SOME APPLICATIONS OF COPULAE TO FINANCE

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Abstract The aim of this thesis is to extend theory and to develop practical applications of copulae in finance. A copula is a dependence function that links random variables – expressed through their marginal distributions – to their joint or multivariate distribution.

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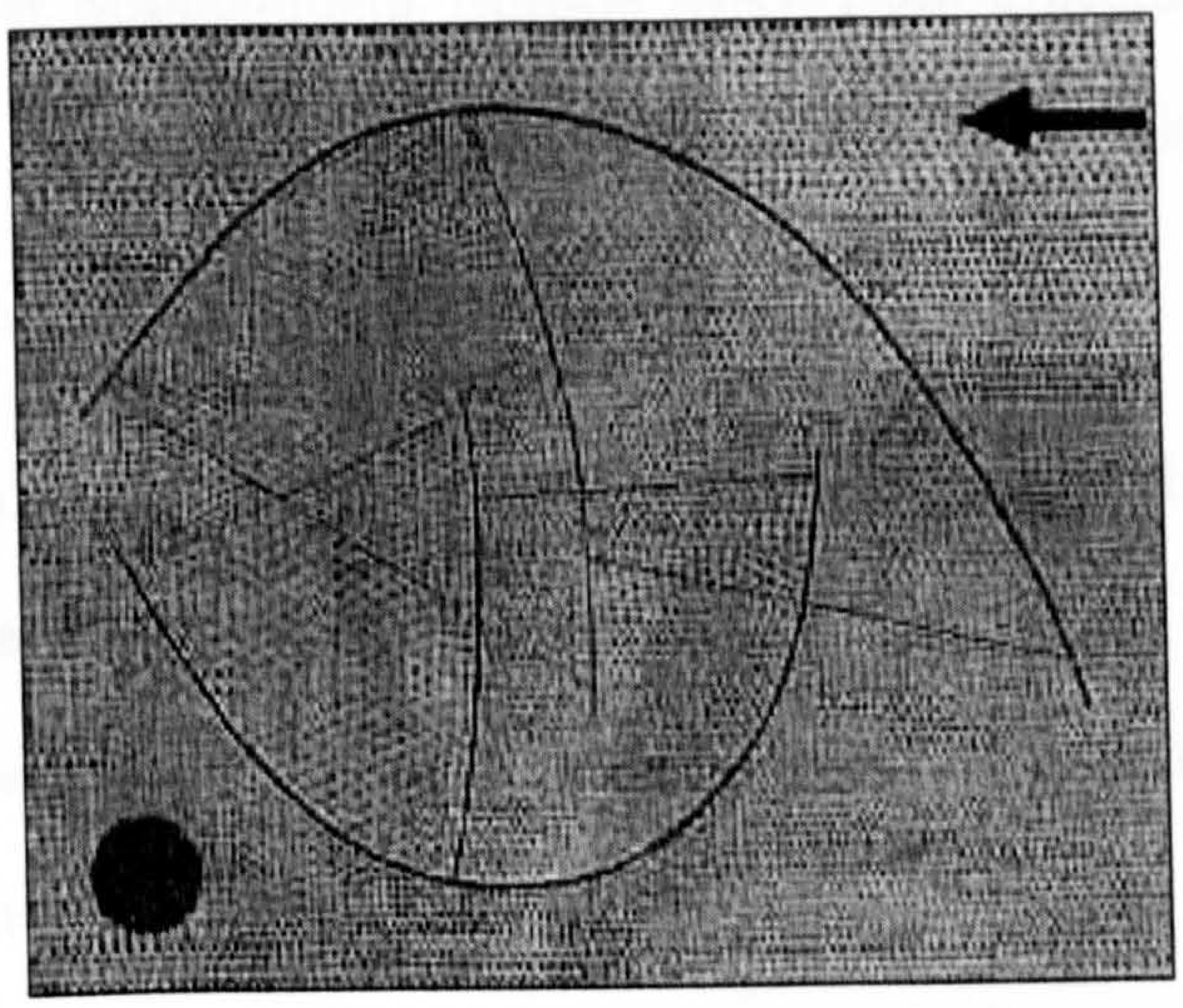

PAUL KLEE, *In copula.*

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Chapter 1

Introduction

1.1 Aim and overview

The aim of this thesis is to extend theory and to develop practical applications of copulae in finance. A copula is a dependence function that links variables – expressed through their marginal distributions – to their joint or multivariate distribution. In the second chapter, we consider the origin of the concept of copulae, the issue of dependence modelling between random variables and develop a general framework for the analysis of copulae.

In the third chapter, we propose a methodology to provide risk measures for portfolios during extreme events. The approach is based on splitting the multivariate extreme value distribution of the assets of the portfolio into two parts: the distributions of each asset and their dependence function – named *copula*. The estimation problem is also investigated. A *trivariate* empirical application for market index portfolios (US, German and Japanese stock markets) is provided. Then, stress-testing values and Monte-Carlo based risk measures – Value-at-Risk and Expected Shortfall respectively – are computed.

The fourth chapter investigates the problem of dynamic dependence using copulae. A general methodology for time series modelling is developed which works down from distributional properties to implied structural models for quantile functions thereby including the standard regression relationship. This general to specific approach is important since it can avoid spurious assumptions such as linearity in the form of the dynamic relationship between variables. It is based on splitting the multivariate distribution of a time series into two parts: (i) the marginal unconditional distribution, (ii) the serial dependence encompassed in a general function, the copula. General properties of the class of copula functions that fulfill the necessary requirements for Markov chain construction are exposed. Special cases for the gaussian copula with AR(p) dependence structure and for Archimedean copulae are presented. We also develop copula based dynamic dependency measures – auto-concordance in place of autocorrelation. Dynamic dependence is more precisely studied for different probability levels by defining quantile regressions derived from the copula structure. Finally, we provide empirical applications using financial returns and transactions based forex data. Our model encompasses the AR(p) model and allows for non-linearity.

In the fifth chapter, we introduce a general class of nonlinear regression quantile models that are again based on a top-down approach. We extend Koenker and Bassett's (1978) original problem of quantile regression by deriving a distribution for the dependent variable Y conditionally on the regressor X. We then deduce a conditional copula based non linear quantile relationship. Some properties of our model are exhibited. Finally, a financial application to the foreign exchange markets is provided.

The sixth chapter concludes.

1.2 Contents and contributions

Chapter 2 is innovative in that its second and third sections provide two new syntheses of copulae. The second section provides an historical perspective about the origin of the concept of copula during the first half of the 20th century. The third section is a short but concise review of the very recent financial literature that uses the copula tool. The definitions, properties and concepts developed in sections 4 to 7 provide the general but necessary background to understand copulae. Other introductions on the subject exist see for instance Joe (1997) and Nelsen (1998). However, the introduction presented here is the first that is dedicated to the use of copulae in the context of financial econometrics. The chapter also provides a new approach that links the mathematical origin with the statistical and financial applications of copulae.

Chapter 3 starts with and introduction (section 2) and some standard definitions of the extreme value theory in its third section. The univariate case is presented (Gumbel (1960)). The multivariate case that introduces the copulae that are eligible for modelling extrema is due to Deheuvels (1978). The author proposed a theorem that constitutes the theoretical starting point of the chapter. Then, three families of multivariate extreme value distributions are described: Gumbel (1960), Hüsler and Reiss (1989), Joe and Hu (1996). The Gumbel copula is the first extreme copula to be used in the financial literature, but only in the bivariate case (Longin and Solnik (2001), Journal of Finance). To our knowledge, the explicit formulation of the Hüsler and Reiss copula is new. The chapter is the first application of the inference for margins method (Joe and Xu (1996)) to extreme value distributions. Section 4 contributes to the literature in two ways: (i) it contains the first trivariate extreme value estimation applied to stock market indices, (ii) it offers a comparison of three non-nested extreme value models and concludes that the Hüsler and Reiss copula we introduced best fits the data (Cox test criterion). In section 5, we present a new concept - the financial failure area - and apply it to stock market returns. Finally, we propose an innovative copula based methodology to compute standard risk measures such as value-at-risk or expected shortfall and we again provide original empirical applications.

The chapter four explores and extends the general framework proposed by Joe (1994, 1997) to model the dynamic dependence of time series whatever their marginal distributions. We provide a new approach to the analysis of the standard linear class of gaussian autoregressive (AR) models by extracting the copula from the model. This allows us to present a new class of models: the gaussian copula AR models that takes into account non-linearity. We then propose a new definition, the intrinsic copula to characterize the minimal representation copula that encompasses all the serial dependence. We provide a theoretical definition of the intrinsic copula. This intrinsic copula furnishes the starting point for model selection. In a non linear time series model the minimal dimension is most naturally provided by identifying the intrinsic copula. Moreover, we show that there are relationships between copulae which order is greater than the order of the intrinsic copula. In section 6, we extend the estimation methods developed by Joe and Xu (1996) to our intrinsic copula defined earlier. We introduce the basis for copula based model selection and show how the copula function can provide a nonlinear autocorrelation function. Section 7 is an empirical contribution to the study of stock market index returns. Four models that assume the same marginal distribution but with four different copulae are estimated and compared. The estimates are computed using two different methods: (i) a non parametric method that has been proposed by Genest and Rivest (1993) - this is its first financial application -, (ii) the maximum likelihood method described in the previous section. Finally, auto-concordance measures are proposed to capture the non-linear serial dependence properties in a time series beyond serial correlation.

Chapter 5 is certainly the more innovative from a theoretical point of view. Indeed, it proposes a generalization of both standard regression quantiles introduced by Koenker and Bassett (1978) and other non-linear regression quantiles (see Engle and Manganelli (2000) or Koenker and Hallock (2001)). The traditional quantile models are reviewed in section 3. The fourth section introduces the definitions and properties that are necessary to understand the link between the functional form of a given copula and the shape of the dependence generated by this copula. Tail dependence measures are also presented in the Section 5. The sixth section is the core of the contribution of the chapter to the literature. We propose a new model: the copula quantile (or c-quantile) regression model. We first define the concept of copula quantile curve. Its properties - positive or negative relationship, symmetry - are then derived. The c-quantile curves are more precisely studied for some copulae, with a particular focus on the archimedean class. We also show that the Koenker and Basset's regression quantiles are encompassed by our model since it corresponds, in our model, to the gaussian copula with gaussian marginal distributions. Finally, the seventh section is an empirical application of our model to foreign exchange rates. We find evidence of a much stronger form of efficiency than implied by standard martingale approach based on the conditional expectation. In some sense, our model also allows us to extend Fama's (1970) definition of efficiency.

Chapter 2

Copulae: history and general framework

2.1 Contents and contributions

This chapter provide two new syntheses of copulae. The first section provides an historical perspective about the origin of the concept of copula during the first half of the 20th century. The second section is a short but concise review of the very recent financial literature that uses the copula tool. The definitions, properties and concepts developed in sections 3 to 6 provide the general but necessary background to understand copulae. Other introductions on the subject exist see for instance Joe (1997) and Nelsen (1998). However, the introduction presented here is the first that is dedicated to the use of copulae in the context of financial econometrics. The chapter also provides a new approach that links the mathematical origin with the statistical and financial applications of copulae.

2.2 Some history about copulae

The question of the distance between two random variables and their distribution functions is the starting point of copula theory in the middle of the 20^{th}

century. A copula function – this concept will be defined later – is a measure of this distance. In the Note B of the fundamental textbook *Généralités sur les probabilités. Variables aléatoires* - FRÉCHET, M. (1937) -, Paul Lévy wrote:

> *If one knows the probability distribution functions of two random variables X and Y, one is still not able to determine their distance*

Fréchet wrote his famous paper[1] "Sur les tableaux de corrélation dont les marges sont données" in 1951. The examples are mainly based on the existence of bounds discovered by Fréchet himself in 1935. The 1951 paper is split into two parts. The first is dedicated to the discrete case. The author asks the following question: how many contingency tables are consistent with marginal distributions arbitrarily given ? Let us think about two variables X and Y that respectively take values $\{x_i\}_{i=1,\ldots,q}$ and $\{y_j\}_{j=1,\ldots,r}$.

Then, the contingency table is:

	x_1	x_2	$\cdots$	x_i	$\cdots$	x_q	
y_1	n_{11}	n_{21}	$\cdots$	n_{i1}	$\cdots$	n_{q1}	N'_1
y_2	n_{12}	n_{22}	$\cdots$	n_{i2}	$\cdots$	n_{q2}	N'_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
y_j	n_{1j}	n_{2j}	$\cdots$	n_{ij}	$\cdots$	n_{qj}	N'_j
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
y_r	n_{1r}	n_{2r}	$\cdots$	n_{ir}	$\cdots$	n_{qr}	N'_r
	N_1	N_2	$\cdots$	N_i	$\cdots$	N_q	N

with $N_i = \sum_{j=1}^{r} n_{ij}$, $N'_j = \sum_{i=1}^{q} n_{ij}$ and $N = \sum_{i=1}^{q} N_i = \sum_{j=1}^{r} N'_j$. Fréchet shows that for some given margins, there exist at least two contingency tables and that the necessary and sufficient condition for the existence of only one is that either

[1]The fact that Fréchet chose a regional journal such that the Annales de l'Université de Lyon provides us information about his personality as noted by Dall'Aglio in 1991:

> *I must add that I listened to a conversation in Esperanto between him and Jimmy Savage [...]*

X or Y is constant. The author explores the following distribution functions:

$$\begin{cases} \mathbf{F}_X(x_i) = \frac{1}{N}\sum\limits_{k<i} N_k \\ \mathbf{F}_Y(y_j) = \frac{1}{N}\sum\limits_{l<j} N'_l \end{cases}$$

and

$$\mathbf{H}(x_i, y_j) = \frac{1}{N}\sum_{k<i}\sum_{l<j} n_{kl} \tag{2.1}$$

and demonstrates that the joint distribution function is bounded:

$$\max\left[\mathbf{F}_X(x_i) + \mathbf{F}_Y(y_j) - 1, 0\right] \leq \mathbf{H}(x_i, y_j) \leq \min\left[\mathbf{F}_X(x_i), \mathbf{F}_Y(y_j)\right] \tag{2.2}$$

Let us consider a simple financial example with two rows and two colums. We postulate we have two groups of firms : the first group belongs to sector G and the second group to sector G'. We are interested in the number of firms that can default for each sector. The number of defaults that can occur is respectively $\{x_1, x_2\}$ for the sector G and $\{y_1, y_2\}$ for the sector G'. Let assume there are eight states of the World. We assume that the information available is only about the margins which are $N_1 = N_2 = 4$, $N'_1 = 5$ and $N'_2 = 3$. For example, N_1 represents the number of states of the World with x_1 defaults in sector G and N'_2 the number of states of the World with y_2 defaults in sector G' . Then the problem can be summarized with the following contingency table:

	x_1	x_2	
y_1	n_{11}	n_{21}	5
y_2	n_{12}	n_{22}	3
	4	4	8

However, as we have no information available about the joint distribution, we exploit the bounds introduced by Fréchet:

$$1 = \max(N_1 - N'_2, 0) \leq n_{11} \leq \min(N_1, N'_1) = 4$$

Then, the number of cases simultaneously corresponding to x_1 defaults in sector G and y_1 defaults in sector G' is bounded between one and four. It means that there exist four contingency tables or dependence structures with the assumed margins:

$$\boxed{\begin{matrix}1 & 4\\ 3 & 0\end{matrix}}\qquad \boxed{\begin{matrix}2 & 3\\ 2 & 1\end{matrix}}\qquad \boxed{\begin{matrix}3 & 2\\ 1 & 2\end{matrix}}\qquad \boxed{\begin{matrix}4 & 1\\ 0 & 3\end{matrix}}$$

The second part of the paper extends the discrete case by focusing on the continuous case. The question becomes: is it possible to find the joint probability distribution $\mathrm{H}(x_1, x_2) = \Pr\{X_1 < x_1 \text{ and } X_2 < x_2\}$ of X_1 and X_2 if we only know their marginal distributions, $\mathrm{F}_1(x) = \Pr\{X_1 < x_1\}$ and $\mathrm{F}_2(x_2) = \Pr\{X_2 < x_2\}$? To answer the question, Fréchet shows that the bivariate distribution is bounded:

$$\mathrm{C}^-(\mathrm{F}_1(x_1), \mathrm{F}_2(x_2)) \leq \mathrm{H}(x_1, x_2) \leq \mathrm{C}^+(\mathrm{F}_1(x_1), \mathrm{F}_2(x_2))$$

with

$$\begin{cases} \mathrm{C}^-(\mathrm{F}_1(x_1), \mathrm{F}_2(x_2)) = & \max(\mathrm{F}_1(x_1) + \mathrm{F}_2(x_2) - 1, 0) \\ \mathrm{C}^+(\mathrm{F}_1(x_1), \mathrm{F}_2(x_2)) = & \min(\mathrm{F}_1(x_1), \mathrm{F}_2(x_2)) \end{cases}$$

Then H_λ defines a class of the possible distribution functions:

$$\begin{aligned} \mathrm{H}_\lambda(x_1, x_2) &= \lambda \mathrm{C}^-(\mathrm{F}_1(x_1), \mathrm{F}_2(x_2)) + (1-\lambda)\mathrm{C}^+(\mathrm{F}_1(x_1), \mathrm{F}_2(x_2)) \\ \text{with } 0 &\leq \lambda \leq 1 \end{aligned}$$

However, as noted by Fréchet in 1958, H_λ does not allow us to characterize the independent case $\mathrm{H}(x_1, x_2) = \mathrm{F}_1(x_1)\mathrm{F}_2(x_2)$.

Under the initiative of Bass, Féron – who was Fréchet's student - extended the problem of the existence of the bounds to the three-dimensional case in 1956. He demonstrated that there exists an upper bound of the form,

$$\mathrm{H}(x_1, x_2, x_3) \leq \min(\mathrm{F}_1(x_1), \mathrm{F}_2(x_2), \mathrm{F}_3(x_3))$$

but that $\max\left(F_1(x_1)+F_2(x_2)+F_3(x_3)-1,0\right)$ is not a distribution function. In contrast to common opinion, we argue that **Féron was the inventor of copulae**. Indeed, in order to prove the existence of the bounds, he makes the following remark:

> *The search for maximal and minimal elements of the distribution functions* $H(x_1,x_2,x_3)$ *with given margins is completely equivalent to the search for the same maximal and minimal elements if the marginal variables are uniformly distributed on the segment* $(0,1)$ *for each axis. Then, if we succeed in demonstrating that the functions* G *have one maximal element and several minimal elements, it will be necessarily the same for the functions* $H(x_1,x_2,x_3)$ *with given margins.*

with $G(u_1,u_2,u_3)$, the *map* of $H(x_1,x_2,x_3)$ on the unit cube is defined as follows:

$$\begin{cases} G(u_1,\infty,\infty) = & u_1 \quad \text{for} \quad 0\le u_1\le 1 \\ G(\infty,u_2,\infty) = & u_2 \quad \text{for} \quad 0\le u_2\le 1 \\ G(\infty,\infty,u_3) = & u_3 \quad \text{for} \quad 0\le u_3\le 1 \end{cases}$$

where $u_1=F_1(x_1)$, $u_2=F_2(x_2)$ and $u_3=F_3(x_3)$. **This map function is of course nothing else but the copula function.**

Another contribution by Féron concerns the formulation of Fréchet's question in the discrete case as a linear programming problem:

$$LP(1)\quad \begin{cases} \max \left[\frac{1}{N}\sum_{k<i}\sum_{l<j} n_{kl} = H(x_i,y_j)\right] \\ \text{s. t.} \begin{cases} N_i = \sum_{j=1}^{r} n_{ij} \\ N'_j = \sum_{i=1}^{q} n_{ij} \\ n_{ij}\ge 0 \end{cases} \end{cases} \tag{2.3}$$

In 1958, Gumbel in the *Compte Rendus de l'Académie des sciences de Paris* went further than the existence of the joint distribution and provided an analytical solution to Fréchet's problem that encompasses the case of independence. In order to do this he used a copula function introduced by Morgenstern in 1956:

$$\mathrm{C}(\mathrm{U},\mathrm{V}) = \mathrm{UV}\,(1+\alpha\,(1-\mathrm{U})\,(1-\mathrm{V})) \quad \text{with} \quad -1<\alpha<1$$

Gumbel provides an example using gaussian distributions for the margins. This leads us to define the following bivariate density:

$$\begin{aligned} \mathrm{h}(x_1,x_2) &= \frac{\partial^2 \mathrm{H}(x_1,x_2)}{\partial x_1 \partial x_2} \\ &= \frac{1}{2\pi}\,(1+\alpha\,(2\Phi(x_1)-1)\,(2\Phi(x_2)-1))\exp\left\{-\frac{1}{2}\left(x_1^2+x_2^2\right)\right\} \end{aligned}$$

More generally, Gumbel proposed a multivariate distribution with the following structure:

$$\mathrm{H}(x_1,x_2,\ldots,x_N) = \prod_{i=1}^{N} \mathrm{F}_i(x_i)\left[1+\alpha\prod_{i=1}^{N}(1-\mathrm{F}_i(x_i))\right]$$

However, as shown by the bivariate example with gaussian margins, the ability to describe dependence is highly restricted since the correlation coefficient must satisfy by construction $|\rho| < \frac{1}{\pi}$, so there was a need to develop new families of copulae that allow for more general dependence. During the second part of the 20th century, many developments on copulae appeared in mathematics (especially in the field of probabilistic metric spaces – see the very important book by Schweizer and Sklar (1983)) and statistical research. We refer to the two main textbooks that summarize the "state of the art" on copulae: Joe (1997) and Nelsen (1998).

2.3 Financial Applications of copulae

The application of copulae to finance is quite recent. Certainly, the most influential paper is "Correlation and dependency in risk management : properties and pitfalls" by EMBRECHTS, MCNEIL and STRAUMANN (1999). Indeed, even if some previous papers introduced the advantages of introducing the concept of copula (for example FREES and VALDEZ (1998)) to financial problems, Embrechts *et al.* were the first to identify the power of the approach. Many of the original influential authors are members of the Eidgenössische Technische Hochschule Zürich and we especially refer the reader to the Masters Thesis of LINDSKOG (2000) for an introduction to copulae. Another research group has been playing an important role in the growing interest of financial researchers in copulae: the Groupe de Recherche Opérationelle of Crédit Lyonnais, a french bank's research department located in Paris. A much cited paper by BOUYÉ, DURRLEMAN, NICKEGHBALI, RIBOULET and RONCALLI (2000) was presented at the 2000 Bachelier Conference in Paris. The authors pointed out the various possible applications of copulae for finance.

PATTON (2001a) explored the use of copulae in econometrics. The author studies the dependence between the Deutsche mark - U.S. dollar and Yen - U.S. dollar exchange rate returns. He finds that the dependence is time-varying and asymmetric. Moreover he shows that the introduction of the euro has implied a change in the dependence structure. In another paper PATTON (2001b) studies a copula based multi-stage maximum likelihood estimator. An application of the estimator to daily Japanese yen - U.S. dollar and euro - U.S. dollar exchange rates is provided. Again, the author finds evidence of asymmetric dependence. An application of this asymmetric dependence to asset allocation for a constant relative risk aversion (CRRA) is provided in PATTON (2001c). In her job market paper, HU (2001) constructs a mixed copula to model positive dependence and provides some possible applications of her model. In this paper, three shapes are defined: L-shape for joint crashes, J-shape for joint booms and U-shape for symmetric comovements. In the same manner, JONDEAU and ROCKINGER (2001) apply bivariate copulae with GARCH-type margins to daily returns of stock-market indices. We note that the asymmetric dependence of financial returns was pointed out before by LONGIN and SOLNIK (2001) and ANG and CHEN (2001).

An important financial area where copulae are very useful is the modelling of joint defaults. The default correlation concept is developed in LI (2000). The copulae have then be used in the two approaches that coexist in the credit litterature: the structural approach *à la* Merton and the intensity approach (GIESECKE (2001)). In the intensity litterature, the margins correspond to the distribution of the survival time of each obligor and the copula function permits to introduce dependence between them. SCHÖNBUCHER and SCHUBERT (2001), develop a copula based dynamic model of the joint defaults in a continuous time framework. They obtain closed formulae for a specific copula family (called Archimedean) and furnish a *modus operandi* to simulate the joint behaviour of the default times. HAMILTON, JAMES and WEBBER (2001), discuss the calibration issue from historical default times. This litterature is of great interest, especially in the context of a growing credit derivatives market with two underlying issues: (i) the pricing of financial products (Collateralized Debt Obligation, Collateralized Fund Obligation, First to default, nth to default, etc) and (ii) their risk management as required by the banking regulations of Basle II. The copulae are also adopted for option pricing in CHERUBINI, U. and E. LUCIANO ((2002a), (2002b)).

2.4 Definitions and properties

In this section, we review the main definitions and properties that are necessary to begin to work with a copula function.

Definition 1 (Nelsen (1998), page 39) *An N-dimensional copula is a function* $\mathbf{C}$ *with the following properties:*

1. $\operatorname{Dom} \mathbf{C} = [0,1]^N$;

2. $\mathbf{C}$ *is grounded and N-increasing.*

3. $C_k(u) = u, \forall\, u \in [0,1], \forall\, k = 1, \ldots, N$ *with* $C_k(u) = \mathbf{C}(1, \ldots, 1, u, 1, \ldots, 1)$ *the k-th margin of the copula*

The first condition indicates that each element of the copula might be a cumulative distribution function that corresponds to a probability level (and thus bounded between 0 and 1). In order to satisfy the same property as a multivariate distribution, the copula has to be grounded, N-increasing and we should know its margins either parametrically or nonparametrically. Indeed, a copula is nothing else but a multivariate distribution defined on uniform margins.

Theorem 1 (Sklar's theorem) *Let* $\mathbf{F}$ *be an N-dimensional distribution function with continuous margins $F_1, \ldots, F_N$. Then* $\mathbf{F}$ *has a unique copula representation:*

$$\mathbf{F}(x_1, \ldots, x_N) = \mathbf{C}(F_1(x_1), \ldots, F_N(x_N)) \tag{2.4}$$

Let $\mathbf{f}$ be the N-dimensional density function of $\mathbf{F}$ defined as follows:

$$\mathbf{f}(x_1, \ldots, x_N) = \frac{\partial \mathbf{F}(x_1, \ldots, x_N)}{\partial x_1 \cdots \partial x_N} \tag{2.5}$$

Then we have

$$\mathrm{f}\left(x_1,\ldots,x_N\right)=\frac{\partial\,\mathrm{C}\left(F_1\left(x_1\right),\ldots,F_N\left(x_N\right)\right)}{\partial\,x_1\cdots\partial\,x_N}$$

With the notation $u_n = F_n\left(x_n\right)$ for $n = 1,\ldots,N$, we obtain

$$\mathrm{f}\left(x_1,\ldots,x_N\right)=\frac{\partial\,\mathrm{C}\left(u_1,\ldots,u_N\right)}{\partial\,u_1\cdots\partial\,u_N}\prod_{n=1}^{N}f_n\left(x_n\right)$$

with f_n the density corresponding to F_n. The term $\frac{\partial\,\mathrm{C}(u_1,\ldots,u_N)}{\partial\,u_1\cdots\partial\,u_N}$ is called the copula density of C and is noted $\mathrm{c}\left(u_1,\ldots,u_N\right)$. Obviously,

$$\mathrm{c}\left(F_1\left(x_1\right),\ldots,F_N\left(x_N\right)\right)=\frac{\mathrm{f}\left(x_1,\ldots,x_N\right)}{\prod_{n=1}^{N}f_n\left(x_n\right)}. \tag{2.6}$$

To illustrate the copula family, two famous copulae, Gaussian[2] and Gumbel[3] are plotted for different values of their parameter corresponding to different degrees of dependence (Figures 2.1 and 2.2). Figure 2.3 (respectively Figure

[2]The bivariate Gaussian copula distribution is

$$C\left(u_1,u_2;\rho\right)=\Phi_\rho\left(\Phi^{-1}\left(u_1\right),\Phi^{-1}\left(u_2\right)\right)$$

with Φ_ρ the bivariate gaussian cdf with correlation parameter ρ, and $\Phi^{-1}\left(.\right)$ the inverse gaussian cdf. The bivariate Gaussian copula density

$$c\left(u_1,u_2;\rho\right)=\frac{\exp\left[\frac{1}{2}\left(x^2+y^2\right)\right]}{\sqrt{1-\rho^2}}\exp\left\{-\frac{x^2+y^2-2\rho xy}{2\left(1-\rho^2\right)}\right\}$$

with $x=\Phi^{-1}\left(u_1\right)$ and $y=\Phi^{-1}\left(u_2\right)$.

[3]The distribution function of the Gumbel copula is

$$C(u,v;\delta)=\exp\left\{-((-\log u)^\delta+(-\log v)^\delta)^{1/\delta}\right\} \tag{2.7}$$

with $\delta\in[1,\infty)$ and its density function is

$$c(u,v;\delta)=C(u,v;\delta)(uv)^{-1}\frac{(\log u\log v)^{\delta-1}}{(-\log u)^\delta+(-\log v)^\delta)^{2-1/\delta}}\left[((-\log u)^\delta+(-\log v)^\delta)^{1/\delta}+\delta-1\right]. \tag{2.8}$$

2.4) plots three different bivariate distributions based on the same Gaussian (respectively Gumbel) copula with different margins.

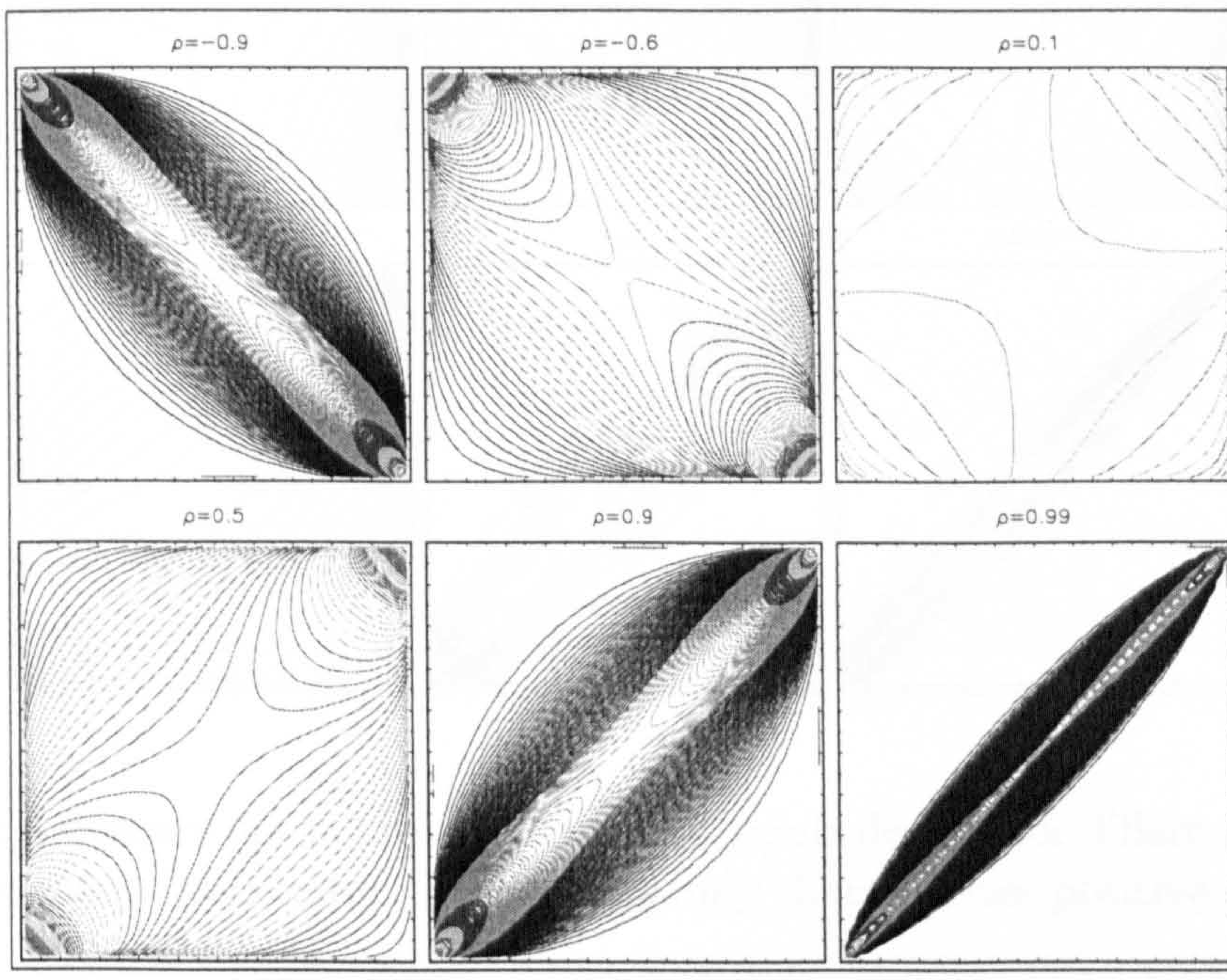

Figure 2.1: Contours of the bivariate Gaussian copula density for different values of the correlation parameter. The two first plots corresponds to negative dependence. The other plots exhibit positive dependence.

Definition 2 *Let* $(\mathbf{C}, \mathbf{D}) \in \mathcal{C}^2$ *with* $\mathcal{C}$ *the set of copulae. One says that* $\mathbf{C}$ *is greater than* $\mathbf{D}$*, noted* $\mathbf{C} \succ \mathbf{D}$*, if*

$$\forall \mathbf{u} \in [0,1]^N, \ \mathbf{C}(\mathbf{u}) \geq \mathbf{D}(\mathbf{u})$$

Theorem 2 (Fréchet Bounds) *Let* $\mathbf{C} \in \mathcal{C}$*. Then,*

$$\mathbf{C}^- \prec \mathbf{C} \prec \mathbf{C}^+$$

where $\mathbf{C}^-$ *and* $\mathbf{C}^+$ *are such that*

$$\mathbf{C}^-(u_1, \ldots, u_N) = \max\left(\sum_{n=1}^{N} u_n - N + 1, 0\right)$$
$$\mathbf{C}^+(u_1, \ldots, u_N) = \min(u_1, \ldots, u_N) \qquad (2.9)$$

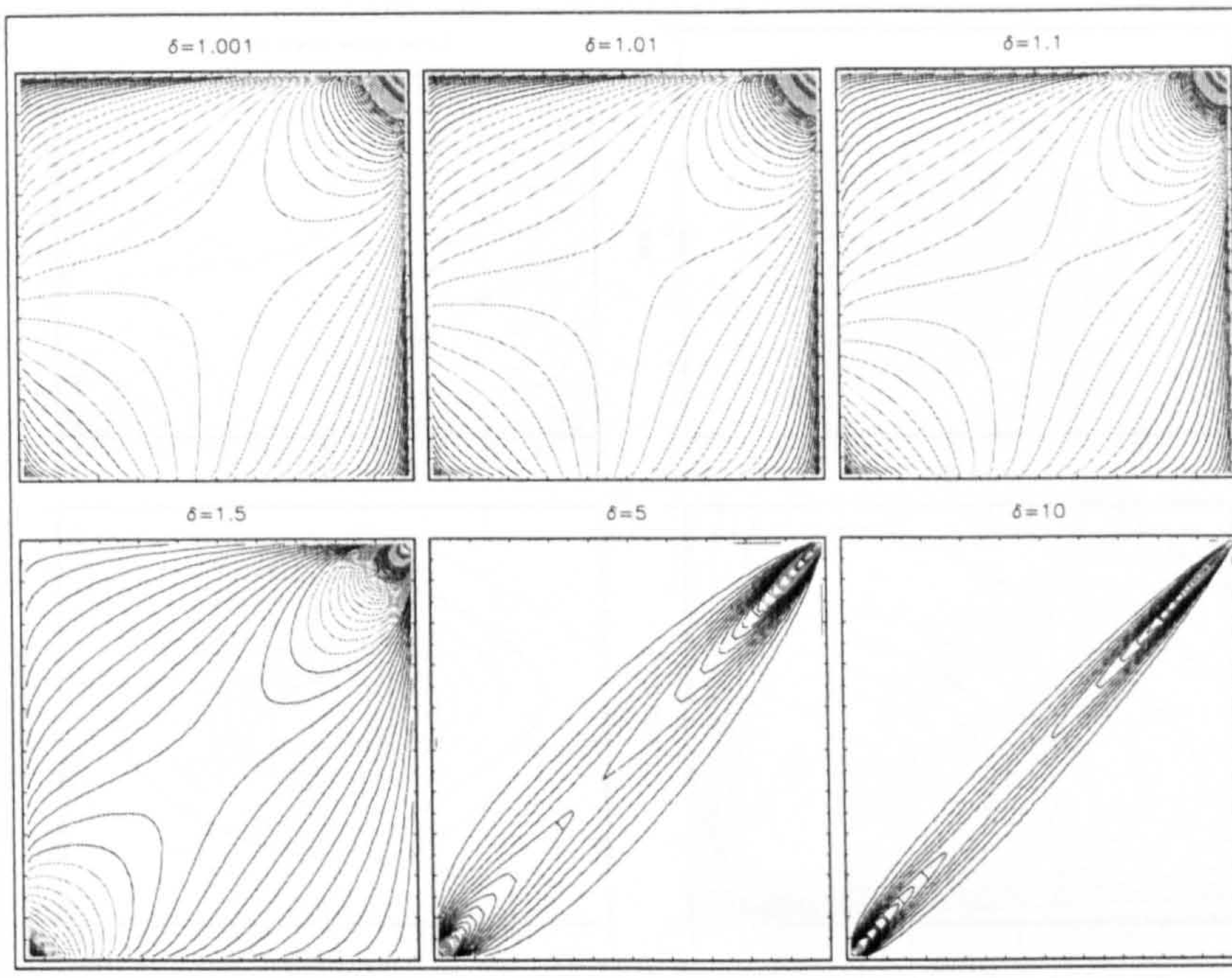

Figure 2.2: Contours of the bivariate Gumbel copula density for different values of the dependence parameter. This copula only characterizes positive dependence.

The concept of order is of interest for copulae since it allows us to quantify the dependence between random variables. Another important copula is the product copula $\mathbf{C}^{\perp}$- that corresponds to independence - such that

$$\mathbf{C}^{\perp}(u_1, \ldots, u_N) = \prod_{n=1}^{N} u_n.$$

The Fréchet bounds and the product copulae are plotted in Figure 2.5.

Theorem 3 ***Copula invariance***

If a vector of random variables $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N)$ *has a copula* $\mathbf{C}$*, then the vector*

$(\mathbf{T}_1(\mathbf{X}_1), \mathbf{T}_2(\mathbf{X}_2), \ldots, \mathbf{T}_N(\mathbf{X}_N))$ *- with* $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_N$ *increasing continuous functions - has the same copula* $\mathbf{C}$.

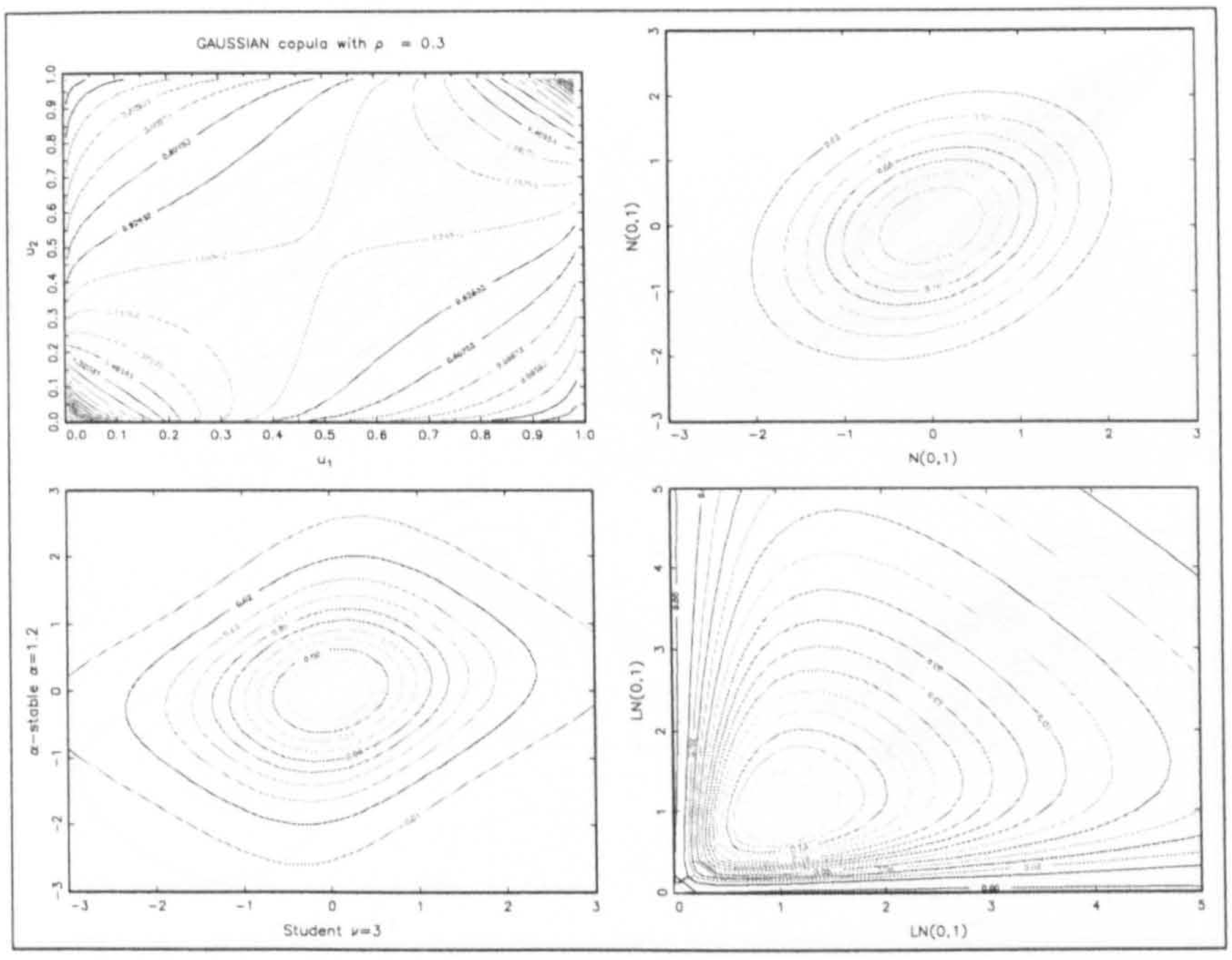

Figure 2.3: Three different bivariate distributions based on the same Gaussian copula with correlation parameter $\rho = 0.3$ but with different margins. The margins are respectively: (i) gaussians, (ii) α-stable and Student, (iii) lognormals.

2.5 General types of dependency

In this section, we review several different types of dependency. Moreover, we indicate how the measures can be linked to the copulae functions.

2.5.1 Concordance and dependence measures

Definition 3 (Nelsen (1998), page 136) *A numeric measure κ of association between two continuous random variables X_1 and X_2 whose copula is $\mathbf{C}$ is a measure of concordance if it satifies the following properties:*

1. *κ is defined for every pair X_1, X_2 of continuous random variables;*

2. $-1 = \kappa_{X,-X} \leq \kappa_{\mathbf{C}} \leq \kappa_{X,X} = 1$;

3. $\kappa_{X_1,X_2} = \kappa_{X_2,X_1}$;

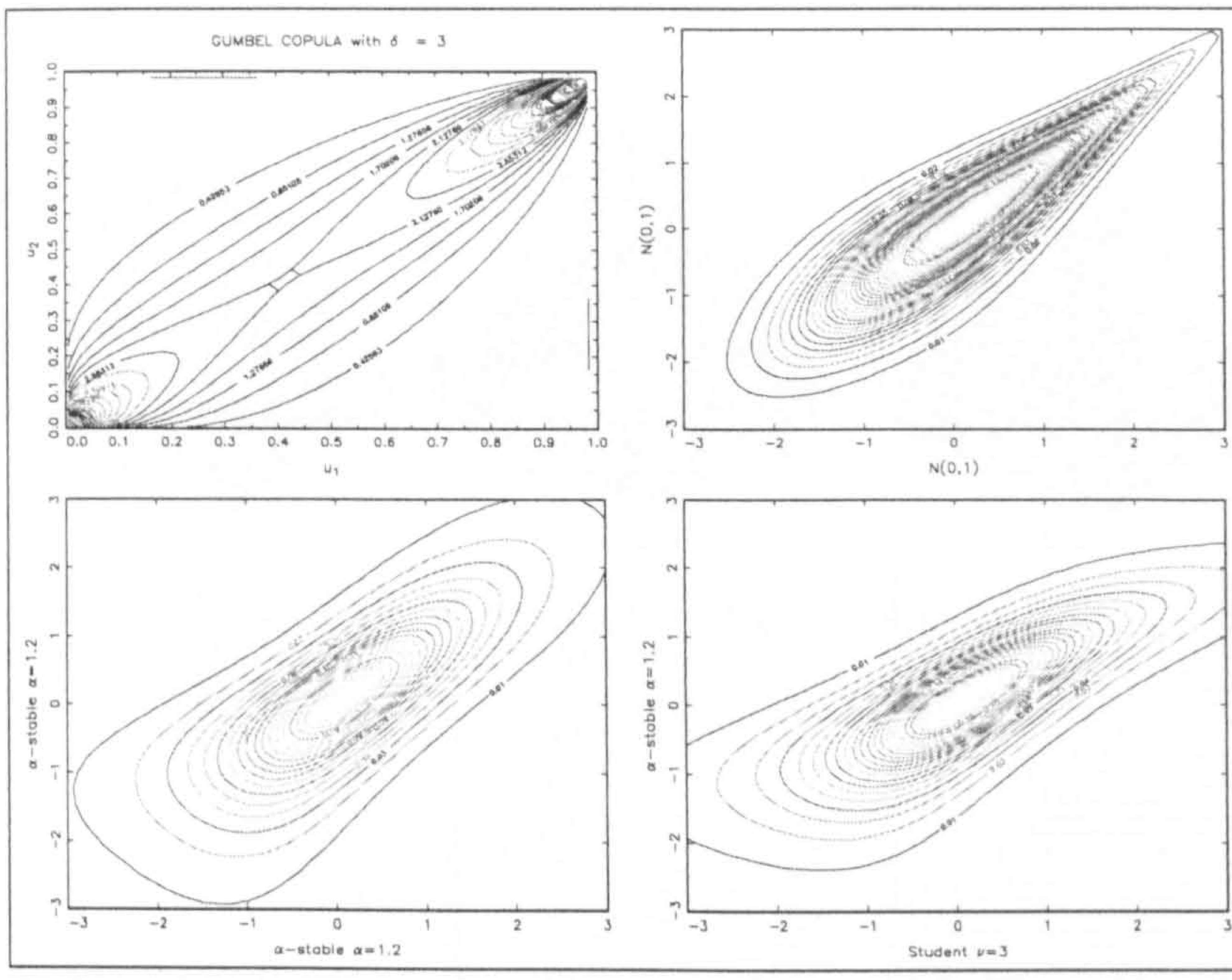

Figure 2.4: Three different bivariate distributions based on the same Gumbel copula with dependence parameter $\delta = 3$ but with different margins. The margins are respectively: (i) gaussians, (ii) α-stables, (iii) α-stable and Student.

4. *if X_1 and X_2 are independent, then* $\kappa_{X_1,X_2} = \kappa_{\mathbf{C}^\perp} = 0$;

5. $\kappa_{-X_1,X_2} = \kappa_{X_1,-X_2} = -\kappa_{X_1,X_2}$;

6. *if* $\mathbf{C}_1 \prec \mathbf{C}_2$, *then* $\kappa_{\mathbf{C}_1} \leq \kappa_{\mathbf{C}_2}$;

7. *if* $\{(X_{1,n}, X_{2,n})\}$ *is a sequence of continuous random variables with copulae* $\mathbf{C}_n$, *and if* $\{\mathbf{C}_n\}$ *converges pointwise to* $\mathbf{C}$, *then* $\lim_{n\to\infty} \kappa_{\mathbf{C}_n} = \kappa_{\mathbf{C}}$.

Three famous concordance measures are the Kendall's tau τ, the Spearman's rho ϱ and the Gini index γ. SCHWEITZER and WOLFF (1981) show that

$$\tau = 4\iint_{[0,1]^2} \mathbf{C}(u_1,u_2)\,\mathrm{d}\mathbf{C}(u_1,u_2) - 1 \tag{2.10}$$

$$\varrho = 12\iint_{[0,1]^2} u_1 u_2\,\mathrm{d}\mathbf{C}(u_1,u_2) - 3 \tag{2.11}$$

$$\gamma = 2\iint_{[0,1]^2} \left(|u_1+u_2-1| - |u_1-u_2|\right)\mathrm{d}\mathbf{C}(u_1,u_2) \tag{2.12}$$

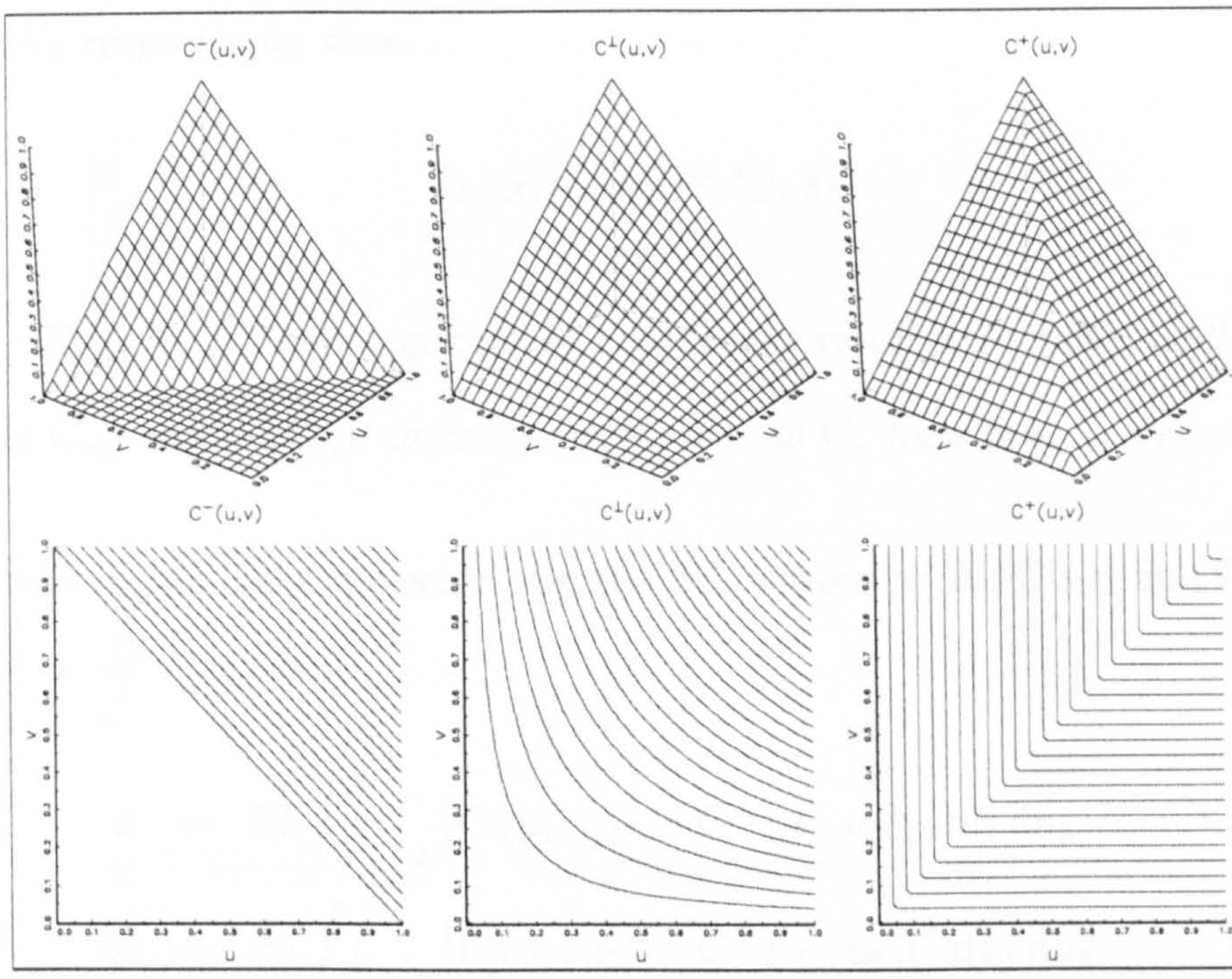

Figure 2.5: $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$ in the bivariate case

Definition 4 (Nelsen (1998), page 170) *A numeric measure δ of association between two continuous random variables X_1 and X_2 whose copula is $\mathbf{C}$ is a measure of dependence if it satifies the following properties:*

1. *δ is defined for every pair X_1, X_2 of continuous random variables;*

2. $0 = \delta_{\mathbf{C}^\perp} \leq \delta_{\mathbf{C}} \leq \delta_{\mathbf{C}^+} = 1$*;*

3. $\delta_{X_1,X_2} = \delta_{X_2,X_1}$*;*

4. $\delta_{X_1,X_2} = \delta_{\mathbf{C}^\perp} = 0$ ***if and only if*** *X_1 and X_2 are independent;*

5. $\delta_{X_1,X_2} = \delta_{\mathbf{C}^+} = 1$ ***if and only if*** *each of X_1 and X_2 is almost surely a strictly monotone function of the other;*

6. *if h_1 and h_2 are almost surely strictly monotone functions on* $\operatorname{Im} X_1$ *and*

$\operatorname{Im} X_2$ *respectively, then*

$$\delta_{h_1(X_1),h_2(X_2)} = \delta_{X_1,X_2}$$

7. *if* $\{(X_{1,n}, X_{2,n})\}$ *is a sequence of continuous random variables with copulae* $\mathbf{C}_n$*, and if* $\{\mathbf{C}_n\}$ *converges pointwise to* $\mathbf{C}$*, then* $\lim_{n\to\infty} \delta_{\mathbf{C}_n} = \delta_{\mathbf{C}}$.

Two famous dependence measures are the Schweitzer or Wolff's σ and Hoeffding Φ^2:

$$\sigma = 12 \iint_{[0,1]^2} \left|\mathbf{C}(u_1,u_2) - \mathbf{C}^{\perp}(u_1,u_2)\right| \mathrm{d}u_1 \, \mathrm{d}u_2 \tag{2.13}$$

$$\Phi^2 = 90 \iint_{[0,1]^2} \left|\mathbf{C}(u_1,u_2) - \mathbf{C}^{\perp}(u_1,u_2)\right|^2 \mathrm{d}u_1 \, \mathrm{d}u_2 \tag{2.14}$$

2.5.2 Other dependence concepts

There are many other dependence concepts, that are useful for financial applications. For example, X_1 and X_2 are said to be positive quadrant dependent (PQD) if

$$\Pr\{X_1 > x_1, X_2 > x_2\} \geq \Pr\{X_1 > x_1\} \Pr\{X_2 > x_2\} \tag{2.15}$$

Suppose that X_1 and X_2 are random variables standing for two financial losses. The probability of simultaneous large losses is greater for dependent variables than for independent ones. In terms of copulae, relation (2.15) is equivalent to

$$\mathbf{C} \succ \mathbf{C}^{\perp} \tag{2.16}$$

The notion of tail dependence is also interesting since it allows to focus on the joint extremal behaviour of random variables. Joe (1997) gives the following definition:

Definition 5 *If a bivariate copula* $\mathbf{C}$ *is such that*[4]

$$\lim_{u \to 1} \frac{\bar{\mathbf{C}}(u,u)}{1-u} = \lambda \tag{2.20}$$

exists, then $\mathbf{C}$ *has upper tail dependence for* $\lambda \in (0,1]$ *and no upper tail dependence for* $\lambda = 0$.

The measure λ is extensively used in extreme value theory. It is the probability that one variable is extreme given that the other is extreme. Let $\lambda(u) = \Pr\{U_1 > u | U_2 > u\} = \frac{\bar{\mathbf{C}}(u,u)}{1-u}$. $\lambda(u)$ can be viewed as a "quantile-dependent measure of dependence" (Coles, Currie and Tawn (1999)). In Figure 2.6, the tail dependence estimates between the daily log-returns of MSCI UK and MSCI France from january 1987 to january 2002 are plotted. It appears that the dependence between negative returns is not well fitted by the Gaussian copula.

2.6 Nonparametric modelling

Deheuvels (1979) introduced the empirical copula functions. For a sample $\{x_t\}_{t=1\ldots T}$, the empirical copula distribution is

$$\hat{\mathbf{C}}\left(\frac{t_1}{T},\ldots,\frac{t_n}{T},\ldots,\frac{t_N}{T}\right) = \frac{1}{T}\sum_{t=1}^{T} 1_{\left[x_1^t \le x_1^{(t_1)},\ldots,x_n^t \le x_n^{(t_n)},\ldots,x_N^t \le x_N^{(t_N)}\right]} \tag{2.21}$$

[4] $\bar{\mathbf{C}}$ is the joint survival function, that is

$$\bar{\mathbf{C}}(u_1,u_2) = 1 - u_1 - u_2 + \mathbf{C}(u_1,u_2) \tag{2.17}$$

Note that it is related to the survival copula $\mathbf{C}_S$

$$\mathbf{C}_S(u_1,u_2) = u_1 + u_2 - 1 + \mathbf{C}(1-u_1,1-u_2) \tag{2.18}$$

in the following way

$$\bar{\mathbf{C}}(u_1,u_2) = \mathbf{C}_S(1-u_1,1-u_2) \tag{2.19}$$

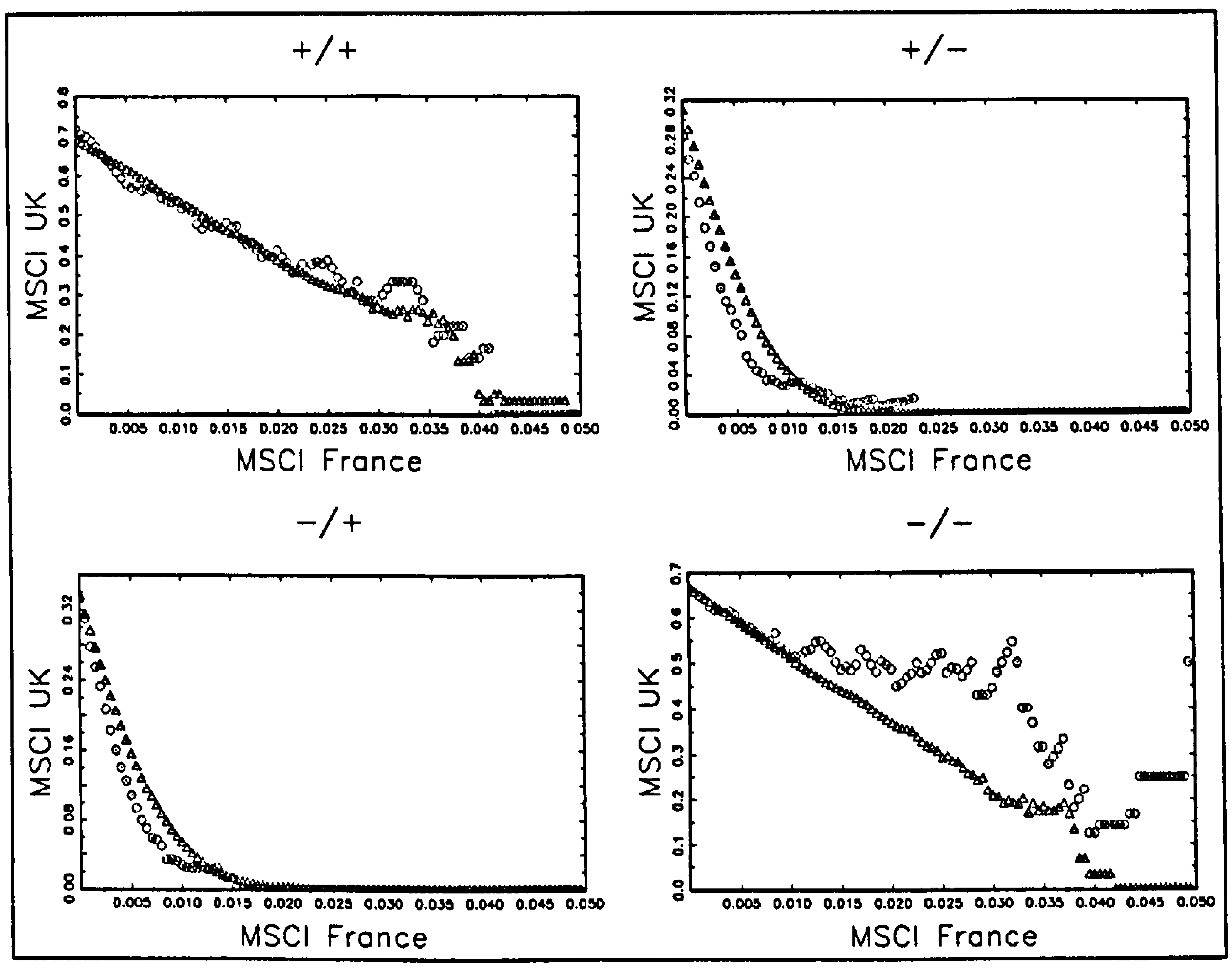

Figure 2.6: Tail dependence estimates between the daily log-returns of MSCI UK and MSCI France from (01/1987-01/2002). The absciss axis correspond to different quantiles. The triangles correspond to tail dependence for simulated bivariate returns under the assumption of a gaussian copula. The circles represent the empirical tail dependence directly extracted from data. The 4 windows correspond to the 4 cases: (i) +/+ a short position in MSCI France and a short position in MSCI UK (ii) +/− a short position in MSCI France and a long position in MSCI UK (iii) −/+ a long position in MSCI France and a short position in MSCI UK (iv)−/− a long position in MSCI France and a long position in MSCI UK

where $x_n^{(t)}$ are the order statistics and $1 \leq t_1, \ldots, t_N \leq T$. The empirical copula frequency corresponds to $\hat{c}\left(\frac{t_1}{T}, \ldots, \frac{t_n}{T}, \ldots, \frac{t_N}{T}\right) = \frac{1}{T}$ if $\left(x_1^{(t_1)}, \ldots, x_N^{(t_N)}\right)$ belongs to $\mathcal{X}$ or 0 otherwise. The relationships between empirical copula distribution and frequency are

$$\hat{\mathrm{C}}\left(\frac{t_1}{T}, \ldots, \frac{t_n}{T}, \ldots, \frac{t_N}{T}\right) = \sum_{i_1=1}^{t_1} \cdots \sum_{i_N=1}^{t_N} \hat{c}\left(\frac{i_1}{T}, \ldots, \frac{i_n}{T}, \ldots, \frac{i_N}{T}\right) \tag{2.22}$$

and

$$\hat{c}\left(\frac{t_1}{T}, \ldots, \frac{t_n}{T}, \ldots, \frac{t_N}{T}\right) = \sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} (-1)^{i_1+\cdots+i_N} \times \hat{\mathrm{C}}\left(\frac{t_1 - i_1 + 1}{T}, \ldots, \frac{t_N - i_N + 1}{T}\right) \tag{2.23}$$

The empirical copula density function for the daily log-returns of MSCI UK and MSCI France is provided in Figure 2.7. Empirical copulae could be used to estimate dependence measures. For example, estimation of Spearman's ϱ is given by

$$\hat{\varrho} = \frac{12}{T^2 - 1} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \left(\hat{\mathrm{C}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) - \frac{t_1 t_2}{T^2}\right) \tag{2.24}$$

Scaillet (2001) derives the asymptotic properties of kernel estimators of copulae for multivariate stationary process. The author also develops independence tests and furnish an empirical application to European and US stock returns.

2.7 Copulae, Likelihood and Estimation

The maximum likelihood estimation (MLE) method is certainly the most widely used estimation method in the statistical litterature. For copulae, three

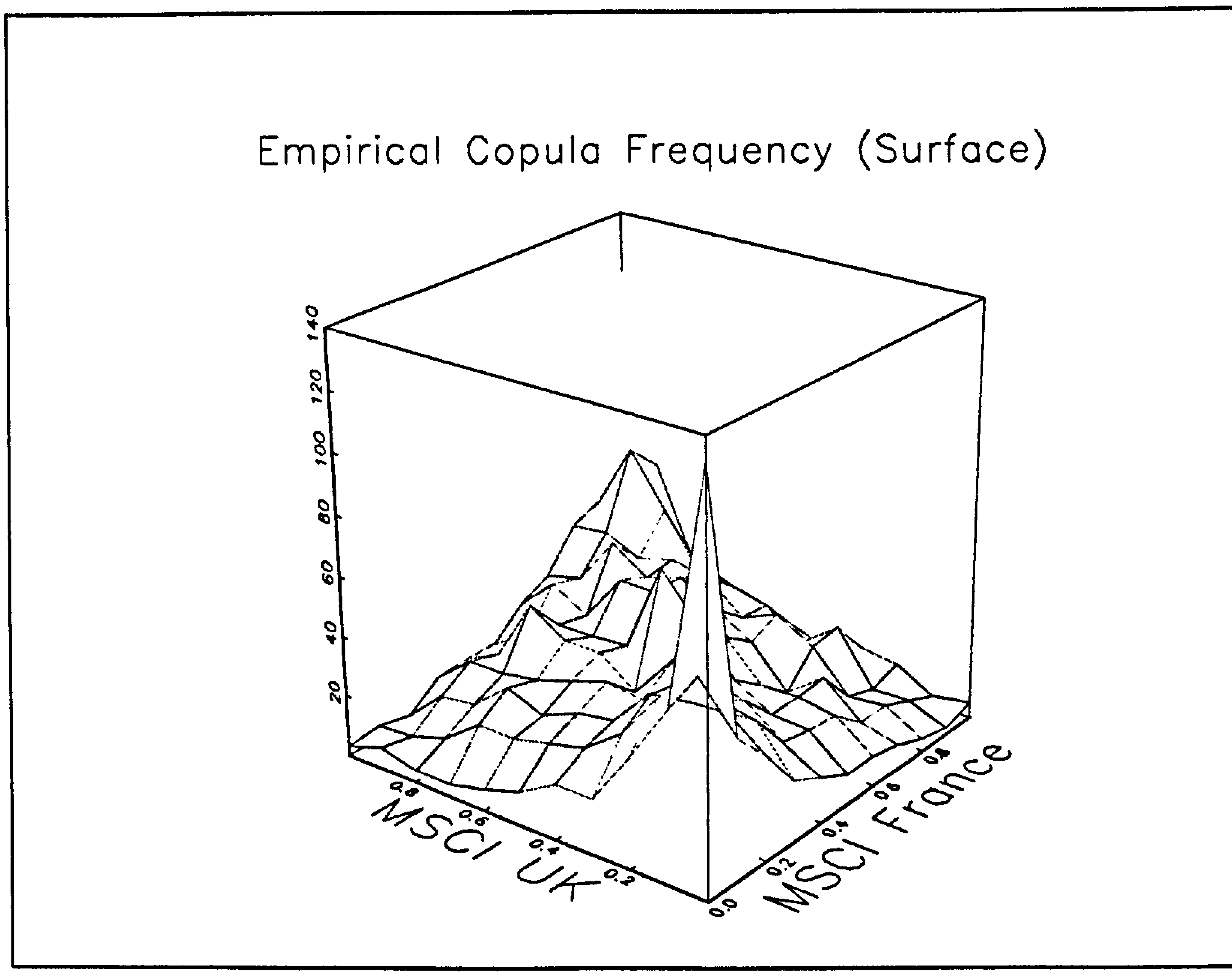

Figure 2.7: Empirical copula density estimation between the daily log-returns of MSCI UK and MSCI France from (01/1987-01/2002).

methods are available (see JOE and XU (1996)):

1. The full MLE method: the parameters of the copula and margins are estimated simultaneously.

2. The Inference Function for Margins (IFM) method: it is a two-step procedure. First, the parameters of the margins are estimated. Second, MLE is applied to the copula.

3. The empirical MLE method: only the parameters of the copula are estimated. The cdfs are obtained by empirically mapping variables to uniforms.

Let us develop these methods. Let $\mathbf{X}$ be a d-dimensional random variable with the following distribution function

$$\mathrm{F}(\mathrm{x};\gamma_1,\ldots,\gamma_d,\delta)=\mathrm{C}\left(F_1(x_1;\gamma_1),\ldots,F_d(x_d;\gamma_d),\delta\right)$$

where γ_j $(j=1,\ldots,d)$ the parameters of the margins and δ the parameters of the copula function. Then, $\mathbf{X}$ has for density

$$\mathrm{f}(\mathrm{x};\gamma_1,\ldots,\gamma_d,\delta)=\mathrm{c}\left(F_1(x_1;\gamma_1),\ldots,F_d(x_d;\gamma_d),\delta\right)\prod_{j=1}^{d}f_j(x_j;\gamma_j) \qquad (2.25)$$

2.7.1 The full MLE method

The log-likelihood of the joint distribution function for a sample of size T is

$$L(\mathrm{x};\gamma_1,\ldots,\gamma_d,\delta)=\sum_{i=1}^{T}\log\mathrm{f}(\mathrm{x}_i;\gamma_1,\ldots,\gamma_d,\delta) \qquad (2.26)$$

The MLE estimates $(\hat{\gamma}_1,\ldots,\hat{\gamma}_d,\hat{\delta})$ maximize L, they are obtained from solving:

$$\boxed{\left(\frac{\partial L}{\partial\gamma_1},\ldots,\frac{\partial L}{\partial\gamma_d},\frac{\partial L}{\partial\delta}\right)=0} \qquad (2.27)$$

2.7.2 The Inference Function for Margins (IFM) : a 2-step method

First, the d log-likelihood functions for the univariate margins are considered:

$$L_j(\gamma_j)=\sum_{i=1}^{T}\log f_j(x_{ij};\gamma_j) \quad \text{for} \quad j=1,\ldots,d \qquad (2.28)$$

and the estimates $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_d$ maximize respectively $L_1,\ldots,L_d$. Second, the log-likelihood of the joint distribution function $L(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_d,\delta)$ is maximized over δ to obtain $\tilde{\delta}$. Finally, the IFM estimates $(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_d,\tilde{\delta})$ are obtained from solving:

$$\boxed{\left(\frac{\partial L_1}{\partial\gamma_1},\ldots,\frac{\partial L_d}{\partial\gamma_d},\frac{\partial L}{\partial\delta}\right)=0} \qquad (2.29)$$

2.7.3 The empirical MLE method : no parametric assumption for marginals

First, the d variables are mapped to uniforms:

$$\mathbf{X} \in \mathbf{R}^d \longmapsto \mathbf{u} \in (0,1)^d$$

Then, the parameters of the copula δ are obtained by maximizing the log-likelihood of the copula cdf:

$$L_c(\delta) = \sum_{i=1}^{T} \log c(\mathbf{u}_i; \delta). \tag{2.30}$$

The estimate $\bar{\delta}$ is obtained by solving:

$$\boxed{\frac{\partial L_c}{\partial \delta} = 0.} \tag{2.31}$$

2.7.4 Comparison of the three methods

The empirical and IFM methods have the advantage to make inference for multivariate models computationally feasible. Anyway, these methods are useful to fix the starting values for the full MLE method. As noted in Joe and Xu (1996), IFM and full MLE are equivalent for the multivariate normal distribution. The authors also provide the general types of conditions that must hold for the IFM estimator to be asymptotically normally distributed. However it is difficult to provide a general framework for the comparison of the IFM and full MLE estimators (problems of tractability). The properties of the empirical MLE estimator are investigated in Genest, Ghoudi and Rivest (1995). They show that this estimator is consistent, asymptotically normal and fully efficient at independence. There also exists a consistent estimator of its asymptotic variance. The three approaches generally may not lead to equivalent estimators and Monte Carlo simulations have often to be used. As an example, the three methods are compared in Figure 2.8 for a Gaussian copula with correlation 0.5. The larger the sample size, the closer the estimates of the three MLE methods. For multidimensional finite sample (typically financial data), there is a trade-off between (i) the efficiency of the estimation and (ii) the tractability of the estimation. Indeed, the full MLE method is more efficient but less tractable than the two others methods.

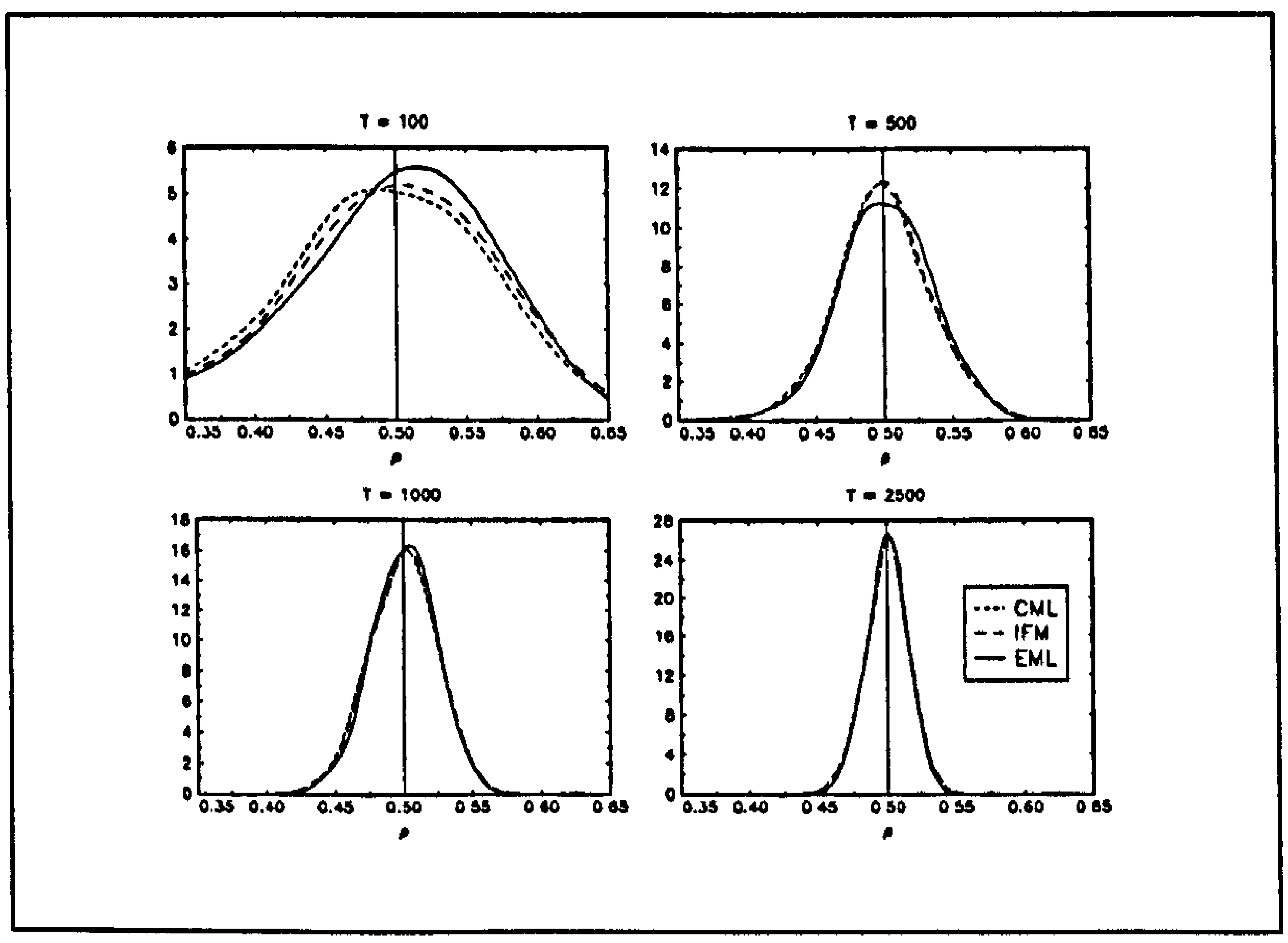

Figure 2.8: Comparison of the estimates for the three MLE methods. A Gaussian copula with $\rho = 0.5$ is simulated and the estimations are performed for different sample sizes.

Chapter 3

Portfolio Risk Management : an Application of Multivariate Extremes

3.1 Contents and contributions

The chapter starts with some standard definitions of the extreme value theory in its second section. The univariate case is presented (Gumbel (1960)). The multivariate case that introduces the copulae that are eligible for modelling extrema is due to Deheuvels (1978). Deheuvels proposed a theorem that constitutes the theoretical starting point of the chapter. Then, three families of multivariate extreme value distributions are described: Gumbel (1960), Hüsler and Reiss (1989), Joe and Hu (1996). The Gumbel copula is the first extreme copula to be used in the financial literature, but only in the bivariate case (Longin and Solnik (2001), Journal of Finance). To our knowledge, the explicit formulation of the Hüsler and Reiss copula is new. The chapter is the first application of the inference for margins method (Joe and Xu (1996)) to extreme value distributions. Section 4 contributes to the literature in two ways: (i) it contains the first trivariate extreme value estimation applied to stock market indices: we provide location, scale and tail parameters estimates of the univariate extreme value distributions for the MSCI US, MSCI Germany and MSCI Japan indices. Moreover, the parameters estimates for three extreme value copulae are computed. This provides a new link between the three markets described above ; (ii) it offers a comparison of three non-nested extreme value models and concludes that the Hüsler and Reiss copula we introduced best fits the data (Cox test criterion). In section 5, we present a new concept - the financial failure area - and apply it to stock market returns. We furnish a new *modus operandi* for risk management with formulae and examples. Finally, we propose an innovative copula based methodology to compute standard risk measures such as value-at-risk or expected shortfall and we again provide original empirical applications.

3.2 Introduction

The behaviour of portfolios during financial crises is an important element of risk management (Basle Committees I and II). The goal of this chapter is to construct a methodology to calculate risk measures – such as value at risk and expected shortfall – which directly come from the extremal dependence structure between portfolio components. This is achieved by considering their multivariate extreme value (MEV) probability distributions.

Extreme value theory (EVT) is now a well developed tool used to model maxima and minima of financial returns. A seminal paper is Embrechts and Schmidli (1994) in an insurance context. Longin (1996) provides a study of stock market extreme returns. An influential book that provides a "state of the art" of the subject is Modelling Extremal Events for Insurance and Finance by Embrechts, Klüppelberg and Mikosch (1997). However, the extension to multivariate modelling is not obvious, as pointed out by Embrechts, de Haan and Huang (2000). However, some examples of MEV theory can be found in the non-financial literature (for example Coles and Tawn (1991), Coles and Tawn (1994), de Hann and de Ronde (1998)). For an overview of the theoretical aspects of the subject, we refer to Resnick (1987).

In the financial literature, some measures for extremal dependence between returns can be found in Straetmans (1999) and Stărică (1999) and Longin (2000) who proposed an approach based on EVT for computing value at risk compatible with extreme events. Longin provides an ad hoc aggregation formula to approximate the value at risk.

As noted above MEV distributions often become analytically intractable. An interesting way to avoid these difficulties is to use a copula function that allows us to split the univariate extremes from their dependence structure. Concerning the application of copulae to joint extreme events, a bivariate case with two assets is presented in Longin and Solnik (2001) who use a Gumbel copula to study the conditional correlation structure of international equity returns. However, as we will show in this chapter, there are many possible copulae to model joint extremal dependence. These copulae may exhibit different dependence structures. For example, the Gumbel copula induces a particular (called compounded) dependence structure if the dimension is higher than two as we will further see in more details.

In the second section, we introduce univariate EVT and review the link between copulae and MEV distributions. Then, we present three copulae that can be used in an extreme value context. In the third section, we describe our estimation methodology and provide an application to the joint dependence of German, Japanese and US market indices during extreme events. In the fourth section, we use the estimated parameters of the MEV distribution to compute risk measures for multi-indices portfolios. Specifically, the risk of the portfolios is studied from two directions: (i) multivariate stress testing and (ii) Monte-Carlo based risk measures. The fifth section concludes.

3.3 Multivariate Extreme Value Theory

In this section, we first briefly introduce univariate EVT. We then state a theorem that tells us that a MEV distribution can be built from univariate extreme value distributions and a specific family of copulae. We present the three copulae that will be used through this chapter. The results for maxima are developed, although equivalent results exist straight forwardly for minima.

3.3.1 Preliminaries

The general context of univariate extreme value theory is easily explained. A very useful result is the Fisher-Tippett theorem which tells us that normalised maxima - under particular conditions - follow one of only three (extreme value) distributions. For i.i.d. random variables (X_n), if there are constants $a_n > 0$, $b_n \in R$ and a non degenerate function G with $a_n^{-1}(\chi^+ - b_n) \xrightarrow{d} G$ where $\chi^+ = \max(X_{1,\dots,}X_n)$, then G corresponds to:

Type I (Gumbel)	$G(x) =$	$\exp(-e^{-x})$	$x \in \mathbb{R}$	
Type II (Frechet)	$G(x) =$	$\begin{cases} 0 \\ \exp(-x^{-\alpha}) \end{cases}$	$\begin{matrix} x \leq 0 \\ x > 0 \end{matrix}$	$\alpha > 0$
Type III (Weibull)	$G(x) =$	$\begin{cases} \exp(-(-x)^{\alpha}) \\ 1 \end{cases}$	$\begin{matrix} x \leq 0 \\ x > 0 \end{matrix}$	$\alpha > 0$

In practice the Von-Mises representation encompases this result and provides a unique distribution for all extremes:

$$G(\gamma;\chi^{+}) = \exp\left\{-\left(1-\tau\frac{\chi^{+}-b}{a}\right)^{1/\tau}\right\} \tag{3.1}$$

with $\left(1-\tau\frac{\chi^{+}-b}{a}\right) > 0$ and $\gamma = (\tau, a, b)$. We recover the three cases as $\tau = 0$ (Gumbel), $\tau = -\alpha^{-1} < 0$ (Frechet) and $\tau = \alpha^{-1} > 0$ (Weibull). This distribution is called the Generalised Extreme Value (GEV) distribution.

The theory of multivariate extremes was introduced by Gumbel (1960) and an overview can be found in Resnick (1987). The main reference given our current objective is Deheuvels (1978) which contains a theorem that allows us to split the problem of characterising multivariate extreme value distributions into two distinct problems:

1. the characterisation of the univariate extreme value distributions
2. the existence of a limiting dependence function (or copula) that links univariate extreme value distributions in order to obtain the multivariate extreme value distribution.

This idea is summarized in the following theorem:

Theorem 4 (Deheuvels (1978)) *Let χ_n^{+} be such that*

$$\chi_n^{+} = \left(\chi_{1,n}^{+},\ldots,\chi_{d,n}^{+}\right) = \left(\bigvee_{k=1}^{n} X_{1,k},\ldots,\bigvee_{k=1}^{n} X_{d,k}\right) \tag{3.2}$$

with $(X_{1,n},\ldots,X_{d,n})$ an i.i.d. sequence of random vectors with distribution function F, marginal distributions $F_1,\ldots,F_d$ and copula C. Then,

$$\lim_{n\to\infty} \Pr\left\{\frac{\chi_{1,n}^{+}-b_{1,n}}{a_{1,n}} \le x_1,\ldots,\frac{\chi_{d,n}^{+}-b_{d,n}}{a_{d,n}} \le x_d\right\} = \mathbf{G}_{\infty}(x_1,\ldots,x_d) \quad \forall\,(x_1,\ldots,x_d) \in \mathbb{R}^N \tag{3.3}$$

with $a_{j,n} > 0, j = 1, \ldots, d, n \geq 1$ iff

1. $\forall j = 1, \ldots, d$, there exist some constants $a_{j,n}$ and $b_{j,n}$ and a non-degenerate limit distribution G_j such that

$$\lim_{n\to\infty} \Pr\left\{\frac{\chi_{1,n}^{+} - b_{j,n}}{a_{j,n}} \leq x_j\right\} = G_j(x_j) \qquad \forall x \in \mathbb{R} \tag{3.4}$$

2. there exists a copula C_∞ such that

$$\mathbf{C}_\infty(u_1, \ldots, u_d) = \lim_{n\to\infty} \mathbf{C}^n\left(u_1^{1/n}, \ldots, u_d^{1/n}\right). \tag{3.5}$$

If the conditions of the previous theorem are fulfilled, we have

$$\mathbf{G}_\infty(x_1, \ldots, x_d) = \mathbf{C}_\infty(G_1(x_1), \ldots, G_d(x_d)) \tag{3.6}$$

The first condition is not specific to the multivariate case and is already present in univariate EVT. It corresponds to an existence condition. The second condition directly informs us about the dependence structure that allows us to obtain MEV distributions with given margins. The link between the one dimensional extremes is obtained by applying the copula function C_∞. The concept of maximum domain of attraction (MDA) is sometimes alternatively used. In the theorem above, each real-valued random variable X_j for $j = 1, \ldots, d$ has its own univariate distribution function F_j. And each maximum χ_j (respectively corresponding to X_j) follows an extreme value distribution G_j (amongst the three already presented: Gumbel, Fréchet and Weibull). We say that F_j belongs to the maximum domain of attraction of G_j. This concept can be extended to the multivariate distribution F that belongs to the MDA of the MEV distribution G_∞. By introducing copula - see Galambos (1978) for more details - the theorem can be restated as follows:

Theorem 5 *$F \in \mathrm{MDA}(\mathrm{G}_\infty)$ iff*

1. $F_j \in \mathrm{MDA}(G_j)\ \forall j = 1, \ldots, d$;

2. $C \in \mathrm{MDA}(\mathrm{C}_\infty)$.

The copula C models the dependence of the original real-valued random variables and C_∞ links their maxima. We will call C_∞ an extreme value copula.

3.3.2 Some families of MEV copulae

There are many candidates for extreme value copula. As indicated by Deheuvel's theorem, an extreme value copula C should satisfy[1] :

$$\mathrm{C}\left(u_1^t, \ldots, u_N^t\right) = \mathrm{C}^t\left(u_1, \ldots, u_N\right) \qquad \forall t > 0 \tag{3.8}$$

A corollary of this condition is that extreme value copulae only model positive dependence and this will influence our modelling strategy as we will see below for the empirical estimation of the copulae parameters. For an overview of extreme value copulae, we refer to Joe (1997). However, many of them are not tractable in high dimensions. For our study, we focus on three copulae: Gumbel, Hüsler and Reiss, and Joe and Hu. Our choice is motivated by the fact that these copulae can be expressed in a recursive form. This property is of particular interest from a numerical point of view as it means that the copula of dimension d can be directly deduced from the copula of dimension

[1] *We note that the multivariate gaussian distribution – that is often an assumption in financial theory – is not a good candidate for our current objective. More precisely, to postulate that a vector follows a multivariate gaussian d.f. is equivalent to assume that: (i) each vector component follows an univariate gaussian distribution, (ii) the link between the components is provided by the gaussian copula. The problem arises from the fact that the gaussian copula belongs to the MDA of the product copula $\mathrm{C}^\perp$ such that*

$$\mathrm{C}^\perp\left(u_1, \ldots, u_n\right) = u_1 \times \ldots \times u_n \tag{3.7}$$

In short, this would lead to assume independence for the extremes.

$(d-1)$. In the following subsections, we will provide the functional forms of the chosen copulae, the formulae that allow us to extend them to higher dimensions and we will discuss the dependence structure they exhibit. Let us denote by $u_d = (u_1, \ldots, u_d) = (G_1(x_1), \ldots, G_d(x_d))$ the d-margins vector and δ_d the extreme dependence parameters vector whose dimension depends on the copula.

3.3.2.1 Gumbel

The bivariate Gumbel copula

$$\mathrm{C}(u_1, u_2; \delta) = \exp\left(-\left(\tilde{u}_1^{\delta} + \tilde{u}_2^{\delta}\right)^{\frac{1}{\delta}}\right) \tag{3.9}$$

with $\delta \in (1, \infty)$. This copula can be extended to a higher dimension by the compound method as follows:

$$\begin{aligned}\mathrm{C}(\mathbf{u}_d; \boldsymbol{\delta}_{d-1}) &= \mathrm{C}(\mathrm{C}(\mathbf{u}_{d-1}; \boldsymbol{\delta}_{d-2}), u_d) \\ &= \exp\left\{-\left[\left(\ldots\left(\ldots\left[\left(\tilde{u}_1^{\delta_{d-1}} + \tilde{u}_2^{\delta_{d-1}}\right)^{\frac{\delta_{d-2}}{\delta_{d-1}}} + \tilde{u}_3^{\delta_{d-2}}\right]^{\frac{\delta_{d-3}}{\delta_{d-2}}}\right.\right.\right.\right. \\ &\qquad \left.\left.\left.\left. + \ldots + \tilde{u}_n^{\delta_{d-n+1}}\right)^{\frac{\delta_{d-n}}{\delta_{d-n+1}}} + \ldots + \tilde{u}_{d-1}^{\delta_2}\right)^{\frac{\delta_1}{\delta_2}} + \tilde{u}_d^{\delta_1}\right]^{\frac{1}{\delta_1}}\right\}\end{aligned} \tag{3.10}$$

and the dependence structure is given by $\boldsymbol{\delta}_{d-1} = (\delta_1, \ldots, \delta_{d-1})$ as follows:

$$\begin{array}{ccccccc} \delta_{d-1} & \rightarrow & (\chi_1, \chi_2) & & & & \\ \vdots & & \vdots & \ddots & & & \\ \delta_n & \rightarrow & (\chi_1, \chi_n) & \cdots & (\chi_2, \chi_3) & & \\ \vdots & & \vdots & & \vdots & \ddots & \\ \delta_1 & \rightarrow & (\chi_1, \chi_d) & \cdots & (\chi_n, \chi_d) & \cdots & (\chi_{d-1}, \chi_d) \end{array} \tag{3.11}$$

with $\infty > \delta_{d-1} \geq \ldots \geq \delta_3 \geq \delta_2 \geq \delta_1 \geq 1$. The parameter δ_{d-1} characterizes the dependence of one pair and the parameter δ_1 of $(d-1)$ pairs. The Gumbel copula employs a few parameters and then induces clustering.

3.3.2.2 Hüsler and Reiss

The bivariate Hüsler-Reiss copula is given (Hüsler and Reiss (1989)) by:

$$\mathbf{C}(u_1, u_2; \delta) = \exp\left\{-\tilde{u}_1 \Phi\left(\delta^{-1} + \frac{1}{2}\delta \ln\left(\frac{\tilde{u}_2}{\tilde{u}_1}\right)\right) - \tilde{u}_2 \Phi\left(\delta^{-1} + \frac{1}{2}\delta \ln\left(\frac{\tilde{u}_1}{\tilde{u}_2}\right)\right)\right\} \tag{3.12}$$

where $\delta \geq 0$ and $\tilde{u}_i = -\ln u_i = -\ln G_i(x_i)$. Although the Gumbel copula is characterised by $(d-1)$ parameters, the multivariate Hüsler-Reiss copula contains $\frac{d(d-1)}{2}$ parameters ($\delta_{i,j}, 1 \leq i < j \leq d$ and $\delta_{i,j} = \delta_{j,i}$). It can be derived recursively[2]:

$$\mathbf{C}(\mathbf{u}_d; \boldsymbol{\delta}_d) = \mathbf{C}(\mathbf{u}_{d-1}; \boldsymbol{\delta}_{d-1}) \times \exp\left\{-\int_0^{-\ln u_d} \boldsymbol{\Phi}_{d-1}\left(\boldsymbol{\kappa}_{d-1}(\mathbf{u}_{d-1}, q); \boldsymbol{\rho}_{d-1}\right) dq\right\} \tag{3.16}$$

[2] *The expression of this copula directly comes from the link between Multivariate Extreme Value (MEV) Distributions and Min-Stable Multivariate Exponential (MSMVE). Indeed, with* $\mathbf{C}$ *an MEV copula, if one can write:*

$$\mathbf{C}(u_1, \ldots, u_n) = \mathbf{D}(\tilde{u}_1, \ldots, \tilde{u}_n) \tag{3.13}$$

with $\tilde{u}_i = -\ln u_i$ *then* $\mathbf{D}$ *is an MSMVE distribution. Let use the definition of the dependence with* $\mathbf{A} = -\ln \mathbf{D}$*, as in* Joe *[1997] (p. 184), the Hüsler-Reiss is defined recursively:*

$$\mathbf{A}(\mathbf{y}_n; \boldsymbol{\delta}_n) = \mathbf{A}(\mathbf{y}_{n-1}; \boldsymbol{\delta}_{n-1}) + \int_0^{y_n} \boldsymbol{\Phi}_{n-1}\left(\boldsymbol{\kappa}_{n-1}(\exp(-\mathbf{u}_{n-1}), q); \boldsymbol{\rho}_{n-1}\right) dq \tag{3.14}$$

and equation (3.16) *follows. In the trivariate case, we have:*

$$\mathbf{C}(\mathbf{u}_3; \boldsymbol{\delta}_3) = \mathbf{C}(\mathbf{u}_2; \boldsymbol{\delta}_2) \times \exp\left\{-\int_0^{-\ln u_3} \boldsymbol{\Phi}_2(\boldsymbol{\kappa}_2(\mathbf{u}_2, q); \rho) dq\right\} \tag{3.15}$$

where $\rho = \rho_{3,1,2} = \frac{\delta_{1,3}\delta_{2,3}}{2}\left(\delta_{1,3}^{-2} + \delta_{2,3}^{-2} - \delta_{1,2}^{-2}\right)$.

with

$$\rho_{d-1}=\begin{pmatrix} 1 & & & & \\ \rho_{d,1,2} & 1 & & & \\ \rho_{d,1,3} & \rho_{d,2,3} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \rho_{d,1,d-1} & \rho_{d,2,d-1} & \cdots & \rho_{d,d-2,d-1} & 1 \end{pmatrix}$$

where $\rho_{d-1,i,j} = \frac{\delta_{i,d-1}\delta_{j,d-1}}{2}\left(\delta_{i,d-1}^{-2} + \delta_{j,d-1}^{-2} - \delta_{i,j}^{-2}\right)$ and

$$\begin{cases} \mathbf{u}_d = (u_1, \ldots, u_d) \\ \delta_d = (\delta_{i,j}, 1 \leq i < j \leq d) \\ \kappa_{d-1}(\mathbf{u}_{d-1}, q) = (\kappa_{1,d}(u_1, q), \ldots, \kappa_{d-1,d}(u_{d-1}, q)) \\ \text{with} \quad \kappa_{i,d}(u_i, q) = \delta_{i,d}^{-1} + \frac{1}{2}\delta_{i,d}\ln\left(-\frac{q}{\ln u_i}\right) \\ \text{for} \quad i = 1, \ldots, d-1 \end{cases}$$

and $\Phi_k(.;\rho)$ corresponds to the multivariate gaussian cumulative function with correlation ρ.

3.3.2.3 Joe and Hu

Another interesting copula has been defined by Joe and Hu (1996):

$$\mathbf{C}(\mathbf{u}_d;\delta_d) = \exp\left\{-\left[\sum_{i=1}^{d}\sum_{j=i+1}^{d}\left[\left(p_i\tilde{u}_i^{\theta}\right)^{\delta_{i,j}} + \left(p_j\tilde{u}_j^{\theta}\right)^{\delta_{i,j}}\right]^{\frac{1}{\delta_{i,j}}} + \sum_{i=1}^{d}\nu_i p_i \tilde{u}_i^{\theta}\right]^{\frac{1}{\theta}}\right\} \tag{3.17}$$

with $p_i = (\nu_i + d - 1)^{-1}$ and where δ_d has the following elements: $\delta_{i,j}$ the pairwise coefficients, ν_i the bivariate and multivariate asymmetry coefficients and θ a common parameter. To extend this copula to higher dimensions, one only has to extend the sum components of the formula. The bivariate margins are given by:

$$\mathbf{C}_{ij}(u_i, u_j) = \exp\left\{-\left[\begin{array}{c}\left[\left(p_i\tilde{u}_i^{\theta}\right)^{\delta_{i,j}} + \left(p_j\tilde{u}_j^{\theta}\right)^{\delta_{i,j}}\right]^{\frac{1}{\delta_{i,j}}} + \\ (\nu_i + d - 2)\, p_i\tilde{u}_i^{\theta} + (\nu_j + d - 2)\, p_j\tilde{u}_j^{\theta}\end{array}\right]^{\frac{1}{\theta}}\right\} \tag{3.18}$$

3.4 Empirical estimation of copulae parameters

Estimation is a two-step procedure. First, the parameters of the marginal distributions are estimated, then the original variables are mapped to uniforms using these estimated parameters and the dependence parameters are then estimated. A detailed description of this procedure can be found in Joe and Xu (1996). In practice, for each margin, a sample of size nT can be divided into T blocks of n observations. Then, T maxima are available: $\chi_n^{+(t)} = \max(X_{n(t-1)+1,\ldots,}X_{nt})$ with $t = 1 \ldots T$. The likelihood function is:

$$L(\gamma;\chi^+) = \prod_{t=1}^{T} g(\gamma;\chi_n^{+(t)}) 1_{\left\{1-\tau\frac{\chi_n^{+(t)}-b}{a}>0\right\}} \tag{3.19}$$

with $g(\gamma;\chi^+) = \frac{1}{a}\left(1-\tau\frac{\chi^+-b}{a}\right)^{\frac{1}{\tau}-1} \exp\left\{-\left(1-\tau\frac{\chi^+-b}{a}\right)^{1/\tau}\right\}$. The log-likelihood estimator for each margin is then:

$$\begin{aligned}
\hat{\gamma} &= \arg\max_{\gamma\in\Theta} \ln L(\gamma;\chi_n^{+(1)},\ldots,\chi_n^{+(T)}) \\
\hat{\gamma} &= \arg\max_{\gamma\in\Theta} \left\{ \begin{array}{c} -T\ln(a) + \left(\frac{1}{\tau}-1\right)\sum_{t=1}^{T} \ln\left(1-\tau\frac{\chi_n^{+(t)}-b}{a}\right) \\ -\sum_{t=1}^{T}\left(1-\tau\frac{\chi_n^{+(t)}-b}{a}\right)^{1/\tau} \end{array} \right\}
\end{aligned} \tag{3.20}$$

where $\chi_n^{+(t)}$ is the maxima of the t^{th} block. The score vector $s(\gamma)$ is as usual:

$$s(\gamma) = \frac{\partial \log g(\gamma;\chi^+)}{\partial\gamma} \tag{3.21}$$

where the derivatives are developed in the Appendix. Finally, to compute the standard errors, an estimator $(Q(\gamma))^{-1}$ of the asymptotic covariance matrix is used :

$$(Q(\gamma)) = T^{-1}\sum_{t=1}^{T} s(\gamma)s(\gamma)^{\top}$$

We apply this approach to daily returns for MSCI US ($MSUS$), MSCI Germany ($MSGE$) and MSCI Japan ($MSJP$) indices. The dataset starts from 1/1/1981 to 1/1/2001. Estimation by blocks, as described above, has been applied and different block sizes have been tested to insure the consistency of the results. The results are presented for a block size of 21 that corresponds to one trading month.

$-\chi^-$	MSGE	MSUS	MSJP
Location parameter $\hat{b}$	0.0263	0.0194	0.0255
	(0.0014)	*(0.0013)*	*(0.0018)*
Scale parameter $\hat{a}$	0.0094	0.0084	0.0109
	(0.0012)	*(0.0011)*	*(0.0015)*
Tail index $\hat{\tau}$	-0.2824	-0.3259	-0.3617
	(0.1023)	*(0.0981)*	*(0.1422)*

Table 3.1: MLE for the parameters of the univariate GEV for the minima

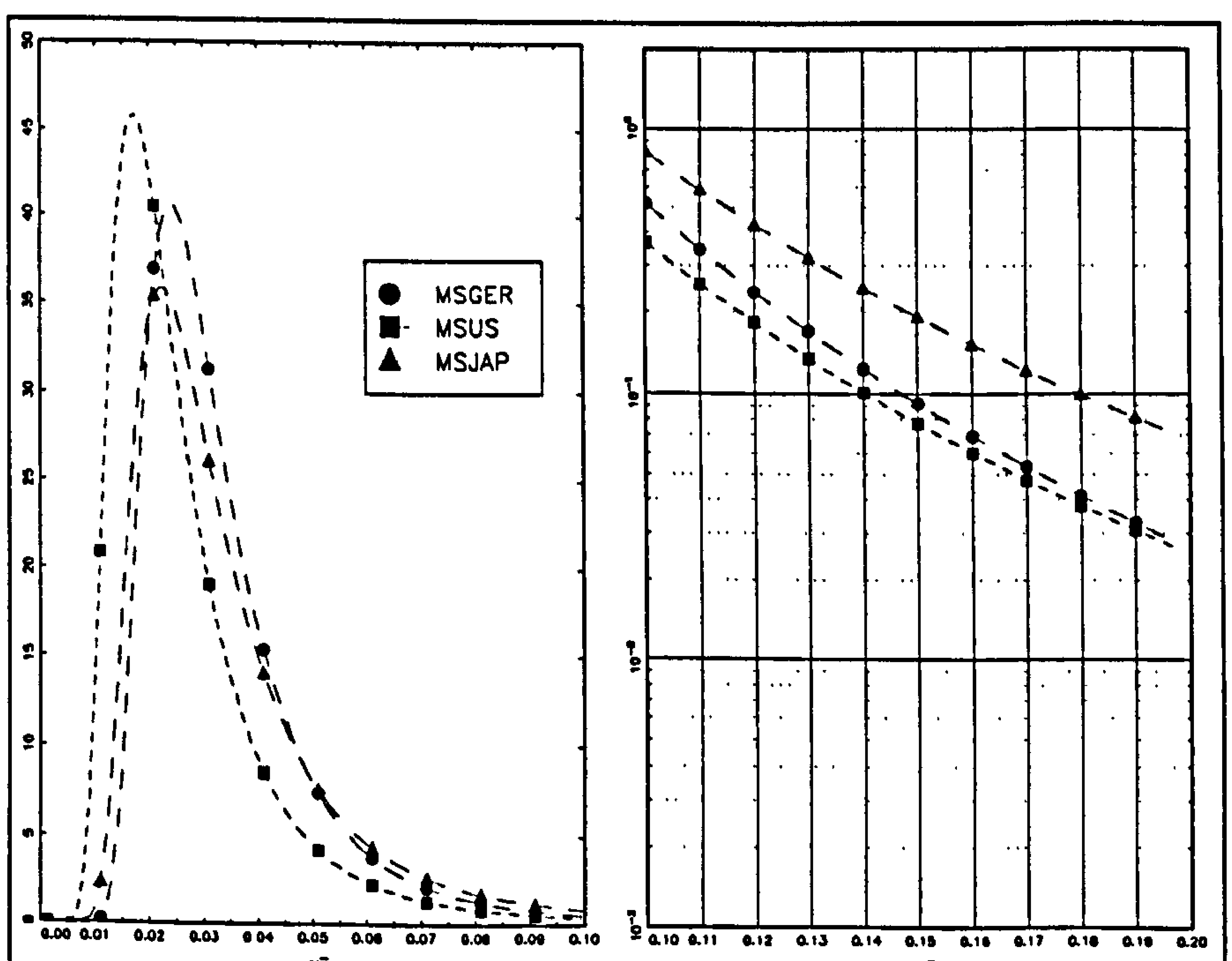

Figure 3.1: Estimated GEV distributions for minima

χ^+	MSGE	MSUS	MSJP
Location parameter $\hat{b}$	0.0281	0.0203	0.0315
	(0.0014)	*(0.0013)*	*(0.0018)*
Scale parameter $\hat{a}$	0.0103	0.0065	0.0118
	(0.0012)	*(0.0011)*	*(0.0015)*
Tail index $\hat{\tau}$	-0.0957	-0.2064	-0.2502
	(0.1023)	*(0.0981)*	*(0.1422)*

Table 3.2: MLE for the parameters of the univariate GEV for the maxima

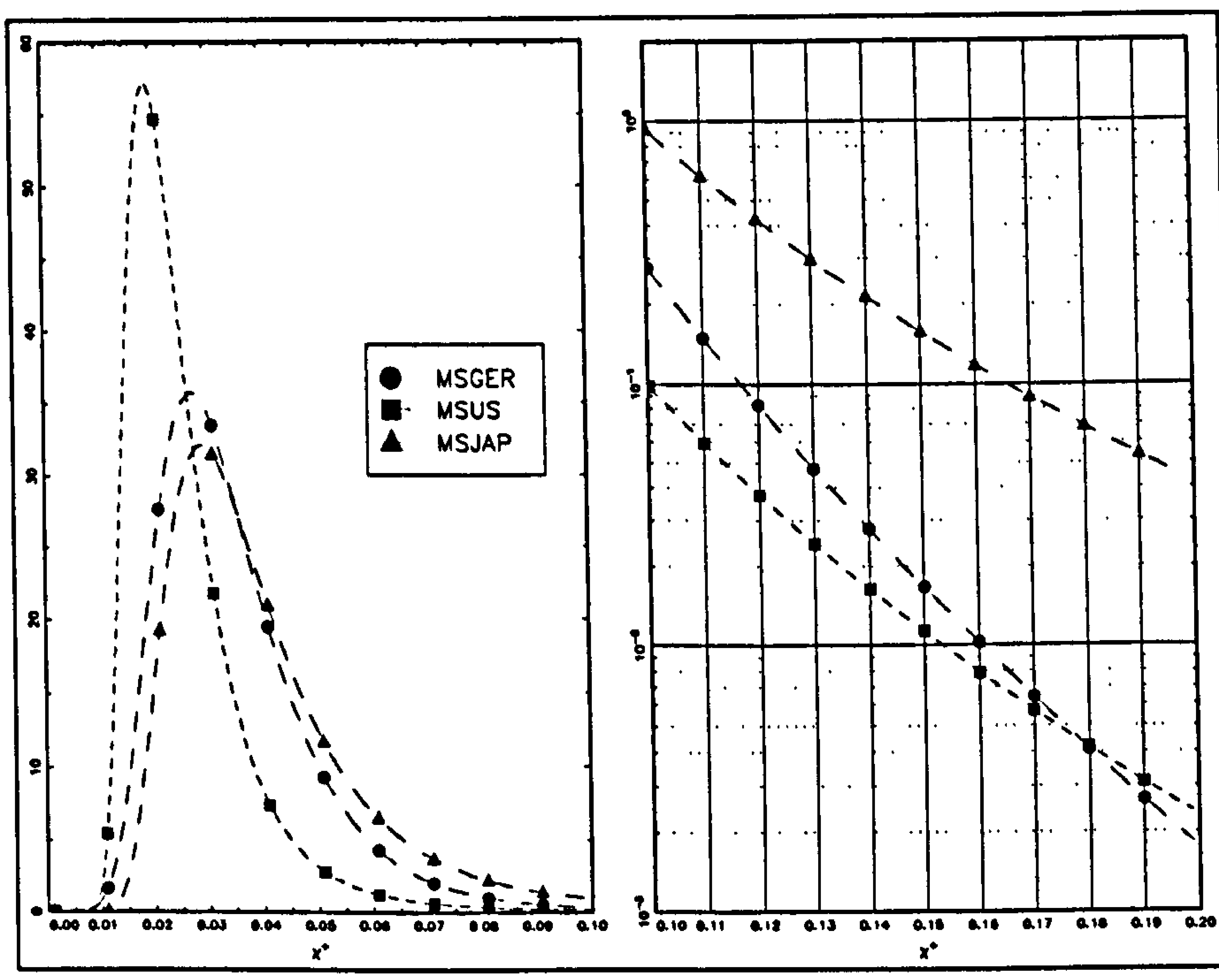

Figure 3.2: Estimated GEV distributions for maxima

From tables 3.1 and 3.2, it appears that extreme returns follow a Fréchet distribution (the tail indices are negative for all market indices). The degree of fatness is given by the absolute level of the tail index. The higher the absolute value of the tail index, the higher the fatness. As confirmed by the figures 3.1 and 3.2 (the left plot is the extreme value density functions, the right plot is a zoom in the tails with log-scale), MSGE has the lowest degree of fatness for both minima and maxima, and MSJP has the greatest degree of fatness for both minima and maxima. The second step of estimation consists of estimating the parameters for different dependence structures. The log-likelihood ℓ of the multivariate extreme distribution is:

$$\begin{aligned} \ell(\chi_1,\ldots,\chi_d;\hat{\gamma},\delta) &= \ln \mathrm{g}(\chi_d,\ldots,\chi_d;\hat{\gamma},\delta) \\ &= \sum_{t=1}^{T} \ln \mathrm{c}(G(\chi_1^{(t)};\hat{\gamma}_1),\ldots,G(\chi_d^{(t)};\hat{\gamma}_d);\delta) \end{aligned} \quad (3.22)$$

where $\chi_i = \left(\chi_i^{(1)},\ldots,\chi_i^{(T)}\right)$ for $i = 1,\ldots,d$, g the asymptotic MEV density and c the associated copula density[3]. A criticism of this estimation methodology might arise from the fact that extrema may not occur simultaneously (same day) in one month. However, we believe that this method is asymptotically valid since the asymptotic MEV distribution is usually found by assuming componentwise extrema. An alternative estimation method called threshold estimation method could be used. It would lead us to use a multivariate generalised pareto distribution that is directly linked to a MEV distribution. We refer to Longin and Solnik (2001) for an application of this technique to financial series. Moreover, the goal of the chapter is to focus on risk management implications rather than estimation methods. The ML estimator of the dependence parameters is:

$$\hat{\delta} = \arg\max_{\delta\in\Delta} \ln \mathrm{g}(\chi_1,\ldots,\chi_d;\hat{\gamma},\delta) \quad (3.24)$$

with Δ the set of dependence parameters. As seen above, extreme value copulae can only model positive dependence. Consequently, if one wants to model

[3] *Let* g *be the* N*-dimensional density function of* $\mathbf{G}$ *defined as follows:*

$$\mathrm{g}(x_1,\ldots,x_N) = \frac{\partial\,\mathrm{G}(x_1,\ldots,x_N)}{\partial x_1\cdots\partial x_N} \quad (3.23)$$

With the notation $u_n = G_n(x_n)$ *for* $n = 1,\ldots,N$, *we have*

$$\mathrm{c}(u_1,\ldots,u_N) = \frac{\partial\,\mathrm{C}(u_1,\ldots,u_N)}{\partial u_1\cdots\partial u_N}$$

with c *the copula density of* $\mathbf{C}$.

the minima and maxima simultaneously, one needs to split the estimation problem. For example, if we are interested in estimating the bivariate dependence parameters for the extrema of three variables, we will have to estimate twelve dependence structures – four for each pair –, as summarized in the following diagram:

(x_1^-, x_2^+)	(x_1^+, x_2^+)
(x_1^-, x_2^-)	(x_1^+, x_2^-)

(x_2^-, x_3^+)	(x_2^+, x_3^+)
(x_2^-, x_3^-)	(x_2^+, x_3^-)

Extending this methodology, it will be necessary to estimate eight different copulae for the trivariate case. The results for the three market indices are reported below. The subscripts 1, 2 and 3 are respectively used for MSGE, MSUS and MSGE.

Let us comment the results of Table 3.3. The abbreviation ldv means "less dependent variable" and corresponds to the number of the variable that exhibits the lowest dependence with the two others extremes. This is motivated by the fact that only two parameters are estimated for the Gumbel copula. One parameter measures the dependence for one pair, the other one – corresponding to a lower dependence – is common for the two remaining pairs. In most cases, the dependence is higher between the extrema of MSGE and MSJP, except for the minima of MSUS and MSJP. The highest extremal dependences appear for: (i) a long position in the MSJP and a short position in the MSGE, and (ii) a short position in the MSUS and a short position in the MSJP. We note that these dependence measures are conditional to the dependence with the extremes (maxima or minima) of the remaining index. Not surprisingly, the dependence hierarchy is the same for all three copulae. Since the Hüsler-Reiss copula is asymmetric, three parametric dependences are possible. Indeed, the choice of the Hüsler-Reiss MEV distribution depends on the two market indices that are firstly selected. The dependence pattern with the higher likelihood has been selected and reported in the table. Some numerical difficulties arose in finding the maximum likelihood parameters for the Hu-Joe copula. This led us to constrain the common dependence parameter θ to 1.

The three copulae are non-nested models. Cox statistics are then computed to test between the alternative specifications (see Cox (1961)). We consider the maximum log-likelihood function evaluated under the two non-nested hypotheses (corresponding to the two specifications of the copula function). Then, The Cox procedure is applied: (i) we compute the likelihood ratio (LR) test statistic, (ii) a consistent estimate of the limit of the LR statistic under the first hypothesis – divided by the sample size – is numerically computed, (iii) this consistent estimate is subtracted from the LR statistic to obtain the adjusted Cox statistic. Other criteria may be used like the extended Wald test or the extended score test. For all trivariate estimations, the Hüsler-Reiss copula hypothesis can not be rejected against both the Gumbel copula hypothesis and the Hu-Joe copula hypothesis (at 5% level).

3.5 Application to risk management

The results above can be applied to risk management in two ways. First, it is possible to compute stress test values that would correspond to the evolution of the portfolio under extremal scenarios. Secondly, the parametric estimates of the MEV distributions can be used to simulate the joint extrema of the portfolio components.

3.5.1 Stress testing scenarios design

Draisma, de Haan and Peng (1997) define a failure area as the set of extrema with a given probability that at least one of them is exceeded. We will adopt a different definition by considering the set that corresponds to a simultaneous exceedence. Formally, this set A_p is:

$$\mathcal{A}_p^{s_1 \ldots s_n} = \{(x_1, \ldots, x_n) \in \mathbb{R}^{s_1} \times \ldots \times \mathbb{R}^{s_n}, \Pr(\chi_1^{s_1} > x_1, \ldots, \chi_n^{s_n} > x_n) = p\} \tag{3.25}$$

with for $i = 1, \ldots, n$, $s_i = +$ for maxima or $-$ for minima. In the bivariate case, four sets need to be defined: $A_p^{++}, A_p^{+-}, A_p^{-+}, A_p^{--}$. In the trivariate case, eight sets are necessary: A_p^{+++}, A_p^{++-}, A_p^{-++}, A_p^{-+-}, A_p^{+-+}, A_p^{+--}, A_p^{--+}, A_p^{---}. More generally, for an n-dimensional problem, the number of sets equals 2^n. The probability involved for the characterisation of the failure area is nothing else but the survival distribution function that can be expressed with copulae (Joe (1997)) as:

$$\Pr(\chi_1^{s_1} > x_1, \ldots, \chi_n^{s_n} > x_n) = 1 + \sum_{M \in \mathcal{M}} (-1)^{|M|} \mathrm{C}_M \left(\mathrm{G}\left(x_j; \gamma_j\right), j \in M; \delta_M \right) \tag{3.26}$$

where $|M|$ denotes the cardinality of M the set of marginal distributions of C.

For $n = 3$, we have[4]:

$$\Pr\left(\chi_1^{s_1} > x_1, \chi_2^{s_2} > x_2, \chi_3^{s_3} > x_3\right) = \tag{3.28}$$

$$1 - \mathrm{C}_{12}\left(\mathrm{G}\left(x_1;\gamma_1^{s_1}\right), \mathrm{G}\left(x_2;\gamma_2^{s_2}\right); \delta_{12}^{s_1 s_2}\right)$$

$$-\mathrm{C}_{13}\left(\mathrm{G}\left(x_1;\gamma_1^{s_1}\right), \mathrm{G}\left(x_3;\gamma_3^{s_3}\right); \delta_{13}^{s_1 s_3}\right) \tag{3.29}$$

$$-\mathrm{C}_{23}\left(\mathrm{G}\left(x_2;\gamma_2^{s_2}\right), \mathrm{G}\left(x_3;\gamma_3^{s_3}\right); \delta_{23}^{s_2 s_3}\right)$$

$$+\mathrm{C}\left(\mathrm{G}\left(x_1;\gamma_1^{s_1}\right), \mathrm{G}\left(x_2;\gamma_2^{s_2}\right), \mathrm{G}\left(x_3;\gamma_3^{s_3}\right); \delta^{s_1 s_2 s_3}\right) \tag{3.30}$$

with C_{ij} the marginal copulae.

From this definition, a natural question arises: which probability level should be chosen ? An elegant answer – often used in the statistical literature and introduced by Gumbel (1958) for extreme value distributions – is to associate a waiting period t to the probability level p such that $t = \frac{1}{p}$. The univariate daily stress test scenarios for different waiting periods are reported in table 3.4.

To illustrate the concept of failure area with two variables, we provide an example for the maxima of two abstract indices with the same univariate stress

[4] *This formula comes directly as*

$$\begin{aligned} \Pr\left(\chi_1^{s_1} > x_1, \chi_2^{s_2} > x_2, \chi_3^{s_3} > x_3\right) &= 1 - \Pr\left(\chi_1^{s_1} \le x_1, \chi_2^{s_2} \le x_2\right) \\ &\quad - \Pr\left(\chi_1^{s_1} \le x_1, \chi_3^{s_3} \le x_3\right) \\ &\quad - \Pr\left(\chi_2^{s_2} \le x_2, \chi_3^{s_3} \le x_3\right) \\ &\quad + \Pr\left(\chi_1^{s_1} \le x_1, \chi_2^{s_2} \le x_2, \chi_3^{s_3} \le x_3\right) \end{aligned} \tag{3.27}$$

testing scenarios (+9%) but with different degrees of dependence (Figure 3.3). The higher the dependence, the lower the distance between the failure area and univariate stress testing scenarios. We then build the trivariate failure areas under the hypothesis of a MEV distribution obtained from the Hüsler and Reiss copula. The associated probability level corresponds to a 50 years waiting period. The failure areas have been obtained by numerically solving equation (3.25). The three dimensional space – each axis corresponds to one index – is split into two parts: a short position in the MSUS (Figure 3.4) and a long position in the MSUS (Figure 3.5). For both figures, each point of the discretized surface is a three dimensional stress testing scenario that corresponds to values of the triplet (MSGE,MSUS,MSJP). The univariate stress testing scenarios (Table 3.4) are also represented. By definition, the trivariate failure areas are included in the parallelepiped arising from these univariate scenarios.

3.5.2 Monte-Carlo based risk measures

Stress testing becomes intractable in higher dimensions because the number of points in the failure areas increases very quickly. Moreover, the failure areas have to be re-built if one wants portfolio values for different probability levels. An alternative is to simulate the variables that follow a MEV distribution. To illustrate this Monte-Carlo approach, we consider the following portfolios:

Portfolio positions	MSGE	MSUS	MSJP
P_1	0	1	1
P_2	1	0	1
P_3	1	1	1
P_4	1	0	-1

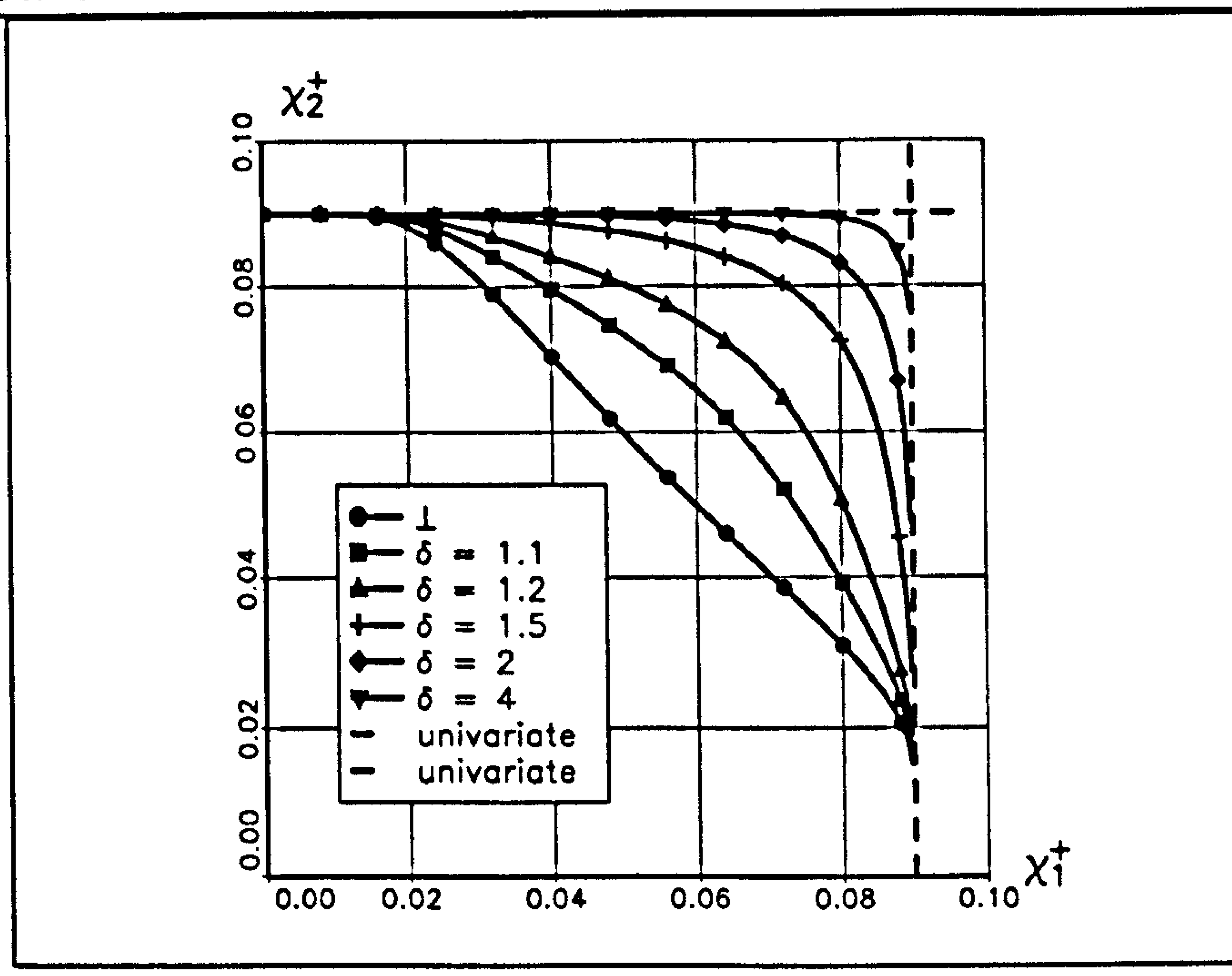

Figure 3.3: Bivariate failure area $\mathcal{A}_p^{++}$ with different parameter values for the Gumbel copula

The following algorithm, based on the conditional distributions, can be used to simulate extrema with a given n-variate copula C:

1. Generate n independent uniform variates $(t_1, \ldots, t_n)$;

2. The n uniform variates are given recursively for $j = n, \ldots, 1$ by:

$$u_j = \mathbf{C}^{-1}\left(t_j \mid u_1, \ldots, u_{j-1}\right) \tag{3.31}$$

   where

$$\begin{aligned} \mathbf{C}\left(u_j \mid u_1, \ldots, u_{j-1}\right) &= \Pr\left\{U_j \le u_j \mid (U_1, \ldots, U_{j-1}) = (u_1, \ldots, u_{j-1})\right\} \\ &= \frac{\partial^{j-1}\mathbf{C}\left(u_1, \ldots, u_j, 1, \ldots, 1\right)/\partial u_1 \ldots \partial u_{j-1}}{\partial^{j-1}\mathbf{C}\left(u_1, \ldots, u_{j-1}, 1, \ldots, 1\right)/\partial u_1 \ldots \partial u_{j-1}} \end{aligned} \tag{3.32}$$

3. The extrema are obtained by inverting the estimated GEV distribution: $\chi_j = G_j^{-1}\left(u_j; \gamma_j\right)$ for $j = 1, \ldots, n$.

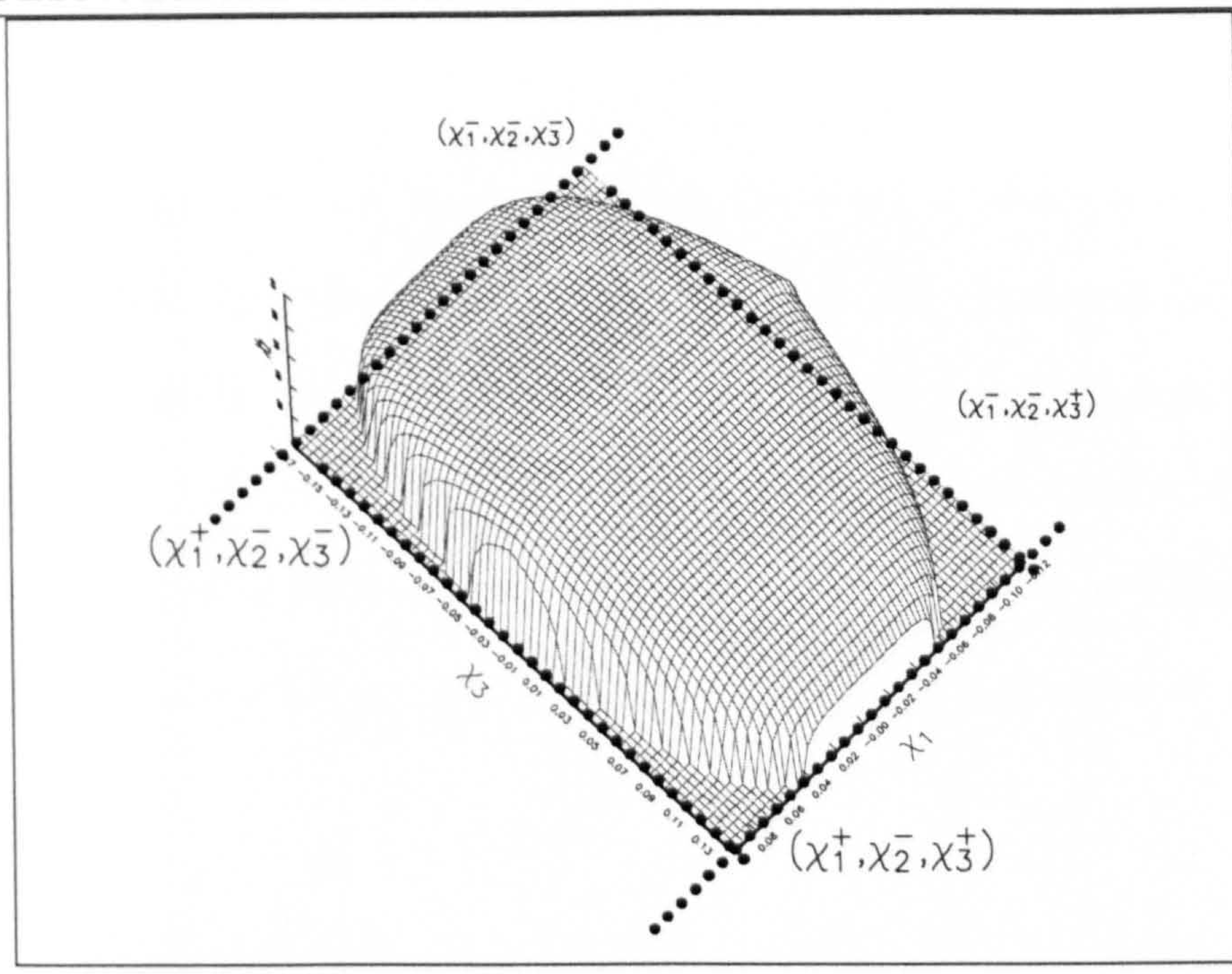

Figure 3.4: Trivariate failure areas $\mathcal{A}_p^{+-+}, \mathcal{A}_p^{+--}, \mathcal{A}_p^{--+}, \mathcal{A}_p^{---}$ for MSGE (χ_1), MSUS (χ_2) and MSJP (χ_3) with a 50 years waiting period (surface). Univariate stress test scenarios are also represented (dotted line)

Fortunately, this algorithm can be simplified for specific copula families. A detailed description for archimedean copulae can be found in Lindskog (2000). From the simulated data, two risk measures are computed: value at risk (VaR) and expected shortfall (ES). Following Artzner, Delbaen, Eber and Heath (1999), these measures are defined as follows:

Definition 6 *For a given probability space $(Om, \mathcal{F}, \mathbb{P})$, the value-at-risk VaR_p of the net worth X with distribution P is such that*

$$VaR_p(X) = -\inf(x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq p) \tag{3.33}$$

with $p \in (0,1)$.

Definition 7 *The expected shortfall ES_p is directly defined from VaR_p as follows:*

$$ES_p = -\mathbb{E}(X \mid X \leq -VaR_p(X)) \tag{3.34}$$

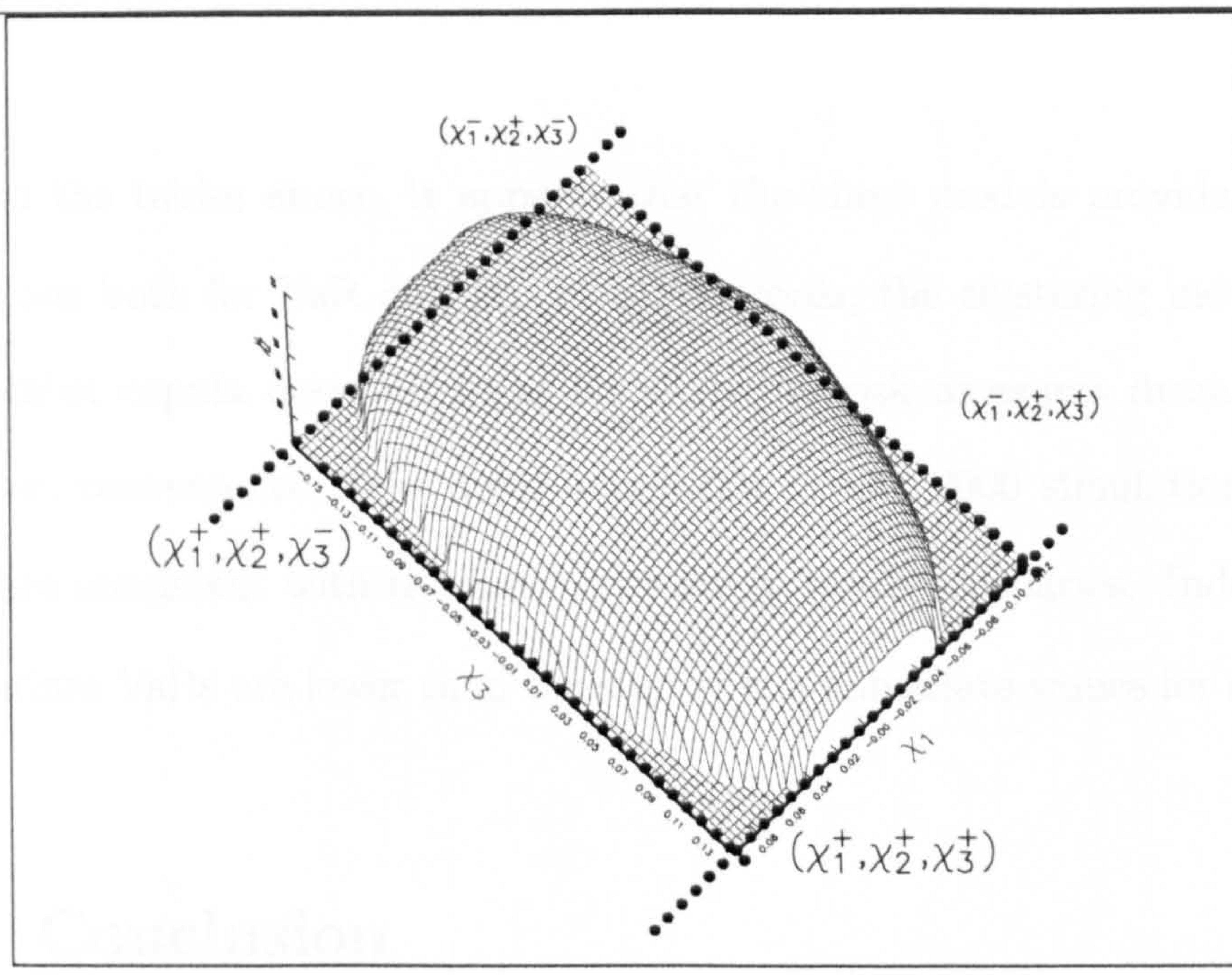

Figure 3.5: Trivariate failure areas $\mathcal{A}_p^{+++}, \mathcal{A}_p^{++-}, \mathcal{A}_p^{-++}, \mathcal{A}_p^{-+-}$ for MSGE (χ_1), MSUS (χ_2) and MSJP (χ_3) with a 50 years waiting period (surface). Univariate stress test scenarios are also represented (dotted line)

ES is a coherent risk measure – see the definition of a coherent measure of risk in Artzner, Delbaen, Eber and Heath (1999) –, but VaR is generally not. The empirical estimation of these quantities is carried out as follows. Let $\{X_i\}_{i=1\ldots n}$ be a vector of n realizations of the random variable X. The order statistics are such that $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. Then the empirical estimators of these measures are given by:

$$\mathrm{VaR}_p^{(n)}(X) = X_{\lfloor np \rfloor :n}$$

$$\mathrm{ES}_p^{(n)}(X) = -\frac{\sum_{i=1}^{\lfloor np \rfloor} X_{i:n}}{\lfloor np \rfloor}$$

with $\lfloor np \rfloor = \max\{j \mid j \leq np,\, j \in \mathbb{N}\}$.

Estimated values of VaR and ES are reported in the Tables 3.5 (50000 simulations) and 3.6 (100000 simulations). Two probability levels are considered, respectively corresponding to 10 and 50 year waiting periods.

From the tables above, it appears that the three models provide similar conclusions both for VaR and ES. In other words, the clustering induced by the Gumbel copula does not seem to affect the risk measures dramatically. Moreover, convergence looks quite acceptable given 50000 simulations. The values are consistent with the univariate stress testing scenarios. Indeed, the multivariate VaRs are lower than the sum of the univariate values for all portfolios.

3.6 Conclusion

In this chapter a methodology for multivariate risk management based on MEV parametric distributions has been investigated following two approaches: stress scenario design and Monte-Carlo based risk measures. It appears that the results are similar for the three copulae examined: the Gumbel, the Hüsler and Reiss, and the Joe and Hu. Analytical formula might be developed by providing bounds for the risk measures. This has been done in a non-extremal context in Durrleman, Nikeghbali and Roncalli (2000) and Embrechts, Hoeing and Juri (2001).

3.7 Appendix

SCORE VECTOR FOR GEV DISTRIBUTION

The score vector described in equation (3.21) is detailed:

$$
\begin{aligned}
\frac{\partial \log g(\gamma; x)}{\partial a} &= -\frac{a + (b-x)\left(\left(\frac{a+\tau(x-b)}{a}\right)^{\frac{1}{\tau}} - 1\right)}{a(a+\tau(x-b))} \\
\frac{\partial \log g(\gamma; x)}{\partial b} &= \frac{\tau - 1 + \left(\frac{a+\tau(x-b)}{a}\right)^{\frac{1}{\tau}}}{(a+\tau(x-b))} \\
\frac{\partial \log g(\gamma; x)}{\partial \tau} &= \frac{\tau(b-x)\left(\left(\frac{a+\tau(x-b)}{a}\right)^{\frac{1}{\tau}} + \tau - 1\right) + (a+\tau(x-b))}{\tau^2(a+\tau(x-b))} \\
&\quad \times \log\left(\frac{a+\tau(x-b)}{a}\right)\left(\left(\frac{a+\tau(x-b)}{a}\right)^{\frac{1}{\tau}} - 1\right) \qquad (3.35)
\end{aligned}
$$

TRIVARIATE DENSITIES

The density for the trivariate Gumbel copula is developed. The copula function can be obtained by compound method:

$$
\begin{aligned}
\mathrm{C}(u_1, u_2, u_3; \delta_1, \delta_2) &= \mathrm{C}(u_3, \mathrm{C}(u_1, u_2; \delta_2); \delta_1) \\
&= \exp\left(-\left[\tilde{u}_3^{\delta_1} + \left(\tilde{u}_1^{\delta_2} + \tilde{u}_2^{\delta_2}\right)^{\frac{\delta_1}{\delta_2}}\right]^{\frac{1}{\delta_1}}\right) \qquad (3.36)
\end{aligned}
$$

By computing the three iterative derivatives, it follows that:

$$
\begin{aligned}
\mathrm{c}(u_1, u_2, u_3; \delta_1, \delta_2) &= \mathrm{c}(u_1, u_2; \delta_2) \times \mathrm{c}(u_3, \mathrm{C}(u_1, u_2; \delta_2); \delta_1) \\
&\quad + (\partial_1 \mathrm{C})(u_1, u_2; \delta_2) \times (\partial_2 \mathrm{C})(u_1, u_2; \delta_2) \\
&\quad \times (\partial_{221} \mathrm{C})(u_3, \mathrm{C}(u_1, u_2; \delta_2); \delta_1) \qquad (3.37)
\end{aligned}
$$

where

$$\left\{\begin{array}{ll}
\mathbf{c}\left(u_1,u_2;\delta\right) & = \mathbf{C}(u_1,u_2;\delta)(u_1u_2)^{-1}\frac{(\ln u_1 \ln u_2)^{\delta-1}}{\vartheta(u_1,u_2;\delta)^{2-1/\delta}} \\
 & \left[\vartheta\left(u_1,u_2;\delta\right)^{1/\delta}+\delta-1\right] \\
\left(\partial_1\mathbf{C}\right)\left(u_1,u_2;\delta\right) & = u_1^{-1}\mathbf{C}(u_1,u_2;\delta)\left(1+(\frac{\tilde{u}_2}{\tilde{u}_1})^{\delta}\right)^{1/\delta-1} \\
\left(\partial_2\mathbf{C}\right)\left(u_1,u_2;\delta\right) & = \left(\partial_1\mathbf{C}\right)\left(u_2,u_1;\delta\right) \\
\left(\partial_{221}\mathbf{C}\right)\left(u_1,u_2;\delta\right) & = u_1^{-1}u_2{}^{-2}\mathbf{C}(u_1,u_2;\delta)\times\tilde{u}_1^{-1+\delta}\,\vartheta\left(u_1,u_2;\delta\right)^{-3+\frac{1}{\delta}}\tilde{u}_2^{-2+\delta} \\
 & \left\{\left(-1+\delta+\vartheta\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\right)\vartheta\left(u_1,u_2;\delta\right)\ln(u_2)\right. \\
 & -\left(\tilde{u}_1^{\delta}+\delta^2\,\vartheta\left(u_1,u_2;\delta\right)-\tilde{u}_1^{\delta}\vartheta\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\right. \\
 & +\delta\left[-2\,\tilde{u}_1^{\delta}+\tilde{u}_1^{\delta}\,\vartheta\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\right. \\
 & \left.+\,\tilde{u}_2^{\delta}-2\,\vartheta\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\tilde{u}_2^{\delta}\right] \\
 & \left.\left.+\,2\,\vartheta\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\tilde{u}_2^{\delta}-\vartheta\left(u_1,u_2;\delta\right)^{\frac{2}{\delta}}\tilde{u}_2^{\delta}\right)\right\} \\
\text{with} & \vartheta\left(u_1,u_2;\delta\right)=\left(\tilde{u}_1^{\delta}+\tilde{u}_2^{\delta}\right)
\end{array}\right.$$

We develop the Hüsler-Reiss copula density for the trivariate case.

$$\frac{\partial\mathbf{C}\left(\mathbf{u}_3;\delta_3\right)}{\partial u_3} = \frac{1}{u_3}\mathbf{C}\left(\mathbf{u}_2;\delta_2\right)\times\Phi_2\left(\kappa_2\left(\mathbf{u}_2,u_3\right);\rho\right) \tag{3.38}$$

$$\times\exp\left\{-\int_0^{-\ln u_3}\Phi_2\left(\kappa_2\left(\mathbf{u}_2,q\right);\rho\right)dq\right\} \tag{3.39}$$

Then,

$$\frac{\partial^2\mathbf{C}\left(\mathbf{u}_3;\delta_3\right)}{\partial u_2\partial u_3}=\frac{1}{u_3}\left(\alpha\left(\mathbf{u}_3;\delta_3\right)+\beta\left(\mathbf{u}_3;\delta_3\right)+\gamma\left(\mathbf{u}_3;\delta_3\right)\right) \tag{3.40}$$

where

$$\left\{\begin{array}{ll}
\alpha\left(\mathbf{u}_3;\delta_3\right) & = \partial_2\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\Phi_2\left(\kappa_2\left(\mathbf{u}_2,u_3\right);\rho\right)\times\theta\left(\mathbf{u}_3;\delta_3\right) \\
\beta\left(\mathbf{u}_3;\delta_3\right) & = \mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\frac{\partial\kappa_{23}(u_2,u_3)}{\partial u_2}\times\Phi\left(\frac{\kappa_{13}(u_1,u_3)-\rho\kappa_{23}(u_2,u_3)}{\sqrt{1-\rho^2}}\right)\times\theta\left(\mathbf{u}_3;\delta_3\right) \\
\gamma\left(\mathbf{u}_3;\delta_3\right) & = \mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\Phi_2\left(\kappa_2\left(\mathbf{u}_2,u_3\right);\rho\right)\times\varsigma_{21}\left(\mathbf{u}_3;\delta_3\right)
\end{array}\right.$$

with

$$\left\{\begin{array}{ll}
\partial_j\mathbf{C}\left(u_i,u_j;\delta\right) & = \frac{\partial\mathbf{C}(u_i,u_j,\delta)}{\partial u_j}=\mathbf{C}\left(u_i,u_j;\delta\right)\times u_j^{-1}\times\Phi\left(\delta^{-1}+\frac{1}{2}\delta\ln\left(\frac{\ln u_j}{\ln u_i}\right)\right) \\
 & \text{for } (i,j)=(1,2),(2,1) \\
\frac{\partial\kappa_{i3}(u_i,u_3)}{\partial u_i} & = -\frac{\delta_{i3}}{2u_i\ln u_i}\text{ for } i=1,2 \\
\theta\left(\mathbf{u}_3;\delta_3\right) & = \exp\left\{-\int_0^{-\ln u_3}\Phi_2\left(\kappa_2\left(\mathbf{u}_2,q\right);\rho\right)dq\right\} \\
\varsigma_{ij}\left(\mathbf{u}_3;\delta_3\right) & = \exp\left\{-\int_0^{-\ln u_3}\frac{\partial\kappa_{i3}(u_i,q)}{\partial u_i}\Phi\left(\frac{\kappa_{j3}(u_j,q)-\rho\kappa_{i3}(u_i,q)}{\sqrt{1-\rho^2}}\right)dq\right\}.
\end{array}\right.$$

Note that this comes partly from $\frac{\partial\Phi_2}{\partial x}(x,y;\rho)=\Phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)$. The copula density can be deduced:

$$\begin{aligned} \mathrm{c}(\mathrm{u}_3;\delta_3) &= \frac{\partial^3 \mathrm{C}(\mathrm{u}_3;\delta_3)}{\partial u_1 \partial u_2 \partial u_3} \\ &= \frac{1}{u_3}\left(\frac{\partial\alpha(\mathrm{u}_3;\delta_3)}{\partial u_1}+\frac{\partial\beta(\mathrm{u}_3;\delta_3)}{\partial u_1}+\frac{\partial\gamma(\mathrm{u}_3;\delta_3)}{\partial u_1}\right) \end{aligned} \tag{3.41}$$

where

$$\begin{cases} \frac{\partial\alpha(\mathrm{u}_3;\delta_3)}{\partial u_1} & = c(u_1,u_2;\delta_{12})\times\Phi_2(\kappa_2(\mathrm{u}_2,u_3);\rho)\times\theta(\mathrm{u}_3;\delta_3) \\ & +\partial_2\mathrm{C}(u_1,u_2;\delta_{12})\times\frac{\partial\kappa_{13}(u_1,u_3)}{\partial u_1} \\ & \times\Phi\left(\frac{\kappa_{23}(u_2,u_3)-\rho\kappa_{13}(u_1,u_3)}{\sqrt{1-\rho^2}}\right)\times\theta(\mathrm{u}_3;\delta_3) \\ & +\partial_2\mathrm{C}(u_1,u_2;\delta_{12})\times\Phi_2(\kappa_2(\mathrm{u}_2,u_3);\rho)\times\varsigma_{12}(\mathrm{u}_3;\delta_3) \\ \frac{\partial\beta(\mathrm{u}_3;\delta_3)}{\partial u_1} & = \partial_1\mathrm{C}(u_1,u_2;\delta_{12})\times\frac{\partial\kappa_{23}(u_2,u_3)}{\partial u_2} \\ & \times\Phi\left(\frac{\kappa_{13}(u_1,u_3)-\rho\kappa_{23}(u_2,u_3)}{\sqrt{1-\rho^2}}\right)\times\theta(\mathrm{u}_3;\delta_3) \\ & +\mathrm{C}(u_1,u_2;\delta_{12})\times\frac{\partial\kappa_{23}(u_2,u_3)}{\partial u_2}\times\frac{\partial\kappa_{13}(u1,u_3)}{\partial u_1} \\ & \times\frac{1}{\sqrt{1-\rho^2}}\times\phi\left(\frac{\kappa_{13}(u_1,u_3)-\rho\kappa_{23}(u_2,u_3)}{\sqrt{1-\rho^2}}\right) \\ & \times\theta(\mathrm{u}_3;\delta_3) \\ & +\mathrm{C}(u_1,u_2;\delta_{12})\times\frac{\partial\kappa_{23}(u_2,u_3)}{\partial u_2} \\ & \times\Phi\left(\frac{\kappa_{13}(u_1,u_3)-\rho\kappa_{23}(u_2,u_3)}{\sqrt{1-\rho^2}}\right)\times\varsigma_{12}(\mathrm{u}_3;\delta_3) \\ \frac{\partial\gamma(\mathrm{u}_3;\delta_3)}{\partial u_1} & = \partial_1\mathrm{C}(u_1,u_2;\delta_{12})\times\Phi_2(\kappa_2(\mathrm{u}_2,u_3);\rho)\times\varsigma_{21}(\mathrm{u}_3;\delta_3) \\ & +\mathrm{C}(u_1,u_2;\delta_{12})\times\frac{\partial\kappa_{13}(u_1,u_3)}{\partial u_1}\times\Phi\left(\frac{\kappa_{23}(u_2,u_3)-\rho\kappa_{13}(u_1,u_3)}{\sqrt{1-\rho^2}}\right)\times\varsigma_{21}(\mathrm{u}_3;\delta_3) \\ & -\mathrm{C}(u_1,u_2;\delta_{12})\times\Phi_2(\kappa_2(\mathrm{u}_2,u_3);\rho) \\ & \times\int_0^{-\ln u_3}\left(\frac{\partial\kappa_{23}(u_2,q)}{\partial u_2}\frac{\partial\kappa_{13}(u_1,q)}{\partial u_1}\Phi\left(\frac{\kappa_{13}(u_1,q)-\rho\kappa_{23}(u_2,q)}{\sqrt{1-\rho^2}}\right)\right)dq\times\varsigma_{21}(\mathrm{u}_3;\delta_3) \end{cases}$$

with the bivariate copula density such that:

$$c(u_1,u_2;\delta) = \frac{\mathrm{C}(u_1,u_2;\delta)}{u_1u_2}\left[\begin{array}{l}\Phi\left(\delta^{-1}+\frac{1}{2}\delta\ln\left(\frac{\tilde{u}_1}{\tilde{u}_2}\right)\right) \\ \times\Phi\left(\delta^{-1}+\frac{1}{2}\delta\ln\left(\frac{\tilde{u}_2}{\tilde{u}_1}\right)\right)\end{array}\right. \\ \left.-\frac{\delta}{2\ln u_2}\phi\left(\delta^{-1}+\frac{1}{2}\delta\ln\left(\frac{\tilde{u}_1}{\tilde{u}_2}\right)\right)\right]$$

The likelihood can then be numerically computed.

For the trivariate Joe-Hu copula, the analytical expression is simpler. For indication, the bivariate copula density is

$$\begin{aligned} c(u_2;\delta_2) &= C(u_2;\delta_2)\,\tilde{u}_2^{-1}u_1u_2^{-1}\xi(u_2;\delta_2)^{-1+\frac{1}{\delta}} \\ &\times\left(\left(\left(p_1\tilde{u}_1^{\theta}\right)^{\delta}+\left(p_2\tilde{u}_2^{\theta}\right)^{\delta}\right)^{-1+\frac{1}{\delta}}\left(p_2\tilde{u}_2^{\theta}\right)^{\delta}+p_2q_2\tilde{u}_2^{\theta}\right) \end{aligned}$$

with $\xi(u_2;\delta_2)=\left(\left(\left(p_1\tilde{u}_1^{\theta}\right)^{\delta}+\left(p_2\tilde{u}_2^{\theta}\right)^{\delta}\right)^{\frac{1}{\delta}}+p_1q_1\tilde{u}_1^{\theta}+p_2q_2\tilde{u}_2^{\theta}\right)$, $p_i=(\nu_i+n-1)^{-1}$ and $q_i=(\nu_i+n-2)$.

	G			HR		
	$\hat{\delta}_1$	$\hat{\delta}_2$	ldv	$\hat{\delta}_{12}$	$\hat{\delta}_{13}$	$\hat{\delta}_{23}$
$(\chi_1^+, \chi_2^+, \chi_3^+)$	1.50	1.88	2	1.76	2.66	2.13
	(0.14)	(0.23)		(0.31)	(0.93)	(0.39)
$(\chi_1^+, \chi_2^+, -\chi_3^-)$	1.32	1.42	2	1.72	1.84	1.67
	(0.14)	(0.19)		(0.30)	(0.41)	(0.29)
$(\chi_1^+, -\chi_2^-, \chi_3^+)$	1.49	1.91	2	1.69	2.84	2.45
	(0.15)	(0.24)		(0.33)	(1.15)	(0.61)
$(\chi_1^+, -\chi_2^-, -\chi_3^-)$	1.52	2.03	1	1.62	1.66	2.88
	(0.15)	(0.25)		(0.30)	(0.35)	(0.65)
$(-\chi_1^-, -\chi_2^-, -\chi_3^-)$	1.51	1.77	2	1.69	2.53	1.34
	(0.15)	(0.22)		(0.45)	(0.81)	(0.24)
$(-\chi_1^-, -\chi_2^-, \chi_3^+)$	1.53	2.00	2	2.70	3.28	1.39
	(0.15)	(0.24)		(1.29)	(0.93)	(0.22)
$(-\chi_1^-, \chi_2^+, -\chi_3^-)$	1.26	1.81	2	1.60	2.79	1.51
	(0.14)	(0.23)		(0.33)	(0.91)	(0.30)
$(-\chi_1^-, \chi_2^+, \chi_3^+)$	1.45	2.05	2	1.53	3.40	2.13
	(0.14)	(0.25)		(0.28)	(0.91)	(0.44)

	HJ			
	$\hat{\delta}_{12}$	$\hat{\delta}_{13}$	$\hat{\delta}_{23}$	$\hat{\theta}$
$(\chi_1^+, \chi_2^+, \chi_3^+)$	1.40	1.95	1.52	1
	(0.29)	(0.52)	(0.31)	1
$(\chi_1^+, \chi_2^+, -\chi_3^-)$	1.25	1.71	1.20	1
	(0.25)	(0.39)	(0.24)	1
$(\chi_1^+, -\chi_2^-, \chi_3^+)$	1.23	2.25	1.72	1
	(0.24)	(0.86)	(0.40)	1
$(\chi_1^+, -\chi_2^-, -\chi_3^-)$	1.35	1.39	2.45	1
	(0.27)	(0.28)	(0.95)	1
$(-\chi_1^-, -\chi_2^-, -\chi_3^-)$	1.45	2.06	1.29	1
	(0.31)	(0.72)	(0.26)	1
$(-\chi_1^-, -\chi_2^-, \chi_3^+)$	2.12	2.19	1.10	1
	(0.80)	(0.84)	(0.18)	1
$(-\chi_1^-, \chi_2^+, -\chi_3^-)$	1.22	2.31	1.12	1
	(0.21)	(0.89)	(0.19)	1
$(-\chi_1^-, \chi_2^+, \chi_3^+)$	1.28	2.59	1.64	1
	(0.26)	(1.04)	(0.39)	1

Table 3.3: MLE for the parameter of the trivariate copulae (**G**: Gumbel, **HR**: Hüsler-Reiss, **HJ**: Hu-Joe)

	Minima			Maxima		
Waiting period	MSGE	MSUS	MSJP	MSGE	MSUS	MSJP
5 years	−6.1%	−5.2%	−7.0%	5.8%	4.2%	7.3%
10 years	−7.6%	−6.8%	−9.3%	6.7%	5.0%	9.0%
50 years	−12.4%	−11.9%	−17.0%	9.2%	7.5%	14.3%

Table 3.4: Univariate daily stress testing scenarii

Copula	**G**				**HR**				**HJ**			
	VaR		ES		VaR		ES		VaR		ES	
Portfolio	10y	50y	10y	50y	10y	50y	10y	50y	10y	50y	10y	50y
P_1	-13.4%	-26.3%	-17.1%	-29.6%	-13.4%	-26.4%	-17.3%	-30.1%	-13.0%	-26.7%	-17.2%	-29.4%
P_2	-15.4%	-25.1%	-18.4%	-31.5%	-14.9%	-25.0%	-18.7%	-30.7%	-15.2%	-17.9%	-18.4%	-30.6%
P_3	-20.6%	-36.0%	-25.1%	-40.7%	-20.4%	-36.4%	-25.5%	-40.5%	-20.8%	-36.2%	-25.4%	-41.0%
P_4	-15.1%	-23.0%	-17.9%	-27.2%	-15.5%	-23.0%	-17.6%	-27.1%	-15.4%	-23.5%	-18.4%	-26.9%

Table 3.5: VaR and ES with 50000 Monte-Carlo simulations

Copula	**G**				**HR**				**HJ**			
	VaR		ES		VaR		ES		VaR		ES	
Portfolio	10y	50y	10y	50y	10y	50y	10y	50y	10y	50y	10y	50y
P_1	-15.2%	-27.3%	-18.3%	-33.8%	-15.3%	-27.2%	-18.1%	-33.3%	-15.3%	-27.5%	-18.2%	-33.6%
P_2	-15.9%	-28.5%	-19.2%	-33.9%	-15.8%	-28.2%	-19.0%	-33.6%	-15.5%	-28.1%	-19.0%	-33.8%
P_3	-22.1%	-38.0%	-26.4%	-46.1%	-22.2%	-38.2%	-26.6%	-46.5%	-22.0%	-37.6%	-26.3%	-45.5%
P_4	-15.8%	-25.4%	-18.4%	-29.4%	-15.9%	-25.9%	-18.6%	-29.5%	-15.5%	-25.4%	-18.2%	-29.1%

Table 3.6: VaR and ES with 100000 Monte-Carlo simulations

Chapter 4

Investigating Dynamic Dependence using copulae

4.1 Contents and contributions

This chapter explores and extends the general framework proposed by Joe (1994, 1997) to model the dynamic dependence of time series whatever their marginal distributions. We provide a new approach to the analysis of the standard linear class of gaussian autoregressive (AR) models by extracting the copula from the model. This allows us to present a new class of models: the gaussian copula AR models that takes into account non-linearity. Two new formulations are then obtained for (i) the regression model, (ii) the autoregression function, that directly comes from the functional form of the dependence. We then propose a new definition, the intrinsic copula to characterize the minimal representation copula that encompasses all the serial dependence. We provide a theoretical definition of the intrinsic copula. This intrinsic copula furnishes the starting point for model selection. Two properties are exhibited and proved in the section. In a non linear time series model the minimal dimension is most naturally provided by identifying the intrinsic copula. Moreover, we show that there are relationships between copulae which order is greater than the order of the intrinsic copula. In section 6, we extend the estimation methods developed by Joe and Xu (1996) to our intrinsic copula defined earlier. We introduce the basis for copula based model selection and show how the copula function can provide a nonlinear autocorrelation function. Section 7 is an empirical contribution to the study of stock market index returns. Four models that assume the same marginal distribution but with four different copulae are estimated and compared. The estimates are computed using two different methods: (i) a non parametric method that has been proposed by Genest and Rivest (1993) - this is its first financial application -, (ii) the maximum likelihood method described in the previous section. The parameters estimates are given for four different copulae (Gaussian, Gumbel, Joe and Frank) under the assumption that the daily log-returns are first order Markov processes for the five following stock indices: Cazenove Small Companies, Barings, S&P 500, Nasdaq 100, MSCI Singapore. Finally, auto-concordance measures are proposed to capture the non-linear serial dependence properties in a time series beyond serial correlation.

4.2 Introduction

The problem of assessing the temporal dependence of financial returns has been an important issue in empirical finance for at least the last three decades, see for instance Campbell and Shiller(1988), Fama and French(1989) and Ang and Bekaert(2001). Much of this work has concentrated on linear predictability through linear autocorrelation analysis although much more general dynamic dependency patterns could exist. Copulae provide a general approach to modelling dependence between random variables since they link univariate margins to their full distribution function and in this chapter we seek to develop this approach to examining general dynamic dependence and make applications to examine the question of financial return predictability.

A time series can be viewed as a single drawing from a multivariate distribution. The goal of this chapter is to split this distribution into two components: the margins and the dependence structure given by the Copula. This framework allows to specify any univariate distribution for the margins and enables us to consider general non-linear relationships for the time series. The question of the departure from linearity is an important issue quite generally; see Teräsvirta, TjØstheim and Granger (1994), and copulae provide a powerful tool to explore this question. Patton (2001) has also recently explored the use of copulae in time series by studying the dependence between the Deutsche mark - U.S. dollar and Yen - U.S. dollar exchange rate returns. He finds that the dependence pattern is time-varying and asymmetric; a structure that would be difficult to isolate using linear techniques.

In the next section, the concept of a copula is briefly introduced and we show how a stationary time series - and more generally a stationary Markov chain - can be constructed from a copula function. The multivariate case is briefly discussed and the expression for the transition density function is given. The fourth section provides a theoretical framework to the concept of copula based regression which we will call c-regression. We then demonstrate that the dependence structure of a $p'th$-order Markov processes can be captured in a copula called intrinsic copula of dimension $(p+1)$. The dimension of this intrinsic copula provides the minimal representation. So while higher order copulae will capture the same structure non-parsimoniously the intrinsic copula has the lowest dimension required to fully capture the time dependence. Empirically it is frequently difficult to identify the correct dynamic order of a multivariate dynamic system even in the linear case. However in the linear case that order of the intrinsic copula will be the McMillan Degree of the System or the size of the minimal state space representation. The extension to the nonlinear case that we could potentially consider through the use of non-gaussian copulae is as far as we know a completely undeveloped area of research except through the use of determining the correlation dimension in chaos theory. In the fifth section, we focus on autoregressive (AR) models based on the multivariate gaussian copula to construct stationary Markov processes of $p'th$-order. An alternative class of copulae - Archimedean - is then used to construct markov models and some of its properties are given. In the sixth section, we consider the maximum likelihood estimation of time series with a given copula based serial dependence assumption and some discussion of model mispecification is provided. In the sixth section, we apply our model to financial examples and consider auto-concordance measures suggested in Joe (1997) based on Kendall's Tau and Spearman's rho. Finally we offer some conclusions.

4.3 Copulae and serial dependence

There are two issues when modelling a one-dimension time series: (i) the choice of the univariate margin and (ii) the structure of the time dependence. Joe (1996) proposes a very general way of obtaining stationary time series models with margins in the convolution-closed infinitely divisible class. He introduces an operator $A(.;\alpha)$ such that for $X \sim F_\theta$, $A(X) \sim F_{\alpha\theta}$ with F_θ such that $\forall(\theta_1, \theta_2) \in R_+^{*2}, F_{\theta_1} * F_{\theta_2} = F_{\theta_1+\theta_2}$ with $*$ the convolution product. He then constructs a time series as follows: $X_t = A_t(X_{t-1}) + \varepsilon_t$ with $\varepsilon_t \sim IIDF_{(1-\alpha)\theta}$ where the autocorrelation $\alpha \in (0,1)$. Our interest will focus on simpler structures such as

$$X_t = \vartheta(X_{t-1}, \ldots, X_{t-p}, \epsilon_t)$$

which are implied by the copula that describes the joint density of the data. Models for the conditional higher order moments of the random variables , corresponding to ARCH processes will also be implied by the assumed copula and we take up that question in the next chapter[1].

Let us start by providing some useful properties. Our aim initially is simply to construct a model for the conditional expectation of the time series where the serial dependence is implied by the associated copula. However, not all copulae are eligible and some structure must be put on the joint density and hence copula to ensure stationarity. Let us assume that $\{X_t\}_{t=1\ldots p+1}$ is a stationary time series generated by a p-order Markov process i.e.

$$X_t = \vartheta(X_{t-1}, \ldots, X_{t-p}, \epsilon_t)$$

for some real-valued function ϑ and ϵ_t ,the innovation which is independent of $\{X_{t-1}, \ldots, X_{t-p}\}$. Let $F = C(F, \ldots, F)$ be a $(p+1)$-variate cumulative density function (cdf) with F absolutely continuous. The copula C has to satisfy certain conditions in order to construct a stationary Markov chain and these have been summarised by Joe in the following proposition.

Proposition 6 (Joe (1997), page 245) *A stationary Markov chain of order p can be constructed from a $(p+1)$-dimensional copula C that satisfies the following conditions:*

1. *the bivariate margins $C_{i,j}(u_i, u_j)$ are such that $C_{i,i+l}(u_i, u_{i+l}) = C_{1,1+l}(u_1, u_{1+l})$ for $l = 1, \ldots, p-1$ and $i = 2, \ldots, p+1-l$*

[1] *For additional results on the dependence for stationary Markov chains, we refer the reader to* Fang, Hu *and* Joe *[1994]*, Hu *and* Joe *[1995]*

2. the higher dimensional margins $C_{i_1,\dots,i_n}(u_1,\dots,u_n)$ *are such that* $C_{i_1,\dots,i_n} =$

$C_{1,i_2-i_1+1,\dots,i_n-i_1+1}$ *for* $1 \leq i_1 < \dots < i_n \leq K$ *and* $3 \leq n \leq p$

3. C *is differentiable in its first* p *arguments*

The two first conditions[2] ensure stability in the dependence structure. Indeed, for a sample $\{x_t\}_{t=1\dots T}$ the serial dependence has to be the same, for example, between (X_t, X_{t+1}, X_{t+5}) for $t = 1, \dots, T-5$. The third condition is essentially a technical condition that allows us to compute the density of the process. In short, we see that a time series model with p lags can be deduced from a $(p+1)$-dimensional copula.

4.4 Copula based regression and model selection

This section introduces two standard concepts with a new angle for econometrics. Indeed, the role played by the copula function in standard regression and auto-regression is exhibited. This general approach permits to define (i) a new class of model: the copula regression model and (ii) a general methodology to measure serial dependence which encompasses the traditional autoregressive models. For simplicity and clarity, the results in this section are presented only in the bivariate case. The extension to the multivariate case is straightforward.

[2] *For a 5-dimensional copula, conditions 1 and 2 become*

$$\begin{cases} C_{12} = C_{23} = C_{34} = C_{45} \\ C_{13} = C_{24} = C_{35} \\ C_{14} = C_{25} \end{cases}$$

and

$$\begin{cases} C_{123} = C_{234} = C_{345} \\ C_{124} = C_{235} \\ C_{134} = C_{245} \\ C_{1234} = C_{2345} \end{cases}$$

4.4.1 Copula based regression

The copula based regression model is presented in the bivariate case: we propose to go back to the regression definition to exhibit the relationship in mean between two random variables X and Y with support S_X and S_Y. The functions F and G (respectively f and g) are the distribution functions (respectively the density functions) of X and Y. Let us note $r(x)$ the regression function defined as follows:

$$\begin{aligned} r(x) &= \mathrm{E}[Y \mid X = x] \\ &= \int_{S_Y} y \times \frac{h(x,y)}{f(x)} dy \end{aligned}$$

with $h(x,y)$ the joint distribution function. From the copula definition, $h(x,y) = f(x) \times g(y) \times c(F(x), G(y))$, the regression function is such that:

$$r(x) = \int_{S_Y} y \times g(y) \times c(F(x), G(y))\, dy \tag{4.1}$$

Under the assumption of uniform marginals[3], we have:

$$r(x) = \int_0^1 y \times c(x,y)\, dy \tag{4.2}$$

4.4.2 Autoregression

By applying the copula based regression to time series, we obtain the relationship between the auto-regression function and the copula that measures the

[3] *An alternative useful formula exists. We have*

$$r(x) = \left[y \times \frac{\partial C(x,y)}{\partial x} \right]_0^1 - \int_0^1 \frac{\partial C(x,y)}{\partial x} dy.$$

From the copula definition $\frac{\partial C(x,1)}{\partial x} = 1$, *then the regression function is*

$$r(x) = 1 - \int_0^1 \frac{\partial C(x,y)}{\partial x} dy.$$

serial dependence. Let us note $\vartheta(x_{t-1})$ the autoregression function defined as follows:

$$\begin{aligned}\vartheta(x_{t-1}) &= \mathrm{E}[X_t \mid X_{t-1} = x_{t-1}] \\ &= \int_{S_X} x_t \times \frac{h(x_{t-1}, x_t)}{f(x_{t-1})} dx_t\end{aligned}$$

with h the joint distribution function. From the copula definition and assuming stationarity, $h(x_t, x_{t-1}) = f(x_t) \times f(x_{t-1}) \times c(F(x_{t-1}), F(x_t))$, the regression function is such that:

$$\vartheta(x_{t-1}) = \int_{S_X} x_t \times f(x_t) \times c(F(x_{t-1}), F(x_t))\, dx_t \tag{4.3}$$

For example, the standard linear auto-regressive models are encompassed by this representation. This will be shown later for some AR models. An important point is that, as pointed out by the auto-regression equation, all the dynamic properties can be deduced from the distributional properties of our model. An important issue is the model selection. Two questions have to be answered: (i) Which copula family is the right one ? Different testing strategies will be used, depending on the fact that alternative models are nested or not; (ii) what is the dimension of the copula ? If the copula dimension is $p+1$, it means that we model a p-order Markov process.

For the first question, we will provide later an example in the financial application section by using the Akaike Information Criterion (AIC). To answer to the second question, the concept of intrinsic copula is presented in the next section. It permits to characterize the maximum lag in the serial dependence of the time series.

4.4.3 Model selection: the intrinsic copula

The aim of this section is to introduce a new concept: the intrinsic copula that is the starting point for model selection. Indeed, this intrinsic copula is defined as the copula that encompasses all the information about the serial dependence.

Definition 8 *For a sample $\{x_t\}_{t=1...T}$ with copula $C(u_1, ..., u_T)$ drawn from a p'th-order stationary Markov process , the intrinsic copula is the minimal representation copula with dimension $(p+1)$ that encompasses all the dependence structure.*

While this minimal order is unique the representation of the model explaining the conditional expectation or any other moment may not be unique as is well known from linear time series analysis where, for instance a given $VARMA$ model may be expressed alternatively in a state space form and there are a range of exchangable models, see Li and Tsay (1998) and Tiao and Tsay(1989). It is clear that in the linear context there is a direct relationship between the McMillan degree of the dynamic system and the order of the intrinsic copula. In practice however, since higher order models will non-parsimoniously capture the same dynamic information it is an empirical issue of how to determine the minimal order. While this may be relatively easily achieved in the linear case it is not in the nonlinear dynamic case and the only corresponding work on identifying minimal dynamic orders in nonlinear or chaotic systems we know of is through the correlation dimension of Grassberger and Procaccia (1983). In fact it may be that the copula approach to this issue is the simplest route to follow in the general case. From Bayes' theorem, the conditional density is a function of the copula density

$$f(x_t|x_{t-1},\ldots,x_{t-p}) = f(x_t)\frac{c(F(x_{t-p}),\ldots,F(x_t))}{c(F(x_{t-p}),\ldots,F(x_{t-1}))} \tag{4.4}$$

The two following properties show that Bayes' theorem provides an elegant way to obtain the copula with the lowest dimension which then captures the general structure of serial dependence within the time series.

Property 1 *For a first-order stationary Markov process , the following relations hold :*

$$c(u_1,\ldots,u_T) = \prod_{t=2}^{T} c^*(u_{t-1},u_t)$$

and

$$c(u_1,\ldots,u_T) = \prod_{t=0}^{\frac{T-k}{k-1}} c(u_{tk-t+1},u_{(t+1)k-t}) \quad \text{for} \quad k \geq 2 \quad \text{and} \quad \left(\frac{T-1}{k-1}\right) \in \mathbb{N} \tag{4.5}$$

where c^* is the intrinsic copula density

Proof

Note that $f(x_1,\ldots,x_T) = f(x_1)\prod_{t=2}^{T} f(x_t \mid x_{t-1})$ and use equation (4.4).

For the second property, write $c(u_{tk-t+1},u_{(t+1)k-t})$ in terms of the intrinsic copula c^* :

$$c(u_{tk-t+1},u_{(t+1)k-t}) = \prod_{i=1}^{k-1} c^*(u_{t(k-1)+i},u_{t(k-1)+i+1})$$

Property 2 *For a p-order Markov process, the following relations hold :*

$$c(u_1,\ldots,u_T) = \frac{\prod_{t=p+1}^{T} c^*(u_{t-p},\ldots,u_t)}{\prod_{t=p+2}^{T} c(u_{t-p},\ldots,u_{t-1})} \tag{4.6}$$

with c^* the intrinsic copula density

Proof

Note that $f(x_1, \ldots, x_T) = f(x_1, \ldots, x_p) \prod_{t=p+1}^{T} f(x_t \mid x_{t-1}, \ldots, x_{t-p})$ and use equation (4.4).

Notice that in the case of a p^{th} order Markov processes there will still be relationships between copulae with order greater than the order of the intrinsic copula. Given the intractable form of a general formula, we prefer to give three examples for $p = 2$ which indicate the main intuition. The following shortcut notation is adopted: $c(u_1, u_2, \ldots, u_k) = c_{12\ldots k}$.

Example 1 *Too much information about serial dependence. We know about copulae with dimension strictly greater than three such as* c_{12345} *. Then as*

$$c_{12345} = \frac{c^*_{123} c^*_{234} c^*_{345}}{c_{23} c_{34}}$$

and $c_{1234} = \frac{c^*_{123} c^*_{234}}{c_{23}}$ *and* $c_{2345} = \frac{c^*_{234} c^*_{345}}{c_{34}}$, *the following relationship arises :*

$$c^*_{234} = \frac{c_{1234} c_{2345}}{c_{12345}}$$

The dependence structure does not have a minimal representation with a four dimensional copula. However, all the information about serial dependence is available and the three dimensional intrinsic copula can be found.

Example 2 *Partial information about serial dependence. We have a knowledge about copulae with dimension strictly lower than three. We have*

$$c_{1234} = \frac{c^*_{123} c^*_{234}}{c_{23}}$$

and extensions of this formul for future periods. All the information about serial dependence is not available and the minimal three dimensional intrinsic copula can not be deduced.

Example 3 *Full and minimal information about serial dependence. We have a knowledge about copula with dimension three. This is the intrinsic copula and all the information about serial dependence is available.*

4.5 Two examples: the Gaussian and archimedean copulae

In this section, we provide the intuition behind copula functions for linear autoregressive models. We present what is the role of the Gaussian copula in linear auto-regression. All these models are encompassed by the copula based representation of the auto-regression described above.

4.5.1 Gaussian Copula and autoregressive models

The gaussian AR(1) and AR(p) models are explored and their dependence structure is characterized using copulae. Notice that any continuous univariate distribution (not necessarily gaussian) will be a candidate to construct a nonlinear time series model based on the gaussian copula.

4.5.1.1 The Gaussian copula regression

Let us consider the regression function assuming a gaussian copula with given margins. The gaussian copula is given by:

$$
\begin{aligned}
c_\rho(F(x), G(y)) &= \frac{\exp\left[\frac{1}{2}\left(\varsigma_x^2(x) + \varsigma_Y^2(y)\right)\right]}{\sqrt{1-\rho^2}} \\
&\quad \times \exp\left\{-\frac{\varsigma_x^2(x) + \varsigma_x^2(x) - 2\rho\varsigma_x(x)\varsigma_Y(y)}{2(1-\rho^2)}\right\}
\end{aligned}
$$

where $\varsigma_X(x) = \Phi^{-1}(F(x))$ and $\varsigma_Y(y) = \Phi^{-1}(G(y))$ with Φ^{-1} the inverse of the gaussian distribution function. We have $g(y) = \varsigma_Y'(y)\varphi(\varsigma_Y(y))$ with φ

the gaussian density function. Then,

$$\begin{aligned}
r_\rho(x) &= \int_{S_Y} y \times g(y) \times c_\rho(F(x), G(y))\, dy \\
&= \int_{S_Y} y \times \varsigma_Y'(y) \times \varphi(\varsigma_Y(y)) \times c_\rho(F(x), G(y))\, dy \\
&= \int_{S_Y} y \times \varsigma_Y'(y) \times \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\varsigma_X^2(x)\right] \\
&\quad \times \frac{\exp\left[\frac{1}{2}\left(\varsigma_X^2(x) + \varsigma_Y^2(y)\right)\right]}{\sqrt{1-\rho^2}} \\
&\quad \times \exp\left\{-\frac{\varsigma_X^2(x) + \varsigma_Y^2(y) - 2\rho\varsigma_X(x)\varsigma_Y(y)}{2(1-\rho^2)}\right\} dy \\
&= \int_{S_Y} \frac{y \times \varsigma_Y'(y)}{\sqrt{1-\rho^2}} \times \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\varsigma_Y(y) - \rho\varsigma_X(x)}{\sqrt{1-\rho^2}}\right)^2\right\} dy.
\end{aligned}$$

Applying the change of variable $Z = \frac{\varsigma_Y(Y) - \rho\varsigma_X(x)}{\sqrt{1-\rho^2}}$, we obtain

$$\begin{aligned}
r_\rho(x) &= \int_{S_Z} \varsigma_Y^{-1}\left(z\sqrt{1-\rho^2} + \rho\varsigma_X(x)\right) \times \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} dz \\
r_\rho(x) &= \mathrm{E}\left[\varsigma_Y^{-1}\left(Z\sqrt{1-\rho^2} + \rho\varsigma_X(x)\right)\right] \qquad (4.7)
\end{aligned}$$

The regression function described above underlies the standard gaussian AR models. We explain this point in the next subsection.

4.5.1.2 AR(1) model

Consider the simple AR(1) model:

$$\left\{\begin{array}{l} x_t = c + \phi x_{t-1} + \varepsilon_t \\ \varepsilon_t \sim IID\mathcal{N}(0, \sigma^2) \end{array}\right. \qquad (4.8)$$

The conditional and unconditional pdfs are well known to be :

$$\left\{\begin{array}{l} x_t | x_{t-1} \sim \mathcal{N}(\phi x_{t-1}, \sigma^2) \\ x_t \sim \mathcal{N}(\frac{c}{1-\phi}, \sigma^2/(1-\phi^2)) \end{array}\right. \qquad (4.9)$$

A time series sample $\{x_t\}_{t=1\ldots T}$ can be viewed as single draw from $x \sim \mathrm{N}(\mu, \Sigma)$ with density

$$f(\mathrm{x}; \mu, \Sigma) = (2\pi)^{-\frac{T}{2}} \left|\Sigma^{-1}\right|^{\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathrm{x}-\mu)^\top \Sigma^{-1} (\mathrm{x}-\mu)\right\} \qquad (4.10)$$

where

$$\mu = \mathrm{E}(\mathrm{x}) = \frac{c}{1-\phi} \tag{4.11}$$

$$\Sigma = \mathrm{E}(\mathrm{x}-\mu)(\mathrm{x}-\mu)^{\mathsf{T}} = \sigma^2 \mathrm{V} = \frac{\sigma^2}{1-\phi^2}\rho \tag{4.12}$$

with

$$\mathrm{V} = \frac{1}{1-\phi^2}\begin{pmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{T-1} \\ \phi & 1 & \phi & \cdots & \phi^{T-2} \\ \phi^2 & \phi & 1 & \cdots & \phi^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{T-1} & \phi^{T-2} & \phi^{T-3} & & 1 \end{pmatrix} \tag{4.13}$$

Proposition 7 *The copula density function corresponding to time series* $\{x_t\}_{t=1\ldots T}$ *from an AR(1) process is*

$$\mathrm{c}\left(u_1,\ldots,u_t,\ldots,u_T;\rho\right) = (1-\phi^2)^{\frac{1-T}{2}} \exp\left(-\frac{1}{2}\varsigma^{\mathsf{T}}\left(\rho^{-1}-\mathbb{I}\right)\varsigma\right) \tag{4.14}$$

with $\varsigma_t = \Phi^{-1}\left(u_t\right)$.

This copula density function is plotted for the AR(1) process (i) in the bivariate case for different correlation levels (Figure 4.1) (ii) in the trivariate case by fixing the value of one margin (Figures 4.2 to 4.4).

Proof

We have $\prod_{t=1}^{T} f_t\left(x_t\right) = (2\pi)^{-\frac{T}{2}}\sigma^{-T}(1-\phi^2)^{\frac{T}{2}} \exp\left\{-\frac{1}{2}\varsigma^{\mathsf{T}}\varsigma\right\}$ with $\varsigma_t = \Phi^{-1}\left(u_t\right)$. From (4.10), the copula density is obtained. As $V^{-1} = L^{\mathsf{T}}L$ with L a lower triangular matrix with diagonal product $\sqrt{1-\phi^2}$, $|\mathrm{V}^{-1}| = 1-\phi^2$. Then note that $(1-\phi^2)\rho^{-1} = V^{-1}$.

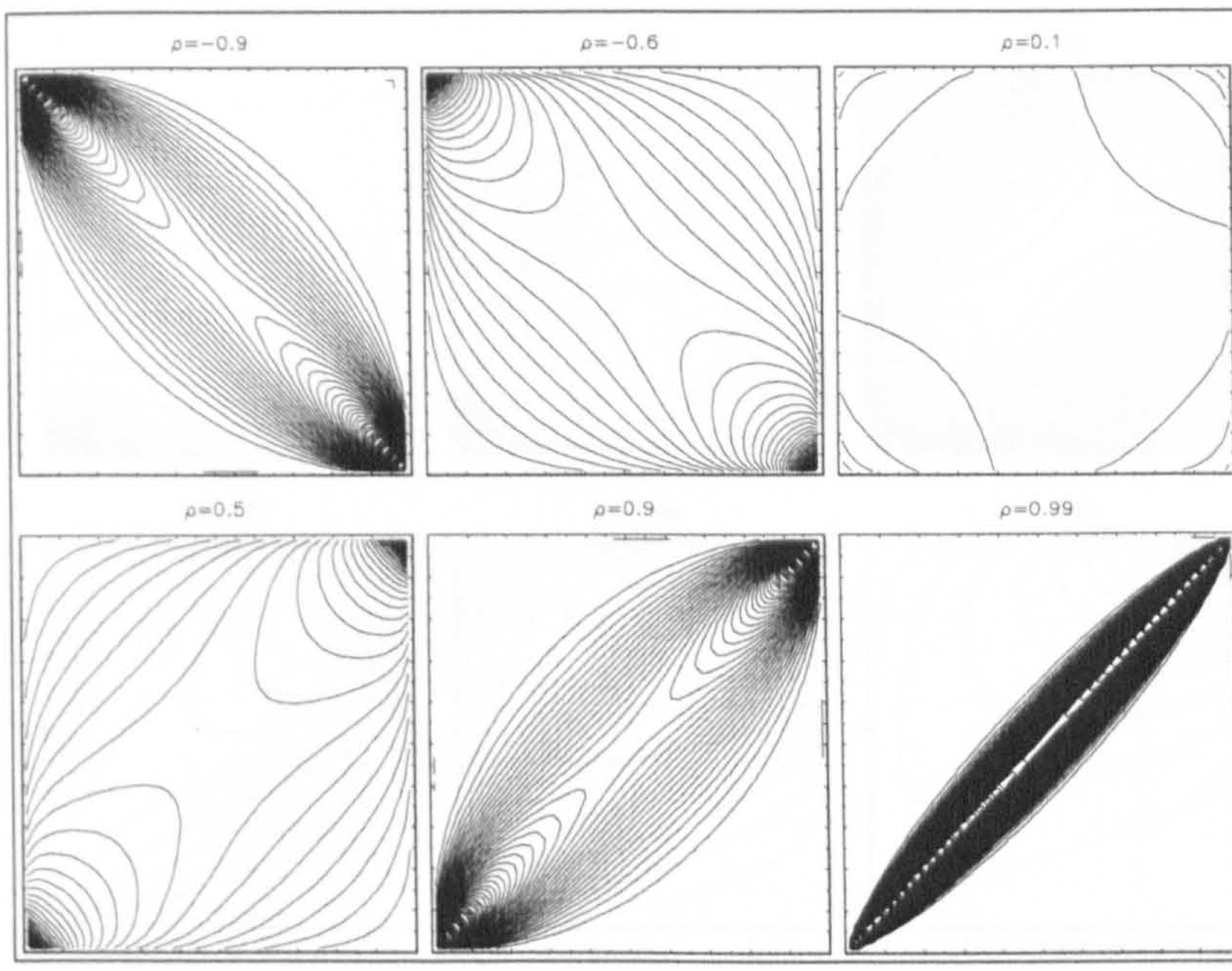

Figure 4.1: Contour plot of the bivariate gaussian copula with different values for the serial dependence parameter ρ

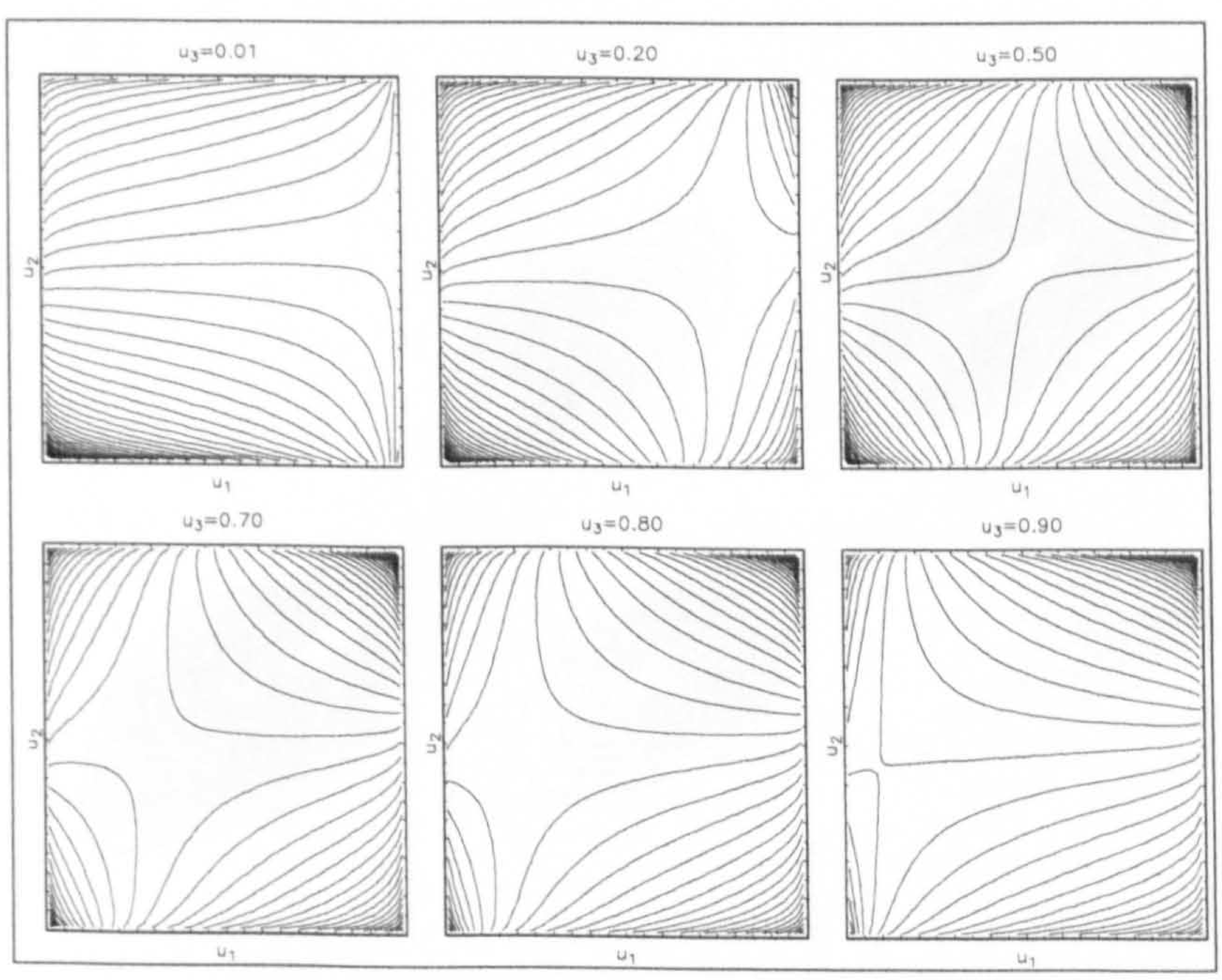

Figure 4.2: Contour slices of the 3−dimensional gaussian copula with an AR(1) correlation structure with $\rho = 0.1$

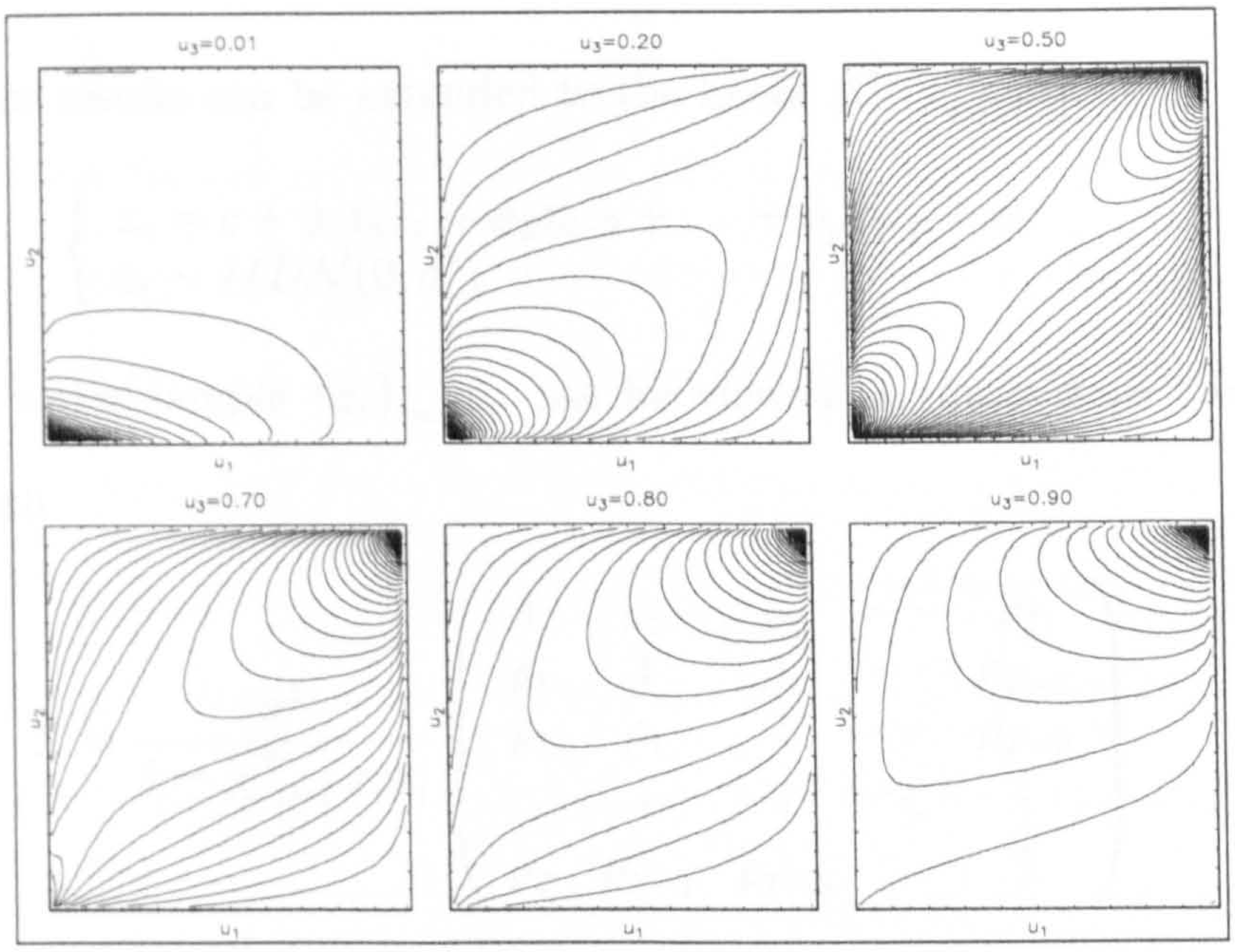

Figure 4.3: Contour slices of the 3−dimensional gaussian copula with an AR(1) correlation structure with $\rho = 0.5$

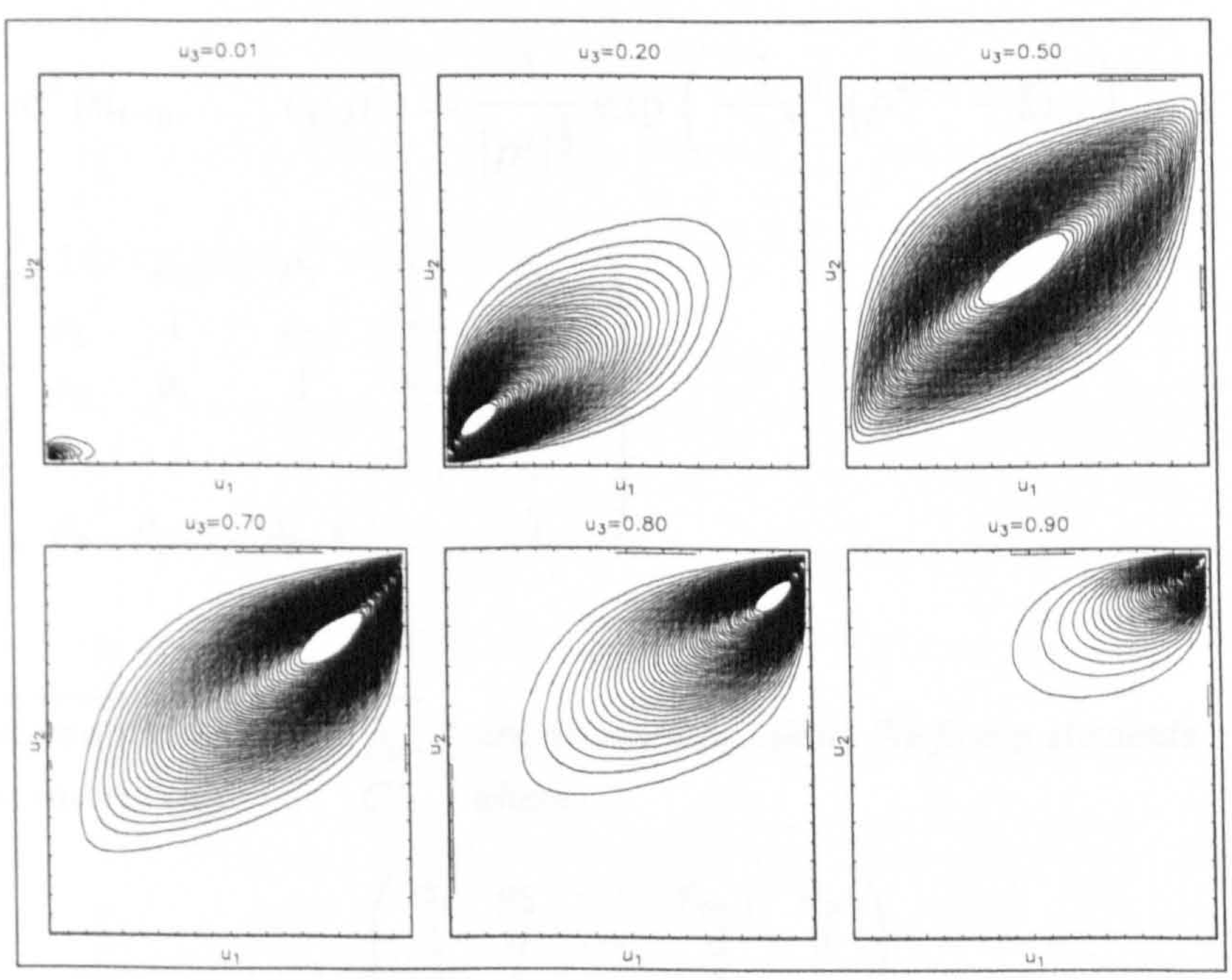

Figure 4.4: Contour slices of the 3−dimensional gaussian copula with an AR(1) correlation structure with $\rho = 0.9$

4.5.1.3 AR(p) model

The previous results can be extended to the linear AR(p) case:

$$\begin{cases} x_t = c + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + \varepsilon_t \\ \varepsilon_t \sim IID\mathcal{N}(0, \sigma^2) \end{cases} \tag{4.15}$$

A time series sample $\{x_t\}_{t=1 \ldots T}$ can be viewed as single draw from $x \sim N(\mu, \Sigma)$ with

$$\Sigma = \frac{\sigma^2}{1 - \sum_{j=1}^{p} \rho_j \phi_j} \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_T \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{T-1} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{T-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_T & \rho_{T-1} & \rho_{T-2} & & 1 \end{pmatrix} \tag{4.16}$$

with ρ_j the autocorrelations[4] that fulfill the Yule-Walker equations:

$$\begin{cases} \rho_0 = 1 \\ \rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} + \ldots + \phi_p \rho_{j-p} \quad \text{for} \quad j = 1, 2, \ldots \end{cases} \tag{4.17}$$

Property 3 *The AR(p) gaussian intrinsic copula density function is given by*

$$\mathbf{c}^* \left(u_{t-p}, \ldots, u_t; \rho^*\right) = \frac{1}{|\rho^*|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \varsigma^\top \left(\rho^{*-1} - \mathbb{I}\right) \varsigma\right) \tag{4.18}$$

with $\rho^* = \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_p \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-1} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_p & \rho_{p-1} & \rho_{p-2} & & 1 \end{pmatrix}$

[4] *The autocorrelations* $(\rho_0, \ldots, \rho_{p-1})$ *are obtained by taking the first p elements of the first column of the matrix* $\left(\mathbb{I}_{p^2} - C \quad C\right)^{-1}$ *where*

$$C = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

4.5.2 An alternative dependence model: the archimedean class

Another famous class of copula functions, the Archimedean class, provides an alternative to the gaussian copula. This class is very useful since each member of this class can be characterised by a simple generator function. However, its extension from the bivariate to the multivariate case often becomes intractable. Indeed, since each correlation parameter of a gaussian copula provides information about the dependence between each pair of random variables so for N variables, there are $N(N-1)/2$ parameters. For Archimedean copulae, the dependence is characterised by only $(N-1)$ parameters. We shall focus on the bivariate case i.e. on first-order Markov processes. Genest and MacKay (1996) provided a definition of this family:

$$\mathrm{C}\left(u_1,\ldots,u_N\right)=\varphi^{-1}\left(\varphi\left(u_1\right)+\ldots+\varphi\left(u_N\right)\right) \tag{4.19}$$

with $\varphi(u)$ a C^2 function with $\varphi(1)=0$, $\varphi'(u)<0$ and $\varphi''(u)>0$ for all $0\leq u\leq 1$. One technique by which to construct multivariate archimedean copulae is the compound method where

$$\begin{array}{l}\mathrm{C}\left(u_1,u_2\right)=\varphi^{-1}\left(\varphi\left(u_1\right)+\varphi\left(u_2\right)\right)\\ \mathrm{C}\left(u_1,u_2,u_3\right)=\mathrm{C}\left(\mathrm{C}\left(u_1,u_2\right),u_3\right)\\ \vdots\\ \mathrm{C}\left(u_1,\ldots,u_N\right)=\mathrm{C}\left(\mathrm{C}\left(u_1,\ldots,u_{N-1}\right),u_N\right)\end{array} \tag{4.20}$$

The function $\varphi(u)$ is called the generator of the copula and essentially identifies the copula function. Genest and Rivest (1993) proposed a method to identify an Archimedean copula by comparing the true value of a function $\lambda(u)$ to its nonparametric estimate, where

$$\begin{aligned}\lambda(u) &= u-\Pr\left\{\mathrm{C}\left(U_1,\ldots,U_N\right)\leq u\right\}\\ &= \sum_{n=1}^{N}(-1)^{n-1}\frac{\varphi^n\left(u\right)}{n!}\omega_{n-1}\left(u\right)\end{aligned} \tag{4.21}$$

Copula	$C(u_1, u_2)$
$C^{\perp}$	$u_1 u_2$
Gumbel	$\exp\left(-\left((-\ln u_1)^{\delta} + (-\ln u_2)^{\delta}\right)^{\frac{1}{\delta}}\right)$
Joe	$1-\begin{pmatrix} (1-u_1)^{\delta} + (1-u_2)^{\delta} \\ -(1-u_1)^{\delta}(1-u_2)^{\delta} \end{pmatrix}^{\frac{1}{\delta}}$
Frank	$-\frac{1}{\delta}\ln\left(1+\frac{(e^{-\delta u_1}-1)(e^{-\delta u_2}-1)}{e^{-\delta}-1}\right)$

Copula	$\varphi(u)$	$\delta \in$	dependence
$C^{\perp}$	$-\ln u$		
Gumbel	$(-\ln u)^{\delta}$	$(1, \infty)$	+
Joe	$-\ln\left(1-(1-u)^{\delta}\right)$	$(1, \infty)$	+
Frank	$-\ln \frac{e^{-\delta u}-1}{e^{-\delta}-1}$	R^*	+ and -

Table 4.1: Three famous bivariate archimedean copulae $\mathbf{C}(u_1, u_2)$ with the generator function $\varphi(u)$ and properties

with

$$\begin{cases} \omega_0(u) = (\varphi\prime(u))^{-1} \\ \omega_n(u) = (\varphi'(u))^{-1}\left(\frac{\partial \omega_{n-1}(u)}{\partial u}\right) \end{cases}$$

We refer to Barbe, Genest, Ghoudi and Rémillard (1996) for the proof. A non parametric estimate of $\lambda(u)$ is given by

$$\hat{\lambda}(u) = u - \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}_{[\eta_i \leq u]} \tag{4.22}$$

with

$$\eta_i = \frac{1}{T-1}\sum_{t=1}^{T} \mathbf{1}_{[x_1^t < x_1^i, \ldots, x_N^t < x_N^i]}$$

4.6 Maximum likelihood estimation and model selection

4.6.1 MLE methods

First, the three maximum likelihood methods already described are presented in the context of time series model estimation. Then, it is shown how the likelihood function for a sample of size T can be described as a function of the intrinsic copula. The case of the gaussian copula is specifically studied.

As noted in Joe and Xu (1996) and Bouyé, Durrleman, Nickeghbali, Riboulet and Roncalli (2000), three maximum likelihood methods are available.

1. The Exact Maximum Likelihood (EML) method: Parameters of the copula and marginals are estimated simultaneously. The time series sample, $x = \{x_t\}_{t=1...T}$, has density

$$f(\mathrm{x};\gamma,\delta) = c(F(x_1;\gamma),\ldots,F(x_T;\gamma),\delta)\prod_{t=1}^{T} f(x_t;\gamma)$$

The log-likelihood of the joint distribution function for a sample of size T is

$$L(\gamma,\delta) = \sum_{t=1}^{T} \log f(\mathrm{x}_t;\gamma,\delta) \tag{4.23}$$

The MLE estimates $(\hat{\gamma},\hat{\delta})$ maximize L, are obtained by solving

$$\left(\frac{\partial L}{\partial \gamma},\frac{\partial L}{\partial \delta}\right)^{\top} = 0.$$

2. The Inference Function for Margins (IFM) method is a two-step procedure. First, parameters of the marginals are estimated. Second, MLE is applied to estimate the dependence parameters of the copula. The log-likelihood functions for the univariate margin L_m is considered:

$$L_m(\gamma) = \sum_{t=1}^{T} \log f(x_t;\gamma) \tag{4.24}$$

The estimates $\tilde{\gamma}$ maximize L_m. The log-likelihood of the joint distribution function $L(\tilde{\gamma},\delta)$ is maximized over δ to obtain $\tilde{\delta}$. Finally, the IFM estimates $(\tilde{\gamma},\tilde{\delta})$ are obtained by solving

$$\left(\frac{\partial L_m}{\partial \gamma},\frac{\partial L}{\partial \delta}\right) = 0.$$

3. The Canonical Maximum Likelihood (CML) method: Only the parameters of the copula are estimated. The empirical cdfs are obtained by mapping variables to uniforms. The margins are mapped to uniforms:

$$\mathbf{X} \in \mathbf{R}^T \longmapsto \mathbf{u} \in (0,1)^T$$

The parameters of the copula δ are obtained by maximizing the log-likelihood of copula cdf

$$L_c(\delta) = \sum_{t=1}^{T} \log c(\mathbf{u}_t; \delta) \tag{4.25}$$

The estimate $\bar{\delta}$ is obtained from solving $\frac{\partial L_c}{\partial \delta} = 0$.

Let (γ, δ) be the vector of parameters to be estimated and (Γ, Δ) the parameter spaces where γ characterizes the margins and δ the dependence. For a sample of size T, the log-likelihood function $\ell_t(\gamma, \delta)$ can be constructed so that $(\hat{\gamma}_{\mathrm{ML}}, \hat{\delta}_{\mathrm{ML}})$ is the Maximum Likelihood (ML) estimator given by

$$(\hat{\gamma}_{\mathrm{ML}}, \hat{\delta}_{\mathrm{ML}}) = \underset{(\gamma,\delta)\in(\Gamma,\Delta)}{\mathrm{ArgMax}} \sum_{t=1}^{T} \ell_t(\gamma, \delta) \tag{4.26}$$

with asymptotic normality:

$$(\hat{\gamma}_{\mathrm{ML}}, \hat{\delta}_{\mathrm{ML}}) \longrightarrow T^{-\frac{1}{2}}\mathcal{N}\left((\gamma_0, \delta_0), \mathcal{I}^{-1}(\gamma_0, \delta_0)\right) \tag{4.27}$$

with $I(\gamma_0, \delta_0)$ the Fisher information matrix.

The likelihood for a sample $\{x_t\}_{t=1\ldots T}$ of a p-order Markov process can be deduced from (4.4):

$$\begin{aligned} L(x_1, \ldots, x_T; \gamma, \delta) &= \mathrm{f}(x_1, \ldots, x_T; \gamma, \delta) \\ &= \prod_{t=1}^{T} f(x_t; \gamma) \frac{\prod_{t=p+1}^{T} \mathrm{c}^*(F(x_{t-p}; \gamma), \ldots, F(x_t; \gamma); \delta)}{\prod_{t=p+2}^{T} c(F(x_{t-p}; \gamma), \ldots, F(x_{t-1}; \gamma); \delta)} \end{aligned} \tag{4.28}$$

and the log-likelihood estimator is then given by

$$\begin{aligned} \ell(x_1,\ldots,x_T;\gamma,\delta) &= \ln \mathrm{f}(x_1,\ldots,x_T;\gamma,\delta) \\ &= \sum_{t=1}^{T} \ln f(x_t;\gamma) + \sum_{t=p+1}^{T} \ln \mathrm{c}^*(F(x_{t-p};\gamma),\ldots,F(x_t;\gamma);\delta) \\ &\quad - \sum_{t=p+2}^{T} \ln c(F(x_{t-p};\gamma),\ldots,F(x_{t-1};\gamma);\delta) \end{aligned} \tag{4.29}$$

In the case of the gaussian copula, we have

$$\begin{aligned} \ell(\gamma,\rho^*) &= \sum_{t=1}^{T} \ln f(x_t;\gamma) - \frac{T-p}{2} \ln|\rho^*| - \frac{1}{2} \sum_{t=p+1}^{T} \varsigma_t^{\top} \left(\rho^{*-1} - \mathbb{I}\right) \varsigma_t \\ &\quad - \frac{T-p-1}{2} \ln|\rho| \qquad (4.30) \\ &\quad - \frac{1}{2} \sum_{t=p+2}^{T} \varsigma_{t-1}^{\top} \left(\rho^{-1} - \mathbb{I}\right) \varsigma_{t-1} \qquad (4.31) \end{aligned}$$

with

$$\begin{cases} Rank(\rho^*) = p+1 \\ Rank(\rho) = p \\ \varsigma_t = \left(\Phi^{-1}\left(F(x_{t-p})\right),\ldots,\Phi^{-1}\left(F(x_t)\right)\right) \\ \varsigma_{t-1} = \left(\Phi^{-1}\left(F(x_{t-p})\right),\ldots,\Phi^{-1}\left(F(x_{t-1})\right)\right) \end{cases}$$

and the ML estimate of ρ is

$$\hat{\rho}^*_{\mathrm{ML}} = \frac{1}{T} \sum_{t=1}^{T} \varsigma_t^{\top} \varsigma_t \tag{4.32}$$

4.6.2 Model selection

The main intuition of our model selection methodology is to compare the values of the maximum likelihood function under the different hypothesis for the copula model. This comparison is performed by using: (a) standard likelihood based tests (Wald test, Lagrange multiplier test or likelihood ratio test) if the models are nested, (b) or extended versions of these tests if the models are non-nested.

For example, for a sample $\{x_t\}_{t=1...T}$ and under the assumption that the serial dependence is characterized by the same copula (e.g. gaussian copula), one might be interested in testing two nested hypothesis:

- H_p the dimension of the intrinsic copula is p with $p < T$
- H_q the dimension of the intrinsic copula is q with $q < p < T$

Then the maximum likelihood values of all the sample (of size T) are computed under H_p and H_q and a standard LR test, for example, can be applied as H_q is nested in H_p. This test can be used iteratively in order to find the dimension of the intrinsic copula. More precisely, the dimension is p if H_p can not be rejected against H_q.

Another test that can be performed for model selection

- H_{C_1} the copula C_1 characterizes the serial dependence of the sample
- H_{C_2} the copula C_2 characterizes the serial dependence of the sample

In this case, the hypothesis will often be non-nested as it will not be tractable to write C_2 as a sub-family of C_1.

4.7 Financial applications

The section provides an empirical financial application of the IFM method to the estimation of the copula regression parameters. First, we find the distribution that best fits the univariate returns. We will see that the Burr3 distribution is a good candidate. However, it is important to note that our methodology does not rest on the use of the Burr3 density. Second, we select the copula that provides the best goodness of fit.

4.7.1 Estimation of the margins

The annualized daily log-returns are considered for five indices: Cazenove small companies (CAZSCOS), Barings (BARINGS), S&P 500 (SP500), Nasdaq 100 (NASDAQ) and MSCI Singapore (MSSING), from January 1983 to March 2000. The sample size is $T = 4499$. A number of condidate marginal distributions were tested (Gaussian, Weibull, Student and Burr3), but the Burr3 was found to the each of the series best. This is shown by the Kolmogorov-Smirnov statistics (KS) in Appendix. The Burr3 has the following distribution :

$$F(x;\gamma) = \left[1 - \frac{1}{1+(x/\tau)^{\alpha}}\right]^{\lambda} \quad \text{with} \quad x \in \mathbb{R}^{+} \tag{4.33}$$

with the parameters $\gamma = (\alpha, \lambda, \tau)$. The probability density function (pdf) is

$$f(x;\gamma) = \frac{\alpha\lambda x^{\alpha\lambda-1}\tau^{\alpha}}{(x^{\alpha}+\tau^{\alpha})^{\lambda+1}} \tag{4.34}$$

The distribution can be split into two parts for positive and negative returns and the corresponding parameters are superscripted + or − depending on which side of the estimated distribution we are considering. The left tails (respectively right tails) are plotted in the Figure 4.13 (respectively Figure 4.14) of the Appendix. The maximum likelihood estimates of the parameters and the Kolmogorov-Smirnov (KS) values[5] for the selected distributions are reported in Table A1 of the Appendix.

4.7.2 Copula model selection

The following procedure is applied for copula selection:

1. Different dimensions are tested for the intrinsic copula. We first assume that the intrinsic copula is a gaussian copula with dimension T (the sample size) and LR tests are iteratively performed to reduce the order of the serial dependence. Finally, it appears that, for all indices, only the one order serial dependence (one lag) is not rejected against the independence hypothesis.

[5] **, **, *** mean that the null hypothesis of the Kolmogorov-Smirnov test (the true distribution equals the estimated one) is respectively rejected at 1%, 5% and 10% level.*

2. On the basis of the above results, the IFM estimation is performed by assuming one lag. Alternative models to the gaussian copula are also estimated. The CML estimates (known to be consistent and efficient as proved in Genest, Ghoudi and Rivest (1995)) are also computed and appear to be very close to the IFM estimates that make us more confident about our results. The parameters using the IFM (respectively CML) method with a Gaussian AR(1) copula and three archimedean copulae are reported in Table 4.2 (respectively 4.3). The Gumbel copula clearly fits the dependence structure in the data best for illiquid markets such as CAZSCOS, BARINGS and MSSING. Not surprisingly, the SP500 is the more liquid market which is also best fit by the Gumbel[6]. Looking at the 95% confidence intervals, we can see that the hypothesis of serial independence can only be rejected for CAZSCOS. A fully non-parametric estimation is also done and it confirms that the Gumbel copula best fits the one lag dependence. Figures 4.5 to 4.9 plot the non-parametric estimator $\hat{\lambda}(u)$ with the fitted ML values of the independent and archimedean copulae for all markets.

4.8 Auto-concordance

One main advantage of using copulae is that all the dependence structure is captured by the copula function itself since it is obviously not held in the marginal distributions. Measures of dependence based on the copula have the advantage that they are also invariant to monotonic transformations of the data. Auto-correlation analysis as an approach to measuring dynamic dependence suffers from the same serious limitations that restrict the use of correlation as a measure of association. In particular the autocorrelogram is designed to detect only linear autoregressive processes. For example Ljung and Box (1979) or Dufour and Roy (1995) statistics are a good indicator of the serial dependence if it is linear with normal innovations. It is therefore natural to

[6] *Notice that a simple model selection can be based on the likelihood values themselves in this case since there is a single parameter in each of the separate families and so comparison by the likelihood value corresponds to selection by Akaike's Information Criterion (AIC).*

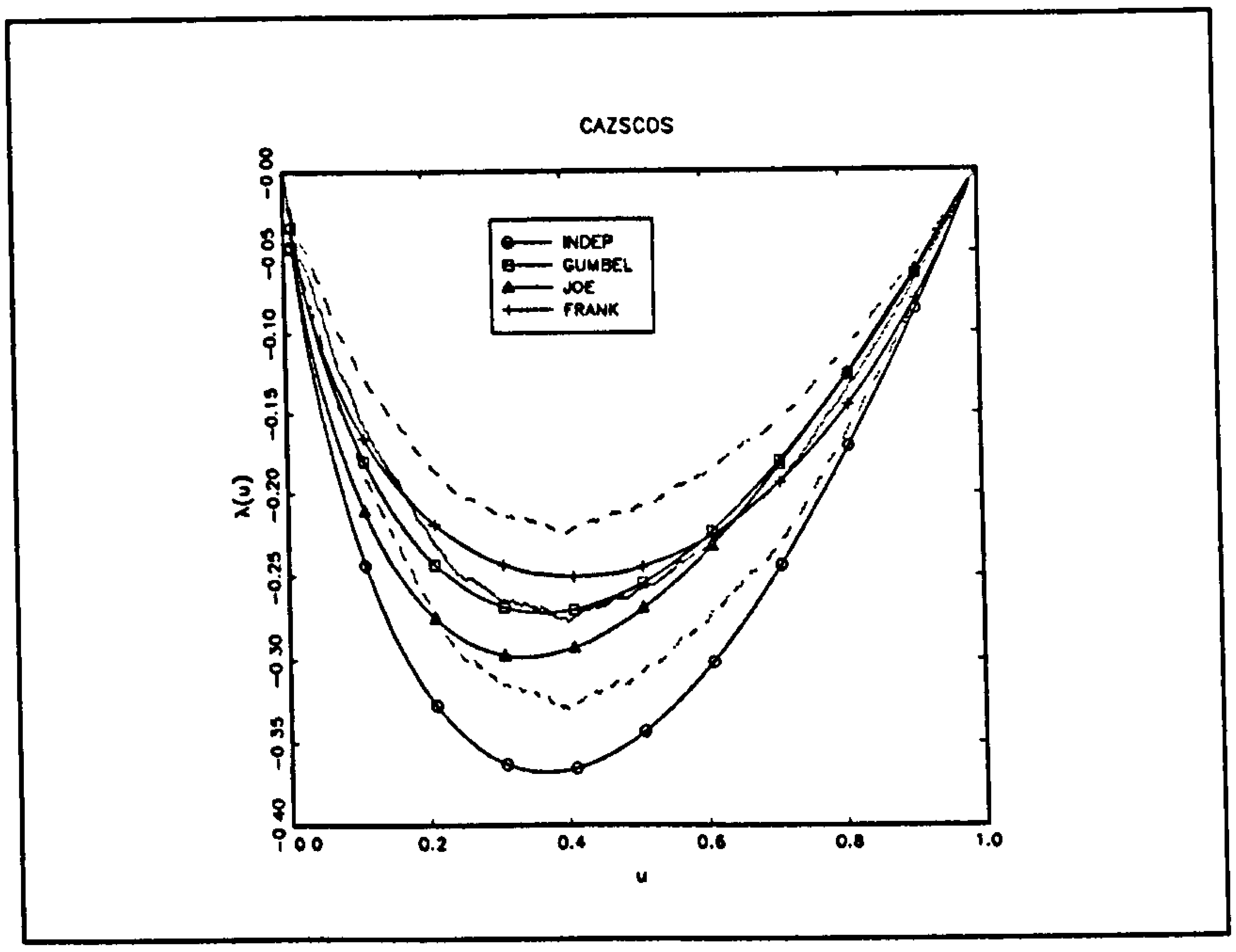

Figure 4.5: Empirical and fitted functions of $\lambda(u)$ for CAZSCOS. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

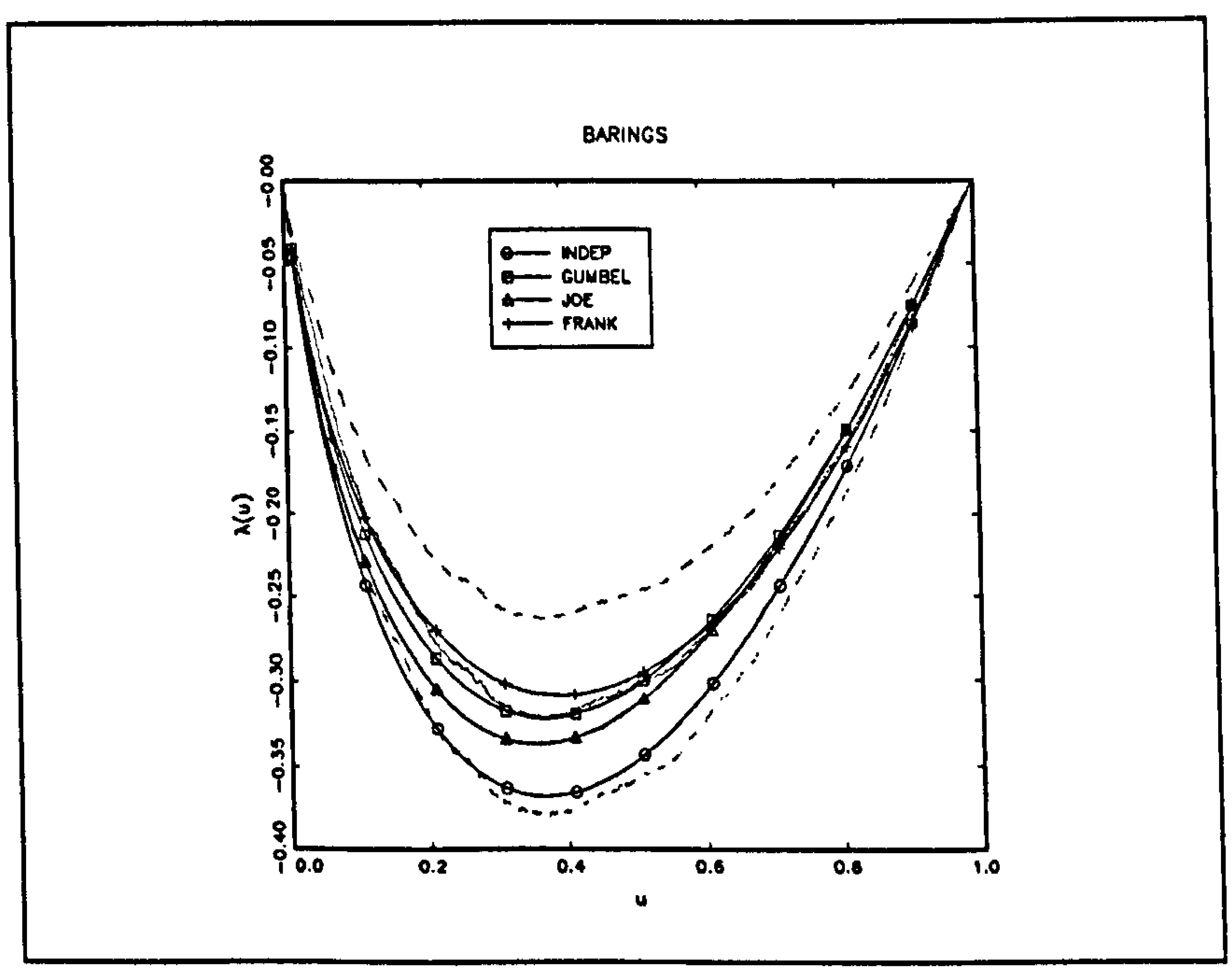

Figure 4.6: Empirical and fitted functions of $\lambda(u)$ for BARINGS. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

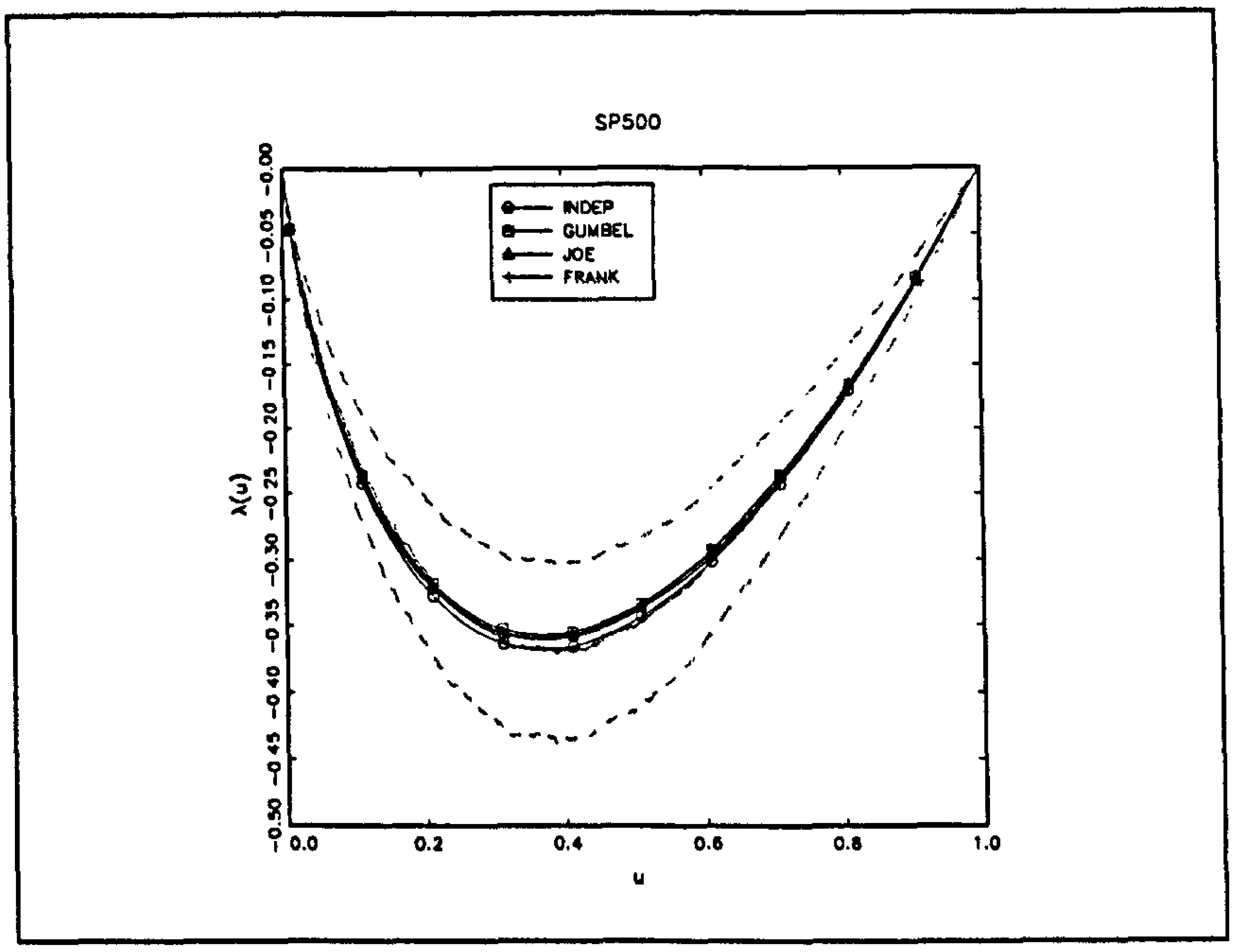

Figure 4.7: Empirical and fitted functions of $\lambda(u)$ for SP500. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

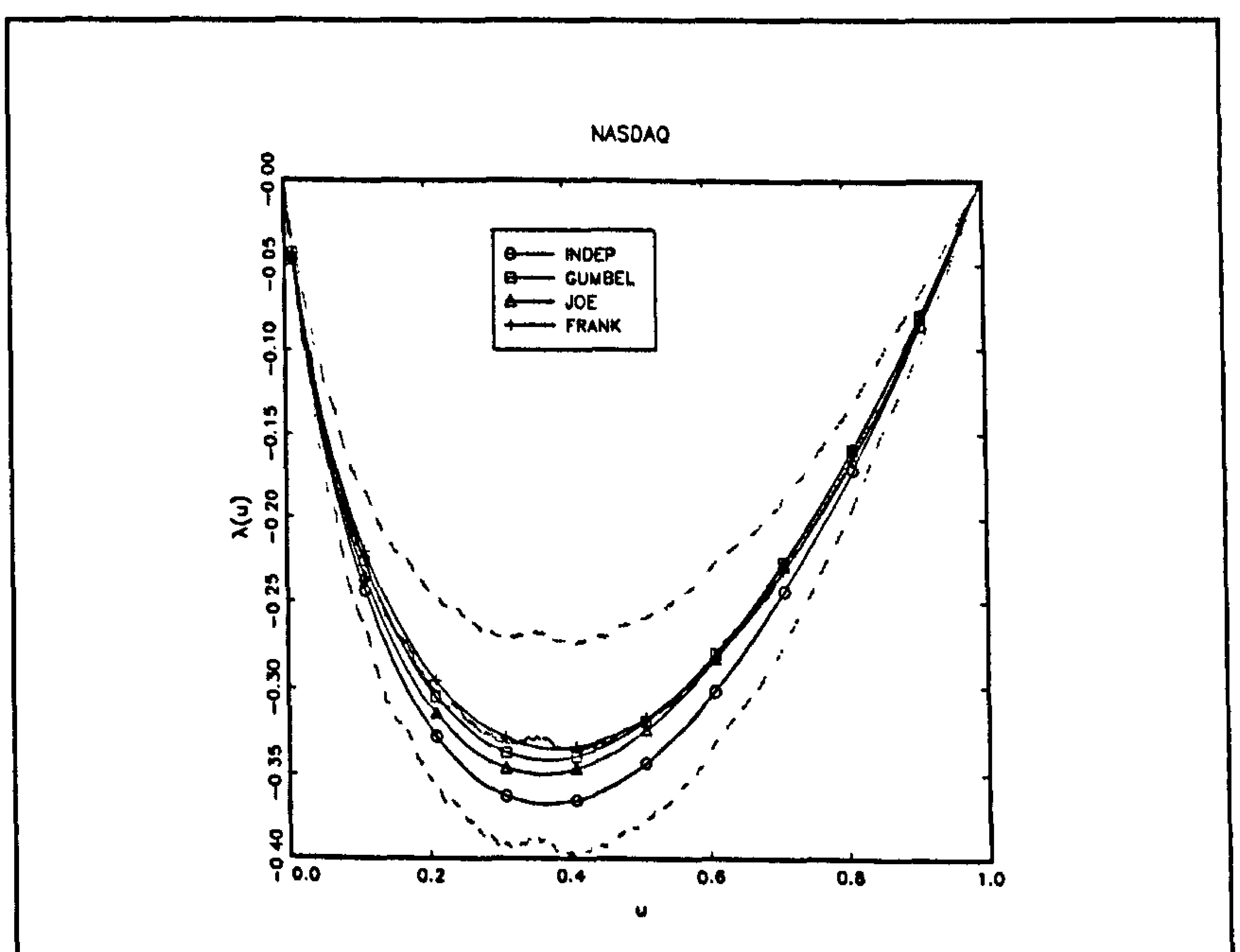

Figure 4.8: Empirical and fitted functions of $\lambda(u)$ for NASDAQ. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

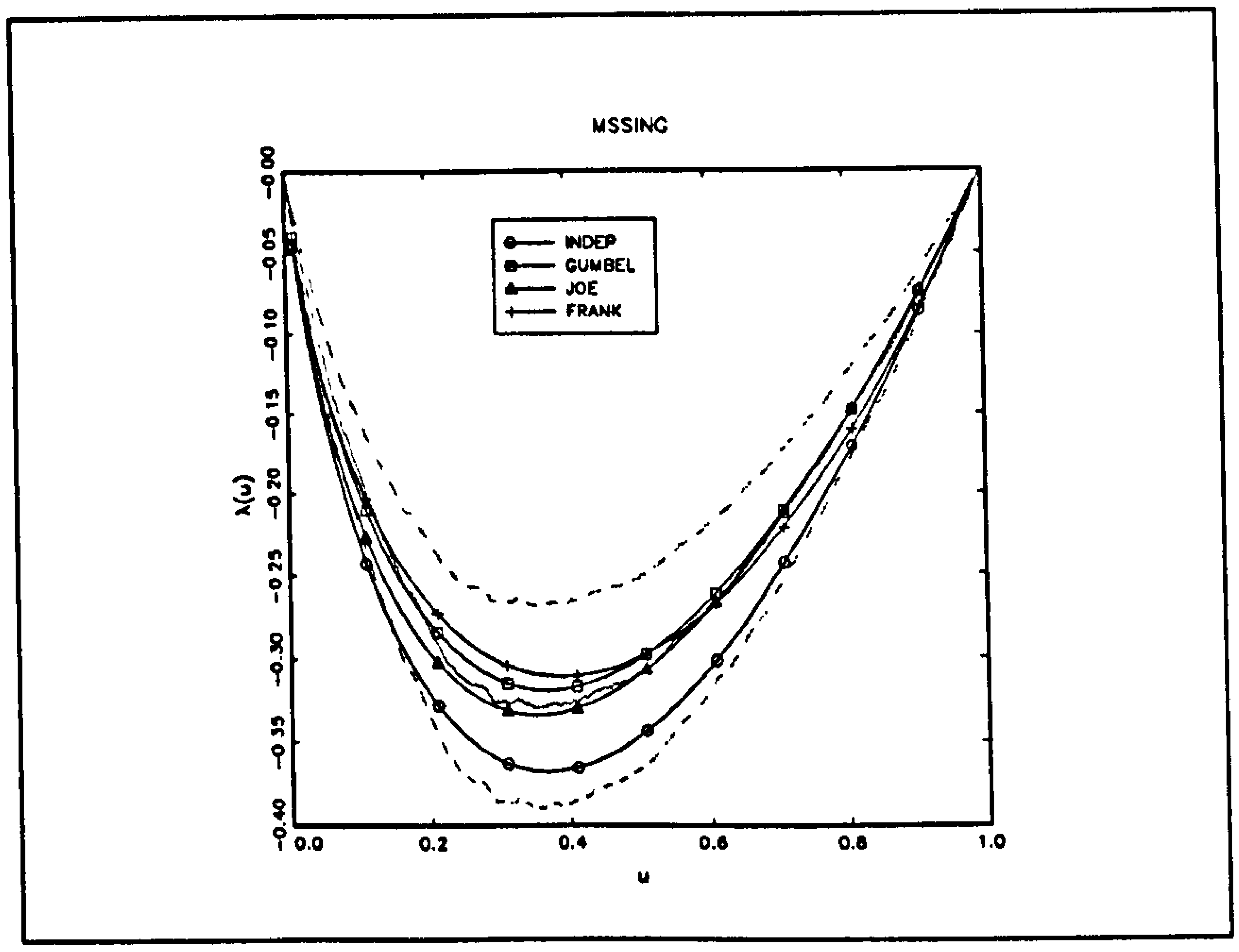

Figure 4.9: Empirical and fitted functions of $\lambda(u)$ for MSSING. The dashed lines are the 95% confidence interval for the empirical $\lambda(u)$.

	CAZSCOS	BARINGS	SP500	NASDAQ	MSSING
GAUSSIAN					
$\delta = \rho$	0.383	0.206	0.034	0.089	0.188
	(0.012)	*(0.014)*	*(0.015)*	*(0.015)*	*(0.014)*
LogLik	-7041.3	-8983.6	-9739.8	-11541.7	-11016.7
GUMBEL					
δ	1.352	1.145	1.030	1.076	1.154
	(0.016)	*(0.012)*	*(0.009)*	*(0.011)*	*(0.012)*
LogLik	-6973.6	-8971.6	-9733.1	-11523.4	-10977.5
JOE					
δ	1.448	1.173	1.039	1.095	1.194
	(0.023)	*(0.017)*	*(0.012)*	*(0.015)*	*(0.018)*
LogLik	-7059.5	-8998.5	-9733.3	-11529.6	-10999.5
FRANK					
δ	2.662	1.266	0.154	0.687	1.206
	(0.099)	*(0.093)*	*(0.093)*	*(0.095)*	*(0.095)*
LogLik	-7022.1	-8987.1	-9740.8	-11532.0	-11015.6
INDEP					
LogLik	-7385.7	-9078.9	-9742.2	-11558.3	-11096.3

Table 4.2: IFM estimates for various copulae under the assumption of a first order Markov process.

	CAZSCOS	BARINGS	SP500	NASDAQ	MSSING
GAUSSIAN					
$\delta = \rho$	0.379	0.206	0.031	0.082	0.192
	(0.012)	*(0.014)*	*(0.015)*	*(0.015)*	*(0.014)*
LogLik	346.2	96.9	2.2	15.0	83.5
GUMBEL					
δ	1.345	1.143	1.027	1.070	1.153
	(0.015)	*(0.012)*	*(0.008)*	*(0.010)*	*(0.012)*
LogLik	414.7	107.9	9.7	33.3	120.2
JOE					
δ	1.441	1.169	1.034	1.084	1.191
	(0.023)	*(0.017)*	*(0.011)*	*(0.015)*	*(0.017)*
LogLik	330.0	80.3	9.5	27.0	96.8
FRANK					
δ	2.604	1.259	0.142	0.654	1.211
	(0.097)	*(0.093)*	*(0.092)*	*(0.092)*	*(0.094)*
LogLik	359.5	92.2	1.2	25.1	82.7

Table 4.3: CML estimates for various copulae under the assumption of a first order Markov process.

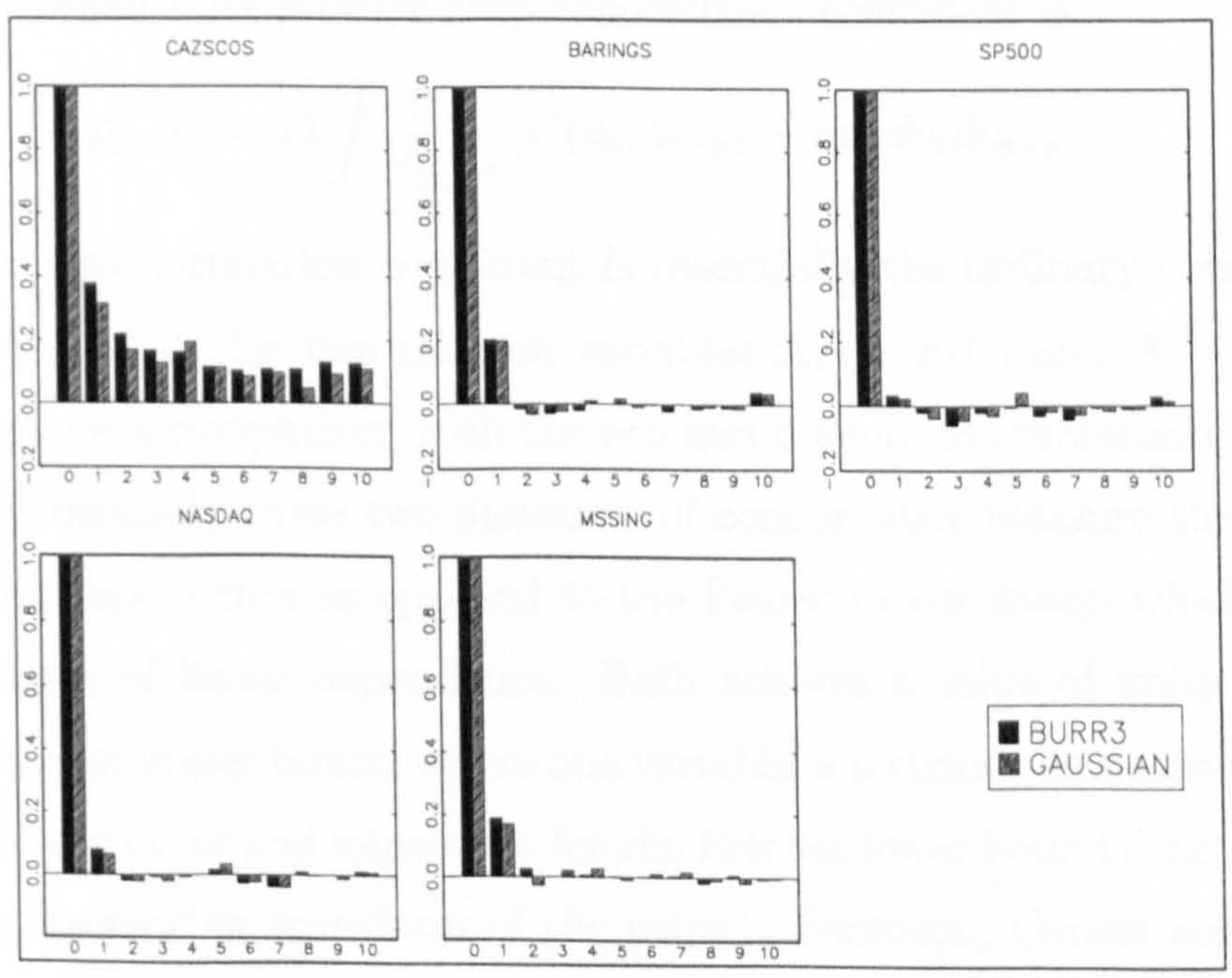

Figure 4.10: Empirical autocorrelation function under alternative hypothesis about margins : Burr3 and Gaussian.

consider extending methods of detecting potentially nonlinear dynamic structure to copula based measures of dependence that will be applicable outside the class of elliptic distributions such as the Gaussian. Figure 4.10 plots (i) the standard autocorrelation function (acf) - corresponding to the hypothesis of gaussian margins and gaussian copula - (ii) an extended version of the acf assuming Burr margins and gaussian copula. Several alternative measures of dependence immediately suggest themselves; in particular auto-concordance measures as opposed to autocorrelation. Two measures of concordance are given by Kendall's Tau and Spearman's rho which may be defined in general in terms of the parameters of the copula. The use of Kendall's tau to measure serial dependence has been introduced by Ferguson, Genest and Hallin (2000), Wang and Wells (2000), Genest, Quessy and Rémillard (2002). To define the auto-concordance coefficients we treat the original variable and its lag as the two random variables in what follows. The two concordance measures we use lead to the Kendall p-order auto-concordance coefficient defined as

$$\tau_C(p) = 4 \int\int_{(0,1)^2} C(u_t, u_{t-p}) dC(u_t, u_{t-p}) - 1 \tag{4.35}$$

and the Spearman rank p-order auto-concordance coefficient is

$$\rho_C(p) = 12 \int\int_{(0,1)^2} \left(C(u_t, u_{t-p}) - uv\right) du_t du_{t-p} \tag{4.36}$$

Spearman's rank correlation coefficient is essentially the ordinary correlation of $\rho(F_1(X_1), F_2(X_2))$ for two random variables $X_1 \backsim F_1(.)$ and $X_2 \backsim F_2(.)$. Notice the explicit comparison with the product copula, uv, representing independence. Essentially these two measures of concordance measure the degree of monotonic dependence as opposed to the Pearson Correlation which measures the degree of linear dependence. Both achieve a value of unity for the bivariate Fréchet upper bound where one variable is a strictly increasing transformation of the other and minus one for the Fréchet lower bound (one variable is a strictly decreasing transform of the other). Ferguson, Genest and Hallin (2000) show that Kendall's statististic is more powerful than Spearman's autocorrelation for samples of moderate size.

Essentially using these auto-concordance measures should enable us to detect monotonic and nonlinear dynamic dependence in non-gaussian assets and

Lag	tau	rho	acf
1	0.26	0.37	0.32
2	0.15	0.22	0.17
3	0.11	0.17	0.12
4	0.11	0.16	0.19
5	0.09	0.13	0.11
6	0.08	0.11	0.09
7	0.08	0.12	0.10
8	0.08	0.12	0.05
9	0.09	0.14	0.09
10	0.09	0.13	0.11
11	0.06	0.10	0.12
12	0.06	0.09	0.04

All entries significant

Table 4.4: Auto-concordance / Autocorrelation CAZSCOS

hence are potentially useful in financial applications. Further measures of copula based dynamic dependence could be based on dynamic tail area dependency measures, we consider this in the next chapter.

The following tables compare the auto-concordance and auto-correlation for the return series described above[7].

The general conclusion we can draw from these results is that within the same general pattern of dependence some potentially important differences emerge between the auto-concordance and auto-correlation coefficients. The same general dependence structure is indicated by both the auto-concordance measures. The distributions of these return series show the classic pattern of relatively small skewness but substantial excess kurtosis indicating the dangers of using autocorrelation analysis. The relative symmetry of these distributions may well explain the lack of any dramatic difference being indicated between the auto-concordance and the auto-correlation coefficients. In order to investigate this further we have applied the same procedures to two duration series drawn from a sample of all transactions from the DM2000-2 electronic order

[7] *Star's(*) indicate values significantly different from zero at a 5% level.*

Lag	tau	rho	acf
1	0.13418*	0.196*	0.201*
2	-0.00163	-0.003	-0.0286
3	-0.0235*	-0.035*	-0.0223
4	-0.0184*	-0.027*	0.0109
5	0.00138	0.0024	0.0157
6	-0.0059	-0.009	-0.0019
7	-0.0131	-0.019	-0.0013
8	-0.008	-0.0127	-0.0084
9	-0.0115	-0.017	-0.0138
10	-0.0237*	0.035*	0.0340*
11	-0.006	-0.009	0.0111
12	-0.004	-0.006	0.0034

Table 4.5: Auto-concordance / Autocorrelation BARINGS

Lag	tau	rho	acf
1	0.071*	0.102*	0.0656*
2	-0.008	-0.012	-0.0178
3	-0.0019	-0.003	-0.0194
4	-0.003	-0.004	0.0004
5	0.009	0.014	0.0339*
6	-0.005	-0.0082	-0.0213
7	-0.022*	-0.032*	-0.0372*
8	0.0087	0.0127	-0.0016
9	0.0054	0.007	-0.0112
10	0.007	0.01	0.0096
11	-0.0018	-0.003	-0.0089
12	0.018	0.027	0.0232

Table 4.6: Auto-concordance / Autocorrelation NASDAQ

Lag	tau	rho	acf
1	0.0155	0.023	0.0229
2	-0.012	-0.018	-0.0372*
3	-0.043*	-0.063*	-0.0459*
4	-0.0145	-0.021	-0.0295*
5	-0.0082	-0.012	0.0378*
6	-0.0173	-0.025	-0.0163
7	-0.027*	-0.040*	-0.0273
8	-0.0029	-0.004	-0.0162
9	-0.011	-0.016	-0.0099
10	0.0144	0.022	0.0141
11	0.0003	0.005	-0.0063
12	0.0179	0.026	0.0066

Table 4.7: Auto-concordance / Autocorrelation SP500

Lag	tau	rho	acf
1	0.1266*	0.182*	0.1721*
2	0.0208*	0.030*	-0.0237
3	0.004	0.005	0.0217
4	0.0032	0.005	0.0301*
5	0.008	0.011	-0.0070
6	0.003	0.005	-0.0123
7	0.005	0.0082	0.0187
8	-0.016	-0.022	-0.0096
9	0.0077	0.011	-0.0162
10	-0.0027	-0.004	-0.0032
11	0.019*	0.028	0.0274
12	0.031*	0.046*	0.0454*

Table 4.8: Auto-concordance / Autocorrelation MSSING

Lag	tau	rho	acf
1	0.171	0.252	0.259
2	0.137	0.203	0.206
3	0.120	0.179	0.322
4	0.11	0.165	0.294
5	0.096	0.144	0.171
6	0.102	0.152	0.212
7	0.106	0.158	0.338
8	0.084	0.126	0.185
9	0.086	0.129	0.165
10	0.108	0.162	0.259
11	0.090	0.135	0.273
12	0.094	0.139	0.179

Table 4.9: Auto-concordance / Autocorrelation DM2000-2 Transactions

book screen trading system for the Dollar DeutscheMark[8]. We consider the question of dynamic dependence within the duration between transactions and also the order flow duration onto the DM2000-2 system and since these must be non-negative their distributions must be asymmetric and lie entirely in the positive quadrant. In fact they are relatively well represented by members of the Weibull distribution. The following two tables provide the auto-concordance and auto-correlation coefficients for these two series of 26578 observations (order entries) and 4404 (transactions).

Unfortunately we see little discrimination between the measures in these last two tables since all entries are significantly different from zero and show essentially the same pattern. We clearly need to find more subtle examples in order to demonstrate the value of the auto-concordance functions. However this does not imply that other dynamic dependency patterns may be discovered using Copulae. The most obvious choice would seem to be looking at dependency and the dynamic evolution in the tails of the distribution of returns.

[8] *Further details of this data set and an analysis of its structure can be found in* HILLMAN *and* SALMON *[2000]*

Lag	tau	rho	acf
1	0.160	0.237	0.267
2	0.157	0.223	0.277
3	0.155	0.230	0.251
4	0.148	0.220	0.234
5	0.142	0.221	0.215
6	0.137	0.203	0.205
7	0.139	0.207	0.250
8	0.137	0.203	0.234
9	0.126	0.187	0.206
10	0.133	0.198	0.212
11	0.124	0.184	0.180
12	0.127	0.190	0.188

Table 4.10: Auto-concordance / Autocorrelation DM2000-2 Order flow entries

4.9 Conclusion

We have taken the first steps in this chapter to develop an empirical methodology to investigate dynamic dependence in non-gaussian time series and financial returns in particular using copula functions. Some properties of the class of copula functions that allow us to construct p-order markov processes have been presented and it is important to note that any density can be assumed for the margins. It is conceptually easy to move from considering the conditional expectation derived from the copula to consider the implied model of conditional volatility and its dynamic strucutre. An important innovation of this chapter consists in the concept of copula based regression function and its associated model selection procedure. A critical assumption of this chapter is that the density of the margins does not change through time. One further extension would be to look at possible regime changes both for the marginal density and the dependence structure itself.

4.10 Appendix

4.10.1 ML Estimates for margins

	CAZSCOS	BARINGS	SP500	NASDAQ	MSSING
GAUSSIAN					
μ	0.088	0.093	0.142	0.219	0.068
	(0.011)	(0.010)	(0.010)	(0.009)	(0.009)
σ	1.727	2.031	2.616	3.768	3.614
	(0.018)	(0.021)	(0.028)	(0.041)	(0.038)
LogLik	-8603.2	-9515.6	-10370.1	-11865.4	-12032.2
KS-Test	0.118*	0.058*	0.078*	0.058*	0.096*
WEIBULL					
a^+	1.042	1.104	1.070	1.134	0.985
	(0.014)	(0.017)	(0.016)	(0.015)	(0.014)
x^+	0.992	1.488	1.795	2.797	2.237
	(0.019)	(0.026)	(0.036)	(0.049)	(0.047)
a^-	0.882	1.029	0.990	1.048	0.932
	(0.014)	(0.016)	(0.015)	(0.018)	(0.014)
x^-	1.047	1.416	1.726	2.806	2.200
	(0.028)	(0.030)	(0.038)	(0.062)	(0.047)
p	0.577	0.527	0.539	0.548	0.518
	(0.008)	(0.007)	(0.007)	(0.008)	(0.008)
LogLik	-7475.6	-9107.8	-9770.4	-11569.1	-11159.5
KS-Test	0.021**	0.011	0.013	0.011	0.015
STUDENT					
ν	3.165	2.119	1.561	0.982	1.172
	(0.126)	(0.068)	(0.043)	(0.022)	(0.028)
LogLik	-7471.2	-9247.9	-10040.3	-12529.6	-11563.5
KS-Test	0.084*	0.076*	0.098*	0.161*	0.099*

	CAZSCOS	BARINGS	SP500	NASDAQ	MSSING
BURR3					
α^+	2.535	3.082	3.050	2.946	2.600
	(0.096)	(0.131)	(0.139)	(0.129)	(0.112)
λ^+	0.404	0.303	0.291	0.340	0.343
	(0.024)	(0.019)	(0.020)	(0.021)	(0.022)
τ^+	1.267	2.263	2.845	3.945	3.275
	(0.050)	(0.078)	(0.104)	(0.143)	(0.134)
α^-	2.173	2.944	2.469	2.784	2.473
	(0.094)	(0.132)	(0.112)	(0.130)	(0.103)
λ^-	0.415	0.306	0.390	0.336	0.354
	(0.029)	(0.021)	(0.028)	(0.025)	(0.022)
τ^-	1.358	2.179	2.306	4.081	3.166
	(0.073)	(0.084)	(0.111)	(0.171)	(0.134)
p	0.577	0.527	0.539	0.548	0.518
	(0.008)	(0.007)	(0.008)	(0.007)	(0.008)
LogLik	-7385.7	-9078.9	-9742.2	-11558.3	-11096.3
KS-Test	0.006	0.007	0.013	0.010	0.006

4.10.2 Inverse of the Burr3 distribution

The Burr3 distribution :

$$G(x;\gamma) = \left[1 - \frac{1}{1+(x/\tau)^{\alpha}}\right]^{\lambda} = u_G \quad \text{with} \quad x \in \mathbb{R}^+$$

with the parameters $\gamma = (\alpha, \lambda, \tau)$. Then it comes that its inverse $G^{[-1]}$ is

$$G^{[-1]}(u_G;\gamma) = \tau \left[\frac{1}{1-u_G^{1/\lambda}} - 1\right]^{1/\alpha} \quad \text{with} \quad u_G \in [0,1] \tag{4.37}$$

A distribution F for positive and negative values is constructed:

$$F\left(x;\gamma^-,\gamma^+\right) = 1-p-\mathbb{I}_{\{x<0\}}\left(1-p\right)G(|x|\,;\gamma^-) + \mathbb{I}_{\{x\geq 0\}}pG(x;\gamma^+) = u_F$$

Then, the quasi-inverse is slightly modified, depending on the sign of x:

$$\begin{aligned} x &< 0, \; u_G = \frac{u_F}{1-p} - 1 \\ x &\geq 0, \; u_G = (1-u_F)(1-p) \end{aligned}$$

4.10.3 Gaussian copula based simulations

We may postulate other distributions for the margins with a dependence structure summarized by a gaussian copula with AR(p) structure correlation matrix.

To illustrate this idea, we simulate two times series of size $T = 200$ with an AR(1) intrinsic gaussian copula but with different univariate margins (standard normal and Student) - see Figure 4.11. A simulation for an AR(2) intrinsic gaussian copula is also reported in Figure 4.12.

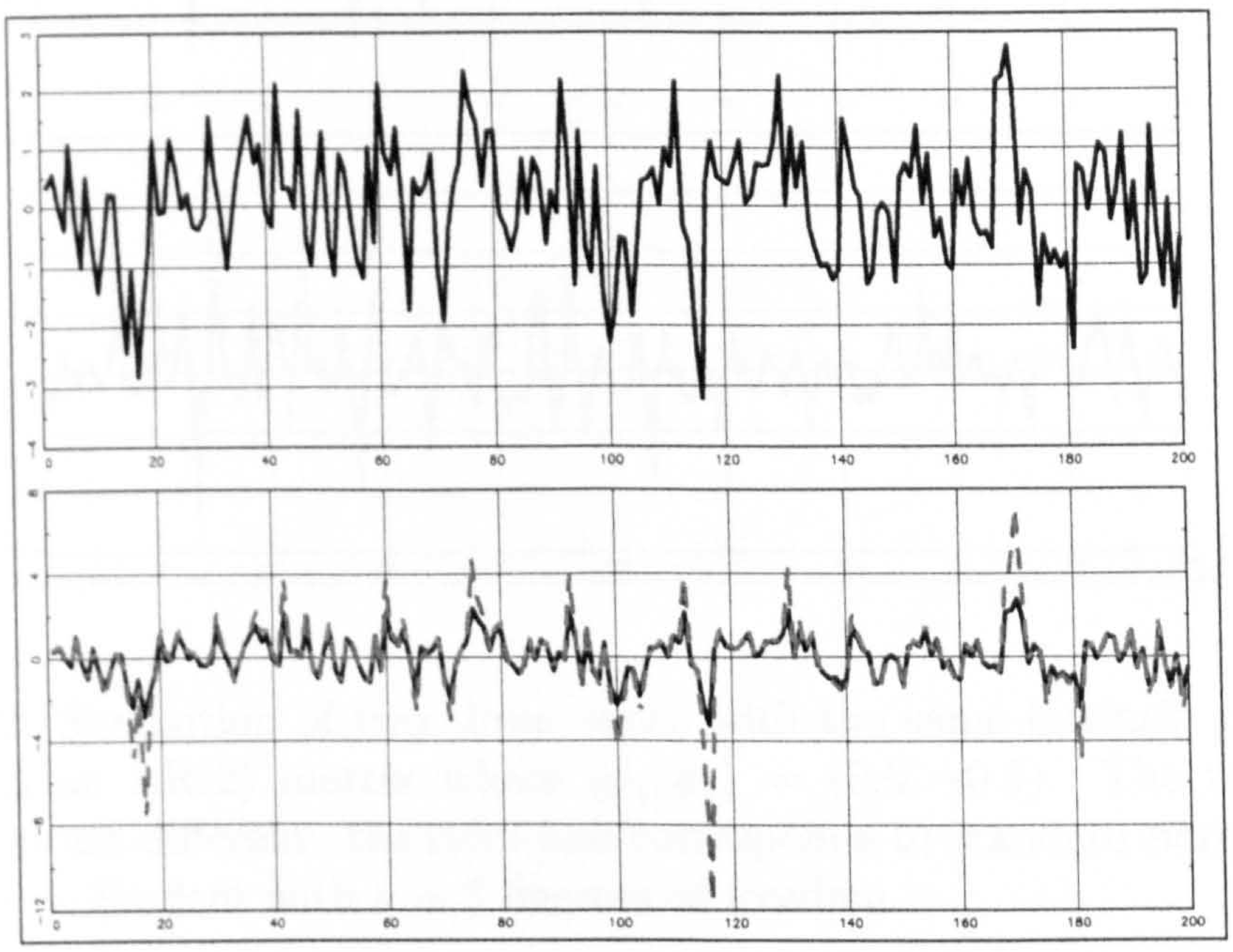

Figure 4.11: Simulation of two times series with the same intrinsic gaussian copula with an AR(1) matrix where $\phi = 0.3$. The marginal distributions are different: the solid line corresponds to standard normal, the dashed line to Student with $\nu = 3$ degrees of freedom.

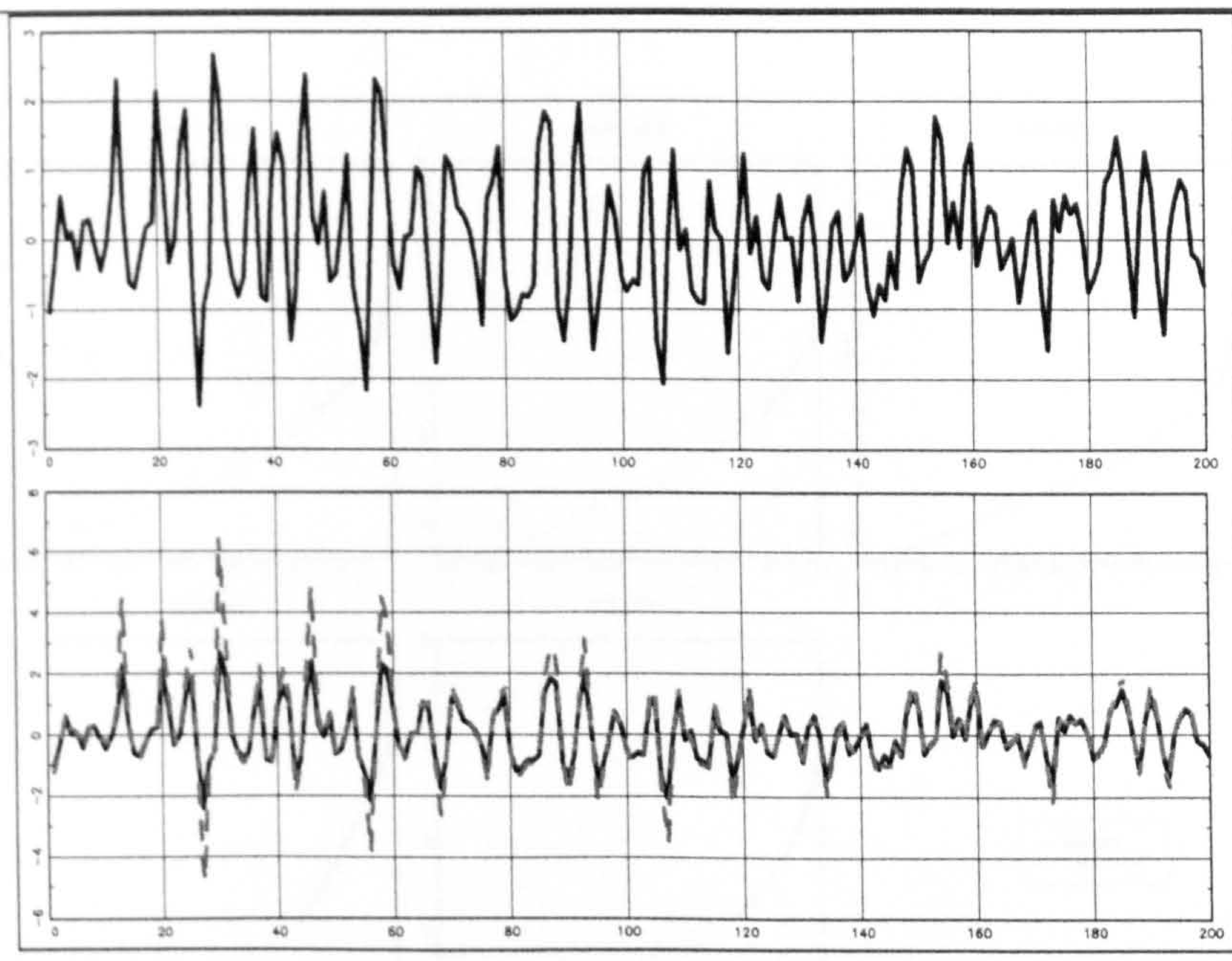

Figure 4.12: Simulation of two times series with the same intrinsic gaussian copula with an AR(2) matrix where $(\phi_1, \phi_2) = (0.6, -0.5)$. The marginal distributions are different: the solid line corresponds to standard normal, the dashed line to Student with $\nu = 3$ degrees of freedom.

4.10.4 Additional figures

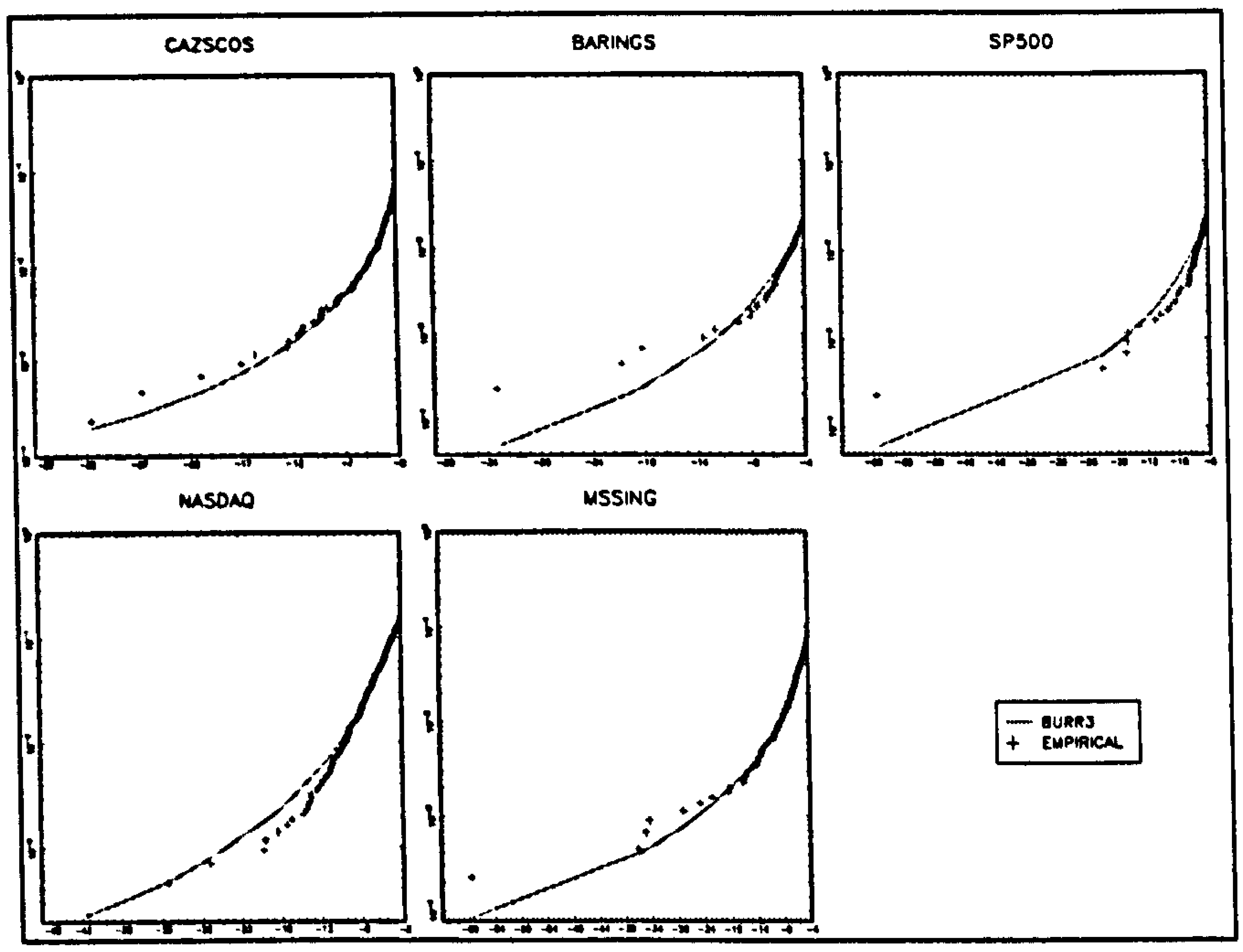

Figure 4.13: Left Tail of the CDF (in log-scale) for the annualized log-returns

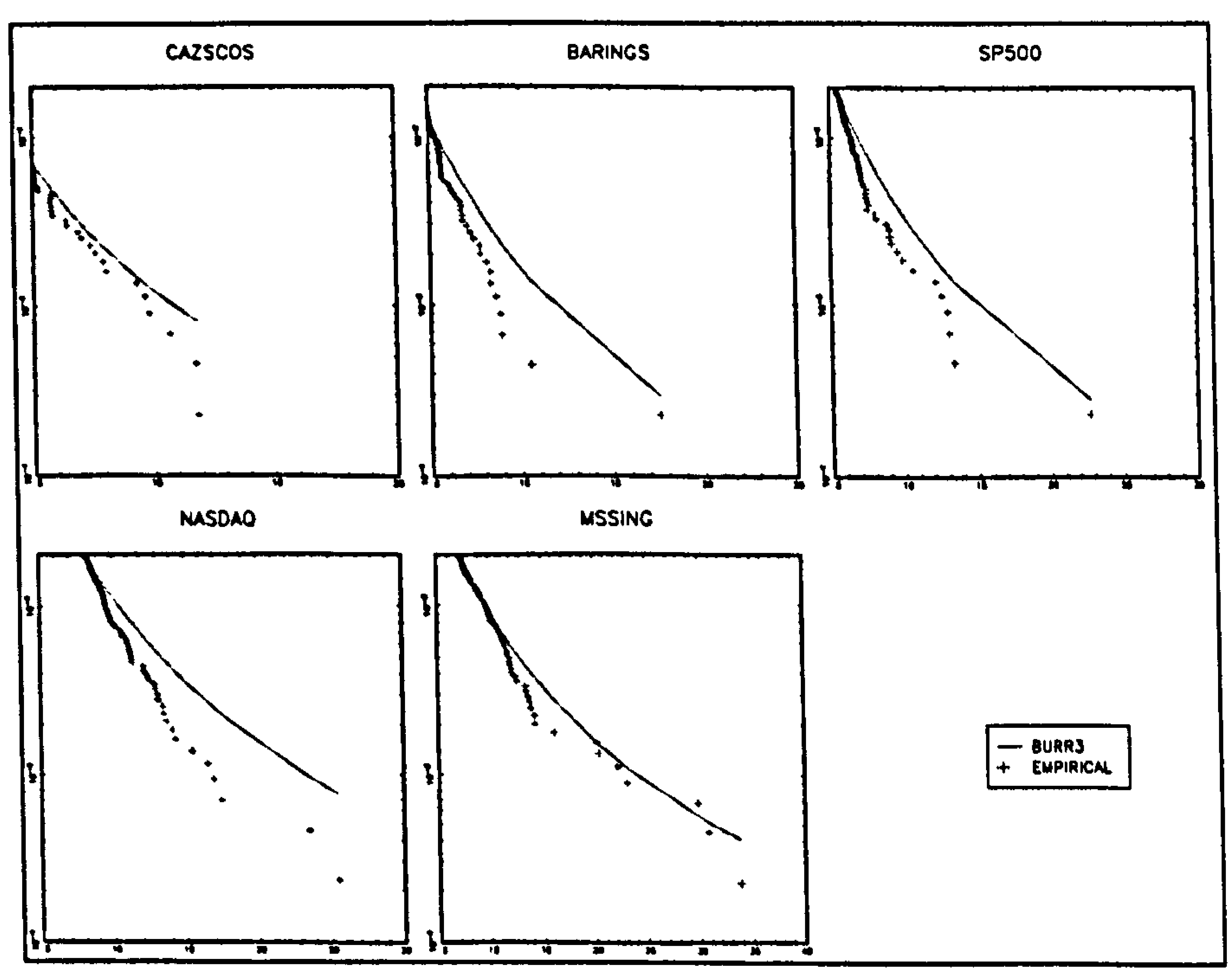

Figure 4.14: Right Tail of the survival CDF (in log-scale) for the annualized log-returns

Chapter 5

Copula quantile curves and quantile regressions

5.1 Contents and contributions

This chapter is certainly the more innovative from a theoretical point of view. Indeed, it proposes a generalization of both standard regression quantiles introduced by Koenker and Bassett (1978) and other non-linear regression quantiles (see Engle and Manganelli (2000) or Koenker and Hallock (2001)). The third section introduces the definitions and properties that are necessary to understand the link between the functional form of a given copula and the shape of the dependence generated by this copula. The idea of positive quadrant dependence comes from Lehmann (1966). The left tail decreasing definition has been introduced by Esary and Proschan (1972) and its associated copula properties can be found in Nelsen (1998). Tail dependence measures are also presented in the Section 4. The fifth section is the core of the contribution of the chapter to the literature. We propose a new model: the copula quantile (or c-quantile) regression model. We first define the concept of copula quantile curve. Its properties - positive or negative relationship, symmetry - are then derived from the properties of the underlying copula (the implicit function theorem is used). The c-quantile curves are more precisely studied for some copulae, with a particular focus on the archimedean class. We also show that the Koenker and Basset's regression quantiles are encompassed by our model since it corresponds, in our model, to the gaussian copula with gaussian marginal distributions. Finally, the sixth section is an empirical application of our model to foreign exchange rates. We find evidence of a much stronger form of efficiency than implied by standard martingale approach based on the conditional expectation. In some sense, our model also allows us to extend Fama's (1970) definition of efficiency.

5.2 Introduction

The problem of characterising the dependence between random variables at a given quantile is an important issue, especially if the distributions of the variables involved are non-elliptic and fat tailed as it is standard with financial returns. Tail area dependency for instance may be quite different to that implied by correlation and may signal where extreme downside protection may be found if two assets do not show positive causal dependency in their extreme quantiles. One goal of this chapter is to introduce a general approach to non-linear quantile regression modelling where the form of the quantile regression is implied by the copula linking the assets involved.

Our starting point is the joint distribution for the variables which will almost certainly be non-gaussian. Working down, in a general to specific manner, this multivariate distribution can be split into two parts - the marginal densities and the dependence function or copula that joins these marginals together to give the joint distribution function. Since the copula function holds all the information on the forms and structure of dependency between the assets we can see that the form of the conditional (non linear) quantile relationship is then implied by the copula function. We refer below to this relationship as a c-quantile regression to distinguish it from a quantile regression which may have been assumed, as is usual, to be linear or estimated non-parametrically.

A second objective of this chapter is to apply the c-quantile idea to assess the form and degree of conditional dependence between foreign exchange rates. Correlation analysis is implicitly based on an assumption of multivariate gaussianity and may give very misleading results when the market is under stress; in particular multivariate gaussianity implies asymptotic tail area independence unless the correlation is unity. It is an important issue in practice to consider how exchange rates are inter-related when forex markets are under stress and by using c-quantiles we can examine the entire conditional distribution at a range of quantile levels rather than just the degree of limiting dependence which is captured by standard tail area dependency measures. Patton (2001) and Hartmann, Straetmans and De Vries (2002) have considered the dependence between exchange rates using related but different techniques to those employed in this chapter. We also consider dynamic dependency both across and within forex markets and show how the c-quantile method provides a different approach to that considered by Engle and Manganelli (2000) who assumed the form of the Conditional Autoregressive Value at Risk or dynamic quantile functions which they proposed for risk management in their CAViaR model. The form of our dynamic c-quantiles follows immediately from the determination of the joint distribution and the copula rather than by assumption. In this way we are also able to examine the question of market efficiency at all quantiles including the tails of the distributions, instead of simply through a mean regression, by exploiting a natural test for independence that follows from the copula.

In the next section, we briefly review regression quantiles as proposed and developed by Koenker and Bassett (1978) and then the concept of copula is defined and the implications for the assessing the forms of dependence between two assets are presented. We then introduce tail dependence measures. In the fifth section, we introduce the copula quantile curves, derive some properties of this c-quantiles curve and provide some examples for particular copulae. Then, the copula quantile regression model is formally defined and we briefly discuss the estimation issue. Then the application to analysing c-quantile regressions and tail area dependence in foreign exchange markets is presented in the sixth section. A final section offers some conclusions.

5.3 Regression Quantiles

Koenker and Bassett introduced linear quantile regression in Econometrica in 1978. We first review how they define quantile regression and the main properties of their model. Let $(y_1, \ldots, y_T)$ be a random sample on Y and $(\mathrm{x}_1, \ldots, \mathrm{x}_T)'$ a random k-vector sample on X.

Definition 9 *The p-th quantile regression is any solution to the following problem:*

$$\min_{\beta\in\mathbb{R}^k}\left(\sum_{t\in T_p} p\left|y_t-\mathrm{x}_t'\beta\right|+\sum_{t\in T_{1-p}}(1-p)\left|y_t-\mathrm{x}_t'\beta\right|\right)$$

with $T_p=\{t: y_t\geq \mathrm{x}_t'\beta\}$ *and* T_{1-p} *its complement. This can be alternatively expressed as* [1]*:*

$$\min_{\beta\in\mathbb{R}^k}\left(\sum_{t=1}^{T}\left(p-\mathbb{I}_{\{y_t\leq \mathrm{x}_t'\beta\}}\right)\left(y_t-\mathrm{x}_t'\beta\right)\right) \tag{5.1}$$

Non-linearity in quantile regression was developed by Powell (1986) using a censored model. The consistency of non-linear quantile regression estimation has been investigated by White (1994), Engle and Manganelli (2000) and Kim and White (2002). For a recent overview of quantile regression see Yu, Lu, and Stander (2001) and Koenker and Hallock (2001). As noted by Buchinsky (1998), quantile regression models have a number of useful properties: (i) with non-gaussian error terms, quantile regression estimators may be more efficient than least-square estimators, (ii) the entire conditional distribution can be characterized, (iii) different relationships between the regressor and the dependent variable may arise at different quantiles. In this chapter, we attempt to resolve one difficulty with using quantile regression, the question of how to specify the form of the quantile regression function. We achieve this by deriving a conditional distribution for Y given X from the copula which then implies the structural form of the quantile regression. For simplicity, our model is developed for the one regressor case, corresponding to a bivariate copula but it may, in principle, be extended to multiple regressors.

[1] *Koenker and Bassett discuss properties of their estimator, especially through the following theorem:*

Theorem 8 *Let* $\beta^*(p,y,X)\in\mathcal{B}^*(p,y,X)$. *Then, the following properties hold:*

1. $\beta^*(p,\kappa y,X)=\begin{cases}\kappa\beta^*(p,y,X) & \text{for } \kappa\in\mathbb{R}^+\\ \kappa\beta^*(1-p,y,X) & \text{for } \kappa\in\mathbb{R}^-\end{cases}$
2. $\beta^*(p,y+X\delta,X)=\beta^*(p,y,X)$ *for* $\delta\in\mathbb{R}^k$
3. $\beta^*(p,y,X\Gamma)=\Gamma^{-1}\beta^*(p,y,X)$ *with* Γ *non-singular* $(k\times k)$ *matrix*

5.4 Copulae and dependence

The goal of this preliminary section is to provide a definition of a copula function and Sklar's theorem which ensures the uniqueness of the copula when the bivariate distribution for two random variables (corresponding in our modelling framework to the dependent variable and a regressor) is given and the margins are continuous. Then, we introduce the concepts of positive quadrant dependence and the left tail decreasing property and show how these two concepts are related. These definitions are the starting point to demonstrating that the concavity (respectively convexity) of the copula in its first argument induces a positive (respectively negative) dependence at each quantile level.

Definition 10 *A bivariate copula is a function* $C : [0,1]^2 \to [0,1]$ *such that:*

1. $\forall (u,v) \in [0,1]^2$,

$$\begin{cases} \mathrm{C}(u,0) = \mathrm{C}(0,v) = 0 \\ \mathrm{C}(u,1) = u \text{ and } \mathrm{C}(1,v) = v \end{cases} \tag{5.2}$$

2. $\forall (u_1,v_1,u_2,v_2) \in [0,1]^4$, $u_1 \leq u_2$ *and* $v_1 \leq v_2$,

$$\mathrm{C}(u_2,v_2) - \mathrm{C}(u_1,v_2) - \mathrm{C}(u_2,v_1) + \mathrm{C}(u_1,v_1) \geq 0 \tag{5.3}$$

Theorem 9 (Sklar's Theorem) *Let* X *and* Y *be two random variables with joint distribution* F*. Then, there exists a unique copula* C *satisfying*

$$\mathrm{F}(x,y) = \mathrm{C}(F_X(x), F_Y(y)) \tag{5.4}$$

if F_X *and* F_Y *are continuous and represent the marginal distribution functions of* X *and* Y *respectively.*

Definition 11 (Order) *Let* $(\mathrm{C},\mathrm{D}) \in \mathcal{C}^2$ *with* $\mathcal{C}$ *the set of copulae. One says that* C *is greater than* D ($C \succ D$ *or* $D \prec C$) *if*

$$\forall (u,v) \in [0,1]^2,\ \mathrm{C}(u,v) \geq \mathrm{D}(u,v)$$

Theorem 10 (Fréchet Bounds) *Let* $C \in \mathcal{C}$*. Then,*

$$\mathrm{C}^- \prec \mathrm{C} \prec \mathrm{C}^+$$

where C^- and C^+ are such that

$$\begin{aligned} \mathrm{C}^-(u,v) &= \max(u+v-1,0) \\ \mathrm{C}^+(u,v) &= \min(u,v) \end{aligned}$$

The concept of order for copulae is important as it allows us to rank the dependence between random variables. One interesting copula is the product copula $C^{\perp}$- that corresponds to independence - so that $C^{\perp}(u,v) = uv$.

Definition 12 (Lehmann (1966)) *The pair (X,Y) is positive quadrant dependent*
$(PQD(X,Y))$ if

$$\Pr\{X \leq x, Y \leq y\} \geq \Pr\{X \leq x\}\Pr\{Y \leq y\} \tag{5.5}$$

In terms of copulae, this definition can be restated $C^{\perp} \prec C$.

Definition 13 (Esary and Proschan (1972)) *Y is left tail decreasing in X $(LTD(Y \mid X))$ if*

$$\forall y,\ \Pr\{Y \leq y \mid X \leq x\} \text{ is a nonincreasing function of } x \tag{5.6}$$

This definition can be equivalently expressed using copulae as:

Theorem 11 (Nelsen (1998))

$$\begin{aligned} \mathrm{LTD}(Y \mid X) &\iff \frac{\mathrm{C}(u,v)}{u} \text{ is nonincreasing in } u \\ &\iff \frac{\partial \mathrm{C}(u,v)}{\partial u} \leq \frac{\mathrm{C}(u,v)}{u} \end{aligned} \tag{5.7}$$

Theorem 12 *Let $C \in \mathcal{C}$. The following holds*

$$\text{If } \forall (u,v) \in [0,1]^2,\ \frac{\partial^2 \mathrm{C}(u,v)}{\partial u^2} \leq 0 \text{ then } \mathrm{C}^{\perp} \prec \mathrm{C} \tag{5.8}$$

$$\text{If } \forall (u,v) \in [0,1]^2,\ \frac{\partial^2 \mathrm{C}(u,v)}{\partial u^2} \geq 0 \text{ then } \mathrm{C} \prec \mathrm{C}^{\perp} \tag{5.9}$$

Proof. We refer to NELSEN (1998), p 151-160, for the proof. The first part is based on the fact that $\frac{\partial^2 \mathrm{C}(u,v)}{\partial u^2} \leq 0 \Rightarrow \mathrm{LTD}(Y \mid X) \Rightarrow \mathrm{PQD}(X,Y)$. ■

The previous theorem tells us that if the copula function is concave in the marginal distribution F_X then the random variables X and Y are positively related i.e. their copula value is greater than given by the independence copula $C^{\perp}$. Conversely, convexity implies a negative relationship i.e. the copula linking X and Y lies below the independence copula $C^{\perp}$. For simplicity, we still have not introduced the parameter(s) of the copula function which in effect measure the degree and different forms of dependence between the variables, let us denote these parameters by $\delta \in \Delta$. Then, through the family of copula functions, we can distinguish three classes:

1. Copulae that only exhibit negative dependence:

$$\forall \delta \in \Delta, \forall (u,v) \in [0,1]^2, \text{ then } \mathbf{C}(u,v;\delta) \prec \mathbf{C}^{\perp}(u,v)$$

2. Copulae that only exhibit positive dependence:

$$\forall \delta \in \Delta, \forall (u,v) \in [0,1]^2, \text{ then } \mathbf{C}^{\perp}(u,v) \prec \mathbf{C}(u,v;\delta)$$

3. Copulae that exhibit both negative and positive dependence depending on the parameter values:

$$\begin{aligned} \forall \delta &\in \Delta^{-}, \forall (u,v) \in [0,1]^2, \text{ then } \mathbf{C}(u,v;\delta) \prec \mathbf{C}^{\perp}(u,v) \\ \forall \delta &\in \Delta^{+}, \forall (u,v) \in [0,1]^2, \text{ then } \mathbf{C}^{\perp}(u,v) \prec \mathbf{C}(u,v;\delta) \end{aligned}$$

In the next section, tail dependence measures are presented and the concept of a quantile curve of Y conditional on X is defined and we derive several results that are directly deduced from the underlying copula properties outlined above.

5.5 Tail Area Dependency

5.5.1 Tail dependence measures

Tail dependence measures, for both the upper tail, λ_U and lower tail λ_L have been developed and discussed for instance in Joe (1997), Mari and Kotz (2001) and Coles, Heffernan and Tawn (2000). Upper tail independence is normally thought to be shown by $\lambda_U = 0$ and a value of $\lambda_U \in (0,1)$ indicates the degree of upper tail dependence with lower tail dependence $\lambda_L \in (0,1)$ similarly defined.

For two random variables, (X, Y) with marginal distributions F_X and F_Y, λ_U and λ_L are linked to the asymptotic behaviour of the copula in the left and right tails. So for the lower tail index we have

$$\begin{aligned}\lambda_L &= \lim_{\alpha\searrow 0}\frac{\mathrm{C}(\alpha,\alpha)}{\alpha} \\ &= \lim_{\alpha\searrow 0}(\Pr\{F_X(X)\leq\alpha|F_Y(Y)\leq\alpha\})\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\lambda_U &= \lim_{\alpha\nearrow 1}\Pr\{Y>F_Y^{-1}(\alpha)|X>F_X^{-1}(\alpha)\} \\ &= \lim_{\alpha\nearrow 1}\frac{\Pr\{Y>F_Y^{-1}(\alpha),X>F_X^{-1}(\alpha)\}}{\Pr\{X>F_X^{-1}(\alpha)\}} \\ &= \lim_{\alpha\nearrow 1}\frac{\bar{\mathrm{C}}(\alpha,\alpha)}{1-\alpha} \\ &= \lim_{\alpha\nearrow 1}\frac{1-2\alpha+\mathrm{C}(\alpha,\alpha)}{1-\alpha}\end{aligned} \tag{5.11}$$

Given the survival copula of two random variables with copula $C(\cdot,\cdot)$ is given by

$$\hat{\mathrm{C}}(u,v) = u+v-1+\mathrm{C}(1-u,1-v)$$

and the joint survival function for two uniform random variables with distribution function given by $C(u,v)$ is given by

$$\bar{\mathrm{C}}(u,v) = 1-u-v+\mathrm{C}(u,v) = \hat{\mathrm{C}}(1-u,1-v)$$

hence it follows that

$$\lim_{\alpha\nearrow 1}\frac{\bar{\mathrm{C}}(\alpha,\alpha)}{1-\alpha} = \lim_{\alpha\nearrow 1}\frac{\hat{\mathrm{C}}(1-\alpha,1-\alpha)}{(1-\alpha)} = \lim_{\alpha\searrow 0}\frac{\hat{\mathrm{C}}(\alpha,\alpha)}{\alpha} \tag{5.12}$$

which implies that the coefficient of upper tail dependence of $C(\cdot,\cdot)$ is the coefficient of lower tail dependence of $\hat{C}(\cdot,\cdot)$.

A major difficulty with interpreting asymptotic tail area dependency however is that independence in the sense of the factorisation of the bivariate distribution in the tails implies $\lambda_U = 0$ but $\lambda_U = 0$ does not imply factorization and hence independence. There may still be dependence in the tails even though there is asymptotic independence. An additional condition must be used to ensure factorization, Ledford and Tawn (1998), for instance, show that we also need to satisfy $\bar{\lambda} = 0$ as a necessary condition, where

$$\begin{aligned}\bar{\lambda} &= \lim_{\alpha\nearrow 1} \frac{2\log \Pr\{X > F_X^{-1}(\alpha)\}}{\log \Pr\{X > F_X^{-1}(\alpha), Y > F_Y^{-1}(\alpha)\}} - 1 \\ &= \lim_{\alpha\nearrow 1} \frac{2\log(1-\alpha)}{\log(1-2\alpha+\mathrm{C}(\alpha,\alpha))} - 1 \\ &= \lim_{\alpha\nearrow 1} \frac{2\log(1-\alpha)}{\log \bar{\mathrm{C}}(\alpha,\alpha)} - 1\end{aligned}$$

If $\bar{\lambda} > 0$ large values occur simultaneously more frequently than if they were independent and if $\bar{\lambda} < 0$ simultaneous large movements occur less frequently than under independence. $\bar{\lambda} = 1$ if and only if $\lambda_u > 0$ while it takes values in $(-1, 1)$ when $\lambda_u = 0$ which enables us to quantify the strength of dependence in the tail. Values of $\bar{\lambda} > 0$, $\bar{\lambda} = 0$, $\bar{\lambda} < 0$ loosely correspond to when the variables are positively associated in the extremes, exactly independent and negatively associated.

The two indices $(\lambda_U, \bar{\lambda})$ are then used to measure extreme upper tail dependence:

1. $(\lambda_U > 0, \bar{\lambda} = 1)$ indicates asymptotic dependence and λ_U measures the degree of upper tail dependence; or

2. $(\lambda_U = 0, \bar{\lambda} < 1)$ indicates asymptotic independence and $\bar{\lambda}$ measures the strength of dependence in the tail.

5.5.2 Parametric and Non-parametric Estimation of Tail Dependency

An archimedean copula is defined as follows

$$\mathrm{C}(u,v) = \phi^{-1}\left[\phi(u) + \phi(v)\right] \tag{5.13}$$

with ϕ a continuous and strictly decreasing function from $[0, 1]$ to $[0, \infty]$ such that $\phi(1) = 0$. ϕ is often called the generator function.The standard lower and upper tail dependence measures for archimedean copulae are defined in general by

$$\begin{cases} \lambda_L = \lim\limits_{\alpha\to 1^-} \frac{1-2\alpha+\phi^{-1}(2\phi(\alpha))}{1-\alpha} \\ \lambda_U = \lim\limits_{\alpha\to 0^+} \frac{\phi^{-1}(2\phi(\alpha))}{\alpha} \end{cases} \tag{5.14}$$

and for the Clayton Joe Copula specifically are given by

$$\begin{cases} \lambda_L = 2^{-1/\delta} \\ \lambda_U = 2 - 2^{1/\theta} \end{cases} \tag{5.15}$$

hence MLE estimates of the parameters of the copula provide direct parametric estimates of the tail area dependency measures and we shall employ these formula below.

Alternatively we can use an empirical or non-parametric copula $C_n(\frac{1}{n}, \frac{i}{n}) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq X_{(i)}, Y_i \leq Y_{(i)})$ to estimate tail area dependency given the order statistics $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ and $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(n)}$. Since we have

$$\lambda_L = \lim_{\alpha \searrow 0} \frac{\mathrm{C}(\alpha, \alpha)}{\alpha}$$

which implies

$$\mathrm{C}(\alpha, \alpha) = \lambda_L \alpha + o(\alpha) \tag{5.16}$$

for $0 \leq \alpha \leq 1$ where $o(\alpha)/\alpha \to 0$ as $\alpha \to 0$. A natural estimator of λ_L is given by the derivative which is approximated by the secant

$$\hat{\lambda}_L^1 = \left(\frac{k}{n}\right)^{-1} \hat{\mathrm{C}}_n\left(\frac{k}{n}, \frac{k}{n}\right)$$

Alternatively a least squares estimator can be applied to (5.16) to give a second estimator

$$\hat{\lambda}_L^2 = \left(\sum_{i=1}^{k}\left(\frac{i}{n}\right)^2\right)^{-1} \sum_{i=1}^{k}\left(\frac{i}{n}.\hat{\mathrm{C}}_n\left(\frac{i}{n}, \frac{i}{n}\right)\right) \tag{5.17}$$

A third estimator is given by recognising that the copula $C(u, v)$ can be approximated by the mixture of the comonotonicity and independence copulae, C^+ and $C^\perp$, giving

$$\mathrm{C}(\alpha, \alpha) = \lambda_L \alpha + (1 - \lambda_L)\alpha^2$$

If we rewrite this as

$$\mathrm{C}(\alpha, \alpha) - \alpha^2 = \lambda_L(\alpha - \alpha^2)$$

and again apply least squares to this expression we get a third estimator

$$\hat{\lambda}_L^3 = \frac{\sum_{i=1}^{k}\left(\hat{\mathrm{C}}_n\left(\frac{i}{n}, \frac{i}{n}\right) - \left(\frac{i}{n}\right)^2\right)\left(\frac{i}{n} - \left(\frac{i}{n}\right)^2\right)}{\sum_{i=1}^{k}\left(\frac{i}{n} - \left(\frac{i}{n}\right)^2\right)^2}$$

As shown by Dobrić and Schmidt (2002) each of these are consistent estimators for λ_L provided k, the number of observations in the lower tail, is chosen so that $k \approx \sqrt{n}$. Our limited experimentation with these estimators suggests that $\hat{\lambda}_L^2$ is the most reliable, with $\hat{\lambda}_L^3$ giving values from time to time outside the feasible range $(0,1)$. Dobrić and Schmidt provide some Monte Carlo evidence on the relative merits of each estimator which depends on the sample size and the true degree of dependence. We use $\hat{\lambda}_L^2$ below to calculate both lower and upper tail dependence using the relationship in (5.12).

5.6 Copulae and Quantiles

First, the copula p-th quantile curve of y conditionally on x or p'th c-quantile curve is defined. Second, its main properties are exhibited. Third, the case of radially symmetric variables is studied. Finally, the quantile curves are developed for three special cases: the Kimeldorf and Sampson, Gaussian and Frank copulae.

5.6.1 Definitions

We restrict the study to monotonic copula for simplicity. Define the probability distribution of y conditional on x by $p(x, y; \delta)$:

$$\begin{aligned}
p(y|x;\delta) &= \Pr\{Y \leq y \mid X = x\} \\
&= \mathrm{E}\left(\mathbb{I}_{\{Y\leq y\}} \mid X = x\right) \\
&= \lim_{\varepsilon\to 0} \Pr\{Y \leq y \mid x \leq X \leq x+\varepsilon\} \\
&= \lim_{\varepsilon\to 0} \frac{\mathrm{F}(x+\varepsilon, y;\delta) - \mathrm{F}(x, y;\delta)}{F_X(x+\varepsilon) - F_X(x)} \\
&= \lim_{\varepsilon\to 0} \frac{\mathrm{C}[F_X(x+\varepsilon), F_Y(y);\delta] - \mathrm{C}[F_X(x), F_Y(y);\delta]}{F_X(x+\varepsilon) - F_X(x)} \\
p(y|x;\delta) &= C_1[F_X(x), F_Y(y);\delta] \qquad (5.18)
\end{aligned}$$

with $C_1(u, v; \delta) = \frac{\partial}{\partial u} C(u, v; \delta)$. Since the distribution functions F_X and F_Y are nondecreasing, $p(y|x;\delta)$ is nondecreasing in y. Using the same argument, $p(y|x;\delta)$ is nondecreasing in x if $C_{11}(u, v; \delta) \leq 0$ and nonincreasing in x if $C_{11}(u, v; \delta) \geq 0$ where $C_{11}(u, v; \delta) = \frac{\partial^2 \mathrm{C}(u,v;\delta)}{\partial u^2}$.

Definition 14 *For a parametric copula $C(.,.;\delta)$, the p-th copula quantile curve of y conditional on x is defined by the following implicit equation*

$$p = C_1[F_X(x), F_Y(y); \delta] \tag{5.19}$$

where $\delta \in \Delta$ the set of parameters.

Under some conditions[2], equation (5.19) can be expressed as follows in order to capture the relationship between X and Y:

$$y = \mathrm{q}(x, p; \delta) \tag{5.20}$$

where $q(x,p;\delta) = F_Y^{[-1]}(D(F_X(x), p; \delta))$ with D the partial inverse in the second argument of C_1 and $F_Y^{[-1]}$ the pseudo-inverse of F_Y. Note that the relationship (5.20) can alternatively be expressed using uniform margins as:

$$v = \mathrm{r}(u, p; \delta). \tag{5.21}$$

with $u = F_X(x)$ and $v = F_Y(y)$.

5.6.2 Properties

Two properties are demonstrated. The first tells us that the quantile curve shifts up with the quantile level. The second indicates that the quantile curve has a positive (respectively negative) slope if the copula function is concave (respectively convex) in its first argument.

Property 4 *If $0 < p_1 \leq p_2 < 1$ then $q(x, p_1; \delta) \leq q(x, p_2; \delta)$.*

Property 5 *Let $x_1 \leq x_2$.*
If $C(u,v)$ is concave in u then $q(x_1, p; \delta) \leq q(x_2, p; \delta)$
If $C(u,v)$ is convex in u then $q(x_1, p; \delta) \geq q(x_2, p; \delta)$

Proof. Given the implicit function theorem, y may be expressed as a function of x and p *i.e.* $y = \mathrm{q}(x,p;\delta)$. Let us rewrite equation (5.19) as $F(x, p, \mathrm{q}(x, p; \delta)) = 0$. Thus,

$$\begin{cases} \frac{\partial F}{\partial x}(x,p,\mathrm{q}(x,p;\delta)) + \frac{\partial F}{\partial y}(x,p,\mathrm{q}(x,p;\delta))\frac{\partial \mathrm{q}}{\partial x}(x,p;\delta) &= 0 \\ \frac{\partial F}{\partial p}(x,p,\mathrm{q}(x,p;\delta)) + \frac{\partial F}{\partial y}(x,p,\mathrm{q}(x,p;\delta))\frac{\partial \mathrm{q}}{\partial p}(x,p;\delta) &= 0 \end{cases}.$$

[2] *Note that C_1 has to be partially invertible in its second argument. If it is not analytically invertible, a numerical root finding procedure can be used.*

Then,

$$\begin{cases} \frac{\partial q}{\partial x}(x,p;\delta) &= -\frac{\frac{\partial F}{\partial x}(x,p,q(x,p;\delta))}{\frac{\partial F}{\partial y}(x,p,q(x,p;\delta))} \\ \frac{\partial q}{\partial p}(x,p;\delta) &= -\frac{\frac{\partial F}{\partial p}(x,p,q(x,p;\delta))}{\frac{\partial F}{\partial x}(x,p,q(x,p;\delta))} \end{cases}.$$

Just note that $F(x,p,y) = C_{1.}[F_X(x), F_Y(y); \delta] - p$, it follows that

$$\begin{cases} \frac{\partial q}{\partial x}(x,p;\delta) &= -\frac{f_X(x)C_{2.}[F_X(x),F_Y(y);\delta]}{f_Y(y)C_{11}[F_X(x),F_Y(y);\delta]} \\ \frac{\partial q}{\partial p}(x,p;\delta) &= \frac{1}{f_Y(y)C_{11}[F_X(x),F_Y(y);\delta]} \end{cases}. \tag{5.22}$$

As $\forall (u,v) \in [0,1]^2$, $C_{11}[u,v;\delta] \geq 0$, $f_X(x) \geq 0$ and $f_Y(y) \geq 0$, this completes the proof. ■

5.6.3 Symmetric case

An interesting case concerns the radial symmetry of X and Y. Indeed, in this case, a remarkable relationship exists between the p-th quantile curve and the $(1-p)$-th quantile curve. First, the definition of radial symmetry is given. Then, a theorem is stated and a corollary that informs us about the slopes of the quantile curves is provided.

Definition 15 *Two random variables X and Y are radially symmetric about (a,b) if*

$$\Pr\{X \leq x - a, Y \leq y - b\} = \Pr\{X \geq x + a, Y \geq y + b\} \tag{5.23}$$

Theorem 13 (Nelsen (1998)) *Let X and Y be respectively symmetric about a and b. They are radially symmetric about (a,b) iff their copula C satisfies:*

$$\mathrm{C}(u,v) = u + v - 1 + \mathrm{C}(1-u, 1-v) \tag{5.24}$$

Corollary 14 *If the copula C satisfies equation (5.19), then*

$$\begin{cases} C_{1.}(u,v;\delta) = & 1 - C_{1.}(1-u, 1-v;\delta) \\ C_{2.}(u,v;\delta) = & C_{2.}(1-u, 1-v;\delta) \\ C_{11}(u,v;\delta) = & C_{11}(1-u, 1-v;\delta) \end{cases}$$

Theorem 15 (Radial symmetry and copula quantile curves) *If two random variables X and Y are radially symmetric about (a,b) then*

$$\mathrm{q}(a-x,p;\delta) + \mathrm{q}(a+x, 1-p;\delta) = 2b \tag{5.25}$$

Proof. From equation (5.23),

$$\Pr\{Y \leq y - b \mid X \leq x - a\} = \Pr\{Y \geq y + b \mid X \geq x + a\}$$

In terms of copula,

$$\begin{aligned} C_{1.}\left[F_X(a-x), F_Y(b-y); \delta\right] &= 1 - C_{1.}\left[F_X(a+x), F_Y(b+y); \delta\right] \\ p(a-x, b-y) &= 1 - p(a+x, b+y) \end{aligned}$$

Then, for $p(a-x, b-y) = p$,

$$\begin{cases} b - y = & \mathrm{q}(a-x, p; \delta) \\ b + y = & \mathrm{q}(a+x, 1-p; \delta) \end{cases}$$

and the proof follows. ■

Note that a direct implication of this theorem is $q\left(a, \frac{1}{2}; \delta\right) = b$.

Corollary 16 *If two random variables X and Y are radially symmetric about (a, b) then*

$$\frac{\partial \mathrm{q}}{\partial x}(a-x, p; \delta) = \frac{\partial \mathrm{q}}{\partial x}(a+x, 1-p; \delta) \tag{5.26}$$

5.6.4 Examples

We first describe a case where the copula quantiles can be derived analytically, this is for the Kimeldorf and Sampson copula. We then describe how to develop c-quantiles for the general class of archimedean copulae and two specific archimedean copulae that we use in the empirical analysis below; the Joe-Clayton Copula (BB7 in Joe (1997)) which was used by Patton (2001) and BB3. We then study two copulae that allow both positive and negative slopes for the quantile curves, depending on the value of their dependence parameter. These are the Gaussian copula where the dependence pattern is measured by correlation but where the marginal distributions may be non-gaussian and the Frank copula. We then show that we have to be careful when selecting copula since some copulae, such as the Frank copula, may not allow us to adequately capture the full range of behaviour in the distribution of the dependent variable Y.

5.6.4.1 Kimeldorf and Sampson copula

Consider the copula given by

$$C(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}} \text{ for } \theta > 0$$

we then have

$$\begin{aligned} C_1(v|u) &= \frac{\partial C(u,v)}{\partial u} \\ &= -\frac{1}{\theta}(u^{-\theta} + v^{-\theta} - 1)^{-\frac{(1+\theta)}{\theta}}(-\theta u^{-(1+\theta)}) \\ &= (1 + u^{\theta}(v^{-\theta} - 1))^{-\frac{(1+\theta)}{\theta}} \end{aligned}$$

solving $p = C_1(v|u)$ for v gives

$$C_1^{-1}(v|u) = v = (p^{\frac{-\theta}{1+\theta}} - 1)u^{-\theta} + 1)^{-\frac{1}{\theta}}$$

which provides us with the c-quantiles relating v and u for different values of p. Using the empirical distribution functions for $u = F_X(x)$ and $v = F_Y(y)$ we can find explicit expressions for the conditional c-quantiles for the variable Y conditional on X.

$$y = F_Y^{-1}\left((p^{-\frac{\theta}{1+\theta}} - 1)F_X(x)^{-\theta} + 1)^{-\frac{1}{\theta}}\right)$$

5.6.4.2 Archimedean Copulae

5.6.4.2.1 General case From $p = \frac{\partial}{\partial u}C(u,v)$, we obtain

$$\begin{aligned} p &= \frac{\phi\prime(u)}{\phi\prime(C(u,v))} \\ p &= \frac{\phi_{\delta}\prime(u)}{\phi\prime(\phi^{-1}[\phi(u) + \phi(v)])} \end{aligned} \tag{5.27}$$

and the quantile regression curve for archimedean copulae can in general be deduced as

$$\begin{aligned} v &= \mathrm{r}(u,p;\delta) \\ v &= \phi^{-1}\left[\phi\left(\phi\prime^{-1}\left(\frac{1}{p}\phi\prime(u)\right)\right) - \phi(u)\right] \end{aligned}$$

Introducing $u = F_X(x)$ and $v = F_Y(y)$, the equation for the c-quantile above becomes

$$y = F_Y^{-1}\left(\phi^{-1}\left[\phi\left(\phi\prime^{-1}\left(\frac{1}{p}\phi\prime(F_X(x))\right)\right) - \phi(F_X(x))\right]\right)$$

5.6.4.2.2 Joe-Clayton (BB7) For the copula defined by

$$C_{\delta,\theta}(u,v) = 1 - \left(1 - \left[\left(1-(1-u)^{\theta}\right)^{-\delta} + \left(1-(1-v)^{\theta}\right)^{-\delta} - 1\right]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}} \tag{5.28}$$

with $\theta \geq 1$ and $\delta > 0$, see Joe (1997), p 153). This two parameter copula is archimedean as

$$C_{\delta,\theta}(u,v) = \phi_{\delta,\theta}^{-1}\left[\phi_{\delta,\theta}(u) + \phi_{\delta,\theta}(v)\right]$$

with

$$\begin{cases} \phi_{\delta,\theta}(s) = \left[1-(1-s)^{\theta}\right]^{-\delta} - 1 \\ \phi_{\delta,\theta}^{-1}(s) = 1 - \left[1-(1+s)^{-\frac{1}{\delta}}\right]^{\frac{1}{\theta}} \\ \phi_{\delta,\theta}{}'(s) = -\left[1-(1-s)^{\theta}\right]^{-\delta-1}\delta\left[-(1-s)^{\theta}\frac{\theta}{-1+s}\right] \end{cases} \tag{5.29}$$

It only allows positive dependence and we can see that

$$\begin{aligned} \lim_{\delta\to\infty} C_{\delta,\theta}(u,v) &= C^{+}(u,v) \\ \lim_{\theta\to\infty} C_{\delta,\theta}(u,v) &= C^{+}(u,v) \end{aligned}$$

An important property is that each parameter respectively measures lower (δ) and upper (θ) tail dependence as we show below. Moreover this copula encompasses two copulae sub-families as for $\theta = 1$ one obtains the Kimeldorf & Sampson (1975) copula:

$$C_{\delta}(u,v) = \left(u^{-\delta} + v^{-\delta} - 1\right)^{-\frac{1}{\delta}},$$

and for $\delta \to 0$ the Joe (1993) copula:

$$C_{\theta}(u,v) = 1 - \left((1-u)^{\theta} + (1-v)^{\theta} - (1-u)^{\theta}(1-v)^{\theta}\right)^{1/\theta}.$$

5.6.4.2.3 BB3 For the BB3 copula defined below (Joe (1997)),

$$C_{\delta,\theta}(u,v) = \exp\left(1 - \left[\delta^{-1}\ln\left(\exp\left(\delta\tilde{u}^{\theta}\right) + \exp\left(\delta\tilde{v}^{\theta}\right) - 1\right)\right]^{\frac{1}{\theta}}\right) \tag{5.30}$$

with $\theta \geq 1$ and $\delta > 0$. This copula is archimedean as

$$C_{\delta,\theta}(u,v) = \phi_{\delta,\theta}^{-1}\left[\phi_{\delta,\theta}(u) + \phi_{\delta,\theta}(v)\right]$$

with

$$\phi_{\delta,\theta}^{-1}(s) = \exp\left(-\left[\delta^{-1}\ln(1+s)\right]^{\frac{1}{\theta}}\right) \tag{5.31}$$

Again this copula allows us to model positive dependence and

$$\begin{aligned} \lim_{\delta\to\infty} C_{\delta,\theta}(u,v) &= C^{+}(u,v) \\ \lim_{\theta\to\infty} C_{\delta,\theta}(u,v) &= C^{+}(u,v) \end{aligned}$$

The lower and upper tail area dependence measures are given by

$$\left\{ \begin{array}{l} \lambda_L = \left\{ \begin{array}{l} 2^{-1/\delta} \text{ if } \theta = 1 \\ 1 \text{ if } \theta > 1 \end{array} \right. \\ \lambda_U = 2 - 2^{1/\theta} \end{array} \right. \tag{5.32}$$

Again each parameter respectively measures lower (δ) and upper (θ) tail dependence.

5.6.4.3 Gaussian copula

The bivariate copula in this case is written:

$$\mathbf{C}(u,v;\rho) = \Phi_2\left(\Phi^{[-1]}(u), \Phi^{[-1]}(v); \rho\right) \tag{5.33}$$

with Φ_2 the bivariate gaussian distribution and Φ the univariate distribution.

$$p = \Phi\left(\frac{\Phi^{[-1]}(v) - \rho\Phi^{[-1]}(u)}{\sqrt{1-\rho^2}}\right)$$

or equivalently solving for v we find the $p'th$ c-quantile curve to be,

$$v = \mathbf{r}(u,p;\rho) = \Phi\left(\rho\Phi^{[-1]}(u) + \sqrt{1-\rho^2}\Phi^{[-1]}(p)\right).$$

The slope of the p-quantile curve is given by:

$$\frac{\partial \mathbf{r}(u,p;\rho)}{\partial u} = \rho\frac{\phi\left(\rho\Phi^{[-1]}(u) + \sqrt{1-\rho^2}\Phi^{[-1]}(p)\right)}{\phi\left(\Phi^{[-1]}(u)\right)}.$$

A positive correlation is characterized by a positive slope and conversely for a negative correlation. Moreover,

$$\frac{\partial \mathrm{r}(u,p;\rho)}{\partial p} = \sqrt{1-\rho^2}\frac{\phi\left(\rho\Phi^{[-1]}(u)+\sqrt{1-\rho^2}\Phi^{[-1]}(p)\right)}{\phi\left(\Phi^{[-1]}(u)\right)}.$$

that is always positive. Then, the higher p the higher the quantile curve. The relationship between y and x for the p-quantile is:

$$y = F_Y^{[-1]}\left[\Phi\left(\rho\Phi^{[-1]}(F_X(x))+\sqrt{1-\rho^2}\Phi^{[-1]}(p)\right)\right]. \tag{5.34}$$

The gaussian copula density with its corresponding quantile curves are plotted in Figure 5.1 (under the assumption of Student margins).

Let assume that X and Y are jointly bivariate gaussian with $\mu_X = E[X]$, $\mu_Y = E[Y]$, $\sigma_X^2 = Var[X]$, $\sigma_Y^2 = Var[Y]$ and $\rho = Corr[X,Y]$. Then, given equation (5.34), the relationship becomes linear and we have

$$y = \mathrm{q}(x_t,p;\rho) = a + bx$$

with slope and intercept values determined by the quantile level;

$$\begin{cases} a = \mu_Y + \sigma_Y\sqrt{1-\rho^2}\Phi^{[-1]}(p) - \rho\frac{\sigma_Y}{\sigma_x}\mu_X \\ b = \rho\frac{\sigma_Y}{\sigma_x} \end{cases}$$

5.6.4.4 Frank copula

This copula is given by

$$\mathrm{C}(u,v;\delta) = -\frac{1}{\delta}\ln\left(1+\frac{(e^{-\delta u}-1)(e^{-\delta v}-1)}{e^{-\delta}-1}\right) \tag{5.35}$$

By computing its first derivative with respect to u, one obtains the copula p-th quantile curve, $p = C_{1.}(u,v;\delta)$ as

$$p = e^{-\delta u}\left(\left(1-e^{-\delta}\right)\left(1-e^{-\delta v}\right)^{-1} - \left(1-e^{-\delta u}\right)\right)^{-1}$$

or equivalently,

$$v = -\frac{1}{\delta}\ln\left(1-\left(1-e^{-\delta}\right)\left[1+e^{-\delta u}\left(p^{-1}-1\right)\right]^{-1}\right).$$

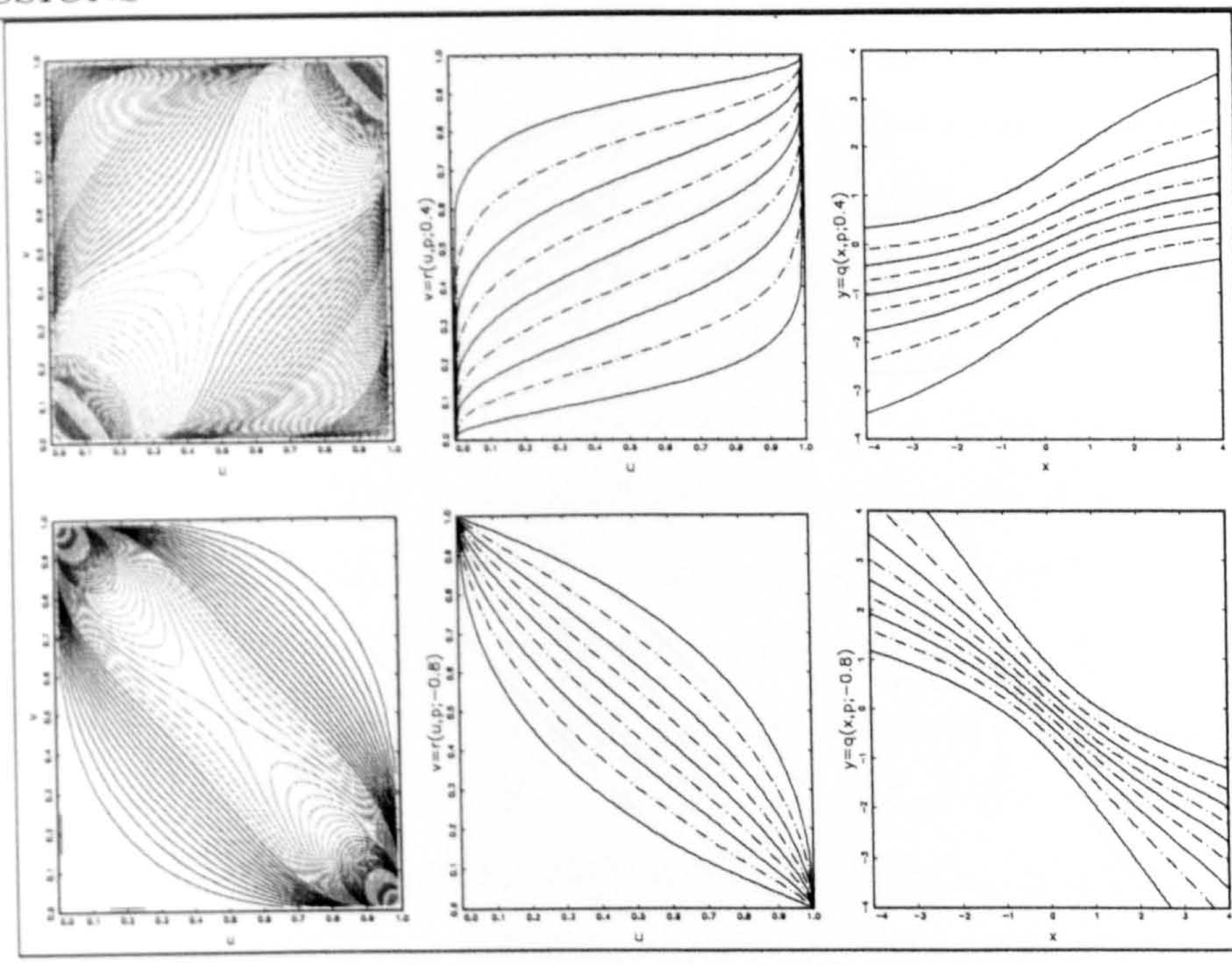

Figure 5.1: Gaussian copula densities, copula p^{th} quantile curves (for $p = .1, .2, \ldots, .9$) for (u, v) and (x, y) under the hypothesis of Student margins ($\nu = 3$) for $\rho = 0.4$ (upper plots) and $\rho = -0.8$ (lower plots)

From the definition of the uniform distribution, one obtains the non-linear relationship between x and y for the p-quantile as:

$$y = F_Y^{[-1]}\left[-\frac{1}{\delta}\ln\left(1-\left(1-e^{-\delta}\right)\left[1+e^{-\delta F_X(x)}\left(p^{-1}-1\right)\right]^{-1}\right)\right] \tag{5.36}$$

We can see that the Frank copula might not always be a good choice as shown in Figure 5.2 the full range of potential values for the variables may not be captured. So for $u \in [0, 1]$,

$$-\frac{1}{\delta}\ln\left(1-\left(1-e^{-\delta}\right)p\right) \leq \mathbf{r}(u,p;\delta) \leq -\frac{1}{\delta}\ln\left(\frac{1-e^{-\delta}}{1+e^{-\delta}\left(p^{-1}-1\right)}\right) \text{ for } \delta > 0$$

and

$$-\frac{1}{\delta}\ln\left(1-\left(1-e^{-\delta}\right)p\right) \geq \mathbf{r}(u,p;\delta) \geq -\frac{1}{\delta}\ln\left(\frac{1-e^{-\delta}}{1+e^{-\delta}\left(p^{-1}-1\right)}\right) \text{ for } \delta < 0.$$

The Frank copula density with its corresponding quantile curves are plotted in Figure 5.1 (under the assumption of Student margins).

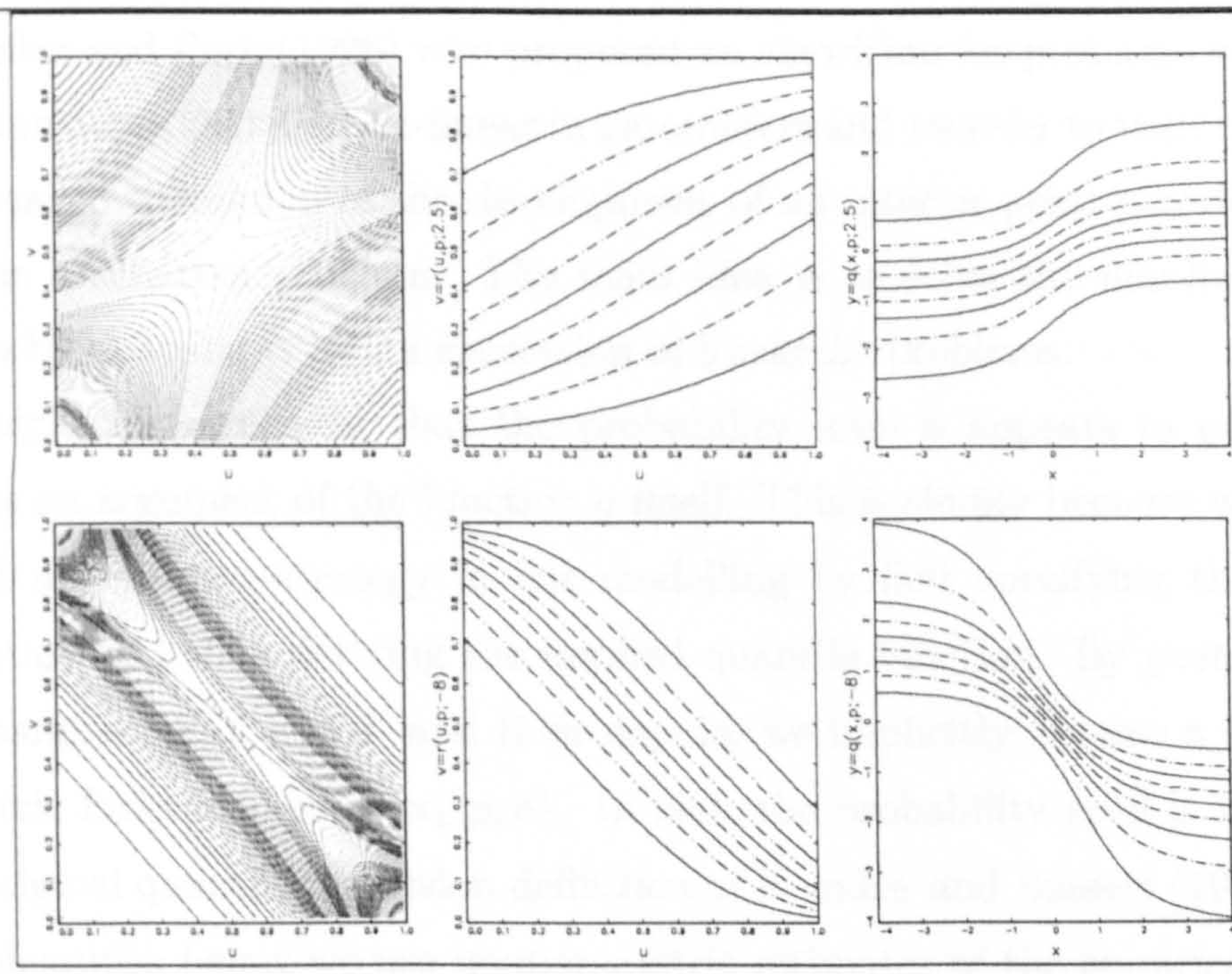

Figure 5.2: Frank copula densities, copula p^{th} quantile curves (for $p = .1, .2, \ldots, .9$) for (u, v) and (x, y) under the hypothesis of Student margins ($\nu = 3$) for $\delta = 2.5$ (upper plots) and $\delta = -8$ (lower plots)

5.6.5 Copula Quantile Regression

Given the development above the concept of a copula quantile regression can be seen to be just a special case of non-linear quantile regression. Let $(y_1, \ldots, y_T)$ be a random sample on Y and $(x_1, \ldots, x_T)$ a random k-vector sample on X.

Definition 16 *The p-th copula quantile regression $q(\mathbf{x}_t, p; \boldsymbol{\delta})$ is a solution to the following problem:*

$$\min_{\delta} \left(\sum_{t \in \mathcal{T}_p} p \left| y_t - \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta}) \right| + \sum_{t \in \mathcal{T}_{1-p}} (1-p) \left| y_t - \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta}) \right| \right) \tag{5.37}$$

with $T_p = \{t : y_t \geq \mathbf{q}(\mathbf{x}_t, p; \delta)\}$ and T_{1-p} its complement. This can be expressed alternatively as:

$$\min_{\delta} \left(\sum_{t=1}^{T} \left(p - \mathbb{I}_{\{y_t \leq \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta})\}} \right) \left(y_t - \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta}) \right) \right) \tag{5.38}$$

This definition indicates that the estimate of the dependence parameter δ is provided by an L^1 norm estimator. This general problem has been investigated by Koenker and Park (1996) who proposed an algorithm for problems with response functions that are non-linear in parameters and we refer to their chapter for a detailed discussion of the development of an interior point algorithm to solve the estimation problem. The main idea is to solve the non-linear L^1 problem by splitting it into a succession of linear L^1 problems.

It might be surprising that the probability level p appears in equation (5.37) as an argument of the function q itself. This is simply because we have adopted a top-down strategy in our modelling by first specifying the joint distribution and then deriving the implied quantile function. By postulating given margins for X and Y and their copula, we implicitly assume a specific parametric functional for $q(\mathrm{x}_t, p; \delta)$. In fact, the probability level is implicit in the original quantile regression definition of Koenker and Bassett (1978). In the applications below we use nonparametric estimates of the empirical marginal distribution functions but estimate the parameters of the copula quantile parameters, δ, as described above. Conditions for the consistency of this semi-parametric approach to the estimation of copula based time series models has been discussed in Chen and Fan (2002).

The c-quantile approach, developed above, enables us to examine the dependency between assets at any given quantile, including extreme quantiles and we can now compare this approach with the standard asymptotic tail area dependency measures. We may in fact not often be interested in dependency in the far extremes where highly infrequent but potentially disasterous joint loss may occur and we may be more interested in the more frequent dependency where large but not extreme loss can arise and in this latter case the c-quantile approach should provide a better measure of association between the assets.

5.7 Application to FX markets

As a demonstration of the methods discussed above we now turn to examine the dependency between exchange rates, both in the extremes and at a range of quantiles describing the conditional distributions. We start by considering the static relationship between the Dollar-Yen (USD/Y), Dollar-Sterling (USD/£) and Dollar-DM (USD/DM) rates using 522 weekly returns from August 1992 to August 2002; the rates themselves are shown in the Figure 5.3 below. The empirical bivariate return distributions are plotted from Figures 5.4 to 5.6 for the three pairs. We then turn to consider the dynamic dependence of conditional quantiles both within and between these rates. All three exchange rates fail univariate normality tests with excess kurtosis and a positive skew except for the USD/£ rate which shows a negative skew over the sample period. For comparison purposes we start by imposing a Gaussian copula to combine these non-gaussian marginals and then examine the sensitivity of our conclusions by using the Joe-Clayton (BB7) copula employed by Patton (2001) and then using daily data with a BB3 copula which is the copula that appears to be supported by the data.

5.7.1 Weekly data

We examine the Gaussian copula first simply because multivariate Gaussianity is a standard assumption in practice (even if this is only implicit through the use of correlation as a measure of dependence) and also because we know that the Gaussian copula implies asymptotic independence and hence it provides a useful basis for a comparison between quantile dependence and the tail area dependence measures.

5.7.1.1 Cross dependence

We compute the nonlinear quantile regression estimates of $\hat{\rho}(p)$ such that:

$$\hat{\rho}(p) = \arg\min \left(\sum_{t=1}^{T} \left(p - \mathbb{I}_{\{S_{1t} \leq \mathrm{q}(S_{2t}, p; \rho, \hat{\theta}_1, \hat{\theta}_2)\}} \right) \left(S_{1t} - \mathrm{q}\left(S_{2t}, p; \rho, \hat{\theta}_1, \hat{\theta}_2 \right) \right) \right) \tag{5.39}$$

Assuming a Gaussian copula the relationship between any two exchange rates S_1 and S_2 at the p'th-quantile is:[3]

$$S_1 = \hat{F}_1^{[-1]} \left[\Phi \left(\hat{\rho}(p) \Phi^{[-1]} \left(\hat{F}_2(S_2) \right) + \sqrt{1 - \hat{\rho}^2(p)} \Phi^{[-1]}(p) \right) \right], \tag{5.40}$$

with $\hat{F}_1$ and $\hat{F}_2$ the empirical marginal distribution functions for the two exchange rates. The estimates of the copula parameter (which in this case is

[3] *Again simply for interest we start by examining the dependence between the level of the exchange rates noting that on the basis of some criteria they may be non-stationary.*

Figure 5.3: Exchange rates

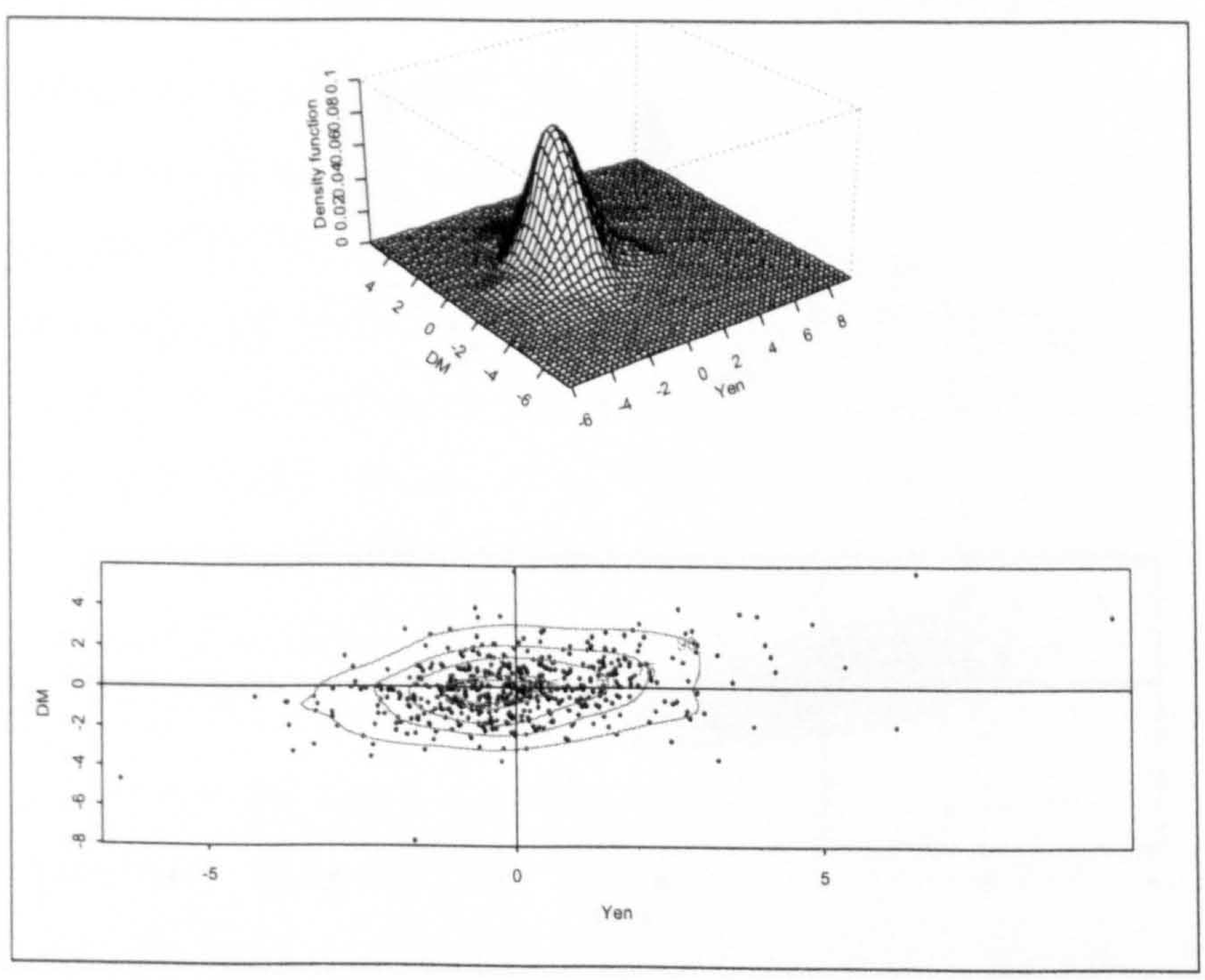

Figure 5.4: Empirical (USD/Y,USD/DM) bivariate return distribution

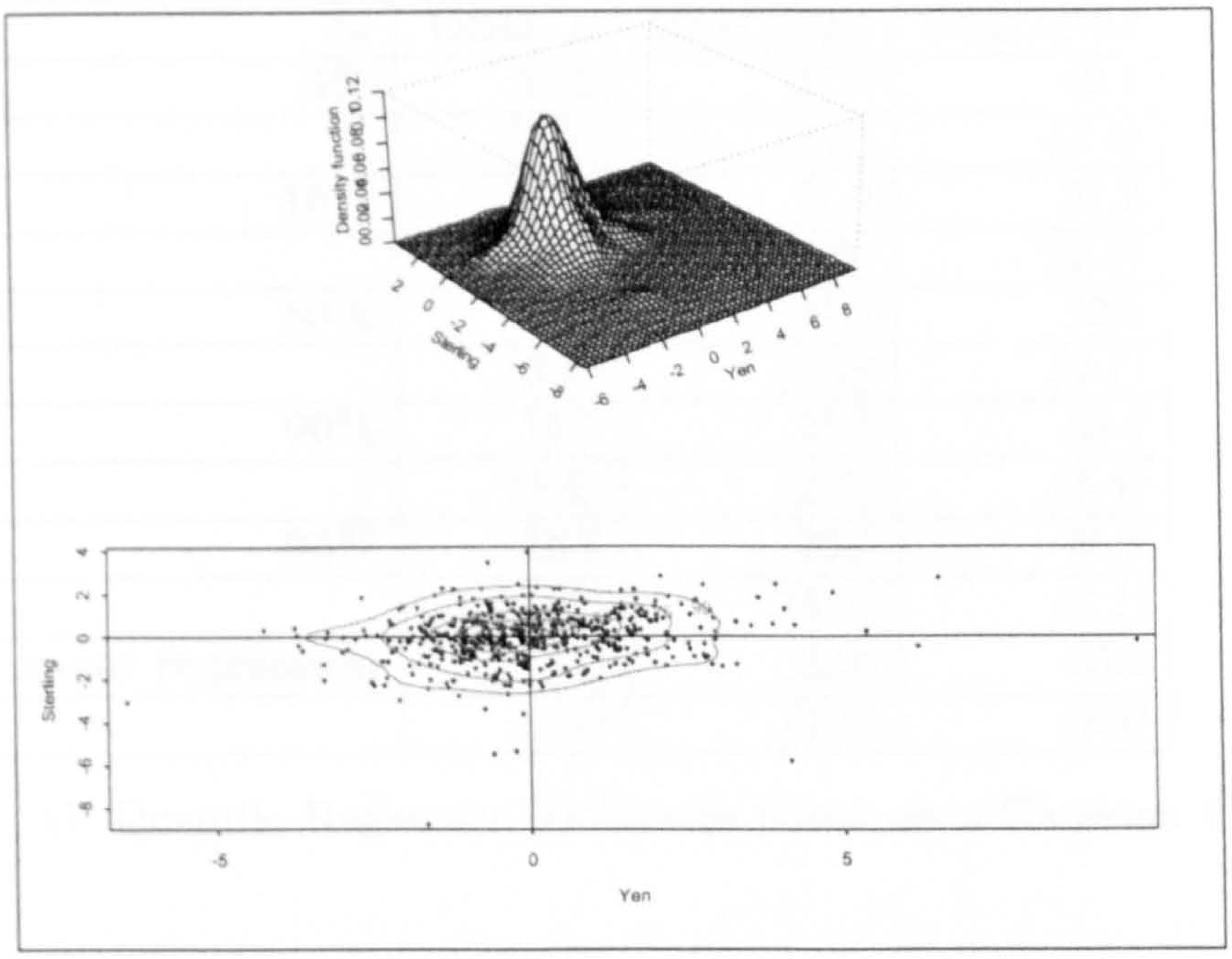

Figure 5.5: Empirical (USD/Y,USD/ £) bivariate return distribution

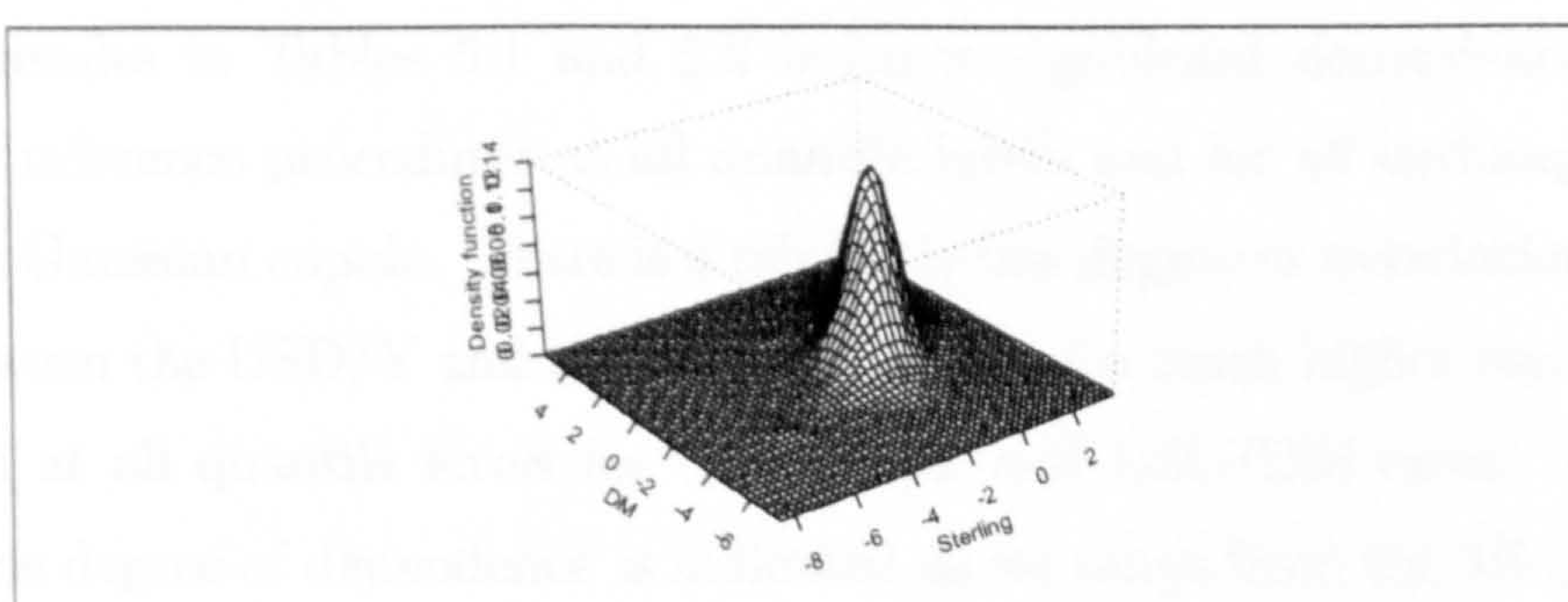

Figure 5.6: Empirical (USD/£,USD/DM) bivariate return distribution

S_1	USD/Y	USD/Y	USD/£
S_2	USD/£	USD/DM	USD/DM
5%	14.2	37.7	49.1
	(5.4)	*(3.5)*	*(4.6)*
10%	16.5	31.9	57.2
	(4.7)	*(4.2)*	*(4.0)*
50%	20.2	32.9	72.0
	(3.8)	*(4.0)*	*(3.1)*
90%	14.1	28.5	63.2
	(5.5)	*(4.7)*	*(3.6)*
95%	13.2	23.3	55.8
	(5.9)	*(5.7)*	*(4.1)*
mean regression	18.3	32.0	65.2
	(4.2)	*(4.2)*	*(3.5)*

Table 5.1: C- Quantile Regression Estimates based on a Gaussian Copula: Levels

just the correlation coefficient) at each quantile level $\hat{\rho}(p)$, expressed in percentage terms, are reported in Tables 5.1 and 5.2 below together with their estimated standard deviations. The mean regression results are also reported for information. The lower p the higher the quantile regression curve.

The results in Tables 5.1 and 5.2 indicate significant dependence using standard inference procedures at all quantile levels and for all exchange rates using the Gaussian copula. There is a relatively low degree of association indicated between the USD/Y and the USD/£ rates and a much higher association indicated at all quantile levels for the USD/£ and USD/DM rates. A fairly symmetric degree of dependence is indicated as we range from the 5% quantile to the 95% quantile with relatively minor differences from the mean regression results. We find the same qualitative conclusions in these two cases when we reverse the causality in Table 5.2. What is striking however are the results for USD/Y and USD/DM where we can see a clear asymmetric structure in the dependency between the lower quantiles and the upper quantiles with much stronger dependency being shown in the lower quantiles when the Yen is the dependent variable (and vice versa in the upper quantiles when the DM is the dependent variable). Use of the mean or median regression in this case could give a substantially misleading idea of the relative joint risks. These results clearly show that there is considerable quantile dependence at both the upper

S_1	USD/£	USD/DM	USD/DM
S_2	USD/Y	USD/Y	USD/£
5%	14.4	21.4	51.2
	(5.9)	*(6.2)*	*(4.0)*
10%	17.5	20.1	57.7
	(4.9)	*(6.6)*	*(3.6)*
50%	20.5	33.4	64.3
	(4.1)	*(4.0)*	*(3.2)*
90%	22.8	37.1	66.1
	(3.7)	*(3.6)*	*(3.1)*
95%	16.9	34.3	51.2
	(5.0)	*(3.9)*	*(4.0)*
mean regression	19.2	32.0	62.1
	(4.4)	*(4.2)*	*(3.3)*

Table 5.2: Reverse C-Quantile Regression Estimates based on a Gaussian Copula: Levels

r_1	USD/Y		USD/Y		USD/£	
r_2	USD/£		USD/DM		USD/DM	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.07	0.00	1.17*	0.00	1.42*	0.19*
10%	1.07	0.00	1.17*	0.00	1.39*	0.21*
50%	1.06	0.03	1.13	0.09	1.21*	0.37*
90%	1.05	0.08	1.10	0.21	1.07	0.53*
95%	1.04	0.09	1.09	0.23	1.06	0.55*

Table 5.3: Joe Clayton C-Quantile Regression estimates: Returns

and lower tails even though we are certain that the standard tail dependence measures would indicate independence since we are using the Gaussian copula in this example. Different information is provided by the quantile dependence measures at fairly extreme quantiles than shown by the (asymptotic) tail area dependence measure.

We briefly compare these Gaussian copula results with those from using the Joe-Clayton copula[4] in Tables 5.3 (returns) and 5.4 (levels) where the stars indicate significance at the 95% level from the value of one for θ (upper tail

[4] *The Joe Clayton Copula was preferred by the data in AIC comparisons with several alternative copulae including the Gaussian Copula.*

S_1	USD/Y		USD/Y		USD/£	
S_2	USD/£		USD/DM		USD/DM	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.01	0.67*	1.39*	0.00	1.04	0.44*
10%	1.02	0.54*	1.39*	0.00	1.03	0.44*
50%	1.00	0.00	1.37*	0.00	1.00	0.35*
90%	1.00	0.00	1.25*	0.17	1.00	0.15
95%	1.00	0.00	1.24*	0.20	1.00	0.11

Table 5.4: Joe Clayton C-Quantile Regression estimates: Levels

dependency) and zero for δ (lower tail dependency). We can see the same indication of upper tail dependence in the (USD/Y,USD/DM) dollar rates in levels and (USD/£,USD/DM) dollar rates in the upper tail in returns but not in the levels in contrast to the Gaussian Copula results. Some lower tail dependence is found for the (USD/£,USD/Y) rates and (USD/£,USD/DM) rates in levels and more strongly in the (USD/£,USD/DM) in returns. Otherwise we find little or no dependence at all with $\hat{\theta}(p)$ being approximately 1 and $\hat{\delta}(p)$ not significantly different from 0 for most quantile levels. The obvious advantage from using the Joe Clayton copula is that we can separate the dependence parameters θ and δ with their distinct interpretations from the correlation which describes the entire dependence structure with the Gaussian copula.

Next we compute the upper and lower tail indices for the returns of the three exchange rates using both the non-parametric estimator λ_L^2 discussed above and then the parametric estimates using these estimated copula parameters. The nonparametric estimates are shown in Table 5.5 and the parametric estimates in Table 5.6 using the relevant formulae for the Joe-Clayton copula (5.15), with the upper tail dependency parameters given above the main diagonal and the lower tail dependency given below.

These two sets of estimates differ in interesting ways; we can clearly see the moderate degree of both higher and lower tail dependence in both the non-parametric and parametric estimates for the (USD/DM) and (USD/£) rates but critically this is not strongly shown at the median parameter estimates.

λ_U^2

		Yen	Sterling	DM
	Yen	-	0.16	0.03
λ_L^2	Sterling	0.09	-	0.32
	DM	0.20	0.38	-

Table 5.5: Upper and Lower Tail index Nonparametric Estimates: Returns

r_t	USD/Y		USD/Y		USD/£	
r_t	USD/£		USD/DM		USD/DM	
p	λ_U	λ_L	λ_U	λ_L	λ_U	λ_L
5%	0.09	0.00	0.19*	0.00	0.37*	0.03*
10%	0.09	0.00	0.19*	0.00	0.35*	0.04*
50%	0.08	0.00	0.15	0.00	0.23*	0.15*
90%	0.06	0.00	0.12	0.04	0.09	0.27*
95%	0.05	0.00	0.11	0.05	0.08	0.28*

Table 5.6: Upper and Lower tail dependency using Joe Clayton C-Quantile Regression parameter estimates: Returns.

In fact the upper tail dependency is shown only at the 5% quantile and not at all at the 95% quantile. Conversely the lower tail dependence suggested by the non-parameteric estimate is only shown at the 95% quantile parameter estimates. The weak relationship between the USD/Y and USD/£ rates is shown effectively at all quantiles. The degree of both lower and upper non-parametric tail dependence between the Yen and the DM rates is not found at any quantile. It is however clear we get substantially more information regarding the joint risk structure from carrying out this analysis using the c-quantile parameter estimates through being able to examine the dependence at all quantiles rather than simply through the mean. The question that is implicitly raised is whether we are really interested in asymptotic dependence or the dependence as shown by the quantile results at the particular level with which the risk manager may be concerned. Coles, Heffernan and Tawn (1999) have also suggested that λ_U (and hence also λ_L) can be viewed as quantile based by varying the level α in (5.10) and (5.11) through the range (0,1) as opposed to the normal limiting values at 0 and 1. It is not however entirely clear if the interpretation of λ_U at a particular α corresponds to a quantile based measure of upper tail dependence instead of simply a measure of quantile dependence. Carrying out their suggestion produces the results shown in Figures 5.7 to 5.12,

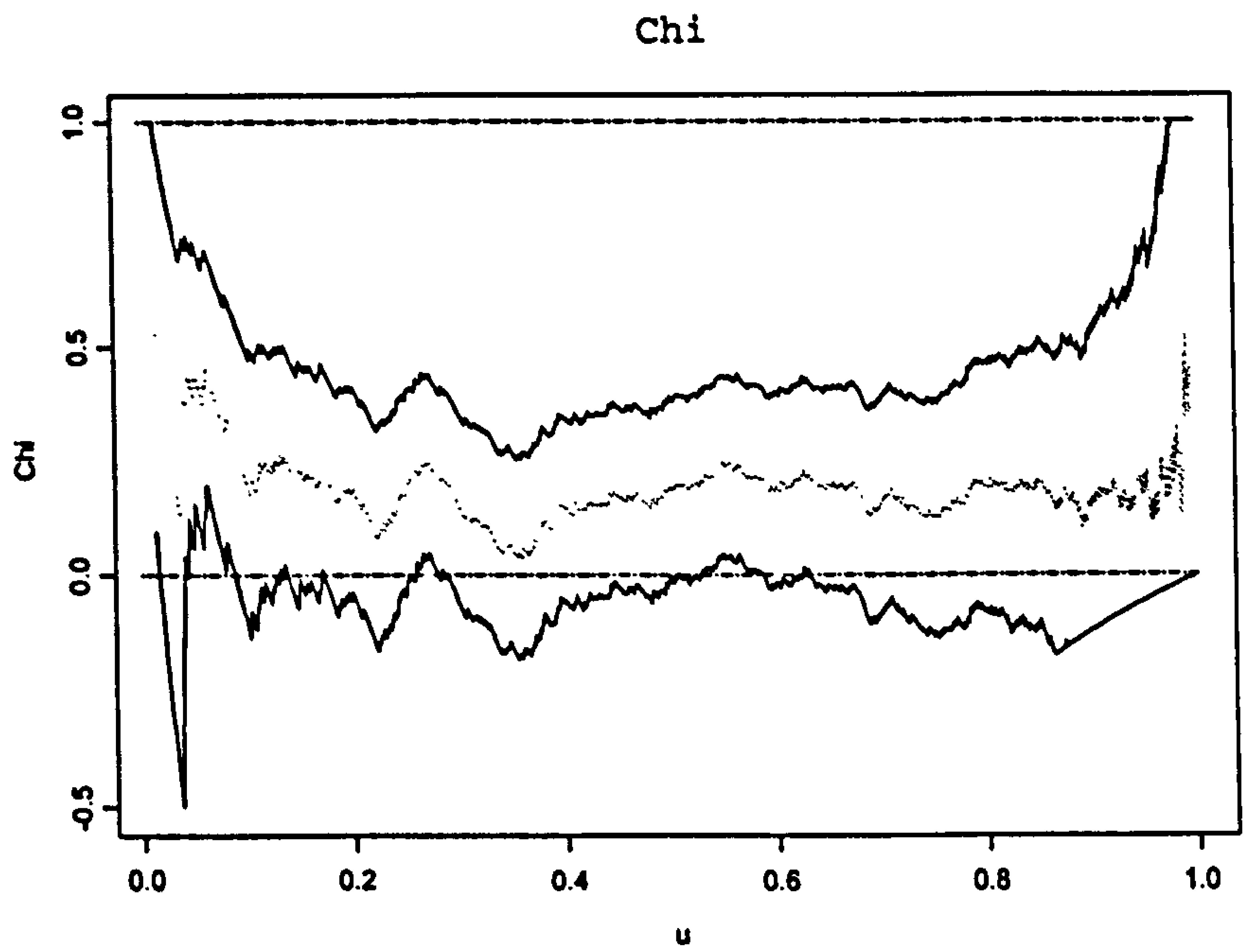

Figure 5.7: (USD/£,USD/Y) estimates of λ for varying α

where their χ and $\bar{\chi}$ statistics and 95% confidence intervals which correspond to our λ_U and $\bar{\lambda}$ statistics evaluated at each α value are shown[5]. The USD/Y and USD/£ rates can be seen from these figures to be effectively independent except as we get close to the upper tail which contradicts our c-quantile results shown above. The USD/Y and USD/DM rates also appear to show weak dependence from these figures with somewhat more upper tail dependence as suggested by the quantile regression results above. The USD/£ and USD/DM results show dependence which appears to decline as we get close to the upper tail and then explodes as we get to the tail, however at this point the confidence intervals are very wide. It would seem that the C-quantile approach is providing an alternative and perhaps more reliable view of tail area and moderate quantile dependence.

[5] *We are grateful to Jan Heffernan for making the SPLUS code for computing these figures publically available.*

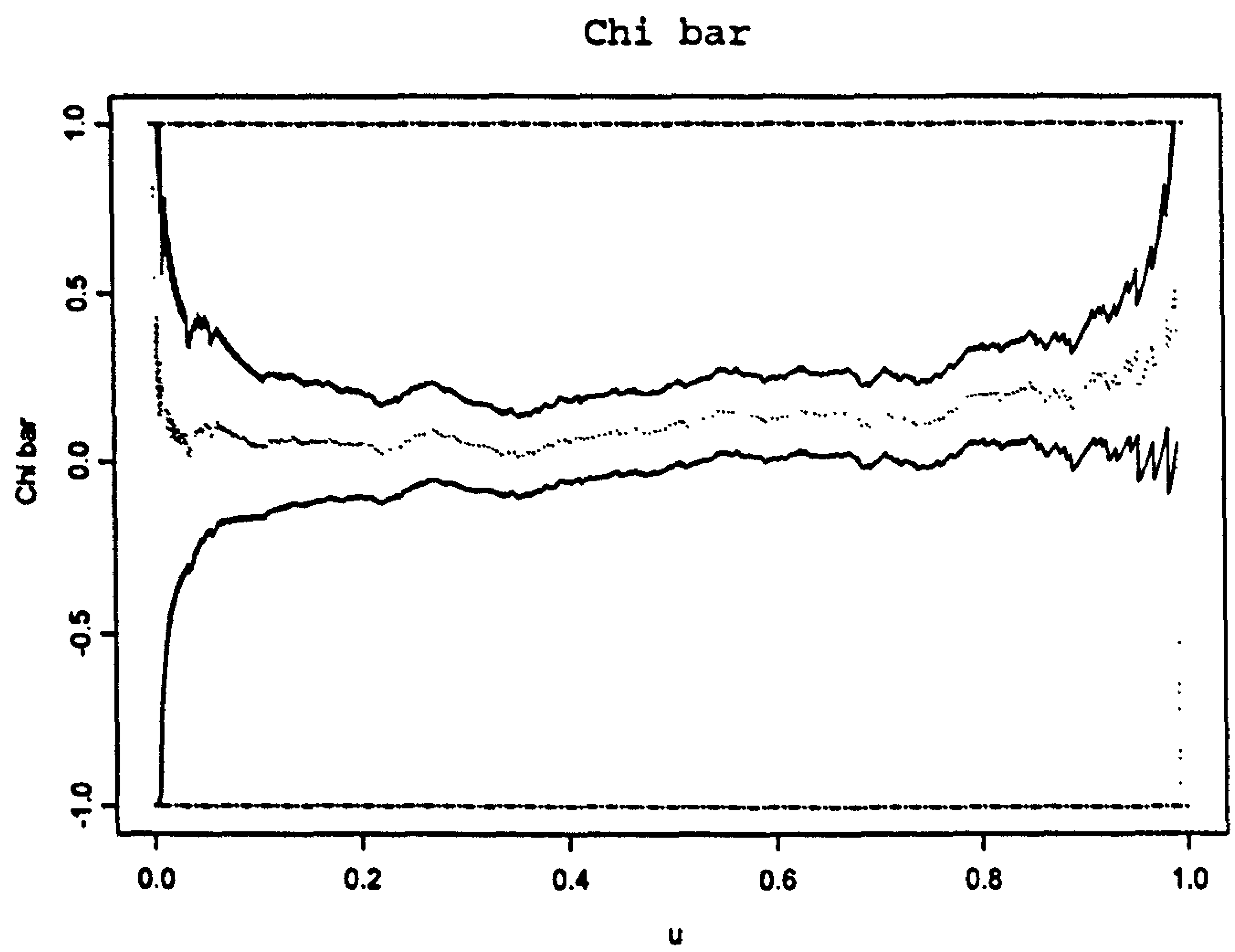

Figure 5.8: (USD/£,USD/Y) estimates of $\bar{\lambda}$

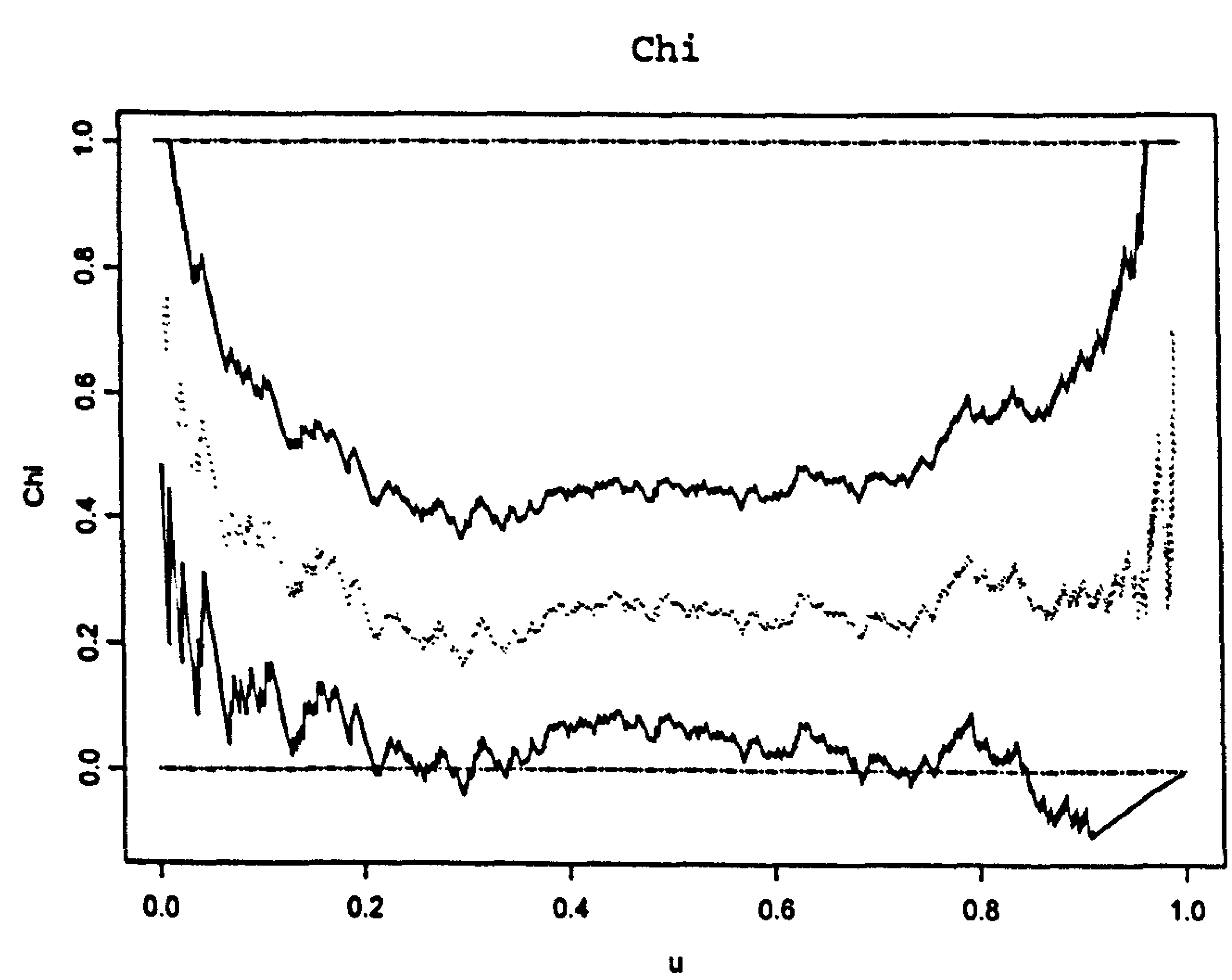

Figure 5.9: (USD/Y,USD/DM) estimates of λ

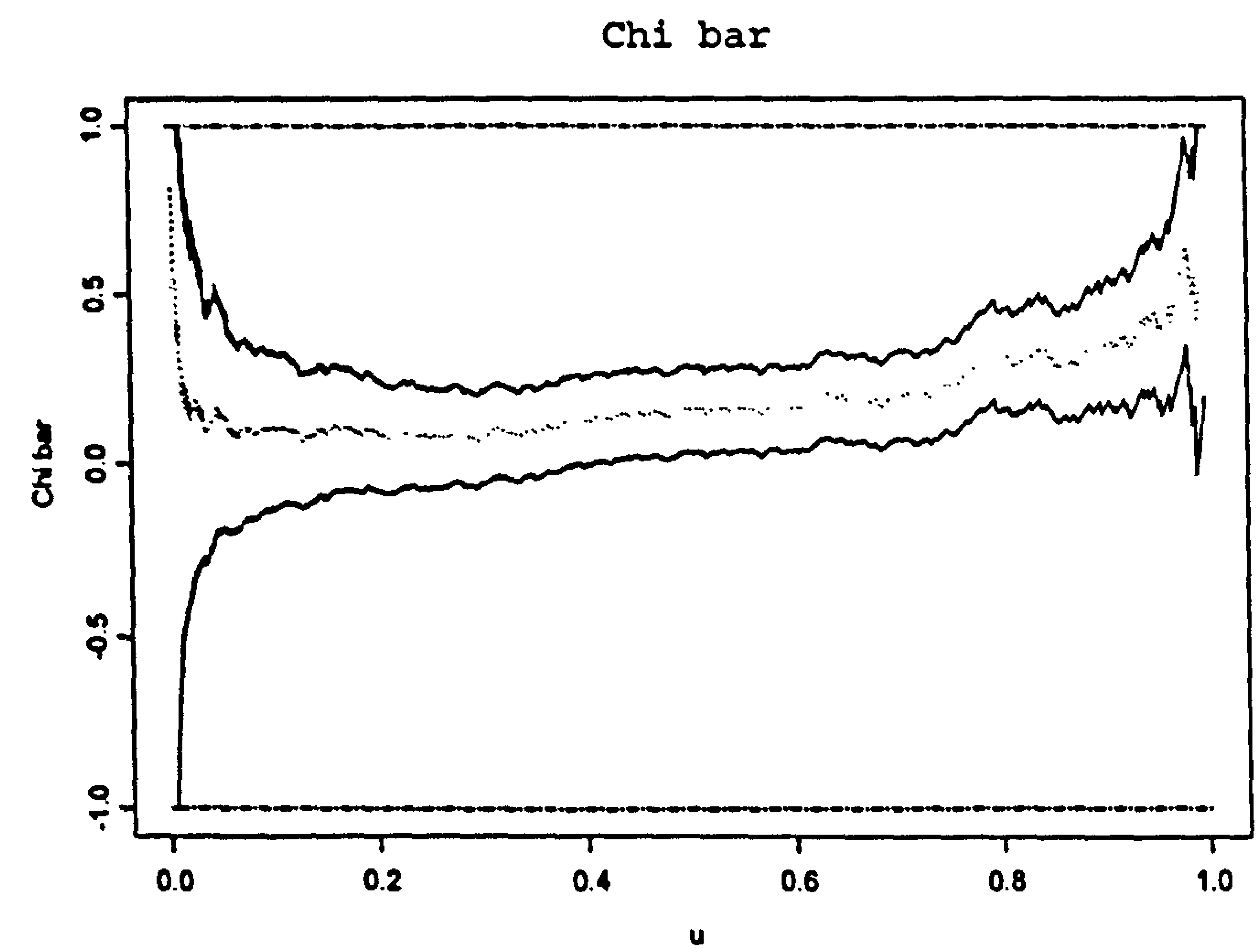

Figure 5.10: (USD/Y,USD/DM) estimates of $\bar{\lambda}$

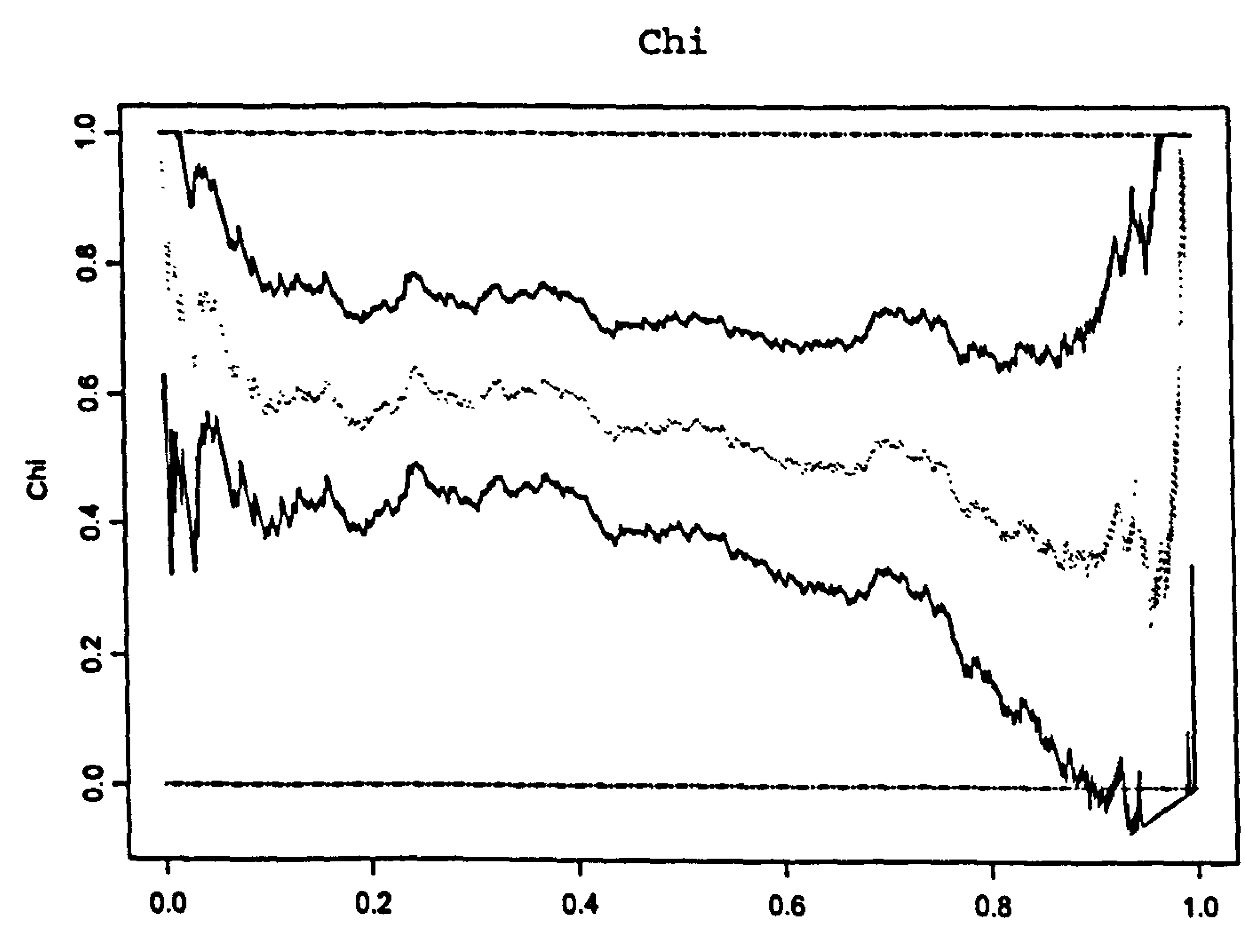

Figure 5.11: (USD/£,USD/DM) estimates of λ

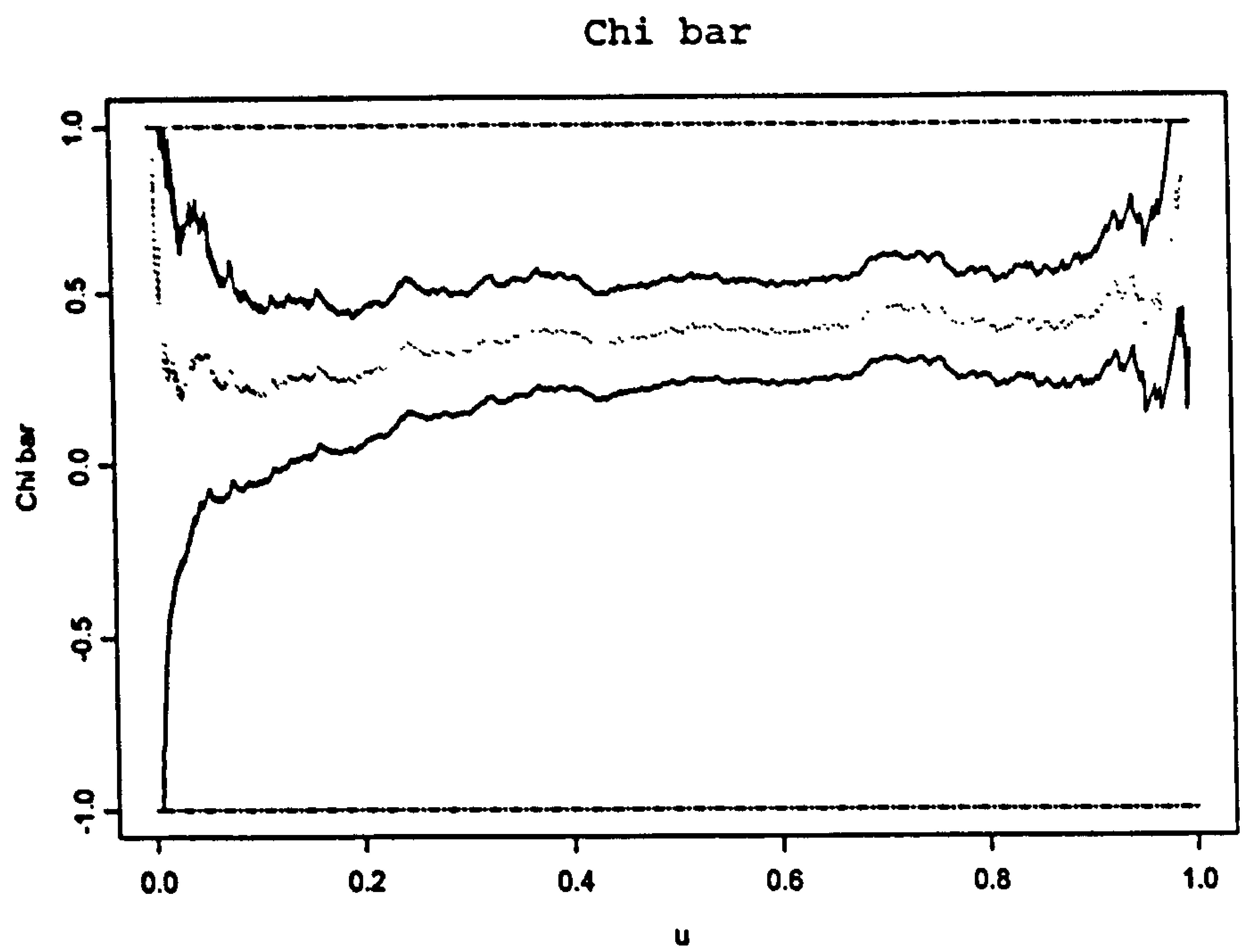

Figure 5.12: (USD/£,USD/DM) estimates of $\bar{\lambda}$

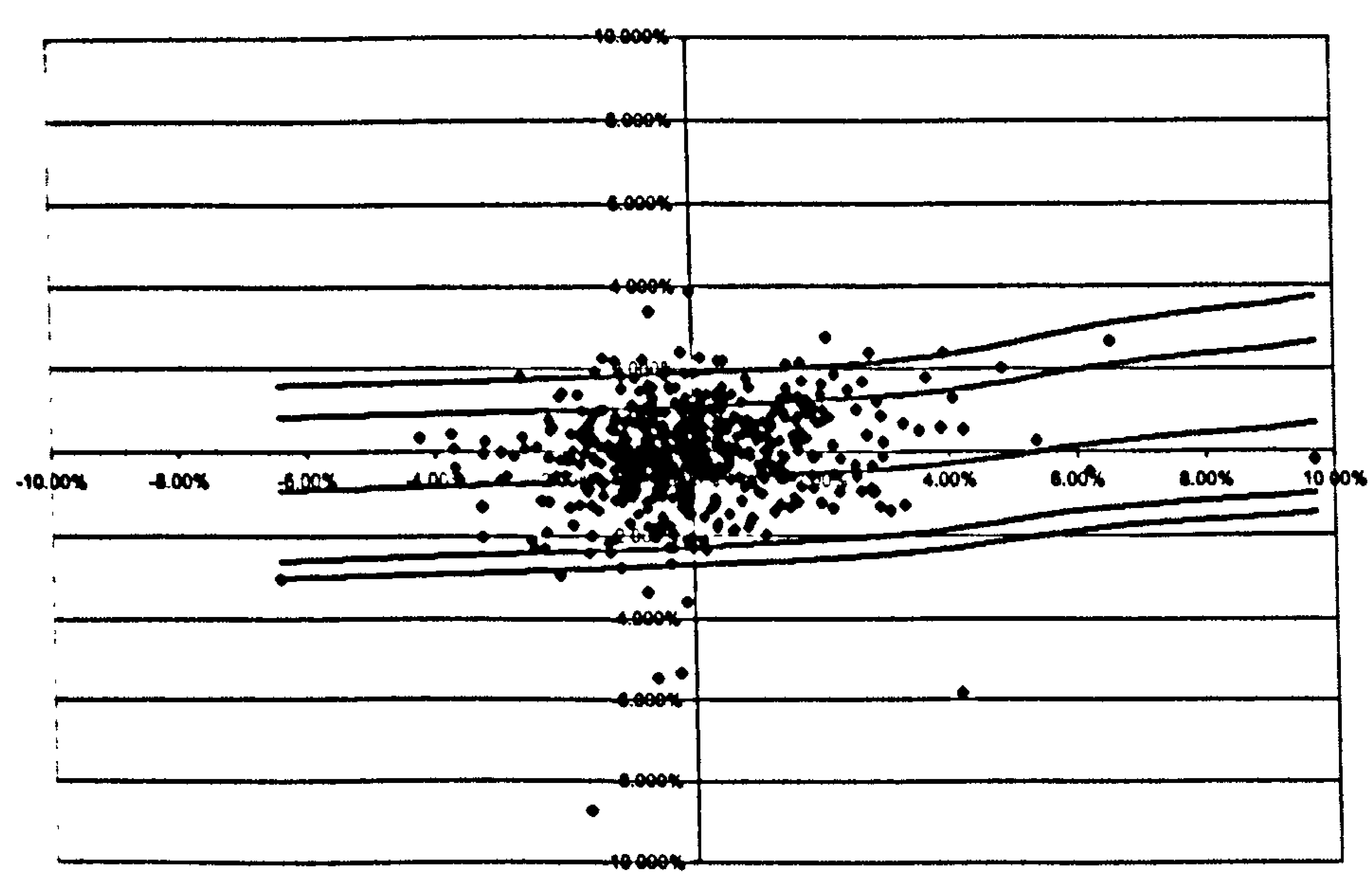

Figure 5.13: Nonlinear quantile regression of USD/Y on $USD/£$ for 5%, 10%, 50%, 90%, 95% probability levels.

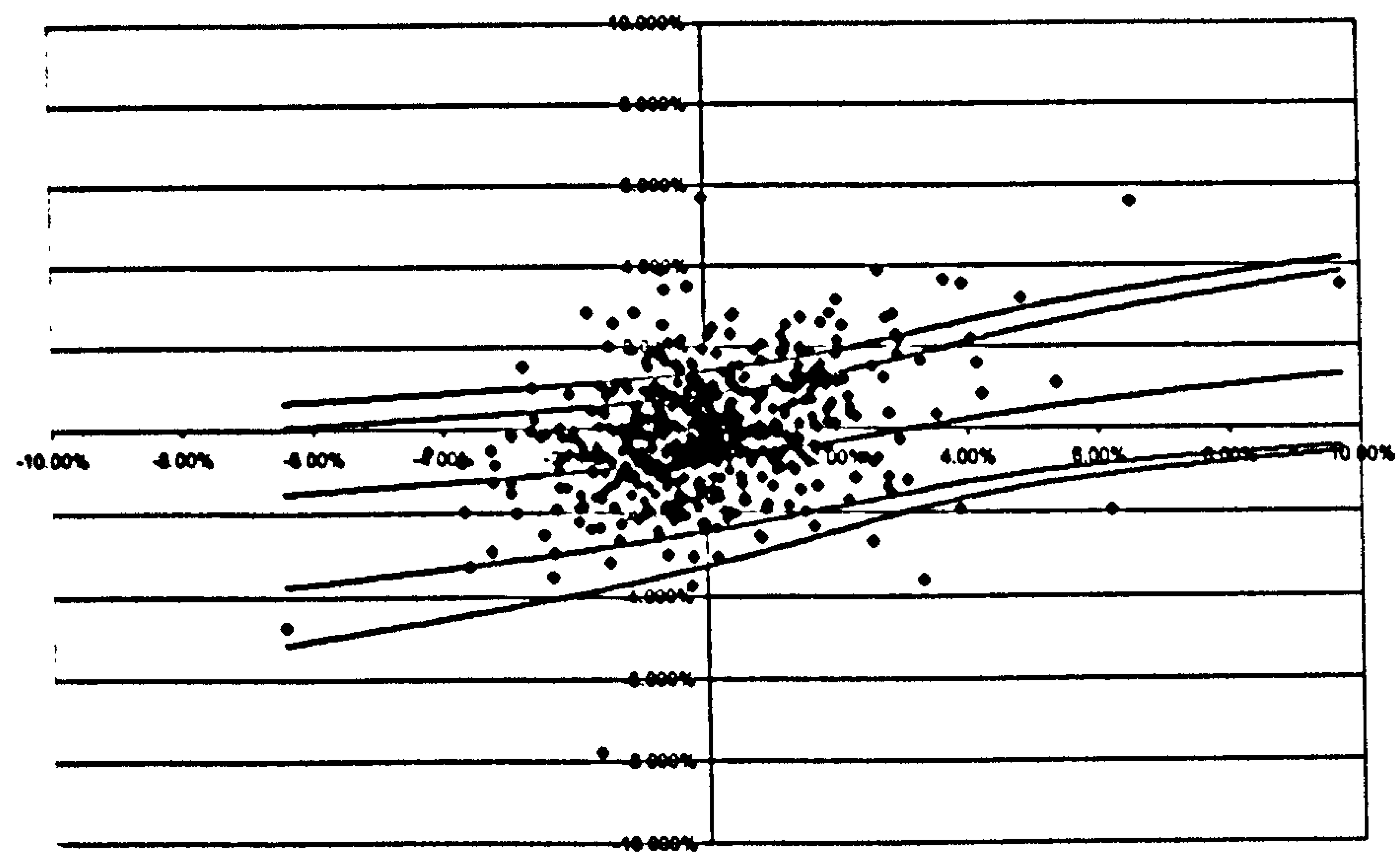

Figure 5.14: Nonlinear quantile regression of USD/Y on USD/DM for 5%, 10%, 50%, 90%, 95% probability levels.

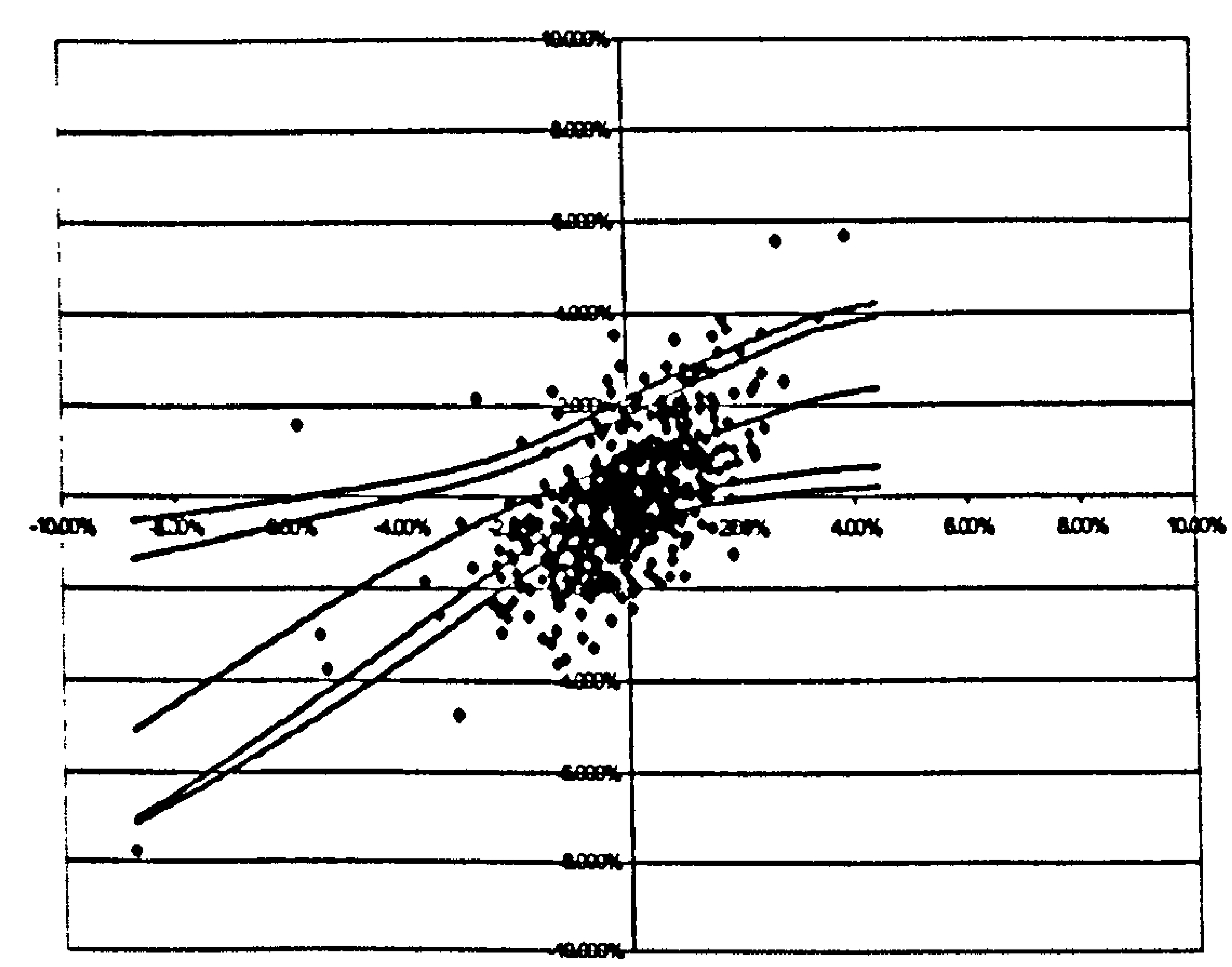

Figure 5.15: Nonlinear quantile regression of $USD/£$ on USD/DM for 5%, 10%, 50%, 90%, 95% probability levels.

r_{t-1}	USD/Y		USD/£		USD/DM	
r_t	USD/Y		USD/Y		USD/Y	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.00	1.00	0.00	1.00	0.00
10%	1.01	0.00	1.00	0.00	1.00	0.00
50%	1.03*	0.00	1.00	0.00	1.00	0.00
90%	1.05*	0.00	1.01	0.00	1.01	0.00
95%	1.05*	0.00	1.01	0.00	1.02	0.00

Table 5.7: C-Quantile Regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

5.7.1.2 Dynamic c-quantiles

We next compute the nonlinear dynamic quantile regression estimates $\left(\hat{\delta}(p), \hat{\theta}(p)\right)$ using the Joe-Clayton Copula concentrating now only on the weekly return data so that:

$$\left(\hat{\delta}(p), \hat{\theta}(p)\right) = \arg\min\left(\sum_{t=1}^{T}\left(p - \mathbb{I}_{\{r_t \leq \mathrm{q}(r_{t-1}, p; \delta, \theta)\}}\right)\left(r_t - \mathrm{q}(r_{t-1}, p; \delta, \theta)\right)\right) \tag{5.41}$$

with

$$\mathrm{q}(r_{t-1}, p; \delta, \theta) = \hat{F}^{[-1]}\left[\phi_{\delta,\theta}^{-1}\left[\phi_{\delta,\theta}\left(\phi_{\delta,\theta}{'}^{-1}\left(\frac{1}{p}\phi_{\delta,\theta}{'}\left(\hat{F}(r_{t-1})\right)\right)\right) - \phi_{\delta,\theta}\left(\hat{F}(r_{t-1})\right)\right]\right] \tag{5.42}$$

with $\phi_{\delta,\theta}$ the generator of the Joe-Clayton copula defined in equation (5.31) and $\hat{F}$ the empirical distribution function of the exchange rate return r_t. The estimates are given in Tables 5.7 to 5.9. The quantile curves corresponding to these estimates are plotted in Figures 5.13, 5.14 and 5.15.

These results show that there is no significant dynamic dependence, either cross rates or within rates, at any quantile level between the returns of the exchange rates in this weekly data. The Clayton Joe parameter estimates indicate independence even in the relative extremes of the joint distribution. This result appears to suggest that forex markets retain efficiency, in a very standard sense, even when the markets are in crisis and in either the upper or lower tail. These quantile results are confirmed, but not quite so clearly, as

r_{t-1}	USD/Y		USD/£		USD/DM	
r_t	USD/£		USD/£		USD/£	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.02	0.00	1.00	0.08	1.00	0.06
10%	1.02	0.00	1.00	0.07	1.00	0.06
50%	1.02	0.00	1.00	0.01	1.00	0.04
90%	1.02	0.00	1.00	0.00	1.02	0.00
95%	1.02	0.00	1.00	0.00	1.03*	0.00

Table 5.8: C-Quantile Regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

r_{t-1}	USD/Y		USD/£		USD/DM	
r_t	USD/DM		USD/DM		USD/DM	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.02	1.00	0.08	1.00	0.06
10%	1.00	0.02	1.00	0.07	1.00	0.06
50%	1.00	0.02	1.00	0.04	1.00	0.04
90%	1.00	0.01	1.00	0.00	1.02	0.00
95%	1.00	0.01	1.00	0.00	1.02	0.00

Table 5.9: C-Quantile Regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

shown in the table 5.10, when we examine the asymptotic tail area dependency measures.

5.7.2 Daily Forex Returns

We next turn to consider the effect of temporal aggregation by examining the dependence between daily Dollar exchange rates for several emerging economies as well as the Yen, DM and Sterling Rates considered above. Table 5.11 presents penalised likelihood statistics, in fact the negative value of Akaike's

Tail Area Dependency	λ_L	λ_U	λ_U	λ_L
Yen	0.017	0.186	0.061	-0.016
DM	0.0	0.011	-0.054	-0.114
Sterling	0.097	0.0	-.0613	0.126

Table 5.10: Tail Area Dependency Measures on lagged own returns

Copula	(USD/DM,USD/£)	(USD/DM,USD/Y)	(USD/Y,USD/£)
Gaussian	136.56	27.09	8.39
Frank	127.93	22.10	7.29
K-Sampson	110.62	18.32	5.66
Gumbel	133.73	32.51	9.77
Galambos	132.68	29.81	7.44
Husler-Reiss	128.96	26.88	6.12
Tawn	133.49	32.96	11.40
BB1	144.76	32.83	9.52
BB2	109.61	17.32	4.66
BB3	171.95*	49.47*	20.26*
BB4	144.00	30.50	7.63
BB5	132.72	31.51	8.77
BB7	141.88	33.40	9.46
Joe	103.29	27.32	6.85

Table 5.11: Penalised Maximum Likelihood Values for daily forex returns

Information Criterion applied to the three major currencies in this set[6]. The copula supported by the sample data in this case is the BB3 copula (see Joe (1997)) is best able to describe these daily exchange rates as opposed to the Joe Clayton for the weekly data used above.

The BB3 copula is similar to the Joe-Clayton copula in that it is able to describe both positive and negative dependence. The estimates of the BB3 copula are estimated for the log-returns of six exchange rates (with respect to USD): the Argentinean peso (ARS), the British pound (GBP), the Chilean peso (CLP), the Colombian peso (COP), the Deutsche mark (DEM), the Japanese yen (JPY). The sample period is 01/01/2000 to 15/12/2002. The log-returns are computed with daily data. Through the estimation of the C-quantile regression curves, lower tail dependence and upper tail dependence for different conditional probability levels and as above the higher the probability level, the lower the C-quantile regression curve. For example, a probability level means that the joint observations below the curve are weighted with 1% and observations above the curve with 99%. The scatterplot can be divided in four quadrants: long/long, long/short, short/short and short/long. Grosso modo, for the C-median regression curve (with a 50% probability level) an up-

[6] *The BB3 copula is in fact preferred by all six of these daily exchange rates but for ease of presentation we have just shown the AIC values for the developed economy rates.*

per tail dependence (respectively lower tail dependence) corresponds to joint observations in the long/long (respectively short/short) quadrant. The results of the estimation are reported for

- (E1) cross-dependence between currency returns for the same day (no lag) (see Appendix),
- (E2) cross-dependence between currency returns with one day lag (see Appendix), and,
- (E3) one day dependence for each currency (see Appendix).

For (E1) and (E2), the results are given in matrix form. The upper (respectively lower) triangle part of the matrix corresponds to the estimates of θ (respectively δ). A matrix is computed for 11 different C-quantile levels: 1%, 5%, 10%, 30%, 50%, 70%, 90%, 95% and 99%. For each quantile level, we obtained $\frac{n(n-1)}{2}$ estimates of θ and δ for the static dependence (E1) and $n(n-1)$ for the dynamic case (E2). Indeed, the dynamic dependence has two characterizations: either between Y_t and X_{t-1}, or between X_t and Y_{t-1}.

Let us comment first the static results. Concerning lower tail dependence, the currencies can be divided into two groups. (1) high lower tail dependence for low quantiles and low lower tail dependence for high quantiles (GBP, DEM), (2) low lower tail dependence for low quantiles and high lower tail dependence for high quantiles (ARS, CLP, COP, JPY). Concerning upper tail dependence, we observe low upper tail dependence for low quantiles and high upper tail dependence for high quantiles – (GBP, DEM) and (JPY,DEM). An important point is that for most of the currencies, the dependence is higher towards the tails (low or high conditional probability) that for median events.

The dynamic results are very interesting since they furnish a subtle measure of the efficiency in the foreign exchange markets. The efficiency of each market alone is measured by (E3). Four types of serial dependence are observed: (1) both upper tail dependence for low quantile levels and lower tail dependence for high quantile levels (ARS), (2) upper tail dependence for low quantile levels (COP), (3) upper tail dependence for high quantile levels (DEM). No dynamic

	GBP				DEM				JPY			
	λ_L	λ_U	$\bar{\lambda}_U$	$\bar{\lambda}_L$	λ_L	λ_U	$\bar{\lambda}_U$	$\bar{\lambda}_L$	λ_L	λ_U	$\bar{\lambda}_U$	$\bar{\lambda}_L$
ARS	0.00	0.00	0.00	0.08	0.00	0.04	0.06	0.03	0.00	0.00	0.03	0.10
GBP					0.34	0.22	0.47	0.54	0.06	0.12	0.29	0.29
DEM									0.14	0.08	0.26	0.30
JPY												
CLP												

	CLP				COP			
	λ_L	λ_U	$\bar{\lambda}_U$	$\bar{\lambda}_L$	λ_L	λ_U	$\bar{\lambda}_U$	$\bar{\lambda}_L$
ARS	0.08	0.00	0.12	0.16	0.00	0.07	0.08	0.06
GBP	0.02	0.00	0.06	0.12	0.01	0.04	0.22	0.03
DEM	0.04	0.01	0.06	0.12	0.05	0.06	0.15	0.03
JPY	0.00	0.07	0.10	0.00	0.10	0.04	0.14	0.08
CLP					0.02	0.03	0.16	0.18

Table 5.12: Asymptotic tail area dependency measures for daily rates

dependence is detected for GBP and JPY and their respective one day lag returns.

Tail dependence measures are computed for the daily data in the Table 5.12.

These final results on asymptotic tail area dependency measures would seem to suggest that currency dependencies are localised to a considerable degree. In particular the developed economies rates appear to be almost completely unaffected in the extremes by the emerging currencies even during this period which includes the crisis in Argentina. The relatively strong dependence shown between the USD/£ and USD/DEM rates reflects almost exactly what we found for the weekly returns above.

5.8 Conclusion

In this chapter we have developed and applied a new approach to measuring tail dependence. The methodology rests on identifying the copula which captures the dependence structure between the series of interest and then deriving the implied conditional quantile regression specification. This enables us to examine the conditional dependence of one variable conditional on the other at a range of quantile levels as opposed to the normal regression relationship which examines the form of dependency at the conditional expectation. In this way we can explore causal dependencies at moderate risk levels that may be more relevant to risk managers than the normal approach given by examining standard asymptotic tail area dependency measures. We have developed several theoretical results describing the properties of C-quantiles and compared their performance with the standard tail area dependency measures.

Our empirical results are indicative of the structure that can be uncovered using copula based quantile regressions. We note that the independence shown between the returns of the own exchange rates applies at all quantiles and hence a much stronger "efficiency" condition seems to apply, even into the tails of the distribution than implied by standard martingale efficiency conditions which involve the conditional expectation.

5.9 Appendix

Static dependence for daily data

The first column corresponds to $X(t)$ and the first line to $Y(t)$.

1.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.17	1.25	1.23	1.20	1.19
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.00	0.00	-	1.02	1.00	1.00
COP	0.00	0.00	0.00	-	1.13	1.10
DEM	0.00	0.43	0.00	0.00	-	1.00
JPY	0.00	0.31	0.00	0.00	0.46	-

Table 5.13: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

5.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.16	1.24	1.21	1.18	1.18
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.00	0.00	-	1.02	1.00	1.00
COP	0.00	0.01	0.00	-	1.12	1.10
DEM	0.00	0.44	0.00	0.00	-	1.00
JPY	0.00	0.32	0.00	0.00	0.46	-

Table 5.14: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

10.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.14	1.22	1.19	1.17	1.16
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.00	0.00	-	1.02	1.00	1.00
COP	0.00	0.01	0.00	-	1.11	1.09
DEM	0.00	0.45	0.00	0.00	-	1.00
JPY	0.00	0.32	0 00	0.00	0.46	-

Table 5.15: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

Cross-dependence and auto-dependence with one day lag

In the left tables, the first column corresponds to $X(t-1)$ and the first line to $Y(t)$. In the right tables the first column corresponds to $Y(t)$ and the first line to $X(t-1)$. The last table corresponds to auto-dependence.

30.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.08	1.13	1.11	1.10	1.10
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.00	0.00	-	1.03	1.00	1.00
COP	0.00	0.01	0.00	-	1.06	1.05
DEM	0.00	0.52	0.00	0.00	-	1.00
JPY	0.00	0.34	0.00	0.00	0.47	-

Table 5.16: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

50.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.02	1.03	1.03	1.03	1.04
GBP	0.00	-	1.00	1.00	1.19	1.05
CLP	0.07	0.00	-	1.03	1.00	1.00
COP	0.00	0.02	0.00	-	1.01	1.01
DEM	0.03	0.30	0.00	0.02	-	1.09
JPY	0.00	0.28	0.03	0.00	0.32	-

Table 5.17: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

70.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.00	1.00	1.00
GBP	0.12	-	1.03	1.05	1.35	1.18
CLP	0.22	0.00	-	1.00	1.01	1.00
COP	0.12	0.00	0.12	-	1.00	1.00
DEM	0.16	0.19	0.01	0.05	-	1.26
JPY	0.18	0.14	0.10	0.03	0.11	-

Table 5.18: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

90.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.00	1.00	1.00
GBP	0.23	-	1.07	1.12	1.49	1.28
CLP	0.34	0.00	-	1.00	1.00	1.00
COP	0.22	0.00	0.20	-	1.00	1.00
DEM	0.27	0.14	0.07	0.07	-	1.37
JPY	0.32	0.05	0.18	0.05	0.01	-

Table 5.19: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

95.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.00	1.00	1.00
GBP	0.26	-	1.08	1.13	1.53	1.30
CLP	0.37	0.00	-	1.00	1.00	1.00
COP	0.25	0.00	0.22	-	1.00	1.00
DEM	0.29	0.13	0.09	0.08	-	1.39
JPY	0.36	0.03	0.21	0.06	0.00	-

Table 5.20: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

99.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.00	1.00	1.00
GBP	0.28	-	1.09	1.15	1.55	1.32
CLP	0.39	0.00	-	1.00	1.00	1.00
COP	0.27	0.00	0.24	-	1.00	1.00
DEM	0.31	0.12	0.10	0.08	-	1.40
JPY	0.39	0.02	0.23	0.06	0.00	-

Table 5.21: BB3 c-quantile Estimates for daily forex returns between X_t and Y_t

1.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.18	1.24	1.17	1.16	1.21
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.00	0.03	-	1.07	1.03	1.02
COP	0.00	0.07	0.00	-	1.09	1.08
DEM	0.00	0.00	0.00	0.00	-	1.00
JPY	0.00	0.00	0.00	0.00	0.01	-

1.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.36	1.00	1.00
GBP	0.00	-	1.03	1.06	1.00	1.00
CLP	0.00	0.00	-	1.09	1.00	1.00
COP	0.00	0.00	0.00	-	1.00	1.00
DEM	0.00	0.00	0.11	0.03	-	1.00
JPY	0.00	0.00	0.00	0.00	0.00	-

Table 5.22: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

5.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.17	1.23	1.16	1.15	1.20
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.00	0.03	-	1.07	1.03	1.02
COP	0.00	0.07	0.00	-	1.09	1.07
DEM	0.00	0.00	0.00	0.00	-	1.00
JPY	0.00	0.00	0.00	0.00	0.01	-
5.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.34	1.00	1.00
GBP	0.00	-	1.03	1.05	1.00	1.00
CLP	0.00	0.00	-	1.09	1.00	1.00
COP	0.00	0.00	0.00	-	1.00	1.00
DEM	0.00	0.00	0.11	0.03	-	1.00
JPY	0.00	0.00	0.00	0.00	0.00	-

Table 5.23: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

10.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.15	1.21	1.14	1.14	1.19
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.00	0.03	-	1.07	1.02	1.02
COP	0.00	0.07	0.00	-	1.08	1.06
DEM	0.00	0.00	0.00	0.00	-	1.00
JPY	0.00	0.00	0.00	0.00	0.00	-
10.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.32	1.00	1.00
GBP	0.00	-	1.03	1.05	1.00	1.00
CLP	0.00	0.00	-	1.08	1.00	1.00
COP	0.00	0.00	0.00	-	1.00	1.00
DEM	0.00	0.00	0.10	0.03	-	1.00
JPY	0.00	0.00	0.00	0.00	0.00	-

Table 5.24: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

30.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.08	1.14	1.08	1.08	1.13
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.00	0.05	-	1.06	1.01	1.00
COP	0.00	0.07	0.00	-	1.04	1.03
DEM	0.00	0.00	0.00	0.00	-	1.00
JPY	0.00	0.00	0.00	0.00	0.00	-

30.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.22	1.00	1.00
GBP	0.00	-	1.01	1.02	1.00	1.00
CLP	0.00	0.00	-	1.04	1.00	1.01
COP	0.00	0.00	0.00	-	1.00	1.00
DEM	0.00	0.01	0.07	0.04	-	1.00
JPY	0.00	0.00	0.00	0.03	0.00	-

Table 5.25: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

50.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.01	1.05	1.01	1.02	1.05
GBP	0.00	-	1.00	1.00	1.00	1.00
CLP	0.05	0.06	-	1.04	1.00	1.00
COP	0.00	0.06	0.04	-	1.00	1.00
DEM	0.00	0.00	0.00	0.01	-	1.00
JPY	0.02	0.00	0.02	0.00	0.00	-

50.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.11	1.00	1.00
GBP	0.00	-	1.00	1.00	1.00	1.03
CLP	0.07	0.03	-	1.00	1.00	1.01
COP	0.07	0.00	0.00	-	1.00	1.00
DEM	0.00	0.03	0.04	0.06	-	1.00
JPY	0.01	0.00	0.04	0.07	0.00	-

Table 5.26: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

70.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.00	1.00	1.00
GBP	0.11	-	1.07	1.03	1.02	1.00
CLP	0.23	0.00	-	1.00	1.00	1.00
COP	0.12	0.01	0.16	-	1.00	1.00
DEM	0.10	0.00	0.01	0.04	-	1.02
JPY	0.22	0.00	0.04	0.04	0.00	-

70.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.01	1.00	1.00
GBP	0.00	-	1.00	1.00	1.08	1.02
CLP	0.20	0.04	-	1.00	1.08	1.00
COP	0.26	0.00	0.01	-	1.06	1.00
DEM	0.00	0.00	0.00	0.00	-	1.00
JPY	0.09	0.09	0.10	0.12	0.06	-

Table 5.27: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

90.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.00	1.00	1.00
GBP	0.20	-	1.15	1.13	1.10	1.08
CLP	0.38	0.00	-	1.00	1.00	1.00
COP	0.24	0.00	0.22	-	1.00	1.00
DEM	0.20	0.00	0.01	0.07	-	1.12
JPY	0.37	0.00	0.05	0.08	0.00	-

90.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.02	1.00	1.00	1.13	1.03
GBP	0.00	-	1.00	1.00	1.16	1.00
CLP	0.34	0.05	-	1.00	1.12	1.00
COP	0.42	0.05	0.03	-	1.17	1.00
DEM	0.00	0.00	0.00	0.00	-	1.01
JPY	0.13	0.20	0.16	0.17	0.12	-

Table 5.28: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

95.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.00	1.00	1.00
GBP	0.22	-	1.17	1.15	1.12	1.10
CLP	0.42	0.00	-	1.00	1.00	1.00
COP	0.27	0.00	0.24	-	1.00	1.00
DEM	0.23	0.00	0.01	0.08	-	1.15
JPY	0.41	0.00	0.06	0.10	0.00	-

95.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.05	1.00	1.00	1.17	1.05
GBP	0.00	-	1.00	1.00	1.18	1.00
CLP	0.38	0.05	-	1.00	1.13	1.00
COP	0.46	0.06	0.03	-	1.20	1.00
DEM	0.00	0.00	0.00	0.00	-	1.01
JPY	0.13	0.23	0.18	0.19	0.14	-

Table 5.29: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

99.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.00	1.00	1.00	1.00	1.00
GBP	0.24	-	1.19	1.17	1.14	1.12
CLP	0.45	0.00	-	1.00	1.00	1.00
COP	0.30	0.00	0.26	-	1.00	1.00
DEM	0.25	0.00	0.01	0.09	-	1.16
JPY	0.44	0.00	0.06	0.11	0.00	-

99.00%	ARS	GBP	CLP	COP	DEM	JPY
ARS	-	1.07	1.00	1.00	1.20	1.06
GBP	0.00	-	1.00	1.00	1.20	1.00
CLP	0.41	0.06	-	1.00	1.14	1.00
COP	0.49	0.07	0.04	-	1.22	1.00
DEM	0.00	0.00	0.00	0.00	-	1.01
JPY	0.13	0.25	0.19	0.20	0.15	-

Table 5.30: BB3 c-quantile Estimates for daily forex returns between Y_t and X_{t-1} (left table) and X_t and Y_{t-1} (right table)

			probability level p								
Y(t-1)	Y(t)		1.0%	5.0%	10.0%	30.0%	50.0%	70.0%	90.0%	95.0%	99.0%
ARS	ARS	θ	1.69	1.66	1.63	1.47	1.30	1.12	1.00	1.00	1.00
1	1	δ	0.00	0.00	0.00	0.00	0.13	0.56	1.09	1.16	1.21
GBP	GBP	θ	1.00	1.00	1.00	1.00	1.00	1.07	1.15	1.17	1.19
2	2	δ	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00
CLP	CLP	θ	1.01	1.01	1.02	1.01	1.00	1.00	1.00	1.00	1.00
3	3	δ	0.00	0.00	0.00	0.01	0.06	0.10	0.14	0.15	0.16
COP	COP	θ	1.22	1.20	1.17	1.09	1.03	1.00	1.00	1.00	1.00
4	4	δ	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
DEM	DEM	θ	1.00	1.00	1.00	1.00	1.00	1.05	1.15	1.17	1.19
5	5	δ	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
JPY	JPY	θ	1.00	1.00	1.00	1.00	1.00	1.01	1.03	1.03	1.04
8	8	δ	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

BB3 c-quantile Estimates for daily forex returns between $Y(t)$ and $Y(t)$

Chapter 6

Conclusion

In this thesis, we first considered the origin of the concept of copulae, the general framework for the analysis of copulae and we provided a review of the literature. We then proposed three applications of copulae to finance.

The third chapter proposes a methodology for developing risk measures for portfolios during extreme events. The approach is based on splitting the multivariate extreme value (MEV) distribution of the assets of the portfolio into two parts: the distributions of each asset and their dependence function – copula. The estimation problem is also investigated. A trivariate MEV empirical application for market index portfolios (US, German and Japanese stock markets) is provided. Then, stress-testing values and Monte-Carlo based risk measures – Value-at-Risk and Expected Shortfall – are computed.

In the fourth chapter, we propose a general to specific approach that can avoid spurious assumptions such as linearity in the form of the dynamic relationship between variables. A new model based on the Gaussian copula with the same dependence structure as an AR(p) with non-gaussian marginals is presented. Copula based dynamic dependency measures – auto-concordance in place of autocorrelation – are developed. An important added value of our model is that it encompasses the AR(p) model and allows non-linearity. Non-linear time dependence functions that generalize the autocorrelation function are also given. We have developed the concept of a C-autoregression whose structure can be deduced from the copula given the marginals. The C-autoregression is a generalization of the standard linear autoregression and provides a general framework for non-linear regression that is compatible with given assumptions on the dependence structure between the random variables.

We introduce a general approach to nonlinear quantile regression modelling, the C-quantile regression, that is based on the identification of the copula function that completely defines the entire dependency structure between the variables of interest. Hence we extend Koenker and Bassett's (1978) original statement of the quantile regression problem by determining a distribution for the dependent variable Y conditional on the regressors X and hence the specification of the quantile regression functions. Some properties of the C-quantiles are then derived. Finally, we develop an empirical application which examines conditional quantile dependency in the foreign exchange market and compare this approach with the standard tail area dependency.

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