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# Orientifolds and K–theory

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**Abstract**

Recently it has been shown that D-branes in orientifolds are not always described by equivariant Real K–theory. In this paper we define a previously unstudied twisted version of equivariant Real K–theory which gives the D-brane spectrum for such orientifolds. We find that equivariant Real K–theory can be twisted by elements of a generalised group cohomology. This cohomology classifies all orientifolds just as group cohomology classifies all orbifolds. As an example we consider the $\Omega \times \mathcal{I}_4$ orientifolds. We completely determine the equivariant orthogonal K–theory $KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q})$ and analyse the twisted versions. Agreement is found between K–theory and Boundary Confromal Field Theory (BCFT) results for both integrally- and torsion-charged D-branes.

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# 1 Introduction

Brane-anti-brane annihilation [4, 9] is the physical manifestation of the equivalence relations that define K–theory. Lower dimensional D-branes can be thought of as non-trivial tachyon bundles on D$\bar{\text{D}}$9-brane pairs in Type IIB [6] or non-BPS D9-branes in Type IIA [20]. As such, the spectrum of stable D-branes is classified by K–theory [10, 6, 20]. In this construction D-branes on orbifolds are described by equivariant K–theory while D-branes in Type I and its T-duals correspond to Real K–theory.[1]

It has been known for some time that certain orbifolds admit discrete torsion [17]. These are classified by the projective representations of the orbifold group $G$, or equivalently by group cohomology[2] $H^2(G, U_1)$. The allowed choices give different closed string backgrounds and hence also different D-brane spectra. It is possible to define twisted equivariant K–theories which describe the D-brane spectrum in orbifolds with discrete torsion [6] (see also [13]. We review this construction section 3.

It was originally thought that for orientifolds of the form $\Omega \times H$, where $H$ is some orbifold group and $\Omega$ worldsheet parity, the D-brane spectrum was classified by Real equivariant K–theory [6], $KR_{\mathbb{Z}_2 \times H}(X) = KO_H(X)^3$. However, it is sometimes possible to define several distinct orientifolds for a fixed group $H$; this is somewhat similar to the discrete torsion in orbifolds. Since the closed string spectra differ for such orientifolds, the stable D-branes in these backgrounds are also distinct. It is then clear that $KR_{\mathbb{Z}_2 \times H}$ cannot describe the stable D-branes in all such backgrounds.

Our work has been motivated by the $\Omega \times \mathcal{I}_4$ orientifolds. These were originally studied by [11] before the discovery of the significance of D-branes [3]. It was found that there were essentially two such models (in the non-compact case); the massless twisted sector was found to contain either a hypermultiplet or a tensor multiplet. More recently these orientifolds were re-considered in D-brane language. In particular the hypermultiplet model was studied by Gimon and Polchinski [14] and the tensor multiplet model by Blum, Zaffaroni and Dabholkar, Park [16, 15]. We will refer to these models throughout the paper as either hyper, tensor multiplet models or GP, BZDP models.[4]

Recently all stable D-branes in the two models have been identified using BCFT techniques [1]. As expected the D-brane spectra are quite different, and preliminary results suggested that $KR_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ corresponds to D-branes in the tensor multiplet model. It was suggested that D-branes in the hypermultiplet model should correspond to a twisted version of $KR_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ in which the anti-linear $\Omega$ should *anti-*

[1]Elements of K–theory are pairs of isomorphism classes of complex bundles on a manifold; in equivariant K–theory a group acts on the bundles with the corresponding map on fibres being linear; Real K–theory or KR–theory is similar to equivariant K–theory but with an element acting anti-linearly (for example by complex conjugation) on the fibres.

[2]We use $U_1$ instead of $U(1)$ for typographical reasons.

[3]We are using the notation where the involution appears as part of the group in $KR$-theory. The equality between the Real and orthogonal K–group follows since the involution is taken to have trivial action on the manifold.

[4]For a recent review of orientifolds see [12].

commute with the linear $\mathcal{I}_4$ on the fibres. This proposal was made by analogy with $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. However, it was unclear whether such an object would in fact form a K–theory. In other words would it satisfy the usual exact sequence and periodicity properties.

The main goal of this paper is to construct twisted KR–theories for all consistent orientifolds with fixed group $G$. In the process we find a generalisation of group cohomology, so-called group cohomology with local coefficients, which classifies orientifolds for a given group $G$. We will find that, just as the different choices in orbifold theories correspond to projective representations, orientifolds are classified by projective Real representations. Given this classification we obtain twisted KR–theories which give the D-brane spectra of the orientifold theories. In particular we will apply this construction to find the twisted KR–theory which classifies stable D-branes in the GP orientifold. This construction guarantees that the twisted KR–theories satisfy the usual K–theory axioms.

In section 2 we compute $KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q})$ and show that it matches exactly with the D-brane spectrum of the BZDP model found in [1]. Section 3 contains most of our results. We start by reviewing the construction of twisted equivariant K–theories in section 3.2. We generalise this construction to the $\Omega \times \mathcal{I}_4$ orientifolds in order to find the KR–theory which corresponds to the hypermultiplet model in section 3.3. We present a general classification of orientifolds in terms of cohomology with local coefficients, and the construction of corresponding twisted KR–theories in section 3.4. In section 4 we give a physical interpretation to the choices allowed for finite abelian orientifolds in terms of phases in front of the different contributions to one-loop partition functions. We conclude and present some open problems in section 5. The paper contains several appendices where some of the technical details of our calculations are presented.

Some work on the classification of orientifolds was carried out in [22]. In [23] cohomology with local coefficients was discussed in a somewhat related, though different, context.

# 2 Computation of $KO_{\mathbb{Z}_2}$

In this section we compute the K–theory relevant to the non-compact BZDP model and show that it agrees exactly with the D-brane spectrum found using BCFT techniques. We do this in two different ways; first we use a long exact sequence similar to the one in [8], then we show that the result can be easily obtained by using the connection between Clifford Algebras and K–theory. The former method's advantage is that it identifies which D-branes carry the same charges. This is particularily useful for torsion charged D-branes. The exact sequence method however, becomes quite cumbersome and it is sometimes difficult to disentangle the results.

Following [8] we define a D$p$-brane to be a $(r, s)$-brane if it has $s/r+1$ Neumann directions on which $\mathcal{I}_4$ does/does not act and $p = r + s$. In [1] the D-brane spectrum of the BZDP orientifold was computed using BCFT. We reproduce it here for convenience

$$\begin{array}{rl} \mathbb{Z}\oplus\mathbb{Z} & r=1,5 \text{ and } s=0,4 \\ \mathbb{Z} & r=-1,3 \text{ and } s=2 \\ \mathbb{Z} & r=1,5 \text{ and } s=1,2,3 \\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & r=-1,0 \text{ and } s=0 \text{ or } r=3,4 \text{ and } s=4 \\ \mathbb{Z}_2 & r=-1,0 \text{ and } s=1 \text{ or } r=3,4 \text{ and } s=3\,. \end{array} \tag{1}$$

The first two types of D-branes are BPS and are respectively, the fractional and stuck branes. The third type of integrally charged D-brane are the non-BPS truncated branes; the torsion charged branes are also non-BPS.

In [1] it was suggested that $KO_{\mathbb{Z}_2}$ should be the K–theory which classifies such D-branes. In particular, in the non-compact theory, an $(r,s)$-brane should correspond to

$$KO_{\mathbb{Z}_2}(\mathbf{R}^{4-s,5-r})\,, \tag{2}$$

where the bundles are taken to have compact support and $\mathbb{Z}_2$ acts as

$$(x_1,\ldots,x_{4-s},x_{5-s},\ldots,x_{9-s-r})\mapsto(-x_1,\ldots,-x_{4-s},x_{5-s},\ldots,x_{9-s-r})\,. \tag{3}$$

As a first step towards computing such KO-theories we note that they are 8-periodic

$$KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q})=KO_{\mathbb{Z}_2}(\mathbf{R}^{p+8,q})=KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q+8})\,, \tag{4}$$

and that for $p=0$ the group action is trivial and we get immediately

$$KO_{\mathbb{Z}_2}(\mathbf{R}^{0,q})=KO(\mathbf{R}^{q})\otimes RO(\mathbb{Z}_2)\,, \tag{5}$$

where the real representation ring of $\mathbb{Z}_2$ is $RO(\mathbb{Z}_2)=\mathbb{Z}\oplus\mathbb{Z}$. Therefore

| $q$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $KO(\mathbf{R}^q)$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| $KO_{\mathbb{Z}_2}(\mathbf{R}^{0,q})$ | $\mathbb{Z}\oplus\mathbb{Z}$ | $\mathbb{Z}_2\oplus\mathbb{Z}_2$ | $\mathbb{Z}_2\oplus\mathbb{Z}_2$ | 0 | $\mathbb{Z}\oplus\mathbb{Z}$ | 0 | 0 | 0 |

(6)

which agrees with the spectrum of $(5-q,4)$-branes.

The other K–groups can be computed using long exact sequences. For general manifolds $Y\subset X$ our K–theory satisfies the usual long exact sequence

$$\cdots\to KO^{-1}_{\mathbb{Z}_2}(Y)\to KO_{\mathbb{Z}_2}(X,Y)\to KO_{\mathbb{Z}_2}(X)\to KO_{\mathbb{Z}_2}(Y)\to\ldots\,, \tag{7}$$

which, due to periodicity is 24–cyclic. With

$$Y=S^{1,0}\times\mathbf{R}^{p,q}\subset D^{1,0}\times\mathbf{R}^{p,q}=X\,, \tag{8}$$

this becomes[5]

$$\cdots \to KO^{-1}(\mathbf{R}^{p+q}) \to KO_{\mathbb{Z}_2}(\mathbf{R}^{p+1,q}) \to KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q}) \to KO(\mathbf{R}^{p+q}) \to \cdots . \tag{9}$$

We have used the fact that $D^{1,0}$ is contractible (in an $\mathbb{Z}_2$–equivariant way)

$$KO_{\mathbb{Z}_2}(D^{1,0} \times \mathbf{R}^{p,q}) = KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q}) \tag{10}$$

and that the $\mathbb{Z}_2$–action is free on $S^{1,0} = \{+1\} \cup \{-1\}$.

$$KO_{\mathbb{Z}_2}(S^{1,0} \times \mathbf{R}^{p,q}) = KO(\mathbf{R}^{p+q}) . \tag{11}$$

Given $KO^*_{\mathbb{Z}_2}(\mathbf{R}^{p,q})$ and $KO^*(\mathbf{R}^{p+q})$ it is then possible to deduce $KO^*_{\mathbb{Z}_2}(\mathbf{R}^{p+1,q})$. Further, for $p = 0$ the forgetting map

$$KO_{\mathbb{Z}_2}(\mathbf{R}^{0,q}) \to KO(\mathbf{R}^{q}) \tag{12}$$

is onto. From this it is easy to show that

| $q$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $KO_{\mathbb{Z}_2}(\mathbf{R}^{1,q})$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

(13)

which is in agreement with the $(5-q, 3)$-brane spectrum in equation (1). Furthermore the maps

$$KO_{\mathbb{Z}_2}(\mathbf{R}^{1,s}) \to KO_{\mathbb{Z}_2}(\mathbf{R}^{0,s}) \tag{14}$$

are one-to-one, indicting that the $\mathbb{Z}_2$ charge of the $(r, 3)$-brane is the same as (part of the) charge of the $(r, 4)$-brane for $r = 3, 4$. One may continue in this way, however it becomes increasingly difficult to solve the extension ambiguities.

Instead we will now use the connection between K–theory and Clifford algebras to compute the K–groups. One finds that [2]

$$KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q}) = KO^{(0,p+1)}(\mathbf{R}^{q}) . \tag{15}$$

In Appendix A we define $KO^{(m,n)}(X)$ and prove the above result. The above formula is useful since the object on the right hand side is purely algebraic and is well known [26]:

$$KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q}) = \begin{array}{c|cccccccc}
q{=}7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q{=}6 & 0 & 0 & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} & 0 \\
q{=}5 & 0 & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 0 \\
q{=}4 & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
q{=}3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q{=}2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & \mathbb{Z}_2 \\
q{=}1 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}_2 \\
q{=}0 & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\hline
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 & p{=}6 & p{=}7
\end{array} \tag{16}$$

[5] $S^{m,n}$ and $D^{m,n}$ are the unit sphere and disk in $\mathbf{R}^{m,n}$ with inherited $\mathbb{Z}_2$–action.

Comparing with equation (1) we see exact agreement between the spectrum of BZDP D-branes and the K–theory predictions.

# 3 Twisting in equivariant K–theory

In this section we construct a K–theory which describes the D-brane spectrum of the GP orientifold. Comparing the D-brane spectrum of the GP model (see appendix C) with the results of the previous section it is clear that this is not described by $KO_{\mathbb{Z}_2}$. Instead, we argue that D-branes in the hyper model are described by a twisted K–theory $KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$. We present a unified picture of twisting equivariant K–theories, which allows for twists involving linear as well as anti-linear group elements. We begin the section by discussing such twisting in the case of $K^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ which gives the D-brane spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion. In the following subsection we obtain $KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$. The construction is then generalised to describe generic non-compact orientifolds; as a by-product we show that these are classified by a cohomology group $H^2\big(*, \widetilde{U}_1\big)$ much in the same way as orbifolds are classified by $H^2(*, U_1)$ [17].

## 3.1 Twisting for ordinary orbifolds

Let us quickly review how group cohomology enters twists of ordinary K–theory, that is how discrete torsion alters the K–groups. This section is not strictly necessary for the understanding of the rest of the paper, we only want to recall how the twisted K–theory appearing in the analysis of WZW models [36, 37] is related to discrete torsion and projective representations.

If spacetime $X$ is a bona fide manifold, i.e. not an orbifold, then the twist is caused by a nontrivial B-field. In such a background D–branes no longer carry ordinary gauge bundles but "twisted bundles", where the transition functions $h_{ij}$ do not close [6, 38, 39]. Instead there is a $U_1$ valued function on triple overlaps such that

$$h_{jk}h_{ik}^{-1}h_{ij} = g_{ijk}\,. \tag{17}$$

Ignoring torsion in $H^3(X,\mathbb{Z})$ the twist class corresponds to the de Rahm class of the flux $\mathrm{d}B$ via the well-known identity[6]

$$[g_{ijk}] \in H^2\left(X; \underline{U_1}\right) \xrightarrow{\sim} H^3(X;\mathbb{Z}) \ni [\mathrm{d}B]\,. \tag{18}$$

Mathematically this corresponds to the statement that you can twist ordinary K–theory $K(X)$ by $H^2\big(X; \underline{U_1}\big)$.

Now orbifold string theory on $X/G$ is really $G$-equivariant theory on the manifold $X$. The relevant K–theory is equivariant K–theory $K_G(X)$ and it can be twisted by classes in the equivariant cohomology group $H^2_G\big(X; \underline{U_1}\big)$. Now for general (topologically nontrivial) manifolds $X$ all possible twists may be difficult to determine, but

[6] $\underline{U_1}$ is the sheaf of $U_1$ valued functions.

there is a rather well-understood subclass of twists. These come from pullbacks via the projection $\pi : X \to \mathrm{pt}$ from $H^2_G(\mathrm{pt}; U_1)$, that is we are interested in the image

$$\pi^* \Big( H^2_G(\mathrm{pt}; U_1) \Big) \subset H^2_G\big(X; \underline{U_1}\big) \,. \tag{19}$$

The advantage of this subclass is that

$$H^2_G(\mathrm{pt}; U_1) = H^2(BG; U_1) = H^2(G, U_1) \,, \tag{20}$$

where $BG$ is the base space of the classifying space $EG$. The connection to group cohomology yields an interpretation of spacetime twists as projective representations. Motivated by this we now turn to the classification of projective representations.

## 3.2 Twisted equivariant complex K–theory

For an orbifold group $G$ which admits projective representations (see below) there are several orbifolds consistent with modular invariance [17]. Typically it is possible to define the action of some generator $g_i$ in several different ways on a $g_j$-twisted sector ($i \neq j$) giving distinct closed string backgrounds. As a result the D-brane spectrum is distinct in each of the orbifolds. $G$-equivariant K–theory will then describe the D-brane spectrum in the orbifold without discrete torsion, and one has to define twisted versions of $K_G$ which describe the D-brane spectrum of the various orbifolds with torsion. These twisted K–theories are the Grothendieck group of isomorphism classes of bundles with a projective representation of $G$ on the fibres rather than a proper representation.

Recall that a projective representation of a finite group $G$ is a representation of the central extension of $G$ by $U_1$ such that $U_1$ acts by multiplication with a phase. In other words a projective representation of $G$ is a choice of $H$ such that the following sequence is exact

$$1 \to U_1 \xrightarrow{i} H \xrightarrow{\pi} G \to 1 \tag{21}$$

and $U_1$ is in the centre of $H$.

Choose a section $s : G \to H$ such that $\pi \circ s = \mathrm{id}_G$. This is always possible since $\pi$ is surjective, but in general $s$ will not be a group homomorphism. The failure to be a group homomorphism

$$s(g_1)s(g_2) = c(g_1, g_2)s(g_1 g_2) \tag{22}$$

defines a function $c : G \times G \to U_1$. An ordinary representation of $H$ $\rho : H \to GL_n$ defines a projective representation of $G$ via $\gamma = \rho \circ s : G \to GL_n$, which will be a "representation up to phases". In particular, as is familiar to physicists, it satisfies

$$\gamma(g_1)\gamma(g_2) = c(g_1, g_2)\gamma(g_1 g_2) \,. \tag{23}$$

Requiring that $s$ or $\gamma$ be associative restricts $c$ to satisfy

$$c(g_1, g_2)\frac{1}{c(g_1, g_2 g_3)}c(g_1 g_2, g_3)\frac{1}{c(g_2, g_3)} = 1 \,. \tag{24}$$

In group cohomology language the left hand side defines the coboundary of a two-cocycle and the above equality says that $c$ is coclosed. Further, given a function $G \to U_1$ (by abuse of notation also called $c$) we can replace $s(g) \to c(g)s(g)$ and then

$$s(g_1)s(g_2) = c(g_1,g_2)s(g_1g_2) \quad \to \quad s(g_1)s(g_2) = \frac{c(g_1g_2)}{c(g_1)c(g_2)}c(g_1,g_2)s(g_1g_2)\,. \tag{25}$$

Hence 2-cocycles that differ by

$$c(g_1)c(g_1g_2)^{-1}c(g_2) \tag{26}$$

correspond to the same extension. The above defines a coboundary of a 1-cochain in group cohomology, and therefore we identify

$$\left\{\begin{array}{c}\text{projective}\\ \text{representations of } G\end{array}\right\} \quad = \quad \left\{\begin{array}{c}\text{central extensions}\\ 1\to U_1 \to H \to G \to 1\end{array}\right\} \quad = \quad H^2\big(G,U_1\big)\,. \tag{27}$$

A twisted $G$-equivariant vector bundle is a vector bundle with the group $G$ acting by a projective representation.[7] Such bundles form a semigroup under the Whitney sum, and as usual we can define the Grothendieck group $K_G^{[H]}(X)$ corresponding to $H \in H^2\big(G,U_1\big)$. By equation ((27)) any representation of $H$ is either a proper or a projective representation of $G$. Succinctly, at the level of K-theory this implies a decomposition

$$K_H(X) = K_G(X) \oplus K_G^{[H]}(X)\,. \tag{28}$$

Since both $K_G$ and $K_H$ satisfy the usual K–theory properties, such as periodicity and long exact sequences, then so does $K_G^{[H]}$.

As an example consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with generators $g_1, g_2$. The D-brane spectrum of these orbifolds is well known [19, 24]. The groups $K_{\mathbb{Z}_2\times\mathbb{Z}_2}$ for Euclidean space have been computed and have been found to agree with the D-brane spectrum of the orbifold with no discrete torsion [19]. Since

$$H^2\big(\mathbb{Z}_2 \times \mathbb{Z}_2, U_1\big) = \mathbb{Z}_2 \tag{29}$$

there is one nontrivial projective representation given by the following choice of normalised (meaning $s(1) = 1$) section:

$$s(g_1)s(g_2) = -s(g_1g_2) = -s(g_2)s(g_1), \quad s(g_1)^2 = s(g_2)^2 = 1\,. \tag{30}$$

This projective representation is a representation of the group $D_8$

$$D_8 = \Big\{\sigma, \gamma_1, \gamma_2 \,\Big|\, \gamma_1\gamma_2 = \sigma\gamma_2\gamma_1,\ \gamma_1^2 = \gamma_2^2 = \sigma^2 = 1\Big\} \tag{31}$$

[7] One of us (BS) learnt about this approach to twisted equivariant K–theory from Burt Totaro in July 2000 [29]. We have later been informed that it has been known to others [30, 31].

where the generator $\sigma$ is represented by $-1$ (and acts trivially on the base space $X$). Since $D_8$ irreducible representations decompose into projective and ordinary representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ the K–theory of $D_8$ splits as in equation (28):

$$K_{D_8}(X) = K_{\mathbb{Z}_2\times\mathbb{Z}_2}(X) \oplus K^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}(X)\,. \tag{32}$$

The above equation not only shows that $K^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is well defined but is also useful in computing twisted equivariant K–groups if the groups act trivially on the base (e.g. $X = \mathrm{pt}$). As usual we have

$$K^i_G(\mathrm{pt}) = K^i(\mathrm{pt}) \otimes R[G]\,, \tag{33}$$

where the representation rings are [35]

$$R[D_8] = \mathbb{Z}^{\oplus 5}\,, \qquad R[\mathbb{Z}_2 \times \mathbb{Z}_2] = \mathbb{Z}^{\oplus 4}\,. \tag{34}$$

Then equation (32) yields[8]

$$\begin{aligned} K^i(\mathrm{pt}) \otimes \mathbb{Z}^5 = \Big(K^i(\mathrm{pt}) \otimes \mathbb{Z}^4\Big) \oplus K^{[D_8],i}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathrm{pt}) \\ \Rightarrow \quad K^{[D_8],i}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathrm{pt}) = K^i(\mathrm{pt}) = \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z} & i \text{ even} \end{cases} \end{aligned} \tag{35}$$

in agreement with the presence of $(2k_1+1; 2k_2, 2k_3, 2k_4)$-branes in the Type IIB orbifold with discrete torsion [24].

We note that the twisted K–theory defined by representations of $Q_8$, the unit quaternions,

$$Q_8 = \Big\{\sigma, \gamma_1, \gamma_2 \,\Big|\, \gamma_1\gamma_2 = \sigma\gamma_2\gamma_1,\ \gamma_1^2 = \gamma_2^2 = \sigma, \sigma^2 = 1\Big\} \tag{36}$$

is the same as the $D_8$-twisted K–theory defined above. This is because the two yield projective representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ that differ by a coboundary (*c.f.* equation (29)). This will be in contrast to the case of Real K–theory, as we will see in the next section.

### 3.3 Defining $KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$

In this subsection we wish to extend the construction of twisted equivariant K–groups to KR–groups. Following the above discussion it seems clear that D-branes in the GP model are described by a K–theory of bundles on which $g$ acts linearly, $\tau$ acts anti-linearly, and the two anti-commute. Consider then $D_8$-equivariant KR theory in which $g$ and $\sigma$ act linearly and $\tau$ acts anti-linearly. Such K–theories have

[8] Below, and throughtout the text, we write equalities between different K-groups. These are meant to indicate that there is an isomorphism between such groups. It should be however, understood that such equalities need not map generators to generators; for example presently in the equality between $K^{[D_8],i}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathrm{pt})$ and $K^i(\mathrm{pt})$ the generator of the twisted group gets mapped to twice the generator of the complex K–theory.

been considered in the mathematical literature [27, 32]. Any representation[9] of $D_8$ is either a representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ or a projective representation ($\tau$ and $g$ anti-commute). At the level of K–theory this is

$$KR_{D_8}(X) = KR_{\mathbb{Z}_2\times\mathbb{Z}_2}(X) \oplus KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}(X)\,. \tag{37}$$

As in the previous section this guarantees that $KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is a K–theory.

In order to confirm that $KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is the K–theory which describes D-branes in the hypermultiplet model we shall compute it on $\mathbf{R}^{0,i}$, on which $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts trivially. There are three irreducible representations of $D_8$ with $\tau$ acting anti-linearly (we denote complex conjugation by $\delta$)

| $g$ | $1$ | $\sigma$ | $g$ | $\tau$ |
|---|---|---|---|---|
| $r_1(g)$ | $(1)$ | $(1)$ | $(1)$ | $(1)\circ\delta$ |
| $r_2(g)$ | $(1)$ | $(1)$ | $(-1)$ | $(1)\circ\delta$ |
| $r_3(g)$ | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ | $\begin{pmatrix}0&i\\-i&0\end{pmatrix}$ | $\begin{pmatrix}1&0\\0&1\end{pmatrix}\circ\delta$ |

(38)

The analogue of equation (33) in KR–theory is [32]

$$KR_G(X) = \Big(A_G \otimes KO(X)\Big) \oplus \Big(B_G \otimes K(X)\Big) \oplus \Big(C_G \otimes KSp(X)\Big)\,, \tag{39}$$

where the representation ring of $G$ decomposes as

$$R[G] = A_G \oplus B_G \oplus C_G \tag{40}$$

according to commuting field $\mathbf{R}, \mathbf{C}, \mathbb{H}$. Both 1-dimensional representations have commuting field $\mathbf{R}$ and one can easily show that the commuting field for $r_3$ is $\mathbf{C}$.[10] Then equations (37) and (39) gives

$$\left.\begin{array}{l} KR_{\mathbb{Z}_2\times\mathbb{Z}_2}(X) = KO(X) \oplus KO(X) \\ KR_{D_8}(X) = KO(X) \oplus KO(X) \oplus K(X) \end{array}\right\} \Rightarrow\; KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}(X) = K(X) \tag{42}$$

In particular for $X = \mathbf{R}^i$ with trivial group action we find that

$$KR^{[D_8],i}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathrm{pt}) = K^i(\mathrm{pt}) = \left\{\begin{array}{ll} 0 & i \text{ odd} \\ \mathbb{Z} & i \text{ even}\end{array}\right. \tag{43}$$

which is in agreement with the presence of $\mathbb{Z}$ charged (2k+1,4)-branes in the GP model [1] (see also (80)).

[9] We use the word representation somewhat loosely here as $\gamma(\tau)$ acts anti-linearly on a vector space. Really we are talking about Real representations; we shall give more precise definitions in the next section.

[10] The (complex) matrices commuting with $r_3$ are

$$\mathbb{F}_3 \stackrel{\text{def}}{=} \mathbf{R}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \mathbf{R}\begin{pmatrix}0&1\\-1&0\end{pmatrix} \simeq \mathbf{C} \tag{41}$$

so the commuting field is $\mathbf{C}$.

Repeating the above construction for the unit quaternions $Q_8$ to obtain $KR^{[Q_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ one comes across a surprise. As we show in appendix C

$$KR^{[Q_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathrm{pt}) = KSp_{\mathbb{Z}_2}(\mathrm{pt}) \neq KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathrm{pt})\,. \tag{44}$$

This stems from the fact that projective representations of $\mathbb{Z}_2\times\mathbb{Z}_2$ which are representations of $D_8$ (with $\tau$ acting anti-linearly) are not equivalent modulo coboundaries to those of $Q_8$. Clearly then we need a generalisation of $H^2(G,U_1)$ to classify all inequivalent such representations.

## 3.4 Classification of Orientifolds

In the previous subsection we have shown that projective representations of $\mathbb{Z}_2\times\mathbb{Z}_2$ in which one of the generators acts anti-linearly differ significantly from linear projective representations. In this subsection we will generalise group cohomology to take into account such differences. This will allow us to classify the analogue of discrete torsion in orientifolds, and to obtain suitable twistings of KR–theory which will describe D-branes in such models.

An orientifold group has linear and anti-linear elements. To keep track of which act linearly and which anti-linearly we will use the notion of an *augmented* group, that is a group together with a homomorphism $\epsilon: G\to\mathbb{Z}_2$. A Real representation of $G$ on some complex vector space $V$ associates to each $g\in G$ a linear or antilinear map $V\to V$ depending on whether $\epsilon(g)=+1$ or $\epsilon(g)=-1$.

As before we want $G$ to act "up to phases". By the same reasoning as in section 3.2 this means we have to find an extension

$$\begin{tikzcd} 1 \arrow[r] & U_1 \arrow[r, "i"] & H \arrow[r, "\pi"] \arrow[dr, "\epsilon' \overset{\mathrm{def}}{=} \epsilon\circ\pi"'] & G \arrow[r] \arrow[d, "\epsilon"] & 1 \\ & & & \mathbb{Z}_2 & \end{tikzcd} \tag{45}$$

However, there are two important differences compared to equation (21):

- $G$ is now an augmented group, and $H$ inherits an augmentation $\epsilon'$.
- The extension is no longer central: complex conjugation acts on the $U_1$.

Since anti-linear elements act by complex conjugation on $U_1$-phases, conditions (24) and (26) have to be modified. In particular the differentials of 1- and 2-cochains are

$$\big(\mathrm{d}c\big)(g_1,g_2) = c(g_1)\frac{1}{c(g_1g_2)}g_1\circ c(g_2) \tag{46a}$$

$$\big(\mathrm{d}c\big)(g_1,g_2,g_3) = c(g_1,g_2)\frac{1}{c(g_1,g_2g_3)}c(g_1g_2,g_3)\frac{1}{g_1\circ c(g_2,g_3)} \tag{46b}$$

where

$$g\circ c(h) \overset{\mathrm{def}}{=} \begin{cases} \overline{c(h)} & \text{if } \epsilon(g)=1 \\ c(h) & \text{if } \epsilon(g)=0 \end{cases} \tag{47}$$

Mathematically this is well-known as group cohomology with local coefficients (by abuse of notation again denoted $H^*(G, F)$) where the group in the first slot, $G$, acts on the second, $F$.[11] We will use $\widetilde{U}_1$ to denote the "$U_1$ with action on it". Then

$$H^2\left(G, \widetilde{U}_1\right) \tag{48}$$

classifies all inequivalent non-compact $G$ orientifolds. Further a non-trivial projective Real representation of $G$ gives a Real representation of some group $H$ and hence an element $[H] \in H^2\big(G, \widetilde{U}_1\big)$, just as in equation (27). This may be used to construct $KR_G^{[H]}$, the K–theory which gives the D-brane spectrum of this particular $G$ orientifold

$$KR_H(X) = KR_G(X) \oplus KR_G^{[H]}(X)\,. \tag{49}$$

In appendix D we compute the cohomology of the most general finite abelian orientifold group, in particular we find

$$H^2\left(\mathbb{Z}_2 \times \mathbb{Z}_2, \widetilde{U}_1\right) = \mathbb{Z}_2 \oplus \mathbb{Z}_2\,. \tag{50}$$

As we have seen the projective Real representations given by $[Q_8]$ and $[D_8]$ are inequivalent, and so they can be taken as the generators of $H^2\big(\mathbb{Z}_2 \times \mathbb{Z}_2, \widetilde{U}_1\big)$. From the explicit 2-cocycles (see appendix B) we can then identify the following inequivalent projective Real representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$,

$$\begin{aligned}
[(\mathbb{Z}_2)^3]: &\quad g^2 = 1,\ \tau^2 = 1,\ g\tau = \tau g\,,\\
[D_8]: &\quad g^2 = 1,\ \tau^2 = 1,\ g\tau = -\tau g\,,\\
[Q_8]: &\quad g^2 = -1,\ \tau^2 = -1,\ g\tau = -\tau g\,,\\
[\mathbb{Z}_2 \times \mathbb{Z}_4]: &\quad g^2 = -1,\ \tau^2 = -1,\ g\tau = \tau g\,.
\end{aligned} \tag{51}$$

As a result there are four inequivalent twisted $KR_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ theories. Those twisted by $(\mathbb{Z}_2)^3[]$ (i.e. untwisted) and $[D_8]$ give the D-brane spectrum of the tensor and hyper models. Space-time filling branes in the $[Q_8]$ twisted theory are classified by $KSp_{\mathbb{Z}_2}$. This describes D-branes in the $\mathcal{I}_4$ orbifold of the Type I theory with $Sp$ gauge group, and a twsited sector tensor multiplet. Similarly then the theory twisted by $[\mathbb{Z}_2 \times \mathbb{Z}_4]$ gives the D-brane spectrum of the $\mathcal{I}_4$ orbifold of the Type I theory with $Sp$ gauge group, and a twisted sector hypermultiplet. In appendix C, generalising [1], we have computed the D-brane spectrum of both the $Sp$ orientifolds, and (partially) matched it with the corresponding twisted KR–theories.

As a further example to illustrate the construction consider orientifolding Type IIB by $\Omega$. One easily shows that

$$H^2\left(\mathbb{Z}_2, \widetilde{U}_1\right) = \mathbb{Z}_2\,. \tag{52}$$

[11] In [33] it was noted that this possibility so far did not appear in the physics literature, but the case at hand shows that it is indeed necessary.

The untwisted KR–theory is simply

$$KR_{\mathbb{Z}_2}(X) = KO(X)\,, \tag{53}$$

which indeed describes D-branes in the Type I $SO$ theory. The KR–theory twisted by the generator of $H^2\big(\mathbb{Z}_2, \widetilde{U}_1\big)$ is

$$KR_{\mathbb{Z}_2}^{[\mathbb{Z}_4]}(X) = KSp(X)\,, \tag{54}$$

which gives the D-brane spectrum of the Type I $Sp$ theory. This example was also discussed in [34].

# 4 Physical interpretation

In the previous section we have shown that for an orientifold group $G$ there are $H^2(G, \widetilde{U}_1)$ different models, and that given a particular such orientifold $[H] \in H^2(G, \widetilde{U}_1)$ we can construct the K–theory $KR_G^{[H]}$ which classifies the stable D-branes in it. In appendix D we have computed $H^2(G, \widetilde{U}_1)$ for the most general finite abelian orientifold group. In this section we analyse the various one-loop partition functions in the orientifold and identify the places where a choice of phase is allowed without spoiling the properties of such partition functions. We will show that the physically acceptable choices are isomorphic to elements of $H^2(G, \widetilde{U}_1)$.

The most general finite abelian orientifold group $G$ is generated by anti-linear elements $t_1, \cdots, t_a$ and linear elements $s_1, \cdots, s_b$. This is equivalent to the orientifold group generated by $t, g_1, \cdots, g_n, h_1, \cdots, h_m$, where $t$ is the only anti-linear element (of even order), the $g_i$ are linear even-order elements and the $h_i$ are linear odd-order elements.[12] In appendix D we show that

$$H^2(G, \widetilde{U}_1) = \Big(\mathop{\oplus}_{i=1}^{n} \mathbb{Z}_2\Big) \oplus \mathbb{Z}_2 \oplus \Big(\mathop{\oplus}_{i<j=1}^{n} \mathbb{Z}_2\Big) = \mathbb{Z}_2^{\oplus n(n+1)/2+1}\,. \tag{55}$$

We will interpret each of the three terms in the middle of the above equation as coming from phase choices in front of the various one-loop partition functions.

Consider first the torus amplitude. In an orbifold given two generators $K$ and $L$ of order $k$ and $l$ it is possible to choose a phase $\omega^p = \exp(2\pi i p/\gcd(k,l))$ with $p = 1, \cdots, \gcd(k,l)$ in front of the torus amplitude

$$\omega^p \mathrm{Tr}^K \left(L e^{-tH_c}\right)\,. \tag{56}$$

The trace is taken over the $K$-twisted sector with $L$ inserted and $H_c$ is the closed string Hamiltonian. This phase effectively changes the action of $L$ on the $K$-twisted groundstate. Then for $K$ and $L$ to remain order $k$ and $l$ respectively, various other parts of the torus amplitude will change their phases. For example we will have

$$\omega^{-p} \mathrm{Tr}^K \left(L e^{-tH_c}\right)\,. \tag{57}$$

[12] This equivalence holds as $< t_1, \cdots, t_a >$ generates the same group as $< t_1, t_1 t_2 \cdots, t_1 t_a >$ and $t_1 t_i$ is a linear even-order element.

Recently it has been shown [22] that orbifolds with discrete torsion different from $\pm 1$ cannot be consistently projected by $\Omega$. The argument also applies to more general anti-linear elements $t$. From it we see that the $\oplus_{i<j=1}^{n} \mathbb{Z}_2$ factor in equation (55) comes from the orbifold discrete torsion which is allowed for an orientifold background. Hence, as in [17], the phase in front of the $g_1$-twisted sector amplitude with $g_2$ inserted is proportional to $c(g_1, g_2)/c(g_2, g_1)$ where $c \in H^2\big(G, \widetilde{U}_1\big)$. In particular it is worth noting that there is no consistent discrete torsion between two odd-order elements.

An anti-linear order two element $\tau \in G$ gives rise to an Orientifold plane coupling to the untwisted sectors.[13] As is well known the overall sign of the normalisation of this crosscap state can be freely chosen; for example this choice of sign in the $\Omega$ orientifold of Type IIB gives the Type I theory with $SO$ or $Sp$ gauge groups. It is easy to show that for anti-linear $\tau \in G$ with $\tau^2 = 1$ $c(\tau, \tau) = \pm 1$[14] and as a result the corresponding Möbius strip amplitude has the phase

$$c(\tau, \tau)\mathrm{Tr}\left(\tau e^{-tH_o}\right) , \tag{60}$$

where $H_o$ is the open string Hamiltonian. Such phase choice is possible for $t \in G$ as well as for $tg_i \in G$ if the order of these elements is $4k + 2$.

For an anti-linear element $\tau \in G$ of order $4k$ the difference of signs between the Klein bottle amplitudes

$$\mathrm{Tr}^{\tau^2}\left(\tau e^{-tH_c}\right) \qquad \text{and} \qquad \mathrm{Tr}^{\tau^2}\left(\tau^{2k+1} e^{-tH_c}\right) \tag{61}$$

gives two different, consistent models (see for example [41])[15]. With a bit more work it is possible to show as above that $c(\tau,\tau)/c(\tau^{2k+1}, \tau^{2k+1}) = \pm 1$, and this is the cocycle contribution which keeps track of this choice.

We can now explain the appearance of $\left(\oplus_{i=1}^{n} \mathbb{Z}_2\right) \oplus \mathbb{Z}_2$ in equation (55). Each element $t, tg_1, \cdots, tg_n$ is even-order and anti-linear. If its order is $4k + 2$ then we may choose a phase proportional to $c(tg_i, tg_i)$ which governs the overall sign of the orientifold plane. On the other hand if it is of order $4k$ we may choose a sign proportional to $c(tg_i, tg_i)/c((tg_i)^{2k+1}, (tg_i)^{2k+1})$ as described in the previous paragraph. Either way each $t, tg_1, \cdots, tg_n$ gives rise to a $\mathbb{Z}_2$ choice. In total this reproduces the $\left(\oplus_{i=1}^{n} \mathbb{Z}_2\right) \oplus \mathbb{Z}_2$ factor in equation (55).

Finally, it is possible to convince oneself that there are no other phase choices that we can make consistently. For example in the $\Omega \times \mathbb{Z}_3$ orientifold we may only choose the overall sign of the O9-planes. One might think that the natural phase

---

[13] For example $\Omega\mathcal{I}_n$ gives rise to an O$(9-n)$-plane described by the crosscap state

$$|O(9-n)\rangle = |C(9-n)\rangle_{\text{NS-NS}} + |C(9-n)\rangle_{\text{R-R}} \ . \tag{58}$$

[14] With $g_1 = g_2 = g_3 = \tau$ the cocycle condition (46b) becomes

$$c(\tau, \tau)c(1, \tau) = c(\tau, 1)\bar{c}(\tau, \tau) \, . \tag{59}$$

Since $c(1, \tau) = c(\tau, 1) = 1$ the above implies $c(\tau, \tau) = \bar{c}(\tau, \tau) = \pm 1$.

[15] We thank A. Uranga for a discussion on this.

$e^{2\pi i/3}$ could appear in the twisted sector crosscaps. However, the square of this phase appears in the untwisted sector Klein bottle with $g$ (the generator of $\mathbb{Z}_3$) or $g^2$ inserted. The action of $g$ on the untwisted sector is unique and hence we cannot pick up a phase here. This argument can be extended to show that indeed the above choices are the only ones which are consistent.

# 5 Conclusion and Outlook

We have computed $KR_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathbf{R}^{p,q})$ which classifies D-branes in the $\mathcal{I}_4\times\Omega$ orientifold with twisted sector tensor multiplet, and found complete agreement with the BCFT results [1]. We have also constructed a twisted version of this KR–theory relevant to the model with twisted sector hypermultiplet. In the process we have identified a type of cohomology which classifies orientifolds, in a similar way to the classification of orbifolds by the second group cohomology [17]. We have presented a unified approach towards twisting complex and Real K–theories. This procedure allows for the straightforward identification of K–groups relevant to orbifolds and orientifolds with discrete torsion. We have found places in the various one-loop diagrams where $\pm 1$ phases may be introduced and have shown that for finite abelian orientifold groups this freedom is precisely described by cohomology with local coefficients.

In compact orientifolds it was shown that not all possible orientifolds are allowed [42]. In particular there are global conditions which only allow configurations with $8k$ hypermultiplets and $32-8k$ tensor multiplets ($k=0,1,2,3,4$). It would be very interesting to show that these results follow from the cohomology we have presented here. Perturbative orientifolds with the same orientifold group differ from one another by the presence of discrete background $B_{\mu\nu}$ fields, it should be possible to make this connection more precisely. In particular it would be interesting to understand better how the ten-dimensional Type I $SO$ and $Sp$ theories are connected. Finally, it should prove instructive to try to obtain the classification by cohomology with local coefficients from considering 'modular transformations' of two-loop non-oriented diagrams, in a similar way to the discrete torsion results found in [17]. We hope to return to these results in the near future [44].

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# A Clifford algebras and $K^{p,q}$

### Review of Clifford algebras

Our notation is based on [27, 25] but without the category–theoretic language, see also [2]. For completeness we review it here:

**Definition 1.** *The Clifford algebra $\mathbf{C}_{\mathbf{R}}^{p,q}$ is the real algebra generated by $\gamma_1, \ldots, \gamma_{p+q}$ subject to the relations*

$$\begin{aligned} \gamma_i\gamma_j &= -\gamma_j\gamma_i && \forall\ i \neq j \\ \gamma_i^2 &= -1 && \forall\ i \in \{1, \ldots, p\} \\ \gamma_i^2 &= +1 && \forall\ i \in \{p+1, \ldots, p+q\} \end{aligned} \tag{62}$$

The Clifford algebras enjoy the following well–known properties:

$$\mathbf{C}_{\mathbf{R}}^{p+n,q+n} \simeq \mathrm{Mat}_{2^n}\left(\mathbf{C}_{\mathbf{R}}^{p,q}\right) \tag{63a}$$

$$\mathbf{C}_{\mathbf{R}}^{p+8,q} \simeq \mathbf{C}_{\mathbf{R}}^{p,q+8} \simeq \mathrm{Mat}_{16}\left(\mathbf{C}_{\mathbf{R}}^{p,q}\right) \tag{63b}$$

So there are only finitely many cases to determine, the complete list is in table 1 (see e.g. [28]). Note that the notation also reflects the multiplication in the Clifford algebra; For example $\mathbf{C}_{\mathbf{R}}^{0,1}$ is the algebra of pairs $(x_1, x_2) \in \mathbf{R} \oplus \mathbf{R}$ with componentwise multiplication. Especially $(1,0)(0,1) = (0,0)$.

| $n$ | $\mathbf{C}_{\mathbf{R}}^{n,0}$ | $\mathbf{C}_{\mathbf{R}}^{0,n}$ |
|---|---|---|
| 0 | $\mathbf{R}$ | $\mathbf{R}$ |
| 1 | $\mathbf{C}$ | $\mathbf{R} \oplus \mathbf{R}$ |
| 2 | $\mathbb{H}$ | $\mathrm{Mat}_2(\mathbf{R})$ |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathrm{Mat}_2(\mathbf{C})$ |
| 4 | $\mathrm{Mat}_2(\mathbb{H})$ | $\mathrm{Mat}_2(\mathbb{H})$ |
| 5 | $\mathrm{Mat}_4(\mathbf{C})$ | $\mathrm{Mat}_2(\mathbb{H}) \oplus \mathrm{Mat}_2(\mathbb{H})$ |
| 6 | $\mathrm{Mat}_8(\mathbf{R})$ | $\mathrm{Mat}_4(\mathbb{H})$ |
| 7 | $\mathrm{Mat}_8(\mathbf{R}) \oplus \mathrm{Mat}_8(\mathbf{R})$ | $\mathrm{Mat}_8(\mathbf{C})$ |

Table 1: List of Clifford algebras

Now a $\mathbf{C}_{\mathbf{R}}^{p,q}$ vector bundle is an ordinary vector bundle $E$ with an action of the Clifford algebra, that is a map $\rho : \mathbf{C}_{\mathbf{R}}^{p,q} \to \mathrm{End}(E)$. With other words the Clifford algebra acts on the fibres of $E$ via $\rho(\gamma_i) : E_x \to E_x$.

You can add $\mathbf{C}_{\mathbf{R}}^{p,q}$ bundles in the usual way, so by the usual Grothendiek construction we get K–theory for bundles with $\mathbf{C}_{\mathbf{R}}^{p,q}$ action, denoted $KO^{(p,q)}(X)$. But those are not very interesting groups: A real bundle with $\mathbf{R}$, $\mathbf{C}$ or $\mathbb{H}$ action is simply a real, complex or quaternionic bundle and the semigroups of bundles and semigroups of bundles with some matrix algebra action are isomorphic[16]. Furthermore if the

[16] This can be seen as follows: By the $\mathrm{Mat}_n(\mathbf{R})$ action one can decompose a bundle $E$ in the sum of $n$ isomorphic bundles $E = \oplus_{i=1}^n E^i$. With the correspondence $E \leftrightarrow E^1$ we can identify the semigroups.

Clifford algebra is the sum of two orthogonal pieces (like $\mathbf{R} \oplus \mathbf{R}$) we can use the action $\rho(1,0)$, $\rho(0,1)$ of the projectors $(1,0)$ and $(0,1)$ to decompose the bundle into the sum of two independent bundles. So from the last column of table 1 we can simply read of the K–groups in table 2.

$$
\begin{array}{ll}
KO^{(0,0)}(X) = KO(X) & KO^{(0,1)}(X) = KO(X) \oplus KO(X) \\
KO^{(0,2)}(X) = KO(X) & KO^{(0,3)}(X) = K(X) \\
KO^{(0,4)}(X) = KSp(X) & KO^{(0,5)}(X) = KSp(X) \oplus KSp(X) \\
KO^{(0,6)}(X) = KSp(X) & KO^{(0,7)}(X) = K(X)
\end{array}
$$

Table 2: List of the $\mathbf{C}_{\mathbf{R}}^{0,q}$ K–groups

More interesting are the groups $KO^{p,q}(X)$ which fit into the long exact sequence associated to the "restriction of scalars" $r : \mathbf{C}_{\mathbf{R}}^{p,q+1} \to \mathbf{C}_{\mathbf{R}}^{p,q}$:

$$
KO^{(p,q+1)}(X \times \mathbf{R}) \xrightarrow{r} KO^{(p,q)}(X \times \mathbf{R}) \to KO^{p,q}(X) \to KO^{(p,q+1)}(X) \xrightarrow{r} KO^{(p,q)}(X) \tag{64}
$$

So $KO^{p,q}(X)$ is represented by formal differences of $\mathbf{C}_{\mathbf{R}}^{p,q+1}$ vector bundles that are the same if considered as $\mathbf{C}_{\mathbf{R}}^{p,q}$ vector bundles. Another way to think about that is the following: A $\mathbf{C}_{\mathbf{R}}^{p,q+1}$ vector bundle structure on a given $\mathbf{C}_{\mathbf{R}}^{p,q}$ vector bundle $E$ is equivalent to a gradation on $E$:

**Definition 2.** *A gradation on a $\mathbf{C}_{\mathbf{R}}^{p,q}$ vector bundle $(E, \rho) \in \mathrm{Vect}_{\mathbf{R}}^{p,q}$ is an involution $\eta \in \mathrm{End}(E)$ such that $\eta^2 = 1$ and $\eta\rho(\gamma_i) = -\rho(\gamma_i)\eta \ \forall\, i \in \{1, \ldots, p+q\}$*

The group $KO^{p,q}(X)$ is then the group generated by triples $(E, \eta_1, \eta_2)$ subject to the relations

- $(E, \eta_1, \eta_2) + (F, \xi_1, \xi_2) = (E \oplus F, \eta_1 \oplus \xi_1, \eta_2 \oplus \xi_2)$
- $(E, \eta_1, \eta_2) = 0$ if $\eta_1$ is homotopic to $\eta_2$ within the gradations of $E$.

From these properties one can deduce the following:

- $(E, \eta_1, \eta_2) + (E, \eta_2, \eta_1) = 0$
- $E \simeq E', \eta_1 \simeq \eta_1', \eta_2 \simeq \eta_2' \;\Rightarrow\; (E, \eta_1, \eta_2) = (E', \eta_1', \eta_2') \in KO^{p,q}(X)$
  i.e. $KO^{p,q}(X)$ depends only on the isomorphism classes of bundle and gradations.
- $(E, \eta_1, \eta_2) + (E, \eta_2, \eta_3) = (E, \eta_1, \eta_3)$
- Every element of $KO^{p,q}(X)$ can be represented by a single triple.

One can recover ordinary $KO$-theory from $KO^{p,q}(X)$ via the following:

$$
KO^{0,0}(X) = KO(X) \tag{65}
$$

To see this let $(E, \eta_1, \eta_2) \in KO^{0,0}(X)$. Then $\eta_1$, $\eta_2$ are just involutions, they do not have to satisfy anything else. Maybe after adding the trivial triple $(X \times \mathbf{R}^n, \mathbf{1}_n, \mathbf{1}_n)$ one can simultaneously diagonalise both involutions and then cancel the eigenspaces with the same eigenvalue. You are left with a difference of triples $(E_1, -\mathbf{1}_k, \mathbf{1}_k) - (E_2, \mathbf{1}_l, -\mathbf{1}_l)$ which you map to $[E_1] - [E_2] \in KO(X)$.

**Computation of $KO_{\mathbb{Z}_2}$**

All the $KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q})$ groups can be determined by using the connection between K–theory and Clifford algebras. The basic idea is to use the following result of [27] (for notation see appendix A)

**Theorem 1.** *Let $G$ be a compact Lie group, $V$ a $G$ vector space with a positive definite form (i.e. the generated Clifford algebra is $\mathbf{C}(V)$ with all $\gamma_i^2 = +1$) and $X$ a Real $G$ space. Then*

$$KR_G^V(X) = KR_G(X \times V) \tag{66}$$

This reduces the calculation to one where the base space is a point, and then use a trick to absorb the $\mathbb{Z}_2$ action in the Clifford algebra.

Of course we want to compute $KO$ and not $KR$, so we want the real version of the above theorem. So let the Real involution act trivially on every space (that is $G$, $X$ and $V$), then

$$KO_G^V(X) = KO_G(X \times V) \tag{67}$$

The correspondence between $KO \leftrightarrow KR$ classes is as usual by complexification ($\rightarrow$) resp. taking the subbundle invariant under the complex conjugation ($\leftarrow$). Note that for the $V$–action to be well–defined on the real subbundle we need that the complex conjugation acts trivially on Clifford algebra, i.e. on $V$.

So let $V = \mathbf{R}^{p,0}$ and let it generate $\mathbf{C}(V) = \mathbf{C}_{\mathbf{R}}^{0,p}$. We then get

$$KO_{\mathbb{Z}_2}(\mathbf{R}^{p,q}) = KO_{\mathbb{Z}_2}(\mathbf{R}^{0,q} \times V) = KO_{\mathbb{Z}_2}^V(\mathbf{R}^{0,q}) \tag{68}$$

Now we have to reinterpret $KO_{\mathbb{Z}_2}^V(X)$ for some $\mathbb{Z}_2$–invariant space $X$. Its elements are tuples $(E, g, \rho; \eta_1, \eta_2)$ where

- $E$ is a real vector bundle over the base space $X$.
- $g : E \to E$ is an involution that acts trivially on the base (i.e. $g \in \mathrm{End}(E)$).
- $g$ acts also on the Clifford algebra via $g : V \to V, v \mapsto -v$.
- An action of the Clifford algebra $\rho : \mathbf{C}_{\mathbf{R}}^{0,p} \to \mathrm{End}(E)$
- Two gradations $\eta_1$, $\eta_2$.

This data is equivalent to the following:

- A real vector bundle $E$ over the base space $X$.
- An action of the Clifford algebra $\tilde{\rho} : \mathbf{C}_{\mathbf{R}}^{0,p+1} \to \mathrm{End}(E)$ defined by

$$\tilde{\rho}(\gamma_i) = \begin{cases} \rho(\gamma_i) & \forall\, i \in \{1, \ldots, p\} \\ g & i = p+1 \end{cases} \tag{69}$$

- $\tilde{\eta}_1, \tilde{\eta}_2 \in \mathrm{End}(E)$ that commute with the $\mathbf{C}_{\mathbf{R}}^{0,p+1}$ action defined by

$$\tilde{\eta}_i = g\eta_i \tag{70}$$

Since the $\tilde{\eta}_i$ *commute* with the Clifford algebra action this is not $KO^{0,p+1}(X)$; The gradations and the Clifford algebra can rather be treated independently.

However we can think of $\tilde{\eta}_i$ as two gradations of some $\mathbf{C}_{\mathbf{R}}^{0,0} = \mathbf{R}$ action on $(E, \rho)$. But then by the analog of equation (65) for "bundles with $\mathbf{C}_{\mathbf{R}}^{0,p+1}$ action" instead of ordinary bundles we find that

$$KO^V_{\mathbb{Z}_2}(X) = KO^{(0,p+1)}(X) \tag{71}$$

# B Understanding $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \widetilde{U}_1)$

Following (appendix D) we have determined $H^2\big(\mathbb{Z}_2 \times \mathbb{Z}_2, \widetilde{U}_1\big) = \mathbb{Z}_2 \times \mathbb{Z}_2$, so there are four different twists by which $KR_{\mathbb{Z}_2\times\mathbb{Z}_2}$ can be twisted. One of them is just untwisted $KR_{\mathbb{Z}_2\times\mathbb{Z}_2}$, and one of them is $KR^{[D_8]}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ of section 3.3. We discuss the remaining two twists in some detail presently. This appendix is designed to familiarise the reader with group cohomology with coefficients in $\widetilde{U}_1$.

The 2-torsion part of $H^2\big(\mathbb{Z}_2 \times \mathbb{Z}_2, \widetilde{U}_1\big)$ comes from $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ via the $\mathbb{Z}_2 \to \widetilde{U}_1 \to \widetilde{U}_1$ coefficient long exact sequence. The advantage of this description is twofold: First there is no complex conjugation action on $\mathbb{Z}_2$ and second determining $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ is a finite combinatorial problem. Elementary calculation shows that

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{72}$$

corresponds to the extensions

$$0 \to \mathbb{Z}_2 \to \quad D_8 \text{ or } Q_8 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_4 \text{ or } \mathbb{Z}_2^3 \quad \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 0 \tag{73}$$

Let us denote the linear generator with $g$ and the anti-linear one with $\tau$, so that

$$\begin{aligned} D_8 =& \Big\{\sigma, \tau, g \,\Big|\, g\tau = \sigma\tau g,\ g^2 = \tau^2 = \sigma^2 = 1\Big\} \\ Q_8 =& \Big\{\sigma, \tau, g \,\Big|\, g\tau = \sigma\tau g,\ g^2 = \tau^2 = \sigma, \sigma^2 = 1\Big\} \end{aligned} \tag{74}$$

where the projection to $\mathbb{Z}_2 \times \mathbb{Z}_2$ is always by putting $\sigma = 1$. To determine the corresponding group 2-cocycle we choose sections $s(\gamma_1) = \tau$, $s(\gamma_2) = g$, and $s(\gamma_1\gamma_2) = \tau g$. For example

$$c_{D_8}(-,-) = \begin{bmatrix} c(\tau,\tau) & c(g,\tau) & c(\tau g,\tau) \\ c(\tau,g) & c(g,g) & c(\tau g,g) \\ c(\tau,\tau g) & c(g,\tau g) & c(\tau g,\tau g) \end{bmatrix} = \begin{bmatrix} + & - & - \\ + & + & + \\ + & - & - \end{bmatrix} \tag{75}$$

There are altogether 16 closed 2-cocycles and one coboundary

$$c(\tau) = c(g) = -1,\ c(\tau g) = +1 \quad \Rightarrow \quad dc(-,-) = \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix} \tag{76}$$

so the quotient consists of the 8 cohomology classes $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now we are really interested in their image in $H^2\big(\mathbb{Z}_2 \times \mathbb{Z}_2, \widetilde{U}_1\big)$, for that we have to mod out the additional coboundary

$$c(\tau) = c(g) = i,\ c(\tau g) = +1 \quad \Rightarrow \quad dc(-,-) = \begin{bmatrix} + & - & - \\ + & - & - \\ + & + & + \end{bmatrix} \tag{77}$$

Therefore the 4 classes are

$$c_{\mathbb{Z}_2^3}(-,-) = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix} \simeq \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix} \simeq \begin{bmatrix} + & - & - \\ + & - & - \\ + & + & + \end{bmatrix} \simeq \begin{bmatrix} + & + & + \\ - & - & + \\ - & - & + \end{bmatrix} \tag{78a}$$

$$c_{D_8}(-,-) = \begin{bmatrix} + & - & - \\ + & + & + \\ + & - & - \end{bmatrix} \simeq \begin{bmatrix} + & + & + \\ - & + & - \\ - & + & - \end{bmatrix} \simeq \begin{bmatrix} + & + & + \\ + & - & - \\ + & - & - \end{bmatrix} \simeq \begin{bmatrix} + & - & - \\ - & - & + \\ - & + & - \end{bmatrix} \tag{78b}$$

$$c_{Q_8}(-,-) = \begin{bmatrix} - & - & + \\ + & - & - \\ - & + & - \end{bmatrix} \simeq \begin{bmatrix} - & + & - \\ - & - & + \\ + & - & - \end{bmatrix} \simeq \begin{bmatrix} - & + & - \\ + & + & + \\ - & + & - \end{bmatrix} \simeq \begin{bmatrix} - & - & + \\ - & + & - \\ + & - & - \end{bmatrix} \tag{78c}$$

$$c_{\mathbb{Z}_2\times\mathbb{Z}_4}(-,-) = \begin{bmatrix} - & + & - \\ + & - & - \\ - & - & + \end{bmatrix} \simeq \begin{bmatrix} - & - & + \\ - & - & + \\ + & + & + \end{bmatrix} \simeq \begin{bmatrix} - & - & + \\ + & + & + \\ - & - & + \end{bmatrix} \simeq \begin{bmatrix} - & + & - \\ - & + & - \\ + & + & + \end{bmatrix} \tag{78d}$$

The new twist classes corresponding to the 2-cocycles $c_{Q_8}$, $c_{\mathbb{Z}_2\times\mathbb{Z}_4}$ have $c(\tau,\tau) = -1$, so the corresponding projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is one with $\tau^2 = -1$.

# C The D-brane spectrum and K–groups for $\Omega \times \mathcal{I}_4$ orientifolds

In [1] the D-brane spectrum of the hyper and tensor multiplet models was computed using BCFT techniques. It is possible to generalise these results to the two $\Omega \times \mathcal{I}_4$ *Sp* orientifolds. The four theories differ from one another by the overall choice of sign in front of the O9- and O5-plane crosscaps.

| Theory | $O9$ | $O5$ |
|---|---|---|
| *So* tensor / BZDP | $-$ | $+$ |
| *So* hyper / GP | $-$ | $-$ |
| *Sp* tensor | $+$ | $-$ |
| *Sp* hyper | $+$ | $+$ |

(79)

We note in particular that the *Sp* tensor model is T-dual (performing T-duality along all four internal directions) to the BZDP model. Since the *Sp* model comes from an orbifold of a non-supersymmetric theory (Type I with *Sp* gauge group) this puts in doubt the possibility of the BZDP model being supersymmetric[17]. Like the GP model the *Sp* hyper model is T-dual to itself.

[17] We would like to thank C. Angelantonj for discussions on this point.

It is then straightforward to repeat the computation of [1] to obtain the D-brane spectrum of the two $Sp$ theories. We list below the D-brane spectrum of all four models.[18]

| Theory | $\mathbb{Z} \oplus \mathbb{Z}$ | BPS $\mathbb{Z}$ | non-BPS $\mathbb{Z}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ |
|---|---|---|---|---|---|
| $SO$ tensor / BZDP | $(1,5;0,4)$ | $(1,3;2)$ | $(1,5;1,2,3)$ | $(-1,0;0),(3,4;4)$ | $(-1,0;1),(3,4,3)$ |
| $SO$ hyper / GP | $(-1,3;2)$ | $(1,5;0,4)$ | $(-1,3;0,1,3,4)$ | $(5,2)$ | $(5;1,3$ |
| $Sp$ tensor | $(1,5;0,4)$ | $(-1,3;2)$ | $(1,5;1,2,3)$ | $(-1,0;4),(3,4;0)$ | $(-1,0;3),(3,4;1)$ |
| $Sp$ hyper | $(-1,3;2)$ | $(1,5;0,4)$ | $(-1,3;0,1,3,4)$ | $(1,2;2)$ | $(1,2;1,3)$ |

(80)

One can compute the various twisted KR theories for $R^{0,q}$ using equation (32). We only have to determine the Real irreducible representations of the groups. For $Q_8$ we find (again $\delta$ denotes complex conjugation)

$$\begin{array}{c||c|c|c|c} g & 1 & \sigma & g & \tau \\ \hline r_1(g) & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix}\circ\delta \\ \hline r_2(g) & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}-1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix}\circ\delta \\ \hline r_3(g) & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}-1&0\\0&-1\end{pmatrix} & \begin{pmatrix}i&0\\0&i\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix}\circ\delta \\ \hline r_4(g) & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}-1&0\\0&-1\end{pmatrix} & \begin{pmatrix}-i&0\\0&-i\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix}\circ\delta \end{array} \qquad (81)$$

One can easily check that the commuting field for $r_3$, $r_4$ is $\mathbb{H}$, therefore by equation (39)

$$KR^{[Q_8],i}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathrm{pt}) = KSp^i(\mathrm{pt}) \oplus KSp^i(\mathrm{pt}) = KSp^i_{\mathbb{Z}_2}(\mathrm{pt})\,. \qquad (82)$$

Explicitly we have for $R^{0,q}$ with trivial group action

| $q$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $KSp_{\mathbb{Z}}(\mathbf{R}^{0,q})$ | $\mathbb{Z} \oplus \mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | 0 |

(83)

which agrees with the spectrum of D$(5-i,4)$-branes in equation (80). Finally, take for the group defining the remaining twist $[D_8Q_8]$

$$D_8Q_8 = \left\{\sigma,\tau,g \,\middle|\, g\tau=\tau g,\ g^2=\tau^2=\sigma,\sigma^2=1\right\} \qquad (84)$$

Its Real irreducible representations are

$$\begin{array}{c||c|c|c|c} g & 1 & \sigma & g & \tau \\ \hline r_1(g) & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix}\circ\tau \\ \hline r_2(g) & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix} & \begin{pmatrix}-1\end{pmatrix} & \begin{pmatrix}1\end{pmatrix}\circ\tau \\ \hline r_3(g) & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}-1&0\\0&-1\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix}\circ\tau \end{array} \qquad (85)$$

[18]Entries in equation (80) are of the form $(r_1,\cdots,r_m;s_1,\cdots,s_n)$ to indicate that all D$(r_i,s_j)$-branes are allowed.

The only higher-dimensional Real representation $r_3$ has commuting field $\mathbf{C}$, and therefore

$$KR^{[D_8 Q_8],i}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathrm{pt}) = K^i(\mathrm{pt}) \tag{86}$$

This matches with the spectrum of D$(5-i,4)$-branes in equation (80).

# D Cohomology for general abelian groups

In this section we want to compute the cohomology group $H^2\big(G,\widetilde{U}_1\big)$ for all finite abelian groups

$$G = \mathbb{Z}_{2r} \times \Big(\mathop{\times}_{i=1}^{n} \mathbb{Z}_{k_i}\Big) \times \Big(\mathop{\times}_{j=1}^{m} \mathbb{Z}_{\ell_j}\Big) \qquad k_i \text{ even},\ \ \ell_j \text{ odd} \tag{87}$$

where the augmentation $\epsilon : G \to \mathbb{Z}_2$ is $-1$ on the generator of the first factor and $+1$ on the other generators. By redefinition of the generators we can always assume that only one generator acts anti-linearly.

Now it is technically easier to use the $\widetilde{\mathbb{Z}} \to \widetilde{\mathbb{R}} \to \widetilde{U}_1$ coefficient long exact sequence (for finite groups $H^i(G,\widetilde{\mathbb{R}}) = 0$)

$$0 \to H^2\left(G,\widetilde{U}_1\right) \xrightarrow{\sim} H^3\left(G,\widetilde{\mathbb{Z}}\right) \to 0 \tag{88}$$

and actually compute $H^3\big(G,\widetilde{\mathbb{Z}}\big)$. Then the computation naturally splits into two steps:

1. Compute the group cohomology for general cyclic groups
2. Put the cohomology groups of the factors together via the Künneth theorem

The former is standard [43]

**Theorem 2.** *Let $G = \mathbb{Z}_k$ be any cyclic group and $M$ an arbitrary $\mathbb{Z}[G]$ module. Then the cohomology groups are*

$$H^i(G,M) = \begin{cases} \ker N & i \textit{ odd} \\ \operatorname{coker} N = M^G/NM & i>0 \textit{ even} \\ M^G & i = 0 \end{cases} \tag{89}$$

*where*[19] *$N : M_G \to M^G$ is the norm map $N(m) = \sum_{i=1}^{k} g^i(m)$.*

[19]The invariants and coinvariants of $M$ are defined as

$$M^G = \{m \in M|\ g(m) = m\} \quad \text{and} \quad M_G = M/\{g(m) - m|\ g \in G, m \in M\} \tag{90}$$

In particular we have

$$
\begin{array}{c|ccccc}
 & i=0 & i=1 & i=2 & i=3 & i=4 \\
\hline
H^i\big(\mathbb{Z}_{2r}, \widetilde{\mathbb{Z}}\big) & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 \\
H^i\big(\mathbb{Z}_{k_j}, \mathbb{Z}\big) & \mathbb{Z} & 0 & \mathbb{Z}_{k_j} & 0 & \mathbb{Z}_{k_j} \\
H^i\big(\mathbb{Z}_{\ell_j}, \mathbb{Z}\big) & \mathbb{Z} & 0 & \mathbb{Z}_{\ell_j} & 0 & \mathbb{Z}_{\ell_j}
\end{array}
\tag{91}
$$

To assemble the cohomology groups of $G$ from the factors we then need the

**Theorem 3 (Künneth theorem).** *Let $G_1$, $G_2$ be groups (such that resolutions are finitely generated, e.g. finite groups). Furthermore let $M_1$ be a $\mathbb{Z}[G_1]$ module and $M_2$ a $\mathbb{Z}[G_2]$ module such that either $M_1$ or $M_2$ is $\mathbb{Z}$-free. Then there is a split exact sequence*

$$
\begin{aligned}
0 \longrightarrow \bigoplus_{p+q=i} \Big(H^p(G_1, M_1) \otimes H^q(G_2, M_2)\Big) \longrightarrow H^i(G_1 \times G_2, M_1 \otimes M_2) \longrightarrow \\
\longrightarrow \bigoplus_{p+q=i+1} \operatorname{Tor}\Big(H^p(G_1, M_1), H^q(G_2, M_2)\Big) \longrightarrow 0
\end{aligned}
\tag{92}
$$

Since in our case the cohomology groups $H^*(\mathbb{Z}_{2r}, \widetilde{\mathbb{Z}})$ are only 2-torsion and the above theorem implies that the sequence splits we see [20] that the cohomology of the general abelian group $G$ is only 2-torsion. This matches the physical expectation that only $\pm 1 \in U_1$ twists are allowed (see section 4).

Moreover the odd order factors in $G$ do not contribute: viewed as a $\mathbb{Z}$-module equation we have $\mathbb{Z}_2 \otimes \mathbb{Z}_\ell = 0 = \operatorname{Tor}(\mathbb{Z}_2, \mathbb{Z}_\ell)$ for $\ell$ odd we see that

$$
H^*\Big(\mathbb{Z}_{2r} \times \mathbb{Z}_\ell, \widetilde{\mathbb{Z}}\Big) = H^*\Big(\mathbb{Z}_{2r}, \widetilde{\mathbb{Z}}\Big) \qquad \ell \text{ odd}
\tag{93}
$$

so we only have to consider the cyclic subgroups of even order. For those the relevant cohomology groups can be summarised as follows:

**Theorem 4.** *For $G_n \stackrel{\text{def}}{=} \mathbb{Z}_{2r} \oplus \Big(\bigoplus_{i=1}^{n} \mathbb{Z}_{k_i}\Big)$ we have*

$$
\begin{array}{c|cccc}
 & i=0 & i=1 & i=2 & i=3 \\
\hline
H^i\big(G_n, \widetilde{\mathbb{Z}}\big) & 0 & \mathbb{Z}_2 & \mathbb{Z}_2^n & \mathbb{Z}_2^{1+n(n+1)/2}
\end{array}
\tag{94}
$$

*Proof.* Induction: It is correct for $n = 0$ (equation (91)). Then by the Künneth theorem

$$
\begin{aligned}
H^0\big(G_{n+1}, \widetilde{\mathbb{Z}}\big) &= 0 && (95a)\\
H^1\big(G_{n+1}, \widetilde{\mathbb{Z}}\big) &= \mathbb{Z}_2 && (95b)\\
H^2\big(G_{n+1}, \widetilde{\mathbb{Z}}\big) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2^n = \mathbb{Z}_2^{n+1} && (95c)\\
H^3\big(G_{n+1}, \widetilde{\mathbb{Z}}\big) &= \mathbb{Z}_2^{1+n(n+1)/2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^n = \mathbb{Z}_2^{1+(n+1)(n+2)/2} && (95d)
\end{aligned}
$$

□

[20] Recall that $\operatorname{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(n,m)}$, $\operatorname{Tor}(\mathbb{Z}_n, \mathbb{Z}) = 0 = \operatorname{Tor}(\mathbb{Z}, \mathbb{Z}_n)$.

Putting everything together we have learnt that

$$H^2\left(\mathbb{Z}_{2r}\oplus\big(\overset{n}{\underset{i=1}{\oplus}}\mathbb{Z}_{k_i}\big)\oplus\big(\overset{m}{\underset{j=1}{\oplus}}\mathbb{Z}_{\ell_j}\big),\widetilde{U}_1\right)=\mathbb{Z}_2^{1+n(n+1)/2}\qquad k_i \text{ even},\ \ell_j \text{ odd}\,. \tag{96}$$

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