FUNDAMENTAL MODE ESTIMATION FOR MODERN CABLE-STAYED BRIDGES CONSIDERING THE TOWER FLEXIBILITY

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ABSTRACT

The design of cable-stayed bridges is typically governed by the dynamic response. This work provides designers with essential information about the fundamental vibration modes proposing analytical expressions based on the mechanical and geometrical properties of the structure. Different bridge geometries are usually considered in the early design stages until the optimum solution is defined. In these design stages the analytical formulation is advantageous as finite element models are not required and modifying the bridge characteristics is straightforward. The influence of the tower flexibility is included in this study, unlike in previous attempts on mode estimation. The dimensions and proportions of the canonical models proposed in the analytical study stem from the previous compilation of the dimensions of a large number of constructed cable-stayed bridges. Five tower shapes, central or lateral cable-system layouts and box- or 'U'-shaped deck sections have been considered. The vibration properties of more than one thousand cable-stayed bridges with main spans ranging from 200 to 800 m long were extracted within an extensive parametric

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analysis. The Vaschy-Buckingham theorem of dimensional analysis was applied to the numerical results in order to propose the formulation for period estimation. Finally, the formulae were validated with the vibration properties of 17 real cable-stayed bridges constructed in different countries. The importance of the tower flexibility is verified and the errors observed are typically below 15 %, significantly improving the estimations obtained by previous research works.

Keywords: cable-stayed bridges, vibration periods, preliminary design, dimensional analysis, tower flexibility, Chinese bridges.

INTRODUCTION

The large flexibility, light weight and reduced damping of cable-stayed bridges are responsible for severe potential oscillations when subjected to dynamic excitations, particularly for large spans (He et al. 2001). Aerodynamic instabilities like flutter, torsional-flutter, or vortex shedding can be catastrophically accentuated for critical wind speeds which are strongly related to the fundamental frequencies of the structure (Selberg 1961; Simiu and Scanlan 1996; Katsuchi et al. 1998; Strømmen 2006; Mannini et al. 2012). The critical speed for flutter is affected by the characteristic closely spaced vertical and torsional frequencies in cable-stayed bridges, particularly if the deck is supported by two lateral cable planes. On the other hand, Eurocode 1 part 1-4 (EN1991-1-4: 2005) and previous research studies (Walshe and Wyatt 1983) propose simplified expressions for the study of vertical deck movements under wind gusts set in terms of the modal properties, among other variables. The crucial importance of the first vibration modes for bridge safety under wind loads is self evident, and these modes also play an important role in the seismic response of cable-stayed bridges (Camara and Astiz 2012).

The study of the vibration properties of a cable-stayed bridge is consequently a key step to address its global dynamic behaviour and possible design weaknesses. Modal coupling is a distinguishing feature of this structural typology, particularly between the transverse flexure of the deck and its torsional response. This coupling differentiates the dynamic behaviour of cable-stayed bridges from suspension bridges (Walther et al. 1988; Abdel-Ghaffar 1991). The first vibration modes involve the excitation of the deck, and they are strongly influenced by the cable-system in vertical direction due to the closely spaced stays and slender decks currently employed in modern designs. However, in transverse direction the cables offer small restraint to the deck and the vibration is dominated by the transverse flexural stiffness of the girder. Transverse vibration modes can be approximated from those of a continuous beam with the same span arrangement (Wyatt 1991). The torsional stiffness may arise from two sources: (i) from the cable-system geometry if differential longitudinal displacements are prevented in the cable-planes due to the tower geometry, this is the case for A- and inverted Y-shaped towers but not for towers with H shape; or (ii) from the deck cross-section in bridges with moderate-to-medium spans and box-shaped girders, which is typical in structures with one Central Cable Plane (CCP) (Virlogeux 1999).

In the early stages of the project different design options are typically considered and engineers need basic information about the natural frequencies of the bridge to obtain the final configuration. Finite Element (FE) models are able to provide accurate solutions but changes in the geometry (e.g. the tower shape or the span distribution) are not easily introduced. In this context, simple expressions to estimate the first vibration modes are very helpful. However, the aim of the analytical estimation is not the substitution of the FE model and it should be developed once the final bridge configuration is achieved. In the last two decades several analytical formulations that predict the vertical, transverse and torsional deck periods have been proposed. The most simple (and also gross) estimation only includes the main span of the cable-stayed bridge and is based on field forced excitation tests conducted in 13 constructed cable-stayed bridges in Japan (Kawashima et al. 1993). A similar approach was adopted by (Guohao 1992). More rigorously, Wyatt (Wyatt 1991) introduced the mechanical properties of the deck, the cable-system and the geometrical configuration of the bridge in the modal estimation. Recently, Gimsing and Georgakis (Gimsing and Georgakis 2011) proposed an idealized model with two springs representing the cables in order to study the vertical and torsional fundamental frequencies of the deck neglecting its stiffness, which is valid for lateral cable arrangements. The resulting ratios between the first vertical and torsional frequencies were close to the observed ones in practice, between 1.5 and 1.6 (Wyatt 1991). However, the tower flexibility is neglected in all the works published to date, assuming that it is infinitely stiff both in transverse (perpendicular to traffic) and longitudinal (parallel to traffic) directions. This was observed to be a source of significant errors in the present study.

This work starts suggesting dimensionless ratios to define reasonable deck and tower sections in cable-stayed bridges with main spans ranging from 200 to 800 m. Next, analytical expressions are provided to estimate the first vibration periods in terms of the mechanical and geometrical properties of the structure. The terms involved in the proposed formulae are obtained from a dimensional analysis that explicitly includes the tower flexibility. Different parameters of the proposed equations are obtained by means of the least squares approach applied to an extensive modal analysis conducted in more than one thousand FE models. These models are parametrically described in terms of the main span length, the width of the deck and the tower height. The resulting expressions are validated with results reported by other research works (Fan et al. 2001; Pridham and Wilson 2005; Ren et al. 2005; Magalhaes et al. 2007; Wu et al. 2008) on 17 constructed cable-stayed bridges, distinguishing the influence of the tower shape among other features. The improved accuracy of the mode estimation proposed in this work is observed in the great majority of the cases, where the averaged errors are below 15 %.

BRIDGE DEFINITION AND PARAMETRIC STUDIES

The proposed bridges have a conventional configuration with two concrete towers and a composite deck. The distribution is completely symmetric in transverse direction (Y) and also in longitudinal direction (X). The back span to main span ratio (L_S/L_P) and the tower height (above the deck level) to main span ratio (H/L_P) are taken from the compilation of 43 constructed cable-stayed bridges. This database is an extension of the work reported by (Manterola 1994). The geometrical ratios of 80 % of the cable-stayed bridges in this database are within the range: $H/L_P = 0.19 - 0.23$ and $L_S/L_P = 0.3 - 0.5$, hereafter referred to as 'conventional' range. The bridges proposed in this work employ the averaged geometrical ratios: $H/L_P = 0.21$ and $L_S/L_P = 0.4$. These aspect ratios are in accordance with the canonical proportions given by (Leonhardt and Zellner 1980) ($H/L_P = 0.2 - 0.25$, $L_S/L_P = 0.42$) and (Como et al. 1985) ($H/L_P = 0.2$, $L_S/L_P = 0.33$) in the 80's.

The cable-system configuration is arranged in a semi-harp layout, which is the normal solution in modern designs to obtain a balance between structural efficiency and ease of construction. Intermediate piers constrain exclusively the vertical movement of the deck (and its torsion) in the side spans whereas the longitudinal and transverse movements are released. The deck-tower connection plays an important role in the dynamic response of the structure (He et al. 2001). Following the current design trend in seismic areas the only movement restrained at this point is the relative deck-tower displacement in transverse direction (Y) ('floating' connection). Figure 1 shows the generic bridge elevation and plan, besides the boundary conditions along the deck and the towers.

The deck cross-section is composite with two longitudinal edge steel girders and one upper concrete slab (open section) in bridges with two Lateral Cable Planes (LCP). In this case the girder depth slightly increases with the main span due to wind considerations, and the relationship between both parameters is taken from (Astiz 2001). On the other hand, the deck adopted in structures with one Central Cable Plane (CCP) shows an 'U'-shaped steel section below the concrete slab. The closed deck section in CCP models helps to withstand the torsion that is not resisted by the cable-system. The depth of the deck in CCP configurations is adopted from the aforementioned database of constructed cable-stayed bridges. Figure 1 includes the description of the deck cross-sections. The stays are proportioned to a consistent level of stress under the deck self-weight and traffic live load (4 kN/m^2) combination: 708 MPa. This value is 40 % of the ultimate stress allowed in the cable steel. Each cable cross-section is obtained by equilibrium considerations between the cable force and the weight of the deck.

Five different tower shapes have been considered and their sections are defined in terms of the tower height (H) based on the dimensions of 20 real cable-stayed bridges. Figures 2 and 3 represent the studied towers, in which the symbols are self-explanatory. The design of the towers in a real project requires a detailed definition of the transition between sections in different parts. This plays an important role in the static and dynamic response of the whole structure (Camara and Astiz 2011). However, this detail level is beyond the scope of the preliminary design stage. Instead, constant sections between different parts of the towers have been adopted in the parametric study of this paper. The only exception to this is found in the towers with lower diamond (YD and AD configurations in Figure 2), where the transition of the sections below the deck level is smooth to avoid an undesirable seismic behaviour (Camara and Astiz 2011). The thickness (t_c) of the tower cross-sections is obtained so that the maximum allowable compression $f^*_{cd} = 10$ MPa is not exceeded when the self-weight, dead load and traffic live load are applied to the structure. Constructability limitations dictate the vertical pier thickness in the lower diamond to be 0.45 m, regardless of the main span length.

The parametric studies of the FE models are based on three independent variables described in Figures 1 to 3: the main span length L_P, the deck width B and the distance between the tower foundation and the deck level H_i. The proportions and sections of the whole bridge are defined in terms of the central span L_P (which is the main variable), except B and H_i. The distance between consecutive cable anchorages is fixed in the central span to 10 m (see Figure 1). Not every main span length is valid in the parametric analysis as the number of stays in one cable plane (N_C in Figure 1) is obviously a natural number. Consequently, the number of cables is the variable modified in the parametric analysis instead of the main span. In accordance with the cable-system arrangement illustrated in Figure 1, the number of cables and the main span length are related through the expression: $N_C = (L_P - 20)/20$. The bridges in this parametric study are obtained by varying the main span from 200 to 800 m each 20 m (i.e. N_C ranges from 9 to 39 cables). The resulting structures have a typical value of the tower height below the deck: $H_i = H/2$; and four reasonable deck widths for each main span length: $B = 20, 25, 30, 35$ m. In order to cover a broader range of possibilities, two extra values of the tower height below the deck are considered: $H_i = H/2.5$ and $H_i = H/1.5$, but in these cases the deck width is fixed to 25 m. Altogether, 1050 FE models (ABAQUS 2012) have been studied.

The elastic properties of the materials have been defined using the relevant Eurocodes. Each stay cable is represented by only one element without flexural stiffness, consequently ignoring local cable modes. The foundation soil is assumed as infinitely stiff and the towers are encastred at their base, which is a reasonable assumption since the first vibration modes mainly involve the deck deformation and are not significantly affected by the response of the foundation.

The analytical definition of the vibration properties presented in this study is also valid for bridges with different sections and materials than those considered in the parametric analysis, provided that they have two towers and symmetrical configurations.

ANALYTICAL EXPRESSIONS FOR MODE ESTIMATION

Dimensional analysis

The first vibration modes in a cable-stayed bridge mainly involve the deformation of the deck, which is constrained to a greater or lesser extent by the towers and the cable-system. The relevance of this constraint depends on the mechanical properties, the geometry and the nature of the mode shape (i.e. transverse, vertical or torsional). The problem can be simplified to a beam (the deck) simply supported at the abutments and spanning a distance $L_P + 2L_S$ [m] with a distributed mass m_d [kg/m] and rigidity EI_d [Nm^2]. The constraint imposed by the towers, the intermediate piers and/or the cable-system may be defined by means of elastic springs with constant K [N/m]. The physical equation that relates the vibration period T [s] in this model with the mechanical properties of the deck and the restraining system is: $f(m_d, EI_d, L_P, K, T) = 0$. This equation depends on three physical units: the mass, the length and the time. According to the Vaschy-Buckingham Π theorem (Buckingham 1914) this physical equation may be rewritten in terms of two dimensionless parameters $g(\Pi_1, \Pi_2) = 0$:

$$\Pi_1 = T_j \sqrt{\frac{EI_{d,j}}{m_d L_P^4}} \tag{1a}$$

$$\Pi_2 = \frac{EI_{d,j}}{K L_P^3} \tag{1b}$$

where T_j is the first vibration period in direction j ($j = Y$ for the transverse mode and $j = Z$ for the vertical one), E is the Young's modulus of the deck section (homogenized in composite girders), $I_{d,j}$ is the moment of inertia of the deck associated with the flexure in direction j. In the case of torsion, expressions (1a) and (1b) are slightly different and will be discussed in the next section.

The dimensionless parameters Π_1 and Π_2 (particularized for the transverse, vertical and torsional vibration modes) are obtained in all the FE models defined in the parametric analysis. Subsequently, the physical equation $g(\Pi_1, \Pi_2) = 0$ is adjusted by the least squares technique in order to obtain analytical expressions to estimate the vibration periods. This approach is presented in the following paragraphs. Another dimensionless parameter (Π_3) could be included to take into account the influence of the tower mass, but it is irrelevant in the fundamental periods.

Fundamental transverse mode

The contribution of the cable-system to horizontal transverse loads is negligible in cable-stayed bridges, in which the transverse movement of the deck is mainly constrained at the abutments and the towers (see the boundary conditions in Figure 1). Wyatt (Wyatt 1991) assumed that the displacement of the towers due to the transverse reaction of the deck is negligible in the fundamental transverse mode. This is true only if the transverse flexural rigidity of the deck is much lower than the tower stiffness, i.e. if the main span length is large. Figure 4 shows the first transverse vibration mode in two cable-stayed bridges, with 300 and 600 m main span. The transverse movement of the towers and their interaction with the deck is clear in the small bridge ($L_P = 300$ m), where the towers act as elastic transverse springs constraining the deck movement. However, this interaction is negligible in the large bridge ($L_P = 600$ m) and the deck behaves in transverse direction like a beam with fixed supports at the abutments and the towers. Consequently, the simplified physical model that describes the transverse response of the deck is a beam elastically supported at the towers level and simply supported at the abutments. To obtain the transverse tower stiffness $K_{t,Y}$ [N/m] a unit load is applied to the FE model of the tower (excluding the deck and the cable-system) as shown in Figure 5(a). The resulting displacement at the deck-tower connection defines the stiffness of the elastic supports in the deck model.

The dimensionless parameters Π_1 and Π_2 are obtained from expression (1), in which: $T = T_Y$ is the first transverse mode obtained in the modal analysis of the studied bridges; $EI_{d,j} = EI_{d,Y}$

is the transverse rigidity of the deck; and $K = K_{t,Y}$ is the transverse tower stiffness obtained in the static analysis described in Figure 5(a). Figure 6(a) plots Π_1 versus Π_2 in all the studied FE models and proposes an optimum nonlinear relationship between both parameters: $g(\Pi_1, \Pi_2) = a_1\Pi_2^{a_2} + a_3 - \Pi_1 = 0$, where the coefficients a_i are obtained by the least squares approach. From this relationship, and considering expressions (1a) and (1b), the analytical estimation of the first transverse period is obtained:

$$T_Y = \sqrt{\frac{m_d L_P^4}{EI_{d,Y}}}(9.54\Pi_2^{0.70} + 0.39) \tag{2}$$

where $\Pi_2 = EI_{d,Y}/(K_{t,Y}L_P^3)$. Note that expression (2) is reduced to the proposal of Wyatt (Wyatt 1991) if the tower stiffness and/or the main span are very large (i.e. if $K_{t,Y}$ or $L_P \rightarrow \infty$ then $\Pi_2 \rightarrow 0$), and hence the contribution of the tower to the first transverse mode is ignored as it was intended.

The aim of this work is the estimation of the vibration periods by means of simple analytical expressions. Equation (2) could be questioned if a FE model of the tower is required to obtain the parameter $K_{t,Y}$. Consequently, an analytical expression is proposed to approximate the transverse tower stiffness. From the static analysis of the tower frame included in Figure 5(a) it may be observed that the stiffness is governed by the following terms:

$$K_{t,Y} = \frac{E_t\bar{I}_{t,Y}}{(H + H_i)^3}\frac{H}{H_i}(m_Y \sin\alpha + b_Y) \tag{3}$$

in which E_t is the Young's modulus of the material employed in the tower, $\bar{I}_{t,Y}$ is the transverse moment of inertia of the tower leg below the deck level (averaged if the section is variable), H and H_i are the tower height above and below the deck respectively, α is the angle of the tower leg with respect to the horizontal line (see Figure 2). Finally, the parameters m_Y and b_Y result from a linear regression of the tower stiffness observed in the FE models. These values are presented in Table 1 and control the transverse tower stiffness depending on its shape. The estimated tower stiffness is larger if the lateral legs are connected at the top (i.e. inverted Y- and A-shaped towers) due to the geometrical constraint exerted by this point in transverse direction. This result is in agreement with (Camara and Astiz 2011).

The transverse period obtained with expression (2) when the approximation of the tower stiffness in equation (3) is employed ($T_Y = T_{app}$) has been compared with the FE model results (T_{FEM}). The error in the estimation of the transverse vibration period is shown in Figure 7(a) for the whole range of main span lengths studied. This error is defined as: $e = 100(T_{app} - T_{FEM})/T_{FEM}$. Only the results of specific tower shapes and cable layouts are presented but similar trends have been observed in other models. The error obtained with the expressions proposed by Wyatt (Wyatt 1991) and Kawashima *et al.* (Kawashima et al. 1993) is included in this figure for comparison. The estimation of the first transverse period has been clearly improved by the present work: the error could reach 60 % with previous approaches but it never exceeds 10 % if equation (2) is employed. The error with the proposed expression is caused primarily by the definition of $g(\Pi_1, \Pi_2) = 0$ (see the dispersion in the least squares fitting in Figure 6(a)). The proposal of Wyatt significantly underestimates the transverse vibration period below 400 m main span. This interesting result is explained by the significant transverse flexibility of the towers and their strong interaction with the deck in small-to-medium bridges, which is included in expression (2) in contrast to Wyatt's study. The proposal of Kawashima *et al.*: $T_Y = L_P^{1.262}/482$ [s] (L_P in [m]), only depends on the main span length and such a simple expression cannot expect to predict accurately the vibration period of a cable-stayed bridge, as shown in Figure 7(a).

Fundamental vertical mode

The deck of modern cable-stayed bridges with closely spaced stays behaves in vertical direction like a beam over elastic foundation (Walther et al. 1988). The constraint exerted by the cable-system to the vertical deck flexure is caused by the axial deformation of the stays and is reduced due to the movement of the tower anchorage area in longitudinal direction (X, parallel to the traffic). This horizontal movement of the tower reduces the structural effectiveness of the cable-system and is counterbalanced by the back span restraint. Wyatt (Wyatt 1991) proposed the estimation of the first vertical vibration period of the deck by neglecting the longitudinal movement of the tower, i.e. by considering that the cable-system is perfectly effective. Only pure fan cable-system configurations with very stiff towers would be strictly covered by Wyatt's assumption. This approach leads to unreasonably stiff vibration periods in conventional bridges with harp- or semi-harp cable layouts, since the longitudinal movement of the tower cannot be totally avoided and its flexibility should be taken into account (besides the effect of the back span cable-system). Figure 8 shows the first vertical vibration mode in a cable-stayed bridge, highlighting the coupling between the vertical deck flexure and the longitudinal movement of the tower.

The physical model to describe the behaviour of the bridge in vertical direction is again represented by a beam (the deck) that is constrained by elastic springs at the cable anchorages with stiffness $K_{ct,Z}$ [N/m]. The cable-system and the tower may contribute to this stiffness. A parametric FE model of a tower and the associated cable-system is developed to obtain $K_{ct,Z}$, as shown in Figure 5(b). In light of the deck deformation in the fundamental vibration mode (shown in Figure 8), a linearly increasing load is applied to the cable anchorages of this model. Only the cables anchored to the abutment and the intermediate piers are considered in the side spans because they concentrate the larger part of the resistance in this area.

Once the elastic supports of the model are defined, the dimensionless parameters Π_1 and Π_2 are analogously obtained from expression (1), in which: $T = T_Z$ is the first vertical mode obtained in the modal analysis; $EI_{d,j} = EI_{d,Z}$ is the vertical rigidity of the deck; and $K = K_{ct,Z}$. Figure 6(b) compares Π_1 versus Π_2 in all the studied FE models, distinguishing between central and lateral cable-system layouts. The optimum nonlinear relationship between the dimensionless parameters that covers both cable configurations is obtained from: $g(\Pi_1, \Pi_2) = a_1 \Pi_2^{a_2} - \Pi_1 = 0$. The analytical estimation of the first vertical period is expressed as:

$$T_Z = \sqrt{\frac{m_d L_P^4}{EI_{d,Z}}}(1.81\,\Pi_2^{0.46}) \tag{4}$$

where $\Pi_2 = EI_{d,Z}/(K_{ct,Z} L_P^3)$.

The constraint of the cable-system and the tower to the vertical movement of the deck ($K_{ct,Z}$) is composed of two counteracting effects: (1) the ideal vertical stiffness of the central span cable-system ($K_{c,Z}$) in which the longitudinal movement of the tower is considered null (Wyatt's assumption), is reduced by (2) the longitudinal flexibility of the tower restrained by the back span anchoring cables, $K_{tr,Z}$. Both systems are connected in series through the anchorage area when the load is applied along the main span, and consequently the global stiffness is:

$$K_{ct,Z} = \frac{1}{\dfrac{1}{K_{c,Z}} + \dfrac{1}{K_{tr,Z}}} \tag{5}$$

The main span cable-system stiffness $K_{c,Z}$ [N/m] is given by Wyatt:

$$K_{c,Z} = \frac{E_s m_d g H}{f_D(L_P^2 + H^2)}(1.2L_P + 47) \tag{6}$$

in which E_s and f_D are referred to the cables and represent respectively the modulus of elasticity and the average stress due to the dead load, $g = 9.81$ [m/s^2] is the gravitational constant. The term $(1.2L_P + 47)$ is a modification factor introduced herein to take into account the linearly distributed load and the point where the vertical displacement is measured in Figure 5(b) (these conditions differ from those considered by Wyatt).

The stiffness $K_{tr,Z}$ [N/m] results from the combination of the tower stiffness in longitudinal direction ($K_{t,X}$) and the stiffness introduced by the back span anchoring cables ($K_{bs,X}$). The horizontal stiffness of the tower is obtained by considering a cantilever beam with a distributed load applied at the cable anchorages, gradually decreasing from the top to the lower anchorage. The stiffness of the back span cables is obtained through Wyatt's expression. The tower and the back span cables are connected in parallel from the point of view of the calculation of the combined stiffness:

$$K_{tr,Z} = K_{t,X} + K_{bs,X} = \frac{60E_t\bar{I}_{t,X}}{21H_A^3 + 40H_A^2H_t - 70H_AH_t^2 + 20H_t^3} + \frac{E_s m_d g L_S^2}{f_D H(L_S^2 + H^2)}\frac{L_S}{N_C} \tag{7}$$

in which E_t is the Young's modulus of the tower, $\bar{I}_{t,X}$ is the moment of inertia of the tower cross-sections (considering one leg) associated with the longitudinal flexure (X) and averaged along the whole tower height, H_A is the length of the anchorage area in the tower (see Figure 2) and $H_t = H + H_i$ is the total height of the tower (from the foundation to the top). The ratio L_S/N_C gives the distance between cable anchorages in the side span. All the parameters have been described in Figures 1-2 and the previous expressions. According to Wyatt, the tower stiffness is infinite and hence: $K_{t,X} = \infty \rightarrow K_{tr,Z} = \infty$ and $K_{ct,Z} = K_{c,Z}$ in expression (5).

The first vertical period obtained in the FE models is compared with the analytical estimations. The errors are included in Figure 7(b) for different cable-stayed bridges. Again, the approach of Wyatt underestimates the vibration period in the whole main span range. It is verified that neglecting the longitudinal movement of the tower results in vertical vibration modes that can be unrealistically stiff due to certain inefficiency of the semi-harp cable-system layout. This important aspect is corrected in expression (5) by reducing the stiffness due to the longitudinal movement of the tower top. The error of the proposed vertical period estimation is introduced by the analytical approximation of the tower and cable-system restraint in equation (5). The analytical and FE results are almost coincident if the exact value of $K_{ct,Z}$ is employed in (4). Kawashima *et al.* also proposed a simple expression for the estimation of the first vertical mode in terms of the main span exclusively: $T_Z = L_P^{0.763}/33.8$ [s] (L_P in [m]). This simple expression is insensitive to many important aspects of the structure and errors above 40 % have been observed.

Fundamental torsional mode

The torsional deformation of the deck in the main span (angle θ in Figure 5(c)) activates different parts of the bridge depending on the cable-system arrangement: (i) in bridges with two lateral cable planes (LCP) the deck torsion is constrained by the differential vertical deflection of the stays; (ii) in bridges with central cable arrangement (CCP) it mobilises the torsional rigidity of the girder. The vibration period of the first torsional mode can be selected by the designer to some extent. If the bridge has two cable planes that converge to the top of inverted Y- or A-shaped towers, purely torsional deck modes require axial extensions of the stays and the associated periods are lower than those in H-shaped towers, where the two shafts allow for longitudinal differential displacements (Walther et al. 1988; Wyatt 1991; Gimsing and Georgakis 2011).

The torsional response of cable-stayed bridges has been studied in the past by distinguishing the type of cable arrangement or the tower shape, nonetheless in this work a unique physical model is proposed in order to obtain a more general analytical expression. This model is represented by a beam (the deck) with distributed mass and torsional rigidity, in which torsion is constrained between supports spaced L_{tor} [m]. The deck is restrained by the tower and the cable-system through elastic torsional springs with stiffness $K_{ct,\theta}$ [Nm/rad]. In analogy to the approach in the preceding sections, this torsional spring stiffness is obtained by means of the FE model in Figure 5(c). In this model the deck is again removed and the cable anchors in the main span are subjected to a gradually increasing load towards the span center (applied in opposite directions depending on the cable plane).

As it may be observed in expression (8), the dimensionless parameters $(\Pi_{\theta 1}, \Pi_{\theta 2})$ are slightly modified to include the radius of gyration and the torsional stiffness of the elastic supports. However, the procedure to obtain the relationship $g(\Pi_{\theta 1}, \Pi_{\theta 2}) = 0$ is analogous. Figure 6(c) shows the dimensionless parameters in the proposed FE models and the nonlinear relationship between them, which in this case is a hyperbolic function: $g(\Pi_{\theta 1}, \Pi_{\theta 2}) = a_1/(\Pi_{\theta 2} + a_2) + a_3 - \Pi_{\theta 1} = 0$. The analytical estimation of the first torsional period is expressed as:

$$T_\theta = \sqrt{\frac{m_d r^2 L_{tor}^2}{GJ_d}} \left(\frac{2.14}{\Pi_{\theta 2} + 1.11} + 0.07 \right) \quad (8)$$

where $\Pi_{\theta 2} = r^2 K_{ct,\theta} L_{tor}/(GJ_d B^2)$. The parameters r and GJ_d are respectively the radius of gyration and the torsional rigidity of the deck (G is the shear modulus and J_d the torsion constant of the deck section), L_{tor} is the length between effective torsional restraints (in this study the torsion is restrained by the intermediate piers at the side-spans, but expression (8) is also valid in other configurations) and B is the deck width.

Note that in the case of CCP models, the contribution of the cable-system and the tower to the torsional response of the deck is negligible and hence: $K_{ct,\theta} = 0$ and $\Pi_{\theta 2} = 0$. With this condition expression (8) is reduced to the classical formula to obtain the torsional period in a simple beam with the torsion totally constrained at the supports (spaced L_{tor}). This is also the approach suggested by Wyatt in CCP bridges.

Equation (9) approximates the value of the torsional spring stiffness ($K_{ct,\theta}$) without the support of a FE model. It is based on the close relationship that exists between the cable-system and the tower response when the deck is subjected to torsional or vertical movements. The ratio $B^2/2$ relates the torsional stiffness to the vertical one (this is derived from Figures 5(b) and 5(c)):

$$K_{ct,\theta} = \frac{1}{\dfrac{A_1}{K_{c,\theta}} + \dfrac{A_2}{K_{tr,\theta}}} = \frac{1}{\dfrac{A_1}{\dfrac{B^2}{2} K_{c,Z}} + \dfrac{A_2}{\dfrac{B^2}{2} K_{tr,Z}}} \tag{9}$$

in which $K_{c,Z}$ and $K_{tr,Z}$ are respectively defined in expressions (6) and (7). Depending on the inclination of the cable planes, the coefficient A_1 modifies the torsional restraint exerted by the cable-system in the central span: $K_{c,\theta}$. In the study of $K_{c,\theta}$ the differential movement of the tower top in longitudinal direction is avoided. This movement is considered in the second term of expression (9), in which the coefficient A_2 affects the contribution of the tower and back span anchoring cables to the torsional stiffness: $K_{tr,\theta}$. For CCP bridges $K_{ct,\theta} = 0$.

A parametric FE analysis has been conducted to obtain the parameters A_1 and A_2 presented in Table 1 for different tower shapes. The influence of the main span length (L_P), the deck width (B) and the deck height above the tower foundation (H_i) on these parameters is small and, consequently, the values have been averaged from the whole set of results. It is remarkable from Table 1 that only bridges with H-shaped towers allow for differential longitudinal movements of the tower shafts, whereas in the rest of the models the torsional movement of the tower is assumed negligible and thus $A_2 = 0$ (the second term in expression (9) vanishes).

The error of expression (8) in the estimation of the first torsional period of the FE models is lower than 10 %, as it is shown in Figure 7(c). Wyatt's proposal for bridges with central cable layouts (CCP) coincides with the one suggested in this work (since $K_{ct,\theta} = 0$) and the accuracy is very high. Considering bridges with lateral cable-system (LCP), Wyatt proposed a relationship between the vertical and torsional periods: $T_\theta \approx (2r/B)T_Z$. This ratio assumes completely free differential movements of the tower shafts in longitudinal direction, which is only reasonable if H-shaped towers without transverse struts are employed. For comparison purposes, this ratio is applied to all the LCP models in this work regardless of the tower shape. It is observed in Figure 7(c) that the torsional period estimated by Wyatt's procedure is unreasonably large in LCP bridges. This is explained because the torsional stiffness due to the tower shape or the transverse struts in the real model is significant. The accuracy of Wyatt's approach is worse than the analytical expression proposed in this work, but it is improved as long as the deck width is increased or the main span length is reduced in H-LCP models. This is due to the minimisation of the transverse strut constraint to the differential longitudinal movements between both shafts. On the other hand, the simple expression proposed by Kawashima *et al.* (Kawashima et al. 1993): $T_\theta = L_P^{0.453}/17.5$ [s] (L_P in [m]), leads to inadmissible underpredictions of the first torsional period, typically above 50 %.

Sensitivity to changes in the geometrical proportions

The results presented so far demonstrate the accuracy of the proposed formulation if the aspect ratios are $H/L_P = 0.21$ and $L_S/L_P = 0.4$. In order to investigate the influence of variations in the bridge proportions, additional analyses have been carried out considering the limits of the range of conventional bridges: $H/L_P = 0.19 - 0.23$ and $L_S/L_P = 0.3 - 0.5$. The model with H-shaped towers is selected in this specific study to include the possibility of differential shaft movements in torsional vibration modes.

The accuracy of the proposed expressions is not significantly affected by changes in the back to main span ratio (L_S/L_P). On the other hand, the errors in the first vertical and torsional periods increase if the tower height to main span ratio is different than 0.21. However, the error remains below 25 % in the range of conventional tower proportions: $H/L_P = 0.19 - 0.23$. The accuracy of the proposed formulation is considerably higher than that provided by previous studies considering different aspect ratios.

VERIFICATION WITH REAL CABLE-STAYED BRIDGES

Finally, the proposed formulae are verified by means of the vibration properties observed in constructed cable-stayed bridges. Table 2 includes the errors in the vibration period estimated with different formulations (T_{app}), in comparison with the real vibration periods (T_r) reported elsewhere: $e = 100(T_{app} - T_r)/T_r$. The bridge properties and the observed vibration periods (either through numerical or field ambient vibration tests) have been taken from the following authors: Quincy Bayview bridge (Pridham and Wilson 2005), International Guadiana bridge (Magalhaes et al. 2007), Megami bridge (Wu et al. 2008), Qingzhou bridge (Ren et al. 2005). The remaining information is extracted from the work of (Fan et al. 2001) and unpublished reports. Unfortunately, some of the required properties are not reported. In these specific cases reasonable values based on engineering judgement and the dimensions of constructed bridges (Figures 1-3) have been assumed. Possible deviations from the actual project conditions may modify the vibration period estimation and, consequently, the present verification simply aims to provide guidance on the expected accuracy.

The proposed formulae yield accurate results in constructed bridges and the errors are below 20 % in Table 2, with the exception of the vertical and torsional periods in three unconventional bridges in which the canonical proportions assumed for the structure are clearly not satisfied: (i) Nanjing Qinhuai bridge (ref. 1) have very short towers ($H/L_P = 0.15$, much lower than the conventional ratio assumed: 0.21); (ii) the side spans in Anqing bridge (ref. 10) are very large in comparison with the main span ($L_S/L_P = 0.56$, larger than the ratio typically employed: 0.4); on the opposite side (iii) Taoyaomen bridge (ref. 11) presents very short side spans ($L_S/L_P = 0.25$). However, the average error (in absolute value) obtained with the proposed expressions is below 15 % (including in the average the unconventional bridges), which is acceptable in the early stages of the project and improves significantly the results reported by Wyatt and Kawashima *et al.* The average deviation of the transverse, vertical and torsional periods obtained with the approach of Kawashima *et al.* is respectively 74.1, 16.6 and 27.8 %, and is not included in Table 2.

Wyatt's proposal underestimates the first transverse vibration period in almost all the studied bridges, whereas the expression proposed in this work improves significantly the results because the tower flexibility is considered. The importance of this effect on the transverse vibration mode is clear in Nanjing Qinhuai, Donghai, Megami and Jintang bridges (references 1, 7, 9 and 14 in Table 2), in which Wyatt's formula leads to unreasonably stiff vibrations. The tower and the cable-system interaction with the deck movement can also explain the accuracy of the vertical and torsional vibration periods with the new formulation. Nonetheless it is recognized that the applicability of Wyatt's formula for torsional periods is extended for comparison purposes and it is not strictly valid beyond H-shaped towers without transverse struts.

CONCLUSIONS

Fundamental vibration modes are very important in the design of cable-stayed bridges. This work proposes analytical expressions to estimate the first transverse, vertical and torsional vibration periods. The proposed formulation is completely defined in terms of the mechanical properties and proportions of the structure and it is based on the results of more than one thousand finite element models. The following conclusions were drawn:

- The tower flexibility is included in the formulation proposed to estimate the vibration periods, which was ignored in previous research works. The interaction between the towers and the deck is particularly important in the response of small-to-medium cable-stayed bridges in transverse direction. This explains the accuracy of the analytical expression proposed in this work to calculate the first transverse mode.
- The new formulation also takes into account the movement of the tower shafts in longitudinal direction when the vertical and torsional vibration periods are calculated. This is of paramount importance in bridges with harp and semi-harp cable layouts. Previous works neglected this effect and the restraint exerted by the back span anchoring cables. The analytical expressions proposed here reduce the estimation errors in light of a large parametric analysis conducted in 1050 finite element models.
- The accuracy of the proposed analytical expressions is verified in 17 real cable-stayed bridges, constructed in different countries. The observed average errors are below 15 %, which is deemed acceptable when the seismic demand and possible aerodynamic insta-

bilities are evaluated to address the viability of a preliminary design. The average results obtained with the analytical formulations proposed by other authors are significantly less accurate. The expressions proposed in this paper are valid for standard cable-stayed bridges with two towers, regardless of the materials conforming the structure, providing that aspect ratios are conventional ($L_S/L_P = 0.3 - 0.5$ and $H/L_P = 0.19 - 0.23$).

- The sections and proportions of cable-stayed bridges with different tower shapes and cable configurations are suggested through dimensionless ratios obtained from the study of a large number of constructed cable-stayed bridges. The detailed structures may represent an appropriate starting point to address the viability of the project.

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NOTATION

Main symbols employed in this paper and corresponding SI units:

B = deck width; [m]

e = error in the vibration period estimation; [%]

E_s = modulus of elasticity of the steel conforming the stays; [N/m^2]

$EI_{d,j}$ = flexure rigidity of the deck in direction j; [Nm^2]

f_D = average stress in the stays due to the dead load; [N/m^2]

GJ_d = torsional rigidity of the deck; [Nm^2]

H = tower height above the deck level; [m]

H_A = length of the anchorage area in the tower; [m]

H_i = distance between the tower foundation and the deck level; [m]

H_{tot} = distance between the tower foundation and the tower top section; [m]

$K_{t,Y}$ = transverse stiffness of the tower; [N/m]

$K_{ct,Z}$ = tower and cable-system constraint to the vertical deck flexure; [N/m]

$K_{c,Z}$ = main span cable-system constraint to the vertical deck flexure; [N/m]

$K_{tr,Z}$ = tower and back span cables constraint to the vertical deck flexure; [N/m]

$K_{t,X}$ = tower stiffness in longitudinal direction; [N/m]

$K_{ct,\theta}$ = tower and cable-system constraint to the deck torsion; [Nm/rad]

L_P = main span length; [m]

L_S = side span length; [m]

L_{tor} = deck length between effective torsional restraints; [m]

m_d = distributed mass of the deck; [kg/m]

N_C = number of stays in one cable plane;

r = deck radius of gyration; [m]

T_Y, T_Z, T_θ = transverse, vertical and torsional vibration period; [s]

α = angle between the tower leg and the transverse horizontal line (Y);

Π = dimensionless parameter in dimensional analysis.

List of Tables

TABLE 1. Parameters employed in the estimation of the transverse tower stiffness, $K_{t,Y}$ (m_Y, b_Y), and the contribution of the tower and cable-system to the torsional mode, $K_{ct,\theta}$ (A_1, A_2), for different tower shapes (keywords described in Figure 2).

	H-LCP	Y-LCP	YD-LCP	A-LCP	AD-LCP	Y-CCP	YD-CCP
m_Y	309	1177	76	2108	205	1177	76
b_Y	0	-573	-23	-1687	-154	-573	-23
A_1	1.0	2.2	2.2	2.1	2.1		
A_2	2.2	0.0	0.0	0.0	0.0		

TABLE 2. Errors [%] obtained with different analytical expressions in the estimation of the vibration periods of real bridges. Main span length L_P in [m]. Concrete towers and composite girders are employed, except in the following cases: (a) steel deck and towers; (b) concrete deck and towers; (c) steel deck and concrete towers. (1) The International Guadiana bridge is located between Spain and Portugal. (2) The deck of Minpu bridge carries two roadway levels.

			Transverse mode		Vertical mode		Torsional mode	
Bridge	System	L_P	This work	Wyatt	This work	Wyatt	This work	Wyatt
1. Nanjing Qinhuai[b] *(China)*	H-LCP	270	-13.4	-38.2	-30.2	62.4	10.4	268.9
2. Quincy Bayview *(USA, 1987)*	H-LCP	274	-1.5	-0.7	-17.4	16.3	-16.2	44.9
3. Guadiana[b] *(Spain[1], 1991)*	A-LCP	324	13.7	7.0	5.4	7.0	9.5	153.8
4. Lianyan *(China, 2006)*	H-LCP	340	-17.6	-22.9	8.9	-1.0	-13.3	-48.2
5. Haihe[b] *(China, 2001)*	H-LCP	364	-9.8	-14.5	-15.0	-20.7	x	x
6. North Runyang[c] *(China, 2005)*	YD-LCP	406	0.6	-16.0	-12.3	-29.5	16.5	47.5
7. Donghai *(China, 2005)*	YD-LCP	420	-0.7	-36.0	4.2	18.5	-17.3	12.4
8. North Hangzhou[c] *(China, 2008)*	AD-LCP	448	-6.5	-18.5	-7.7	-6.3	15.5	37.7
9. Megami[a] *(Japan, 2005)*	H-LCP	480	6.1	-44.0	-10.8	-2.6	-4.5	44.8
10. Anqing[b] *(China, 2003)*	YD-LCP	510	9.3	-6.6	-22.8	-34.7	20.8	60.7
11. Taoyaomen[c] *(China, 2003)*	AD-LCP	580	8.2	-4.4	-24.1	46.8	19.2	117.0
12. Xupu *(China, 1997)*	A-LCP	590	-4.8	-6.4	-5.4	13.3	-12.3	36.2
13. Qingzhou *(China, 2002)*	AD-LCP	605	10.2	7.0	-9.2	-2.4	-8.2	112.7
14. Jintang[a] *(China, 2009)*	YD-LCP	620	1.5	-36.2	-1.1	1.9	14.2	110.2
15. Second Nanjing[c] *(China, 2001)*	YD-LCP	628	x	x	-13.0	8.2	16.7	120.3
16. Third Nanjing[c] *(China, 2005)*	A-LCP	648	-4.7	-4.7	-9.3	8.2	-0.8	-2.7
17. Minpu[c,2] *(China, 2009)*	H-LCP	708	-8.1	-12.3	-12.0	18.9	-17.5	33.5
	Average error $\|e\|$		7.3	(-)17.2	12.3	17.6	13.3	78.2

List of Figures

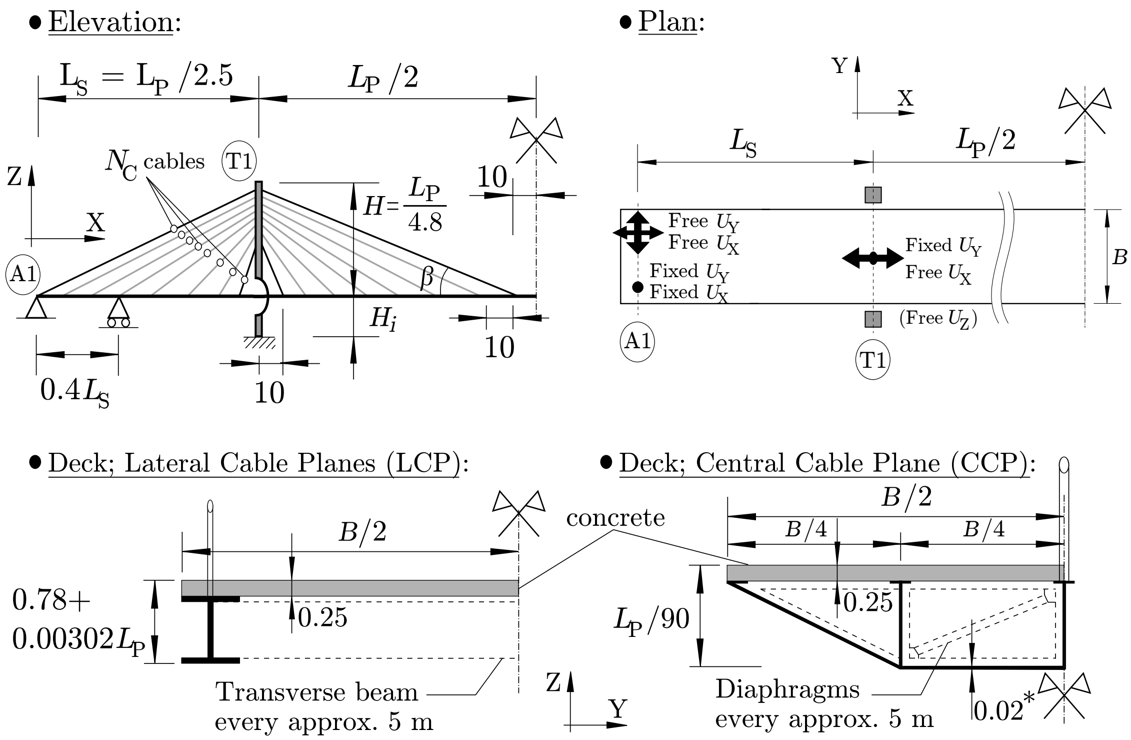


FIG. 1. Schematic bridge elevation and plan with the support conditions, besides the composite deck cross-sections employed in lateral (LCP) and central (CCP) cable configurations. Measurements in meters. Global axes are included. (*) Plate thickness should be larger at localized areas, 2 cm is a mean value for preliminary designs.

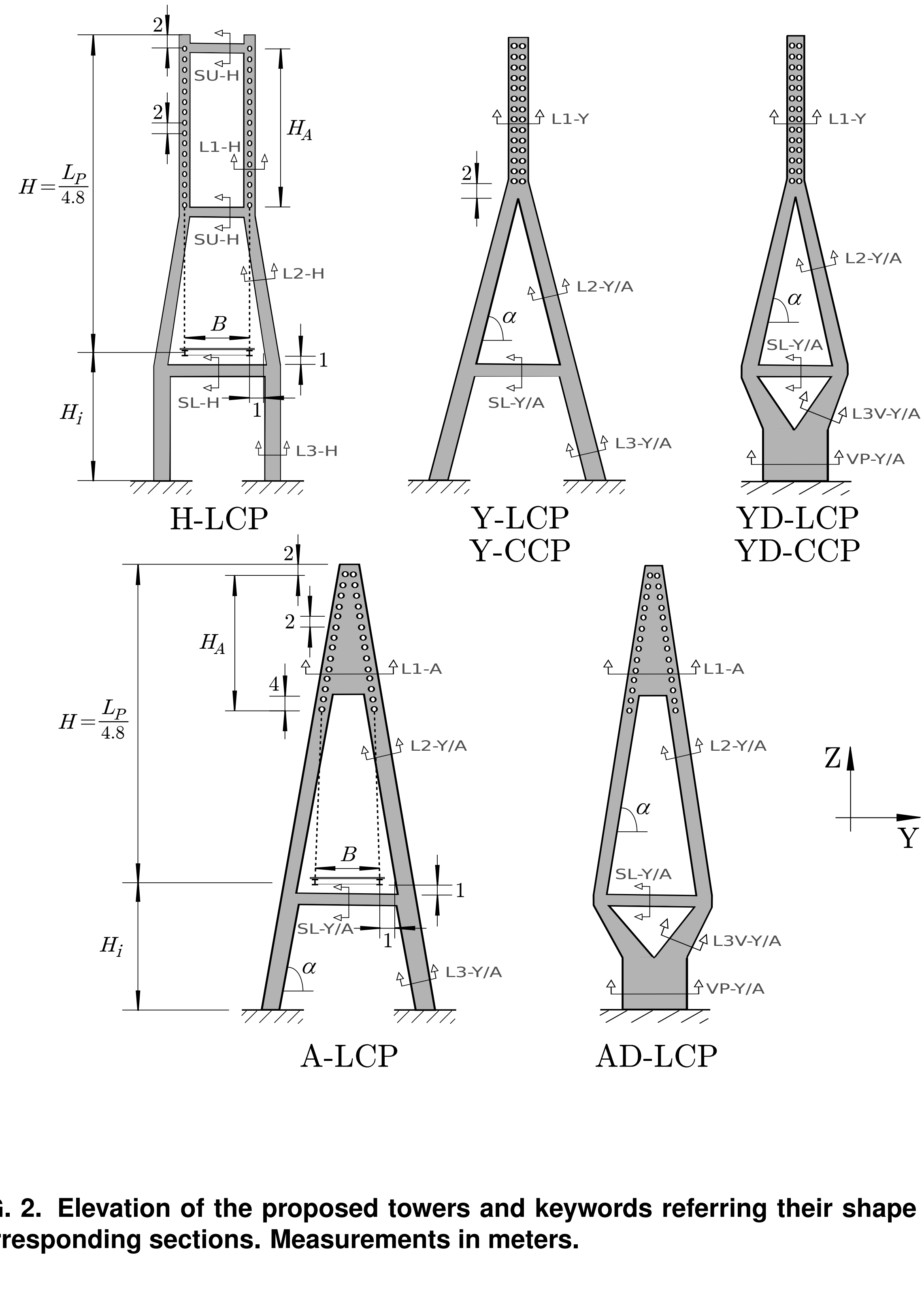


FIG. 2. Elevation of the proposed towers and keywords referring their shape and corresponding sections. Measurements in meters.

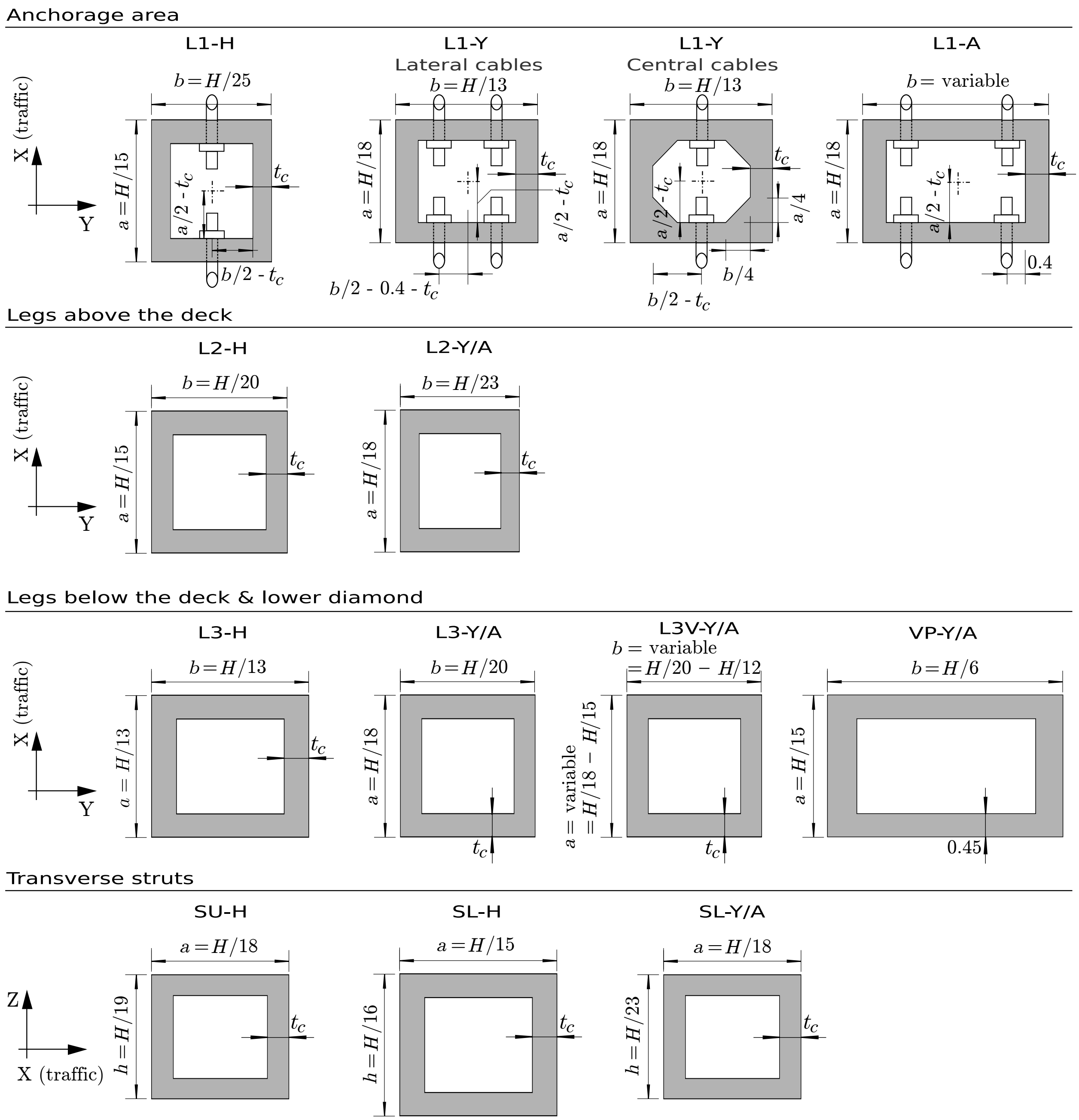


FIG. 3. Definition of tower sections. Measurements in meters.

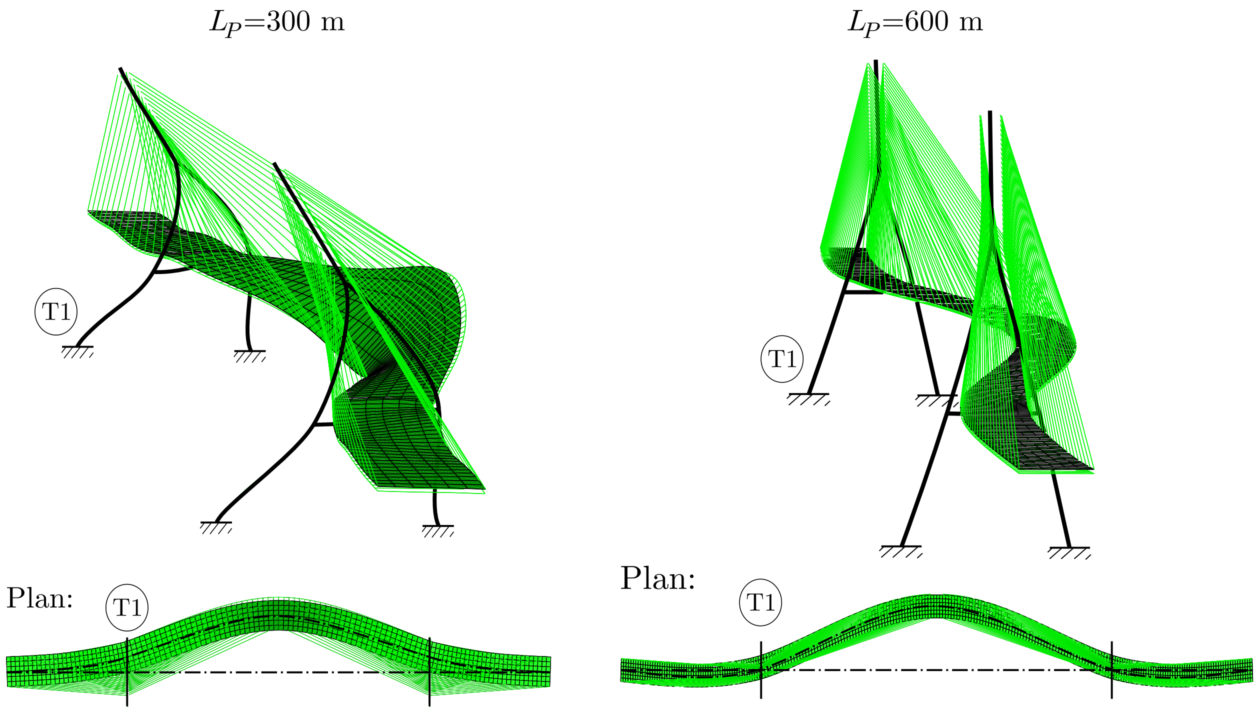


FIG. 4. First transverse vibration mode in Y-LCP models ($B = 25$ m, $H_i = H/2$ m) with a main span of 300 and 600 m.

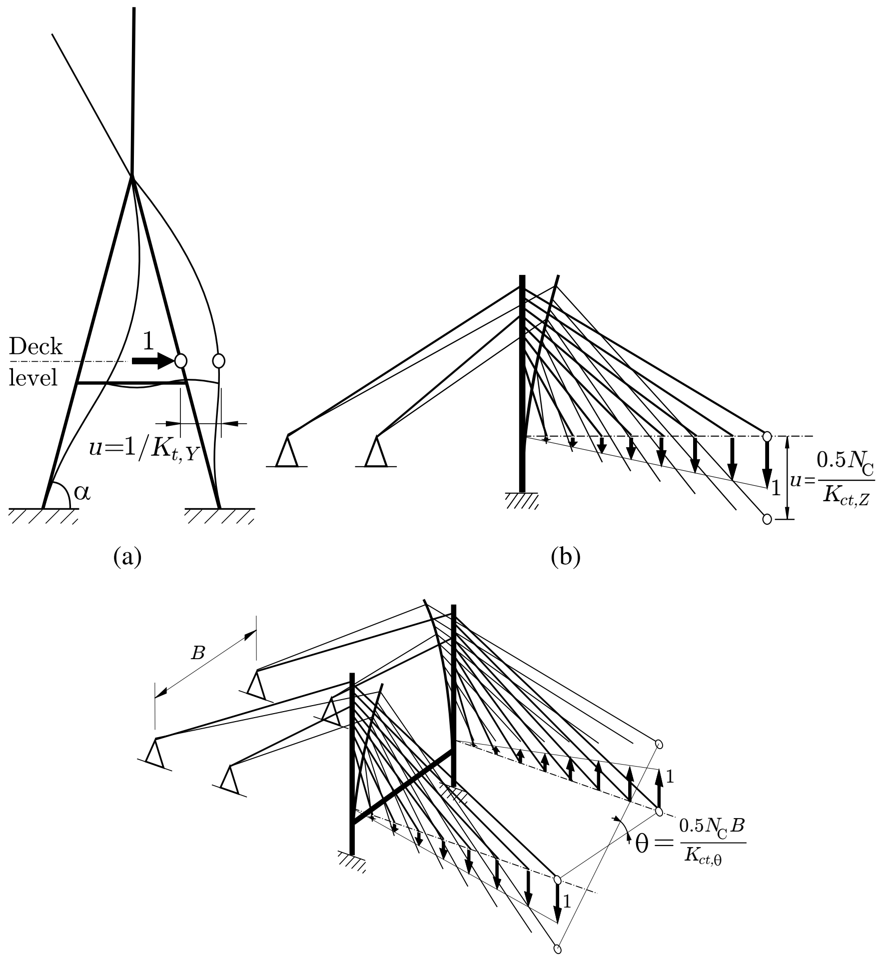
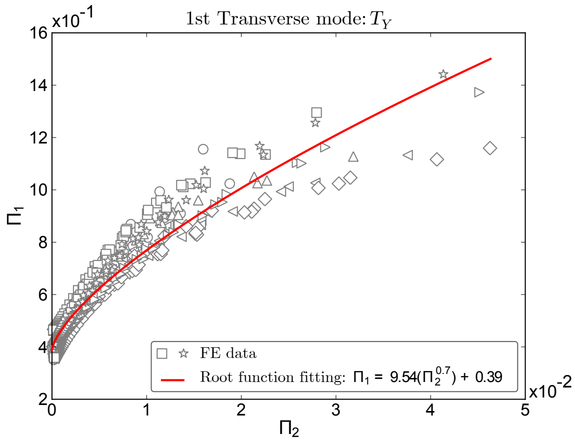
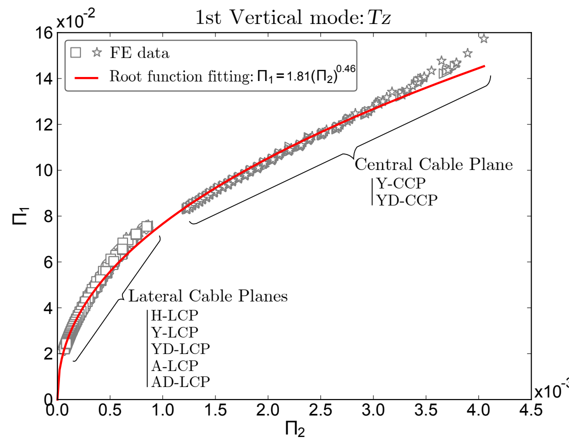


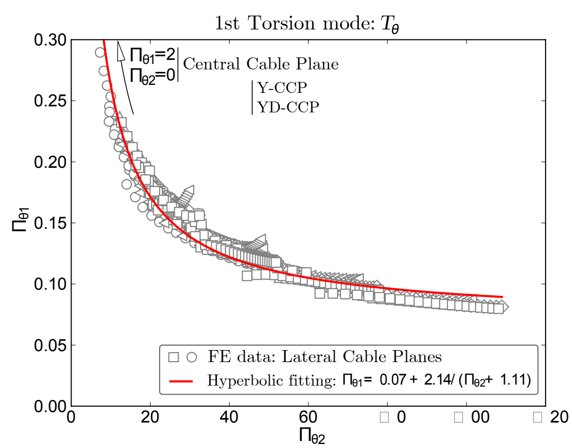
FIG. 5. Simplified FE models to define the influence of the tower and/or the cable-system on the deck deformation; (a) flexure in transverse direction (contribution of the tower); (b) flexure in vertical direction (contribution of the tower and the cable-system); (b) torsion (contribution of the tower and the cable-system). The deck is excluded from these models (in Figure 5(a) the cable-system is also removed).



(a)



(b)



(c)

FIG. 6. Least squares fitting to obtain the relationship between the dimensionless parameters $g(\Pi_1, \Pi_2)$ in the fundamental; (a) transverse mode; (b) vertical mode; (c) torsional mode. Bridge keywords described in Figure 2.

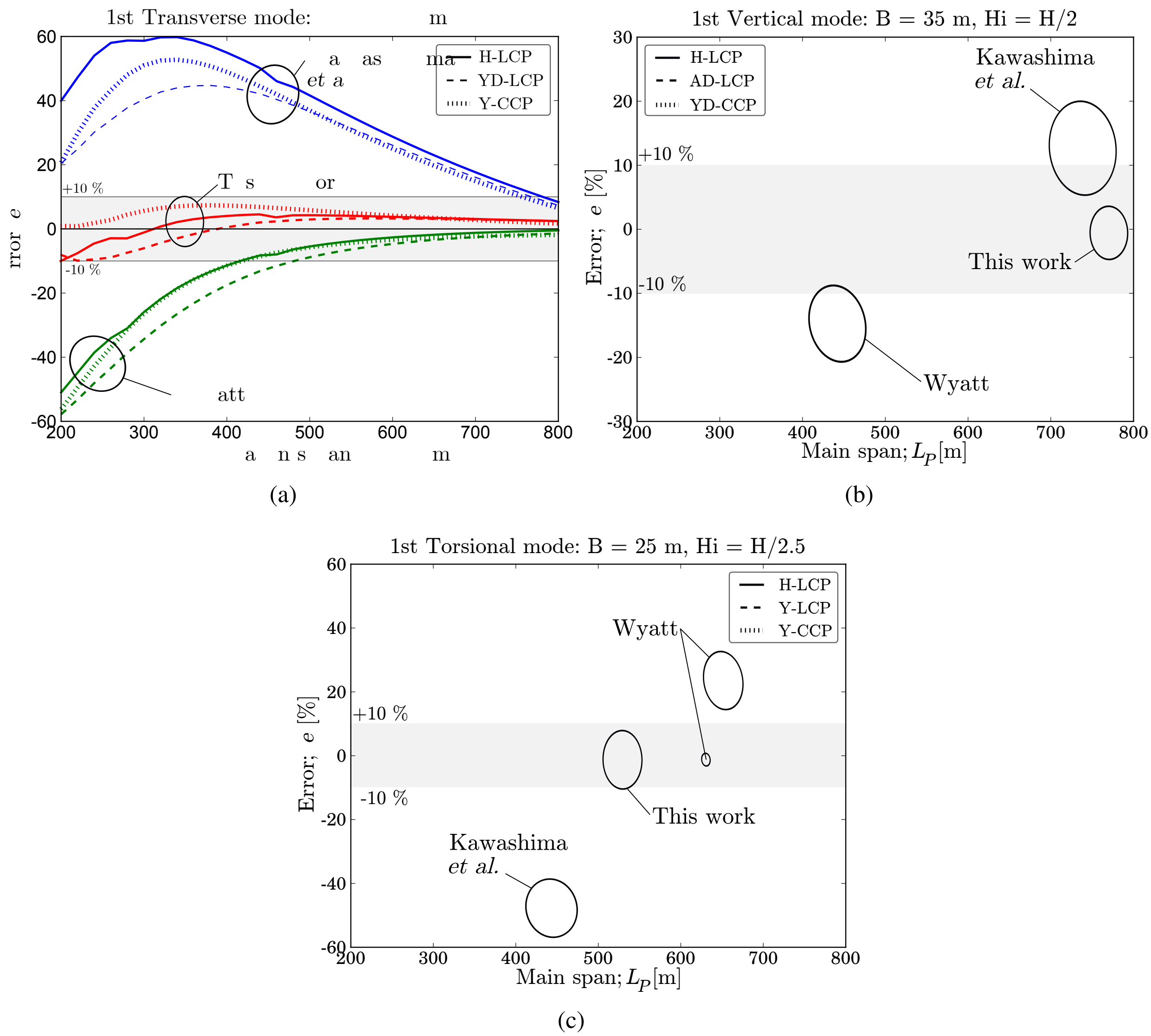


FIG. 7. Error obtained with the analytical expressions proposed by several authors in the estimation of the fundamental; (a) transverse period; (b) vertical period; and (c) torsional period. The reference 'exact' value is obtained from the FE models. Bridge keywords described in Figure 2.

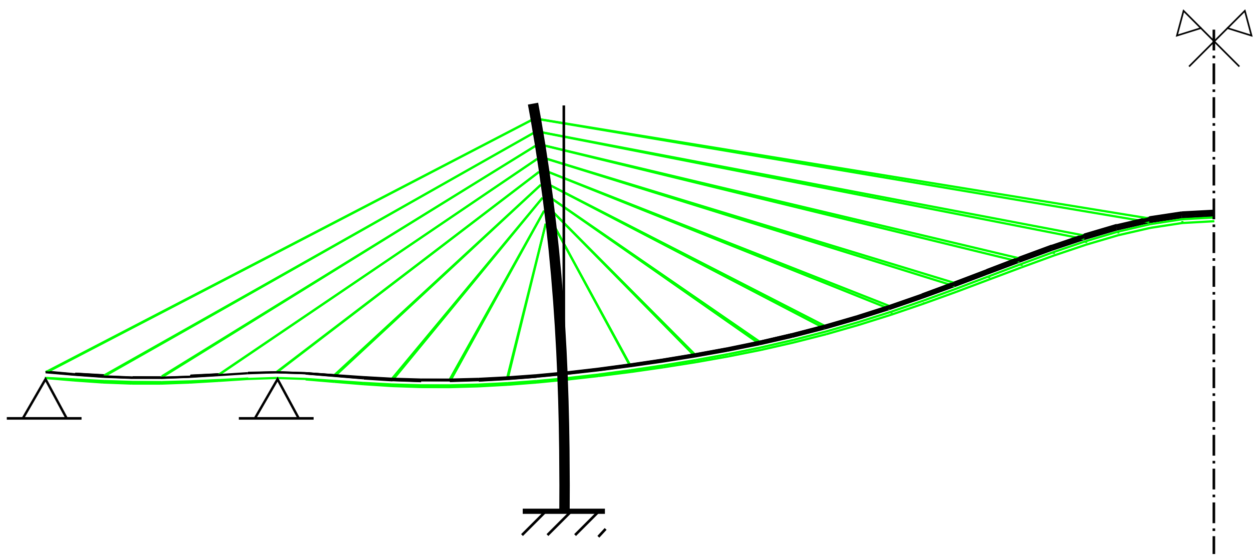


FIG. 8. First vertical vibration mode in the Y-LCP model ($B = 25$ m, $H_i = H/2$ m) with 200 m main span.