Asymptotics for Risk Capital Allocations based on Conditional Tail Expectation

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Abstract. An investigation of the limiting behavior of a risk capital allocation rule based on the Conditional Tail Expectation (CTE) risk measure is carried out. More specifically, with the help of general notions of Extreme Value Theory (EVT), the aforementioned risk capital allocation is shown to be asymptotically proportional to the corresponding Value-at-Risk (VaR) risk measure. The existing methodology acquired for VaR can therefore be applied to a somewhat less well-studied CTE. In the context of interest, the EVT approach is seemingly well-motivated by modern regulations, which openly strive for the excessive prudence in determining risk capitals.

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1. Introduction

Let X denote an insurance risk. Speaking more formally, X is a non-negative random variable defined on the probability space $(\Omega, \mathcal{F}, \Pr)$ and possessing a distribution function $F(x) := \Pr(X \leq x)$ and a tail function $\overline{F}(x) := 1 - F(x),\ x \in \mathbf{R}$. A risk measure is generally formulated as a functional, Q, from the space of distribution functions to $[0, \infty]$. Similarly (see, e.g., Bühlmann, 1980), we can consider the functional Q as from $\mathcal{X}$, the space of insurance risks, to $[0, \infty]$, which we indeed often do in the sequel.

Certainly, $\overline{F}$ establishes a meaningful ordering of $\mathcal{X}$ and, hence, it can be interpreted as a risk measure. However, for the sake of risk capital determination, it is desirable for the risk measure $Q[F]$ to take on monetary units. Thus, $Q[F] = \overline{F}(x)$ is naturally replaced with, e.g., its inverse, bringing us to the notion of Value-at-Risk (VaR). Namely, let $q \in (0, 1)$ denote the confidence level required by regulations. Then

$$\mathrm{VaR}_q[X] := \inf\{x : F(x) \geq q\}$$

establishes arguably the most popular risk measure, which has been a cornerstone of the financial risk measurement of the last century. We note in passing that the Solvency II Accord designed by the EU Commission sets $q = 0.995$ over a one-year time horizon.

Noticeably, the recent financial instability and, as a result, regulators' inclination to excessive prudence in determining risk capital requirements have to a certain extent enfeebled VaR's status. In this respect, the so-called tail-based risk measurement has emerged as a natural tool for quantifying insurance risks while emphasizing the adverse effect of low probability but high severity tail events. Thereby, a more pessimistic Conditional Tail Expectation (CTE) risk measure is defined, for $q \in (0, 1)$, as

$$\mathrm{CTE}_q[X] := \mathbf{E}[X | X > \mathrm{VaR}_q[X]].$$

CTE is known as a coherent risk measure over the space of continuous random variables. It belongs to both the distorted and weighted risk measures (see Wang, 1996; Dhaene *et al.*, 2006; Furman and Zitikis, 2008a). Practically, CTE has already replaced VaR in regulatory requirements of, e.g., Canada, Israel and Switzerland. We note in passing that the current practice in the aforementioned countries is $q = 0.99$ over a one-year time horizon.

Let $X_i \in \mathcal{X},\ i = 1, \ldots, d$, denote $d \in \mathbf{N}$ insurance risks and let $S_d := X_1 + \cdots + X_d$ denote the aggregate risk. Then evaluating $\mathrm{VaR}_q[S_d]$ and $\mathrm{CTE}_q[S_d]$ is a somewhat basic phase of the modern risk capital framework. Indeed, while it is of pivotal importance to determine the overall risk capital requirement for an insurance company, it is of consequent interest to decompose the aforementioned capital into the associated risk sources. To this end, the functional Q is naturally generalized beyond the conditional state independence, to a risk capital allocation functional, A, from the space of the Cartesian product of $\mathcal{X}$ with itself to $[0,\infty]$, and such that $A[X_i, X_i] = Q[X_i],\ i = 1,\dots,d$; see, e.g., Furman and Zitikis (2008b). It should be noted that apart from purely regulatory interest, the functional A is often employed for, e.g., profitability analysis, pricing and quality control.

Various functional forms of A have been proposed in the literature, with the allocation based on the CTE risk measure, formulated as

$$\mathbf{E}[X_i|S_d > \mathrm{VaR}_q[S_d]], \qquad i = 1,\dots,d, \tag{1.1}$$

being arguably the most popular. See Section 6.3 of McNeil *et al.* (2005) for related discussions on this allocation as a consequence of the Euler principle, as well as Dhaene *et al.* (2011) for optimality studies of interest. Although (1.1) is quite elegant and satisfies many desirable properties, its analytic tractability for generally distributed and possibly dependent $X_1, \dots, X_d$ remains seldom feasible. To emphasize the point, we refer the reader to Panjer and Jia (2001), Landsman and Valdez (2003), Valdez and Chernih (2003), Cai and Li (2005), Furman and Landsman (2005, 2006, 2008), Chiragiev and Landsman (2007), Vernic (2006, 2011) and Dhaene *et al.* (2008) for analytic expressions for (1.1) under specific multivariate distributions of a multi-line business and/or a portfolio of risks.

In this paper, we follow a different route. Namely, as the excessive prudence of the current regulatory framework requires a confidence level close to 1, the notion of Extreme Value Theory (EVT) becomes appropriate. We therefore study the asymptotic behavior of capital allocations defined in (1.1) as $q \uparrow 1$, when $X_1, \dots, X_d$ are asymptotically dependent or asymptotically independent. Following Section 5.2 of McNeil *et al.* (2005), the asymptotic independence between two random variables X_i and X_j with distribution functions F_i and F_j is defined as

$$\lim_{q\uparrow 1} \Pr\left(F_j(X_j) > q | F_i(X_i) > q\right) = 0,$$

while the asymptotic dependence is defined via this relation with a positive limit. However, in this paper we slightly relax the notion of asymptotic dependence and define it as

$$\liminf_{q\uparrow 1} \Pr\left(F_j(X_j) > q | F_i(X_i) > q\right) > 0. \tag{1.2}$$

The notion of asymptotic dependence in higher dimensions is an obvious generalization of the two-dimensional definition above. It is known that, for both Fréchet and Gumbel cases, the CTE and VaR of a single risk are proportional for a high confidence level (see Asimit and Badescu, 2010). Therefore, not surprisingly our main results show that capital allocations, as described by (1.1), are asymptotically proportional to $\mathrm{VaR}_q[S_d]$ with a readily calculable coefficient of proportionality. This allows for the utilization of the VaR-related machinery when dealing with the risk capital allocation based on CTE.

The remainder of the paper is organized as follows. The main results under asymptotic dependence and asymptotic independence are formulated and proved in Sections 2 and 3, respectively. Relevant examples are discussed in Section 4, while certain simulation studies verifying the accuracy of the main results are carried out in Section 5. Finally, Section 6 concludes the paper.

2. Main results under asymptotic dependence

From now on, we consider a multi-line insurance business consisting of d non-negative risk variables $X_1, \ldots, X_d$. Denote $\underline{X} = (X_1, \ldots, X_d)$ and $S_d = \sum_{i=1}^{d} X_i$. Unless otherwise stated, all limit relationships hold as $t \to \infty$ or $q \uparrow 1$, letting the relations speak for themselves. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(\cdot) \sim cb(\cdot)$ to mean strong equivalence, i.e., $\lim a(\cdot)/b(\cdot) = c$ for some positive constant c, and we write $a(\cdot) \asymp b(\cdot)$ to mean weak equivalence, i.e., $0 < \liminf a(\cdot)/b(\cdot) \leq \limsup a(\cdot)/b(\cdot) < \infty$. We also denote $\liminf a(\cdot)/b(\cdot) \geq 1$ and $\limsup a(\cdot)/b(\cdot) \leq 1$ by $a(\cdot) \gtrsim b(\cdot)$ and $a(\cdot) \lesssim b(\cdot)$, respectively.

To develop the main results of this paper we extensively employ the EVT techniques. A distribution function F is said to belong to the *Maximum Domain of Attraction* (MDA) of a non-degenerate distribution function G, written as $F \in \mathrm{MDA}(G)$, if there are some $a_n > 0$ and $b_n \in \mathbf{R}$ for $n \in \mathbf{N}$ such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x).$$

Due to the Fisher-Tippett theorem (see Fisher and Tippett, 1928, and Gnedenko, 1943), G is of one of the following three types:

Fréchet type: $\Phi_\alpha(x) = \exp\left\{-x^{-\alpha}\right\}, \quad x > 0, \alpha > 0,$

Gumbel type: $\Lambda(x) = 1 - \exp\left\{-\mathrm{e}^{-x}\right\}, \quad -\infty < x < \infty,$

Weibull type: $\Psi_\alpha(x) = \exp\left\{-(-x)^{\alpha}\right\}, \quad x \leq 0, \alpha > 0.$

Since distributions from $\mathrm{MDA}(\Psi_\alpha)$ have finite upper endpoints while we are interested in risk variables with unbounded supports, in this paper we shall consider the Fréchet and Gumbel cases only.

Another important notion that is crucial for establishing our main results is *vague convergence*. Let $\{\mu_n, n \geq 1\}$ be a sequence of measures on a locally compact Hausdorff space $\mathbb{B}$ with countable base. Then μ_n converges vaguely to some measure μ, written as $\mu_n \xrightarrow{v} \mu$, if for all continuous functions f with compact support we have

$$\lim_{n\to\infty} \int_{\mathbb{B}} f \, \mathrm{d}\mu_n = \int_{\mathbb{B}} f \, \mathrm{d}\mu.$$

A thorough background on vague convergence is given by Kallenberg (1983) and Resnick (1987).

2.1. **Fréchet case.** The next assumption is sufficient for our first main result.

Assumption 2.1. *Let* $\underline{X}$ *be a non-negative random vector with marginal distributions* $F_1, \ldots, F_d$ *such that*

$$\lim_{t\to\infty} \frac{\Pr(X_1 > tx_1, \ldots, X_d > tx_d)}{\bar{F}_1(t)} := H_F(\underline{x})$$

exists for all $\underline{x} = (x_1, \ldots, x_d) \in [0,\infty]^d \setminus \{\underline{0}\}$*, where* $H_F(\cdot)$ *is assumed to be a non-degenerate function and* $\underline{0}$ *is the vector of zeroes.*

This assumption implies that the marginal distribution functions are tail equivalent. That is,

$$0 < \lim_{t\to\infty} \frac{\bar{F}_j(t)}{\bar{F}_i(t)} < \infty, \tag{2.1}$$

for all $1 \leq i, j \leq d$. In addition, there exists some $\alpha > 0$ such that, for each $1 \leq i \leq d$, the distribution function of X_i is regularly varying with index α, written as $X_i \in \mathcal{R}_{-\alpha}$,

$$\lim_{t\to\infty} \frac{\bar{F}_i(tx)}{\bar{F}_i(t)} = x^{-\alpha}, \qquad x > 0. \tag{2.2}$$

Note that $X_i \in \mathcal{R}_{-\alpha}$ is equivalent to $X_i \in \mathrm{MDA}(\Phi_\alpha)$ (see Resnick, 1987). Moreover,

$$\frac{\Pr\left((X_1/t, \ldots, X_d/t) \in \cdot\right)}{\bar{F}_1(t)} \xrightarrow{v} \mu(\cdot) \tag{2.3}$$

holds on $[0,\infty]^d \setminus \{\underline{0}\}$, where the measure μ is given by

$$\mu\left((x_1,\infty] \times \cdots \times (x_d,\infty]\right) := H_F(\underline{x}). \tag{2.4}$$

The function H_F satisfies certain properties (see Resnick, 1987), and one of the most important is continuity on the set $(0,\infty)^d$, but not necessarily on the boundary of its domain. The non-degeneracy assumption ensures that the measure $\mu(\cdot)$ does not put any mass on the boundary of the domain.

Recall Proposition 0.8(vi) of Resnick (1987), which in our current context can be easily restated as:

Lemma 2.1. *Let X_1 and X_2 be two random variables, both belonging to $\mathcal{R}_{-\alpha}$ for some $\alpha > 0$. Then, for $0 < c < \infty$, we have $\Pr(X_2 > t) \sim c \Pr(X_1 > t)$ if and only if $\mathrm{VaR}_q[X_2] \sim c^{1/\alpha}\mathrm{VaR}_q[X_1]$.*

With the help of Lemma 2.1 we can easily verify that Assumption 2.1 describes an asymptotic dependence case.

Lemma 2.2. *Under Assumption 2.1 the components of $\underline{X} = (X_1, \ldots, X_d)$ are pairwise asymptotically dependent.*

Proof. As an illustration we consider relation (1.2) for $(i, j) = (1, 2)$. By relations (2.1) and (2.2) and Lemma 2.1, there is some $c_2 > 0$ such that $\mathrm{VaR}_q[X_2] \sim c_2^{1/\alpha}\mathrm{VaR}_q[X_1]$. As $q \uparrow 1$, or, equivalently, as $t = \mathrm{VaR}_q[X_1] \to \infty$, we have

$$\begin{aligned}
\Pr(F_2(X_2) > q | F_1(X_1) > q) &= \frac{\Pr(F_1(X_1) > q, F_2(X_2) > q)}{\Pr(F_1(X_1) > q)} \\
&\geq \frac{\Pr(X_1 > \mathrm{VaR}_q[X_1], X_2 > \mathrm{VaR}_q[X_2])}{\Pr(X_1 \geq \mathrm{VaR}_q[X_1])} \\
&\gtrsim \frac{\Pr\left(X_1 > t, X_2 > 2c_2^{1/\alpha} t\right)}{\Pr(X_1 > t)} \\
&\to H_F\left(1, 2c_2^{1/\alpha}, 0, \ldots, 0\right) > 0.
\end{aligned}$$

This proves relation (1.2) for $(i, j) = (1, 2)$. □

We are now able to provide asymptotic expressions for the risk capital allocation for a multi-line insurance business. Noticeably, switching the context to a portfolio consisting of a_i units of X_i for all $1 \leq i \leq d$, similar results as given in the next theorem can be obtained by replacing the measure μ from (2.4) with

$$\begin{aligned}
\mu_{\underline{a}}((x_1, \infty] \times \cdots \times (x_d, \infty]) &= \lim_{t \to \infty} \frac{\Pr(a_1 X_1 > t x_1, \ldots, a_d X_d > t x_d)}{\Pr(a_1 X_1 > t)} \\
&= H_F(x_1/a_1, \ldots, x_d/a_d)\, a_1^{-\alpha}.
\end{aligned}$$

Theorem 2.1. *Let $\underline{X}$ be a random vector satisfying Assumption 2.1. If $X_1 \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$ then we have, for all $1 \leq k \leq d$,*

$$\mathbf{E}[X_k | S_d > \mathrm{VaR}_q[S_d]] \sim C_k \mathrm{VaR}_q[S_d],$$

where, with μ defined by (2.4),

$$C_k = \frac{\frac{1}{\alpha-1}\mu(\underline{x} : x_k > 1) + \int_0^1 \mu\left(\underline{x} : x_k > z, \sum_{i=1}^d x_i > 1\right) \mathrm{d}z}{\mu\left(\underline{x} : \sum_{i=1}^d x_i > 1\right)}.$$

Due to Assumption 2.1, it follows that

$$\Pr\left(S_d > t\right) \sim \mu\left(\underline{x} : \sum_{i=1}^{d} x_i > 1\right)\bar{F}_1(t); \tag{2.5}$$

for details, see Proposition 7.3 of Resnick (2007). Thus, $S_d \in \mathcal{R}_{-\alpha}$, which gives that $\sum_{i=1}^{d} C_k = \frac{\alpha}{\alpha-1}$; see Balkema and de Haan (1974), Alink *et al.* (2005), or Asimit and Badescu (2010).

Note 2.1. *After the majority of this work had been done we became aware of a forthcoming paper by Joe and Li (2011). Their Remark 2.3(2) suggests that our Theorem 2.1 is a consequence of their Theorem 2.2. However, this statement is not true because their proof is based on the vague convergence property (see relation 2.3), which can be used only for μ-continuous sets. The reasoning behind applying the vague convergence property is given in the proof of Theorem 2.1, for completeness. It is conveyed that the asymptotic independence case (see Theorem 3.1) requires an altered argumentation.*

Proof. We first note that

$$\begin{aligned}\mathbf{E}\left[X_k|S_d > t\right] &= \int_0^\infty \Pr\left(X_k > z|S_d > t\right)\mathrm{d}z \\ &= \int_0^t \frac{\Pr\left(X_k > z, S_d > t\right)}{\Pr\left(S_d > t\right)}\mathrm{d}z + \int_t^\infty \frac{\Pr(X_k > z)}{\Pr\left(S_d > t\right)}\mathrm{d}z \\ &= I_1(t) + I_2(t).\end{aligned} \tag{2.6}$$

For the first part of (2.6), $I_1(t)$, we have

$$I_1(t) = t\int_0^1 \frac{\Pr(X_k > tz, S_d > t)}{\Pr(S_d > t)}\mathrm{d}z \sim t\int_0^1 \frac{\mu(\underline{x} : x_k > z, \sum_{i=1}^d x_i > 1)}{\mu(\underline{x} : \sum_{i=1}^d x_i > 1)}\mathrm{d}z, \tag{2.7}$$

which is a consequence of relations (2.3) and (2.5), the Dominated Convergence Theorem and Proposition A2.12 of Embrechts *et al.* (1997). Note that the latter proposition can be applied since $\mu\left(\partial\{\underline{x} : x_k > z, \sum_{i=1}^d x_i > 1\}\right) = 0$ for all $z \geq 0$. In fact, Assumption 2.1 implies that

$$\mu\left\{x_i = z, (x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_d) \in A\right\} = 0$$

for all $z > 0$ and any relatively compact set A in $[0,\infty]^{d-1} \setminus \{\underline{0}\}$. This is still true for $z = 0$ under the additional condition that the set A is bounded away from $\{\underline{0}\}$. Moreover, the proof of Theorem 3.2 of Kortschak and Albrecher (2009) justifies the fact that no mass is put by the measure μ over the line $\sum_{i=1}^d x_i = 1$ and any neighborhood around ∞.

For the second part of relation (2.6), $I_2(t)$, we have

$$I_2(t) = t\frac{\Pr(X_k > t)}{\Pr(X_1 > t)}\frac{\Pr(X_1 > t)}{\Pr(S_d > t)}\int_1^\infty \frac{\Pr(X_k > tz)}{\Pr(X_k > t)}\mathrm{d}z \sim \frac{t}{\alpha - 1}\frac{\mu(\underline{x} : x_k > 1)}{\mu\left(\underline{x} : \sum_{i=1}^d x_i > 1\right)}, \quad (2.8)$$

where in the last step above we have applied the relations (2.5) and

$$\Pr(X_k > t) \sim \mu(\underline{x} : x_k > 1)\Pr(X_1 > t)$$

due to (2.3), and made use of the Dominated Convergence Theorem as justified by Proposition 0.8 of Resnick (1987).

Plugging (2.7) and (2.8) into (2.6) yields $\mathbf{E}[X_k|S_d > t] \sim C_k t$. The proof is complete.

□

As a consequence of Theorem 2.1, we now express the capital allocations in terms of the VaR of the reference risk. The proof simply uses Lemma 2.1 and relation (2.5) and for this reason is omitted.

Corollary 2.1. *Under the conditions of Theorem 2.1, it holds for all* $1 \leq k \leq d$ *that*

$$\mathbf{E}[X_k|S_d > \mathrm{VaR}_q[S_d]] \sim C_k \mu^{1/\alpha}\left(\underline{x} : \sum_{i=1}^d x_i > 1\right)\mathrm{VaR}_q[X_1],$$

where the constants C_k *are defined as in Theorem 2.1.*

2.2. **Gumbel case.** The Gumbel case is further investigated in the presence of asymptotic dependence. Since we are only interested in risks with unbounded supports, all individual risks are assumed to have an infinite upper endpoint. It is well known (see Embrechts *et al.*, 1997) that if $F \in \mathrm{MDA}(\Lambda)$, then there exists a positive, measurable function $a(\cdot)$ such that

$$\lim_{t\to\infty}\frac{\bar{F}(t + a(t)s)}{\bar{F}(t)} = \mathrm{e}^{-s} \quad (2.9)$$

for any $s \in \mathbf{R}$. In addition, the latter holds locally uniformly in s (see Resnick, 1987). Recall that the auxiliary function $a(\cdot)$ satisfies $a(t) = o(t)$ and is such that the relation

$$\lim_{t\to\infty}\frac{a(t + a(t)x)}{a(t)} = 1 \quad (2.10)$$

holds locally uniformly in x. Moreover, this auxiliary function can be chosen as the mean excess function of F, i.e.,

$$a(t) = \int_t^\infty \frac{\bar{F}(s)}{\bar{F}(t)}\mathrm{d}s.$$

See Section 3.3 of Embrechts *et al.* (1997) for more details.

The next assumption is sufficient for our main results of this subsection.

Assumption 2.2. *Let* $\underline{X} = (X_1, \ldots, X_d)$ *be a non-negative random vector with marginal distributions* $F_1, \ldots, F_d$ *such that*

$$\lim_{t\to\infty} \frac{\Pr(X_1 > t + a(t)x_1, \ldots, X_d > t + a(t)x_d)}{\bar{F}_1(t)} := H_G(\underline{x})$$

exists for all $\underline{x} \in \mathbf{R}^d$, *where* $H_G(\cdot)$ *is assumed to be a non-degenerate function.*

Assumption 2.2 implies that

$$\frac{\Pr\left(\left(\frac{X_1-t}{a(t)}, \ldots, \frac{X_d-t}{a(t)}\right) \in \cdot\right)}{\bar{F}_1(t)} \xrightarrow{v} \nu(\cdot) \tag{2.11}$$

holds on $[-\infty, \infty]^d \setminus \{-\underline{\infty}\}$ for the measure ν given by

$$\nu\left((x_1, \infty] \times \cdots \times (x_d, \infty]\right) := H_G(\underline{x}),$$

where $\{-\underline{\infty}\} = (-\infty, \ldots, -\infty)$. Moreover, the distribution functions are tail equivalent with

$$\lim_{t\to\infty} \frac{\bar{F}_i(t)}{\bar{F}_1(t)} = \nu(\underline{x} : x_i > 0), \qquad 1 \le i \le d, \tag{2.12}$$

and, therefore, all belong to MDA(Λ). One choice for the function $a(\cdot)$ from Assumption 2.2 is given by the auxiliary function corresponding to the random variable X_1 as described in relation (2.9). In fact, all marginal distributions admit the same auxiliary function.

We now show that the non-degeneracy of H_G in Assumption 2.2 ensures that the random vector $\underline{X}$ has the asymptotic dependence property, which is a key property in describing the tail probability of S_d (see Section 4.1 of Klüppelberg and Resnick, 2008).

Lemma 2.3. *Under Assumption 2.2 the components of* $\underline{X} = (X_1, \ldots, X_d)$ *are pairwise asymptotically dependent.*

Proof. As in the proof of Lemma 2.2, we consider relation (1.2) for $(i, j) = (1, 2)$. By (2.12) and (2.9) with F replaced by F_1, we have

$$\lim_{t\to\infty} \frac{\bar{F}_2(t + a(t)s)}{\bar{F}_1(t)} = \nu(\underline{x} : x_2 > 0)\mathrm{e}^{-s}.$$

Fix some s large enough such that the right-hand side above is smaller than 1. Then it holds for all large t that $\bar{F}_2(t + a(t)s) < \bar{F}_1(t)$, from which it follows that

$$\mathrm{VaR}_{F_1(t)}[X_2] \le t + a(t)s.$$

Analogously to the proof of Lemma 2.2, as $q \uparrow 1$, or, equivalently, as $t = \mathrm{VaR}_q[X_1] \to \infty$,

$$\begin{aligned}
\Pr\left(F_2(X_2) > q | F_1(X_1) > q\right) &\geq \frac{\Pr\left(X_1 > \mathrm{VaR}_q[X_1], X_2 > \mathrm{VaR}_q[X_2]\right)}{\Pr\left(X_1 \geq \mathrm{VaR}_q[X_1]\right)} \\
&\geq \frac{\Pr\left(X_1 > t, X_2 > \mathrm{VaR}_{F_1(t)}[X_2]\right)}{\Pr\left(X_1 \geq t\right)} \\
&\geq \frac{\Pr\left(X_1 > t, X_2 > t + a(t)s\right)}{\Pr\left(X_1 \geq t\right)} \\
&\to \nu(\underline{x} : x_1 > 0, x_2 > s) > 0,
\end{aligned}$$

where in the last step we used the fact $\Pr\left(X_1 \geq t\right) \sim \Pr\left(X_1 > t\right)$ due to Corollary 1.6 of Resnick (1987). This proves relation (1.2) for $(i, j) = (1, 2)$. □

Note 2.2. *Due to Assumption 2.2, it holds that*

$$\Pr\left(S_d > dt\right) \sim \nu\left(\underline{x} : \sum_{i=1}^{d} x_i > 0\right) \bar{F}_1(t), \tag{2.13}$$

which is a consequence of the vague convergence in (2.11) and Proposition 4.1 of Klüppelberg and Resnick (2008). Note that Assumption 2.2 allows us to withdraw the equal marginal distributions assumption from their proposition. Thus,

$$\lim_{t\to\infty} \frac{\Pr\left(S_d > t + a\left(t/d\right) ds\right)}{\Pr\left(S_d > t\right)} = \mathrm{e}^{-s} \tag{2.14}$$

holds for any $s \in \mathbf{R}$ and relation (2.9) is satisfied by S_d with an auxiliary function $\tilde{a}(t) := a(t/d)d$. The latter implies that $S_d \in \mathrm{MDA}(\Lambda)$, which gives that

$$\mathbf{E}\left[S_d | S_d > t\right] \sim t. \tag{2.15}$$

Relation (2.15) agrees with the next theorem, which provides the asymptotic expressions for the capital allocations for a multi-line insurance business of d asymptotically dependent risks that belong to $\mathrm{MDA}(\Lambda)$.

Theorem 2.2. *Let $\underline{X}$ be a random vector satisfying Assumption 2.2. Then it holds for all $1 \leq k \leq d$ that*

$$\mathbf{E}\left[X_k | S_d > \mathrm{VaR}_q[S_d]\right] \sim \frac{1}{d}\mathrm{VaR}_q[S_d].$$

Proof. The first step of the proof is developed as

$$\mathbf{E}\left[X_k | S_d > dt\right] = \left(\int_0^t + \int_t^\infty\right) \Pr\left(X_k > z | S_d > dt\right) \mathrm{d}z = I_1(t) + I_2(t). \tag{2.16}$$

The second integral can be easily reduced to

$$I_2(t) \leq \int_t^\infty \frac{\Pr(X_k > z)}{\Pr\left(S_d > dt\right)} \mathrm{d}z. \tag{2.17}$$

The change of variables $z = t + a(t)v$ leads to

$$\int_t^\infty \frac{\Pr(X_k > z)}{\Pr(X_k > t)} \mathrm{d}z = a(t) \int_0^\infty \frac{\Pr\left(X_k > t + a(t)v\right)}{\Pr(X_k > t)} \mathrm{d}v = o(t)$$

due to the Dominated Convergence Theorem, relations (2.9) and $a(t) = o(t)$. Thus, the latter relation, together with (2.13) and (2.17), concludes that

$$I_2(t) = o(t). \tag{2.18}$$

It only remains to investigate the first term of (2.16), which is equal to

$$\begin{aligned} I_1(t) &= t - \left(\int_0^{t+(d-1)a(t)s} + \int_{t+(d-1)a(t)s}^t \right) \Pr\left(X_k \le z | S_d > dt\right) \mathrm{d}z \\ &= t - I_{11}(t,s) - I_{12}(t,s), \end{aligned} \tag{2.19}$$

where s is a negative number.

Recall that $a(\cdot)$ is a positive function. It holds for every $s < 0$ that

$$\begin{aligned} \frac{I_{11}(t,s)}{t} &\le \Pr\left(X_k \le t + (d-1)a(t)s | S_d > dt\right) \\ &\le \frac{\Pr\left(\sum_{i=1,i\neq k}^d X_i > (d-1)t - (d-1)a(t)s\right)}{\Pr\left(S_d > dt\right)} \\ &\sim \frac{\nu\left(\underline{x} : \sum_{i=1,i\neq k}^d x_i > 0\right)}{\nu\left(\underline{x} : \sum_{i=1}^d x_i > 0\right)} \mathrm{e}^s, \end{aligned} \tag{2.20}$$

as a result of (2.13) and (2.14). Thus,

$$\lim_{s\downarrow -\infty} \limsup_{t\to\infty} \frac{I_{11}(t,s)}{t} = 0.$$

As before, the change of variables $z = t + (d-1)a(t)v$ yields

$$\begin{aligned} I_{12}(t,s) &= (d-1)a(t) \int_s^0 \Pr\left(X_k \le t + (d-1)a(t)v | S_d > dt\right) \mathrm{d}v \\ &\le (d-1)a(t) \frac{\int_s^0 \Pr\left(\sum_{i=1,i\neq k}^d X_i > (d-1)t - (d-1)a(t)v\right) \mathrm{d}v}{\Pr\left(S_d > dt\right)} \\ &\le (d-1)a(t)\,|s|\, \frac{\Pr\left(\sum_{i=1,i\neq k}^d X_i > (d-1)t\right)}{\Pr\left(S_d > dt\right)} \\ &\sim (d-1)a(t)\,|s|\, \frac{\nu\left(\underline{x} : \sum_{i=1,i\neq k}^d x_i > 0\right)}{\nu\left(\underline{x} : \sum_{i=1}^d x_i > 0\right)} = o(t), \end{aligned} \tag{2.21}$$

due to $a(t) = o(t)$ and relations (2.13) and (2.14).

A combination of relations (2.19), (2.20) and (2.21) yields that $I_1(t) \sim t$. Plugging this and (2.18) into (2.16) concludes the proof. □

Recall that every distribution F from MDA(Λ) with an infinite upper endpoint also has a rapidly varying tail (see Embrechts *et al.*, 1997, page 148), written as $F \in \mathcal{R}_{-\infty}$, characterized by

$$\lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = \begin{cases} 0, & x > 1, \\ \infty, & 0 < x < 1. \end{cases}$$

The following result is similar to Lemma 2.1 but for the rapid variation case:

Lemma 2.4. *Let X_1 and X_2 be two random variables with distribution functions F_1 and F_2 in $\mathcal{R}_{-\infty}$. If $\bar{F}_1(lt) \asymp \bar{F}_2(t)$ for some $l > 0$ then, for every $m > 0$, as $q = 1 - p \uparrow 1$,*

$$\mathrm{VaR}_q[X_1] \sim l\mathrm{VaR}_{1-mp}[X_2]. \tag{2.22}$$

Proof. Let $0 < \varepsilon < 1$ be arbitrarily fixed. As $p \downarrow 0$, there is some function $b(p) \to 0$ such that

$$\begin{aligned} \bar{F}_1\left((1+\varepsilon)l\mathrm{VaR}_{1-mp}[X_2]\right) &= b(p)\bar{F}_1\left((1+\varepsilon/2)l\mathrm{VaR}_{1-mp}[X_2]\right) \\ &\asymp b(p)\bar{F}_2\left((1+\varepsilon/2)\mathrm{VaR}_{1-mp}[X_2]\right) \leq b(p)mp. \end{aligned}$$

Similarly, as $p \downarrow 0$, there is some function $c(p) \to \infty$ such that

$$\begin{aligned} \bar{F}_1\left((1-\varepsilon)l\mathrm{VaR}_{1-mp}[X_2]\right) &= c(p)\bar{F}_1\left((1-\varepsilon/2)l\mathrm{VaR}_{1-mp}[X_2]\right) \\ &\asymp c(p)\bar{F}_2\left((1-\varepsilon/2)\mathrm{VaR}_{1-mp}[X_2]\right) \geq c(p)mp. \end{aligned}$$

Thus, for all $0 < p < 1$ sufficiently close to 0, we have

$$\bar{F}_1\left((1+\varepsilon)l\mathrm{VaR}_{1-mp}[X_2]\right) < p < \bar{F}_1\left((1-\varepsilon)l\mathrm{VaR}_{1-mp}[X_2]\right).$$

It follows that

$$(1-\varepsilon)l\mathrm{VaR}_{1-mp}[X_2] \leq \mathrm{VaR}_q[X_1] \leq (1+\varepsilon)l\mathrm{VaR}_{1-mp}[X_2].$$

By the arbitrariness of ε, this leads to relation (2.22). □

Similar to the Fréchet case, Theorem 2.2 allows us to express the capital allocations in terms of the reference risk measure, $\mathrm{VaR}_q[X_1]$. Finally, relation (2.13), Lemma 2.4 and the fact that $S_d \in \mathcal{R}_{-\infty}$ imply the following result:

Corollary 2.2. *Under the conditions of Theorem 2.2, as $q = 1 - p \uparrow 1$ it holds that*

$$\mathbf{E}\left[X_k | S_d > \mathrm{VaR}_q[S_d]\right] \sim \mathrm{VaR}_{1-p/K}[X_1], \tag{2.23}$$

where $K = \nu\left(\underline{x} : \sum_{i=1}^{d} x_i > 0\right)$.

Lemma 2.4 shows that $\mathrm{VaR}_q[X_1]$ as a function of p is slowly varying as $p \downarrow 0$ (see also Proposition 0.8(v) of Resnick, 1987). Therefore, the right-hand side of (2.23) can be changed to $\mathrm{VaR}_{1-mp}[X_1]$ for every constant $m > 0$. However, in view of relation (2.13), the most rational choice for m should be $m = 1/K$.

We have thus obtained an appealing expression for the CTE-based risk capital allocations for a multi-line insurance business consisting of asymptotically dependent risks that are not extremely heavy tailed. The fact that all risks belong to MDA(Λ) allows us to conclude that the marginal risk capitals under CTE are equal for conservative scenarios. It is interesting that this remains true without assuming exchangeable risks, a situation in which the allocations are obviously equal at any degree of safeness.

3. Main results under asymptotic independence

There has been a particular interest in understanding the tail behavior of the sum of asymptotically independent random variables with heavy tails (see Albrecher *et al.*, 2006, Ko and Tang, 2008, Asmussen and Rojas-Nandayapa, 2008, Geluk and Tang, 2009, and Mitra and Resnick, 2009). A similar pattern for the CTE-capital allocations is expected. The results are less homogeneous under asymptotic independence, and various assumptions are provided in this case. The Fréchet and Gumbel cases exhibit fundamentally different behaviors for the capital allocations and, therefore, analyses are made separately.

3.1. Fréchet case.

We first restrict our attention to a portfolio of risks with regularly varying tails. Two overlapping sets of assumptions are made, but neither of them is a consequence of the other. The first assumption is a natural extension of Assumption 2.1 under the asymptotic independence setting.

Assumption 3.1. *Let $\underline{X}$ be a non-negative random vector with marginal distributions $F_1, \dots, F_d$ such that*

$$\lim_{t\to\infty} \frac{\Pr(X_1 > tx_1, \dots, X_d > tx_d)}{\bar{F}_1(t)} := \mu_I\left((x_1, \infty] \times \cdots \times (x_d, \infty]\right)$$

exists for all $\underline{x} \in [0,\infty]^d \setminus \{\underline{0}\}$, where μ_I is a Radon measure such that $\mu_I((0,\infty]^d) = 0$.

Since the limiting measure is Radon, Assumption 3.1 suggests that all marginal distributions are regularly varying tailed with possibly different indexes, α_i, such that $\alpha_1 = \wedge_{1\le i\le n}\alpha_i$. If these indexes are not equal, then the measure μ_I does not put mass on the vast majority of the boundary of its domain, $[0,\infty]^d \setminus \{\underline{0}\}$. In such a case, the problem becomes trivial and only components with indexes equal to α_1 have contributions to extreme events related to the aggregate portfolio. Therefore, without loss of generality it is further assumed that all marginal distributions belong to MDA(Φ_α), since otherwise the main results remain unchanged. Then, for each $1 \le i \le d$, there exists a positive constant c_i such that

$$\bar{F}_i(t) \sim c_i \bar{F}_1(t) \tag{3.1}$$

and

$$\frac{\Pr\left((X_1/t,\ldots,X_d/t) \in \cdot\right)}{\Pr(X_1 > t)} \xrightarrow{v} \mu_I(\cdot) \tag{3.2}$$

holds on $[0,\infty]^d \setminus \{\underline{0}\}$. Furthermore, the measure μ_I puts mass only on the coordinate axes due to the fact that $\mu_I((0,\infty]^d) = 0$. Thus, for all $z > 0$,

$$\mu_I\left(\underline{x} : x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_d = 0,\ x_i \in (z,\infty]\right) = c_i z^{-\alpha}.$$

It is interesting to outline the link between Assumption 3.1 and asymptotic independence. Under Assumption 3.1, following the proof of Lemma 2.2 we can easily verify that the components of $\underline{X}$ are pairwise asymptotically independent. As for the inverse statement, for simplicity consider $\underline{X} = (X_1, X_2)$. Assume that X_1 and X_2 belong to MDA(Φ_α), have strongly equivalent tails as in (3.1) and are asymptotically independent. For this case, the asymptotic independence is equivalent to

$$\lim_{t\to\infty} \Pr\left(X_2 > t | X_1 > t\right) = 0.$$

Then for all positive x_1 and x_2 we have

$$\frac{\Pr(X_1 > tx_1, X_2 > tx_2)}{\Pr(X_1 > t)} \le \frac{\Pr(X_1 \wedge X_2 > tx_1 \wedge tx_2)}{\Pr(X_1 > t)} = o(1).$$

Thus,

$$\lim_{t\to\infty} \frac{\Pr(X_1 > tx_1, X_2 > tx_2)}{\Pr(X_1 > t)} = \begin{cases} x_1^{-\alpha}, & x_1 > 0, x_2 = 0, \\ c_2 x_2^{-\alpha}, & x_1 = 0, x_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Assumption 3.1 describes a situation in which

$$\Pr\left(S_d > t\right) \sim \bar{F}_1(t) \sum_{i=1}^{d} c_i; \tag{3.3}$$

see Lemma 2.1 of Davis and Resnick (1996) or Proposition 7.3 of Resnick (2007).

The first main result of this subsection is now given for a multi-line insurance business consisting of asymptotically independent risks with regularly varying tails.

Theorem 3.1. *Let* $\underline{X}$ *be a random vector satisfying Assumption 3.1. If* $X_1 \in \mathcal{R}_{-\alpha}$ *with* $\alpha > 1$*, then for all* $1 \le k \le d$ *we have*

$$\mathbf{E}\left[X_k | S_d > \mathrm{VaR}_q[S_d]\right] \sim \frac{\alpha}{\alpha - 1} \frac{c_k}{\sum_{i=1}^d c_i} \mathrm{VaR}_q[S_d],$$

where the constants c_i *are given by (3.1).*

Proof. Similar to the proof of Theorem 2.1,

$$\begin{aligned} \mathbf{E}\left[X_k | S_d > t\right] &= \int_0^t \frac{\Pr\left(X_k > z, S_d > t\right)}{\Pr\left(S_d > t\right)} \mathrm{d}z + \int_t^\infty \frac{\Pr(X_k > z)}{\Pr\left(S_d > t\right)} \mathrm{d}z \\ &= I_1(t) + I_2(t). \end{aligned} \tag{3.4}$$

Now, for any $0 < z \le 1$ we have

$$\begin{aligned} \frac{\Pr(X_k > tz, S_d > t)}{\Pr(X_1 > t)} &= \frac{\Pr(X_k > t)}{\Pr(X_1 > t)} + \frac{\Pr(tz < X_k \le t, S_d > t)}{\Pr(X_1 > t)} \\ &\sim c_k + \mu_I \left(\underline{x} : z < x_k \le 1, \sum_{i=1}^d x_i > 1 \right) \\ &= c_k, \end{aligned}$$

where Proposition A2.12 of Embrechts *et al.* (1997) and the vague convergence property in (3.2) are applied over the μ_I-negligible set $\left\{ \underline{x} : z < x_k \le 1, \sum_{i=1}^d x_i > 1 \right\}$. The latter, relation (3.3) and the Dominated Convergence Theorem yield that

$$I_1(t) = t \int_0^1 \frac{\Pr(X_k > tz, S_d > t)}{\Pr(S_d > t)} \mathrm{d}z \sim \frac{c_k}{\sum_{i=1}^d c_i} t. \tag{3.5}$$

As in the proof of Theorem 2.1, the second term in (3.4) satisfies

$$I_2(t) \sim \frac{1}{\alpha - 1} \frac{c_k}{\sum_{i=1}^d c_i} t. \tag{3.6}$$

Putting together (3.4), (3.5) and (3.6), we obtain

$$\mathbf{E}\left[X_k | S_d > t\right] \sim \frac{\alpha}{\alpha - 1} \frac{c_k}{\sum_{i=1}^d c_i} t,$$

which completes the proof. □

An even simpler expression for the CTE-capital allocations is given without proof in the next corollary, which is a consequence of Theorem 3.1, Lemma 2.1 and relation (3.3).

Corollary 3.1. *Under the conditions of Theorem 3.1, it holds for all* $1 \le k \le d$ *that*

$$\mathbf{E}\left[X_k|S_d > \mathrm{VaR}_q[S_d]\right] \sim \frac{\alpha}{\alpha-1} c_k \left(\sum_{i=1}^{d} c_i\right)^{\frac{1}{\alpha}-1} \mathrm{VaR}_q[X_1],$$

where the constants c_i *are given by (3.1).*

Our next assumption is motivated by the work of Asimit and Badescu (2010).

Assumption 3.2. *Let* $\underline{X}$ *be a nonnegative random vector with marginal distribution functions* $F_1, \ldots, F_d$*. Assume that there are some measurable and bounded functions* $h_i(\cdot) : (0, \infty) \to (0, \infty)$ *such that, for distinct* $i, j \in \{1, \ldots, d\}$*, the relation*

$$\Pr\left(X_j > t | X_i = x\right) \sim \bar{F}_j(t) h_i(x) \tag{3.7}$$

holds uniformly for $0 \le x < \infty$*. In addition, for* $d \ge 3$*, it is also assumed that*

$$\Pr\left(X_j > t, X_k > t | X_i = x\right) = o\left(\bar{F}_j(t) + \bar{F}_k(t)\right) h_i(x) \tag{3.8}$$

holds uniformly for $0 \le x < \infty$ *for distinct* $i, j, k \in \{1, \ldots, d\}$*.*

The uniformity of relation (3.7) is understood as

$$\lim_{t\to\infty} \sup_{0 \le x < \infty} \left| \frac{\Pr\left(X_j > t | X_i = x\right)}{\bar{F}_j(t) h_i(x)} - 1 \right| = 0$$

while the uniformity of relation (3.8) as

$$\lim_{t\to\infty} \sup_{0 \le x < \infty} \frac{\Pr\left(X_j > t, X_k > t | X_i = x\right)}{\left(\bar{F}_j(t) + \bar{F}_k(t)\right) h_i(x)} = 0.$$

The recent work of Li *et al.* (2010) discussed the verification of the uniformity of relation (3.7) and the boundedness of the functions $h_i(\cdot)$. Clearly, the uniformity property implies that $\mathbf{E}\left[h_i(X)\right] = 1$.

For the dependence structures discussed by Asimit and Badescu (2010) and Li *et al.* (2010) with strongly equivalent tails, both Assumptions 3.1 and 3.2 are satisfied. It is not difficult to prove that a bivariate random vector with strongly equivalent tails and dependence structure given by the Marshall-Olkin copula (see, e.g., Nelsen, 1999, page 46) satisfies only Assumption 3.1. It will be later seen in Note 3.1 that within a particular (yet fairly general) setting, Assumption 3.2 may provide a refinement for the capital allocations given by Theorem 3.1. Therefore, neither of Assumptions 3.1 and 3.2 is dominated by the other one.

We first establish that random vectors satisfying Assumption 3.2 have pairwise asymptotically independent components.

Lemma 3.1. *Relation (3.7) implies that X_i and X_j are asymptotically independent.*

Proof. By definition,

$$\begin{aligned}
\Pr\left(F_j(X_j) > q | F_i(X_i) > q\right) &\leq \frac{\Pr\left(F_i(X_i) > q, F_j(X_j) \geq q\right)}{\Pr\left(F_i(X_i) > q\right)} \\
&= \frac{\int_{\mathrm{VaR}_q[X_j]^-}^{\infty} \Pr\left(F_i(X_i) > q | X_j = x\right) \Pr\left(X_j \in \mathrm{d}x\right)}{\Pr\left(F_i(X_i) > q\right)}.
\end{aligned}$$

Hence, by the uniformity of (3.7), the right-hand side above converges to

$$\lim_{q \uparrow 1} \int_{\mathrm{VaR}_q[X_j]^-}^{\infty} h_j(x) \Pr\left(X_j \in \mathrm{d}x\right) = 0.$$

Therefore, X_i and X_j are asymptotically independent. □

The next result is crucial for developing our second main result of this subsection.

Lemma 3.2. *Let $\underline{X}$ be a random vector satisfying Assumption 3.2 such that each component has a regularly varying tail. Then the relation*

$$\Pr\left(\sum_{i=1, i \neq k}^{d} X_i > t \,\middle|\, X_k = x \right) \sim h_k(x) \sum_{i=1, i \neq k}^{d} \bar{F}_i(t)$$

holds uniformly for $0 \leq x < \infty$ for all $1 \leq k \leq d$.

Proof. The proof below proceeds for $d \geq 3$ and $k = 1$, but it can easily be adjusted so that it is valid for $d \geq 2$ and $k \neq 1$. We first derive an asymptotic lower bound. Since each X_i is nonnegative, by Bonferroni's inequality we have, uniformly for $0 \leq x < \infty$,

$$\begin{aligned}
& \Pr\left(\sum_{i=2}^{d} X_i > t \,\middle|\, X_1 = x \right) \\
\geq \; & \Pr\left(\bigvee_{i=2}^{d} X_i > t \,\middle|\, X_1 = x \right) \\
\geq \; & \sum_{i=2}^{d} \Pr\left(X_i > t | X_1 = x\right) - \sum_{2 \leq i \neq j \leq d} \Pr\left(X_i > t, X_j > t | X_1 = x\right) \\
\sim \; & h_1(x) \sum_{i=2}^{d} \bar{F}_i(t),
\end{aligned}$$

where in the last step we applied both (3.7) and (3.8). Next, we derive the corresponding asymptotic upper bound. For an arbitrarily fixed constant $0<\varepsilon<1$, we have

$$\begin{aligned}
&\Pr\left(\sum_{i=2}^{d} X_i > t \,\middle|\, X_1 = x\right) \\
\leq\;& \Pr\left(\bigvee_{i=2}^{d} X_i > (1-\varepsilon)t \,\middle|\, X_1 = x\right) + \Pr\left(\sum_{i=2}^{d} X_i > t, \bigvee_{i=2}^{d} X_i \leq (1-\varepsilon)t \,\middle|\, X_1 = x\right) \\
=\;& I_1(t,\varepsilon) + I_2(t,\varepsilon). \qquad (3.9)
\end{aligned}$$

Let $\alpha_i > 0$ be the regularly varying index of $\bar{F}_i$. Then relation (3.7) gives that

$$I_1(t,\varepsilon) \leq \sum_{i=2}^{d} \Pr\left(X_i > (1-\varepsilon)t | X_1 = x\right) \sim h_1(x) \sum_{i=2}^{d} (1-\varepsilon)^{-\alpha_i} \bar{F}_i(t). \qquad (3.10)$$

Similarly, by (3.8),

$$\begin{aligned}
I_2(t,\varepsilon) &= \Pr\left(\sum_{i=2}^{d} X_i > t, \bigvee_{j=2}^{d} X_j > \frac{t}{d-1}, \bigvee_{i=2}^{d} X_i \leq (1-\varepsilon)t \,\middle|\, X_1 = x\right) \\
&\leq \sum_{j=2}^{d} \Pr\left(\sum_{i=2}^{d} X_i > t, X_j > \frac{t}{d-1}, X_j \leq (1-\varepsilon)t \,\middle|\, X_1 = x\right) \\
&\leq \sum_{j=2}^{d} \Pr\left(\sum_{i=2, i\neq j}^{d} X_i > \varepsilon t, X_j > \frac{t}{d-1} \,\middle|\, X_1 = x\right) \\
&\leq \sum_{2\leq i\neq j\leq d} \Pr\left(X_i > \frac{\varepsilon t}{d-2}, X_j > \frac{t}{d-1} \,\middle|\, X_1 = x\right) \\
&= o\left(h_1(x) \sum_{i=2}^{d} \bar{F}_i(t)\right). \qquad (3.11)
\end{aligned}$$

Substituting (3.10) and (3.11) into (3.9) and noticing the arbitrariness of ε, we obtain

$$\Pr\left(\sum_{i=2}^{d} X_i > t \,\middle|\, X_1 = x\right) \lesssim h_1(x) \sum_{i=2}^{d} \bar{F}_i(t).$$

The proof is complete. □

Now we are ready to state the second main result of this subsection.

Theorem 3.2. *Let $\underline{X}$ be a random vector satisfying Assumption 3.2 such that each component X_i has a regularly varying tail with index $\alpha_i > 1$. In addition, if each $h_i(\cdot)$ is a regularly varying function, then*

$$\Pr\left(S_d > t\right) \sim \sum_{i=1}^{d} \bar{F}_i(t) \qquad (3.12)$$

and

$$\mathbf{E}\left[X_k|S_d>t\right]\sim\frac{\mathbf{E}\left[X_kh_k(X_k)\right]\sum_{i=1,i\neq k}^{d}\bar{F}_i(t)+t\bar{F}_k(t)\frac{\alpha_k}{\alpha_k-1}}{\sum_{i=1}^{d}\bar{F}_i(t)}. \tag{3.13}$$

All examples presented in the work of Li *et al.* (2010) give the regular variation property of the functions $h_i(\cdot)$. If relation (3.1) holds for each $1\le i\le d$, then one may use Theorem 3.2 to recover Corollary 3.1.

Note 3.1. *Let us assume that there exist finite constants c_i, $1\le i\le d$, such that $\bar{F}_i(t)\sim c_i\bar{F}_1(t)$ for all $1\le i\le d-1$ and $\bar{F}_d(t)\sim t\bar{F}_1(t)c_d$. Thus, $\alpha_i=\alpha_1$ for all $1\le i\le d-1$ and $\alpha_d=\alpha_1-1$. Suppose $\alpha_1>2$ so that X_d has a finite mean as well. In addition, $\Pr\left(S_d>t\right)\sim\bar{F}_d(t)$, which together with Lemma 2.1, implies that $\mathrm{VaR}_q[S_d]\sim\mathrm{VaR}_q[X_d]$. Some algebraic manipulations lead to*

$$\lim_{t\to\infty}\mathbf{E}\left[X_k|S_d>t\right]=\mathbf{E}\left[X_kh_k(X_k)\right]+\frac{\alpha_1}{\alpha_1-1}\frac{c_k}{c_d}$$

for all $1\le k\le d-1$ and

$$\mathbf{E}\left[X_d|S_d>t\right]\sim t\frac{\alpha_d}{\alpha_d-1}.$$

Obviously, these relations generate the asymptotic capital allocation estimates

$$\lim_{t\to\infty}\mathbf{E}\left[X_k|S_d>\mathrm{VaR}_q[S_d]\right]=\mathbf{E}\left[X_kh_k(X_k)\right]+\frac{\alpha_1}{\alpha_1-1}\frac{c_k}{c_d}$$

for all $1\le k\le d-1$ and

$$\mathbf{E}\left[X_d|S_d>\mathrm{VaR}_q[S_d]\right]\sim\frac{\alpha_d}{\alpha_d-1}\mathrm{VaR}_q\left(X_d\right). \tag{3.14}$$

On the contrary, in this particular setting, Theorem 3.1 gives that

$$\mathbf{E}\left[X_k|S_d>\mathrm{VaR}_q[S_d]\right]=o\left(\mathrm{VaR}_q[X_d]\right),\qquad 1\le k\le d-1,$$

while the d^{th} capital allocation is given as in relation (3.14). The advantage of using Theorem 3.2 over Theorem 3.1 becomes transparent.

Proof. The proof is provided only for $k=1$ as the extensions to all other values of k are obvious. Similar to the proof of Theorem 2.1,

$$\begin{aligned}\mathbf{E}\left[X_1|S_d>t\right] &= \int_{0^-}^{\infty}x\Pr\left(X_1\in \mathrm{d}x|S_d>t\right)\\ &= \frac{\int_{0^-}^{\infty}x\Pr\left(S_d-X_1>t-x|X_1=x\right)F_1(\mathrm{d}x)}{\int_{0^-}^{\infty}\Pr\left(S_d-X_1>t-x|X_1=x\right)F_1(\mathrm{d}x)}.\end{aligned} \tag{3.15}$$

We first investigate the numerator from (3.15). By Lemma 3.2, for every $\varepsilon > 0$, there is some $t_0 > 0$ such that, for all $t \geq t_0$ and $x > 0$,

$$(1-\varepsilon)h_1(x)\sum_{i=2}^{d}\bar{F}_i(t) \leq \Pr\left(S_d - X_1 > t | X_1 = x\right) \leq (1+\varepsilon)h_1(x)\sum_{i=2}^{d}\bar{F}_i(t). \tag{3.16}$$

According to this constant t_0, for all $t \geq t_0$ we split the integral into three parts as

$$\sum_{i=1}^{3} I_i(t) = \left(\int_{0-}^{t-t_0} + \int_{t-t_0}^{t} + \int_{t}^{\infty}\right) x \Pr\left(S_d - X_1 > t - x | X_1 = x\right) F_1(\mathrm{d}x). \tag{3.17}$$

Clearly,

$$I_3(t) = \int_t^{\infty} x F_1(\mathrm{d}x) \tag{3.18}$$

and

$$I_2(t) \leq \int_{t-t_0}^{t} x F_1(\mathrm{d}x) = o\left(I_3(t)\right). \tag{3.19}$$

By (3.16),

$$1-\varepsilon \leq \frac{I_1(x)}{\sum_{i=2}^{d}\int_{0-}^{t-t_0} x h_1(x)\bar{F}_i(t-x)F_1(\mathrm{d}x)} \leq 1+\varepsilon. \tag{3.20}$$

Define a distribution F_* by

$$F_*(\mathrm{d}x) = \frac{x h_1(x)}{\mathbf{E}\left[X_1 h_1(X_1)\right]} F_1(\mathrm{d}x),$$

where $\mathbf{E}\left[X_1 h_1(X_1)\right]$ is a finite positive normalizing constant. Clearly, F_* has a regularly varying tail as well. Thus, by Lemma 1.3.1 of Embrechts *et al.* (1997) (see also Proposition 1.2 or Theorem 2.1 of Cai and Tang, 2004),

$$\begin{aligned}
&\int_{0-}^{t-t_0} x h_1(x)\bar{F}_i(t-x)F_1(\mathrm{d}x) \\
&= \mathbf{E}\left[X_1 h_1(X_1)\right]\left(\int_{0-}^{\infty} - \int_{t}^{\infty} - \int_{t-t_0}^{t}\right)\bar{F}_i(t-x)F_*(\mathrm{d}x) \\
&= (1+o(1))\,\mathbf{E}\left[X_1 h_1(X_1)\right]\left(\bar{F}_i(t)+\bar{F}_*(t)\right) - \mathbf{E}\left[X_1 h_1(X_1)\right]\bar{F}_*(t) - o(1)\bar{F}_*(t) \\
&= (1+o(1))\,\mathbf{E}\left[X_1 h_1(X_1)\right]\bar{F}_i(t) - o(1)\bar{F}_*(t).
\end{aligned}$$

Substituting this into (3.20), then substituting (3.18), (3.19) and (3.20) into (3.17) and noticing the arbitrariness of ε, we obtain

$$\begin{aligned}
\sum_{i=1}^{3} I_i(t) &\sim \mathbf{E}\left[X_1 h_1(X_1)\right]\sum_{i=2}^{d}\bar{F}_i(t) + \int_t^{\infty} x F_1(\mathrm{d}x) \\
&\sim \mathbf{E}\left[X_1 h_1(X_1)\right]\sum_{i=2}^{d}\bar{F}_i(t) + t\frac{\alpha_1}{\alpha_1 - 1}\bar{F}_1(t). \qquad (3.21)
\end{aligned}$$

Using a similar argumentation and keeping in mind $\mathbf{E}[h_1(X_1)] = 1$, one sees that the denominator on the right-hand side of (3.15) can be approximated as

$$\int_{0-}^{\infty} \Pr\left(S_d - X_1 > t - x | X_1 = x\right) F_1(\mathrm{d}x) \sim \sum_{i=1}^{d} \bar{F}_i(t), \tag{3.22}$$

which is equivalent to (3.12). By substituting (3.21) and (3.22) into (3.15), we obtain (3.13) and the proof is complete. □

3.2. **Gumbel case.** In the second part of this section, some asymptotic results are obtained for the case in which the risks belong to MDA(Λ) and are unbounded and asymptotically independent. The class MDA(Λ) essentially contains all distributions with rapidly varying tails. Moderately heavy-tailed distributions such as lognormal and Weibull as well as light-tailed distributions such as exponential and gamma are members of MDA(Λ). For more information, see Embrechts *et al.* (1997).

It has been seen that asymptotic capital allocations are closely related to the tail behavior of the aggregate risk. A moderately heavy-tailed distribution from the Gumbel family is also known as a distribution from the subexponential class $\mathcal{S}$ for which two independent, identically distributed and nonnegative copies, X_1 and X_2, satisfy

$$\Pr(X_1 + X_2 > t) \sim 2\Pr(X_1 > t).$$

For a distribution function $F \in \mathrm{MDA}(\Lambda)$, conditions on its auxiliary function $a(\cdot)$ under which $F \in \mathcal{S}$ are available in Goldie and Resnick (1988) and Hashorva *et al.* (2010). The mainstream study of the tail probability of the sum of asymptotically dependent and asymptotically independent random variables has focused on the subexponential case. Mitra and Resnick (2009) investigated the problem from a different perspective. They considered dependent random variables from MDA(Λ) but not necessarily subexponential.

Following the work of Mitra and Resnick (2009) we propose a set of conditions below:

Assumption 3.3. *Let $\underline{X}$ be a positive random vector with marginal distributions $F_1, \ldots, F_d$. Assume that $F_1 \in \mathrm{MDA}(\Lambda)$ with an auxiliary function $a(\cdot)$ as defined in (2.9). In addition, for each $1 \le i \le d$, there exists a non-negative constant c_i such that $\bar{F}_i(t)/\bar{F}_1(t) \to c_i$. Furthermore, assume that, for all $1 \le i \ne j \le d$,*

$$\lim_{t\to\infty} \frac{\Pr\left(X_i > t, X_j > a(t)x\right)}{\bar{F}_1(t)} = 0 \qquad \text{for all } x > 0, \tag{3.23}$$

and

$$\lim_{t\to\infty} \frac{\Pr\left(X_i > L_{ij}t, X_j > L_{ij}t\right)}{\bar{F}_1(t)} = 0 \qquad \text{for some } L_{ij} > 0. \tag{3.24}$$

If $c_i > 0$ for all $1 \le i \le d$, then Assumption 3.3 indeed describes an asymptotic independence case. For simplicity, we only verify this for $d = 2$. Note that relation (3.23) with $(i, j) = (1, 2)$ trivially implies relation (3.25) below since $a(t) = o(t)$.

Lemma 3.3. *Let $\underline{X} = (X_1, X_2)$ be a random vector with marginal distributions F_1 and F_2. Assume that $F_1 \in \mathrm{MDA}(\Lambda)$ with an auxiliary function $a(\cdot)$, that $\bar{F}_2(t)/\bar{F}_1(t) \to c_2$ for some constant $c_2 > 0$ and that the relation*

$$\lim_{t\to\infty} \frac{\Pr(X_1 > t, X_2 > tx)}{\bar{F}_1(t)} = 0 \tag{3.25}$$

holds for some $0 < x < 1$. Then X_1 and X_2 are asymptotically independent.

Proof. Note that both F_1 and F_2 belong to $\mathrm{MDA}(\Lambda)$ since $c_2 > 0$. Thus, for every $s \in \mathbf{R}$,

$$\lim_{t\to\infty} \frac{\bar{F}_2(t + a(t)s)}{\bar{F}_1(t)} = c_2 \mathrm{e}^{-s}.$$

Fix some s small enough such that the right-hand side above is larger than 1. Then it holds for arbitrarily fixed $y \in (x, 1)$ and for all large t that

$$\bar{F}_2(tx) \ge \bar{F}_2(ty + a(ty)s) > \bar{F}_1(ty).$$

It follows that $\mathrm{VaR}_{F_1(ty)}[X_2] > tx$. Analogously to the proof of Lemma 2.2, as $q \uparrow 1$, or, equivalently, as $t = \mathrm{VaR}_q[X_1] \to \infty$,

$$\begin{aligned} \Pr(F_2(X_2) > q | F_1(X_1) > q) &\le \frac{\Pr(X_1 \ge \mathrm{VaR}_q[X_1], X_2 \ge \mathrm{VaR}_q[X_2])}{\Pr(X_1 > \mathrm{VaR}_q[X_1])} \\ &\le \frac{\Pr\left(X_1 \ge t, X_2 \ge \mathrm{VaR}_{F_1(ty)}[X_2]\right)}{\bar{F}_1(t)} \\ &\le \frac{\Pr(X_1 = t) + \Pr(X_1 > t, X_2 > tx)}{\bar{F}_1(t)} \to 0, \end{aligned}$$

where in the last step we used the fact $\Pr(X_1 = t) = o\left(\bar{F}_1(t)\right)$ due to Corollary 1.6 of Resnick (1987). Therefore, X_1 and X_2 are asymptotically independent. □

Corollary 2.2 of Mitra and Resnick (2009) shows that, under Assumption 3.3,

$$\Pr(S_d > t) \sim \bar{F}_1(t) \sum_{i=1}^{d} c_i. \tag{3.26}$$

This result aligns with other asymptotic results for the sum of asymptotically independent subexponential random variables and provides a conspicuous step ahead in understanding the extreme behavior of the sum of dependent random variables for the Gumbel case.

The next lemma provides useful information for the proof of Theorem 3.3 below.

Lemma 3.4. *Let $\underline{X}$ be a random vector satisfying Assumption 3.3. Then, for every $y \in \mathbf{R}$,*

$$\lim_{t\to\infty} \frac{\Pr\left(X_k > a(t)x, S_d - X_k > t + a(t)y\right)}{\bar{F}_1(t)} = \begin{cases} \mathrm{e}^{-y} \sum_{i=1, i\neq k}^{d} c_i, & x \leq 0, \\ 0, & x > 0, \end{cases} \tag{3.27}$$

and

$$\lim_{t\to\infty} \frac{\Pr(X_k \leq a(t)x, S_d > t + a(t)y)}{\bar{F}_1(t)} = \begin{cases} \mathrm{e}^{-y} \sum_{i=1, i\neq k}^{d} c_i, & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{3.28}$$

Proof. Our first remark is that relation (3.27) holds for non-positive values of x due to (3.26). In addition, the proof of Corollary 2.2 of Mitra and Resnick (2009) shows that, for every $x > 0$,

$$\Pr\left(X_k > a(t)x, S_d - X_k > t\right) = o\left(\bar{F}_1(t)\right). \tag{3.29}$$

Now, relation (2.10) implies that, for arbitrarily fixed $0 < \varepsilon < 1$ and all large t,

$$(1-\varepsilon)a(t) \leq a\left(t + a(t)y\right) \leq (1+\varepsilon)a(t).$$

Thus, for any $x > 0$ and $y \in \mathbf{R}$,

$$\begin{aligned} & \frac{\Pr\left(X_k > a(t)x, S_d - X_k > t + a(t)y\right)}{\bar{F}_1(t)} \\ \leq\ & \frac{\Pr\left(X_k > a\left(t + a(t)y\right)\frac{x}{1+\varepsilon}, S_d - X_k > t + a(t)y\right)}{\bar{F}_1(t)} \\ \sim\ & \frac{\Pr\left(X_k > a\left(t + a(t)y\right)\frac{x}{1+\varepsilon}, S_d - X_k > t + a(t)y\right)}{\bar{F}_1\left(t + a(t)y\right)} \mathrm{e}^{-y} \to 0, \end{aligned}$$

which is a consequence of relations (2.9) and (3.29). Therefore, (3.27) is proved.

The proof of the second constituent is facilitated by some vague convergence properties. Let $M < y$ be fixed. Then, by relation (3.27),

$$\frac{\Pr\left(\left(\frac{X_k}{a(t)}, \frac{S_d - X_k - t}{a(t)}\right) \in \cdot\right)}{\bar{F}_1(t)} \xrightarrow{v} \mu_k(\cdot)$$

holds on $[-\infty, \infty] \times [M, \infty]$ for the measure μ_k given by

$$\mu_k(\mathrm{d}x, \mathrm{d}y) := \sum_{i=1, i\neq k}^{d} c_i \mathrm{e}^{-y} \epsilon_0(\mathrm{d}x)\mathrm{d}y,$$

where $\epsilon_0(\cdot)$ denotes the Dirac measure. It is useful to note that in order to fully justify the latter vague convergence property, one should obtain similar results to (3.27) for other compact sets, which can be simply verified since (3.26) holds.

Note that (3.28) is trivial for non-positive values of x and for this reason we assume $x > 0$. Denote $A := \{(x_1, x_2) : x_1 \leq x, x_1 + x_2 > y\}$. The measure μ_k puts mass over the set A only on the line $\{x_1 = 0, x_2 > y\}$ and, therefore, $\mu_k(\partial A) = 0$. Thus, Proposition A2.12 of Embrechts *et al.* (1997) allows us to generate the conclusion

$$\begin{aligned}
\lim_{t\to\infty} \frac{\Pr(X_k \le a(t)x, S_d > t + a(t)y)}{\bar{F}_1(t)} &= \lim_{t\to\infty} \frac{\Pr\left(\left(\frac{X_k}{a(t)}, \frac{S_d - X_k - t}{a(t)}\right) \in A\right)}{\bar{F}_1(t)} \\
&= \mu_k(A) = \mathrm{e}^{-y} \sum_{i=1, i\ne k}^{d} c_i.
\end{aligned}$$

The proof is complete. □

Now, we are able to provide the last main result of this section.

Theorem 3.3. *Let $\underline{X}$ be a random vector satisfying Assumption 3.3. Then the relation*

$$\mathbf{E}\left[X_k | S_d > \mathrm{VaR}_q[S_d]\right] = (C_k + o(1))\, \mathrm{VaR}_q[S_d] \tag{3.30}$$

holds for all $1 \le k \le d$, where $C_k = c_k \left(\sum_{i=1}^d c_i\right)^{-1}$.

Note that the constant C_k above can take value 0 for which reason we did not write relation (3.30) as an equivalence.

Proof. Similar to the proof of Theorem 2.2,

$$\begin{aligned}
\mathbf{E}\left[X_k | S_d > t\right] &= \left(\int_0^{a(t)} + \int_{a(t)}^{t} + \int_t^{\infty}\right) \Pr\left(X_k > z | S_d > t\right) \mathrm{d}z \\
&= I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

Now, $I_1(t) \le a(t) = o(t)$. The change of variables $z = t + a(t)s$ yields that

$$\begin{aligned}
I_3(t) &= a(t) \int_0^{\infty} \Pr\left(X_k > t + a(t)s | S_d > t\right) \mathrm{d}s \\
&\le \frac{a(t)}{\Pr\left(S_d > t\right)} \int_0^{\infty} \Pr\left(X_k > t + a(t)s\right) \mathrm{d}s \\
&\sim \left(\sum_{i=1}^{d} c_i\right)^{-1} a(t) \int_0^{\infty} \mathrm{e}^{-s} \mathrm{d}s \\
&= o(t),
\end{aligned}$$

as a result of relations (2.9) and (3.26), the Dominated Convergence Theorem and the fact that $a(t) = o(t)$. It remains to justify

$$I_2(t) = (C_k + o(1))\, t. \tag{3.31}$$

Some useful bounds for $I_2(t)$ are as follows:

$$(t - a(t)) \Pr\left(X_k > t | S_d > t\right) \le I_2(t) \le t \Pr\left(X_k > a(t) | S_d > t\right). \tag{3.32}$$

The left-hand side above is equal to

$$(t-a(t))\Pr\left(X_k>t|S_d>t\right)=(t-a(t))\,\frac{\Pr(X_k>t)}{\Pr(S_d>t)}=(C_k+o(1))\,t,$$

due to relation (3.26). A similar argumentation and relation (3.28) help us to find the asymptotic behavior of the right-hand side of (3.32), as

$$\lim_{t\to\infty}\Pr\left(X_k>a(t)|S_d>t\right)=C_k.$$

Thus, relation (3.31) holds, which completes the proof. □

As before, Theorem 3.3 indicates that capital allocations can be related only to the reference risk measure, $\mathrm{VaR}_q[X_1]$. Finally, relation (3.26), Lemma 2.4 and the fact that $S_d\in\mathcal{R}_{-\infty}$ conclude the following result:

Corollary 3.2. *Let $\underline{X}$ be a random vector satisfying Assumption 3.3. Then, as $q=1-p\uparrow 1$,*

$$\mathbf{E}\left[X_k|S_d>\mathrm{VaR}_q[S_d]\right]=(C_k+o(1))\,\mathrm{VaR}_{1-p/\sum_{i=1}^d c_i}[X_1].$$

The same as in Corollary 2.2, the right-hand side above can be changed to $\mathrm{VaR}_{1-mp}[X_1]$ for every constant $m>0$, but the most rational choice for m should be $m=1/\sum_{i=1}^d c_i$.

4. Examples

We first provide some examples under which Assumptions 2.1, 2.2 and 3.1 are satisfied. We start out by giving some information regarding the copula concept (further details can be found in Nelsen, 1999). It is well known that the dependence structure associated with the distribution of a random vector can be characterized in terms of a *copula.* A bivariate copula is a two-dimensional distribution function defined on $[0,1]^2$ with uniformly distributed marginals. Due to Sklar's theorem (see Sklar, 1959), if (X_1,X_2) has continuous marginal distributions, then there exists a unique copula, C, such that

$$\Pr(X_1\le x_1,X_2\le x_2)=C\left(\Pr(X_1\le x_1),\Pr(X_2\le x_2)\right).$$

Similarly, the *survival copula*, $\widehat{C}$, is defined as the copula corresponding to the joint tail function satisfying

$$\Pr(X_1>x_1,X_2>x_2)=\widehat{C}\left(\Pr(X_1>x_1),\Pr(X_2>x_2)\right).$$

Note that every copula satisfies

$$(u_1+u_2-1)\vee 0\le C(u_1,u_2)\le u_1\wedge u_2.$$

Recall that $W(u_1,u_2) := (u_1+u_2-1)\vee 0$ and $M(u_1,u_2) := u_1\wedge u_2$ are known as the *comonotonic* and *counter-monotonic* copulae, which respectively correspond to the strongest and weakest possible dependence structures that may occur between two random variables. The comonotonic (respectively, counter-monotonic) dependence structure arises when one random variable is a non-decreasing (respectively, non-increasing) function of the other.

An appealing class of copulae is the Archimedean one. By definition, an *Archimedean copula* C is given by

$$C(u_1,u_2)=\varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right),$$

where $\varphi:[0,1]\longmapsto[0,\infty]$, called the generator of $C(u_1,u_2)$, is a strictly decreasing and convex function with $0<\varphi(0)\leq\infty$ and $\varphi(1)=0$. The function $\varphi^{-1}(\cdot)$ is the pseudo-inverse of $\varphi(\cdot)$, and by convention $\varphi^{-1}(t)=0$ if $t>\varphi(0)$. A *strict generator* satisfies $\varphi(0)=\infty$.

Juri and Wüthrich (2003) developed a set of sufficient conditions on the generator of an Archimedean copula under which the joint concomitant extreme events are characterized. That is, if

$$\lim_{u\downarrow 0}\frac{\varphi(1-xu)}{\varphi(1-u)}=x^{\beta},\qquad 1<\beta<\infty, \tag{4.1}$$

then

$$\lim_{u\downarrow 0}\frac{\widehat{C}(ux_1,ux_2)}{u}=x_1+x_2-\left(x_1^{\beta}+x_2^{\beta}\right)^{1/\beta}.$$

Now, under the assumption that both risks belong to MDA(Φ_α) and have strongly equivalent tail probabilities,

$$\lim_{t\to\infty}\frac{\Pr(X_2>t)}{\Pr(X_1>t)}=c \tag{4.2}$$

with $c=c_F>0$, we have

$$\lim_{t\to\infty}\frac{\Pr(X_1>tx_1,X_2>tx_2)}{\Pr(X_1>t)}=x_1^{-\alpha}+c_Fx_2^{-\alpha}-\left(x_1^{-\alpha\beta}+c_F^{\beta}x_2^{-\alpha\beta}\right)^{1/\beta}. \tag{4.3}$$

As expected, if condition (4.1) is satisfied and both risks belong to MDA(Λ) such that relation (4.2) holds with $c=c_G>0$, then

$$\lim_{t\to\infty}\frac{\Pr(X_1>t+a(t)x_1,X_2>t+a(t)x_2)}{\Pr(X_1>t)}=\mathrm{e}^{-x_1}+c_G\mathrm{e}^{-x_2}-\left(\mathrm{e}^{-\beta x_1}+c_G^{\beta}\mathrm{e}^{-\beta x_2}\right)^{1/\beta},$$

where the auxiliary function $a(\cdot)$ is defined as in (2.9) corresponding to X_1.

Similarly, assume now that the survival copula is Archimedean with a strict generator satisfying

$$\lim_{u\downarrow 0}\frac{\varphi(xu)}{\varphi(u)}=x^{-\beta},\qquad 0\leq\beta\leq\infty, \tag{4.4}$$

where $\beta = \infty$ implies that

$$\lim_{u \downarrow 0} \frac{\varphi(xu)}{\varphi(u)} = \begin{cases} \infty, & x < 1, \\ 0, & x > 1. \end{cases}$$

Now,

$$\lim_{u \downarrow 0} \frac{\widehat{C}(ux_1, ux_2)}{u} = \begin{cases} 0, & \beta = 0, \\ \left(x_1^{-\beta} + x_2^{-\beta}\right)^{-1/\beta}, & 0 < \beta < \infty, \\ x_1 \wedge x_2, & \beta = \infty. \end{cases} \quad \text{for all } x_1, x_2 > 0. \tag{4.5}$$

The work of Juri and Wüthrich (2003) justifies the above for $0 < \beta < \infty$, while Lemma 4.1 below proves (4.5) for $\beta = 0$ and $\beta = \infty$. The same paper shows that Assumptions 2.1, 2.2 and 3.1 are satisfied for finite positive values of β, but one can easily extend this to the remaining cases. Specifically, if relation (4.2) holds with $c = c_F > 0$ and $X_1 \in \mathrm{MDA}(\Phi_\alpha)$, then

$$\lim_{t \to \infty} \frac{\Pr(X_1 > tx_1, X_2 > tx_2)}{\Pr(X_1 > t)} = \begin{cases} \left(x_1^{\alpha\beta} + c_F^{\beta} x_2^{\alpha\beta}\right)^{-1/\beta}, & \text{if } 0 < \beta < \infty, \\ x_1^{-\alpha} \wedge c_F x_2^{-\alpha}, & \text{if } \beta = \infty, \end{cases}$$

and

$$\lim_{t \to \infty} \frac{\Pr(X_1 > tx_1, X_2 > tx_2)}{\Pr(X_1 > t)} = \begin{cases} x_1^{-\alpha}, & \text{if } \beta = 0, x_1 > 0, x_2 = 0, \\ c_F x_2^{-\alpha}, & \text{if } \beta = 0, x_1 = 0, x_2 > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

In addition, if relation (4.2) holds with $c = c_G > 0$ and $X_1 \in \mathrm{MDA}(\Lambda)$, then

$$\lim_{t \to \infty} \frac{\Pr(X_1 > t + a(t)x_1, X_2 > t + a(t)x_2)}{\Pr(X_1 > t)} = \begin{cases} \left(\mathrm{e}^{\beta x_1} + c_G^{\beta} \mathrm{e}^{\beta x_2}\right)^{-1/\beta}, & \text{if } 0 < \beta < \infty, \\ \mathrm{e}^{-x_1} \wedge c_G \mathrm{e}^{-x_2}, & \text{if } \beta = \infty. \end{cases}$$

Finally, note that a non-strict generator for a survival Archimedean copula gives (4.6) since it is certain that concomitant extreme events are impossible to occur.

A comprehensive list of copulae that satisfy the conditions explained in (4.1) and (4.4) is included in Charpentier and Segers (2009). The vast majority of Archimedean copulae that satisfy (4.4) with $\beta = 1$, together with relation (4.2) with $c = c_F > 0$ and $X_1 \in \mathrm{MDA}(\Phi_\alpha)$, gives (4.6), which designs the framework defined in Assumption 3.1. Additional examples outside the Archimedean world can be found in Asimit and Jones (2008) and Kortschak and Albrecher (2009).

The next lemma develops some asymptotic results that are useful in justifying the extreme behavior of a bivariate random vector with an Archimedean survival copula.

Lemma 4.1. *Let $C(\cdot,\cdot)$ be a copula such that the survival copula is Archimedean with a strict generator satisfying (4.4). Then*

$$\lim_{u\downarrow 0}\frac{\widehat{C}(ux_1,ux_2)}{u}=\begin{cases}0, & \beta=0,\\ x_1\wedge x_2, & \beta=\infty,\end{cases}\tag{4.7}$$

holds for any positive x_1 and x_2, and

$$\lim_{u\downarrow 0}\frac{\widehat{C}\left(u,f(u)\right)}{f(u)}=1\tag{4.8}$$

is true for any $0<\beta\le\infty$ and any positive measurable function $f(u)=o(u)$.

Proof. We first prove relation (4.7) for $\beta=0$. Recall that (4.4) holds locally uniformly. Therefore, for any $0<\varepsilon<1/2, x_1,x_2>0$ and u sufficiently small, we have

$$\varphi(ux_1)+\varphi(ux_2)>2(1-\varepsilon)\varphi(u).$$

Now, $\varphi(0)=\infty$ implies that $\varphi^{-1}(\cdot)$ is rapidly varying at ∞ (see Proposition 0.8(v), Resnick, 1987). Thus,

$$\frac{\widehat{C}(ux_1,ux_2)}{u}=\frac{\varphi^{-1}\left(\varphi(ux_1)+\varphi(ux_2)\right)}{u}\le\frac{\varphi^{-1}\left(2(1-\varepsilon)\varphi(u)\right)}{\varphi^{-1}\left(\varphi(u)\right)}\to 0,\qquad u\downarrow 0.$$

The case in which $\beta=\infty$ follows in a similar manner. Proposition 0.8(v) of Resnick (1987) implies that $\varphi^{-1}(\cdot)$ is slowly varying at ∞. As before, we obtain that

$$\frac{\widehat{C}(ux_1,ux_2)}{u}\ge\frac{\varphi^{-1}\left(2\varphi\left(ux_1\wedge ux_2\right)\right)}{\varphi^{-1}\left(\varphi\left(ux_1\wedge ux_2\right)\right)}\left(x_1\wedge x_2\right)\to x_1\wedge x_2,\qquad u\downarrow 0,\tag{4.9}$$

since $\varphi(ux_1)\vee\varphi(ux_2)\le\varphi\left(ux_1\wedge ux_2\right)$ holds for all $u,x_1,x_2>0$. The upper bound is straightforward since $C(x_1,x_2)\le M(x_1,x_2)$, which concludes the first part of this lemma.

To verify relation (4.8), notice that $f(u)=o(u)$ yields

$$\frac{\widehat{C}\left(u,f(u)\right)}{f(u)}\le\frac{u\wedge f(u)}{f(u)}\to 1.$$

First assume that $0<\beta<\infty$. For any $\varepsilon>0$ and u sufficiently small, we have $\varphi\left(f(u)/\varepsilon\right)>\varphi(u)$. Thus,

$$\frac{\widehat{C}\left(u,f(u)\right)}{f(u)}\ge\frac{\varphi^{-1}\left(\varphi\left(\frac{f(u)}{\varepsilon}\right)+\varphi\left(f(u)\right)\right)}{f(u)}\to\left(\varepsilon^{\beta}+1\right)^{-1/\beta},\qquad u\downarrow 0,$$

due to (4.5). By taking $\varepsilon\downarrow 0$, the lower and upper bounds coincide in this setting. For $\beta=\infty$, the lower bound follows by a similar reasoning as used in relation (4.9), which concludes (4.8). The proof is complete. □

Examples regarding Assumption 3.2 have been already discussed. Multiple examples regarding Assumption 3.3 have been provided by Mitra and Resnick (2009). We now indicate a wide class of distributions under which conditions required by Assumption 3.3 are verified.

We consider an asymmetric class of copulae studied by Khoudraji (1995) (see also Genest *et al.*, 1998). If $C(\cdot,\cdot)$ is an Archimedean copula then

$$C_{k,l}(u_1,u_2) := u_1^{1-k}u_2^{1-l}C(u_1^k,u_2^l), \qquad k,l \in (0,1), \tag{4.10}$$

defines another copula, which we call a *transformed asymmetric Archimedean copula*. It can be easily seen that for any dependence structure, the copula $C_{k,l}(\cdot,\cdot)$ describes an asymptotically independent scenario in the upper tail.

Proposition 4.1. *Let (X_1,X_2) be a bivariate non-negative random vector whose marginal distributions are identical to $F \in \mathrm{MDA}(\Lambda)$ with auxiliary function $a(\cdot)$ as described in (2.9). The survival copula is assumed to be given by a transformed asymmetric Archimedean copula $C_{k,l}(\cdot,\cdot)$ as defined in (4.10).*

(i) If φ is a strict generator satisfying (4.4) with $0<\beta\le\infty$ and the distribution F is such that

$$\lim_{t\to\infty}\frac{\bar{F}^b\left(xa(t)\right)}{\bar{F}^c(t)} = \begin{cases} 0, & b>c, \\ \infty\,, & b\le c, \end{cases} \tag{4.11}$$

holds for all $b,c,x>0$, then conditions (3.23) and (3.24) hold.

(ii) If φ is a non-strict generator, then conditions (3.23) and (3.24) hold for any distribution function $F \in \mathrm{MDA}(\Lambda)$ with $a(t)\to\infty$.

The assumption of a non-strict generator can be easily relaxed. Specifically, conditions (3.23) and (3.24) still hold whenever $\Pr(X_1>t_0,X_2>t_0)=0$ for some $t_0>0$ together with the compulsory condition of strongly equivalent tails with an unbounded auxiliary function. This is the case if the underlying survival copula is given by an Archimedean copula or a transformed asymmetric Archimedean copula with non-strict generators. The counter-monotonic dependence structure reflects a similar extreme behavior.

Proof. A non-strict generator excludes joint extreme events with probability one, which together with the fact that $a(\cdot)$ is unbounded, indicates that, for any $x_1,x_2,L>0$ and all large t,

$$\Pr\left(X_1>t,X_2>x_2a(t)\right)=\Pr\left(X_1>x_1a(t),X_2>t\right)=\Pr\left(X_1>La(t),X_2>La(t)\right)=0.$$

Evidently, conditions (3.23) and (3.24) are satisfied in this case.

It is further assumed that φ is a strict generator. We start out the verification of condition (3.23) by noticing that

$$\Pr(X_1 > x_1, X_2 > x_2) = \bar{F}^{1-k}(x_1)\bar{F}^{1-l}(x_2)\widehat{C}\left(\bar{F}^k(x_1), \bar{F}^l(x_2)\right)$$

and that

$$\widehat{C}\left(\bar{F}^k(t), \bar{F}^l(xa(t))\right) \sim \begin{cases} \bar{F}^l(xa(t)), & l > k, \\ \bar{F}^k(t), & l \leq k, \end{cases}$$

where the latter is a direct implication of relations (4.8) and (4.11). Thus,

$$\Pr(X_1 > t, X_2 > xa(t)) \sim \begin{cases} \bar{F}^{1-k}(t)\bar{F}(xa(t)) = o\left(\bar{F}(t)\right), & l > k, \\ \bar{F}(t)\bar{F}^{1-l}(xa(t)) = o\left(\bar{F}(t)\right), & l \leq k, \end{cases}$$

which implies (3.23).

To verify condition (3.24), without loss of generality we assume $k \leq l$. If $k < l$ then

$$\bar{F}^l(La(t)) = o\left(\bar{F}^k(La(t))\right) \qquad \text{for any } L > 0,$$

which, together with (4.8) and (4.11), gives

$$\begin{aligned} \Pr(X_1 > La(t), X_2 > La(t)) &= \bar{F}^{2-k-l}(La(t))\,\widehat{C}\left(\bar{F}^k(La(t)), \bar{F}^l(La(t))\right) \\ &\sim \bar{F}^{2-k}(La(t)) \\ &= o\left(\bar{F}(t)\right). \end{aligned}$$

If $l = k$ then

$$\begin{aligned} \Pr(X_1 > La(t), X_2 > La(t)) &= \bar{F}^{2-2k}(La(t))\,\widehat{C}\left(\bar{F}^k(La(t)), \bar{F}^k(La(t))\right) \\ &\sim 2^{-1/\beta}\bar{F}^{2-k}(La(t)) \\ &= o\left(\bar{F}(t)\right), \end{aligned}$$

where the second step is due to (4.5) and the last step due to (4.11). Thus, the proof is complete. □

Two moderately heavy-tailed distributions satisfy the sufficient condition defined in (4.11). The first example is the lognormal distribution with parameters $\mu \in \mathbf{R}$ and $\sigma > 0$,

$$\bar{F}(x) = \bar{F}_{SN}\left(\frac{\log x - \mu}{\sigma}\right), \qquad x > 0,$$

where $F_{SN}(\cdot)$ denotes the distribution function of the standard normal distribution. The auxiliary function is given by $a(x) = \frac{\sigma^2 x}{\log x - \mu}$ (see Embrechts *et al.*, 1997, page 150). The second example is a distribution function F with a tail

$$\bar{F}(x) = \begin{cases} \mathrm{e}^{-(\log x)^{\gamma}}, & x > 1, \\ 0, & x \leq 1, \end{cases}$$

where $\gamma > 1$. Its auxiliary function is given by $a(x) = \frac{x}{\gamma(\log x)^{\gamma-1}}$. For these two examples, the verification of (4.11) is straightforward and therefore is omitted.

We cannot draw a conclusion that all distributions from the intersection $\mathrm{MDA}(\Lambda) \cap \mathcal{S}$ satisfy the requirements of Proposition 4.1. Recall that a Weibull distribution has a tail

$$\bar{F}(x) = \mathrm{e}^{-cx^{\tau}}, \qquad x \geq 0,\ c > 0,\ 0 < \tau < 1,$$

with an auxiliary function $a(x) = c^{-1}\tau^{-1}x^{1-\tau}$ (see Embrechts *et al.*, 1997, page 150). Hence, it holds for all $k, l, x > 0$ that

$$\lim_{t\to\infty} \frac{\bar{F}^{k}\left(xa(t)\right)}{\bar{F}^{l}(t)} = \infty.$$

5. Simulation study and numerical results

We perform a simulation study on the results derived in Corollary 2.1. A portfolio of two risks is considered and the individual loss random variables, X_1 and X_2, are Pareto distributed with distribution function

$$F(x; \lambda, \alpha) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \qquad x \geq 0,$$

where λ equals $100,000$ for X_1 and $150,000$ for X_2, while α is the same for both X_1 and X_2 and it will be assigned values 2.5 and 3. The portfolio dependence structure is assumed to be given by the Gumbel copula

$$C(u_1, u_2) = \exp\left\{-\left((-\log u_1)^{\beta} + (-\log u_2)^{\beta}\right)^{1/\beta}\right\}, \qquad \beta \geq 1.$$

This copula belongs to the Archimedean family with a strict generator $\varphi(u) = (-\log u)^{\beta}$. Clearly, the Gumbel copula has the asymptotic dependence property for β values greater than 1 since relation (4.1) is satisfied. The measure μ in relation (2.4) is defined with $H_F(\cdot)$ calculated in (4.3) and relation (4.2) holds with $c = c_F = (3/2)^{\alpha}$. The parameter β is chosen to be 2, 3 and 5. Note that the strength of dependence for the Gumbel copula increases as β increases. The asymptotic constants appearing in Corollary 2.1, C_1 and C_2, are numerically computed by using the formulae $C_1 + C_2 = \alpha/(\alpha - 1)$ and

$$C_1 = \frac{\frac{1}{\alpha-1} + \int_0^1 \mu\left((x_1, x_2) : x_1 > z, x_1 + x_2 > 1\right) \mathrm{d}z}{\mu\left((x_1, x_2) : x_1 + x_2 > 1\right)}.$$

TABLE 1. Capital allocation ratio estimates with $\alpha = 2.5$

	Ratio for X_1			Ratio for X_2		
q	$\beta = 2$	$\beta = 3$	$\beta = 5$	$\beta = 2$	$\beta = 3$	$\beta = 5$
0.99	1.0705	1.0733	1.0748	1.0730	1.0747	1.0751
	(0.0059)	(0.0045)	(0.0045)	(0.0053)	(0.0049)	(0.0042)
0.995	1.0500	1.0536	1.0548	1.0522	1.0543	1.0551
	(0.0074)	(0.0067)	(0.0066)	(0.0076)	(0.0067)	(0.0065)
0.999	1.0231	1.0243	1.0263	1.0261	1.0263	1.0264
	(0.0127)	(0.0130)	(0.0126)	(0.0136)	(0.0130)	(0.0129)
0.9995	1.0195	1.0198	1.0194	1.0210	1.0194	1.0194
	(0.0232)	(0.0232)	(0.0193)	(0.0229)	(0.0219)	(0.0191)

TABLE 2. Capital allocation ratio estimates with $\alpha = 3$

	Ratio for X_1			Ratio for X_2		
q	$\beta = 2$	$\beta = 3$	$\beta = 5$	$\beta = 2$	$\beta = 3$	$\beta = 5$
0.99	1.0826	1.0879	1.0905	1.0881	1.0904	1.0917
	(0.0040)	(0.0038)	(0.0038)	(0.0040)	(0.0038)	(0.0039)
0.995	1.0618	1.0660	1.0675	1.0650	1.0681	1.0686
	(0.0044)	(0.0043)	(0.0044)	(0.0049)	(0.0047)	(0.0044)
0.999	1.0320	1.0360	1.0357	1.0357	1.0374	1.0365
	(0.0086)	(0.0108)	(0.0094)	(0.0087)	(0.0102)	(0.0097)
0.9995	1.0246	1.0276	1.0274	1.0254	1.0271	1.0280
	(0.0135)	(0.0138)	(0.0130)	(0.0143)	(0.0134)	(0.0134)

Each analysis is performed for 100 samples consisting of $5,000,000$ simulations from (X_1, X_2). The ratios between the capital allocations, estimated from the empirical distribution of the simulated samples of size $5,000,000$ and from the approximation provided by Corollary 2.1, are calculated for all 100 samples. The averages and standard deviations are then tabulated for various values of the parameters α and β, as discussed previously, as well as four different confidence levels: $q = 99\%$, 99.5%, 99.9% and 99.95%. These results are presented in Tables 1 and 2.

The speed of convergence increases, as the strength of dependence relaxes. Heavier tails, i.e., smaller values for α, entail faster convergence of our ratios to 1. As expected, the variances of the ratios increase for large values of q, but they are still at reasonable levels.

In the last part of this section, we use our results to evaluate the capital requirements for a Swiss-based insurance company. According to the Swiss Solvency Test (SST) guidelines, the capital requirement for an insurance company that operates in Switzerland is given by the CTE-based risk capital corresponding to a 99% level of confidence over a one-year horizon. Unlike the SST, Solvency II, which designs the regulatory requirements for insurance firms that operate in the European Union, sets out qualitative and quantitative requirements for Solvency Capital that ensures an insurance firm to be able to meet its obligations over the next 12 months with a probability of at least 99.5%. For both SST and Solvency II, the risk-based economic capital is defined by the excess of the capital given by the chosen risk measure over the best estimate of the liabilities or the expected amount of liabilities under usual circumstances.

Let us assume that a Swiss-based insurance company holds a portfolio of two dependent business lines, as assumed in the beginning of this section. By the Swiss Solvency Test guidelines, the total risk capital requirement is set to $\mathrm{CTE}_{0.99}[X_1+X_2]-\mathbf{E}[X_1+X_2]$, that is

$$\mathbf{E}\left[X_1+X_2|X_1+X_2>\mathrm{VaR}_{0.99}[X_1+X_2]\right]-\mathbf{E}[X_1+X_2],$$

while the individual capital requirements are given by

$$\mathbf{E}\left[X_i|X_i>\mathrm{VaR}_{0.99}[X_i]\right]-\mathbf{E}[X_i]=\frac{\alpha}{\alpha-1}\mathrm{VaR}_{0.99}[X_i],\qquad i=1,2. \tag{5.1}$$

Due to the SST, the insurer should allocate the risk capital for the two business lines as follows:

$$\mathbf{E}\left[X_i|X_1+X_2>\mathrm{VaR}_{0.99}[X_1+X_2]\right]-\mathbf{E}[X_i],\qquad i=1,2. \tag{5.2}$$

Table 3 elucidates relations (5.1) numerically. All tables herein consider four different confidence levels, i.e., $q=99\%$, 99.5%, 99.9% and 99.95%. The 99% level is recommended by the SST, while the remaining calculations may help in understanding the effect of more conservative regulatory requirements.

Further, Tables 4 and 5 elucidate relations (5.2), which are to this end calculated for each business line for various confidence levels q and values of the parameter β. In this respect, it is well known that risk aggregation should reduce the overall risk of a multi-line

TABLE 3. Individual capital requirements with varying α and q

Confidence Level	Required Capital for X_1 ($\alpha = 2.5$)	Required Capital for X_2 ($\alpha = 2.5$)	Required Capital for X_1 ($\alpha = 3$)	Required Capital for X_2 ($\alpha = 3$)
99%	884,929	1,327,393	546,238	819,357
99.5%	1,220,922	1,831,383	727,205	1,090,808
99.9%	2,474,822	3,712,233	1,350,000	2,025,000
99.95%	3,318,799	4,978,198	1,739,882	2,609,822

TABLE 4. Individual capital requirements with $\alpha = 2.5$

Confidence Level	Required Capital for X_1	Required Capital for X_2	Diversification Effect	β
99%	769,290	1,195,565	88.81%	2
99.5%	1,086,689	1,687,470	90.89%	2
99.9%	2,271,198	3,523,221	93.65%	2
99.95%	3,068,469	4,758,830	94.34%	2
99%	799,973	1,215,167	91.09%	3
99.5%	1,129,022	1,714,514	93.16%	3
99.9%	2,357,006	3,578,039	95.93%	3
99.95%	3,183,540	4,832,343	96.61%	3
99%	812,321	1,223,545	92.02%	5
99.5%	1,146,058	1,726,073	94.10%	5
99.9%	2,391,539	3,601,470	96.86%	5
99.95%	3,229,849	4,863,765	97.55%	5

business and thus results in the *diversification effect.* The phenomenon is conveniently quantified by the ratio

$$\frac{\mathrm{CTE}_{0.99}[X_1 + X_2] - \mathbf{E}[X_1 + X_2]}{\mathrm{CTE}_{0.99}[X_1] + \mathrm{CTE}_{0.99}[X_2] - \mathbf{E}[X_1 + X_2]},$$

and it is included in both tables.

It can be seen that there is a significant drop in risk capital requirements as α increases, which is due to the reduction in the degree of heavy-tailedness. In addition, the change in capital requirements is more pronounced as the confidence level becomes less severe.

Table 5. Individual capital requirements with $\alpha = 3$

Confidence Level	Required Capital for X_1	Required Capital for X_2	Diversification Effect	β
99%	467,939	725,971	87.43%	2
99.5%	639,531	991,330	89.71%	2
99.9%	1,230,061	1,904,559	92.88%	2
99.95%	1,599,744	2,476,258	93.71%	2
99%	485,866	737,220	89.56%	3
99.5%	663,397	1,006,305	91.84%	3
99.9%	1,274,365	1,932,359	95.01%	3
99.95%	1,656,843	2,512,087	95.84%	3
99%	493,058	741,936	90.44%	5
99.5%	672,971	1,012,584	92.71%	5
99.9%	1,292,139	1,944,015	95.89%	5
99.95%	1,679,751	2,527,109	96.72%	5

The first business line always requires less risk capital than the second business line as expected. Also, an increase in the strength of dependence reduces the diversification effect, which makes intuitive sense since higher β values imply that the risks are more positively dependent, i.e., closer to be comonotonic. We refer the reader to Dhaene *et al.* (2009), who investigated the influence of the dependence between losses on the diversification benefit that arises from merging these losses.

6. Conclusions

In this paper we considered the problem of allocating the aggregate risk of a multi-line insurance business consisting of dependent risks to the various sources. We fixed the risk measure to be Conditional Tail Expectation and we employed the machinery of the Extreme Value Theory, in general, and vague convergence, in particular. The allocation phenomenon is of immense importance in view of the increasing risk awareness, as well as because of the indisputable utility of the, e.g., profitability studies and quality control in insurance context (see, e.g., Valdez and Chernih, 2003). The proposed approach to tackle the problem is adequately motivated by the high confidence levels being required by regulations.

Our main results, under both asymptotic dependence and asymptotic independence, reduce the problem to calculating the Value-at-Risk of the aggregate risk of the multi-line business of interest. More specifically, for risks having similar extreme behaviors, we showed the following:

i) Extremely heavy-tailed risks that are asymptotically dependent in the upper tail require allocations asymptotically proportional to the VaR-based aggregate capital, where proportions are sensitive to the strength of the upper tail dependence and the weight of each individual risk.
ii) Moderately heavy-tailed or light-tailed risks that are asymptotically dependent in the upper tail require asymptotically equal capital allocations;
iii) Asymptotic independence in the upper tail requires capital allocations that are asymptotically proportional to the weight of each individual risk in the portfolio.

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