Asimit, A. V., Vernic, R. & Zitikis, R. (2016). Background Risk Models and Stepwise Portfolio Construction. Methodology and Computing in Applied Probability, 18(3), pp. 805-827. doi: 10.1007/s11009-015-9458-3

Background risk models and stepwise portfolio construction

Alexandru V. Asimit • Raluca Vernic • Ričardas Zitikis

Abstract Assuming the multiplicative background risk model, which has been a popular model due to its practical applicability and technical tractability, we develop a general framework for analyzing portfolio performance based on its subportfolios. Since the performance of subportfolios is easier to assess, the herein developed stepwise portfolio construction (SPC) provides a powerful alternative to a number of traditional portfolio construction methods. Within this framework, we discuss a number of multivariate risk models that appear in the actuarial and financial literature. We provide numerical and graphical examples that illustrate the SPC technique and facilitate our understanding of the herein developed general results.

Keywords Portfolio construction · Background risk · Systemic risk · Laplace transform · Risk management · Capital allocation

Mathematics Subject Classification 62H05 · 91B30 · 44A10

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1 Introduction

Suppose we are dealing with the portfolio $\mathbf{R} = (R_1, \ldots, R_n)$ of n risks. We wish to know the distribution of $\mathbf{R}$, and we also wish to assess how the distribution changes when some risks are excluded and/or new ones added. In this paper we explore this problem and provide a solution in the form of a technique that we call stepwise portfolio construction (SPC). Naturally, if the risks $R_1, \ldots, R_n$ were independent, then the problem of assessing the portfolio distribution based on subportfolios would be simple: the entire portfolio's probability density function (pdf) $p_{\mathbf{R}}(\mathbf{r})$ would be the product of the marginal pdf's of $R_1, \ldots, R_n$. The case that we tackle in this paper is much more complex.

Some dependence structure needs to be assumed. To accommodate many practically relevant scenarios, we adopt a popular and practically well-tested background risk model (BRM). The SPC technique that we develop reduces portfolio risk assessment to that of its subportfolios, which can in turn be reduced to further subportfolios until individual risks are reached; hence, the the name of the technique. We note at the outset that the herein developed SPC technique is very different from the similarly sounding 'two-step,' 'two-stage,' and 'multi-stage' procedures that have been used in portfolio construction (cf Marasović and Babić 2011; Yau et al. 2011).

The SPC technique is not restricted to investment or insurance portfolios, which have been extensively explored using methods such as constrained and unconstrained optimization, finite-sample and bootstrap based (eg Meucci 2007; Michaud and Michaud 2008; Buch et al. 2011; see also Bai et al. 2012; Bennett and Zitikis 2014; Stefanovits et al. 2014; You and Li 2014; and references therein). Indeed, the SPC technique can be applied in many other areas, including enterprise risk management (ERM) that has recently been actively researched from various points of view by many authors (eg Fraser and Simkins 2010; Olson and Wu 2010; Segal 2011; McNeil 2013; Ferrari and Migliavacca 2014; Louisot and Ketcham 2014). In particular, ERM crucially relies on one's ability to integrate (usually dependent) risks and to also aggregate individual risk metrics into one enterprise-wide risk metric. The SPC technique developed in this paper is well suited for such tasks, and we shall illustrate it numerically and graphically.

The rest of the paper is organized as follows. In Section 2 we recall the fundamental for this paper BRM that has been widely used in areas such finance, economics, and management science, due to its technical tractability and practical relevance. In Section 3 we discuss a special but highly significant BRM case, and provide corresponding SPC results alongside their numerical and graphical illustrations. In Sections 4–6 we develop more general and practically relevant BRM's and their corresponding SPC results, with further numerical and graphical illustrations. In Section 7 we give a brief overview of our main contributions.

2 The background risk model

An ambitious project, called Solvency II, was started more than a decade ago in an attempt to harmonize regulatory environments within the European Union's (EU) insurance industry. Its legal framework is specified by European Commission (2009). Various quantitative impact studies (QIS) are being performed, such as feedback from insurance and reinsurance companies to constantly augmented Solvency II specifications. A most recent study, known as QIS 5 (cf European Commission 2010), summarizes the most probable recommendations that will lead to the implementation of Solvency II project (eg Cruz 2009; Sandström 2010; Chan-Lau 2013).

We now turn our attention to recommendations given to the Insurance Group (IG) regulation, which provides the ideal framework for illustrating the usefulness of BRM. Namely, the IG's are composed of multiple legal entities that operate in different insurance markets, but here we focus on IG's with multiple subsidiaries in different EU jurisdictions. Diversification across IG's represents a risk management tool, often used to abate the capital requirements, that is, to achieve capital efficiency. We refer to Asimit et al. (2013) for a discussion of this problem in the case of two subsidiaries.

Hence in this paper we work under BRM, also known as systemic risk model, which we rigorously define as follows: there is an underlying risk Y, and there are (independent or dependent) stand-alone risks $X_1, \ldots, X_n$, which are independent of Y. Every individual risk R_k is a function of X_k and Y, and since we work under the multiplicative BRM, our mathematical model is as follows:

$$\mathbf{R} = \left(\mu_1 + \sigma_1 \frac{X_1}{Y}, \ldots, \mu_n + \sigma_n \frac{X_n}{Y}\right),$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ are parameter vectors, X_k's are stand-alone risks, and Y is background or systemic risk. To avoid unnecessary – at least from the practical point of view – technicalities, we assume that the random variables (rv's) under consideration have densities: g of Y, and $p_{\mathbf{X}}$ of $\mathbf{X} = (X_1, \ldots, X_n)$. We assume that $X_1, \ldots, X_n$ have the same marginal distributions, but they may or may not be independent. Since business lines, assets, and so on, do not usually follow identical distributions, we have accommodated this by employing the parameter-vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$. This multiplicative model has been very popular in the literature (cf eg Tsetlin and Winkler 2005; Franke et al. 2006, 2011; and references therein) due to reasons such as practical relevance and mathematical tractability.

For all our purposes, we can and thus do work under the assumption $\boldsymbol{\mu} = (0, \ldots, 0)$ because the results that we shall obtain can easily be transformed into the general case $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. Hence, for the rest of this paper, we shall deal exclusively with the risk-vector

$$\mathbf{Z} = \left(\frac{X_1/\lambda_1}{Y}, \dots, \frac{X_n/\lambda_n}{Y} \right), \tag{2.1}$$

where $\lambda_k = 1/\sigma_k$ is a convenient re-parametrization, meaning that from now on we shall work with the parameter-vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ instead of $\boldsymbol{\sigma}$. Note that the pdf $p_{\mathbf{Z}}$ of $\mathbf{Z}$ can be expressed in terms of the pdf $p_{\mathbf{X}}$ of $\mathbf{X}$ using the formula

$$p_{\mathbf{Z}}(\mathbf{z}) = \left(\prod_{k=1}^{n} \lambda_k \right) \int_0^\infty p_{\mathbf{X}}(y\lambda_1 z_1, \dots, y\lambda_n z_n) y^n g(y) dy. \tag{2.2}$$

To summarize our terminology:

- Z_k's are individual risks, which can be viewed as risks corresponding to individual business lines, assets, etc. The risks are dependent due to reasons such as laws, regulations, general economic conditions, etc.

- X_k's are stand-alone risks, which are associated with individual business lines, assets, etc, assuming no background (ie systemic) risk. Yet, X_k's may be dependent because, for example, business lines can be dependent by the very nature of business; we shall consider independent and dependent cases.

- Y is background or systemic risk, which may be associated with supervisory and regulatory bodies, general economic conditions, etc, that affect stand-alone risks X_k, thus giving rise to the individual risks $Z_k = (X_k/\lambda_k)/Y$.

For applications of BRM in insurance, we refer to Tsanakas (2008) and references therein. Bai et al. (2012) explore finite-sample statistical inference within the BRM and apply their results for the analysis of financial data. Chan-Lau (2013) provides an in-depth discussion of BRM from a practical perspective. Hashorva and Ji (2014) explore several background risk models (ie random shifting and scaling) focusing on credibility theory, collective risk models, and extreme value models. Merz and Wüthrich (2014) use BRM to study optimal insurance designs and, in particular, risk sharing between insureds and insurers. You and Li (2014) explore BRM within the context of capital allocations in the case of dependent (eg exchangeable) risks and connect their research with copulas (cf eg McNeil et al. 2005; Jaworski et al. 2010; Jaworski et al. 2013; Durante et al. 2014; and references therein). The impact of background risk on portfolio diversification has been explored and discussed by Busse et al. (2014), where we also find an extensive list of references on the topic.

3 Portfolio of Paretian risks

To illustrate the above introduced general BRM, and to also get some sense of how the SPC works, we begin with the classical multivariate Pareto distribution of type II (cf Arnold 1983), which is usually denoted by $\mathrm{MP}^{(n)}\mathrm{II}\,(\boldsymbol{\lambda}, \alpha)$ with parameter $\alpha > 0$. The joint de-cumulative distribution function (ddf) of $\mathbf{Z}_0 \sim \mathrm{MP}^{(n)}\mathrm{II}\,(\boldsymbol{\lambda}, \alpha)$ is

$$S_{\mathbf{Z}_0}\left(\mathbf{z} \mid \boldsymbol{\lambda}, \alpha\right) = \left(1 + \sum_{k=1}^{n} \lambda_k z_k\right)^{-\alpha}, \tag{3.1}$$

for all $\mathbf{z} \geq \mathbf{0}$, and the corresponding joint pdf

$$p_{\mathbf{Z}_0}\left(\mathbf{z} \mid \boldsymbol{\lambda}, \alpha\right) = \left(\prod_{i=1}^{n} \lambda_i (n + \alpha - i)\right)\left(1 + \sum_{k=1}^{n} \lambda_k z_k\right)^{-(n+\alpha)}. \tag{3.2}$$

We next present an alternative formula for the pdf of $\mathbf{Z}_0$ that plays a pivotal role in developing SPC for various multivariate models to be discussed later in this paper. Namely, let E_1 be the exponential rv with mean 1, whose pdf is e^{-x}, and let $Y_0(\alpha)$ be the gamma rv with shape and rate parameters $\alpha > 0$ and $\beta = 1$, respectively, that is, its pdf is

$$g_{\mathrm{GA}}(y \mid \alpha) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}, \quad y > 0.$$

We can express $\mathbf{Z}_0 \sim \mathrm{MP}^{(n)}\mathrm{II}\,(\boldsymbol{\lambda}, \alpha)$ as the vector $\mathbf{Z}$ defined by equation (2.1), where the stand-alone risks $X_1, \ldots, X_n$ are independent and each of them follows the exponential distribution with mean 1, and the background risk Y is the gamma rv $Y_0(\alpha)$ independent of all X_i's.

The following theorem, which is due to Vernic (2011), serves an initial building block for our subsequent general models and gives a recurrence relation upon which we can build SPC-type results for evaluating risk measures and capital allocations (cf Asimit et al. 2013).

Theorem 3.1 (Vernic 2011) *Let $n \geq 2$ and $\mathbf{Z}_0 \sim \mathrm{MP}^{(n)}\mathrm{II}\,(\boldsymbol{\lambda}, \alpha)$. When there are at least two unequal λ_k's, say $\lambda_i \neq \lambda_j$, then the pdf $p_{Z_+}(z \mid \boldsymbol{\lambda}, \alpha)$ of the aggregate loss $Z_+ = \sum_{i=1}^{n} Z_{0,i}$ is given by*

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, \alpha) = \frac{1}{\lambda_j - \lambda_i}\left(\lambda_j p_{Z_{(j)+}}(z \mid \boldsymbol{\lambda}_{(j)}, \alpha) - \lambda_i p_{Z_{(i)+}}(z \mid \boldsymbol{\lambda}_{(i)}, \alpha)\right) \tag{3.3}$$

for all $z \geq 0$. When all λ_k's are equal, say to λ, then the pdf is given by

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, \alpha) = \frac{\lambda^n z^{n-1}}{(n-1)!} \, \frac{\prod_{k=1}^{n}(n + \alpha - k)}{(1 + \lambda z)^{n+\alpha}}.$$

We have used the following notations: Given $\mathbf{z} = (z_1, \ldots, z_n)$, the vector $\mathbf{z}_{(i)}$ stands for $\mathbf{z}$ with the coordinate z_i deleted, that is, $\mathbf{z}_{(i)} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$. Furthermore, $z_+ = \sum_{i=1}^{n} z_i$ and $z_{(i)+} = z_+ - z_i$. We shall later use the notation $\mathbf{z}_{(i,m)}$ for the vector $\mathbf{z}$ with its two coordinates z_i and z_m deleted, and we shall use the notation $z_{(i,j)+} = z_+ - z_i - z_j$.

3.1 Numerical illustration

A usual feature of real-life data sets is that they are highly confidential. Therefore, to illustrate SPC in a practically relevant context, we have 'abstracted' certain data that we have dealt with while consulting. This approach also makes our task manageable within the space limits of this paper. We have chosen to work with the tail value at risk (TVaR), which is also known as the conditional value at risk (CVaR) or conditional tail expectation (CTE). It is a risk measure required by Solvency II in the insurance sector (eg Cruz 2009; Sandström 2010) and by Basel accords in the financial sector (eg Cannata and Quagliariello 2011; Sawyer 2012; Ozdemir and Miu 2013).

Specifically, given a rv Z with cdf F_Z, its tail-value-at-risk $\mathrm{TVaR}_p[Z]$ is the conditional expectation $\mathbf{E}[Z|Z > \mathrm{VaR}_p[Z]]$, where $\mathrm{VaR}_p[Z]$ is the value-at-risk, also known in the statistical literature as the p^{th} quantile of Z and denoted by $F_Z^{-1}(p)$. Hence, the TVaR corresponding to the aggregate risk $Z_+ = Z_1 + \cdots + Z_n$ is

$$\mathrm{TVaR}_p[Z_+] = \mathbf{E}[Z_+|Z_+ > \mathrm{VaR}_p[Z_+]] = \frac{\mathbf{E}[Z_+\mathbf{1}\{Z_+ > \mathrm{VaR}_p[Z_+]\}]}{S_{Z_+}(\mathrm{VaR}_p[Z_+])},$$

where S_{Z_+} is the ddf of Z_+ and $\mathbf{1}\{A\}$ is the indicator function of event A. This risk measure naturally extends to capital allocations. Namely, the contribution of risk Z_l to the aggregate risk Z_+ can be measured by

$$\mathrm{TVaR}_p[Z_l, Z_+] = \mathbf{E}[Z_l \mid Z_+ > \mathrm{VaR}_p[Z_+]] = \frac{\mathbf{E}[Z_l\mathbf{1}\{Z_+ > \mathrm{VaR}_p[Z_+]\}]}{S_{Z_+}(\mathrm{VaR}_p[Z_+])}.$$

We shall next employ the SPC technique to calculate these quantities in the case $n = 3$. Of course, with the help of recurrence relations, we can tackle any dimensionality.

Hence, assume that we are dealing with three business lines, and let $\mathbf{Z} \sim \mathrm{MP}^{(3)}\mathrm{II}(\boldsymbol{\lambda}, \alpha)$ for some $\alpha > 1$. Given the recurrence relation of Theorem 3.1, we start with Z that follows the univariate Pareto distribution of the second kind, that is, $Z \sim \mathrm{MP}^{(1)}\mathrm{II}(\lambda, \alpha)$. We have

$$\mathrm{TVaR}_p[Z] = \frac{\alpha \mathrm{VaR}_p[Z] + \lambda^{-1}}{\alpha - 1}$$

and

$$\mathbf{E}[Z\,\mathbf{1}\{Z > s\}] = \frac{\alpha s + \lambda^{-1}}{(\alpha-1)(\lambda s + 1)^{\alpha}}$$

for all $s \geq 0$. With these formulas and the recurrence relation of Theorem 3.1, we obtain the following formula in the bivariate case $\mathbf{Z} \sim \mathrm{MP}^{(2)}\mathrm{II}(\boldsymbol{\lambda}, \alpha)$:

$$\mathbf{E}[Z_i\mathbf{1}\{Z_+ > s\}] = \frac{1}{(\alpha-1)\lambda_i(\lambda_2-\lambda_1)^2}\Big(\lambda_i^2(\lambda_j s+1)^{-\alpha+1} - \lambda_j(\lambda_i s + 1)^{-\alpha}[(\alpha(\lambda_i - \lambda_j) + \lambda_i)\lambda_i s + 2\lambda_i - \lambda_j]\Big)$$

when $\lambda_1 \neq \lambda_2$ and $i \neq j \in \{1, 2\}$, where $Z_+ = Z_1 + Z_2$. When $\lambda_1 = \lambda_2 = \lambda$, we have

$$\mathbf{E}[Z_i \mathbf{1}\{Z_+ > s\}] = \frac{\lambda}{2(\alpha-1)(\lambda s+1)^{\alpha+1}}\left(\alpha(\alpha+1)s^2 + 2(\alpha+1)\frac{s}{\lambda} + \frac{2}{\lambda^2}\right)$$

for $i = 1, 2$. Using these formulas, we can now in turn derive formulas in the tri-variate case $\mathbf{Z} \sim \mathrm{MP}^{(3)}\mathrm{II}(\boldsymbol{\lambda}, \alpha)$. In Figure 3.1 we have depicted TVaR_p as a function of p for various

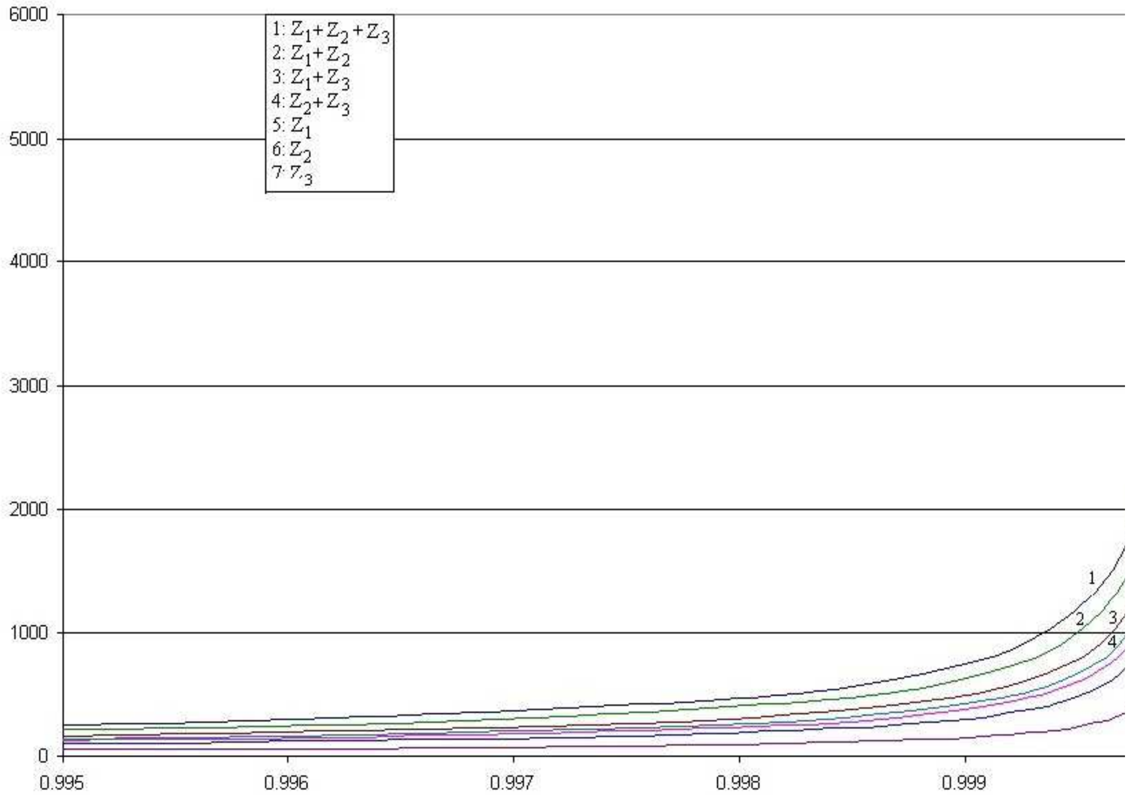

Figure 3.1: TVaR_p as a function of p for various risks originating from the tri-variate $\mathbf{Z} \sim \mathrm{MP}^{(3)}\mathrm{II}((0.8, 1, 2), 1.5)$.

aggregate and individual risks when $\boldsymbol{\lambda} = (0.8, 1, 2)$ and $\alpha = 1.5$, and in Table 3.1 we have reported TVaR_p and p values for pre-specified VaR_p values.

4 Portfolio of $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, g)$ risks

4.1 An alternative view of the earlier model

Formulas (3.1) and (3.2) are fundamental, and they are almost always given as definitions of the $\mathrm{MP}^{(n)}\mathrm{II}(\boldsymbol{\lambda}, \alpha)$ model. They do not, however, directly lead to SPC results, and for this reason we next give a theorem that provides an alternative reformulation of the model suitable for developing SPC.

	VaR = 20		VaR = 50		VaR = 100		VaR = 1000	
Risks	p	TVaR	p	TVaR	p	TVaR	p	TVaR
Z_1	0.9857	62.50	0.9962	152.50	0.9986	302.50	0.99995	3002.50
Z_2	0.9896	62.00	0.9973	152.00	0.9990	302.00	0.99997	3002.00
Z_3	0.9962	61.00	0.9990	151.00	0.9996	301.00	0.99999	3001.00
$Z_1 + Z_2$	0.9702	63.20	0.9919	153.18	0.9971	303.17	0.99991	3003.17
$Z_1 + Z_3$	0.9788	62.68	0.9943	152.67	0.9979	302.67	0.99993	3002.67
$Z_2 + Z_3$	0.9830	62.22	0.9955	152.22	0.9984	302.22	0.99995	3002.21
$Z_1 + Z_2 + Z_3$	0.9617	63.49	0.9896	153.46	0.9962	303.45	0.99987	3003.44

Table 3.1: TVaR_p and p values for pre-specified VaR_p values in the case of several individual and aggregate risks originating from $\mathbf{Z} \sim \mathrm{MP}^{(3)}\mathrm{II}((0.8, 1, 2), 1.5)$.

Theorem 4.1 *The joint pdf of* $\mathbf{Z}_0 \sim \mathrm{MP}^{(n)}\mathrm{II}\,(\boldsymbol{\lambda}, \alpha)$ *can be written as*

$$\begin{aligned} p_{\mathbf{Z}_0}(\mathbf{z} \mid \boldsymbol{\lambda}, \alpha) &= \left(\prod_{k=1}^{n} \lambda_k\right) \int_0^\infty \exp\left\{ -y \sum_{k=1}^{n} \lambda_k z_k \right\} y^n g_{\mathrm{GA}}(y \mid \alpha)\, dy \\ &= \mathbf{E}[Y_0^n(\alpha)] \left(\prod_{k=1}^{n} \lambda_k\right) S_{E_1/Y_n(\alpha)}\left(\sum_{k=1}^{n} \lambda_k z_k\right), \end{aligned} \tag{4.1}$$

where $S_{E_1/Y_n(\alpha)}$ *is the ddf of the ratio* $E_1/Y_n(\alpha)$ *with* $Y_n(\alpha)$ *denoting the size-biased background risk* $Y_0(\alpha)$ *whose pdf is*

$$g_{\mathrm{GA},n}(y \mid \alpha) = \frac{y^n g_{\mathrm{GA}}(y \mid \alpha)}{\mathbf{E}[Y_0^n(\alpha)]}. \tag{4.2}$$

The proof of the theorem is relegated to Appendix A. We note that the procedure of weighting distributions as we have done in formula (4.2) is a powerful tool for generating new distributions and tackling other problems (cf Patil and Ord 1976; Patil and Rao 1978; Patil 2002; also Furman and Zitikis 2008a, 2008b; and references therein).

The joint ddf of $\mathbf{Z}_0$ is given by the formula

$$S_{\mathbf{Z}_0}(\mathbf{z} \mid \boldsymbol{\lambda}, \alpha) = S_{E_1/Y_0(\alpha)}\left(\sum_{k=1}^{n} \lambda_k z_k\right), \tag{4.3}$$

which immediately follows from equation (4.1). The right-hand side of equation (4.3) suggests a number of possible generalizations. For example, we may choose any rv ξ instead of the ratio $E_1/Y_0(\alpha)$ and then define a multivariate ddf by the formula $S_\xi(\sum_{k=1}^n \lambda_k z_k)$. The latter ddf can further be extended to $\mathbf{E}[S_\xi(\sum_{k=1}^n T_k z_k)]$ for some non-negative rv's $T_1, \ldots, T_n$, and this model will naturally appear later in this paper.

4.2 Model $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, g)$ and its SPC

We obtain the first important generalization of Theorem 4.1 by replacing the gamma pdf $g_{\mathrm{GA}}(y \mid \alpha)$ by generic pdf g, thus allowing for various background-risk choices. Namely, we say that $\mathbf{Z}_1 = (Z_{1,1}, \ldots, Z_{1,n}) \sim \mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, g)$ when $\mathbf{Z}_1$ can be expressed as $\mathbf{Z}$ defined by equation (2.1) with the stand-alone risks $X_1, \ldots, X_n$ being independent and each following the exponential distribution with mean 1, and with the background risk $Y > 0$ being absolutely continuous (ie having pdf g) and independent of all X_i's. The joint ddf of $\mathbf{Z}_1$ is given by the formula

$$S_{\mathbf{Z}_1}(\mathbf{z} \mid \boldsymbol{\lambda}, g) = S_{E_1/Y}\bigg(\sum_{k=1}^{n} \lambda_k z_k\bigg),$$

and its pdf by

$$\begin{aligned} p_{\mathbf{Z}_1}(\mathbf{z} \mid \boldsymbol{\lambda}, g) &= \bigg(\prod_{k=1}^{n} \lambda_k\bigg) \int_0^\infty \exp\bigg\{ - y \sum_{k=1}^{n} \lambda_k z_k \bigg\} y^n g(y) dy \\ &= \mathbf{E}[Y^n] \bigg(\prod_{k=1}^{n} \lambda_k\bigg) S_{E_1/Y_n(g)}\bigg(\sum_{k=1}^{n} \lambda_k z_k\bigg), \end{aligned} \tag{4.4}$$

where $Y_n(g)$ denotes the size-biased background risk Y, that is, the pdf of $Y_n(g)$ is

$$g_n(y) = \frac{y^n g(y)}{\mathbf{E}[Y^n]}. \tag{4.5}$$

The following theorem establishes SPC for the just introduced BRM and thus, in turn, generalizes Theorem 3.1.

Theorem 4.2 *Let $n \geq 2$ and $\mathbf{Z}_1 \sim \mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, g)$. When there are at least two unequal λ_k's, say $\lambda_i \neq \lambda_j$, then the pdf $p_{Z_+}(z \mid \boldsymbol{\lambda}, g)$ of $Z_+ = \sum_{i=1}^n Z_{1,i}$ can be expressed by*

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, g) = \frac{1}{\lambda_j - \lambda_i} \Big(\lambda_j p_{Z_{(j)+}}(z \mid \boldsymbol{\lambda}_{(j)}, g) - \lambda_i p_{Z_{(i)+}}(z \mid \boldsymbol{\lambda}_{(i)}, g) \Big) \tag{4.6}$$

for all $z \geq 0$. When all λ_k's are equal, say to λ, then the pdf is

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, g) = \frac{\lambda^n z^{n-1}}{(n-1)!} \int_0^\infty e^{-\lambda z y} y^n g(y) dy. \tag{4.7}$$

The proof of the theorem is relegated to Appendix A.

5 Portfolio of $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \pi, g)$ risks

We can depart from the exponential distribution – though keeping the above developed form of recurrence relations and thus of SPC – by considering completely monotone functions $C : (0, \infty) \to [0, \infty)$, which are such that $C(x) = \int_{[0,\infty)} e^{-tx}\, \pi(dt)$ for some measures π on $[0, \infty)$. Since the functions that we deal with are ddf's, we always have $C(0) = 1$ and thus, in turn, all measures π that we consider are probability measures. When choosing C or, alternatively, π for portfolio modeling purposes, we may wish, or need, to impose certain shape constraints on them. We note, however, that shape relationships between C and π can be quite complex, as seen from the recent works of Sendov and Zitikis (2014), and Sendov and Shan (2015). Our next BRM follows.

5.1 Model $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \pi, g)$ and its SPC

We say that $\mathbf{Z}_2 = (Z_{2,1}, \ldots, Z_{2,n}) \sim \mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \pi, g)$ when $\mathbf{Z}_2$ can be expressed as $\mathbf{Z}$ defined by equation (2.1) with $Y > 0$ being a rv with the pdf g and independent of the vector $\mathbf{X} = (X_1, \ldots, X_n)$ whose joint ddf is given by

$$S_{\mathbf{X}}(\mathbf{x} \mid \pi) = \int_{[0,\infty)} \exp\Big\{ -t\sum_{k=1}^{n} x_k \Big\} \pi(dt) \tag{5.1}$$

for all $\mathbf{x} \geq \mathbf{0}$ with a probability measure π on $[0, \infty)$. Hence, the joint ddf of the vector $\mathbf{Z}_2$ is

$$S_{\mathbf{Z}_2}(\mathbf{z} \mid \boldsymbol{\lambda}, \pi, g) = \int_{[0,\infty)} S_{E_1/Y}\Big(t\sum_{k=1}^{n} \lambda_k z_k \Big)\pi(dt) = \mathbf{E}\Big[S_{E_1/Y}\Big(T\sum_{k=1}^{n} \lambda_k z_k \Big)\Big],$$

where T is a rv with the probability law π. The joint pdf of $\mathbf{Z}_2$ can be expressed as

$$\begin{aligned} p_{\mathbf{Z}_2}(\mathbf{z} \mid \boldsymbol{\lambda}, \pi, g) &= \Big(\prod_{k=1}^{n} \lambda_k \Big) \int_{[0,\infty)} t^n \int_0^\infty \exp\Big\{ -yt\sum_{k=1}^{n} \lambda_k z_k \Big\} y^n g(y) dy \pi(dt) \\ &= \mathbf{E}[Y^n] \Big(\prod_{k=1}^{n} \lambda_k \Big) \mathbf{E}\Big[T^n S_{E_1/Y_n(g)}\Big(T\sum_{k=1}^{n} \lambda_k z_k \Big)\Big], \end{aligned} \tag{5.2}$$

where $Y_n(g)$ is a size-biased rv whose pdf is given by formula (4.5). In the next subsection we shall discuss assumption (5.1) in detail. At the moment, we only note that when π is concentrated at the point 1, then $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \pi, g)$ reduces to $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, g)$.

The next theorem establishes SPC for our current model.

Theorem 5.1 *Let $n \geq 2$ and $\mathbf{Z}_2 \sim \mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \pi, g)$. When there are at least two unequal λ_k's, say $\lambda_i \neq \lambda_j$, then the pdf $p_{Z_+}(z \mid \boldsymbol{\lambda}, g)$ of $Z_+ = \sum_{i=1}^{n} Z_{2,i}$ is*

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, \pi, g) = \frac{1}{\lambda_j - \lambda_i} \Big(\lambda_j p_{Z_{(j)+}}(z \mid \boldsymbol{\lambda}_{(j)}, \pi, g) - \lambda_i p_{Z_{(i)+}}(z \mid \boldsymbol{\lambda}_{(i)}, \pi, g) \Big) \tag{5.3}$$

for all $z \geq 0$. When all λ_k's are equal, say to λ, then the pdf is

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, \pi, g) = \frac{\lambda^n z^{n-1}}{(n-1)!} \int_0^\infty t^n \int_0^\infty e^{-t\lambda z y} y^n g(y) dy \pi(dt). \tag{5.4}$$

The proof of the theorem is relegated to Appendix A.

5.2 Laplace transform of π: examples

We start our discussion of assumption (5.1) by rewriting the joint survival function $S_{\mathbf{X}}(\mathbf{x} \mid \pi)$ in terms of the Laplace transform $\mathcal{L}_\pi$ of the measure π, that is, we have the equation

$$S_{\mathbf{X}}(\mathbf{x} \mid \pi) = \mathcal{L}_\pi\left(\sum_{k=1}^{n} x_k\right). \tag{5.5}$$

Given a probability measure π, we can now consult a handbook or text on Laplace transforms (eg Abramowitz and Stegun, 1972; Schilling et al. 2010; Widder, 1945) and have an expression for $S_{\mathbf{X}}(\mathbf{x} \mid \pi)$.

We next present several illustrative examples showing that the herein proposed risk model is quite flexible, and that the stand-alone risks X_k can exhibit various degrees of heavy tailness, such as

- heavy yet lighter than Pareto tails (Examples 5.1 and 5.2).
- Pareto-like tails (Examples 5.3, 5.4, 5.5 and 5.6).

In the examples, we shall also give formulas of the corresponding Laplace transforms $\mathcal{L}_\pi$, which play a pivotal role in our numerical explorations in the following subsection.

Example 5.1 Assume that π follows the inverse gamma law, that is, has the pdf

$$h_{\text{IGA}}(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}, \quad x > 0,$$

for some parameters $\alpha > 0$ and $\beta > 0$. It is a special case of the log-exponential family (Furman and Zitikis 2009). The corresponding Laplace transform is

$$\mathcal{L}_\pi(x) = \frac{2\beta^{\alpha/2}}{\Gamma(\alpha)} x^{\alpha/2} K_\alpha\left(2\sqrt{\beta x}\,\right),$$

where K_α is the modified Bessel function of the second kind (eg Abramowitz and Stegun 1972)

$$K_\alpha(y) = \frac{\Gamma\big(\alpha + 1/2\big)}{\sqrt{\pi}} (2y)^\alpha \int_0^\infty \frac{\cos t}{\big(t^2 + y^2\big)^{\alpha+1/2}}\, dt = \sqrt{\frac{\pi}{2}}\, \frac{e^{-y}}{\sqrt{y}} \big(1 + o(1)\big), \quad y \to \infty.$$

Consequently,

$$S_{X_1}(x \mid \alpha, \beta) = \frac{\sqrt{\pi}\beta^{(2\alpha-1)/4}}{\Gamma(\alpha)} x^{(2\alpha-1)/4} e^{-2\sqrt{\beta x}} \big(1 + o(1)\big), \quad x \to \infty.$$

Hence, the stand-alone risk X_1 has all finite moments.

Example 5.2 Let the measure π be inverse Gaussian with the pdf

$$h_{\text{IGAUSS}}(x \mid \mu, \sigma) = \sqrt{\frac{\sigma}{2\pi}}\, x^{-3/2} \exp\left\{ -\frac{\sigma(x-\mu)^2}{2\mu^2 x} \right\}, \quad x > 0,$$

for some parameters $\mu > 0$ and $\sigma > 0$. This is a classical example of the exponential family (eg Jørgensen 1997). The Laplace transform of this distribution is (Seshadri 1993, p. 41)

$$\mathcal{L}_\pi(x) = \exp\left\{ \frac{\sigma}{\mu} - \sqrt{\frac{\sigma^2}{\mu^2} + 2\sigma x} \right\}.$$

Consequently,

$$S_{X_1}(x \mid \mu, \sigma) = e^{\sigma/\mu - \sqrt{2\sigma x}} \big(1 + o(1)\big), \quad x \to \infty.$$

Hence, X_1 has all finite moments.

Example 5.3 When the measure π is the gamma law with the pdf

$$h_{\text{GA}}(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0,$$

for some parameters $\alpha > 0$ and $\beta > 0$, which is yet another example of the exponential family, then the Laplace transform is

$$\mathcal{L}_\pi(x) = \frac{1}{(1 + x/\beta)^\alpha}.$$

A simple computation yields the asymptotic formula

$$S_{X_1}(x \mid \alpha, \beta) = \frac{\beta^\alpha}{x^\alpha} \big(1 + o(1)\big), \quad x \to \infty,$$

and so X_1 may or may not have a finite mean, depending on the value of $\alpha > 0$.

Example 5.4 Here we explore the half-normal law, which is a special case of the class of folded distributions that have emerged as excellent models for insurance data (Brazauskas and Kleefeld 2011, 2014; Scollnik 2014) and have also been recently used to understand the 'trends in disguise' phenomenon (Brazauskas et al. 2015). Hence, we assume that π is the half-normal law, whose pdf is

$$h_{\text{HNORM}}(x \mid \sigma) = \frac{2}{\pi\sigma} \exp\left\{ -\frac{x^2}{\pi\sigma^2} \right\}, \quad x > 0,$$

for some parameter $\sigma > 0$. The corresponding Laplace transform is

$$\mathcal{L}_\pi(x) = \exp\left\{ \frac{\pi\sigma^2}{4} x^2 \right\} \operatorname{erfc}\left(\frac{\sqrt{\pi}\sigma}{2} x \right),$$

where erfc is the complementary error function

$$\operatorname{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-t^2}\, dt = \frac{1}{\sqrt{\pi} y} e^{-y^2} \big(1 + o(1)\big), \quad y \to \infty.$$

The latter asymptotic formula gives

$$S_{X_1}(x|\sigma) = \frac{2}{\pi\sigma x}\big(1+o(1)\big), \quad x \to \infty.$$

From this expression we see that X_1 has infinite mean, and we refer to Nešlehová et al. (2006) for uses of infinite-mean distributions for modeling operational risks, as well as to Mainik and Embrechts (2013) for further related notes.

Example 5.5 When the measure π is the Rayleigh law with the pdf

$$h_{\text{RLGH}}(x \mid \sigma) = \frac{x}{\sigma^2}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \quad x > 0,$$

for some parameter $\sigma > 0$, then the Laplace transform is

$$\mathcal{L}_\pi(x) = 1 - \sqrt{\frac{\pi}{2}}\sigma x \exp\left\{\frac{\sigma^2}{2}x^2\right\}\text{erfc}\left(\frac{\sigma}{\sqrt{2}}x\right).$$

Using the asymptotic expansion

$$\text{erfc}(y) = \frac{1}{\sqrt{\pi}y}e^{-y^2} - \frac{1}{2\sqrt{\pi}y^3}e^{-y^2}\big(1+o(1)\big), \quad y \to \infty,$$

we have

$$S_{X_1}(x \mid \sigma) = \frac{1}{\sigma^2 x^2}\big(1+o(1)\big), \quad x \to \infty,$$

and so X_1 has a finite mean but the second and higher order moments are infinite.

Example 5.6 When the measure π is the Maxwell-Boltzmann law with the pdf

$$h_{\text{MB}}(x \mid \sigma) = \sqrt{\frac{2}{\pi}}\sigma^{-3}x^2\exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \quad x > 0,$$

for some parameter $\sigma > 0$, then the Laplace transform is

$$\mathcal{L}_\pi(x) = \exp\left\{\frac{\sigma^2}{2}x^2\right\}\Big(1+\sigma^2x^2\Big)\text{erfc}\left(\frac{\sigma}{\sqrt{2}}x\right) - \sigma\sqrt{\frac{2}{\pi}}\,x.$$

Using the asymptotic formula

$$\text{erfc}(y) = \frac{1}{\sqrt{\pi}y}e^{-y^2} - \frac{1}{2\sqrt{\pi}y^3}e^{-y^2} + \frac{3}{4\sqrt{\pi}y^5}e^{-y^2}\big(1+o(1)\big), \quad y \to \infty,$$

and some tedious algebra, we obtain

$$S_{X_1}(x \mid \sigma) = \sqrt{\frac{8}{\pi}}\,\frac{1}{\sigma^3x^3}\big(1+o(1)\big), \quad x \to \infty.$$

Consequently, X_1 has finite mean and variance, but the third and higher order moments are infinite.

5.3 Numerical illustration

Here we provide a numerical example based on the $\mathrm{VaR}_{0.95}$ risk measure for the aggregate risk $Z_+ = Z_1 + \cdots + Z_n$ based on π's of the six examples in the previous subsection. To demonstrate the technique clearly, we keep the complexities at a reasonable level by assuming that

- all λ_k's are equal to 1;
- the stand-alone risks are independent; and
- the background risk Y is either exponential with mean 1, ie pdf $g_{\mathrm{E}}(x \mid 1) = \exp\{-x\}$, or gamma with shape and scale parameters 2 and 1, ie pdf $g_{\mathrm{GA}}(x \mid 2, 1) = x \exp\{-x\}$, respectively.

We find it also useful to view the background risk as a 'competing risk' in the terminology of reliability engineering and survival analysis (cf eg Bebbington et al. 2008; and references therein). In particular, we learn from the literature that the two distributions – exponential and gamma – have very different hazard rate functions: constant in the exponential case and increasing in the gamma case when the shape parameter is greater than 1, which is the case we consider. As a result of this, we shall see distinct diversification effects for the two distributions, which corroborates the fact that portfolio construction is influenced not only by the stand-alone risks but also by the background (or systemic, competing, etc) risk. We refer to Busse et al. (2014) and references therein for an in-depth discussion of the impact of background risk on portfolio diversification.

To be able to compare our findings under various scenarios, we set the mean and the 0.95-value-at-risk of the stand-alone risks at 600 and 2,000, respectively. Since under this set-up the underlying distribution should have at least two parameters, we now focus on Examples 5.1–5.3 and will later explore the remaining one-parameter distributions of Examples 5.4–5.6. Numerical computations have yielded the following values:

$$\begin{aligned} \alpha_{\mathrm{IGA}} &= 0.040980 \quad \text{and} \quad \beta_{\mathrm{IGA}} = 0.000068 \\ \mu_{\mathrm{IGAUSS}} &= 0.007363 \quad \text{and} \quad \sigma_{\mathrm{IGAUSS}} = 0.025166 \\ \alpha_{\mathrm{GA}} &= 4.152880 \quad \text{and} \quad \beta_{\mathrm{GA}} = 1,891.73 \end{aligned}$$

Via equation (5.4), we numerically find $\mathrm{VaR}_{0.95}[Z_+]$ for the noted three distributions (Examples 5.1–5.3). The results are reported in Table 5.1 for the exponential background risk, ie pdf $g_{\mathrm{E}}(x \mid 1)$. The diversification effect, which is a standard measure in risk management, is given by

$$\text{Div Eff} = \left(1 - \mathrm{VaR}_{0.95}[Z_+] / \sum_{i=1}^{n} \mathrm{VaR}_{0.95}[Z_{2i}]\right) 100\% = \left(1 - \frac{\mathrm{VaR}_{0.95}[Z_+]}{n\mathrm{VaR}_{0.95}[Z_{21}]}\right) 100\%.$$

	Inverse Gamma		Inverse Gaussian		Gamma	
n	VaR	Div Eff	VaR	Div Eff	VaR	Div Eff
1	4,130.63	0%	3,122.29	0%	11,161.47	0%
2	9,786.57	-18.46%	6,420.97	-2.82%	22,718.27	-1.77%
3	15,574.78	-25.69%	9,723.03	-3.80%	34,279.46	-2.37%
4	21,400.72	-29.52%	13,025.99	-4.30%	45,841.83	-2.68%
5	27,242.33	-31.90%	16,329.33	-4.60%	57,404.69	-2.86%
10	56,522.01	-36.84%	32,847.81	-5.20%	115,221.25	-3.23%

Table 5.1: $\mathrm{VaR}_{0.95}[Z_+]$ and diversification effects for various values of n under the exponentially distributed background risk.

Note the always negative diversification effects in the table. This can indeed happen for the VaR risk measure because it is not always sub-additive, that is, the bound $\mathrm{VaR}_p[\xi + \eta] \leq \mathrm{VaR}_p[\xi] + \mathrm{VaR}_p[\eta]$ may not hold for some risks ξ and η. For detailed discussions, properties and pitfalls concerning portfolio diversification with emphasis on VaR aggregation in heavy-tailed populations, we refer to Embrechts and Puccetti (2010), Embrechts et al. (2013), Mainik and Embrechts (2013), and references therein.

It is expected that the multiplicative BRM leads to a less risky distribution when Y has a heavier tail than the so far explored exponential. To see this phenomenon, we now assume that Y follows the gamma distribution, ie pdf $g_{\mathrm{GA}}(x \mid 2, 1)$. From Table 5.2 we observe

	Inverse Gamma		Inverse Gaussian		Gamma	
n	VaR	Div Eff	VaR	Div Eff	VaR	Div Eff
1	1,393.84	0%	683.61	0%	2,154.45	0%
2	3,314.62	-18.90%	1,327.24	2.92%	4,048.40	6.05%
3	5,288.93	-26.48%	1,963.83	4.24%	5,912.07	8.53%
4	7,279.24	-30.56%	2,598.30	4.98%	7,766.25	9.88%
5	9,276.28	-33.10%	3,231.87	5.45%	9,616.26	10.73%
10	19,291.79	-38.41%	6,395.54	6.44%	18,846.02	12.53%

Table 5.2: $\mathrm{VaR}_{0.95}[Z_+]$ and diversification effects for various values of n under the gamma distributed background risk.

reduced VaR levels and increased (and even positive) diversification effects.

The next natural task is to assess which of the above three models is more suitable for our data. We may do so by looking at the risk-ratio distributions, say those of X_1/X_2 or Z_{21}/Z_{22}, which are equally distributed. Note that the risk-ratio distribution removes the effect of the background risk and tells us how to identify the 'best possible' model, and it also helps to check the earlier imposed independence assumption on X_i's. In Figure 5.1 we

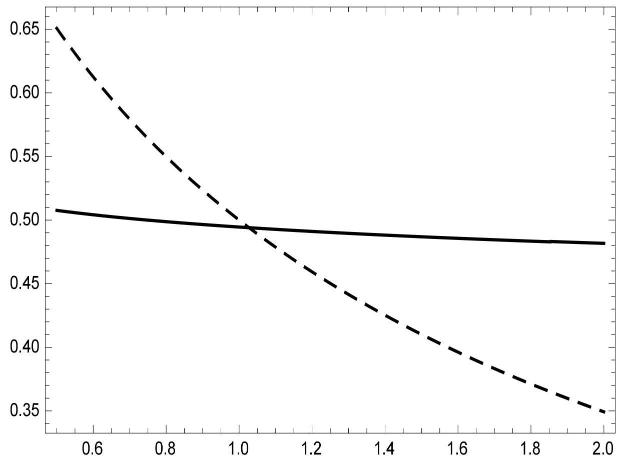

Figure 5.1: Survival functions for the ratio X_1/X_2 in the case of inverse gamma (solid) and inverse Gaussian (dashed) distributions.

have depicted the survival functions of the ratio X_1/X_2 in the case of the inverse gamma and inverse Gaussian distributions, setting the same parameters as in Table 5.1. Both graphs go through the point $(1, 1/2)$ due to the fact that X_i's are assumed to be independent. In these calculations, we have not included the third (ie gamma) distribution because it has a very similar behavior to that in the inverse Gaussian case.

Next, we produce similar analyses of the remaining three examples, that is, of Examples 5.4–5.6. Since each of the three distributions has only one parameter, the only assumption that we now impose is $\mathrm{VaR}_{0.95}[X_1] = 2,000$. As before, the background risk Y is assumed to be exponentially distributed with mean 1, and the stand-alone risks are independent. The numerically obtained parameter values are:

$$\sigma_{\text{HNORM}} = 0.006341, \quad \sigma_{\text{RLGH}} = 0.002077, \quad \sigma_{\text{MB}} = 0.001303.$$

Once again, numerical values of $\mathrm{VaR}_{0.95}[Z_+]$ have been obtained via equation (5.4) and, together with the corresponding diversification effects, reported in Table 5.3.

Similar to the results reported in Table 5.2, the heavier tailed gamma BRM reduces the VaR levels and increases diversification effects if compared with the exponential case. We see this phenomenon from Table 5.4 for all three distributions of Examples 5.4–5.6. Finally, we have depicted the survival functions of the ratio X_1/X_2 under the half-normal and Rayleigh distributions in Figure 5.2. We have not included there the Maxwell-Boltzmann case because it has a very similar behavior to that of Rayleigh.

	Half Normal		Rayleigh		Maxwell-Boltzmann	
n	VaR	Div Eff	VaR	Div Eff	VaR	Div Eff
1	8,644.81	0%	10,674.49	0%	11,370.56	0%
2	18,258.04	-5.60%	21,862.25	-2.40%	23,146.32	-1.78%
3	27,929.99	-7.69%	33,064.57	-3.25%	34,927.40	-2.39%
4	37,362.62	-8.05%	44,271.09	-3.68%	46,709.94	-2.70%
5	46,990.92	-8.71%	55,479.39	-3.95%	58,493.10	-2.89%
10	95,160.88	-10.08%	111,529.33	-4.48%	117,411.71	-3.26%

Table 5.3: $\mathrm{VaR}_{0.95}[Z_+]$ and diversification effects for various values of n under the exponentially distributed background risk.

	Half Normal		Rayleigh		Maxwell-Boltzmann	
n	VaR	Div Eff	VaR	Div Eff	VaR	Div Eff
1	1,863.86	0%	2,107.08	0%	2,180.97	0%
2	3,792.86	-1.75%	4,028.29	4.41%	4,096.90	6.08%
3	5,691.27	-1.78%	5,927.65	6.23%	5,982.72	8.56%
4	7,611.31	-2.09%	7,820.46	7.21%	7,859.17	9.91%
5	9,531.45	-2.28%	9,710.43	7.83%	9,731.51	10.76%
10	19,132.62	-2.65%	19,146.94	9.13%	19,073.23	12.55%

Table 5.4: $\mathrm{VaR}_{0.95}[Z_+]$ and diversification effects for various values of n under the gamma distributed background risk.

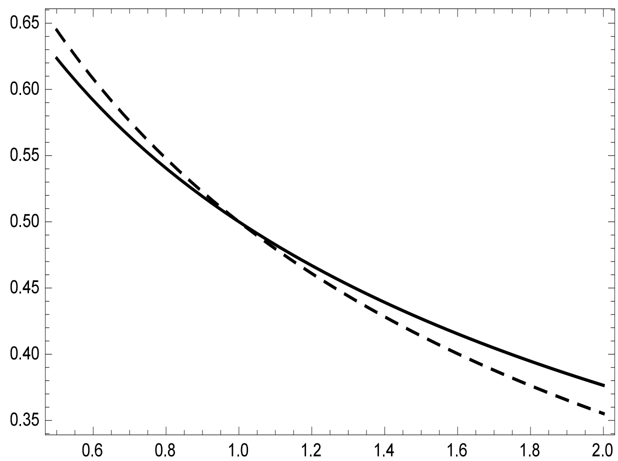

Figure 5.2: Survival functions for the ratios of two risks in the case of half normal (solid) and Rayleigh (dashed) distributions.

6 Portfolio of $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \nu_n, g)$ risks

Following Guillén et al. (2013), we say that a random vector $\mathbf{Z}_3 = (Z_{3,1}, \ldots, Z_{3,n})$ follows the multivariate beta distribution of type II, denoted by $\mathrm{MB}^{(n)}\mathrm{II}\left(\boldsymbol{\lambda}, \{p_k\}_{k=1}^n, q_0\right)$ when it can be expressed as $\mathbf{Z}$ defined by equation (2.1) with $X_i = Y_0(p_i)$, $i = 1, 2, \ldots, n$, being independent gamma rv's with shape parameters $p_i > 0$, $i = 1, 2, \ldots, n$, and the background variable $Y = Y_0(q_0)$ following the gamma distribution with shape parameter $q_0 > 0$ and independent of $X_1, \ldots, X_n$. In what follows, we shall extend the model by considering generic probability measure ν_n instead of the probability distribution $\{p_k\}_{k=1}^n$.

6.1 Model $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \nu_n, g)$ and its SPC

We say that $\mathbf{Z}_4 \sim \mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \nu_n, g)$ when $\mathbf{Z}_4$ can be expressed as $\mathbf{Z}$ defined by equation (2.1) with $Y > 0$ being a rv with pdf g and independent of the vector $\mathbf{X} = (X_1, \ldots, X_n)$ whose joint ddf is given by

$$S_{\mathbf{X}}(\mathbf{x} \mid \nu_n) = \int_{[0,\infty)^n} \exp\left\{ -\sum_{k=1}^n t_k x_k \right\} \nu_n(d\mathbf{t}) \tag{6.1}$$

for all $\mathbf{x} \geq \mathbf{0}$ with a probability measure ν_n on $[0, \infty)^n$. Then the joint ddf of $\mathbf{Z}_4$ is

$$S_{\mathbf{Z}_4}(\mathbf{z} \mid \boldsymbol{\lambda}, \nu_n, g) = \int_{[0,\infty)^n} S_{E_1/Y}\left(\sum_{k=1}^n t_k \lambda_k z_k \right) \nu_n(d\mathbf{t}) = \mathbf{E}\left[S_{E_1/Y}\left(\sum_{k=1}^n T_k \lambda_k z_k \right) \right],$$

where the random vector $\mathbf{T} = (T_1, \ldots, T_n)$ follows the probability measure ν_n. Hence, the joint pdf of $\mathbf{Z}_4$ is

$$\begin{aligned} p_{\mathbf{Z}_4}(\mathbf{z} \mid \boldsymbol{\lambda}, \nu_n, g) &= \left(\prod_{k=1}^n \lambda_k \right) \int_{[0,\infty)^n} \left(\prod_{k=1}^n t_k \right) \int_0^\infty \exp\left\{ -y \sum_{k=1}^n t_k \lambda_k z_k \right\} y^n g(y) dy \nu_n(d\mathbf{t}) \\ &= \mathbf{E}[Y^n] \left(\prod_{k=1}^n \lambda_k \right) \mathbf{E}\left[\left(\prod_{k=1}^n T_k \right) S_{E_1/Y_n(g)}\left(\sum_{k=1}^n T_k \lambda_k z_k \right) \right], \end{aligned} \tag{6.2}$$

where $Y_n(g)$ is a size-biased rv with the pdf given by formula (4.5).

Note 6.1 In general, the vector $\mathbf{T}$ can be discrete, absolutely continuous, with dependent and independent coordinates T_k. For example, when every coordinate T_k takes on value 1 almost surely, then the distribution of $\mathbf{Z}_4$ reduces to that of $\mathbf{Z}_1$, and thus $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \nu_n, g)$ reduces to $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, g)$. If there is a rv T such that every T_k is equal to T almost surely, then we obtain $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \pi, g)$ with π denoting the probability law of T.

Using equations (4.4) and (6.2), we express the pdf of $\mathbf{Z}_4$ in terms of the pdf of $\mathbf{Z}_1$ as follows:

$$p_{\mathbf{Z}_4}(\mathbf{z} \mid \boldsymbol{\lambda}, \nu_n, g) = \mathbf{E}\left[p_{\mathbf{Z}_1}(\mathbf{z} \mid \mathbf{T} \circ \boldsymbol{\lambda}, g) \right], \tag{6.3}$$

where $\mathbf{T} \circ \boldsymbol{\lambda}$ is the Hadamard (ie element-wise) product of the vectors $\mathbf{T}$ and $\boldsymbol{\lambda}$. Since we have already established a recurrence relation for $p_{\mathbf{Z}_1}(\mathbf{z} \mid \boldsymbol{\xi}, g)$ irrespective of the vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$, as long as its coordinates are positive, we combine Theorem 4.2 with equation (6.3) and establish SPC for the model $\mathrm{BRM}^{(n)}(\boldsymbol{\lambda}, \nu_n, g)$.

Theorem 6.1 *Let $T_1, \dots, T_n$ be independent and have continuous cdf's. Then for every pair $i \neq j$ and irrespective of whether λ_i and λ_j are equal or not, the pdf of $Z_+ = \sum_{i=1}^n Z_{4,i}$ is*

$$\begin{aligned} &p_{Z_+}(z \mid \boldsymbol{\lambda}, \nu_n, g) \\ &= \mathbf{E}\bigg[\frac{T_j\lambda_j}{T_j\lambda_j - T_i\lambda_i} p_{Z_{(j)+}}(z \mid (\mathbf{T} \circ \boldsymbol{\lambda})_{(j)}, g)\bigg] - \mathbf{E}\bigg[\frac{T_i\lambda_i}{T_j\lambda_j - T_i\lambda_i} p_{Z_{(i)+}}(z \mid (\mathbf{T} \circ \boldsymbol{\lambda})_{(i)}, g)\bigg]. \quad (6.4) \end{aligned}$$

The proof of the theorem is relegated to Appendix A. Reflecting upon Theorem 6.1, we see that equation (6.4) holds whenever $\lambda_j T_j \neq \lambda_i T_i$. This allows us to consider the case when all T_k's are equal, say to T, provided that $\lambda_j \neq \lambda_i$, which we earlier needed to assume for the validity of equation (4.6). Note also that when all T_k's are equal to T, then T_j and T_i disappear from the two fractions inside the expectations on the right-hand side of equation (6.4). Hence, we can take the expectation sign next to the two densities, which turns them into $p_{Z_{(j)+}}(z \mid \boldsymbol{\lambda}_{(j)}, \pi, g)$ and $p_{Z_{(i)+}}(z \mid \boldsymbol{\lambda}_{(i)}, \pi, g)$. It now remains to notice that when all T_k's are equal to T, then $p_{Z_+}(z \mid \boldsymbol{\lambda}, \nu_n, g)$ becomes equal to $p_{Z_+}(z \mid \boldsymbol{\lambda}, \pi, g)$. We have arrived at equation (5.3), which we earlier established directly.

6.2 Modeling stand-alone risks

With the model of Guillén et al. (2013) in mind, we now restrict ourselves to the class of those measures ν_n that can be written as the product of some probability measures $\pi_1, \dots, \pi_n$. Under this assumption, equation (6.1) reduces to the product

$$S_{\mathbf{X}}(\mathbf{x} \mid \nu_n) = \prod_{k=1}^n C_k(x_k)$$

of completely monotone functions C_k. In other words, each C_k is the Laplace transform $C_k(x) = \int_{[0,\infty)} e^{-tx}\pi_k(dt)$ of a probability measure π_k. Note, for example, that when all π_k's are concentrated at point 1, then the model reduces to $\mathrm{MP}^{(n)}\mathrm{II}(\boldsymbol{\lambda}, \alpha)$, which we discussed at the very beginning of this paper. The following two illustrative examples advance our understanding of $S_{\mathbf{X}}(\mathbf{x} \mid \nu_n)$.

Example 6.1 To get the model of Guillén et al. (2013), but under the restriction that all p_k's are in the interval $(0, 1]$, we choose the probability measures π_k so that each function $C_k(x)$ is the ddf of the gamma rv $Y_0(p_k)$, that is,

$$C_k(x) = S_{\mathrm{GA}}(x \mid p_k) = \frac{1}{\Gamma(p_k)} \int_x^\infty y^{p_k-1}e^{-y}dy \quad (6.5)$$

for all $x > 0$. Since $p_k \in (0, 1]$, function (6.5) is completely monotone. By the Bernstein theorem (cf Schilling et al. 2010), there is a unique measure π_k such that

$$S_{\text{GA}}(x \mid p_k) = \int_{[0,\infty)} e^{-xt} \pi_k(dt).$$

The measure π_k is absolutely continuous, that is, $\pi_k(dt) = h(t \mid p_k)dt$, with the pdf $h(t \mid p_k)$ vanishing for all $t \leq 1$ and equal to

$$\frac{1}{t(t-1)^{p_k}\Gamma(1-p_k)\Gamma(p_k)}$$

for all $t > 1$. Note that $h(t \mid p_k)$ is the pdf of $1/\xi_k$, where the rv ξ_k follows the beta distribution with the parameters p_k and $1 - p_k$.

Note 6.2 When the measure π_k has the pdf $h(t \mid \alpha, \beta)$ that vanishes for all $t \leq \beta$ and is equal to

$$\frac{1}{t(t/\beta-1)^{\alpha}\Gamma(1-\alpha)\Gamma(\alpha)}$$

for all $t > \beta$, where $\alpha \in (0, 1]$ and $\beta > 0$ are parameters, then each stand-alone risk X_k is gamma distributed with the parameters α and β. The just defined pdf $h(t \mid \alpha, \beta)$ is that of β/ξ, where ξ follows the beta distribution with the parameters α and $1 - \alpha$. Finally, we can express the joint ddf of the stand-alone risks $X_1, \dots, X_n$ by $S_{\mathbf{X}}(\mathbf{x}) = S_{\text{GA}}(\sum_{i=1}^n x_i \mid \alpha, \beta)$.

Example 6.2 The Pareto of type II ddf is given by the formula

$$S_{\text{PAR}}(x \mid \alpha, \beta) = \frac{1}{(1 + x/\beta)^{\alpha}}$$

for all $x > 0$, where $\alpha > 0$ and $\beta > 0$ are parameters. The ddf is completely monotone, and thus there is a unique measure π such that

$$S_{\text{PAR}}(x \mid \alpha, \beta) = \int_{[0,\infty)} e^{-xt} \pi(dt).$$

Given our earlier investigations (cf Example 5.3), we know that π is the gamma probability measure $\pi(dt) = h_{\text{GA}}(t \mid \alpha, \beta)dt$.

7 Concluding notes

Numerous works have been devoted to constructing and optimizing portfolios of risks, which could, for example, be investments, insurance policies, or enterprise business lines. While silo-type assessment of individual risks is important and frequently serves a first step in developing portfolios within risk tolerance and with desired rewards, the decision-maker's ultimate goal is nevertheless to maximize the performance of entire portfolio. For this reason in particular, in this paper we have explored a powerful method, which we call stepwise portfolio construction, for achieving the aforementioned goals when individual risks follow the multiplicative BRM, which has received considerable attention in the literature. In particular, our results allow us to see how the portfolio distribution changes when (dependent) risks are added to, or excluded from, the portfolio. For example, starting with individual risk distributions, we can derive the distribution of any subportfolio at any level of risk integration. To illustrate our general considerations, we have discussed a number of parametric models of practical relevance, which may exhibit light, Paretian, or non-Paretian heavy tails.

Acknowledgements

We are grateful to an anonymous referee for constructive criticism and suggestions that helped us to reshape the paper and clarify a number of aspects. Research has been supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

References

Abramowitz M, Stegun, IA (1972) Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. (9th printing.) New York: Dover.

Arnold, BC (1983) Pareto Distributions. Fairland: International Co-operative Publishing House.

Asimit V, Badescu A, Tsanakas A (2013) Optimal risk transfers in insurance groups. European Actuarial Journal, 3, 159–190.

Asimit AV, Furman E., Tang Q, Vernic R (2011) Asymptotics for risk capital allocations based on conditional tail expectation. Insurance: Mathematics and Economics, 49, 310–324.

Asimit AV, Vernic R, Zitikis R (2013) Evaluating risk measures and capital allocations based on multi-losses driven by a heavy-tailed background risk: the multivariate Pareto-II model. Risks, 1, 14–33.

Bai Z, Hui Y, Wong WK, Zitikis R (2012) Prospect performance evaluation: making a case for a non-asymptotic UMPU test. Journal of Financial Econometrics, 10, 703–732.

Bebbington M, Lai CD, Zitikis R (2008) Reduction in mean residual life in the presence of a constant competing risk. Applied Stochastic Models in Business and Industry, 24, 51–63.

Bennett CJ, Zitikis R (2014) Estimation of optimal portfolio weights under parameter uncertainty and user-specified constraints: a perturbation method. Journal of Statistical Theory and Practice, 8 (3, Special Issue on "Advances in Interdisciplinary Statistics and Combinatorics"), 423–438.

Brazauskas V, Jones BL, Zitikis R (2015) Trends in disguise. Annals of Actuarial Science, 9, 58–71.

Brazauskas V, Kleefeld A (2011) Folded- and log-folded-t distributions as models for insurance loss data. Scandinavian Actuarial Journal, 2011, 59–74.

Brazauskas V, Kleefeld A (2014) Authors' reply to "Letter to the Editor: Regarding folded models and the paper by Brazauskas and Kleefeld (2011)" by Scollnik. Scandinavian Actuarial Journal, 2014, 753–757.

Buch A, Dorfleitner G, Wimmer M (2011) Risk capital allocation for RORAC optimization. Journal of Banking and Finance, 35, 3001–3009.

Busse M, Dacorogna M, Kratz M (2014) The impact of systemic risk on the diversification benefits of a risk portfolio. Risks, 2, 260–276.

Cannata F, Quagliariello M (2011) Basel III and Beyond. London: Risk Books.

Chan-Lau JA (2013) Systemic Risk Assessment and Oversight. London: Risk Books.

Cruz M (2009) The Solvency II Handbook. London: Risk Books.

Durante F, Fernández-Sánchez J, Pappadá R (2014) Copulas, diagonals, and tail dependence. Fuzzy Sets and Systems (in print).

Embrechts P, Puccetti G (2010) Risk aggregation. In Jaworski P, Durante F, Härdle W, Rychlik, T (eds), Copulas Theory and its Applications (pp. 111–126). Berlin: Springer.

Embrechts P, Puccetti G, Rüschendorf L (2013) Model uncertainty and VaR aggregation. Journal of Banking and Finance, 37, 2750–2764.

European Commission (2009) Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the Taking-up and Pursuit of the Business of Insurance and Reinsurance (Solvency II). Official Journal of the European Union L335.

European Commission (2010) QIS5 Technical Specifications. Available at the address: `http://ec.europa.eu/internal_market/insurance/docs/solvency/qis5/201007/technical_specifications_en.pdf`.

Ferrari R, Migliavacca A (2014) Tsunami Surfing. Torino: Anteprima.

Franke G, Schlesinger H, Stapleton RC (2006) Multiplicative background risk. Management Science, 52, 146–153.

Franke G, Schlesinger H, Stapleton RC (2011) Risk taking with additive and multiplicative background risks. Journal of Economic Theory, 146, 1547–1568.

Fraser J, Simkins B (2010) Enterprise Risk Management: Today's Leading Research and Best Practices for Tomorrow's Executives. Hoboken: Wiley.

Furman E, Zitikis R (2008a) Weighted premium calculation principles. Insurance: Mathematics and Economics 42, 459–465.

Furman E, Zitikis R (2008b) Weighted risk capital allocations. Insurance: Mathematics and Economics 43, 263–269.

Furman E, Zitikis R (2009) Weighted pricing functionals with applications to insurance: an overview. North American Actuarial Journal, 13, 483–496.

Guillén M, Sarabia JM, Prieto F (2013) Simple risk measure calculations for sums of positive random variables. Insurance: Mathematics and Economics, 53, 273–280.

Hashorva E, Ji L (2014) Random shifting and scaling of insurance. Risks, 2, 277–288.

Jaworski P, Durante F Härdle W (2013) Copulae in Mathematical and Quantitative Finance. Berlin: Springer.

Jaworski P, Durante F, Härdle W, Rychlik T (2010) Copulas Theory and its Applications. Berlin: Springer.

Jørgensen B (1997) The Theory of Dispersion Models. New York: Chapman and Hall.

Louisot JP, Ketcham CH (2014) ERM – Enterprise Risk Management: Issues and Cases. Chichester: Wiley.

Mainik G, Embrechts P (2013) Diversification in heavy-tailed portfolios: properties and pitfalls. Annals of Actuarial Science, 7, 26–45.

Marasović B, Babić Z (2011) Two-step multi-criteria model for selecting optimal portfolio. International Journal of Production Economics, 134, 58–66.

McNeil AJ (2013) Enterprise risk management (editorial). Annals of Actuarial Science, 7, 1–2.

McNeil AJ, Frey R, Embrechts P (2005) Quantitative Risk Management. Princeton: Princeton University Press.

Merz M, Wüthrich MV (2014) Demand of insurance under the cost-of-capital premium calculation principle. Risks, 2, 226–248.

Meucci A (2007) Risk and Asset Allocation. (Third Printing.) Berlin: Springer.

Michaud RO, Michaud RO (2008) Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation. (Second edition.) New York: Oxford University Press.

Nešlehová J, Embrechts P, Chavez-Demoulin V (2006) Infinite mean models and the LDA for operational risk. Journal of Operational Risk, 1, 3–25.

Olson DL, Wu D (2010) Enterprise Risk Management Models. New York: Springer.

Ozdemir B, Miu P (2013) Adapting to Basel III and the Financial Crisis: Re-engineering Capital, Business Mix, and Performance Management Practices. London: Risk Books.

Patil GP (2002) Weighted distributions. In El-Shaarawi AH and Piegorsch WW (eds), Encyclopedia of Environmetrics, Volume 4 (pp. 2369–2377). Chichester: Wiley.

Patil GP, Ord KJ (1976) On size-biased sampling and related form-invariant weighted distributions. Sankhyā, Ser. B 38, 48–61.

Patil GP, Rao CR (1978) Weighted distributions and size-biased sampling with applications to wildlife populations and human families. Biometrics 34, 179–189.

Sandström A (2010) Handbook of Solvency for Actuaries and Risk Managers: Theory and Practice. Boca Raton: Chapman and Hall.

Schilling RL, Song R, Vondraček Z (2010) Bernstein Functions. Theory and Applications. Berlin: De Gruyter.

Segal S (2011) Corporate Value of Enterprise Risk Management: The Next Step in Business Management. Hoboken: Wiley.

Sendov HS, Zitikis R (2014) The shape of the Borwein-Affleck-Girgensohn function generated by completely monotone and Bernstein functions. Journal of Optimization Theory and Applications, 160, 67–89.

Sendov HS, Shan S (2015) New representation theorems for completely monotone and Bernstein functions with convexity properties on their measures. Journal of Theoretical Probability (to appear).

Seshadri V (1993) The Inverse Gaussian Distribution: A Case Study in Exponential Families. New York: Oxford University Press.

Sawyer N (2012) Basel III: Addressing the Challenges of Regulatory Reform. Risk Books, London.

Scollnik DPM (2014) Letter to the Editor: regarding folded models and the paper by Brazauskas and Kleefeld (2011). Scandinavian Actuarial Journal, 2014, 278–281.

Stefanovits D, Schubiger U, Wüthrich MV (2014) Model risk in portfolio optimization. Risks, 2, 315–348.

Tsanakas A (2008) Risk measurement in the presence of background risk. Insurance: Mathematics and Economics, 42, 520–528.

Tsetlin I, Winkler RL (2005) Risky choices and correlated background risk. Management Science, 51, 1336–1345.

Vernic R (2011) Tail conditional expectation for the multivariate Pareto distribution of the second kind: another approach. *Methodology and Computing in Applied Probability*, 13, 121–137.

Widder DV (1945) The Laplace Transform. Princeton: Princeton University Press.

Yau S, Kwon RH, Rogers JS, Wu D (2011) Financial and operational decisions in the electricity sector: contract portfolio optimization with the conditional value-at-risk criterion. International Journal of Production Economics, 134, 67–77.

You Y, Li X (2014) Optimal capital allocations to interdependent actuarial risks. Insurance: Mathematics and Economics, 57, 104–113.

A Appendix: proofs

Proof of Theorem 4.1. Since $E_1,\dots,E_n$ are independent exponential rv's with means 1, their joint pdf is $\exp\big\{-\sum_{k=1}^n x_k\big\}$, and so formula (2.2) implies the first equation of (4.1). To prove the second equation of (4.1), we write

$$\begin{aligned}
S_{E_1/Y_n(\alpha)}\Big(\sum_{k=1}^n \lambda_k z_k\Big) &= \mathbf{P}\Big[E_1 > \Big(\sum_{k=1}^n \lambda_k z_k\Big)Y_n(\alpha)\Big] \\
&= \mathbf{E}\Big[\exp\Big\{-\Big(\sum_{k=1}^n \lambda_k z_k\Big)Y_n(\alpha)\Big\}\Big] \\
&= \frac{1}{\mathbf{E}[Y_0^n(\alpha)]}\int_0^\infty \exp\Big\{-y\sum_{k=1}^n \lambda_k z_k\Big\}y^n g_{\mathrm{GA}}(y\mid\alpha)dy. \qquad \text{(A.1)}
\end{aligned}$$

We have arrived at equation (4.1) and finished the proof of Theorem 4.1. ■

Proof of Theorem 4.2. Consider first the case when $\lambda_i \neq \lambda_j$ and start with the equation

$$p_{Z_+}(z\mid\boldsymbol{\lambda},g) = \int_{\mathbf{D}^z_{(i)}} p_{\mathbf{Z}_1}(z_1,\dots,z_{i-1},z-z_{(i)+},z_{i+1},\dots,z_n\mid\boldsymbol{\lambda},g)d\mathbf{z}_{(i)}, \qquad \text{(A.2)}$$

where $\mathbf{D}^z_{(i)} = \{\mathbf{z}_{(i)} \geq \mathbf{0} \mid z \geq z_{(i)+}\}$. Consequently,

$$\begin{aligned}
p_{Z_+}(z\mid\boldsymbol{\lambda},g) &= \Big(\prod_{k=1}^n \lambda_k\Big)\int_{\mathbf{D}^z_{(i)}}\int_0^\infty \exp\Big\{-\Big(\sum_{\substack{k=1\\k\neq i}}^n \lambda_k z_k + \lambda_i\big(z-z_{(i)+}\big)\Big)y\Big\}y^n g(y)dyd\mathbf{z}_{(i)} \\
&= \Big(\prod_{k=1}^n \lambda_k\Big)\int_0^\infty y^n g(y)\int_{\mathbf{D}^z_{(i,j)}} \exp\Big\{-\Big(\sum_{\substack{k=1\\k\neq i,j}}^n \lambda_k z_k + \lambda_i\big(z-z_{(i,j)+}\big)\Big)y\Big\} \\
&\quad\times \int_0^{z-z_{(i,j)+}} \exp\left\{-\left(\lambda_j-\lambda_i\right)yz_j\right\}dz_j d\mathbf{z}_{(i,j)}dy,
\end{aligned}$$

where $\mathbf{D}^z_{(i,j)} = \{\mathbf{z}_{(i,j)} \geq \mathbf{0} \mid z \geq z_{(i,j)+}\}$. Continuing with the above equations, we have

$$\begin{aligned}
p_{Z_+}(z\mid\boldsymbol{\lambda},g) &= \Big(\prod_{k=1}^n \lambda_k\Big)\int_0^\infty y^n g(y)\int_{\mathbf{D}^z_{(i,j)}} \exp\Big\{-\Big(\sum_{\substack{k=1\\k\neq i,j}}^n \lambda_k z_k + \lambda_i\big(z-z_{(i,j)+}\big)\Big)y\Big\} \\
&\quad\times \frac{1-\exp\left\{-\left(\lambda_j-\lambda_i\right)\left(z-z_{(i,j)+}\right)y\right\}}{\left(\lambda_j-\lambda_i\right)y}d\mathbf{z}_{(i,j)}dy \\
&= \frac{\prod_{k=1}^n \lambda_k}{\lambda_j-\lambda_i}\int_0^\infty y^{n-1}g(y)\int_{\mathbf{D}^z_{(i,j)}}\Big[\exp\Big\{-\Big(\sum_{\substack{k=1\\k\neq i,j}}^n \lambda_k z_k + \lambda_i\big(z-z_{(i,j)+}\big)\Big)y\Big\} \\
&\quad - \exp\Big\{-\Big(\sum_{\substack{k=1\\k\neq i,j}}^n \lambda_k z_k + \lambda_j\big(z-z_{(i,j)+}\big)\Big)y\Big\}\Big]d\mathbf{z}_{(i,j)}dy \\
&= \frac{1}{\lambda_j-\lambda_i}\Big(\lambda_j p_{Z_{(j)+}}(z\mid\boldsymbol{\lambda}_{(j)},g) - \lambda_i p_{Z_{(i)+}}(z\mid\boldsymbol{\lambda}_{(i)},g)\Big).
\end{aligned}$$

This establishes equation (4.6).

When all λ_i's are equal to λ, then

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, g) = \lambda^n \int_{\mathbf{D}^z_{(i)}} \int_0^\infty \exp\Big\{ -\Big(\lambda \sum_{\substack{k=1 \\ k \neq i}}^n z_k + \lambda\left(z - z_{(i)+}\right)\Big) y \Big\} y^n g(y) dy d\mathbf{z}_{(i)}$$

$$= \lambda^n \int_0^\infty e^{-\lambda z y} y^n g(y) \int_{\mathbf{D}^z_{(i)}} d\mathbf{z}_{(i)} dy$$

$$= \frac{\lambda^n z^{n-1}}{(n-1)!} \int_0^\infty e^{-\lambda z y} y^n g(y) dy,$$

where the last equation follows from $\int_{\mathbf{D}^z_{(i)}} d\mathbf{z}_{(i)} = z^{n-1}/(n-1)!$, which has been derived by Vernic (2011). This establishes equation (4.7) and concludes the proof of Theorem 4.2. ■

Proof of Theorem 5.1. With the help of equations (4.4) and (5.2), the pdf $p_{\mathbf{Z}_2}(\mathbf{z} \mid \boldsymbol{\lambda}, \pi, g)$ can be expressed in terms of $p_{\mathbf{Z}_1}(\mathbf{z} \mid \boldsymbol{\lambda}, g)$ as follows:

$$p_{\mathbf{Z}_2}(\mathbf{z} \mid \boldsymbol{\lambda}, \pi, g) = \mathbf{E}[Y^n]\Big(\prod_{k=1}^n \lambda_k\Big) \mathbf{E}\Big[T^n S_{E_1/Y_n(g)}\Big(T \sum_{k=1}^n \lambda_k z_k\Big)\Big]$$

$$= \mathbf{E}\big[p_{\mathbf{Z}_1}(\mathbf{z} \mid T\boldsymbol{\lambda}, g)\big]. \tag{A.3}$$

Hence,

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, \pi, g) = \int_{\mathbf{D}^z_{(i)}} p_{\mathbf{Z}_2}\left(z_1, \ldots, z_{i-1}, z - z_{(i)+}, z_{i+1}, \ldots, z_n \mid \boldsymbol{\lambda}, \pi, g\right) d\mathbf{z}_{(i)}$$

$$= \mathbf{E}\Big[\int_{\mathbf{D}^z_{(i)}} p_{\mathbf{Z}_1}\left(z_1, \ldots, z_{i-1}, z - z_{(i)+}, z_{i+1}, \ldots, z_n \mid T\boldsymbol{\lambda}, g\right) d\mathbf{z}_{(i)}\Big]$$

$$= \mathbf{E}\big[p_{Z_+}(z \mid T\boldsymbol{\lambda}, g)\big], \tag{A.4}$$

where the last equation follows from equation (A.2). Note that equation (A.4) holds irrespective of whether λ_k's are equal or not.

When there are at least two unequal λ_k's, say $\lambda_i \neq \lambda_j$, then using equations (4.6) and (A.4), we obtain

$$p_{Z_+}(z \mid \boldsymbol{\lambda}, \pi, g) = \frac{1}{\lambda_j - \lambda_i}\Big(\lambda_j \mathbf{E}\big[p_{Z_{(j)+}}\left(z \mid T\boldsymbol{\lambda}_{(j)}, g\right)\big] - \lambda_i \mathbf{E}\big[p_{Z_{(i)+}}\left(z \mid T\boldsymbol{\lambda}_{(i)}, g\right)\big]\Big)$$

$$= \frac{1}{\lambda_j - \lambda_i}\Big(\lambda_j p_{Z_{(j)+}}\left(z \mid \boldsymbol{\lambda}_{(j)}, \pi, g\right) - \lambda_i p_{Z_{(i)+}}\left(z \mid \boldsymbol{\lambda}_{(i)}, \pi, g\right)\Big).$$

This establishes equation (5.3).

When all λ_k's are equal to λ, then using formula (4.7) on the right-hand side of equation (A.4), we obtain equation (5.4) and conclude the proof of Theorem 5.1. ■

Proof of Theorem 6.1. Without any assumptions on $\mathbf{T}$ and $\boldsymbol{\lambda}$, we have equations:

$$
\begin{aligned}
p_{Z_+}(z \mid \boldsymbol{\lambda}, \nu_n, g) &= \int_{\mathbf{D}^z_{(i)}} p_{\mathbf{Z}_4}(z_1, \ldots, z_{i-1}, z - z_{(i)+}, z_{i+1}, \ldots, z_n \mid \boldsymbol{\lambda}, \nu_n, g) d\mathbf{z}_{(i)} \\
&= \int_{\mathbf{D}^z_{(i)}} \mathbf{E}\big[p_{\mathbf{Z}_1}(z_1, \ldots, z_{i-1}, z - z_{(i)+}, z_{i+1}, \ldots, z_n \mid \mathbf{T} \circ \boldsymbol{\lambda}, g)\big] d\mathbf{z}_{(i)} \\
&= \mathbf{E}\bigg[\int_{\mathbf{D}^z_{(i)}} p_{\mathbf{Z}_1}(z_1, \ldots, z_{i-1}, z - z_{(i)+}, z_{i+1}, \ldots, z_n \mid \mathbf{T} \circ \boldsymbol{\lambda}, g) d\mathbf{z}_{(i)}\bigg] \\
&= \mathbf{E}\big[p_{Z_+}(z \mid \mathbf{T} \circ \boldsymbol{\lambda}, g)\big]. \qquad \text{(A.5)}
\end{aligned}
$$

Next we use equation (4.6) with $\mathbf{T} \circ \boldsymbol{\lambda}$ instead of $\boldsymbol{\lambda}$, for which we need to ensure that $\lambda_j T_j \neq \lambda_i T_i$, but this holds irrespective of the values of λ_i and λ_j because all T_k's are assumed to have continuous cdf's. Hence, from equation (4.6) we obtain

$$
p_{Z_+}(z \mid \mathbf{T} \circ \boldsymbol{\lambda}, g) = \frac{1}{\lambda_j T_j - \lambda_i T_i} \Big(\lambda_j T_j p_{Z_{(j)+}}(z \mid (\mathbf{T} \circ \boldsymbol{\lambda})_{(j)}, g) - \lambda_i T_i p_{Z_{(i)+}}(z \mid (\mathbf{T} \circ \boldsymbol{\lambda})_{(i)}, g) \Big). \qquad \text{(A.6)}
$$

Using equations (A.6) and (A.5), we obtain equation (6.4) and finish the proof of Theorem 6.1. ■