an important impact.

Table 2: Approximate Values of $\Pr(X > 0.25 + x, Y > 0.25 + y \mid X > 0.25, Y > 0.25)$

x	y	Probability
0.1	0.1	0.2603
0.1	0.2	0.1456
0.1	0.3	0.0826
0.2	0.2	0.0953
0.2	0.3	0.0606
0.3	0.3	0.0427

We now illustrate the used of Theorem 1 in analyzing the conditional joint distribution of returns on two stocks when both are subject to large losses. Let X represent minus the daily log return for a given stock, and let Y represent minus the daily log return for another stock. Assume that (X, Y) is elliptically distributed with mean vector $(0, 0)$, standard deviation vector $(0.01, 0.01)$, $\alpha = 4$ and $\rho = 0.5$. These parameters were chosen arbitrarily, but are intended to be plausible. We are interested in the conditional distribution of (X, Y) given that a significant loss as occurred on both stocks (perhaps due to a market crash). Specifically, we condition on the event that minus the

log return on both stocks exceeds 0.25. That is, both stocks have decreased in value by at least (approximately) 22 percent. Table 2 shows several probabilities obtained from the conditional distribution of interest using the result of Theorem 1 (a). Calculations such as this allow one to correctly capture the impact of the dependence structure when analyzing the severity investment losses under extreme market conditions.

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