Axisymmetric Rayleigh-Benard Convection

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Finally I dedicate this Thesis to my wife Nasrin, for very many years of patience and understanding.

Abstract

This thesis considers axisymmetric Rayleigh-Benard convection in an infinite horizontal layer of fluid heated from below.

Major emphasis is placed on a study of the effect of rotation of the layer, where both the stationary and overstable cases are analysed.

In Chapter 2, a numerical solution of the linearised equations which govern the non-rotating fluid with rigid boundaries, is presented.

In Chapter 3, the non-rotating layer is perturbed by making the elevation of the lower plane a small slowly varying function of the radial coordinate. The modified amplitude equation is found and at the central axis the matching with a local solution in terms of Bessel functions is carried out.

In Chapter 4, the effect of rotation is incorporated and the numerical scheme of Chapter 2, is modified to solve the appropriate linearised equations.

In Chapter 5, the non-linear amplitude equation is derived for the rotating layer with rigid boundaries in the case when the system is subject to the exchange of stabilities. The matching process with a solution in terms of Bessel functions near the axis of rotation is described in Chapter 6, and is shown to lead to the possibility of 'phase-winding' effects associated with variations in the wavelength of convection.

In Chapter 7, it is shown that when the rotating layer is subject to overstability a pair of amplitude equations governs the motion away from the axis of rotation. Again one of the main interests lies in how the solution matches with that valid in the neighbourhood of the axis.

CHAPTER 1

Introduction

This is a study of axisymmetric Rayleigh- Benard convection, based on the formal extension of the conventional linearised stability theory.

As a starting point for this discussion, we have available several previous studies. The theoretical foundations of the onset of the thermal instability in an infinite horizontal layer of fluid heated uniformly from below, were laid by Rayleigh (1916), who proved the validity of the principle of the exchange of stabilities for the case of two free boundaries. The proof in the general case for this problem is due to Pellew and Southwell (1940), and Chandrasekhar (1954) has also discussed the characteristic value problem for high order differential equations which arise in studies of the linear theory of hydrodynamic stability.

It is well known that the resulting cellular convection flow is not uniquely determined by the equations of motion and boundary conditions if the layer is infinite in horizontal extent. An infinite degeneracy was first found in the early linear theories which apply for infinitesimal flow amplitude. Malkus and Veronis (1958) gave a complete discussion of finite amplitude cellular convection in a layer of fluid with stress-free boundaries and later Schlüter, Lortz and Busse (1965) generalised the method of Malkus and Veronis, for the steady state by considering the whole manifold of solutions. Furthermore, they treated the problem for both rigid and free boundaries.

Kuppers and Lortz (1969) and Kuppers (1970) have also used the same approach to consider the case when the layer is rotating about a vertical axis. Newell and Whitehead (1969) have shown how a continuous finite band-width of modes can be readily incorporated into a description of Rayleigh-Benard convection with free boundaries.

1.1. Effect of Lateral Walls

The linear stability of a quiescent, three dimensional rectangular box of fluid heated uniformly from below is considered by Davis (1967), where the influence of lateral walls on the convection process is determined. Later, Drazin (1975) was able to illustrate some of Davis's ideas analytically by a linear analysis of a simplified two-dimensional model with rigid side walls and free upper and lower boundaries. Brown and Stewartson (1977) have now incorporated non linear effects in a theoretical study of convection in a shallow rectangular box and their results also confirm Davis's prediction that the preferred motion is one with rolls parallel to the shorter side of the box.

Non-linear aspects of the problem with lateral walls were first described in the paper of Segel (1969) using a multiple-scale perturbation analysis. Segel (1969) shows that distant side-walls cause a slow amplitude modulation of the cellular convection and this idea was used in a description of the effect of inperfectly insulated distant side walls on the transition to finite amplitude convection by Daniels (1977, 1978). Here the aspect ratio, 2L, is large compared with 1, and the upper and lower surfaces are taken to be free. Hall and Wallton(1977) have considered a similar problem for finite values of L. More recently, the effect of perfectly insulated side walls on the wavelength of the convection pattern has been investigated by Cross, Daniels Hohenberg and Siggia (1980); variations in wavelength are represented by a class of phase-winding solutions which evolve as the Rayleigh number increases above its critical value. An alternative geometrical effect has been studied by Eagles (1980) who considered the modifications to the classical Benard problem obtained by making the elevation of the lower plane a small slowly varying function of the horizontal coordinate. Convection then occurs first in the region of maximum depth, since the value of a 'local Rayleigh number' is greatest there.

Various cylindrical situations have been considered by Zierep (1958, 1959, 1961, 1963) and theoretical predictions of the roll pattern and critical Rayleigh number based on linearised theory for a circular cylinder with a stress-free outer wall by Charlson and Sani (1970, 1971). Joseph (1971) has also given an analytic solution of the linearised equation for axisymmetric flow in a cylinder with a rigid outer wall in terms of Bessel functions.

Non-linear numerical and theoretical studies of unicellular motion have been described by Liang, Vidal and Acrivos (1969) and Jones, Moore and Weiss (1976). Brown and Stewartson (1978,1979) have made a theoretical study of finite amplitude convection in a shallow cylinder of fluid bounded by stress-free planes and have demonstrated how the onset of convection in the form of concentric rolls can be described by matching the solution of a non-linear amplitude equation in the main body of the layer to a linearised solution in terms of Bessel functions near the centre. The singularity which develops in the amplitude equation as the centre of cylinder is approached, leads to an unexpectedly large cell amplitude there and also to an effective increase in the critical Rayleigh number (over that predicted by linear theory) at which finite amplitude convection spreads throughout the cylinder.

Experimentally, Schmidt and Milverton (1935) established preliminary results for the classical Benard problem confirming the general prediction of the linear stability theory for the onset of steady convection and more recently Rossby (1969) has considered the onset of convection in both rotating and non-rotating cylinders. Much experimental work has been done in this area. Axisymmetric solutions, in the form of concentric rolls, have been observed experimentally by Koschmieder (1966) and Hoard, Robertson and Acrivos (1970).

1.2. Effect of Rotation:

In the classical problem of Benard it has been shown that the principle of the exchange of stabilities is valid and theory, in agreement with experiments, shows that when the critical adverse temperature gradient is surpassed, a cellular pattern of motion must ensue. But when external forces are present the principle of the exchange of stabilities is not always valid, and depending on circumstances, instability can set in by the alternative mode of overstability and oscillations of increasing amplitude. Chandrasekhar (1953) has shown that overstability can arise when the fluid is subject to rotation and the theoretical expectations are further described in the paper by Chandrasekhar and Elbert (1955). A linearised solution in terms of Bessel functions for the onset of stationary convection in an infinite cylindrical geometry was first considered by Muller (1965).

Non-linear effects in the rotating case were first considered by Veronis (1959) who assumed disturbances of uniform amplitude in an infinite rectangular Cartesian framework with, stress-free, upper and lower surfaces. His results established the possibility of non-linear motions at subcritical values of the Rayleigh number, and these were further discussed in a subsequent paper (Veronis 1966). The solution of the non-linear steady state equations in an infinite horizontal rotating fluid layer has also been investigated for large Prandtl numbers by Kuppers and Lortz. (1969) and for finite Prandtl numbers by Kuppers (1970).

The effect of side walls on the rotating linear Benard problem has been considered by Davies, Jones and Gilman (1971) and later by Daniels for both the exchange case (1977a) and the overstable case (1980). For stress-free horizontal surfaces, the forms of the amplitude equations for both stationary and over stable motions, incorporating spatial modulation and non-linear effects, were given by Daniels (1978) in order to discuss a finite amplitude motion in the form of two-dimensional rolls in a rotating system bounded by distant side walls. More recently, Daniels (1980) has studied the effect of centrifugal acceleration on axisymmetric convection in a shallow rotating cylinder or annulus. Only the case of stationary convection was considered. In our study we shall neglect this effect, although it is expected that it could be incorporated in the analysis in a fairly straight-forward manner.

The experimental results of Nakagawa and Frenzen (1955) and later Goroff (1960), have confirmed the theoretical work of Chandrasekhar (1953), the experiments showing that either the stationary or oscillatory instability sets in as convection of increasing amplitude as the Rayleigh number increases beyond its critical value. More recently Rossby (1969) has also confirmed the existence of the subcritical instability predicted by Veronis (1959), and has provided a comprehensive set of results for widely varying values of the Rayleigh number, Taylor number and Prandtl number. Axisymmetric solutions, in the form of concentric rolls, have been observed experimentally in the rotating case by Koschmieder (1967).

In the present study attention is also restricted to axisymmetric solutions of the governing equations in the cylindrical geometry.

The classical theory of Benard convection in an infinite layer is based upon the Oberbeck-Boussinesq approximation in which the thermal expansion of the fluid is neglected except where coupled with the gravitational acceleration, g. The resulting equations of motion are

$$\nabla . \underline{u}^* = 0,$$

$$\rho \left\{\frac{\partial \underline{u}^*}{\partial t^*} + (\underline{u}^* . \nabla)\, \underline{u}^*\right\} = -\rho\, g\, \underline{k} - \nabla p^* + \mu . \nabla^2\, \underline{u}^*, \tag{1.1}$$

$$\frac{\partial \theta^*}{\partial t^*} + (\underline{u}^* . \nabla)\, \theta^* = \kappa\, \nabla^2\, \theta^*,$$

with $\nabla^2 = \frac{\partial^2}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial}{\partial r^*} + \frac{\partial^2}{\partial z^{*2}}$, $\nabla \cdot \underline{u}^* = \frac{\partial u^*}{\partial r^*} + \frac{u^*}{r^*} + \frac{\partial w^*}{\partial z^*}$,
where $\underline{u}^*$, p^*, θ^* and ρ are the velocity, pressure, temperature and density, μ is the coefficient of viscosity and κ is thermal diffusivity; $\underline{k}$ is the vertical unit vector and r^*, z^* are the cylindrical polar co-ordinates. The equation of state is taken as

$$\rho = \rho_0 \{1 - \alpha_0(\theta^* - \theta_0^*)\}, \tag{1.2}$$

where α_0 is a constant and ρ_0 is the constant density at temperature θ_0^* which is taken as the temperature of the lower surface of the layer.

In Chapter 2, the equations (1.1) are linearised with respect to small disturbances to obtain the classical Benard problem, first discussed by Rayleigh (1916). A numerical method of solution based on integration across the layer is used to obtain the neutral stability curve for the case when both bounding surfaces are rigid and the results are compared with those already obtained by Chandrasekhar (1961), using an alternative scheme of solution. The same numerical method forms the basis of the method used in later Chapters (4,6) to investigate the properties of the rotating Benard problem.

In Chapter 3, a modified linear Benard problem is considered in which the lower boundary is taken as

$$z^* = d\ \varepsilon^2 G(\varepsilon \frac{r^*}{d}),$$

where ε is a small parameter and d is the depth of the layer. We refer to this as the non-parallel plane problem. The linear amplitude equation is found in the region $r^*/d = O(\varepsilon)$ and the outer solution is matched with an appropriate solution near the centre, where $r^*/d = O(1)$, given in terms of Bessel functions. It is found that the matching conditions are different from those which apply to the non-linear problem investigated by Brown and Stewartson (1978).

In Chapter 4, the numerical solution of Chapter 2 is extended to include the effect of rotation (see Chandrasekhar 1961). The neutral stability curve is calculated for various speeds of rotation, measured by a Taylor number, T, in the case when convection arises through an exchange of stabilities and both bounding surfaces are rigid. Both the main eigenfunction and a related function needed in the calculations of Chapter 5 are determined at the marginal state.

In Chapter 5, the non-linear equations governing stationary convection in the rotating layer near the critical Rayleigh number are expanded in a sequence of inhomogeneous linear equations dependent upon the solution of the linear stability problem (Chapter 4). The non-linear amplitude equation is derived for the case of rigid boundaries and it is found that for certain ranges of the speed of rotation and Prandtl number of the fluid, subcritical instability is possible, as in the model problem with stress-free boundaries considered by Veronis (1959).

An asymptotic analysis for $T \gg 1$ is carried out in order to provide a check on the numerical computation of the amplitude equation coefficients.

The solution of Chapter 5 provides an 'outer solution' of the problem. This must be matched to a solution (which can be expressed in terms of Bessel functions) near the axis of rotation in order to complete the analysis and determine the boundary condition for the amplitude function involved in the outer solution. It is found that the matching procedure at the centre is essentially unaltered from that described by Brown and Stewartson (1978) for the non-rotating layer with stress-free boundaries. Thus the main features of the Brown and Stewartson study apply to the rotating case also.

The matching allows a phase winding effect (see Cross, Daniels, Hohanberg and Siggia 1980) which is analysed in detail and indicates a possible variation in wavelength of the roll pattern. This is characterised by the "phase winding parameter" which arises from a non zero value of the amplitude function at the origin in the matching process. Finally, supercritical and subcritical solutions of the amplitude equation are found numerically and the possible roll patterns are determined for several rotation rates.

In Chapter 7, we consider the non-linear structure of flow at the onset of convection in a rotating fluid layer with stress-free boundaries, when the disturbance leads to overstability. As in the investigation of Chapter 6 and of the exchange case in the non-rotating layer by Brown and Stewartson (1978), the main interest concerns the way in which a linearised solution at the centre matches, and provides boundary conditions for the solution of the non-linear amplitude equations that govern the solution in the outer zone. The details of the analysis are found to be quite different from those of the exchange case and the complete description of the non-linear equilibrium state requires a subdivsion of the outer zone into three regions, which are considered in § 7.4. In the equilibrium state, the amplitude of the motion at the centre is found to be unusually large, of $O(|\ln(R-R_c)|^{-\frac{1}{2}})$, where R_c is the critical Rayleigh number. This chapter is the subject of a joint paper with Dr. P.G. Daniels.

CHAPTER 2

Linear Benard Convection in Cylindrical Geometry

Let (r^*, ϕ^*, z^*) denote cylindrical coordinates and consider a horizontal layer of fluid which is confined between two rigid boundaries $z^* = 0$ and $z^* = d$. Gravity acts in the negative z^* direction and the flow field extends from $r^* = 0$ to $r^* = \infty$. The lower surface is kept at a constant temperature θ_0^* and the upper surface at a constant temperature θ_1^* such that $\theta_0^* > \theta_1^*$. The velocity component v^* is taken to be zero throughout and an axisymmetric motion is assumed so that $\frac{\partial}{\partial \phi^*} = 0$. The governing equations for this system are given by the Oberbeck-Boussinesq approximation (1.1). The dimensionless coordinates, velocity components, temperature and pressure are defined as follows:

$$(u^*, w^*) = \kappa(u, w)(r, z, t)/d, \tag{2.1}$$

$$\theta^* = \theta_0^* + (\theta_1^* - \theta_0^*)z/d + \kappa\nu\theta(r, z, t)/\alpha_0 g d^3, \tag{2.2}$$

$$p^* = p_0^* - \rho_0 g z^* + \tfrac{1}{2}\rho_0 g \alpha_0 (\theta_1^* - \theta_0^*) z^{*2}/d + \rho_0 \kappa^2 p(r, z, t)/d^2 \tag{2.3}$$

$$r = r^*/d, \; z = z^*/d, \; t = \kappa t^*/d^2, \tag{2.4}$$

where p_0 is a constant basic pressure and ρ_0 is the constant density at temperature θ_0^* and α_0 is the coefficient of thermal expansion of the fluid, while g is the acceleration due to gravity and κ is the thermal diffusivity of the fluid. The velocity components are u^*, w^* and t^* is time, θ^* is the temperature, p^* is the pressure.

2.1. Perturbation equations

The full set of equations of motion (1.1), with (2.1) - (2.4), are linearised for a small amplitude of disturbance and may be written in the form

$$\frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} = 0, \qquad (2.1.1)$$

$$\frac{\partial u}{\partial t} - \sigma\left(\nabla^2 u - \frac{u}{r^2}\right) + \frac{\partial p}{\partial r} = 0, \qquad (2.1.2)$$

$$\frac{\partial w}{\partial t} - \sigma(\nabla^2 w + \theta) + \frac{\partial p}{\partial z} = 0, \qquad (2.1.3)$$

$$\frac{\partial \theta}{\partial t} - \nabla^2 \theta - Rw = 0, \qquad (2.1.4)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \qquad (2.1.5)$$

where

$$R = g\alpha_0 d^3(\theta_0^* - \theta_1^*)/\kappa\nu,$$

is the Rayleigh number,

$$\sigma = \nu/\kappa$$

is the Prandtl number and ν is the kinematic viscosity which is assumed to be constant.

The boundary conditions are

$$\theta = u = w = 0 \text{ at } z = 0,\ 1. \qquad (2.1.6.)$$

2.2. The equations governing the marginal state and eigenvalue problem

A steady solution of the linear equations (2.1.1) - (2.1.5) for axisymmetric flow has been considered by Zierep (1959) where the lower and upper planes were taken to be stress-free, while in our study the boudaries are assumed rigid.

Let us now suppose that the perturbations have the forms:

$$u = J_o'(\alpha r)f(z)\ \exp(i\omega t),$$

$$w = J_o(\alpha r)g(z)\ \exp(i\omega t), \qquad (2.2.1)$$

$$\theta = J_o(\alpha r)h(z)\ \exp(i\omega t),$$

where J_o is the zeroth order Bessel function and ω is a complex number in general and α is a real number. We set $\omega = 0$ on the assumption that the principle of the exchange of stabilities is valid (Chandrasekhar 1953). Substitution of (2.2.1) into equations (2.1.1) - (2.1.5), leads to a set of ordinary differential equations as follows:

$$Dg - \alpha f = 0, \qquad (2.2.2)$$

$$D^2h - \alpha^2 h + Rg = 0, \qquad (2.2.3)$$

$$D^4 g - 2\alpha^2 D^2 g + \alpha^4 g - \alpha^2 h = 0, \qquad (2.2.4)$$

where

$$D = \frac{d}{dz}. \qquad (2.2.5)$$

The boundary conditions are

$$h = f = g = 0 \quad \text{at } z = 0,1. \tag{2.2.6}$$

Eliminating f(z) and g(z) between these equations we obtain

$$(D^2 - \alpha^2)^3 h + R\alpha^2 h = 0, \tag{2.2.7}$$

$$h = D^2 h = D(D^2 - \alpha^2) h = 0 \quad \text{at } z = 0, 1. \tag{2.2.8}$$

We have a sixth order ordinary differential equation and we have to satisfy six boundary conditions, three at $z = 0$ and three at $z = 1$. The boundary conditions are homogeneous and so we have an eigenvalue problem for R(for given α). The lowest eigenvalue for R, i.e. the minimum with respect to α of the eigenvalues so obtained, is the critical Rayleigh number at which instability sets in.

The desired solution of(2.2.7)- (2.2.8) can be made unique by the normalisation condition

$$h = 1 \quad \text{at } z = \tfrac{1}{2} \tag{2.2.9}$$

2.3. Numerical Solution of the Eigenvalue Problem and Neutral Curve

Although the order of the equation (2.2.7) is six, it is convenient to introduce a system of twelve first order differential equations. This is done by introducing the twelve variables.

$$(y_1, y_2, \ldots, y_7, \ldots, y_{12}) = (h, h^{(1)}, \ldots, h^{(5)}, \frac{\partial h}{\partial R}, \frac{\partial h^{(1)}}{\partial R}, \ldots, \frac{\partial h^{(5)}}{\partial R}), \qquad (2.3.1)$$

where

(n) $n = 1, 2, \ldots, 5.$

denotes the order of differentiation respect to the variable, z.

From (2.2.7) and (2.3.1)

$$y_1 = h$$

$$y_i' = y_{i+1} \quad (i = 1, 2, \ldots, 5),$$

$$y_6' = \alpha^2(\alpha^4 - R)\, y_1 - 3\alpha^4 y_3 + 3\alpha^2 y_5.$$

The assumption made in (2.3.1) implies that

$$y_7 = \frac{\partial h}{\partial R},$$

$$y_i' = y_{i+1} \quad (i = 7, 8, \ldots, 12),$$

$$y_{12}' = -\alpha^2 y_1 + (\alpha^6 - \alpha^2 R) y_7 - 3\alpha^2 y_9 + 3\alpha^2 y_{11}.$$

A convenient form of this formulation is

$$D \underline{Y} = \underline{B} \, \underline{Y} \qquad (2.3.2)$$

where $\underline{Y} = (y_i)^{tr}$ $(i = 1, 2, \ldots, 12)$,

and, tr, denotes transpose, $\underline{B}$ is a matrix of order (12, 12),

D is $\frac{d}{dz}$.

The boundary conditions are

$$y_1 = y_3 = y_4 - \alpha^2 y_2 = 0 \quad \text{at } z = 0, 1. \qquad (2.3.3)$$

The method involved use of a fourth order Runge-Kutta scheme, where three linearly independent integrals of the equation (2.3.2) were found, each satisfying the boundary conditions (2.3.3) at z = 0. These were called $\underline{Y}^{\{1\}}$, $\underline{Y}^{\{2\}}$, $\underline{Y}^{\{3\}}$ with initial values $y_k^{\{i\}}$, where $y_k^{\{i\}}$ denotes the kth component of $\underline{Y}^{\{i\}}$. Because of linearity

$$\underline{Y}^{\{4\}} = \underline{Y}^{\{1\}} + C_1 \underline{Y}^{\{2\}} + C_2 \underline{Y}^{\{3\}}, \qquad (2.3.4)$$

is the eigenfunction and satisfies in (2.3.3) at z = 0, for all values of C_1 and C_2.

We now choose C_1 and C_2 such that the first two conditions of (2.3.3) are satisfied at z = 1, in other words

$$\left.\begin{aligned} y_1^{\{1\}} + C_1 y_1^{\{2\}} + C_2 y_1^{\{3\}} &= 0, \\ y_3^{\{3\}} + C_1 y_3^{\{2\}} + C_2 y_3^{\{3\}} &= 0, \end{aligned}\right\} \text{at } z = 1 \qquad (2.3.5)$$

where

$$y_1^{\{2\}} \; y_3^{\{3\}} - y_1^{\{3\}} \; y_3^{\{2\}} \neq 0 \text{ at } z = 1. \qquad (2.3.6)$$

This condition is checked in the course of the computations.

Let $$Q = \alpha^2 y_2^{\{4\}} - y_4^{\{4\}} \text{ at } z = 1. \qquad (2.3.7)$$

Then the condition that Q vanishes at z = 1 is equivalent to the usual secular determinant which leads to a relation between α and R. Applying Newton's method we obtain the new approximation for R,

$$R_{new} = R_{old} - Q/\frac{\partial Q}{\partial R}, \qquad (2.3.8)$$

and the iteration scheme is started from an initial guess R_0. This iteration method, thus provides the value of R (for given α).

Two different step sizes (δz = 0.025, 0.0125) were used to check the accuracy of the scheme (see Table 2.1).

As already stated, we are interested in the critical Rayleigh number, R_c, for the onset of instability and this is determined by the condition $\frac{dR}{d\alpha} = 0$; the corresponding value of α is denoted by α_c. We used a minimization routine to find (α_c, R_c) with α_c = 3.1163 and R_c = 1707.763.

z	h(z) (δz = 0.025)	h(z) (δz = 0.0125)
0.0	0.0	
0.025	0.06837345	0.06837346
0.05	0.13701257	0.137013265
0.075	0.20597489	0.20597500
0.1	0.27512703	0.27512716
0.125	0.34418078	0.34418093
0.15	0.41272558	0.41272573
0.175	0.48025764	0.48025779
0.2	0.54620623	0.54620638
0.225	0.60995719	0.60995734
0.25	0.67087398	0.67087413
0.275	0.72831657	0.72831671
0.3	0.78165812	0.78165824
0.325	0.83029979	0.83029999
0.35	0.8736377	0.8736886
0.375	0.91130444	0.91130452
0.4	0.94271809	0.94271815
0.425	0.96755090	0.96755090
0.45	0.98550528	0.98550531
0.475	0.99636534	0.99636536
0.5	1.	1

Table 2.1: Numerical results with different step sizes.

$\alpha = 3.116$, $R = 1707.762$

Comparison of the results given in Table 2.2 with those obtained by a different method by Chandrasekhar (1961) shows adequate agreement between the two methods. In later work we shall use the constants α_c and R_c which are to be interpreted as the final values obtained above. Finally, the neutral curve for marginal stability is constructed in the (α, R) plane in figure 2.1., and typical temperature profile and velocity profiles are given in figures 2.2 , 2.3 , and 2.4 , where the normalization condition (2.2.9) has been adopted.

α	R	Q
1.0050	5807.125	-0.3×10^{-18}
2.0010	2176.320	-0.6×10^{-18}
3.0060	1710.921	-0.2×10^{-17}
3.1163	1707.763	-0.5×10^{-17}
4.00099	1879.627	-0.4×10^{-16}
5.3680	2746.389	-0.2×10^{-15}
5.99	3405.826	-0.4×10^{-14}
6.016	2437.547	-0.6×10^{-13}
7.001	4920.349	-0.8×10^{-13}
8.001	7087.065	-0.1×10^{-12}

Table 2.2. The exact eigenvalue for the first even mode of instability.

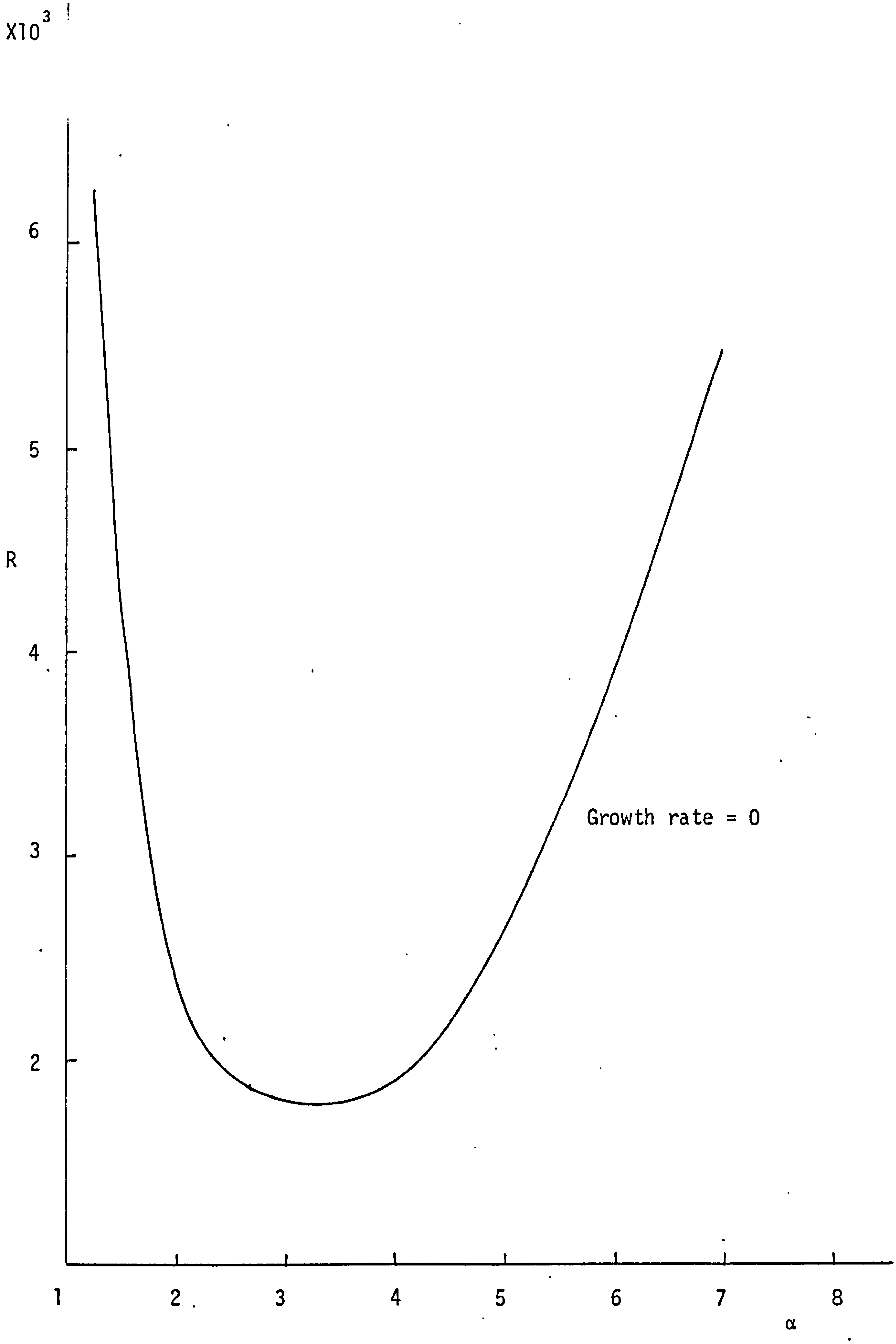

Figure 2.1: Neutral curve based on numerical results.

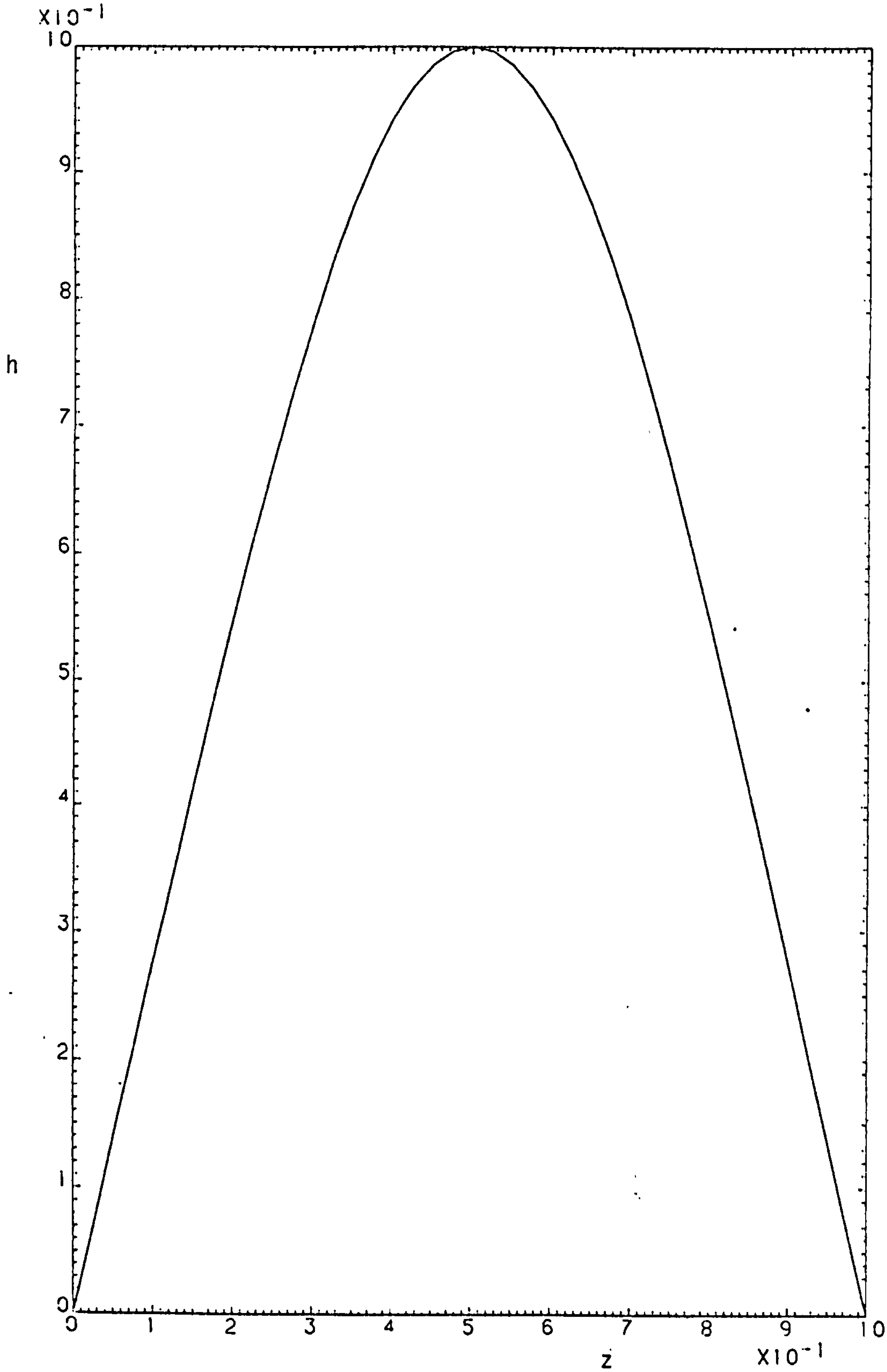

Figure 2.2: Temperature profile for α = 3.1163, R = 1707.763.

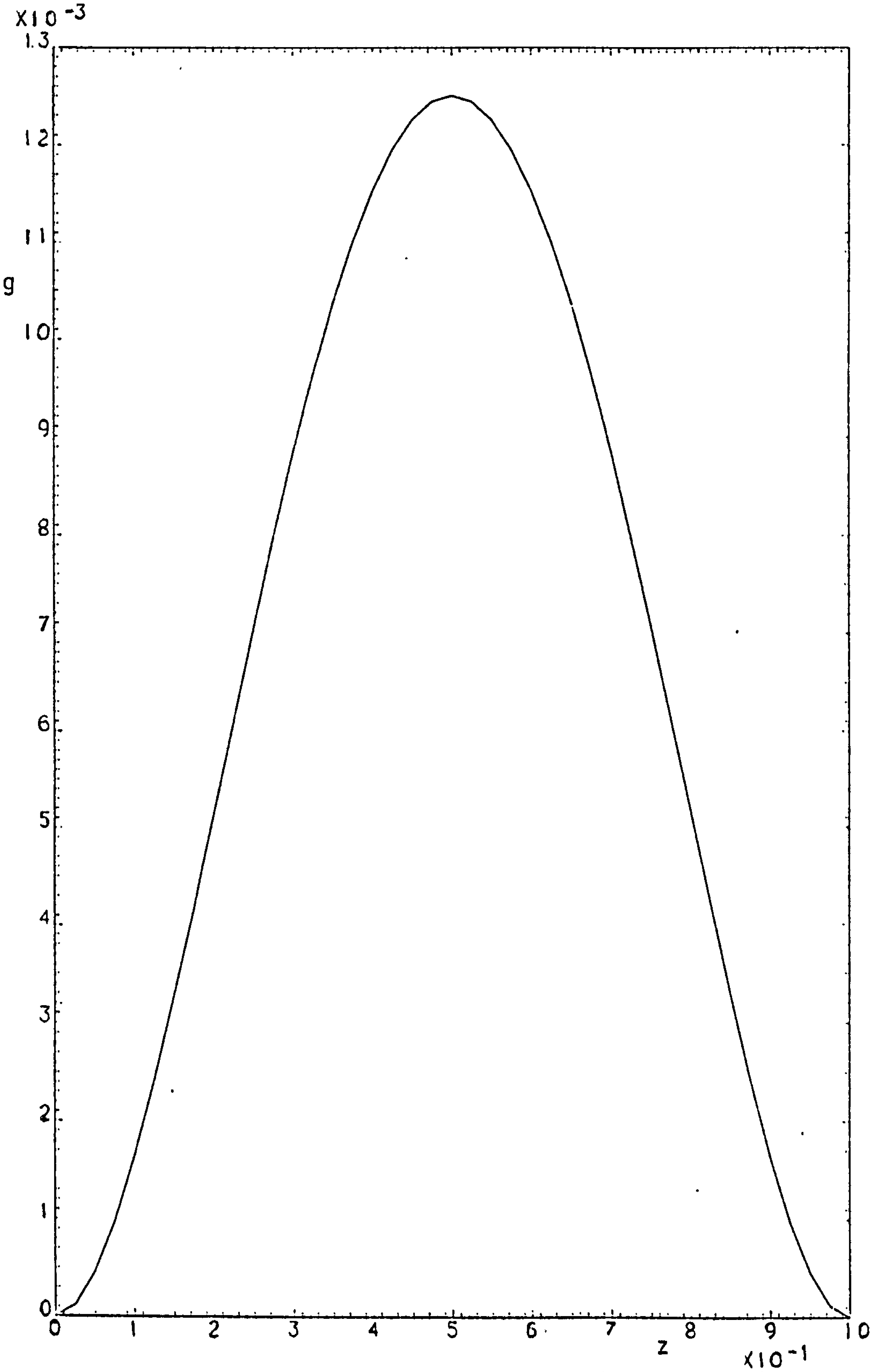

Figure 2.3: Velocity profile (w component) for α = 3.1163, R = 1707.763.

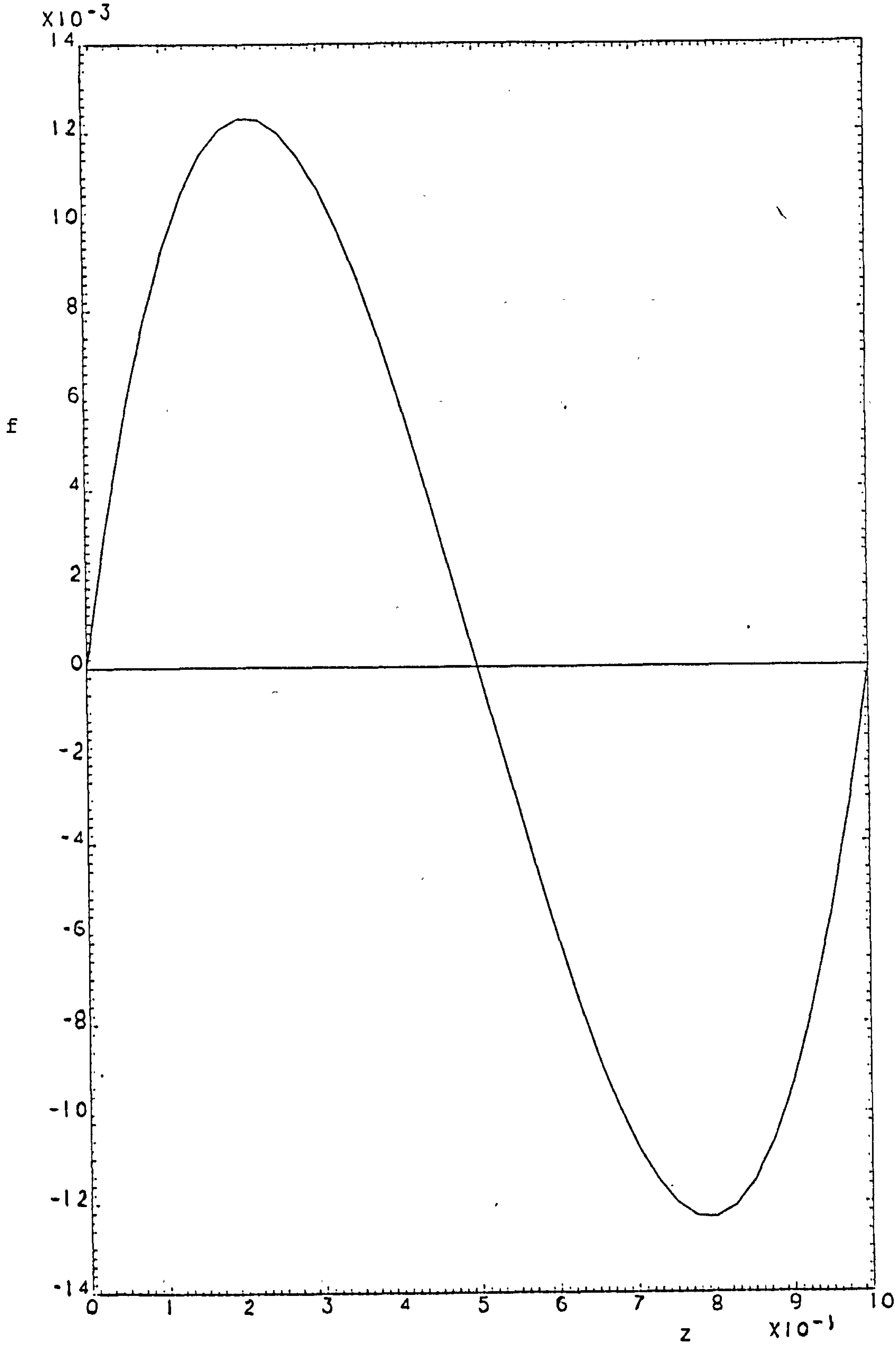

Figure 2.4: Velocity profile (u component) for $\alpha = 3.1163$, $R = 1707.763$.

2.4. Numerical Solution of Inhomogeneous Equation

In the preceding section the eignfunction, $h(z)$, satisfying (2.2.7) - (2.2.8) is obtained numerically for different values of α and R. Here we are interested in finding $\frac{\partial h}{\partial \alpha}$ with $\alpha = \alpha_c$ and $R = R_c$, since we shall use this in chapter 3. We start by deriving the system for which $\frac{\partial h}{\partial \alpha}$ is the solution.

This is done by differentiation of (2.2.3) - (2.2.4) with respect to α, and we set

$$\frac{dR}{d\alpha} = 0.$$

Let

$$\frac{\partial h}{\partial \alpha} = H_o, \quad \frac{\partial g}{\partial \alpha} = G_o, \quad \frac{\partial f}{\partial \alpha} = F_o. \tag{2.4.1}$$

From (2.2.3) - (2.2.4) and (2.4.1),

$$D^2 H_o - \alpha_c^2 H_o + R_c G_o = 2\alpha_c h, \tag{2.4.2}$$

$$D^4 G_o - 2\alpha_c^2 D^2 G_o + \alpha_c^4 G_o - \alpha_c^2 H_o = 4\alpha_c D^2 g - 4\alpha_c^3 g + 2\alpha_c h. \tag{2.4.3}$$

The boundary conditions are

$$G_o = D G_o = H_o = 0 \quad \text{at } z = 0, 1 \tag{2.4.4}$$

If, in addition, h is assumed to satisfy (2.2.9) then $H_o = 0$ at $z = \frac{1}{2}$, and this renders the solution of (2.4.2) - (2.4.4) unique. Otherwise, the solution of H_o can contain an arbitrary multiple of h(z).

In contrast to (2.2.2) - (2.2.4) , the system (2.4.2) - (2.4.4) is inhomogeneous. By construction, it must have the solution $H_o = \frac{\partial h}{\partial \alpha}$, and in combination with the basic linear system (2.2.2) - (2.2.4) gives an integral relation between g and h as follows.

We multiply equation (2.4.2) by h and (2.2.3) by H_o; subtraction and integration from z = 0 to z = 1, now yield

$$\int_o^1 (gH_o - h\, G_o)\, dz = -\, 2\alpha_c/R_c \int_o^1 h^2 dz, \qquad (2.4.5)$$

and from a similar treatment of the other equations

$$-\int_o^1 (gH_o - h\, G_o) dz = 2 \int_o^1 g(2D^2 g - 2\alpha_c^2\, g+h)/\alpha_c dz. \qquad (2.4.6)$$

Combining (2.4.5) and (2.4.6) we find that

$$I_1 = \int_o^1 \{2g(2D^2 g - 2\alpha_c^2\, g+h)/\alpha_c - 2\alpha_c h^2/R_c\}\, dz = 0 \qquad (2.4.7)$$

where

$$g = (\alpha_c^2\, h - D^2 h)/_{R_c}$$

The condition (2.4.7) is the consistency condition which is satisfied by h(z) when $\alpha = \alpha_c$ and $R = R_c$, and this was checked by Simpson's method in the numerical calculations.

Eliminating, $G_0(z)$ between (2.4.2) and (2.4.3), or from differentiation of (2.2.7) - (2.2.8) with respect to α, we obtain:

$$(D^2-\alpha_c^2)^3 H_0+R_c\alpha_c^2 H_0=2\alpha_c\{3D^4h-6\alpha_c^2D^2h+(3\alpha_c^4 - R_c)h\}, \quad (2.4.8)$$

$$H_0 = D^2H_0 = 0. \text{ at } z = 0, 1, \quad (2.4.9)$$

$$D(D^2-\alpha_c^2)H_0 = 2\alpha_c h \text{ at } z = 0, 1, \quad (2.4.10)$$

and also we set $H_0 = 0$ at $z = \frac{1}{2}$; the solution for H_0 is even about $z = \frac{1}{2}$.

The numerical method used here is similar to that used for (2.2.7), but in this case (inhomogeneous) we have to add one particular solution of (2.4.8) to the complementary solutions, and this is done by setting

$$\underline{Y} = \underline{Y}_c + \underline{Y}_p$$

where

$$\underline{Y}_c = \underline{Y}^{\{1\}} + A\,\underline{Y}^{\{2\}} + B\,\underline{Y}^{\{3\}},$$

the constants A and B are found from the condition (2.4.9) at z = 1, and the automatic satisfaction of the boundary condition (2.4.10) at z = 1, provides a check on the calculations.

It should be noted that in order to apply the fourth order Runge-Kutta scheme to (2.4.8) with step size δz, the function, h, on the right-hand side of the equation (2.4.8) is first calculated at intermediate mesh points equivalent to the division of one step into intervals $\delta z/6$, $\delta z/3$, $\delta z/2$. The solution for H_o is included in figure 4.4 below.

CHAPTER 3

Non-Parallel Plane Problem in Cylindrical Geometry

In this chapter we are interested in the Benard convection problem associated with the lower plane being of the form $z = \varepsilon^2 G(\varepsilon r)$, in cylindrical polar coordinates (r, ϕ, z).

A two-dimensional version of this problem has been analysed by Eagles (1980). We refer to the new problem as the non-parallel plane problem in contrast to the parallel plane problem of chapter 2, where the lower plane is given by $z = 0$; in both cases, the upper boundary is $z = 1$. It is assumed that $G(\varepsilon r)$ is bounded and for ε sufficiently small the surfaces do not intersect (Figure 3.1). We choose $G(0) = 0$ and G positive for $r = \infty$, the excess of the Rayleigh number above R_c is assumed $O(\varepsilon^2)$, and the deviation of the lower surface from the planar case is $O(\varepsilon^2)$.

The space co-ordinates, velocity components, pressure and temperature are made dimensionless with respect to the fluid depth, d, at $z = 0$, in the usual manner ((2.1) - (2.4)) and the velocity component v is taken to be zero.

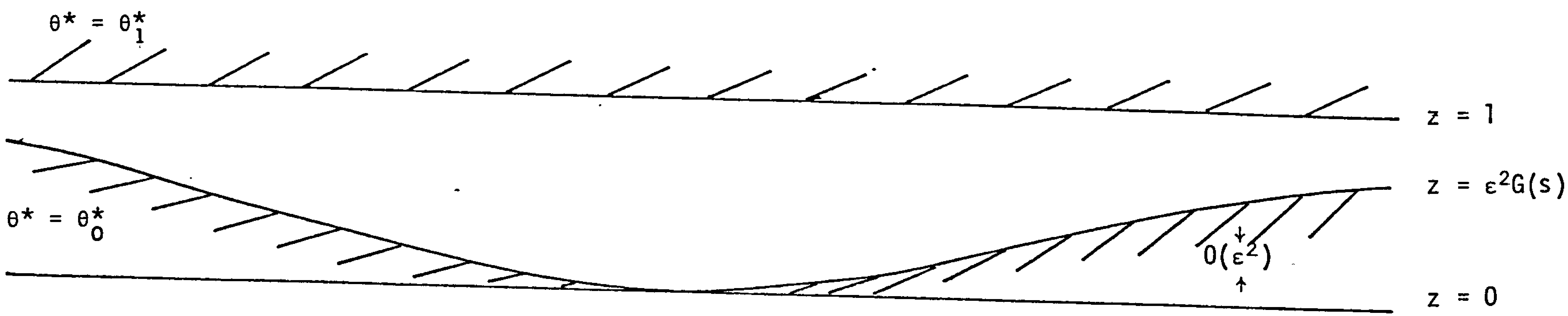

Figure 3.1: Sketch of the configuration.

3.1. The Governing Equations of Motion

The full set of equations in the Oberbeck-Boussinesq approximation for viscous, incompressible, axisymmetric flow can be expressed as follows:

$$u_r + \frac{u}{r} + w_z = 0, \qquad (3.1.1)$$

$$u_t - \sigma(\nabla^2 u - \frac{u}{r^2}) + p_r = -(uu_r + wu_z) \qquad (3.1.2)$$

$$w_t - \sigma(\nabla^2 w + \theta) + p_z = -(uw_r + ww_z), \qquad (3.1.3)$$

$$\theta_t - \nabla^2\theta - Rw = -(u\theta_r + w\theta_z), \qquad (3.1.4)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \qquad (3.1.5)$$

where

$$u_r = \frac{\partial u}{\partial r},\ u_z = \frac{\partial u}{\partial z},\ \ldots,\ \text{etc.}$$

We define the slow variable, s, by $s = \varepsilon r$, where ε is a small parameter.

The boundary conditions are

$$u = w = \theta = 0 \quad \text{at } z = 1,\ \varepsilon^2 G(s). \qquad (3.1.6)$$

3.2. Analysis of the Base Flow and The Steady-State Non-Periodic Solution

For steady flow in the parallel plane problem, which is given by $\varepsilon = 0$, in the above, there is a solution of the form

$$\dot{u} = 0, \quad \dot{w} = 0,$$
$$\theta = 0, \quad p = \text{constant}$$

for $\varepsilon \neq 0$ we denote the velocity components of the steady base flow by (U_s, W_s) and pressure, temperature by p_s, θ_s respectively.

The boundary conditions for the base flow are

$$\begin{aligned} U_s &= W_s = \theta_s = 0 \quad \text{at} \quad z = 1, \\ U_s &= W_s = 0, \ \theta_s = R\varepsilon^2 G(s) \ \text{at} \ z = \varepsilon^2 G. \end{aligned} \tag{3.2.1}$$

We also add the condition that

$$U_s \to 0 \text{ and } W_s \to 0 \text{ as } s \to \infty. \tag{3.2.2}$$

For ε small but different from zero we expand the perturbations U_s, W_s, θ_s and p_s in powers of ε and write

$$\begin{aligned} U_s &= \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 \ \ldots\ldots., \\ W_s &= \varepsilon W_1 + \varepsilon^2 W_2 + \varepsilon^3 W_3 \ \ldots\ldots., \\ \theta_s &= \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \varepsilon^3 \theta_3 \ \ldots\ldots., \\ p_s &= \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 \ \ldots\ldots., \end{aligned} \tag{3.2.3}$$

The functions U_i, W_i, θ_i, p_i for i = 1, 2, ..., are considered to depend on the two variables z and s.

Substituting the form of expansions (3.2.3) into (3.1.1) - (3.1.4), replacing $\frac{\partial}{\partial r}$ by $\varepsilon \frac{\partial}{\partial s}$, and equating powersof ε, we obtain a set of partial differential equations as follows.

From (3.1.1), we find that

$$\frac{\partial W_1}{\partial z} = 0, \; U_i + s \left(\frac{\partial U_i}{\partial s} + \frac{\partial W_{i+1}}{\partial z} \right) = 0 \; (i = 1, 2, 3). \quad (3.2.4)$$

From (3.1.2) and (3.1.3)

$$\frac{\partial^2 U_1}{\partial z^2} = 0, \quad \sigma \frac{\partial^2 U_{i+1}}{\partial z^2} - \frac{\partial p_i}{\partial s} = 0 \quad (i = 1, 2), \quad (3.2.5)$$

$$\frac{\partial p_i}{\partial z} - \sigma\, \theta_1 = 0, \quad \frac{\partial p_i}{\partial z} - \sigma\left(\theta_i + \frac{\partial^2 W_i}{\partial z^2}\right) = 0 \; (i=2,3). \quad (3.2.6)$$

Finally from (3.1.4)

$$\theta_1 = 0, \; \frac{\partial^2 \theta_i}{\partial z^2} + R\, W_i = 0 \; (i = 2, 3), \quad (3.2.7)$$

$$\frac{\partial^2 \theta_2}{\partial s^2} + \frac{1}{s} \frac{\partial \theta_2}{\partial s} + \frac{\partial^2 \theta_4}{\partial z^2} + RW_4 - W_2 \frac{\partial \theta_2}{\partial z} = 0. \quad (3.2.8)$$

Now we define the boundary conditions on U_i, θ_i, W_i,

(i = 1, 2, 3, ...)

From (3.1.6)

$$U_i = W_i = \theta_i = 0 \quad \text{at } z = 1. \qquad (3.2.9)$$

The boundary conditions on $z = \varepsilon^2 G$, for U_i, W_i, θ_i ($i = 1, 2, \ldots$), are given by expansion of U_s, W_s, θ_s about $z = 0$, and the details of this work aregiven for U_s only; we have

$$U_s(s, \varepsilon^2 G) = \varepsilon\, U_1(s,0)+\varepsilon^2\, U_2(s,0)+\varepsilon^3 (GU_1z + U_3)_{z=0} + \varepsilon^4(G\, U_{2_z}+U_4)_{z=0} + \ldots . \qquad (3.2.10)$$

Therefore, since $U_s\ (s, \varepsilon^2\, G) = 0$,

$$U_1 = U_2 = 0, \qquad \text{on } z=0 \qquad (3.2.11)$$
$$GU_{1_z}+U_3=GU_{2_z}+U_4=0,$$

where

$$U_{iz} = \frac{\partial u_i}{\partial z} \quad (i = 1, 2, 4).$$

Similar arguments provide boundary conditions for W_i and θ_i on $z = 0$.

The results are

$$W_1 = W_2 = G\, W_{1_z} + W_3 = G\, W_{2z} + W_4 = 0, \qquad \text{on } z = 0$$
$$\theta_1 = G\theta_{1z} + \theta_3 = G\theta_{2z}+\theta_4=\theta_2-RG = 0, \qquad (3.2.12)$$

where

$$\theta_{iz} = \frac{\partial \theta_i}{\partial z} \quad \text{and} \quad W_{iz} = \frac{\partial w_i}{\partial z} \quad (i = 1, 2).$$

The solutions of the equations (3.2.4) - (3.2.8) subject to the boundary conditions (3.2.10) - (3.2.12) are listed below

$$U_1 = U_2 = 0,$$

$$W_1 = W_2 = W_3 = 0, \tag{3.2.13}$$

$$\theta_1 = \theta_3 = p_1 = 0,$$

$$\theta_2 = R(1 - z)\, G, \tag{3.2.14}$$

$$p_2 = \sigma R\, G(z - z^2/_2) + B(s), \tag{3.2.15}$$

$$U_3 = R(z^3 - z^4/_{24} - z/_8)\,\frac{dG}{ds} + \sigma^{-1}(z^2 - z)/_2\,\frac{dB}{ds}, \tag{3.2.16}$$

$$W_4 = Rf_1(z)\,\left(\frac{d^2G}{ds^2} + \frac{1}{s}\,\frac{dG}{ds}\right) + \sigma^{-1}\, f_2(z)\left(\frac{d^2B}{ds^2} + \frac{1}{s}\,\frac{dB}{ds}\right), \tag{3.2.17}$$

where f_1 and f_2 are given by

$$f_1 = z^5/_{120} + z^2/_{16} - z^4/_{24},$$

$$f_2 = z^2/_2 - z^3/_3. \tag{3.2.18}$$

In obtaining these solutions, p_2 was first found in the form (3.2.15), where B(s) is an unknown function at this stage.

In order that $U_3 \to 0$ as $s \to \infty$, we see from (3.2.16) that

$$\frac{dG}{ds} \to 0 \text{ as } s \to \infty , \qquad (3.2.19)$$

and

$$\frac{dB}{ds} \to 0 \text{ as } s \to \infty . \qquad (3.2.20)$$

From the condition $W_4 = 0$ on $z = 1$, and (3.2.19) we find that

$$\frac{dB}{ds} + (7\sigma R/_{20}) \frac{dG}{ds} = 0, \qquad (3.2.21)$$

and thus

$$B = (-\sigma 7R/_{20}) G(s) + \text{const.} \qquad (3.2.22)$$

Substituting (3.2.22) into (3.2.15) - (3.2.17), provides the explicit forms of p_2, U_3, W_4. Summarising, the expansions for the base flow, pressure and temperature can be written as follows:

$$U_s = \varepsilon^3 R(1-z^4/_{24} + z^3/_{16} - 7z^2/_{40}+z/_{20})\frac{dG}{ds} +.., \qquad (3.2.23)$$

$$W_s = \varepsilon^4(z^5 - 5z^4 + 7z^3 - 3z^2)(\frac{d^2G}{ds^2} + \frac{1}{s}\frac{dG}{ds})R/_{120}+... (3.2.24)$$

$$\theta_s = R\varepsilon^2(1 - z) G + ... , \qquad (3.2.25)$$

$$p_s = R\sigma\varepsilon^2(z - z^2/_2 - 7/_{20}) G + ..., \qquad (3.2.26)$$

It is noted that as $s \to \infty$, we have zero fluid velocity and just a linear temperature variation.

3.3. The Disturbance Equations in Matrix Form

We continue with the case where the equation of the lower boundary is

$$z = \varepsilon^2 G(s)$$

where, $s = \varepsilon r$. In equations (3.1.1) - (3.1.4) we set

$$\begin{aligned} u &= U_s + \hat{U}\,(r, z, t), \\ W &= W_s + \hat{W}\,(r, z, t), \\ \theta &= \theta_s + \hat{\theta}\,(r, z, t), \\ p &= p_s - \hat{p}\,(r, z, t). \end{aligned} \tag{3.3.1}$$

(where the minus sign with the perturbed pressure is merely for convenience) to obtain the equations for small disturbances $\hat{U}$, $\hat{W}$, $\hat{\theta}$, $\hat{p}$; these are assumed to be suffciently small for non linear products of these terms to be neglected in the governing equations. The functions U_s, W_s, θ_s, p_s are the steady solutions of (3.1.1) - (3.1.4) which are given in (3.2.23) - (3.2.26).

Upon substitution of (3.3.1) into (3.1.1) - (3.1.4) we then obtain the linearised system,

$$\frac{\partial \hat{U}}{\partial r} + \frac{\hat{U}}{r} + \frac{\partial \hat{W}}{\partial z} = 0, \tag{3.3.2}$$

$$\frac{\partial \hat{U}}{\partial t} + U_s \frac{\partial \hat{U}}{\partial r} + \hat{U} \frac{\partial U_s}{\partial r} + \hat{W} \frac{\partial U_s}{\partial z} + W_s \frac{\partial \hat{U}}{\partial z} = \hat{p}_r + \sigma\left(\nabla^2 \hat{U} - \frac{\hat{U}}{r^2}\right) \tag{3.3.3}$$

$$\frac{\partial \hat{W}}{\partial t} + U_s \frac{\partial \hat{W}}{\partial r} + \hat{U} \frac{\partial W_s}{\partial r} + \hat{W} \frac{\partial W_s}{\partial z} + W_s \frac{\partial \hat{W}}{\partial z} = \hat{p}_z + \sigma\left(\nabla^2 \hat{W} + \hat{\theta}\right) \tag{3.3.4}$$

$$\frac{\partial \hat{\theta}}{\partial t} + U_s \frac{\partial \hat{\theta}}{\partial r} + \hat{U} \frac{\partial \theta_s}{\partial r} + \hat{W} \frac{\partial \theta_s}{\partial z} + W_s \frac{\partial \hat{\theta}}{\partial z} = R \hat{W} + \nabla^2 \hat{\theta} \tag{3.3.5}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

and

$$\hat{U} = \hat{\theta} = \hat{W} = 0 \text{ at } z = 1, \varepsilon^2 G(s). \tag{3.3.6}$$

We now introduce the notation

$$\tilde{U} = \frac{\partial \hat{U}}{\partial z}, \quad \tilde{\theta} = \frac{\partial \hat{\theta}}{\partial z} \tag{3.3.7}$$

and in equation (3.3.3) we write

$$\nabla^2\hat{U} = \frac{\partial^2\hat{U}}{\partial r^2} + \frac{1}{r}\frac{\partial\hat{U}}{\partial r} + \frac{\partial\tilde{U}}{\partial z}. \tag{3.3.8}$$

Then (3.3.3) expresses $\frac{\partial\tilde{U}}{\partial z}$ in terms of $\hat{U}$, $\tilde{U}$, $\hat{W}$, $\hat{p}$ and their derivatives with respect to r and t. We next note that the derivative with respect to z of (3.3.2), may be written as

$$\frac{\partial^2\hat{W}}{\partial z^2} = -\left(\frac{\tilde{U}}{r} + \frac{\partial\tilde{U}}{\partial r}\right). \tag{3.3.9}$$

The equations (3.3.2) - (3.3.6) may be written in the following forms:

$$\frac{\partial\tilde{U}}{\partial z} = \sigma^{-1} W_s \tilde{U} - \sigma^{-1}\frac{\partial\hat{p}}{\partial r} + \left(\sigma^{-1}U_s\frac{\partial}{\partial r} + \sigma^{-1}\frac{\partial U_s}{\partial r} + \frac{1}{r^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\hat{U} + \sigma^{-1}\frac{\partial U_s}{\partial z}\hat{W} + \sigma^{-1}\frac{\partial\hat{U}}{\partial t} \tag{3.3.10}$$

$$\frac{\partial\hat{p}}{\partial z} = \sigma\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)\tilde{U} + \left(\frac{\partial W_s}{\partial r} - \frac{W_s}{r} - W_s\frac{\partial}{\partial r}\right)\hat{U} + \left(U_s\frac{\partial}{\partial r} + \frac{\partial W_s}{\partial z} - \sigma\frac{\partial^2}{\partial r^2} - \frac{\sigma}{r}\frac{\partial}{\partial r}\right)\hat{W} + \frac{\partial\hat{W}}{\partial t} + \sigma\hat{\theta}$$

$$\frac{\partial\tilde{\theta}}{\partial z} = W_s\tilde{\theta} + \frac{\partial\theta_s}{\partial r}\hat{U} + \frac{\partial\theta_s}{\partial z}\hat{W} + \left(U_s\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\hat{\theta} + \frac{\partial\hat{\theta}}{\partial t} - R\hat{W},$$

$$\frac{\partial\hat{U}}{\partial z} = \tilde{U}$$

$$\frac{\partial \hat{W}}{\partial z} = - \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) \hat{U},$$

$$\frac{\partial \hat{\theta}}{\partial z} = \tilde{\theta}.$$

In our analysis we shall ignore powers of ε^n for $n \geq 3$. With this assumption and from (3.2.23) - (3.2.26) we observe that

$$U_s = 0, \; W_s = 0, \quad \theta_s = \varepsilon^2 R \, G(1 - z). \tag{3.3.11}$$

Now we introduce the extended flow vector

$$\underline{U} = \left[\tilde{U}, \frac{\partial \hat{p}}{\partial r}, \frac{\partial \tilde{\theta}}{\partial r}, \hat{U}, \frac{\partial \hat{W}}{\partial r}, \frac{\partial \hat{\theta}}{\partial r} \right]^{tr} \tag{3.3.12}$$

where tr denotes the transpose and $\tilde{U}$, $\tilde{\theta}$ are given in (3.3.7). Substitution of (3.3.11) in equations (3.3.10) and differentiation of

$\frac{\partial \hat{p}}{\partial z}$, $\frac{\partial \hat{\theta}}{\partial z}$, $\frac{\partial \hat{W}}{\partial z}$, $\frac{\partial \tilde{\theta}}{\partial z}$ with respect to the variable, r, shows that the equations (3.3.10) may be written as

$$\frac{\partial \tilde{U}}{\partial z} = \sigma^{-1} \frac{\partial \hat{p}}{\partial r} - L \, \tilde{U} + \sigma^{-1} \frac{\partial \hat{U}}{\partial t},$$

$$\frac{\partial^2 \hat{p}}{\partial z \partial r} = \sigma(L(\tilde{U} + \frac{\partial \hat{W}}{\partial r}) + \frac{\partial \hat{\theta}}{\partial r}) + \frac{\partial^2 \hat{W}}{\partial r \partial t},$$

$$\frac{\partial^2 \tilde{\theta}}{\partial z \partial r} = -R \frac{\partial \hat{W}}{\partial r} - L (\frac{\partial \hat{\theta}}{\partial r}) + \frac{\partial^2 \hat{\theta}}{\partial r \partial t} + \frac{\partial \hat{W}}{\partial r} \cdot \frac{\partial \theta_s}{\partial z}, \tag{3.3.13}$$

$$\frac{\partial \hat{U}}{\partial z} = \tilde{U},$$

$$\frac{\partial^2 \hat{W}}{\partial z \partial r} = -L\hat{U},$$

$$\frac{\partial^2 \hat{\theta}}{\partial r \partial z} = \frac{\partial \tilde{\theta}}{\partial r},$$

where L is a linear operator

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}. \tag{3.3.14}$$

This formulation enables us to write the equations (3.3.13) in matrix form, which is convenient for later manipulation. The form is similar to that used by Eagles (1980). Now we replace $\hat{p}$ by $\sigma \hat{P}$ in (3.3.13) and it is easy to show that

$$\frac{\partial \underline{U}}{\partial z} = \underline{A} \; \underline{U} + \underline{B} \; \frac{\partial \underline{U}}{\partial t}, \tag{3.3.15}$$

where

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & -L & 0 & 0 \\ L & 0 & 0 & 0 & L & 1 \\ 0 & 0 & 0 & 0 & X & -L \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -L & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad (3.3.16)$$

and

$$X = (1 + \varepsilon^2 G)R \; .$$

Also

$$\underline{B} = \begin{bmatrix} \underline{0} & \begin{matrix} \sigma^{-1} & 0 & 0 \\ 0 & \sigma^{-1} & 0 \\ 0 & 0 & 1 \end{matrix} \\ \underline{0} & \underline{0} \end{bmatrix} \qquad (3.3.17)$$

where $\underline{0}$ is the zero matrix of order (3,3). We can write the matrix $\underline{A}$ in the form

$$\underline{A} = \underline{A}_1 - L\,\underline{A}_2 + R\,(1 + \varepsilon^2 G)\,\underline{A}_3 \; , \qquad (3.3.18)$$

where

$$\underline{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & & & \\ 0 & 0 & 0 & & \underline{0} & \\ 0 & 0 & 1 & & & \end{bmatrix}$$

$$\underline{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ & \underline{0} & & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix} \qquad (3.3.19$$

$$\underline{A}_3 = \begin{bmatrix} & & & 0 & 0 & 0 \\ & \underline{0} & & 0 & 0 & 0 \\ & & & 0 & -1 & 0 \\ & \underline{0} & & & \underline{0} & \end{bmatrix}$$

From(3.3.6)

$$\hat{U} = \frac{\partial\hat{W}}{\partial r} = \frac{\partial\hat{\theta}}{\partial r} = 0 \qquad \text{at } z = 1.$$

On $z = \varepsilon^2 G$, we have, for example, $\frac{\partial\hat{W}}{\partial r} = 0$, so that

$$\frac{\partial\hat{W}}{\partial r} = \frac{\partial\hat{W}(r,0)}{\partial r} + \varepsilon^2 G \frac{\partial^2\hat{W}(r,0)}{\partial z \partial r} + \ldots\ldots, \tag{3.3.20}$$

and so to the order of approximation considered here,

$$\frac{\partial\hat{W}}{\partial r} = 0 \qquad \text{at } z = 0. \tag{3.3.21}$$

The complete set of boundary conditions can be conveniently labelled as follows:

γ_1 : The last three components of $\underline{U}$ are zero at $z = 1$ and $z = 0$,

or alternatively

$$\gamma_1 : \underline{C}\ \underline{U} = 0 \quad \text{at } z = 0 \text{ and } z = 1 \tag{3.3.22}$$

where

$$\underline{C} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & I_3 \end{bmatrix}$$

and I_3 is a unit matrix of order (3,3).

3.4. Expansion Procedure and Outer Solution

If $G(s) = 0$ (or $\varepsilon = 0$) then we have the standard linear parallel plane problem of chapter 2, so that we may expect that the critical disturbance of the plane problem will have a corresponding perturbed solution in the non-parallel case.

We assume $G(s)$ remains $O(1)$ for $r = O(\varepsilon^{-1})$, and look for a steady solution of (3.3.15) in this region of the form

$$\underline{U} = e^{i\alpha_c r}\, \underline{V}\,(z, s) + \text{C.C.}, \tag{3.4.1}$$

with $$R = R_c + \varepsilon^2\beta_o, \tag{3.4.2}$$

where ε is fixed by the size of the depression in the lower surface ($z = \varepsilon^2 G$), and β_o is an arbitrary parameter which represents an $O(\varepsilon^2)$ variation in R about R_c, and the symbol C.C. denotes the complex conjugate. Now we expand the complex function $\underline{V}$ in powers of ε:

$$\underline{V} = \varepsilon \underline{F}_1(z,s) + \varepsilon^2 \underline{F}_2(z,s) + \varepsilon^3 \underline{F}_3(z,s) + \ldots\ldots \tag{3.4.3}$$

On substituting (3.4.1) - (3.4.3) in (3.3.15) - (3.3.18), and equating powers of ε^n, we obtain a set of partial differential equations:

$$L_o \underline{F}_1 = 0; \qquad \gamma_1 \qquad (3.4.4)$$

$$L_o \underline{F}_2 = -i\alpha_c \underline{A}_2 L_1 \underline{F}_1 \; ; \gamma_1 \qquad (3.4.5)$$

$$L_o \underline{F}_3 = (R_c G+\beta_o) \underline{A}_3 \underline{F}_1 - i\alpha_c \underline{A}_2 L_1 . \underline{F}_2 - \underline{A}_2 \underline{L}_2 \underline{F}_1 ; \qquad (3.4.6)$$

$$\underline{C} (\underline{F}_3 + G \underline{F}_{1_z}) = 0 \text{ at } z = 0, \; \underline{C} \underline{F}_3 = 0 \text{ at } z = 1$$

where

$$L_o = \frac{\partial}{\partial z} - (\underline{A}_1 + \alpha_c^2 \underline{A}_2 + R_c \underline{A}_3), \qquad (3.4.7)$$

and

$$L_1 = \frac{\partial}{\partial s} + \frac{1}{2s},$$

$$L_2 = \frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} - \frac{1}{s^2}, \qquad (3.4.8)$$

are linear operators and γ_1 is given by (2.3.22).

In (3.4.4.) - (3.4.6), the boundary conditions on the lower surface, $z = \varepsilon^2 G$, are obtained by means of a Taylor expension of $\underline{V}$ about z = 0, using the fact that the last three components of $\underline{V}$ are zero on $z = \varepsilon^2 G$. From (3.4.3.) we obtain on z = 0

$$f_{1j} = f_{2j} = 0, \qquad j = 4, 5, 6$$

$$G \frac{\partial f_{1j}}{\partial z} + f_{3j} = 0. \qquad (3.4.9)$$

On z = 1,

$$f_{ij} = 0 \qquad (i = 1, 2, 3)$$

$$(j = 4, 5, 6)$$

where f_{ij} denotes the jth component of $\underline{F}_i$ (z, s).

3.5. Adjoint Operator and Adjoint Eigenfunction

To solve equations (3.4.4) to (3.4.6) we introduce the idea of the adjoint operator (see for example Eagles (1980)).

The adjoint operator is defined by

$$\underline{\ell}^* = \frac{d}{dz} + \underline{A}^{tr},$$

and

$$\underline{\ell}^* \underline{X} = 0, \gamma_2 \qquad (3.5.1)$$

where the boundary conditions denoted by γ_2 are that the first three components of $\underline{X}$ are zero at $z = 0$ and $z = 1$, and, tr, denotes the transpose.

Let $\underline{X}$ be the nontrivial solution of (3.5.1), then

$$\underline{\ell}\ \underline{Y} = \underline{W}(z) \quad ; \gamma_1 \tag{3.5.2}$$

has a solution if and only if

$$\int_0^1 \underline{X}\ \underline{W}\ dz = 0, \tag{3.5.3}$$

where $\underline{\ell} = \frac{d}{dz} - \underline{A}.$

γ_1 is given in (3.3.22) and $\underline{X}$ is an adjoint eigenfunction. The condition (3.5.3) is called the adjoint condition and we shall use it below.

The proof of (3.5.3) in conjunction with (3.5.1) and (3.5.2) is first given.

Let $\underline{\ell}\ \underline{F} = 0 \ ; \gamma_1$ (3.5.4)

and $\underline{\ell}^*\ \underline{E} = 0 \ ; \gamma_2$ (3.5.5)

where $\underline{F} \neq 0, \underline{E} \neq 0.$

The inner product of this vector space is defined by

$$<E^{tr}, \underline{\ell}\ \underline{Y}> = <\underline{E}^{tr}, \underline{W}> = \int_0^1 \underline{E}^{tr}\ \underline{W}\ dz. \tag{3.5.6}$$

But $\frac{d\underline{Y}}{dz} - \underline{A}\ \underline{Y} = \underline{W}\,(z)$

from (3.5.6.) and integration by parts gives

$$\int_0^1 \underline{E}^{tr}\ \underline{W}\ dz = \underline{E}^{tr}\ \underline{Y}\Big|_0^1 - \int_0^1 \{\frac{d\underline{E}^{tr}}{dz}\ \underline{Y} + \underline{E}^{tr}\ \underline{A}\ \underline{Y}\}\ dz. \qquad (3.5.7)$$

Since the integral in (3.5.7) is a constant and not a vector, it will have the same value as its transpose, which is

$$I = \int_0^1 \{\frac{d\underline{E}^{tr}}{dz}\ \underline{Y} + \underline{E}^{tr}\ \underline{A}\ \underline{Y}\}^{tr}\ dz.$$

From matrix algebra we find that

$$I = \int_0^1 \underline{Y}^{tr}\ (\frac{d\underline{E}}{dz} + \underline{A}^{tr}\ \underline{E})\ dz = \langle \underline{Y}^{tr}, \underline{\ell}^*\ \underline{E} \rangle. \qquad (3.5.8)$$

From (3.5.5), $\underline{\ell}^*\ \underline{E} = 0$, therefore

$$\int_0^1 \underline{E}^{tr}\ \underline{W}\ dz = \underline{E}^{tr}\ \underline{Y}\ \Big|_0^1, \qquad (3.5.9)$$

where $\underline{E}^{tr}\ \underline{Y}\ \Big|_0^1$, denotes the matrix product

$$\sum_{J=1}^{6} e_j\ y_j\Big|_0^1,$$

and the values of e_j and y_j are given by the boundary conditions, γ_1, and γ_2 which imply that

$$\underline{E}^{tr}\ \underline{Y}\ dz\ \Big|_0^1 = \int_0^1 \underline{E}^{tr}\ \underline{W}\ dz = 0\ .$$

3.6. Solution of the Disturbance Equations and Derivation of the Amplitude Equation.

A general solution of (3.4.4) can be written as

$$\underline{F}_1(z,s) = \bar{A}(s)\ \underline{f}_1(z), \tag{3.6.1}$$

where

$$L_o\ \underline{f}_1(z) = 0;\ \gamma_1 \tag{3.6.2}$$

and $\underline{f}_1(z)$ is the critical eigenfunction of the standard parallel plane problem, α_c and R_c being given by Table 2.2. Let f_{1j} ($j = 1, 2, \ldots 6.$) denote the jth component of $\underline{f}_1(z)$, the solution f_{1j} is given in (2.2.2) - (2.2.4), and thus,

$$f_{14} = f(z),\ f_{15} = g(z),\ f_{16} = h(z). \tag{3.6.3}$$

The solution given in (3.6.1) also contains an amplitude function, $\bar{A}(s)$, which is determined by asolvability condition obtained at higher order ($O(\varepsilon^2)$). Using (3.6.1), equation (3.4.5) becomes

$$L_o\ \underline{F}_2 = -i\alpha_c\ \underline{A}_2\ L_1\ \{\bar{A}(s)\}\ \underline{f}_1\ ;\ \gamma_1 \tag{3.6.4}$$

and has a solution for $\underline{F}_2$, if the adjoint condition:

$$i\alpha_c\ L_1\{\bar{A}(s)\} \int_0^1 \underline{f}^{a,tr}(\underline{A}_2\ \underline{f}_1)\ dz = 0, \tag{3.6.5}$$

is satisfied. This is the case, since from the parallel plane problem (Eagles 1980)

$$\int_0^1 \underline{f}^{a,tr}(\underline{A}_2\underline{f}_1)\, dz = 0. \tag{3.6.6}$$

In fact (3.6.6) is equivalent to the solvability condition which we derived in (2.4.7). From the boundary conditions on $\underline{F}_2$, (3.4.5), and the right-hand side of (3.4.5) $\underline{F}_2$ may be expressed as

$$\underline{F}_2(z, s) = -i\left(\frac{d\bar{A}}{ds} + \frac{\bar{A}}{2s}\right)\underline{f}_2 + \bar{A}_1\,\underline{f}_1, \tag{3.6.7}$$

where $\bar{A}_1(s)$ is an unknown function at this stage, and

$$L_0\,\underline{f}_2 = 2\alpha_c\,\underline{A}_2\,\underline{f}_1 ;\quad \gamma_1. \tag{3.6.8}$$

It should be noted that the system (3.6.8) is equivalent to the inhomogeneous system (2.4.1) - (2.4.3), and we find that

$$f_{24} = F_0(z),\; f_{25} = G_0(z),\; f_{26} = H_0(z), \tag{3.6.9}$$

where the numerical solutions of H_0, G_0, F_0 are given in § 2.4. From (3.6.1) and (3.6.7), equation (3.4.6) may be written as

$$L_0\underline{F}_3 = -i\alpha_c\,\underline{A}_2\,l_1(-i\,\underline{f}_2\,L_1\,\bar{A} + \bar{A}_1\underline{f}_1) - \underline{A}_2\,\underline{f}_1 L_2\bar{A} + \underline{\bar{A}}_3\underline{f}_1(R_cG(s)+\beta_0)\bar{A}, \tag{3.6.10}$$

$\underline{C}\,(\underline{F}_3 + G\,\underline{F}_{1z}) = 0$ at $z = 0$, $\underline{C}\,\underline{F}_3 = 0$ at $z = 1$

where L_1 and L_2 are given in (3.4.8).

A solution for $\underline{F}_3$ exists if the adjoint condition is satisfied

$$-\alpha_c L_1(L_1\bar{A}) \int_o^1 \underline{f}^{a,tr}(\underline{A}_2\ \underline{f}_2)dz+(\beta_o+R_cG)\bar{A}\int_o^1 \underline{f}^{a,tr}(\underline{A}_3\underline{f}_1)dz=\underline{f}^{a,tr}\underline{F}_3\Big|_o^1. \tag{3.6.11}$$

From the boundary condition, (3.6.10), at z = 0, we may set

$$\underline{F}_3 = -G\ \bar{A}\ \frac{d\underline{f}_1}{dz} + \bar{A}_2(s)\ \underline{f}_1. \tag{3.6.12}$$

Hence

$$\underline{f}^{a,tr}\ \underline{F}_3\Big|_o^1 = \{\sum_{k=1}^{6} f_k\ \frac{df_{1k}}{dz}\}\ G\ \bar{A}\ _{z=0} \tag{3.6.13}$$

where $f_k(z)$ is the kth component of the adjoint eigenfunction and f_{1k} is the kth component of $\underline{f}_1(z)$. The equation (3.6.11) may be rearranged as

$$a(\frac{d^2\bar{A}}{ds^2} + \frac{1}{s}\frac{d\bar{A}}{ds} - \frac{\bar{A}}{4s^2})+\bar{A}\{(b - \hat{c}R_c)G+b\ \beta_o\} = 0 \tag{3.6.14}$$

and this is the amplitude equation for $\bar{A}(s)$.

Here

$$a = -\alpha_c \int_o^1 \underline{f}^{a,tr}\ (\underline{A}_2\ \underline{f}_2)dz,$$

$$b = \int_o^1 \underline{f}^{a,tr}\ (\underline{A}_3\ \underline{f}_1)\ dz, \tag{3.6.15}$$

$$c = \{\sum_{k=4}^{6} f_k\ \frac{df_{1k}}{dz}\}_{z=0}$$

Now let

$$A_0(s) = \bar{A}\, s^{\frac{1}{2}}. \qquad (3.6.16)$$

From (3.6.14) we see that

$$\frac{d^2 A_0}{ds^2} + (\delta_1 + \delta_2 G)\, A_0 = 0, \qquad (3.6.17)$$

where $\delta_1 = \frac{b\beta_0}{a} > 0$ and $\delta_2 = \frac{b - cR_c}{a} < 0$.

The equation (3.6.17) is similar to that obtained by Eagles (1980) in the two dimensional case where the boundary conditions are defined at $s = \pm\infty$. In our case, we have no information about the boundary condition at the centre, $s = 0$, and we shall investigate this condition in the next section by using a matching procedure.

3.7. The Inner Solution and Matching Procedure

In the neighbourhood of $r = 0$ the function $G(s)$ tends to zero and we look for a linearised solution of (2.1.1) - (2.1.4), in which $R = R_c$ and the components of the disturbance are given in terms of Bessel functions. An equivalent solution has been found in the stress-free case by Brown and Stewartson (1978).

One solution is

$$\theta = h(z)\, J_0(\alpha_c r), \quad u = f(z)\, J_0'(\alpha_c r), \quad w = g(z)\, J_0(\alpha_c r)$$

and a second solution can be found by writing

$$\begin{aligned} u &= \bar{F}(z)\, J_0'(\alpha r) + r f J_0''(\alpha r), \\ w &= \bar{G}(z)\, J_0(\alpha r) + r g J_0'(\alpha r), \\ \theta &= \bar{H}(z)\, J_0(\alpha r) + r h J_0'(\alpha r). \end{aligned} \tag{3.7.1}$$

On substituting (3.7.1) into equations (2.1.1) - (2.1.4), with $R = R_c$ and $\alpha = \alpha_c$, we obtain a set of differential equations which are given as follows,

$$\begin{aligned} D^2\bar{H} - \alpha^2 \bar{H} + R\bar{G} &= 2\alpha h, \\ D^4\bar{G} - 2\alpha^2 D^2\bar{G} + \alpha^4 \bar{G} - \alpha^2 \bar{H} &= 4\alpha D^2 g - 4\alpha^3 g + 2\alpha h, \end{aligned} \tag{3.7.2}$$

with boundary conditons

$$D\bar{G} = \bar{G} = \bar{F} = 0 \text{ at } z = 0, 1. \tag{3.7.3}$$

Comparing these equations and boundary conditions with those which we obtained in (2.4.1) - (2.4.3) we see that we may take

$$\bar{H} = \frac{\partial h}{\partial \alpha},\ \bar{F} = \frac{\partial f}{\partial \alpha},\ \bar{G} = \frac{\partial g}{\partial \alpha}.$$

Discarding other solutions which are exponentially large as $r \to \infty$, the general solution for θ in the inner zone may now be written in the form

$$\theta_I = \lambda J_0(\alpha_c r)h + \mu\{J_0(\alpha_c r)\bar{H} + rJ_0'(\alpha_c r)h\}, \qquad (3.7.4)$$

where the numerical solutions of h and $\bar{H}$ are given in § 2.3., and § 2.4., and λ, μ, are arbitrary constants. θ_I denotes the inner solution and we use θ_0 to denote the solution already found. In order to match (3.7.4) with the outer solution (3.4.1), we need the behaviour of the amplitude function $A_0(s)$ as $s \to 0$, which is found to be

$$A_0 \sim \bar{a} + \bar{b}\, s + \ldots\ldots \ (s \to 0), \qquad (3.7.5)$$

where $\bar{a}$, $\bar{b}$ are arbitrary constants and $s = \varepsilon r$.

From (3.4.1), (3.6.1) and (3.6.16) in the outer region where $G(s) \neq 0$, θ_0 is given by

$$\theta_0 \sim \frac{e^{i\alpha_c r}}{s^{\frac{1}{2}}}\{h A_0 - i\varepsilon \frac{\partial A_0}{\partial s}\bar{H} + \ldots\ldots\} + C.C. \qquad (3.7.6)$$

Now the asymptotic expansion of (3.7.4) for large r is

$$\theta_I \sim \left(\frac{2}{\pi\alpha_c r}\right)^{\frac{1}{2}}\left\{\left(\lambda-\frac{3\mu}{8\alpha_c}\right)h \cos \hat{r} + \mu\bar{H}\cos \hat{r} - \mu\, rh \sin \hat{r}\right\}, \qquad (3.7.7)$$

where

$$\hat{r} = \alpha_c r - \frac{\pi}{4}.$$

On substituting (3.7.5) into (3.7.6) we obtain

$$\theta_0 \sim e^{i\alpha_c r}/s^{\frac{1}{2}}\{\bar{a}h+\varepsilon\bar{b}rh - i\bar{b}\varepsilon\bar{H}) + C.C, \qquad (3.7.8)$$

and so comparing (3.7.7) with (3.7.8), we see that a match of the terms in, h and rh is secured if, respectively,

$$\bar{a} = \left(\frac{\varepsilon}{2\pi\alpha_c}\right)^{\frac{1}{2}}\left(\lambda - \frac{3\mu}{8\alpha_c}\right)e^{-\frac{i\pi}{4}}, \qquad (3.7.9)$$

$$\varepsilon^{\frac{1}{2}}\, \bar{b}-(\tfrac{1}{2}\pi\alpha_c)^{-\frac{1}{2}}\, \mu\, e^{\frac{i\pi}{4}} = 0.$$

It should also be noted that the terms in $\bar{H}$ then match automatically. Therefore, from matching condition (3.7.9) we can list the boundary conditions for the amplitude equation as:

$$A_0(0) = b_1\, e^{-\frac{i\pi}{4}},$$

$$(3.7.10)$$

$$A_0'(0) = b_2\, e^{\frac{i\pi}{4}},$$

where b_1 and b_2 are real constants. Note that these conditions are quite different from those which apply to the non-linear problem studied by Brown and Stewartson (1978).

3.8. Investigation of the Amplitude Equation

As a check on the analysis thus far, we start with $G(s) = 0$, and look for a solution of the system

$$\begin{aligned} &A_0''(s) + \delta_1 A_0(s) = 0 \qquad (0 < s < \infty), \\ &A_0(0) = b_1 e^{-i\frac{\pi}{4}} \\ &A_0'(0) = b_2 e^{i\frac{\pi}{4}} \end{aligned} \tag{3.8.1}$$

where $\delta_1 > 0$ and b_1, b_2 are real quantities.

From (3.8.1) we find that

$$A_0 = b_1 e^{-i\frac{\pi}{4}} \cos \delta_1^{\frac{1}{2}} s + b_3 e^{i\frac{\pi}{4}} \sin \delta_1^{\frac{1}{2}} s. \tag{3.8.2}$$

The outer solution (3.7.6) may now be expressed as

$$\theta_0 \sim \frac{1}{s^{\frac{1}{2}}} \{D_1 \cos(\alpha_1 r - \tfrac{\pi}{4})(h - \varepsilon\delta_1^{\frac{1}{2}} \bar{H}) + D_2 \cos(\alpha_2 r - \tfrac{\pi}{4})(h + \varepsilon\delta_1^{\frac{1}{2}} \bar{H})\} \tag{3.8.3}$$

where

$$\begin{aligned} &D_1 = (b_1 - b_3)/2, \quad D_2 = (b_1 + b_3)/2, \\ &\alpha_1 = \alpha_c - \delta_1^{\frac{1}{2}} \varepsilon, \quad \alpha_2 = \alpha_c + \delta_1^{\frac{1}{2}} \varepsilon, \\ &b_2 = \delta_1^{\frac{1}{2}} b_3. \end{aligned} \tag{3.8.4}$$

Now using the asymptotic expansion of the Bessel function $J_0(\alpha r)$, for large r, we see that the result (3.8.3) is indeed consistent with the asymptotic form of an arbitrary linear combination constructed from the two roots $\alpha = \alpha_1, \alpha_2$ and corresponding eigenfunctions $h_1(z)$, $h_2(z)$ of the system (2.2.7) - (2.2.8) when $R > R_c$, which can be written as

$$\theta_0 \sim D_1 h_1 J_0 (\alpha_1 r) + D_2 h_2 J_0 (\alpha_2 r). \tag{3.8.5}$$

This result confirms that the linear Benard convection problem in the parallel plane case can be expressed in terms of Bessel functions every where, as the analysis of § 2 suggests. Also it suggests that the matching conditions (3.7.9) are the correct ones.

Now suppose $G(s) \neq 0$, and define

$$G_1(s) = -\delta_2 G(s), \tag{3.8.6}$$

so that the amplitude equation is given by

$$A_0'' + (\delta_1 - G_1) A_0 = 0$$

where $G_1 \geqslant 0$.

In view of the conditions (3.7.10) we set

$$A_0 = e^{-\frac{i\pi}{4}} (A_1 + i A_2) \tag{3.8.8}$$

where $A_1(s)$ and $A_2(s)$ are assumed real functions.

Then

$$A_1(0) = b_1, \; A_2(0) = 0$$
$$A_1'(0) = 0, \; A_2'(0) = b_2. \tag{3.8.9}$$

Let

$$\tilde{A} = A_1 + i\,A_2,$$

and

$$\tilde{A} = \bar{R}(s)\, e^{i\phi(s)}.$$

Then $\bar{R}$ and ϕ satisfy the equations

$$\bar{R}'' + (\delta_1 - G_1)\bar{R} - \phi'^2 \bar{R} = 0$$
$$2\,\phi'\bar{R}' + \phi''\,\bar{R} = 0 \tag{3.8.10}$$

with boundary conditions

$$\bar{R}(0) = b_1, \; \phi(0) = 0,$$
$$\bar{R}'(0) = 0, \phi'(0) = \frac{b_2}{b_1}. \tag{3.8.11}$$

From (3.8.10)

$$\phi' = c/\bar{R}^2 \tag{3.8.12}$$

where c, is an arbitrary real constant, and the boundary conditions (3.8.11) at s = 0, imply that

$$c = b_1 \; b_2 .$$

Hence

$$\phi = b_1 b_2 \int \frac{ds}{\bar{R}^2}. \tag{3.8.13}$$

Now if we impose the condition that $\bar{R} \to 0$ $(s \to \infty)$, then we need to choose c = 0, and so then

$$\bar{R}'' + (\delta_1 - G_1)\bar{R} = 0 ,$$

$$\bar{R}' (0) = 0, \tag{3.8.14}$$

$$\bar{R} \to 0 \; (s \to \infty).$$

This system is the same as the linear form considered by Eagles (1980), where the function $G_1(s)$ is taken to be

$$G_1(s) = \tanh^2 \frac{s}{\sqrt{2}}. \tag{3.8.15}$$

For $\delta_1 = \frac{1}{2}$, there is a solution of (3.8.14)

$$\bar{R} = b_1 \operatorname{sech} \frac{s}{\sqrt{2}} \tag{3.8.16}$$

in which $\bar{R} \to 0$ as $s \to \infty$, representing a distribution of convection cells concentrated near the centre.

The value of δ_1 corresponds to an effective increase in the critical Rayleigh number caused by the decrease in depth of the layer as $s \to \infty$ and we expect the solution (3.8.16) to develop into a finite amplitude solution of the full non-linear equation for $\delta_1 > \frac{1}{2}$ (see Eagles 1980). Further increase in δ_1 beyond $G_1(\infty) = 1$, is likely to introduce the possibility of further solutions of (3.8.16) which have non-zero amplitude at infinity and solutions with b_2 non-zero must also then be envisaged.

We can define a local Rayleigh number by

$$R_L \approx 1 - 3\varepsilon^2 G(s)\, R,$$

where

$$G(s) \begin{cases} \text{Positive} & s > 0, \\ 0 & s = 0. \end{cases}$$

It is well known that the base flow in the parallel plane problem ($G(s) = 0$, or $\varepsilon = 0$), is unstable for $R > R_c = 1707.763$ (Table 2.2.), and that for $R > R_c$ a pattern of convection cells or rolls is set up. In the non-parallel plane case we see that the local Rayleigh number is larger near $s = 0$ than at $s = \infty$, so that the convection cells occur in the centre more readily than away from the centre and the positive value of $G(s)$ for $s > 0$, causes an effective increase in the critical Rayleigh number over that for the plane case where $G(s) = 0$.

Finally, it should be pointed out that the boundary conditions at s = 0 for the linear amplitude equation (3.6.17) differ from those which apply to the non-linear problem investigated by Brown and Stewartson (1978). We shall consider the effect of non-linear terms in Chapter 5.

Rotating Linear Benard Convection in Cylindrical Geometry

In this chapter we shall consider the effect of rotation on the linear Benard problem studied in Chapter 2.

In the stress-free case the effect of rotation has been considered by Muller (1965), who found a steady state solution in terms of Bessel functions for the linear equations that govern axisymmetric motion. The present aim is to use a numerical approach to obtain a solution for the case when the boundaries are rigid.

Let (r^*, ϕ^*, z^*) denote cylindrical polar coordinates with z^* - axis perpendicular to the two parallel planes $z^* = 0$ and $z^* = d$. These planes are rotated about the z^* axis with a constant angular velocity, Ω. The gap between the planes is filled with a fluid with constituent properties as given in Chapter 2. The relevant equations of motion and heat transfer in the O-B approximation are given in (1.1). The space coordinates, velocity components, pressure and temperature are non-dimensionalised as follows:

$$(u^*, w^*) = \kappa(u, w)(r, z, t)/d,$$

$$v^* = \Omega r^* + \kappa v(r, z, t)/d,$$

$$\theta^* = \theta_0^* + (\theta_1^* - \theta_0^*)z^*/d + \kappa\nu\theta(r, z, t)/\alpha_0 \, gd^3,$$

$$p^* = p_0^* - \rho_0 \, g \, z^* + \tfrac{1}{2}\rho_0 \, g \, \alpha_0(\theta_1^* - \theta_0^*)z^{*2}/d + \tfrac{1}{2}\rho_0 \Omega^2 r^{*2} + \frac{\rho_0 \kappa^2}{d^2} P(r, z, t),$$

and r, z, t are non-dimensional coordinates and time, defined by

$$r = r^*/d, \quad z = z^*/d, \quad t = \kappa t^*/d^2.$$

The governing equations in the Oberbeck-Boussinesq approximation are then given by

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{4.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \sigma T^{\frac{1}{2}} v + \sigma\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{4.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\sigma T^{\frac{1}{2}} u + \sigma\left(\nabla^2 u - \frac{v}{r^2}\right) \tag{4.3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \sigma(\nabla^2 w + \theta), \tag{4.4}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r} + w\frac{\partial \theta}{\partial z} = R\,w + \nabla^2\theta \tag{4.5}$$

where $\nabla^2 = \frac{\partial^2}{\partial^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$,

and the Prandtl number, σ, Rayleigh number, R, and Taylor number, T, are defined by

$$\sigma = {}^{\nu}/_{\kappa}, \quad R = \alpha_0 g d^3(\theta_0^* - \theta_1^*)/_{\kappa\nu}, \quad T = 4\Omega^2 d^4/_{\nu^2} \tag{4.6}$$

The boundary conditions are

$$\theta = u = v = w = 0 \text{ at } z = 0,\ z = 1. \tag{4.7}$$

In this chapter attention is restricted to the linear form of the equations (4.1) - (4.5); solutionsof the full non-linear system are discussed in Chapters 5 - 7.

4.1. The Analysis into Normal Modes

Following the procedure of Chapter 2, we analyse the disturbances u, v, w, θ in terms of Bessel functions, and consider perturbations characterized by a particular wave number α. Thus we set

$$(u, v) = J_0'(\alpha r)\ (f', k)\ e^{i\omega t},$$
$$(w, \theta) = J_0(\alpha r)(\alpha f, h)\ e^{i\omega t}, \tag{4.1.1}$$

where f, k, h, are functions of the variable z, and ' denotes the differentiation of a function with respect to the appropriate variable. In contrast to the simple Benard problem (Chapter 2), the principle of the exchange of stabilities is not generally valid in the rotating case (Chandrasekhar 1953), when either stationary or overstable convection can arise first as the Rayleigh number increases, depending on the values of the Prandtl number, σ, and Taylor number, T. Here we fix attention on the stationary case and therefore set $\omega = 0$, in order to determine the neutral stability curve.

On substituting (4.1.1) into the linear form of equations (4.1) - (4.5) we obtain:

$$D^2 h - \alpha^2 h - \alpha R f = 0,$$
$$D^2 k - \alpha^2 k - T^{\frac{1}{2}} f = 0, \tag{4.1.2}$$
$$D^4 f - 2\alpha^2 D^2 f + \alpha^4 f + \alpha h + T^{\frac{1}{2}} DK = 0,$$

where

$$D = \frac{d}{dz} .$$

The boundary conditions (4.7) become

$$f = Df = h = K = 0 \quad \text{at } z = 0, 1. \tag{4.1.3}$$

Eliminating K(z) and f(z) between the equations (4.1.2), we find that

$$\ell_1\{h\} = 0, \tag{4.1.4}$$

and

$$\left.\begin{array}{l} h = D^2h = D(D^2-\alpha^2)h = 0, \\ D^3 \{D^4-3\alpha^2D^2 + R + 2\alpha^4\} h = 0 \end{array}\right\} \quad \text{at } z = 0, 1 \tag{4.1.5}$$

where ℓ_1 is the linear operator,

$$\ell_1 = (D^2 - \alpha^2)\{(D^2 - \alpha^2)^3 - T D^2 + \alpha^2 R\}. \tag{4.1.6}$$

The equation (4.1.4) and the boundary conditions (4.1.5) constitute an eigenvalue problem for R, for given α and T. The problem of determining the critical Rayleigh number for the onset of instability as stationary convection, at a given Taylor number T, reduces to that of finding the lowest value of R as a function of α .

The desired solution of (4.1.4)-(4.1.5) can be made unique by adding the normalization condition

$$h = 1 \quad \text{at } z = \tfrac{1}{2}. \tag{4.1.7}$$

4.2. Numerical Solution of the Eigenvalue Problem

The method used here is similar to that developed for the non-rotating problem in § 2.3, but in view of the order of the operator (4.1.6.) we now set

$$(y_1, y_2, \ldots, y_9, \ldots, y_{16}) = (h, h', \ldots, h^{(7)}, \frac{\partial h}{\partial R}, \ldots, \frac{\partial h^{(7)}}{\partial R}), \quad (4.2.1)$$

where

$$(n) \quad n = 1, 2, \ldots\ldots, 7.$$

denotes the order of differentiation of the appropriate function with respect to the variable, z.

From equations (4.1.4) and (4.2.1) we find that

$$y_i' = y_{i+1} \qquad (i = 1, 2, \ldots, 7.)$$

$$y_8' = \alpha^4(R-\alpha^4)\, y_1+\alpha^2(T+4\alpha^4-R)y_3-(T+6\alpha^4)y_5+4\,\alpha^2 y_7,$$

$$y_i' = y_{i+1} \qquad (i = 9, \ldots\ldots, 15),$$

$$y_{16}' = \alpha^4 y_1-\alpha^2 y_3+4\alpha^2 y_{15}-(T+6\alpha^4)y_{13}+\alpha^2(T+4\alpha^4-R)y_{11}+\alpha^4(R-\alpha^4)y_9. \qquad (4.2.2)$$

The equations (4.2.2) can be expressed in matrix form as

$$\underline{Y}' = \underline{A}\ \underline{Y}, \qquad (4.2.3)$$

Where $\underline{A}$ is a matrix of order (16, 16) and

$$\underline{Y} = \left[y_1, y_2, \ldots\ldots, y_{16} \right]^{tr}.$$

The boundary conditions are

$$y_1 = y_3 = 0,$$

$$y_8 - 3\,\alpha^2 y_6 + \alpha^2 (2\alpha^2 + R) y_4 = 0, \quad \text{at } z = 0,\ 1 \tag{4.2.4}$$

$$y_4 - \alpha^2\, y_2 = 0.$$

In order to solve the equations (4.2.3) - (4.2.4), using a fourth-order Runge-Kutta scheme (see § 2.3), starting values, $y_i(0)$ for i = 1, 2. ...., 16 are specified and each $y_i(z)$ calculated at equally spaced values of z up to z = 1.

Three different step sizes (δz = 0.05, 0.025, 0.0125) were used to check the accuracy of the scheme, as in the earlier computation of § 2.3.

Four linearly independent solutions $\underline{Y}^{\{1\}}$, $\underline{Y}^{\{2\}}$, $\underline{Y}^{\{3\}}$, $\underline{Y}^{\{4\}}$ of (4.2.3), each satisfying the boundary conditions (4.2.4.) at z = 0 are computed. Then the function

$$\underline{Y}^{\{5\}} = \underline{Y}^{\{1\}} + a_1\, \underline{Y}^{\{2\}} + a_2\, \underline{Y}^{\{3\}} + a_3\, \underline{Y}^{\{4\}}, \tag{4.2.5}$$

satisfies the conditions (4.2.4) at $z = 0$, for all values of a_i $(i = 1, 2, 3)$. We now choose a_i such that the first three conditions of (4.2.4) are satisfied at $z = 1$.

We then set

$$Q_1 = \alpha^2 y_2^{\{5\}} - y_4^{\{5\}} \quad \text{at } z = 1 \tag{4.2.6}$$

where $y_i^{\{5\}}$ denotes the ith component of $\underline{Y}^{\{5\}}$ and the final boundary condition, $Q_1 = 0$ at $z = 1$, leads to a relation between α and R for given T. To obtain $Q_1 = 0$, Newton's method is applied in the form

$$R_{new} = R_{old} - Q_1 / \frac{\partial Q_1}{\partial R}. \tag{4.2.7}$$

This iteration method thus provides the values of R on the neutral stability curve at given values of α and T. The lowest value of R and the associated value of α, at fixed T, provides the critical Rayleigh number and wave number for a given rotation rate. The numerical results are given in Table 4.1., and comparison of these results with those obtained by a different method by Chandrasekhar (1961) shows adequate agreement between the two methods.

T	R_c	α_c
10	1712.675	3.121
100	1756.348	3.161
500	1940.199	3.319
1000	2151.341	3.484
2000	2530.125	3.747
5000	3468.493	4.266
10^4	4712.04	4.789
3×10^4	8324.61	5.795
10^5	16719.42	7.172

Table 4.1 : Critical Rayleigh number for the case when both bounding surfaces are rigid and the onset of instability is as stationary convection.

Table 4.1 shows that the system is more stable as the speed of rotation increases, since R_c is an increasing function of T. Finally, the profiles of h, f, k are shown in Figures 4.1 - 4.3.

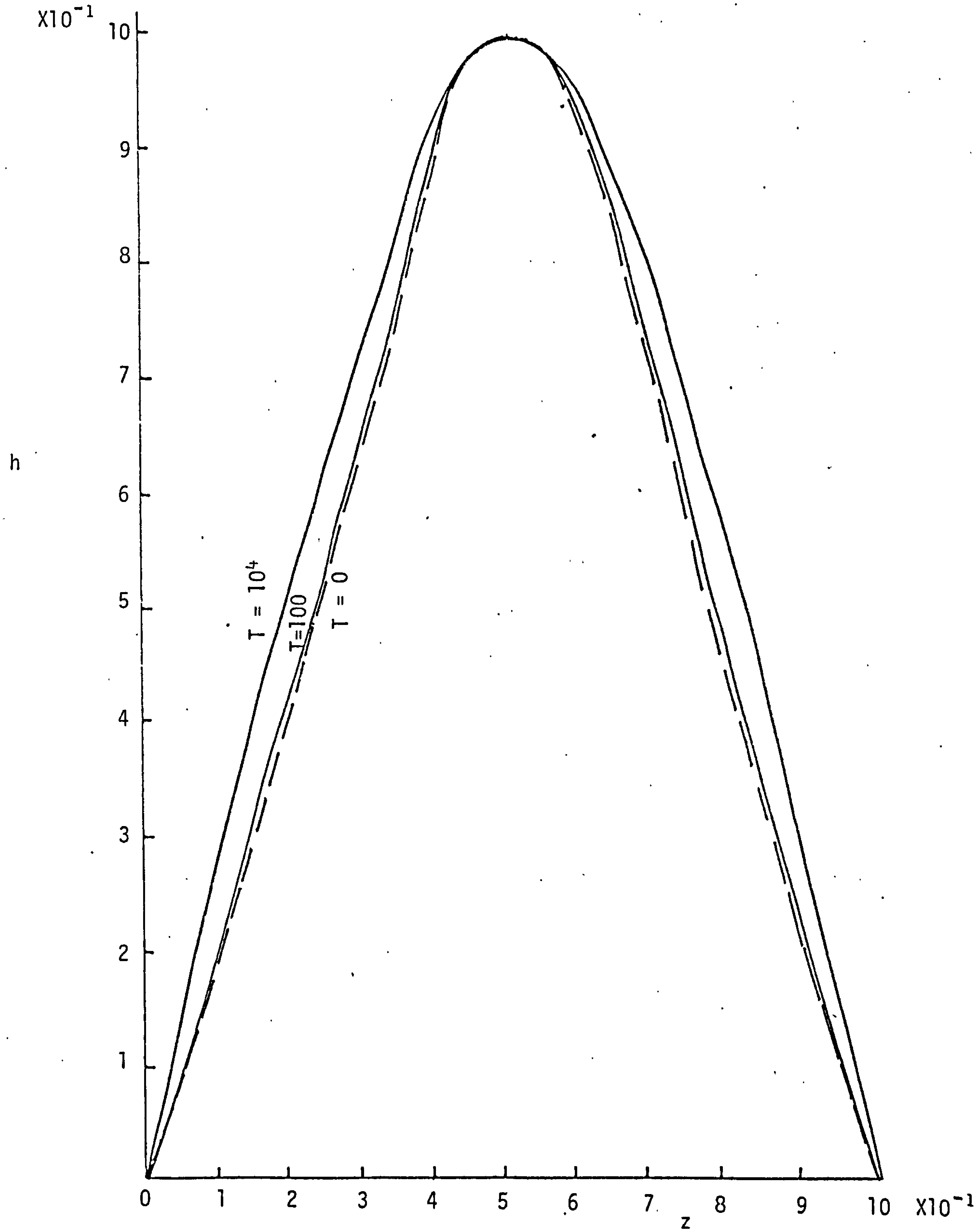

Figure 4.1: Typical temperature profiles for different rotation rates.

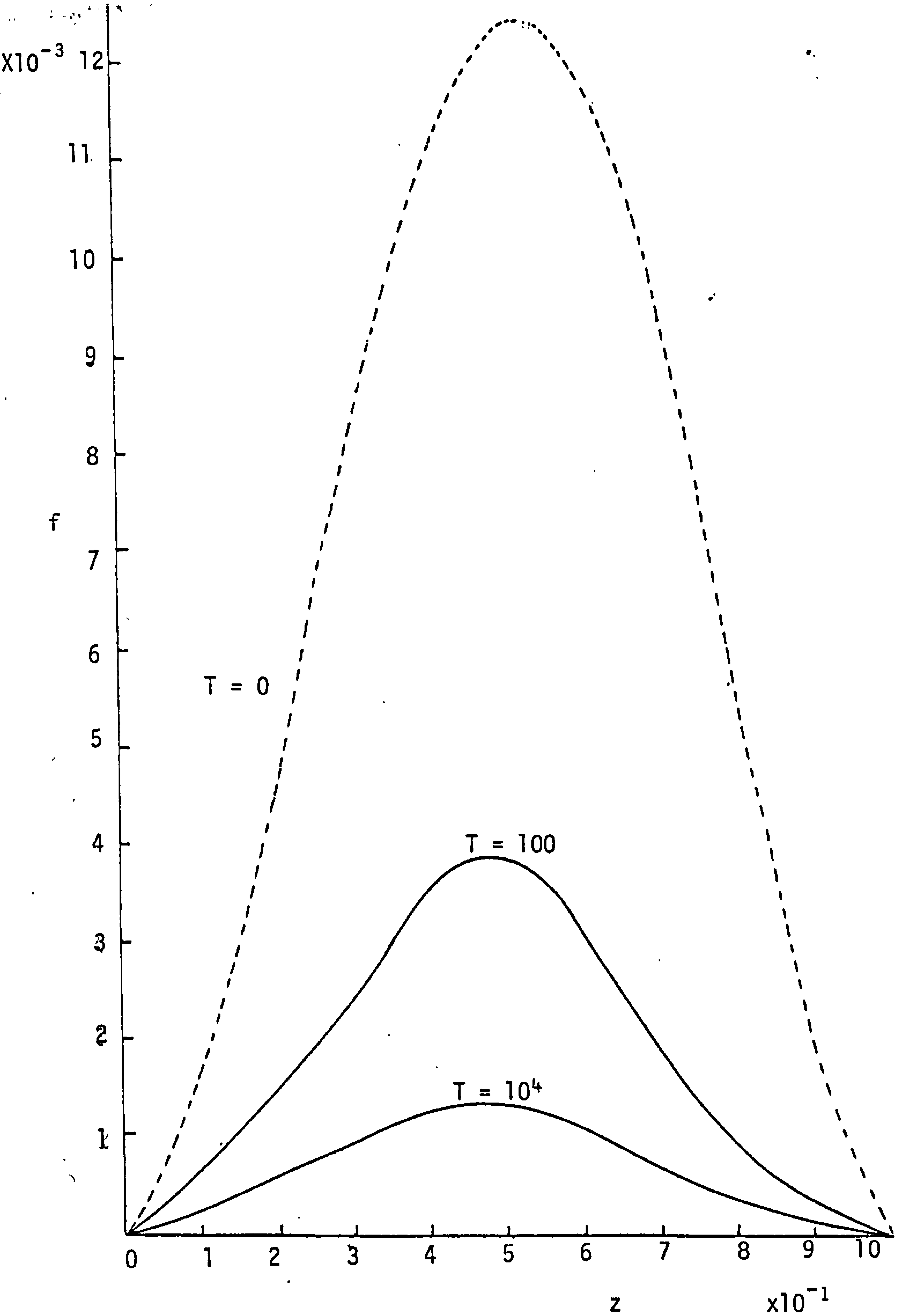

Figure 4.2: Typical velocity profiles (w component) for different rotation rates.

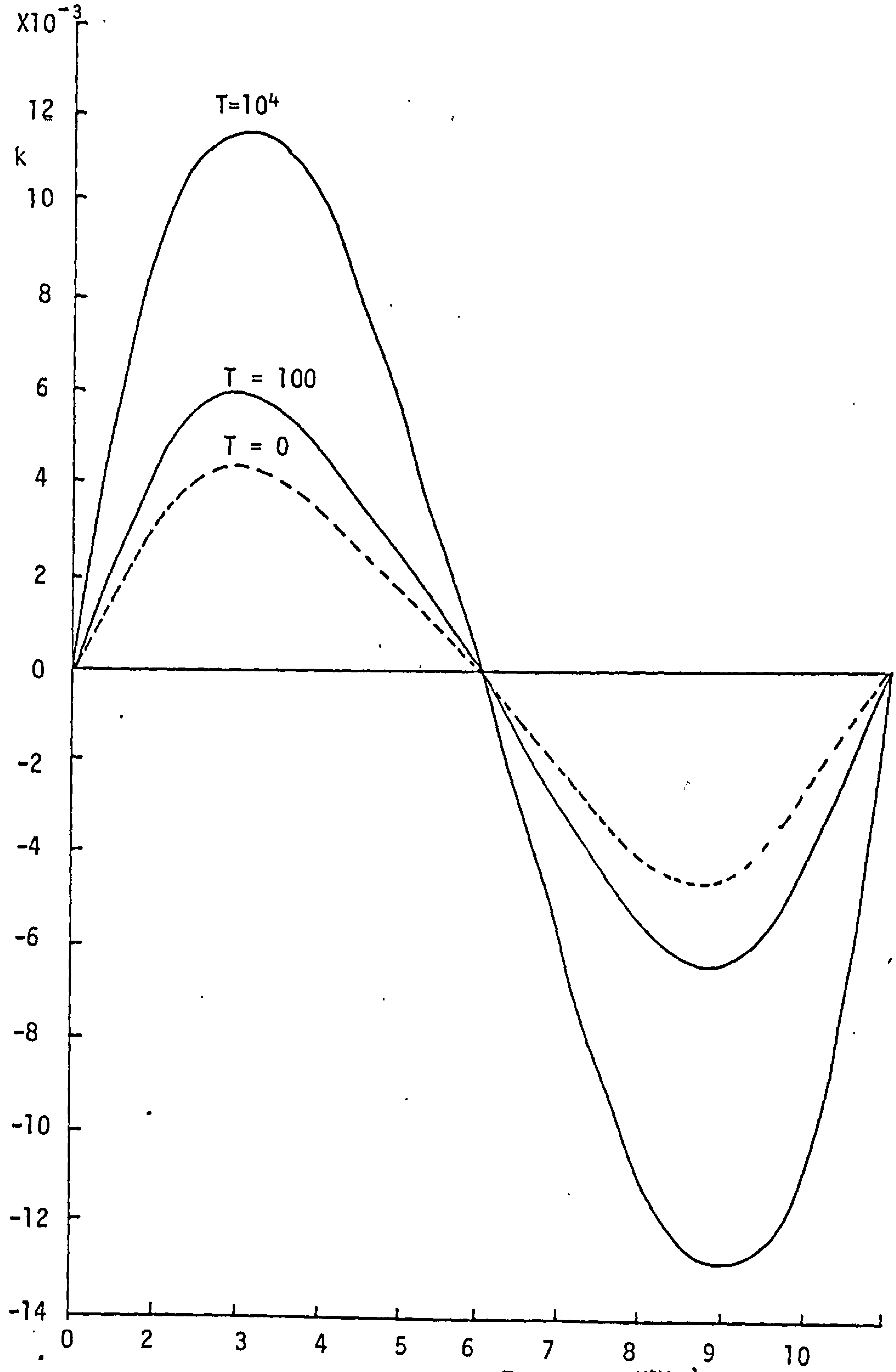

Figure 4.3: Typical velocity profiles (v component) for different rotation rates.

4.3 Numerical Solution of the Inhomogeneous Equation

In the preceding chapter the eigenfunction $h(z)$, satisfying (4.1.4) - (4.1.5) is obtained for different values of α, R and T. Here we wish to find $\frac{\partial h}{\partial \alpha}$ at $\alpha = \alpha_c$ and $R = R_c$ (for given T). The method of solution of this problem is similar to that of § 2.4.

Let
$$\frac{\partial h}{\partial \alpha} = H, \quad \frac{\partial f}{\partial \alpha} = F, \quad \frac{\partial k}{\partial \alpha} = K. \tag{4.3.1}$$

From (4.1.2)

$$\begin{aligned} &D^2H - \alpha^2 H - \alpha RF = 2\alpha h + Rf, \\ &D^2K - \alpha^2 K - T^{\frac{1}{2}} DF = 2\alpha k, \\ &D^4F - 2\alpha^2 D^2F + \alpha^4 F + \alpha H + T^{\frac{1}{2}}DK = 4\alpha D^2 f - 4\alpha^3 f - h. \end{aligned} \tag{4.3.2}$$

The boundary conditions are

$$F = DF = K = H = 0 \quad \text{at } z = 0, 1. \tag{4.3.3}$$

If, in addition, h is assumed to satisfy (4.1.7) then

$$H = 0 \quad \text{at} \quad z = \tfrac{1}{2} \tag{4.3.4}$$

and this renders the solution of (4.3.2) unique. Otherwise, the solution for H can contain an arbitrary multiple of h.

The consistency condition is similar to (2.4.7) and is given by

$$I_2 = \int_0^1 (2\alpha h^2/R - 2\alpha k^2 - 4\alpha f D^2 f + 4\alpha^3 f^2 + 2hf)dz = 0, \quad (4.3.5)$$

with $\alpha = \alpha_c$ and $R = R_c$.

Eliminating F and K between (4.3.2), (or by direct differentiation of (4.1.4) - (4.1.5) with respect to α) we obtain:

$$\ell_1\{H\} = 2\alpha\{D^2(4D^4 - 12\alpha^2D^2 + T + 12\alpha^4 - R) + 2\alpha^2(R - 2\alpha^4)\}h, \quad (4.3.6)$$

where ℓ_1 is given in (4.1.6) and also

$$H = D^2H = 0,$$

$$D(D^2-\alpha^2)H = 2\alpha h, \quad (4.3.7)$$

$$D^3\{D^4 - 3\alpha^2D^2 + R + 2\alpha^4\}H = 2\alpha D^2(3D^3 - 4\alpha^2)h. \text{ at } z = 0,1$$

The solution for H is even about $z = \frac{1}{2}$.

The numerical method used here is similar to that in §2.4.

We set

$$\underline{Y} = \underline{Y}_c + \underline{Y}_p$$

where $\underline{Y}_p$ is a particular solution of (4.3.6) and

$$\underline{Y}_c = \underline{Y}^{\{1\}} + b_1 \underline{Y}^{\{2\}} + b_2 \underline{Y}^{\{3\}} + b_3 \underline{Y}^{\{4\}}.$$

Each $\underline{Y}^{\{i\}}$(i = 1, 2, ..4) is the complementary solution given in (4.2.5), the constants b_i(i = 1, 2, 3) are found from the first three conditions (4.3.7) at z = 1, and automatic satisfaction of the last boundary condition (4.3.7) at z = 1, provides a check on the calculation. As we have already mentioned (§ 2.4), in order to apply the fourth-order Runge-Kutta scheme to (4.3.6) with step size, δz, the function, h, on the right-hand side of equation (4.3.6) is first calculated at intermediate mesh points equivalent to the division of one step into intervals $\frac{\delta z}{6}, \frac{\delta z}{3}, \frac{\delta z}{2}$. The profiles of H, F, K are shown in Figures 4.4 - 4.6.

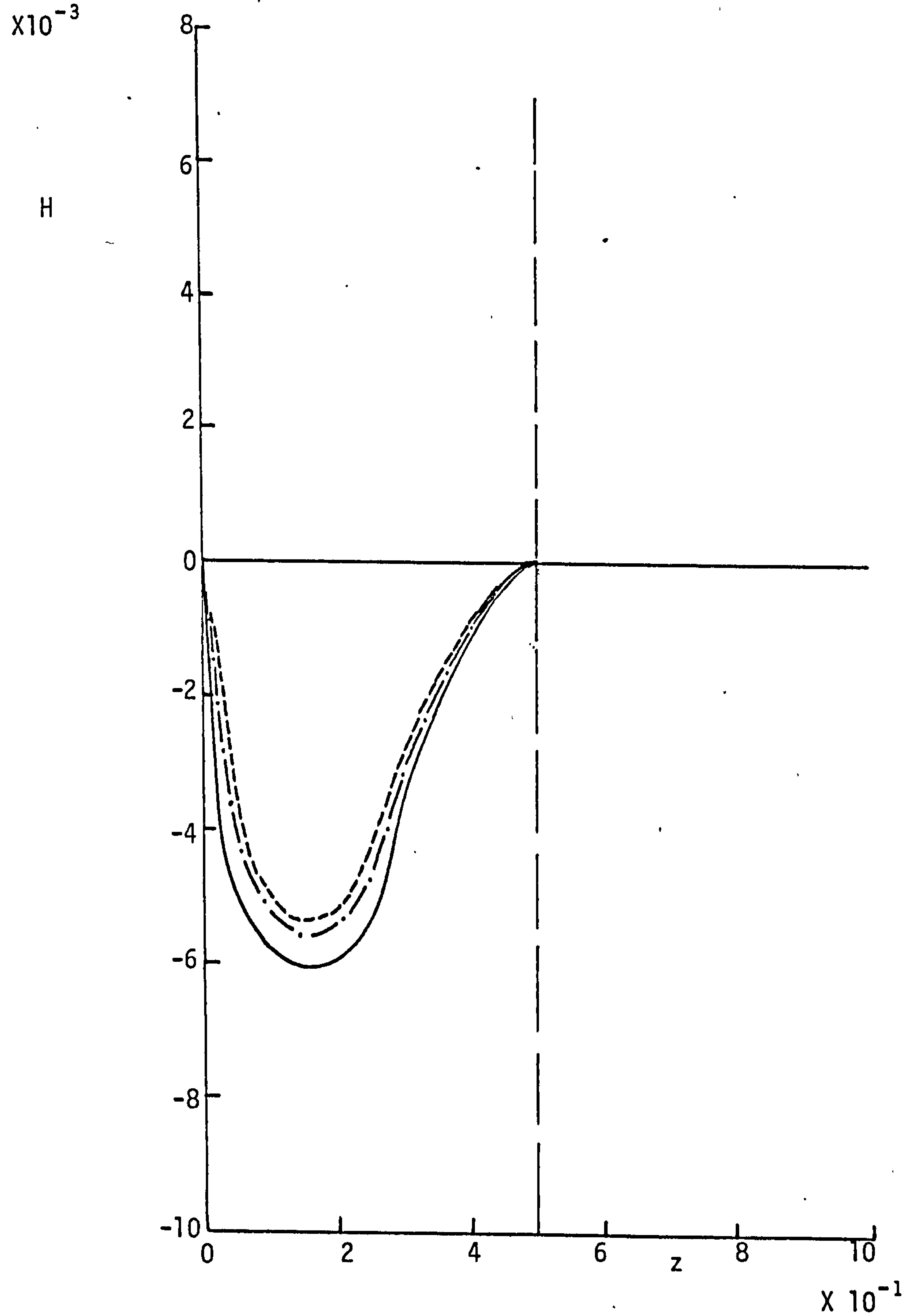

Figure 4.4: Profiles of H for different rotation rates. Here H is an even function about $z = \frac{1}{2}$

-------------	T = 0
—·—·—	T = 100
———	$T = 10^4$

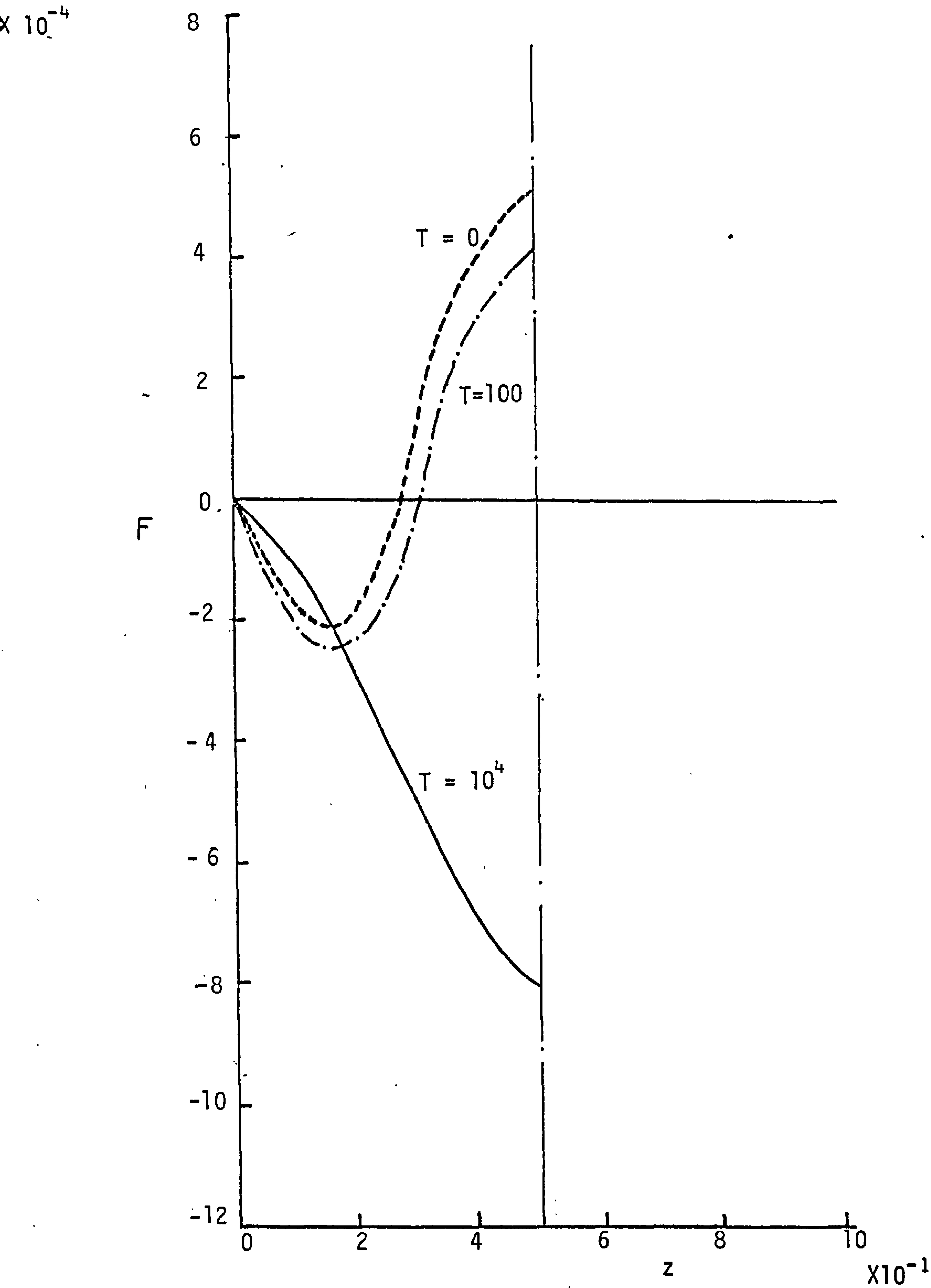

Figure 4.5: Profiles of F for different rotation rates. Here F is an even function about $z = \frac{1}{2}$.

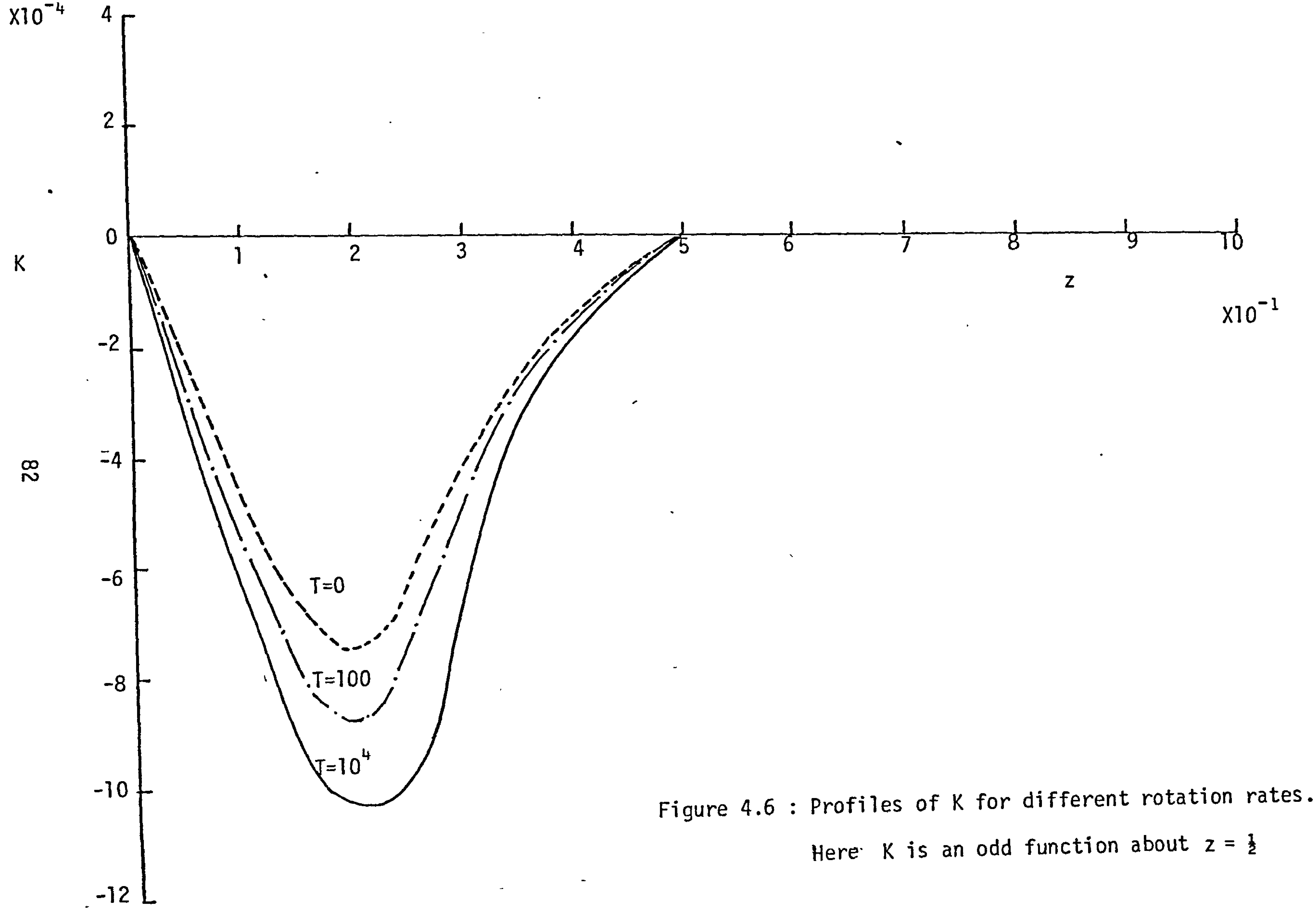

Figure 4.6 : Profiles of K for different rotation rates.
Here K is an odd function about $z = \frac{1}{2}$

CHAPTER 5

Rotating Non-linear Benard Convection : Exchange of Stabilities.

The linear theory for an infinite rotating layer of fluid which is bounded between two rigid planes $z = 0$ and $z = 1$, is given in Chapter 4. Now we consider the non-linear structure of the flow at the onset of convection in a rotating layer when the disturbance has an axisymmetric form and the exchange of stabilities occurs. In this investigation the analysis is based on the linear stability problem (Chapter 4).

The governing equations of motion and heat transfer are given by the Oberbeck-Boussinesq system (4.1) - (4.5), and the boundary conditions are given by (4.7).

5.1. Expansion Procedure and Outer Solution

The critical Rayleigh number, R_c, is calculated for different values of the Taylor number T, in table 4.1.

For consideration of slightly supercritical flows we set

$$R = R_c + \beta\, \varepsilon^2 \qquad (5.1.1),$$

where β is a constant factor introduced for convenience, and use a multiple-scale method in which slow spatial and time variables are defined by

$$s = \varepsilon r, \ \tilde{\tau} = \varepsilon^2 t \qquad (5.1.2),$$

and $\varepsilon \ll 1$.

Rotating non-linear Benard convection with stress-free boundaries was considered by Veronis (1959), and the possibility of non-linear motions at subcritical values of the Rayleigh number were discussed in a subsequent paper (Veronis 1966). For stress-free horizontal surfaces, the forms of the non-linear amplitude equations, incorporating spatial modulation, for both stationary and over stable motions, in a rotating system bounded by distant side-walls, were given by Daniels (1978). More recently, Daniels (1980) has made a theoretical study of the effect of centrifugal acceleration on axisymmetric 'exchange convection' in a shallow rotating cylinder or annulus where upper and lower surfaces are assumed to be stress-free. Here the boundaries ($z = 0$, $z = 1$) are taken to be rigid.

We eliminate the pressure between equations (4.1) - (4.5), and away from the centre the solution may be expanded in the form

$$\begin{bmatrix} u \\ v \\ w \\ \theta \end{bmatrix}(r,z,t) = \varepsilon \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_1 \end{bmatrix} + \varepsilon^2 \begin{bmatrix} u_2 \\ v_2 \\ w_2 \\ \theta_2 \end{bmatrix} + \varepsilon^3 \begin{bmatrix} u_3 \\ v_3 \\ w_3 \\ \theta_3 \end{bmatrix} + \ldots \qquad (5.1.3)$$

Here u_i, v_i, w_i, θ_i ($i = 1, 2, 3$) are functions of r, z, s and τ. On substituting (5.1.1) - (5.1.3) into the continuity equation (4.1), the vorticity equation derived by eliminating the pressure, and the heat transfer equation (4.5) and equating terms of order ε, ε^2, ε^3, we obtain a set of linear partial-differential equations as follows:

$$\begin{aligned} &u_{ir} + w_{iz} = \phi_{1i}, \\ &v_{ir} + v_{izz} - T^{\frac{1}{2}} u_i = \phi_{2i}, \\ &\theta_{irr} + \theta_{izz} + R_c w_i = \phi_{3i}, \\ &u_{irrz} + u_{izzz} - w_{irrr} - w_{irzz} - \theta_{ir} + T^{\frac{1}{2}} v_{iz} = \phi_{4i}, \end{aligned} \tag{5.1.4}$$

where $u_{ir} = \frac{\partial u_i}{\partial r}$, $u_{iz} = \frac{\partial u_i}{\partial z}$, ..., etc

and ϕ_{1i}, ϕ_{2i}, ϕ_{3i}, ϕ_{4i} are functions to be defined below.

The boundary conditions are

$$u_i = v_i = w_i = \theta_i = 0 \text{ at } z = 0,1; \; i = 1, 2, 3. \tag{5.1.5}$$

At order $\varepsilon (i = 1)$ we find that

$$\phi_{i1} = 0 \; ; \; i = 1, 2, 3, 4$$

and the solutions for u_1, v_1, w_1, θ_1 which satisfy the boundary conditions (5.1.5) are given by

$$\begin{aligned} (u_1, v_1) &= \{ A_o (s, \tau) e^{i\alpha r} + A_o^* (s, \tau) e^{-i\alpha r} \} (Df_1, k_1), \\ (w_1, \theta_1) &= i \{ A_o (s, \tau) e^{i\alpha r} - A_o^* (s, \tau) e^{-i\alpha r} \} (-\alpha f_1, h_1), \end{aligned} \tag{5.1.6}$$

where f_1, k_1, h_1 are functions of z and A_o is an amplitude function which is assumed to be a complex function of s and τ. A_o^* is the complex conjugate of A_o. From the equations (5.1.4, i = 1) and (5.1.6), we find that

$$D^2h_1 - \alpha^2 h_1 - \alpha R f_1 = 0,$$
$$D^2k_1 - \alpha^2 k_1 - T^{\frac{1}{2}}Df_1 = 0, \qquad (5.1.7)$$
$$D^4f_1 - 2\alpha^2D^2f_1+\alpha^4f_1 + \alpha h_1 + T^{\frac{1}{2}}Dk_1 = 0$$

where $D = \dfrac{d}{dz}$.

The critical wave number, α_c, and Rayleigh number, R_c, must be determined from solution of these equations which satisfies the conditions

$$h_1 = f_1 = Df_1 = k_1 = 0 \quad \text{at } z = 0,1, \qquad (5.1.8)$$

and at which $\dfrac{dR}{d\alpha} = 0$.

Eliminating, k_1 and f_1, between the equations (5.1.7), we have

$$\ell_1\{h_1\} = 0, \qquad (5.1.9)$$

where ℓ_1 is the linear operator (4.1.6), and the boundary conditions are

$$\left.\begin{array}{l} h_1 = D^2h_1 = D(D^2-\alpha^2)h_1 = 0, \\ D^3 \{ D^4-3\alpha^2D^2+R+2\alpha^4\}h_1 = 0. \end{array}\right\} \text{at } z = 0,1 \qquad (5.1.10)$$

The required solution of (5.1.9)-(5.1.10) is given by the critical eigenfunction $h(z)$, for $\alpha = \alpha_c$ and $R = R_c$, as determined numerically in § 4.2, and it will be assumed that this is normalised by (4.1.7). The functions f_1 and k_1 are given in terms of h_1 by (4.1.2). The condition $\dfrac{dR}{d\alpha} = 0$ is equivalent to the consistency condition (4.3.5), which we shall use in the subsequent analysis.

At order ε^2 ($i = 2$) the functions $\phi_{12}, \phi_{22}, \phi_{32}, \phi_{42}$ are found to be

$$\phi_{12} = -f_1' \{ e^{i\alpha r} B_1 + C.C \}, \tag{5.1.11}$$

$$\phi_{22} = i\alpha \{ A_o^2 E_1 e^{2i\alpha r} - B_2 k_1 e^{i\alpha r} \} + C.C \tag{5.1.12}$$

$$\phi_{23} = \alpha \{ E_3 A_o^2 e^{2i\alpha r} - 2E_4 |A_o|^2 + B_2 e^{i\alpha r} h_1 \} + C.C \tag{5.1.13}$$

$$\phi_{24} = i e^{i\alpha r} \{ (B_2 + \frac{\partial A_o}{\partial s})(\alpha^3 f_1 - \alpha f_1'') + \frac{\partial A_o}{\partial s} h_1 \} + i\alpha E_2 A_o^2 e^{2i\alpha r} + C.C \tag{5.1.14}$$

where C.C denotes the complex conjugate and E_i ($i = 1, 2, 3, 4$) are functions of z which are given in terms of the basic eigen-function and their derivatives with respect to variables, z, as follows:

$$E_1 = (f_1' k_1 - f_1 k_1')/\sigma, \quad E_2 = (f_1' f_1'' - f_1 f_1''')/\sigma,$$
$$E_3 = f_1 h_1' - h_1 f_1', \quad E_4 = f_1' h_1 + h_1' f_1 \tag{5.1.15}$$

also

$$B_1 = \frac{\partial A_o}{\partial s} + \frac{A_o}{s} ; \quad B_2 = 2 \frac{\partial A_o}{\partial s} + \frac{A_o}{s} . \tag{5.1.16}$$

The particular solutions for u_2, v_2, w_2, θ_2 are,

$$u_2 = i \{ (f_1' B_1/\alpha + \frac{\partial F_2}{\partial z}) e^{i\alpha r} + f_{21}' A_o^2 e^{2i\alpha r} \} + C.C \tag{5.1.17}$$

$$w_2 = \alpha \{ F_2 e^{i\alpha r} + 2 f_{21} A_o^2 e^{2i\alpha r} \} + C.C, \tag{5.1.18}$$

$$v_2 = i \{ K_2 e^{i\alpha r} + k_{21} A_o^2 e^{2i\alpha r} \} + C.C, \tag{5.1.19}$$

$$\theta_2 = \{ H_2 e^{i\alpha r} + h_{21} A_o^2 e^{2i\alpha r} + |A_o|^2 h_{22} \} + C.C \tag{5.1.20}$$

where F_2, H_2, K_2, are functions of z, s and τ, while f_{21}, k_{21}, h_{21}, h_{22} are functions of z only. The functions F_2, H_2, K_2 satisfy equations of the form (5.1.7) but with three zero right-hand sides replaced by $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$, respectively, where

$$\bar{E}_1 = -\alpha B_2 h_1 ,$$

$$\bar{E}_2 = -\alpha B_2 k_1 + T^{\frac{1}{2}} f_1' B_1/\alpha, \tag{5.1.21}$$

$$\bar{E}_3 = (h_1 + \alpha^3 f_1)\frac{\partial A_o}{\partial s} -(2\alpha f_1'')\frac{\partial A_o}{\partial s} -(B_1/\alpha)f_1^{(4)}+\alpha^3 B_2 f_1 .$$

Multiplication of the equation (5.1.7) for h_1, by H_2 and that for H_2, by h_1, subtraction and integration from z = 0 to z = 1, now yields

$$-\alpha_c R_c \int_0^1 (F_2 h_1 - f_1 H_2)dz = \int_0^1 \bar{E}_1 h_1 dz, \tag{5.1.22}$$

and from similar treatment of the other equations,

$$-T^{\frac{1}{2}} \int_0^1 (k_1 F_2' - f_1' K_2)dz = \int_0^1 \bar{E}_2 k_1 dz, \tag{5.1.23}$$

and

$$T^{\frac{1}{2}} \int_0^1 (K_2' f_1 - k_1' F_2)dz + \alpha_c \int_0^1 (H_2 f_1 - h_1 F_2)dz = \int_0^1 \bar{E}_3 f_1 dz. \tag{5.1.24}$$

Combining these three results we find that in order that the solution at order ϵ^2 be consistent we must have

$$\int_0^1 (R_c^{-1}\,\bar{E}_1\, h_1 - \bar{E}_3\, f_1)\, dz = 0. \tag{5.1.25}$$

Substitution from (5.1.21) shows that this reduces to

$$\left(\frac{\partial A_o}{\partial s} + \frac{A_o}{2s}\right) I_2 = 0, \tag{5.1.26}$$

and is satisfied, since $I_2=0$ by (4.3.5).

We may set

$$H_2 = \frac{\partial A_o}{\partial s}\, T_{31} + \frac{A_o}{s}\, T_{32},$$

$$F_2 = \frac{\partial A_o}{\partial s} T_{21} + \frac{A_o}{s} T_{22},\quad K_2 = \frac{\partial A_o}{\partial s}\, T_{11} + \frac{A_o}{s} T_{12} \tag{5.1.27}$$

and the functions T_{12}, T_{22} and T_{32} then satisfy equations of the form (5.1.7) but with the three zero right-hand sides replaced by

$$\bar{g}_1 = -\alpha_c h_1\ ,$$

$$\bar{g}_2 = T^{\frac{1}{2}}\, f_1'/\alpha_c - \alpha_c k_1, \tag{5.1.28}$$

$$\bar{g}_3 = \alpha_c^3\, f_1 - f_1^{(4)}/\alpha_c.$$

The functions T_{31}, T_{21}, T_{11} also satisfy these equations, but now the right-hand sides of (5.1.7) are replaced by

$$\bar{g}_{11} = -2\alpha_c h_1,$$

$$\bar{g}_{22} = -2\alpha_c k_1 + T^{\frac{1}{2}} f_1'/\alpha_c, \tag{5.1.29}$$

$$\bar{g}_{33} = -2\alpha_c f_1^{(2)}/\alpha_c - 3\alpha_c^3\, f_1 - f_1^{(4)}/\alpha_c\,.$$

The boundary conditions are

$$\left.\begin{aligned} T_{32} = T_{22} = T'_{22} = T_{12} = 0, \\ T_{31} = T_{21} = T'_{21} = T_{11} = 0 \end{aligned}\right\} \text{ at } z = 0, 1 \quad (5.1.30)$$

and it is easy to show that

$$T_{31} = -H, \quad (5.1.31)$$

and

$$T_{32} = -H/2, \quad (5.1.32)$$

where the numerical solution of H is given in §4.3.

The functions f_{21}, k_{21}, h_{21} satisfy the equations

$$\begin{aligned} & h_{21}^{(2)} - 4\alpha_c^2 h_{21} - 2\alpha_c R_c f_{21} = -\alpha_c E_3, \\ & k_{21}^{(2)} - 4\alpha_c^2 k_{21} - T^{\frac{1}{2}} f'_{21} = \alpha_c E_1 \\ & f_{21}^{(4)} - 8\alpha_c^2 f_{21}^{(2)} + 16\alpha_c^4 f_{21} + 2\alpha_c h_{21} + T^{\frac{1}{2}} k'_{21} = \alpha_c E_2, \end{aligned} \quad (5.1.33)$$

where $E_i (i = 1, 2, 3)$ are given in (5.1.15), and

$$f_{21} = h_{21} = k_{21} = f'_{21} = 0 \text{ at } z = 0, 1. \quad (5.1.34)$$

The numerical solution of the system (5.1.33) - (5.1.34) is found by use of the Runge-Kutta scheme.

Finally, the function h_{22} satisfies

$$h_{22}^{(2)} = 2\alpha_c E_4, \tag{5.1.35}$$

$$h_{22} = 0 \quad \text{at } z = 0, 1$$

and the solution is also found numerically.

Thus all functions which are involved at order ε and ε^2 are now determined and the amplitude equation for $A_o(s,\tilde{\tau})$ can be derived.

5.2 Derivation of the Amplitude Equation

At the order ε^3 (i = 3) the solutions for u_3, v_3, w_3, θ_3, which satisfy the boundary conditions (5.1.5), are given by

$$\mu_3 = \bar{C}_1\, e^{i\alpha r} + \bar{C}_2\, e^{2i\alpha r} + \bar{P}_{1oz}\, e^{i\alpha r} + \bar{P}_{11z}\, e^{2i\alpha r} + C.C$$

$$w_3 = -i\alpha' \{ \bar{P}_{1o}\, e^{i\alpha r} + 2\bar{P}_{11}\, e^{2i\alpha r} + 3\, \bar{P}_{12}\, e^{3i\alpha r} \} + C.C$$

$$v_3 = \bar{P}_{2o}\, e^{i\alpha r} + \bar{P}_{21}\, e^{2i\alpha r} + \bar{P}_{22}\, e^{3i\alpha r} + C.C \tag{5.2.1}$$

$$\theta_3 = i' \{ \bar{P}_{3o}\, e^{i\alpha r} + \bar{P}_{31}\, e^{2i\alpha r} + \bar{P}_{32}\, e^{3i\alpha r} \} - C.C$$

where $\bar{C}_1$, $\bar{C}_2$ are functions of $s, \tilde{\tau}, z$ and each $\bar{P}_{ij}$ (i = 1, 2 , 3 and j = 0, 1, 2) is a function of z, s and $\tilde{\tau}$.

On sustitution of (5.2.1) into (5.1.4), (for i = 3) and equating the coefficients of $e^{i\alpha r}$ on both sides of the equations, we find that, the functions $\bar{P}_{1o}$, $\bar{P}_{2o}$ and $\bar{P}_{3o}$ satisfy equations of the form (5.1.7), but the three zero right-hand sides replaced by $\bar{X}_1$, $\bar{X}_2$ and $\bar{X}_3$ (see Appendix I), where the functions $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$ depend on the functions found numerically in §4.2 and §5.1, and on the amplitude function A_o and its partial derivatives.

The consistency condition at this stage which provides the amplitude equation is

$$\int_o^1 (R_c^{-1}\, \bar{X}_1\, h_1 - \bar{X}_2 k_1 - f_1\, \bar{X}_3)dz = 0. \qquad (5.2.2)$$

From this condition we find that

$$a_1 \frac{\partial A_o}{\partial \tilde{\tau}} - a_2 A_o |A_o|^2 - \frac{a_3}{s^2} A_o - \frac{a_4}{s} \frac{\partial A_o}{\partial s} - a_5 \frac{\partial^2 A_o}{\partial s^2} + \beta a_6 A_o = 0, \qquad (5.2.3)$$

where the coefficients $a_i (i = 1, \ldots, 6)$ have to be calculated from the formula

$$a_i = \int_o^1 \bar{D}_i(z)\, dz, \qquad (5.2.4)$$

and the functions $\bar{D}_i$ $(i = 1, 2, \ldots, 6)$ are given in Appendix I.

The amplitude equation (5.2.3) is similar to that of Daniels (1978) for a bounded rotating system with upper and lower surfaces stress-free, although there the curvature terms are neglected. Brown and Stewartson (1978) have found the same equation in a shallow cylinder of fluid bounded by stress-free planes. In contrast to the equation (3.6.17), where $T = 0$, the equation (5.2.3) is a general form of the amplitude equation in which non-linear effects are included.

It can be shown (see appendix II) that, as in the non-rotating case (3.6.17) the coefficients a_3, a_4, a_5 are related by

$$a_4 = a_5 \ , \ a_4 = -4a_3 . \qquad (5.2.5)$$

The numerical results for the amplitude coefficients for different values of the Taylor number are given in Table 5.1 and figures 5.5-5.8 below.

It should be noted that the coefficient of the non-linear term, a_2, changes sign for different values of the Prandtl number and Taylor number (Table 5.2, Figure 5.1) so that, as in the stress-free case (Veronis 1959, Daniels 1978) subcritical instability can occur.

T	a_1	a_2	a_3	a_4	a_5	a_6
10	0.71667	-0.427×10^{-4}	-0.345	1.38	1.38	-0.545×10^{-2}
100	0.7084	-0.409×10^{-4}	-0.343	1.372	1.372	-0.538×10^{-2}
500	0.6784	-0.350×10^{-4}	-0.337	1.348	1.348	-0.513×10^{-2}
1000	0.6515	-0.302×10^{-4}	-0.330	1.32	1.32	-0.488×10^{-2}
2000	0.6162	-0.243×10^{-4}	-0.322	1.288	1.288	-0.452×10^{-2}
5000	0.5662	-0.167×10^{-4}	-0.308	1.234	1.234	-0.39×10^{-2}
10000	0.5343	-0.118×10^{-4}	-0.299	1.196	1.196	-0.339×10^{-2}
30000	0.5017	-0.629×10^{-5}	-0.289	1.095	1.115	-0.263×10^{-2}

Table 5.1: Coefficients of the amplitude equation for different Taylor numbers.

Prandtl number, $\sigma = 1$

σ \ T	10	100	500	1000	2000	5000	10000	30000
0.1	-0.209 E-3	-0.145 E-4	0.393 E-4	0.142 E-3	0.206 E-3	0.187 E-3	0.130 E-3	0.561 E-4
0.2	-0.818 E-3	-0. 65 E-4	-0.166 E-4	0.121 E-4	0.319 E-4	0.328 E-4	0.226 E-4	0.887 E-4
0.3	-0.587 E-3	-0.507 E-4	-0.27 E-4	-0.121 E-4	-0.52 E-5	0.42 E-5	0.272 E-5	0.106 E-5
0.4	-0.508 E-4	-0.458 E-4	-0.307 E-4	-0.207 E-4	-0.119 E-4	-0.581 E-5	-0.425 E-5	-0.296 E-5
0.5	-0.47 E-4	-0.43 E-4	-0.32 E-4	-0.247 E-4	-0.17 E-4	-0.104 E-4	-0.747 E-5	-0.438 E-5
0.6	-0.451E-4	-0.42 E-4	-0.33 E-4	-0.269 E-4	-0.20 E-4	-0.129 E-4	-0.923 E-5	-0.515 E-5
0.7	-0.44 E-4	-0.418 E-4	-0.34 E-4	-0.28 E-4	-0.218 E-4	-0.145 E-4	-0.102 E-4	-0.56 E-5
0.8	-0.434 E-4	-0.41 E-4	-0.345 E-4	-0.29 E-4	-0.22 E-4	-0.154 E-4	-0.109 E-4	-0.69 E-5
0.9	-0.429 E-4	-0.41 E-4	-0.346 E-4	-0.297 E-4	-0.23 E-4	-0.16 E-4	-0.114 E-4	-0.61 E-5
1	-0.427 E-4	-0.409 E-4	-0.35 E-4	-0.302 E-4	-0.24 E-4	-0.167 E-4	-0.118 E-4	-0.629 E-5

Table 5.2: Numerical results of a_2 (coefficient of non-linear term in amplitude equation) for different Taylor number and Prandtl number.

Where $E - n = 10^{-n}$, $n = 1, 2, 3, \ldots\ldots, 6$.

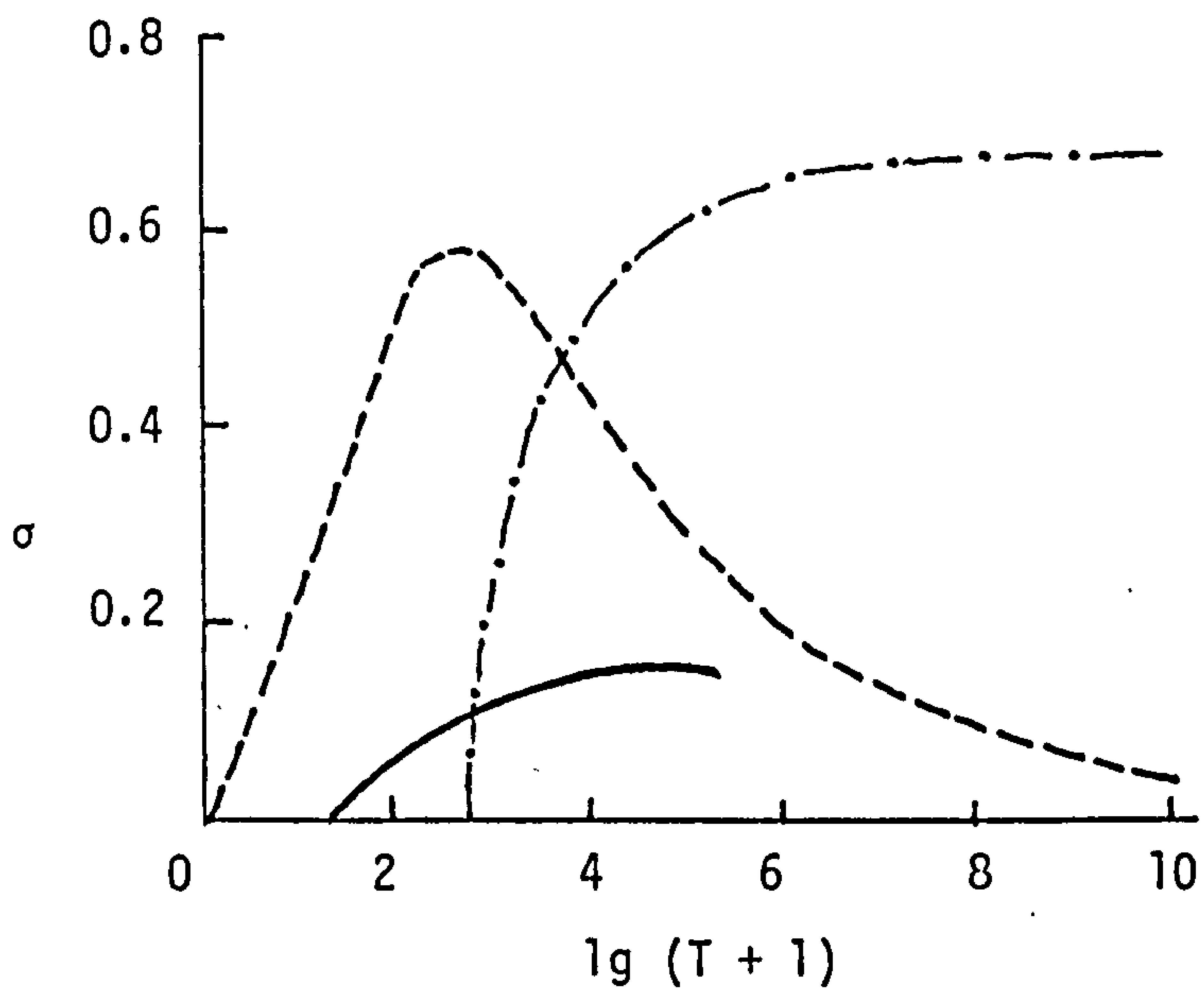

Figure 5.1: Stability boundaries in the σ, T plane. Subcritical instability in the exchange case occurs below the boundary shown:

———————— rigid-rigid case (Present work)

---------- free-free case (Daniels 1978)

Note that the effect of rigid horizontal planes is to substantially reduce the region of subcritical instability.

In the free-free case overstability is preferred below the boundary —·———·—·————·— Chandrasekhar (1961) ; the corresponding boundary in the rigid-rigid case is not available, although it is known, for example, that there is a slight preference for overstability when $\sigma = 0.025$, $T = 10^4$ (Chandrasekhar 1961).

The Modified Amplitude Equation

In view of (5.2.5) and the transformation

$$A = A_o |\bar{\xi}|^{\frac{1}{2}} s^{\frac{1}{2}} e^{i\lambda_o}, \tag{5.2.6}$$

the amplitude equation (5.2.3) may be reduced to the form

$$\frac{\partial A}{\partial \tau} = \frac{\partial^2 A}{\partial s} + A - \frac{1}{s} A|A|^2 \operatorname{sgn}(\bar{\xi}), \tag{5.2.7}$$

where

$$\begin{aligned} \bar{\xi} &= -a_2/a_4, \\ \tau &= (a_4/a_1)\tilde{\tau}, \\ \beta &= -a_4/a_6 \end{aligned} \tag{5.2.8}$$

and the coefficients a_1, a_2, a_4, a_6 are given in Table 5.1, while λ_o is an arbitrary constant. The transformation (5.2.6) removes the curvature terms from the equation (5.2.3) as in the non-rotating study of Brown and Stewartson (1978). In order to consider the subcritical case.

We take

$$\beta = \frac{a_4}{a_6} < 0$$ and this changes the

sign of the A term in equation (5.2.7).

5.3. Asymptotic Analysis for $T \gg 1$: Leading Order Structure

The numerical results for the critical Rayleigh number and wave number and the coefficients of the amplitude equation given in Tables 4.1 and 5.1 can be checked to some extent by comparison with an asymptotic approach for high Taylor number $T \gg 1$.

It emerges that there are three different regions when $T \gg 1$ (see Figure 5.2). Further, symmetry properties about $z = \frac{1}{2}$ can be used to restrict attention to $0 \leqslant z \leqslant \frac{1}{2}$. The inner boundary layer region (III) has the familiar scaling associated with small Ekman number flow near a rigid boundary, the thickness, in terms of T, being $O(T^{-\frac{1}{4}})$ (see Greenspan 1968). It also emerges that there is a middle layer region (II) where $z = O(T^{-\frac{1}{6}})$ and finally, concentration of viscous action into narrow layers means that elsewhere the fluid behaves in an essentially inviscid manner, region (I).

We shall solve equations (5.1.4, i = 1, 2) in these three different regions and use the method of matched asymptotic expansions (Van Dyke 1964) to connect the solution in the various regions.

We set

$$\alpha = \alpha_0 T^{\frac{1}{6}} + \alpha_1 T^{\frac{1}{12}} + \ldots\ldots, \qquad T \gg 1$$

$$R = R_0 T^{\frac{2}{3}} + R_1 T^{\frac{7}{12}} + \ldots\ldots, \qquad (5.3.1)$$

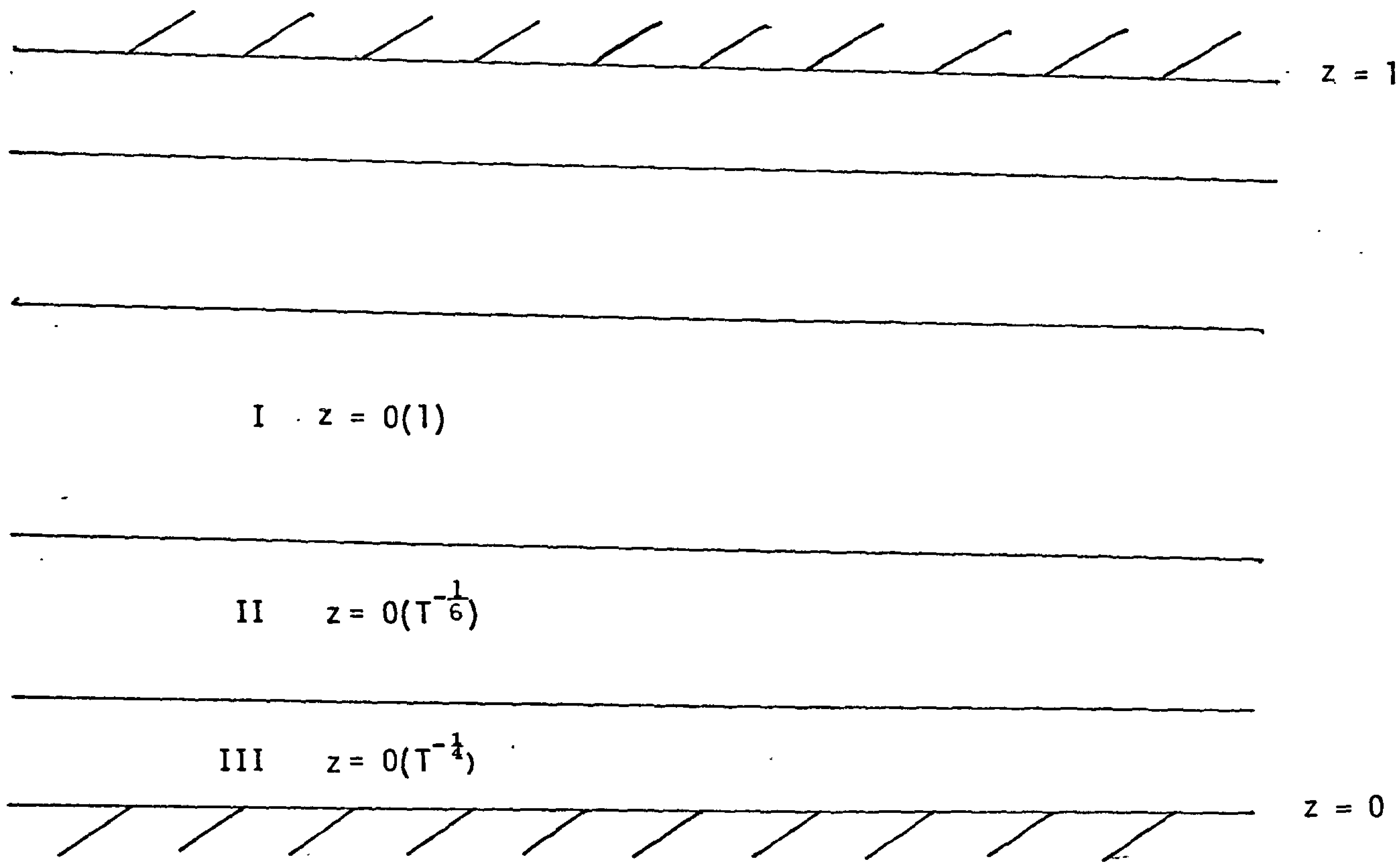

Figure 5.2 : Structure of flow between two rigid planes.

where α and R are the wave number and Rayleigh number respectively and α_i, R_i $(i = 0, 1)$ are unknown values which have to be found. The leading order scalings in (5.3.1) are suggested by comparable results for the stress-free case (Daniels 1978). The second order terms do not appear in the stress-free case, but in general, expansions in powers of $T^{-\frac{1}{12}}$ are suggested by the relative scaling of regions II and III. The asymptotic expansions in the three different regions for h(z), the solution of (5.1.9), are given by

$$\text{In I} : h(z) = h_0 + T^{-\frac{1}{12}} h_1 + T^{-\frac{1}{6}} h_2 + \ldots\ldots, \qquad (5.3.2)$$

where z is O (1) and $h_i (i=0, 1, 2, \ldots)$ is a function of z.

$$\text{In II}: h(z_1) = T^{-\frac{1}{12}} H_0 + T^{-\frac{1}{6}} H_1 + T^{-\frac{1}{4}} H_2 + \ldots\ldots, \qquad (5.3.3)$$

where z_1 is O(1), $z = T^{-\frac{1}{6}} z_1$, and each H_i $(i=0,1,2,..)$ is a function of z_1.

$$\text{In III}: h(z_2) = T^{-\frac{1}{6}} \bar{H}_0 + T^{-\frac{1}{4}} \bar{H}_1 + T^{-\frac{1}{3}} \bar{H}_2 + \ldots\ldots, \qquad (5.3.4)$$

where z_2 is O(1), $z = T^{-\frac{1}{4}} z_2$ and each $\bar{H}_i (i=0,1,2,..)$ is a function of z_2.

Again, the relative scaling of regions II and III suggests that the expansions of h in each region proceed in powers of $T^{-\frac{1}{12}}$.

On substitution of (5.3.1) - (5.3.2) into (5.1.9) with $T \gg 1$ we find that

$$L_3 \{h_o\} = 0, \qquad (5.3.5)$$

where

$$L_3 = \frac{d^2}{dz^2} - \alpha_o^2(\alpha_o^4 - R_o). \qquad (5.3.6)$$

In view of the normalisation condition (4.1.7) and the boundary conditions imposed by matching with II and the symmetry of h about $z = \frac{1}{2}$ ($h_o(0) = 0$, $\frac{dh_o}{dz} = 0$ and $h_o = 1$ at $z = \frac{1}{2}$) the appropriate 'first mode' solution of (5.3.5) is

$$h_o = \sin \pi z, \qquad (5.3.7)$$

where

$$\pi^2 + \alpha_o^2(\alpha_o^4 - R_o) = 0. \qquad (5.3.8)$$

From substitution of (5.3.1) and (5.3.3) in (5.1.9) with $z = z_1 T^{-\frac{1}{6}}$ we find that

$$L_4 \{H_o\} = 0, \qquad (5.3.9)$$

where

$$L_4 = \frac{d^4}{dz_1^4} - \alpha_o^2 \frac{d^2}{dz_1^2}. \qquad (5.3.10)$$

The required solution of the equation (5.3.9) is

$$H_o = A' + B' z_1 + C' e^{-\alpha_o z_1}, \qquad (5.3.11)$$

where A', B', C' are arbitrary constants.

From substitution of (5.3.1) and (5.3.4) with $z = T^{-\frac{1}{4}} z_2$ in (5.1.9) - (5.1.10), we find that

$$L_5 \{ \bar{H}_o \} = 0, \qquad (5.3.12)$$

with

$$\bar{H}_o = \bar{H}_o^{(2)} = \bar{H}_o^{(3)} = \bar{H}_o^{(7)} = 0 \quad \text{at } z_2 = 0; \qquad (5.3.13)$$

where

$$L_5 = \frac{d^4}{dz_2^4} \{ \frac{d^4}{dz_2^4} + 1 \}, \qquad (5.3.14)$$

and

$$\bar{H}_o^{(n)} = \frac{d^n \bar{H}_o}{dz_2^n}. \quad (n = 2, 3, 7)$$

The required solution of (5.3.12) is given by

$$\bar{H}_o = \lambda_{1o} + \lambda_{2o} z_2 + \lambda_{3o} z_2^3 + e^{\frac{-z_2}{\sqrt{2}}} (E_{1o} \cos \frac{z_2}{\sqrt{2}} + E_{2o} \sin \frac{z_2}{\sqrt{2}}), \qquad (5.3.15)$$

where λ_{io}, E_{ko} ($i=1, 2, 3, 4$ and $k = 1, 2$) are arbitrary constants. Using the boundary conditions (5.3.13) the solution of (5.3.15) becomes

$$\bar{H}_o = \lambda_{1o} + \lambda_{2o} z_2 + \lambda_{1o} z_2^2/2 - \lambda_{1o} (\cos \frac{z_2}{\sqrt{2}} - \sin \frac{z_2}{\sqrt{2}}) e^{\frac{-z_2}{\sqrt{2}}}. \qquad (5.3.16)$$

The next step is to complete the matching of the solutions in regions I, II and III.

From (5.3.11) and matching with (I), i.e. $z_1 \to \infty$ and $z \to 0$, we have $B' = 0$. (5.3.17)

Matching between II and III, i.e., $z_1 \to 0$ and $z_2 \to \infty$, also implies that

$$A' + C' = 0, \tag{5.3.18}$$

so that

$$H_o = A' (1 - e^{-\alpha_o z_1}). \tag{5.3.19}$$

Here it should be noted that the assumption that $A' + C' \neq 0$, so that $h = O(T^{-\frac{1}{12}})$ in region III, leads to an inconsistency. Also, in (5.3.16) we require

$$\lambda_{1o} = 0 \text{ and } \lambda_{2o} = \alpha_o A',$$

and hence

$$\bar{H}_o = \alpha_o A' z_2. \tag{5.3.20}$$

Therefore, the leading order structure is now given by (5.3.7), (5.3.19) and (5.3.20) and we shall find higher order solutions in the following section, which determine A' and the values of α_1 and R_1.

5.4. Asymptotic Analysis for $T \gg 1$

Higher Order Structure

In region I at second order, we find that

$$L_3\{h_1\} = \alpha_o \{\alpha_1(6\alpha_o^4 - 2R_o) - R_1\alpha_o\}h_o, \qquad (5.4.1)$$

where L_3 and h_o are given in (5.3.6) and (5.3.7) respectively. The general solution of (5.4.1) which is even about $z = \frac{1}{2}$ is given by

$$h_1 = B\sin\pi z - \frac{\alpha_o}{2\pi}\{\alpha_1(6\alpha_o^4 - 2R_o) - R_1\alpha_o\}(z-\tfrac{1}{2})\cos\pi z \qquad (5.4.2)$$

and from normalization condition (4.1.7)

$$B = 0$$

Therefore, h(z) in region I can be expressed as

$$h(z) = \sin\pi z - \{\frac{\alpha_o}{2\pi}(\alpha_1(6\alpha_o^4 - 2R_o) - R_1\alpha_o)(z-\tfrac{1}{2})\cos\pi z\}T^{-\frac{1}{12}} + \ldots \qquad (5.4.3)$$

In region II at second order, we find that

$$L_4\{H_1\} = 2\alpha_o\alpha_1 H_o^{(2)}, \qquad (5.4.4)$$

where L_4 and H_o are given in (5.3.10) and (5.3.20). The required solution of (5.4.4) is given by

$$H_1 = A_1 + B_1 z_1 + (C_1 + A'\alpha_1 z_1)e^{-\alpha_o z_1} \qquad (5.4.5)$$

Matching between region I and region II, i.e., $z_1 \to \infty$, $z \to 0$, now implies that

$$B_1 = \pi \tag{5.4.6}$$

and also

$$A' = \frac{\alpha_o}{4\pi} \{\alpha_1 (6\alpha_o^4 - 2R_o) - R_1 \alpha_o\} \tag{5.4.7}$$

In region III at second order, we obtain from (5.1.9), (5.1.10), (5.3.4)

$$L_5\{\bar{H}_1\} = 0, \tag{5.4.8}$$

with

$$\bar{H}_1 = \bar{H}_1^{(2)} = \bar{H}_1^{(3)} = H_1^{(7)} = 0 \text{ at } z_2 = 0 \tag{5.4.9}$$

and, from third order,

$$L_5\{\bar{H}_2\} = 4\alpha_o^2 \bar{H}_o^{(6)} + \alpha_o^2 \bar{H}_o^{(2)}, \tag{5.4.10}$$

where

$$\bar{H}_2 = \bar{H}_2^{(2)} = 0$$

$$\bar{H}_2^{(3)} - \alpha_o^2 \bar{H}_o' = 0 \quad \text{at } z_2 = 0 \tag{5.4.11}$$

$$\bar{H}_2^{(7)} - 3\alpha_o^2 \bar{H}_o^{(5)} = 0.$$

From fourth order,

$$L_5\{\bar{H}_3\} = 4\alpha_o^2 \bar{H}_1^{(6)} + 8\alpha_o\alpha_1 \bar{H}_o^{(6)} + \alpha_o^2 \bar{H}_1^{(2)} + 2\alpha_o\alpha_1 \bar{H}_o^{(2)}, \tag{5.4.12}$$

with

$$\bar{H}_3 = \bar{H}_3^{(2)} = 0,$$

$$\bar{H}_3^{(3)} = \alpha_o^2 \bar{H}_1' + 2\alpha_o\alpha_1 \bar{H}_o', \quad \text{at } z_2 = 0 \tag{5.4.13}$$

$$\bar{H}_3^{(7)} = 3\alpha_o^2 \bar{H}_1^{(5)} + 6\alpha_o\alpha_1 \bar{H}_o^{(5)}.$$

The general solution of (5.4.8), (5.4.9) is given by

$$\bar{H}_1 = \lambda_{11} + \lambda_{21}\, z_2 + \frac{\lambda_{11}}{2} z_2^2 - \lambda_{11}\left(\cos\frac{z_2}{\sqrt{2}} - \sin\frac{z_2}{\sqrt{2}}\right)e^{-z_2/\sqrt{2}}, \quad (5.4.14)$$

where λ_{i1} (i=1, 2) is an arbitrary constant.

Matching with region II implies that

$$\lambda_{11} = -A'\alpha_o^2\ , \quad (5.4.15)$$

and

$$B_1 + A'\alpha_1 - \alpha_o\, C_1 = \lambda_{21}, \quad (5.4.16)$$

$$C_1 + A_1 = 0. \quad (5.4.17)$$

Details of the solution for $\bar{H}_2$ are not required in order to determine α_1 and R_1, the necessary conditions actually arising from the solution for $\bar{H}_3$, which, from (5.4.13), can be expressed as

$$\bar{H}_3 = \lambda_{13} + \lambda_{23}\, z_2 + \lambda_{33}\, z_2^2 + \lambda_{43}\, z_2^3 + \left(E_{13}\cos\frac{z_2}{\sqrt{2}} + E_{23}\sin\frac{z_2}{\sqrt{2}}\right)e^{\frac{-z_2}{\sqrt{2}}} + \text{P.I} \quad (5.4.18)$$

where λi_3, E_{k3}(i=1,2,.,4 and k = 1,2) are arbitrary constants and P.I, denotes a particular integral which is given by

$$P.I = -\frac{\alpha_o^4}{4!}A' z_2^4 + A'\frac{3\alpha_o^4}{2\sqrt{2}} z_2 \, e^{-\frac{z_2}{\sqrt{2}}} \sin\frac{z_2}{\sqrt{2}}. \qquad (5.4.19)$$

Now we apply the boundary conditions (5.4.13) and list only those conditions which are necessary for our purpose of determining α_1 and R_1:

$$6\lambda_{33} + \frac{\sqrt{2}}{2}(E_{13} + E_{23}) - A'\alpha_o^2(5\sqrt{2}\ \alpha_o^4 + 2\alpha_1) - \alpha_o^2\ \lambda_{21} = 0 \qquad (5.4.20)$$

$$E_{13} + E_{23} = 9A'\alpha_o^4/2, \qquad (5.4.21)$$

also matching with region II, i.e. $z_1 \to 0$, $z_2 \to \infty$, gives

$$\lambda_{33} = (-\alpha_o^3/6)C_1 + (\alpha_o^2\alpha_1 A')/2. \qquad (5.4.22)$$

From (5.4.6), (5.4.16) and (5.4.20) - (5.4.22), we find that

$$A' = (\pi\sqrt{2})/(2\alpha_o^2), \qquad (5.4.23)$$

and then (5.4.7), implies that

$$\alpha_o^2\ \sqrt{2}\ \{\alpha_1(6\alpha_o^4 - 2R_o) - R_1\ \alpha_o\} - 4\pi^2 = 0. \qquad (5.4.24)$$

A second relation between α_i, $R_i (i=0, 1)$ comes from the fact that $\frac{dR}{d\alpha} = 0$ and this fixes the value of α_i, $R_i (i = 0, 1)$ uniquely as follows:

From (5.3.1), the asymptotic expansion of $\frac{dR}{d\alpha}$ is given by

$$\frac{dR}{d\alpha} = T^{\frac{1}{2}} \cdot \left\{ \frac{dR_o}{d\alpha_o} + T^{-\frac{1}{12}}\left(\frac{dR_1}{d\alpha_o} - \frac{dR_o}{d\alpha_o} \cdot \frac{d\alpha_1}{d\alpha_o}\right) + \ldots\ldots \right\} \qquad (5.4.25)$$

If $\frac{dR}{d\alpha} = 0$, from first and second orders we have

$$\frac{dR_o}{d\alpha_o} = 0, \qquad (5.4.26)$$

$$\frac{dR_1}{d\alpha_o} = \frac{dR_o}{d\alpha_o} \cdot \frac{d\alpha_1}{d\alpha_o} \,. \qquad (5.4.27)$$

From (5.4.26) and (5.3.8) we obtain

$$\alpha_o = \left(\frac{\pi^2}{2}\right)^{\frac{1}{6}}, \quad R_o = 3\left(\frac{\pi^2}{2}\right)^{\frac{2}{3}}, \qquad (5.4.28)$$

and from (5.4.22), (5.3.8) and (5.4.27), we find that

$$\begin{aligned} \alpha_1 &= (-\pi^2 \sqrt{2})/(3\alpha_o^7), \\ R_1 &= (-4\pi^2)/(\alpha_o^4 \sqrt{2}). \end{aligned} \qquad (5.4.29)$$

Thus the asymptotic expansion for the Rayleigh number and wave number, when $T \gg 1$, are given by

$$\begin{aligned} R &\approx 3\left(\frac{\pi^2}{2}\right)^{\frac{2}{3}} T^{\frac{2}{3}} - \pi^{\frac{2}{3}}\, 2^{\frac{13}{6}}\, T^{\frac{7}{12}} + \ldots\ldots, \\ \alpha &\sim \left(\frac{\pi^2}{2}\right)^{\frac{1}{6}} T^{\frac{1}{6}} - 3^{-1}\left(2^{\frac{5}{3}}\, \pi^{-\frac{1}{3}}\right) T^{\frac{1}{12}} + \ldots \end{aligned} \qquad (5.4.30)$$

We shall use the results (5.4.30) below.

It is noted that similar expansions are derived by Homsy and Hudson (1969), although their approach avoids consideration of the middle layer which is essentially a thermal boundary layer.

The numerical results (Table 4.1) are compared with the asymptotic results (5.4.30) in Figures (5.3) - (5.4).

Of course, the asymptotic expansions of functions $f_1(z)$ and $k_1(z)$, which satisfy equations (5.1.7), can also be calculated from the known form of $h(z)$.

The asymptotic expansion of the functions H, h_{21} and h_{22}, which satisfy equations (4.3.2), (5.1.33) and (5.1.35) respectively are given by

$$\text{in I :} \begin{cases} H(z) = T^{-\frac{1}{6}} h_{oo} + T^{-\frac{1}{4}} h_{11} + \ldots\ldots, \\ h_{21}(z) = \bar{h}_{oo} + T^{-\frac{1}{12}} \bar{h}_{11} + \ldots\ldots, \qquad T \gg 1 \\ h_{22}(z) = T^{-\frac{1}{3}} \hat{h}_{oo} + T^{-\frac{5}{12}} \hat{h}_{11} + \ldots, \end{cases} \tag{5.4.31}$$

where $z = 0(1)$.

$$\text{in II:} \begin{cases} H(z_1) = T^{-\frac{1}{4}} H_{oo} + T^{-\frac{1}{3}} H_{11} + \ldots, \\ h_{21}(z_1) = T^{-\frac{1}{12}} \bar{H}_{oo} + T^{-\frac{1}{6}} \bar{H}_{11} + \ldots, \qquad T \gg 1 \\ h_{22}(z_1) = T^{-\frac{1}{2}} \hat{H}_{oo} + T^{-\frac{7}{12}} \hat{H}_{11} + \ldots, \end{cases} \tag{5.4.32}$$

where $z = T^{-\frac{1}{6}} z_1$ and $z_1 = 0(1)$.

$$
\text{in III:}\begin{cases}
H(z_2) = T^{-\frac{1}{3}} f_{oo} + T^{-\frac{5}{12}} f_{11} + \dots, \\
h_{21}(z_2) = T^{-\frac{1}{6}} \bar{f}_{oo} + T^{-\frac{1}{4}} \bar{f}_{11} + \dots, \\
h_{22}(z_2) = T^{-\frac{7}{12}} \hat{f}_{oo} + T^{-\frac{4}{3}} \hat{f}_{11} + \dots,
\end{cases} \qquad T \gg 1 \qquad (5.4.33)
$$

where $z = T^{-\frac{1}{4}} z_2$, $z_2 = O(1)$.

From the equations (4.3.2), (5.1.34) and (5.1.36) and (5.4.31)-(5.4.33) and the results of § 5.3 and § 5.4, the analytical forms of the functions h_{ii}, $\bar{h}_{ii}$, $\hat{h}_{ii}$, H_{ii}, $\bar{H}_{ii}$, $\hat{H}_{ii}$ and f_{ii}, $\hat{f}_{ii}$, $\bar{f}_{ii}$ (i = 0, 1, 2) can be found in each region. Thus, the asymptotic expansions of all functions which are involved in the integrand (5.2.4) are found.

5.5. Asymptotic Expansions of the Coefficients of the Amplitude Equation for $T \gg 1$

From (5.2.4) and appendix I, the integrand $\bar{D}_i(z)$ $i=1,2,..6$, can now be expanded asymptoticaly for $T \gg 1$.

We may set

$$a_i = T^{-\frac{1}{4}} \int_0^{\Delta} \bar{D}_{i3}(z_2)dz_2 + T^{-\frac{1}{6}} \int_{c_1}^{\delta} \bar{D}_{i2}(z_1)dz_1 + \int_{c_2}^{\frac{1}{2}} \bar{D}_{i1}(z)dz, \qquad (5.5.1)$$

$$c_1 = T^{-\frac{1}{12}}\Delta, \quad c_2 = T^{-\frac{1}{6}}\delta \qquad (i = 1, 2, 3, \ldots, 6)$$

where Δ and δ are large parameters, introduced to ensure convergence of the integrals and $\bar{D}_{ij}$ ($i = 1, \ldots, 6$ and $j = 1, 2, 3$) are expansions of $\bar{D}_i$ ($i = 1, 2, \ldots, 6$) in each region. Substitution of the asymptotic expansions of $\bar{D}_{ij}$ into (5.5.1) and retaining terms of leading order and second order we find that

$$a_1 \sim (3 - \sigma^{-1})/6 + (9 + \sigma^{-1})\, T^{-\frac{1}{12}}/9\pi + \ldots\ldots,$$

$$a_2 \sim (-36\alpha_o^4)^{-1}\, T^{-\frac{2}{3}} + (8\pi^2\sqrt{2})^{-1}\, T^{-\frac{3}{4}} + \ldots\ldots, \qquad T \gg 1$$

$$a_3 \sim -\frac{1}{4} - \frac{\alpha_o}{36\pi}\, T^{-\frac{1}{12}} + \ldots\ldots, \qquad (5.5.2)$$

$$a_6 \sim (-6\alpha_o^2)^{-1}\, T^{-\frac{1}{3}} - (36\,\alpha_o\pi)^{-1}\, T^{-\frac{5}{12}} + \ldots,$$

where α_o is given in (5.4.28) and

$$a_4 = a_5 = -4a_3, \quad \text{(see 5.2.5)}$$

It should be noted that the major contributions are from the inviscid region and only the leading terms in (5.5.2) are the same as in the stress-free case (Daniels 1978). The asymptotic behaviour is shown in figures (5.5) - (5.8) where a comparison is made with the numerical results given in Table 5.1.

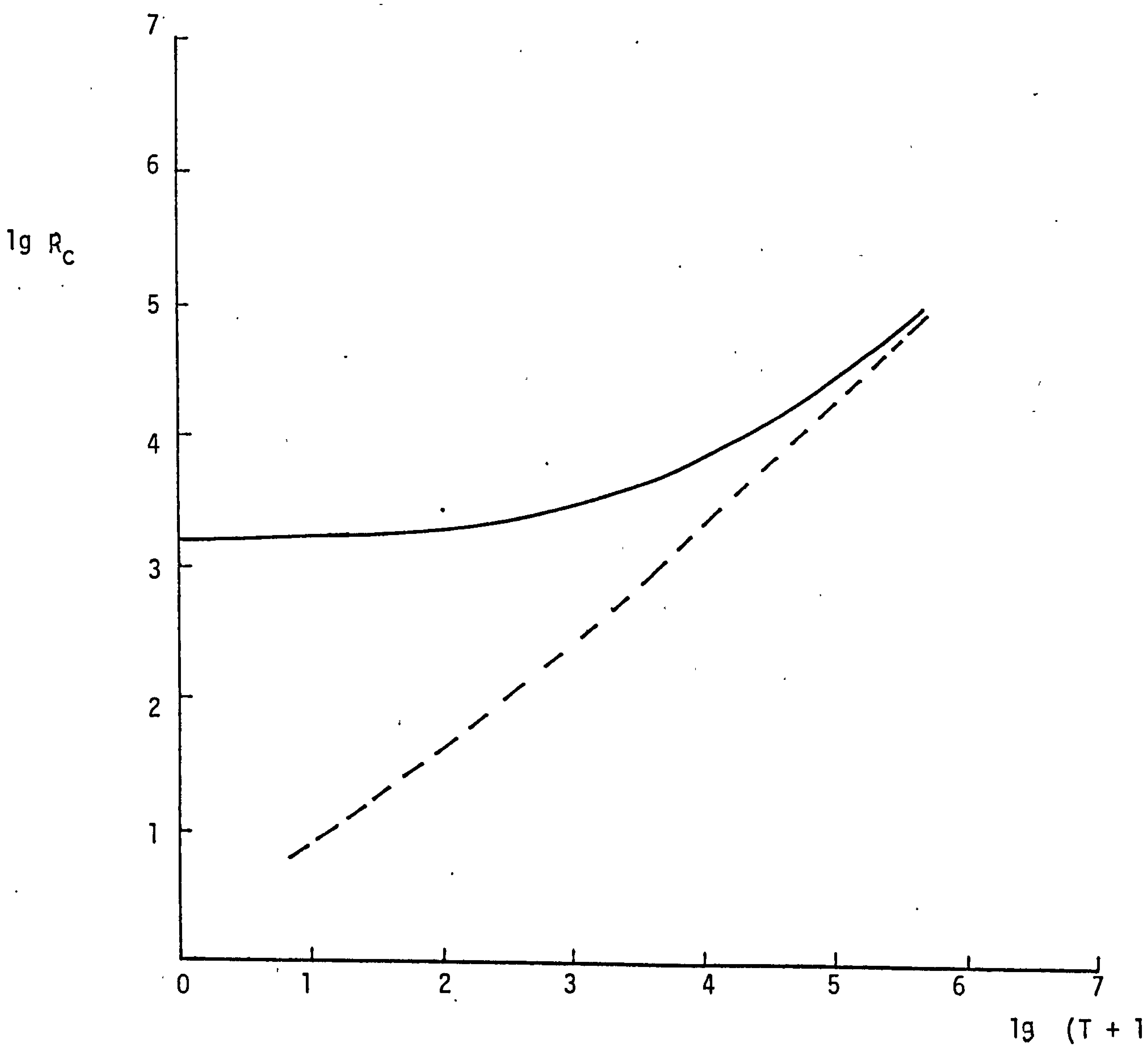

Figure 5.3: The (R_c, T) - relation.

——————— Numerical Result.

- - - - - - - Asymptotic Result

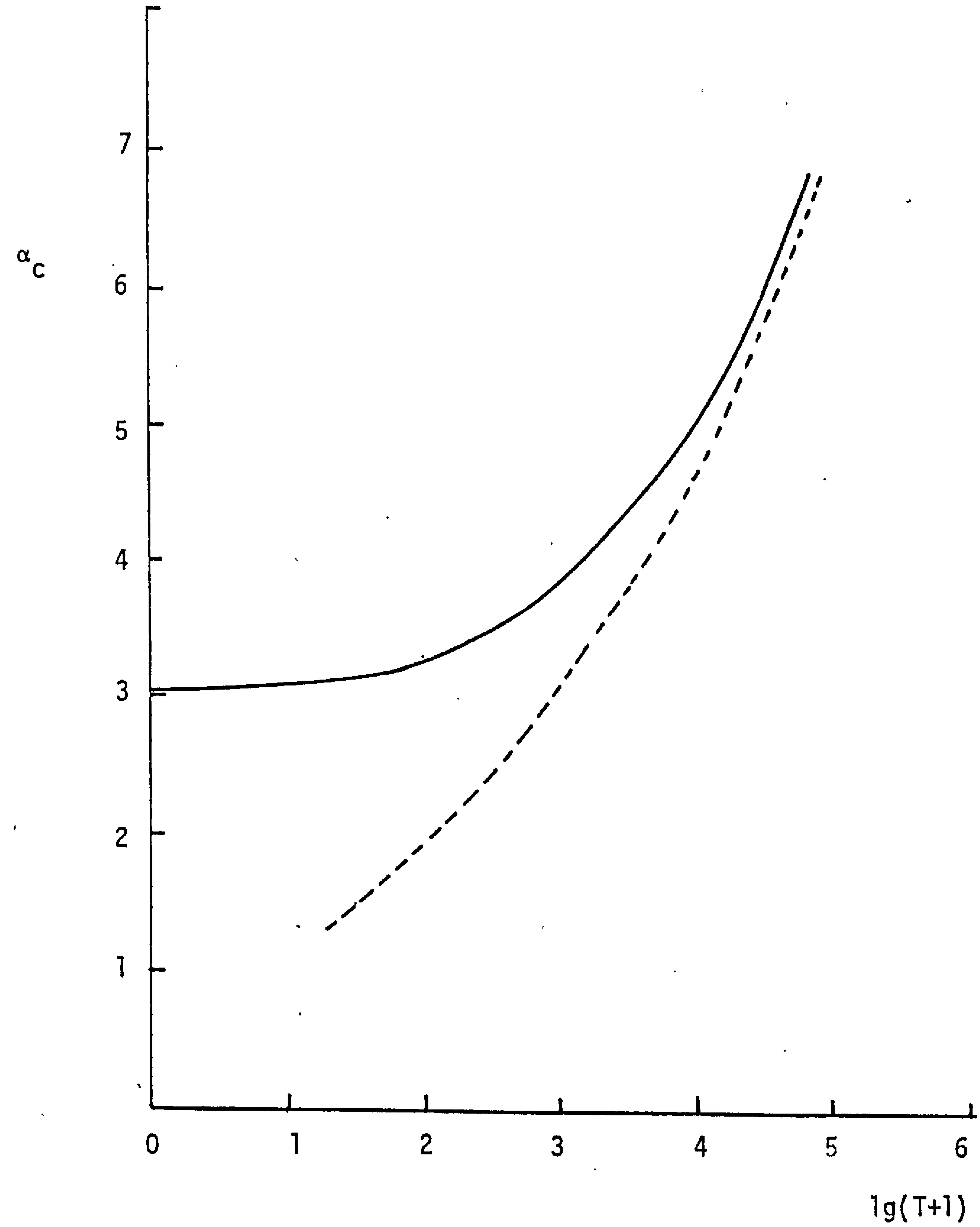

Figure 5.4: The (α_c, T)-relation.

------------ Asymptotic result

———————— Numerical result

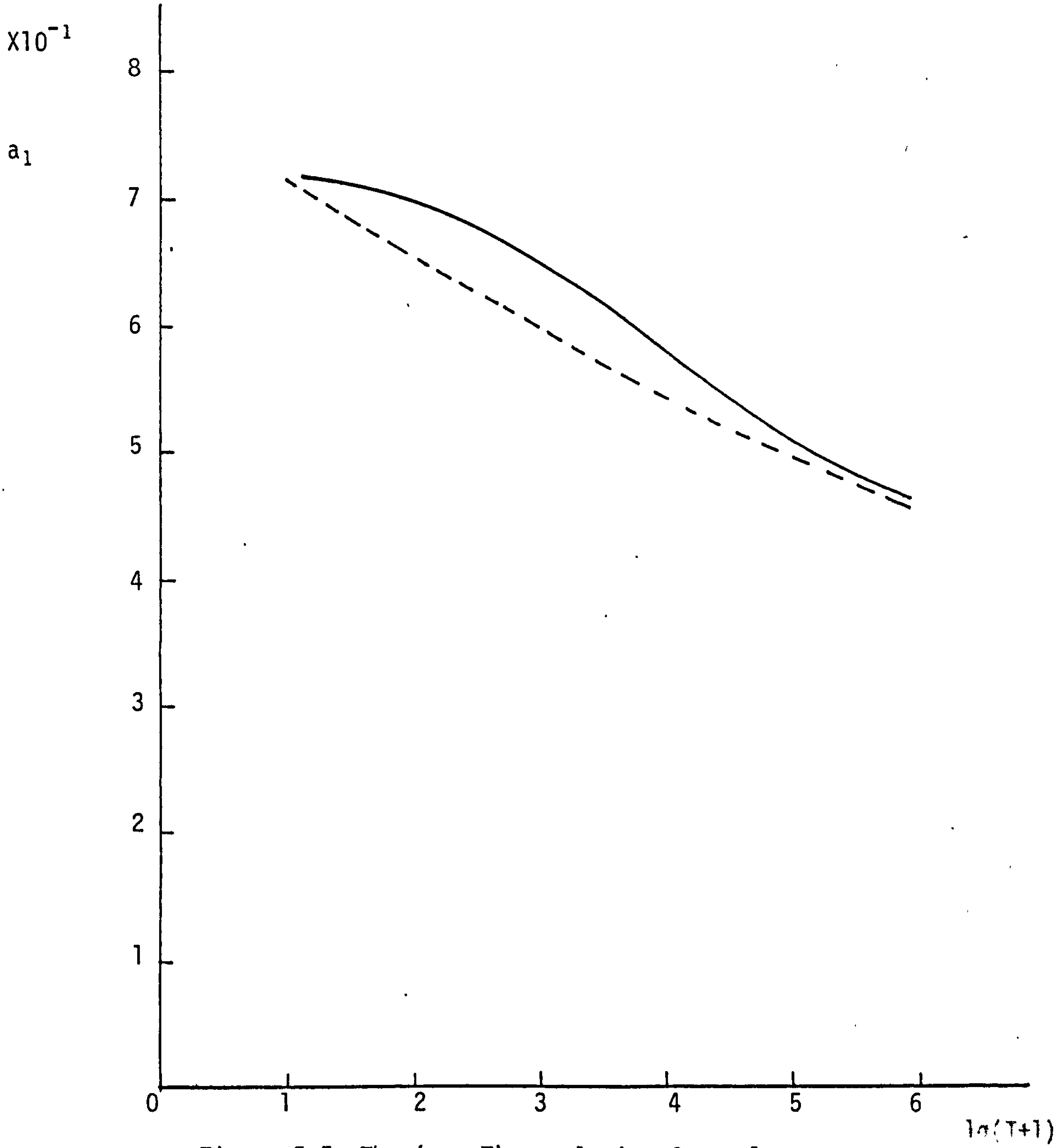

Figure 5.5: The (a_1, T) - relation for σ=1.

------------Asymptotic result.

———— Numerical result.

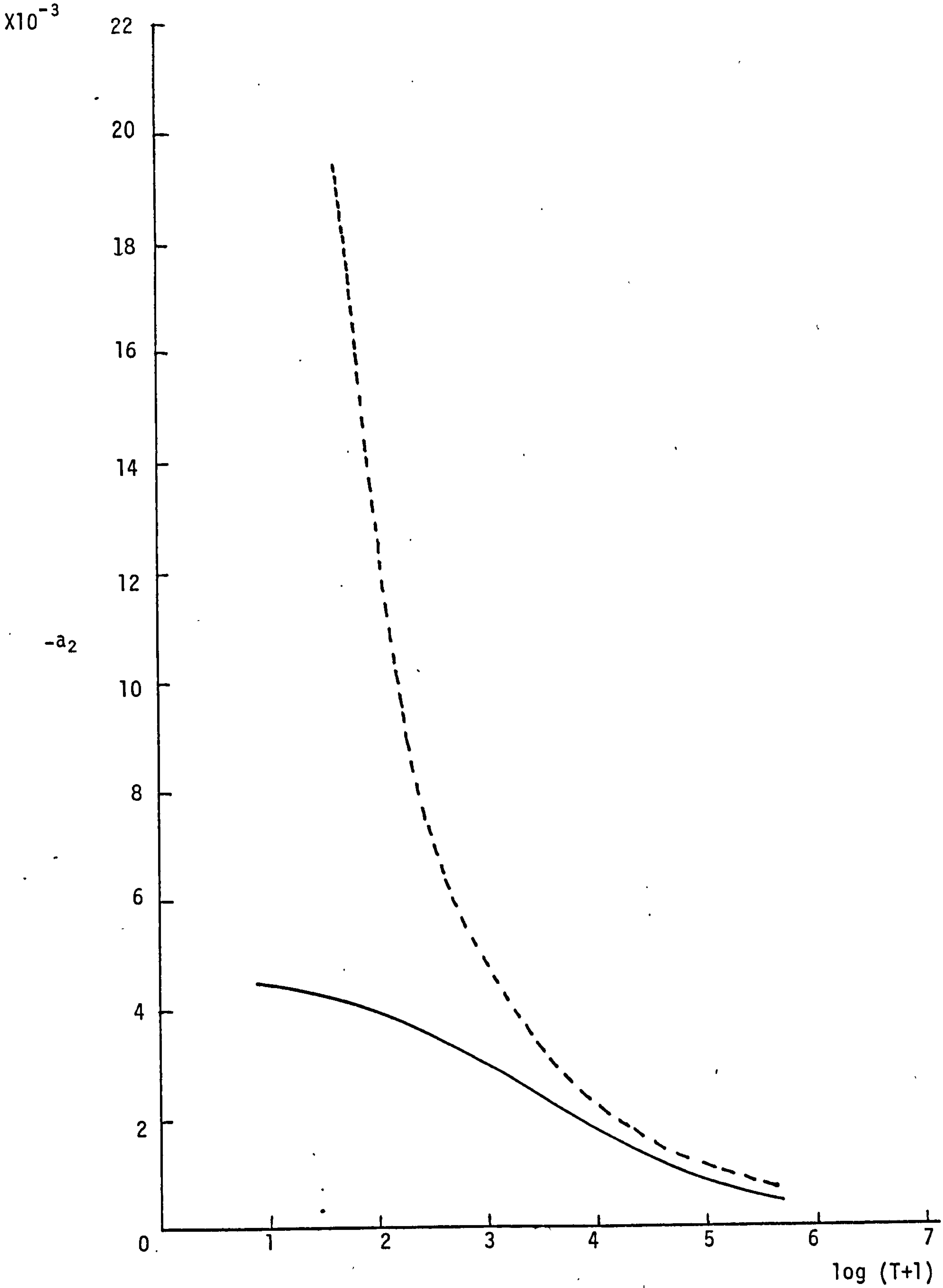

Figure 5.6: The (a_2,T) relation for $\sigma = 1$

---------------- Asymptotic Result

________________ Numerical Result

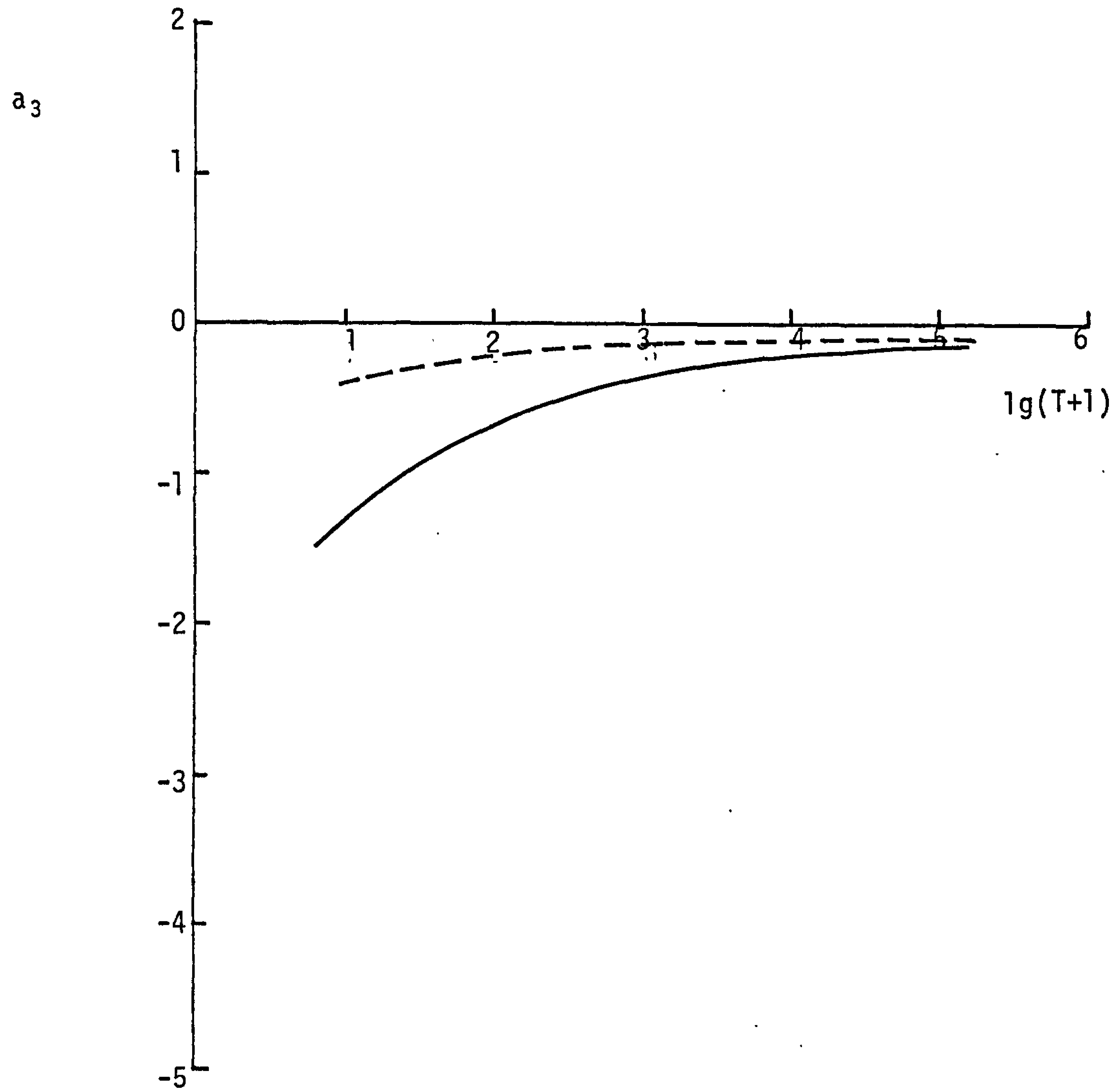

Figure 5.7: The (a_3,T)-relation for $\sigma=1$.

------------- Asymptotic result.

———————— Numerical result.

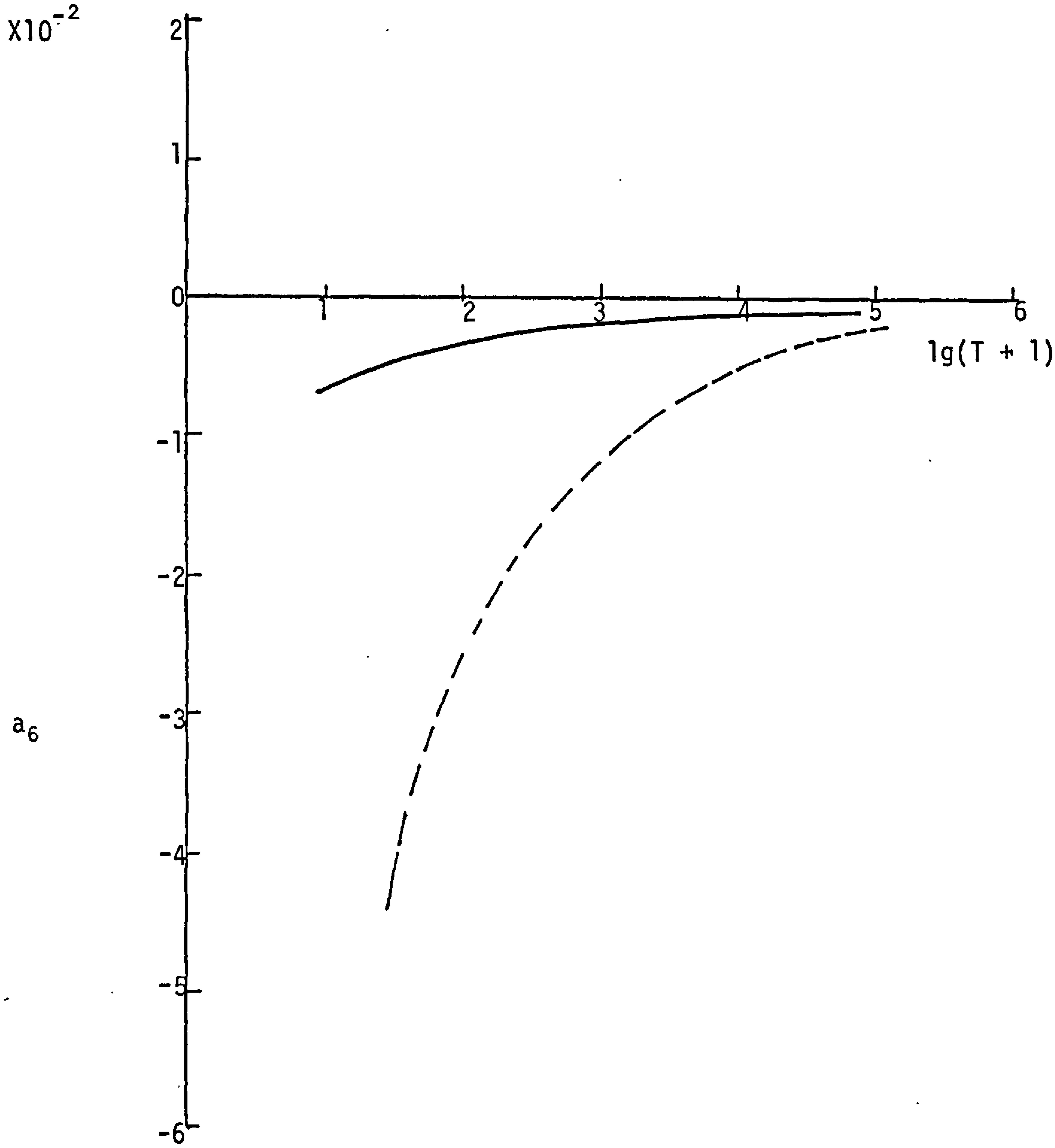

Figure 5.8: The (a_6, T) - relation for σ = 1

-------------- Asymptotic Result

______________ Numerical Result.

CHAPTER 6

Non Linear Rotating Benard Problem:

Matching at the Axis of Rotation and Solution of the Amplitude Equation

For stress-free horizontal boundaries the amplitude equation is also of the form (5.2.7) and is, in fact, of the same form as that derived by Brown and Stewartson (1978) for the non-rotating problem with stress-free boundaries. A linear inner solution for the stress-free case can also be constructed in terms of Bessel functions of the same form as that found by Brown and Stewartson:

$$\theta = (aJ_o(\alpha_c r)+brJ_o'(\alpha_c r))\sin\pi z$$

except that the characteristic equation relating α_c and R_c now also depends on the Taylor number, T. Thus matching between the inner and outer zones in the stress-free rotating problem follows exactly as in the non-rotating problem studied by Brown and Stewartson and leads to a boundary condition for A at s = 0 of the form $A(0, \tau) = 0$. We now establish that a similar condition applies in the case of rigid boundaries.

6.1. Inner Solution and Matching Process

Following the method used in section 3.7, in the neighbourhood of r = 0 we work in terms of the variables r and τ and consider a solution of (4.1) - (4.5) in which the non-linear terms are neglected. In view of (4.1.1) one solution is given by

$$(u, v) = J_o' (\alpha_c r)(f', k),$$
$$(w, \theta) = J_o (\alpha_c r)(\alpha_c f, h), \qquad (6.1.1)$$

where f, k, h are given in (4.1.1), with $\alpha = \alpha_c$ and $R = R_c$. J_o is the zeroth order Bessel function. A second solution (cf. (3.7.1)) is

$$\theta = H(z)\, J_o(\alpha_c r) + rh(z)\, J_o' (\alpha_c r), \qquad (6.1.2)$$

where

$$H = \frac{\partial h}{\partial \alpha},$$

and similar expressions can be found for the other components of the motion (cf.(3.7.1)). Discarding other solutions which are exponentially large as $r\to\infty$, the general solution for θ in the inner zone may now be written in the form

$$\theta_I = \lambda\, J_o(\alpha_c r)h + \mu \{ HJ_o(\alpha_c r) + rJ_o'(\alpha_c r)h\}, \qquad (6.1.3)$$

(see 3.7.4), where the numerical solutions for h and H are given in § 4.2, § 4.3, and λ, μ are functions of t. θ_I denotes the inner solution and we use θ_O to denote the outer solution already found. In order to match (6.1.3) with the outer solution (5.1.3, θ component), we need the behaviour of the amplitude function A (which satisfies (5.2.7)), as $s\to 0$, which is found to be

$$A \sim a + bs + \text{sgn}(\bar{\xi})\, a|a|^2\, s \ln s + O(s^2 \ln s)(s \to 0), \quad (6.1.4)$$

where a and b are functions of τ.

From (5.1.3), (5.1.6) and (5.2.7) in the outer region θ_0 is given by

$$\theta_0 = (|\bar{\xi}|s)^{-\frac{1}{2}} \{\varepsilon(Ae^{i\hat{r}} + C.C)h + \varepsilon^2(\frac{\partial A}{\partial s} e^{i\hat{r}} + C.C)H + \dots\} \quad (6.1.5)$$

$$s = \varepsilon r, \ \hat{r} = \alpha_c r - \frac{\pi}{4}$$

where we take $\lambda_0 = \frac{\pi}{4}$ in (5.2.6) for convenience

$$\theta_0 = (\frac{\varepsilon}{r|\bar{\xi}|})^{\frac{1}{2}}\{((A+C.C)\cos\hat{r} + i(A-C.C)\sin\hat{r})h + \varepsilon((\frac{\partial A}{\partial s} + C.C)\cos\hat{r} + i(\frac{\partial A}{\partial s} - C.C)\sin\hat{r})H + \dots\} \quad (6.1.6)$$

where C.C denotes complex conjugate. Now the asymptotic expansion of (6.1.3) for large r is

$$\theta_I \sim (2/\pi\alpha_c r)^{\frac{1}{2}}\{(\lambda - \frac{3\mu}{8\alpha_c})h \cos\hat{r} + \mu H\cos\hat{r} - \mu r \sin\hat{r}\} \quad (r \gg 1) \quad (6.1.7)$$

from (6.1.4) and comparing (6.1.7) and (6.1.6) we see that a match of the terms $\frac{h}{r^{\frac{1}{2}}} \cos\hat{r}$, $hr^{\frac{1}{2}} \cos\hat{r}$, $\frac{h}{r^{\frac{1}{2}}} \sin\hat{r}$, $hr^{\frac{1}{2}} \sin\hat{r}$ is secured if respectively,

$$\begin{aligned}
& a - a^* = 0 \\
& (a + a^*)\varepsilon^{\frac{1}{2}} - (\frac{2|\xi|}{\pi\alpha_c})^{\frac{1}{2}}(\lambda - \frac{3\mu}{8\alpha_c}) = 0, \\
& b + b^* + |a|^2 (a + a^*) \,\text{sgn}\,(\bar{\xi}) \ln\varepsilon = 0, \qquad (6.1.8) \\
& i(b - b^*)\varepsilon^{\frac{3}{2}} + \mu\,(\frac{2|\bar{\xi}|}{\pi\alpha_c})^{\frac{1}{2}} = 0.
\end{aligned}$$

Here * denotes complex conjugate. It should also be noted that the terms in $\frac{H}{r^{\frac{1}{2}}} \sin\hat{r}$, $\frac{H}{r^{\frac{1}{2}}} \cos\hat{r}$ match automatically.

From the matching conditions (6.1.8), we find that a is real and

$$a^3 \sim -\tfrac{1}{2}\mathrm{Sgn}(\bar{\zeta})(b+b^*)/\ln\varepsilon \cdot \qquad (6.1.9)$$

First suppose that $|b| = O(1)$, which is consistent with the assumption that $|A|$ in (5.2.7) is $O(1)$ when

$$s = O(1); \text{ then}$$

$$a = O(-\ln\varepsilon)^{-\frac{1}{3}}. \qquad (6.1.10)$$

Thus equation (5.2.7) should be solved in the form of a series for A in ascending powers of $(-\ln\varepsilon)^{-\frac{1}{3}}$ with the first approximation having

$$a = 0, \qquad (6.1.11)$$

so that in the limit $\varepsilon \to 0$, the boundary condition for A is given by

$$A(0,\tau) = 0, \qquad (6.1.12)$$

This condition is the same as that in the case of a non-rotating layer with stress-free boundaries, as studied by Brown and Stewartson (1978).

6.2 Phase Winding Effects

In section (6.1) we restricted ourselves to the assumption that (6.1.10) implies that $a = 0$ in the limit $\varepsilon \to 0$. Here we extend the work of § 6.1, to consider the result that $a = O(-\ln\varepsilon)^{-\frac{1}{3}}$ in more detail. In order to do this, the formal structure of the solution of the amplitude equation (5.2.7) is considered in various regions (Figure 6.1) corresponding to different orders of magnitude of s. For example, it is clear that if $a=O((-\ln\varepsilon)^{-\frac{1}{3}})$, the expansion (6.1.4) is not uniformly valid and may fail when $s=O((-\ln\varepsilon)^{-\frac{1}{3}})$. As we have already mentioned, the amplitude function, A, should be expanded in the form of a series in ascending powers of $(-\ln\varepsilon)^{-\frac{1}{3}}$, and this is done in three different regions I, II, III (Figure 6.1) as follows.

To fix ideas, let $\mathrm{sgn}(\bar{\xi}) = 1$ in (5.2.7) and also assume steady flow, so that $\frac{\partial A}{\partial \tau} = 0$. Then

in I: $A(s) = A_{o1} + \bar{\varepsilon} A_{o2} + \bar{\varepsilon}^2 A_{o3} + \ldots\ldots.$ (6.2.1)

where $\bar{\varepsilon} = (-\ln\varepsilon)^{-\frac{1}{3}}$, $s = O(1)$ (6.2.2)

and each A_{oi} $(i = 1, 2, \ldots)$ is a function of s.

In II: $A(s_1) = \bar{\varepsilon} A_{10} + \bar{\varepsilon}^2 A_{20} + \bar{\varepsilon}^3 A_{30} + \ldots,$ (6.2.3)

where $s = \bar{\varepsilon}\, s_1$, $s_1 = O(1)$ (6.2.4)

and each A_{io} $(i = 1, 2, \ldots)$ is a function of s_1.

In III, where $r = O(1)$ or $s = O(\varepsilon)$, (linear zone) the solution is given in terms of Bessel functions, $J_o(\alpha_c r)$, the θ component being given by (6.1.2).

In the main zone, I, it is actually convenient to reformulate the expansion (6.2.1) by setting

$$A = R(s)\, e^{i\bar{\theta}(s)}; \qquad (6.2.5)$$

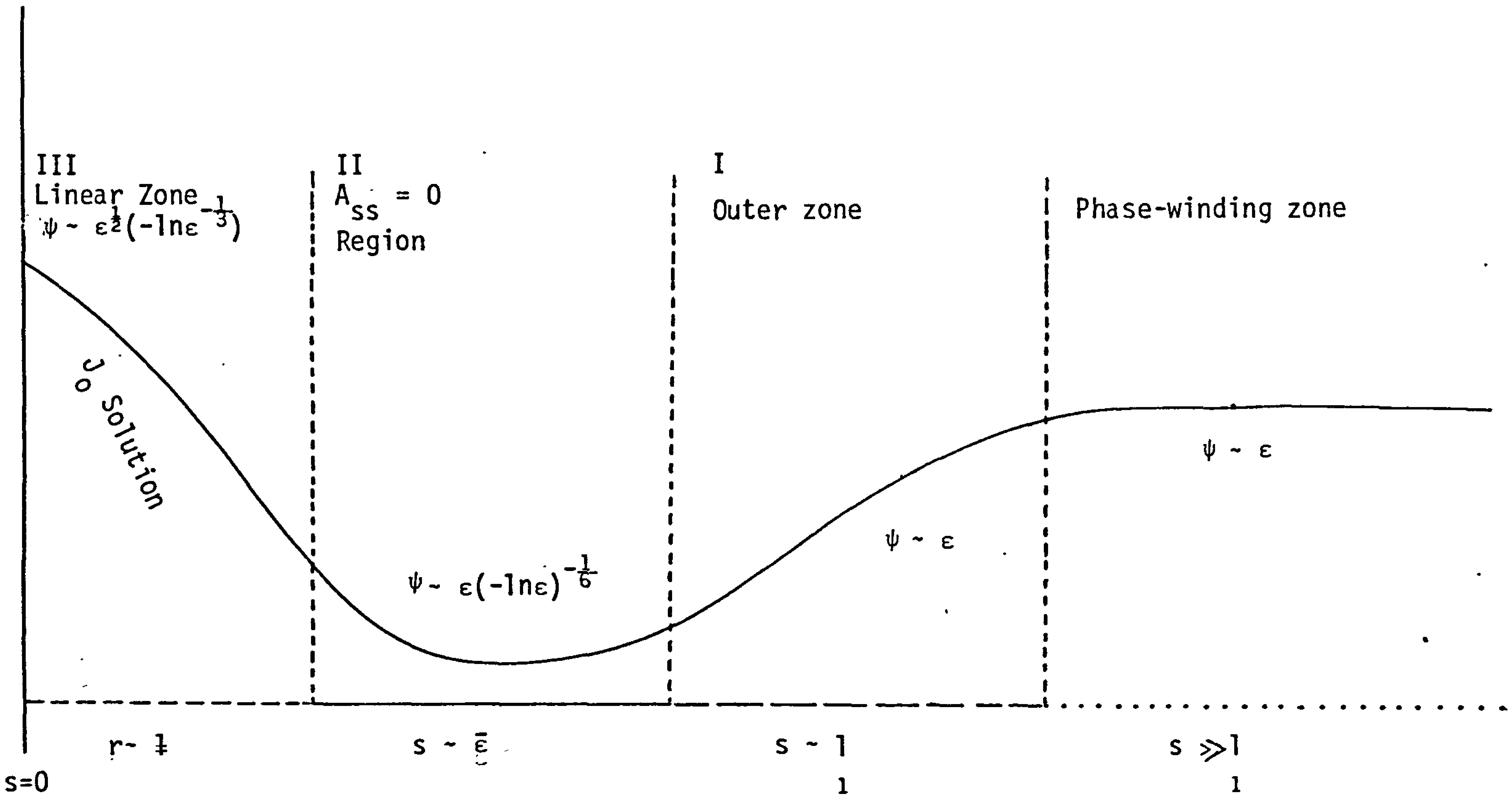

Figure 6.1: Actual Structure of Solution.
$\bar{\varepsilon} = (-\ln \varepsilon)^{-\frac{1}{3}}$.

then from (6.2.1) and (5.2.7), we find that for the supercritical case

$$\frac{d^2R}{ds^2} + ((1 - (\frac{d\bar{\theta}}{ds})^2) R = R^3/s, \qquad (6.2.6)$$

$$\frac{d\bar{\theta}}{ds} = c/R^2 , \qquad (6.2.7)$$

where c is an arbitrary constant.

The expansions of $R, \bar{\theta}$, c in terms of $\bar{\varepsilon}$, equivalent to (6.2.1) above, are then given by

$$R(s) = R_o + \bar{\varepsilon} R_1 + \bar{\varepsilon}^2 R_2 + \ldots\ldots\ldots,$$

$$\bar{\theta}(s) = \gamma + \bar{\varepsilon} \theta_o + \bar{\varepsilon}^2 \theta_1 + \ldots\ldots\ldots, \qquad (6.2.8)$$

$$c = \bar{\varepsilon} c_1 + \bar{\varepsilon}^2 c_2 + \ldots\ldots\ldots\ldots\ldots\ldots,$$

where R_i, θ_i $(i = 0, 1, \ldots)$ are functions of s and each $c_i (i = 1, 2,..)$ is a constant. The leading term, γ, in the expansion for $\bar{\theta}$ is constant since if this were not the case the form (6.2.8) combined with the requirements that $R_o \to 0$ as $s \to 0$ would lead to an unacceptable behaviour of $\bar{\theta}$ at the origin.

Upon substitution of (6.2.8) into (6.2.6) and (6.2.7), and from leading and higher orders, we find that

$$O(1): \quad \frac{d^2R_o}{ds^2} + R_o = R_o^3/s\,, \qquad (6.2.9)$$

$$O(\bar{\varepsilon}): \quad \frac{d^2R_1}{ds^2} + R_1 = 3R_o^2R_1/s\,, \qquad (6.2.10)$$

$$\frac{d\theta_o}{ds} = c_1/R_o^2\,, \qquad (6.2.11)$$

$$O(\bar{\varepsilon}^2): \quad \frac{d^2R_2}{ds^2} + R_2 = R_o\left(\frac{d\theta_o}{ds}\right)^2 + 3R_o(R_oR_2 + R_1^2)/s. \qquad (6.2.12)$$

From (6.1.12) we require $R_o(0) = 0$; then from (6.2.9), the behaviour of R_o as $s \to 0$ is given by

$$R_o \sim a_1(s - s^3/6 + a_1^2\, s^4/12 + \ldots\ldots), \qquad (s \to 0) \quad (6.2.13)$$

a_1 being the gradient of R_o at the origin.

Equation (6.2.11), and (6.2.13) imply that

$$\theta_o \sim -c_1/(a_1^2 s) \qquad (s \to 0). \qquad (6.2.14)$$

We note that we now have

$$A = e^{i\gamma}\{R_o + \bar{\varepsilon}(R_1 + i\theta_o R_o) + \ldots\} \qquad (6.2.15)$$

in the region where $s = O(1)$, and comparing (6.2.1) with (6.2.15) we see that

$$A_{o1} = R_o\, e^{i\gamma} \qquad (6.2.16)$$

$$A_{o2} = e^{i\gamma}(R_1 + i\,\theta_o\, R_o), \qquad (6.2.17)$$

where γ is an arbitrary constant.

Now consider region II, where substitution of (6.2.3) into (5.2.7), gives, from leading order and higher orders,

$$
\begin{aligned}
O(\bar{\varepsilon}):&\quad \frac{d^2A_{1o}}{ds_1^2} = 0,\\
O(\bar{\varepsilon}^2):&\quad \frac{d^2A_{2o}}{ds_1^2} = 0,\\
O(\bar{\varepsilon}^3):&\quad \frac{d^2A_o}{ds_1^2} + A_{1o} = 0,\\
O(\bar{\varepsilon}^4):&\quad \frac{d^2A_{4o}}{ds_1^2} + A_{2o} = A_{1o}|A_{1o}|^2/s_1^2.
\end{aligned}
\qquad (6.2.18)
$$

From the leading order solution it is clear that the expansion of A begins

$$A \sim \bar{\varepsilon}\,(\bar{a} + \bar{b}\, s_1) + \ldots\ldots, \qquad (6.2.19)$$

where $\bar{a}$, $\bar{b}$ are arbitrary complex constants.

Matching with the linear zone (III), gives

$$
\begin{aligned}
&\bar{a} - \bar{a}^* = 0,\\
&\bar{b} + \bar{b}^* - (\bar{a} + \bar{a}^*)|\bar{a}|^2 \doteq 0;
\end{aligned}
\qquad (6.2.20)
$$

it also suggests that the constant λ and μ in (6.1.3) have the form

$$\lambda = \bar{\varepsilon}^{\frac{1}{2}}\,\bar{\varepsilon}\,\bar{\lambda}\ , \quad \mu = \varepsilon^{\frac{3}{2}}\,\bar{\mu}$$

where $\bar{\lambda}$, $\bar{\mu}$ are O(1) constants.

Comparing (6.2.15) and (6.2.19) we see that a match of the two regions I and II is achieved if

$$\bar{b} = e^{i\gamma} a_I , \qquad (6.2.21)$$

and

$$R_1(0)\cos\gamma + \frac{c_1}{a_1}\sin\gamma = \bar{a},$$
$$R_1(0)\sin\gamma - \frac{c_1}{a_1}\cos\gamma = 0. \qquad (6.2.22)$$

Elimination of $R_1(0)$ between these equations gives

$$c_1 = \bar{a}\, a_1 \sin\gamma, \qquad (6.2.23)$$

and from (6.2.20) - (6.2.23), we find that

$$c_1 = a_1^{\frac{4}{3}} \sin\gamma \cos\gamma^{\frac{1}{3}} , \qquad (6.2.24)$$

and this is what we shall refer to as the phase winding parameter, since its value determines the variation in the phase of the complex amplitude function A with s. This, in turn, corresponds to a variation in the wavelength of the roll pattern. It is easy to show from (6.2.24) that

$$|c_1| < 3^{\frac{1}{2}}\; 2^{-\frac{4}{3}}\; a_1^{\frac{4}{3}} ,$$

so that the phase winding is restricted by an upper bound which depends on the slope of the main O(1) solution for R_o at the origin. The value of this slope is determined in section 6.3 below.

In the outer zone, I, we now have

$$A = (R_o + O(\bar{\varepsilon}))\, e^{i(\gamma + \bar{\varepsilon}\,\theta_o + \ldots)}.$$

where

$$\theta_o = c_1 \int \frac{ds}{R_o^2}. \qquad (6.2.25)$$

If $R_o \sim s^{\frac{1}{2}}$ as $s \to \infty$, (corresponding to a finite amplitude motion with $A_o = O(1)$) then

$$\theta_o \simeq c_1 \ln s$$

so that the phase winding effect due to a non-zero value of c_1 leads to an extra roll in a distance given by

$$\bar{\varepsilon} \ln s = O(1)$$

or

$$s \sim \exp\{(-\ln \varepsilon)^{\frac{1}{3}}\} \qquad (6.2.26)$$

The possible variation in wavelength of the roll pattern characterised by the phase winding parameter c_1, arises from the non-zero value of A at the origin in the matching process, and has been discussed in detail for a two-dimensional layer by Cross, Daniels, Hohenberg, Siggia (1980). Here the effect is only evident over a long length scale (6.2.26), because of the small magnitude of $A(0) = a = O(\bar{\varepsilon})$.

It should be noted that as $\epsilon \to 0$, the present structure is consistent with the initial onset of motion in the form of the linear Bessel function profile, $J_o(\alpha_c r)$, everywhere. For, as $\epsilon \to 0$, the outer zones I, II move off to infinity, due to their radial scaling with $\epsilon (r = O(\epsilon^{-1})$ and $r = O(\epsilon^{-1}(-\ln\epsilon)^{-\frac{1}{3}}))$ and the solution where $r = O(1)$ is dominated by the form (6.1.1)

6.3. Solution of the Amplitude Equation

In this section we consider numerical and asymptotic solutions of the amplitude equation (5.2.7). In view of the argument in §6.2 and from the leading order terms in (6.2.1), we can write

$$A(s) = e^{i\gamma} R_o(s).$$

For supercritical Rayleigh numbers R_o satisfies

$$\frac{d^2 R_o}{ds^2} + R_o - \mathrm{sgn}(\bar{\xi})\, R_o^3/s = 0, \qquad (6.3.1)$$

and for subcritical Rayleigh numbers, R_o satisfies

$$\frac{d^2 R_o}{ds^2} - R_o - \mathrm{sgn}(\bar{\xi})\, R_o^3/s = 0. \qquad (6.3.2)$$

Also $\quad R_o(0) = 0.$

Supercritical Solutions

(i) $\bar{\xi} > 0$

Here $\mathrm{sgn}(\bar{\xi}) = 1$, and R_o can be expanded as a power series in s as $s \to 0$:

$$R_o = a_1 \left(s - s^3/6 + a_1^2\, s^4/12 + s^5/120 + \cdots\right) \qquad (6.3.3)$$

where a_1 is an arbitrary constant and

$$\frac{dR_o}{ds} = a_1 \quad (s \to 0) \qquad (6.3.4)$$

The value of a_1 plays a crucial role in the analysis of section 6.2, its value providing an upper bound on the amount of phase winding, or wavelength variation. If we regard a_1 as known, (6.3.1), (6.3.3) and (6.3.4) constitute an initial value problem for R_o. In this way solutions for R_o can be computed by marching from the origin,using the Taylor series expension (6.3.3) for the first few steps, and observing the behaviour of R_o as $s \to \infty$. A fourth order Runge-Kutta scheme was used (cf.(2.3)) and several profiles of R_o for different values of a_1 are given in figure 6.2. If a_1 is too large, R_o tends to infinity at a finite value of s, while if a_1 is too small the solution for R_o passes through zero and oscillates as $s \to \infty$. The required solution, which is likely to represent a stable flow pattern is the intermediate one for which

$$R_o \sim s^{\frac{1}{2}} \text{ as } s \to \infty\,, \qquad (6.3.5)$$

Figure 6.2 : Profiles of R_o(supercritical solution for $\bar{\xi} > 0$)

and an iteration process based on successive adjustment of the value of a_1 was used to obtain this solution which is shown in Fig. 6.2. The required value of a_1 was found to be

$$a_1 \doteq 0.88659. \qquad (6.3.6)$$

We now discuss in some detail the asymptotic expansion of R_o when s is large.

As can be seen from figure 6.2, there are three possible behaviours of R_o which correspond to the curves (a), (b), (c) in figure 6.2. The appropriate asymptotic forms of each curve are as follows

$$\text{(a)} \quad : \quad R_o \sim s^{\frac{1}{2}} - \frac{1}{8} s^{-\frac{3}{2}} \qquad (s \to \infty), \qquad (6.3.7)$$

$$\text{(b)} \quad : \quad R_o \sim \sqrt{2}\, s_o^{\frac{1}{2}} (s_o - s)^{-1} \qquad (s \to s_o), \qquad (6.3.8)$$

$$\text{(c)} \quad : \quad R_o \sim a \sin(f(s)) \qquad (s \to \infty). \qquad (6.3.9)$$

In (6.3.9), a is an arbitrary constant and

$$f(s) \sim s \quad (s \to \infty)$$

indicating a balance between the linear terms in (6.3.1). However, the nonlinear term does have a significant effect on the asymptotic structure and it emerges that, f(s) may be written as

$$f(s) = s + b \ln s + c\,, \qquad (6.3.10)$$

and

$$R_o = a \sin\{f\} + \frac{1}{s} h_1\{f\} + \frac{1}{s^2} h_2\{f\} + \ldots \quad (s \to \infty) \qquad (6.3.11)$$

On substitution of (6.3.10) - (6.3.11) in (6.3.1), and from leading order and higher order terms, we find that

$$O(1): a \sin f - a \sin f = 0 \quad .$$

$$O(\tfrac{1}{s}): L_6\{h_1\} = a^3 \sin^3 f + 2a\, b \sin f \qquad (6.3.12)$$

$$O(\tfrac{1}{s^2}): L_6\{h_2\} = 3a^2 h_1 \sin^2 f + 2\frac{dh_1}{df} - 2b\frac{d^2h_1}{df^2} + a\, b \cos f + a\, b^2 \sin f \qquad (6.3.13)$$

where $L_6 = \frac{d^2}{df^2} + 1.$ (6.3.14)

From the right-hand side of the equation (6.3.12) and the form of the operator (6.3.14) it can be seen that the general solution of (6.3.12) contains a secular term. This must be avoided by choosing the coefficient of sin f on the right-hand side of (6.3.12) equal to zero. It is then found that

$$b = -\tfrac{3}{8} a^2. \qquad (6.3.15)$$

The particular solution of (6.3.12) is given by

$$h_{1p} = C_1 \sin 3f \qquad (6.3.16)$$

where

$$C_1 = a^3/32,$$

and $h_1(f) = a_1 \sin f + b_1 \cos f + a^3/32 \sin 3f.$ (6.3.17)

From substitution of (6.3.17) in the right-hand side of (6.1.13), and equating the coefficients of sin f and cos f to zero we find that

$$a_1 = \frac{3}{16} a^3, \; b_1 = \frac{51}{256} a^5. \tag{6.3.18}$$

Hence the asymptotic expension of R_o is given by

$$R_o \sim a \sin f + a^3/_s\{\frac{3}{16} \sin f + \frac{51}{256} a^2 \cos f + \frac{\sin 3f}{32}\} + \ldots . \tag{6.3.19}$$

$$(s \to \infty)$$

where

$$f(s) = s - \tfrac{3}{8} a^2 \ln s + c, \tag{6.3.20}$$

and a, c are arbitrary constants.

The result (6.3.20) indicates how the wavelength of oscillation varies with s. If ω is the local wavelength near $s = s_o$, so that

$$s_o + b \ln s_o = n\pi$$

$$(\omega + s_o) + b \ln(\omega + s_o) = n\pi + \pi \tag{6.3.21}$$

then

$$\omega \sim \pi(1 - b s_o), (s_o \to \infty) \tag{6.3.22}$$

The numerical results (Fig. 6.2, c) exhibit this behaviour, as shown in Table 6.1. The values of b obtained using (6.3.21) and the values of the amplitude of the oscillation a (see Table 6.1) are also found to be consistent with (6.3.15).

s	ω		b	a
	Numerical results	Asymptotic results	$(\pi-\omega)/\ln(1+\frac{\omega}{s})$	
6.1	3.6	3.36	-0.49	-1.08
9.4	3.3	3.29	-0.49	1.06
12.72	3.2	3.25	-0.47	-1.06
15.9	3.2	3.22	-0.43	1.06
19.2	3.2	3.21	-0.42	-1.058
22.4	3.19	3.20	-0.42	1.057
25.6	3.18	3.19	-0.41	-1.056
28.8	3.17	3.18	-0.41	1.055
31.9	3.17	3.18	-0.41	-1.054
35.1	3.16	3.17	-0.405	1.054
38.3	3.16	3.17	-0.395	-1.054
41.5	3.15	3.16	-0.393	1.054
44.6	3.145	3.156	-0.3925	-1.053
47.8	3.144	3.154	-0.3923	1.053
50.9	3.1445	3.153	-0.3922	-1.053
54.1	3.1443	3.149	-0.3921	1.052
57.3	3.1443	3.145	-0.3921	-1.052
60.4	3.1443	3.145	-0.3921	1.052

Table 6.1: Comparison between numerical results and asymptotic results for $a_1 = 0.8$

(ii) $\bar{\xi} < 0$

In this case numerical solutions of the equation (6.3.1) can be found as in section (i) although all solutions for different values of a_1 now oscillate as $s\to\infty$; (Fig 6.3).

The only possible behaviour of R_0 as $s\to\infty$ is given by

$$R_0 = a\sin f + a^3/s\{-\frac{3}{16}\sin f + \frac{51}{265}a^2\cos f - \frac{\sin 3f}{32}\} + \ldots \quad (6.3.23)$$
$$(s\to\infty)$$

where

$$f(s) = s + b\ln s + C$$

and

$$b = \tfrac{3}{8}a^2. \quad (6.3.24)$$

Again the numerical results and asymptotic results are found to be in good agreement.

Figure 6.3: Profiles of R_o (supercritical solution for $\bar{\xi} < 0$)

Subcritical Solutions

(i) $\xi > 0$

Numerical solutions can be obtained using the method of section (i), but now all solutions of R_o in equation (6.3.2), for different values of a_1, are unbounded (Fig. 6.4).

It is easy to show that R_o are increasing functions of s.

From (6.3.2)

$$R_o^{2'} = R_o^{2'}(0) + R_o^2 + \int_0^s \frac{R_o^3 R_o'}{s} \, ds \qquad (6.3.25)$$

and

$$R_o'' = R_o \left(1 + R_o^2/s\right) \qquad (6.3.26)$$

therefore R_o' is positive and R_o'' is also positive. Hence R_o and R_o' are increasing functions of s. The solution for R_o becomes unbounded in the form (6.3.8) at a finite value of $s = s_o$.

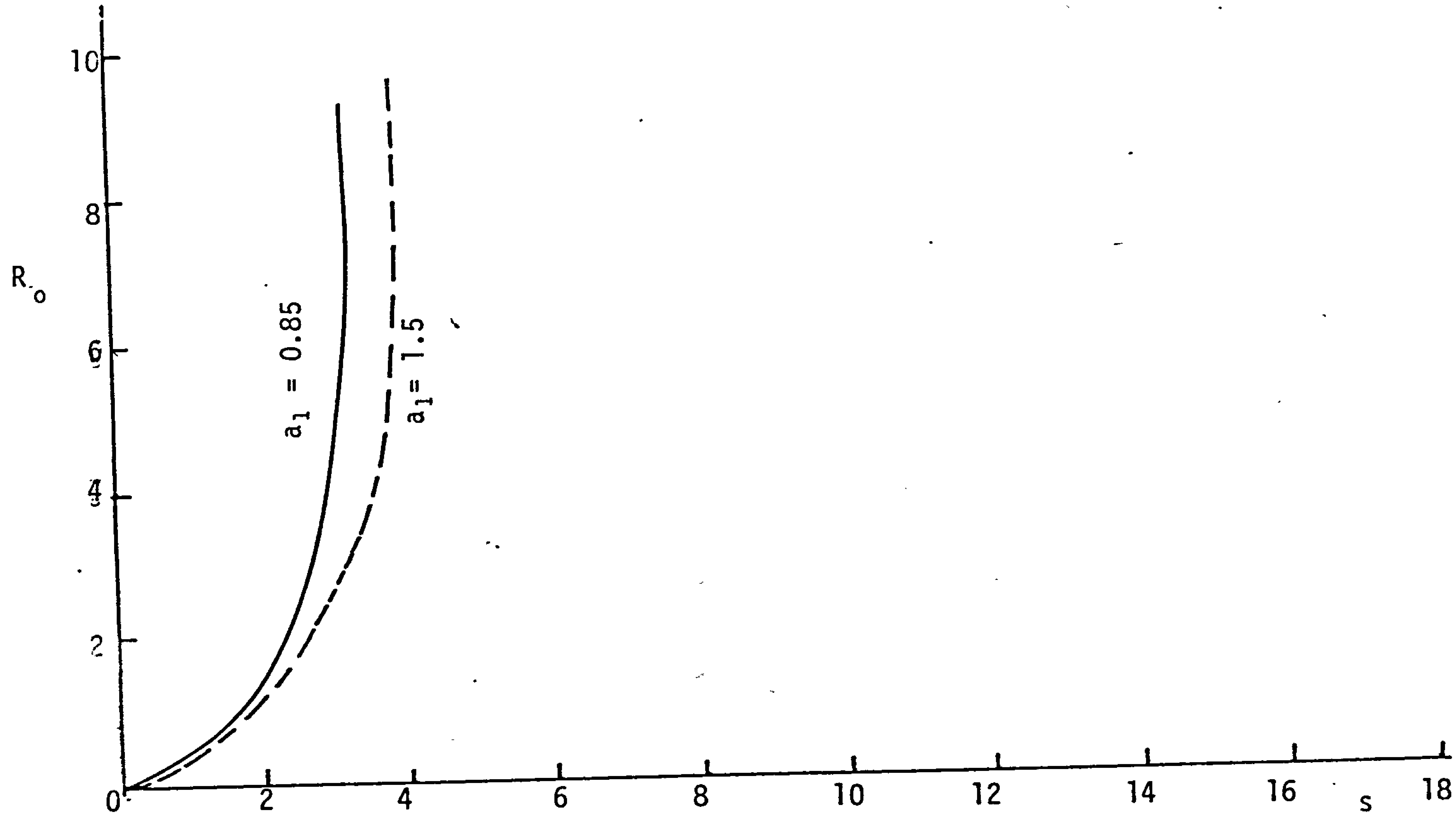

Figure 6.4: Profiles of R_o (Subcritical Solution for $\bar{\xi} > 0$).

(ii) $\bar{\xi} < 0$

The numerical solutions for different values of a_1 are given in Figures 6.5 - 6.7, but it seems likely that all such solutions correspond to unstable motions.

In general, the asymptotic form of R_o as $s \to \infty$ is given by

$$R_o \sim s^{\frac{1}{2}} + a \cos (\sqrt{2}\, s + b), \qquad (6.3.27)$$

although there is a family of eigenfunctions which decay exponentially as $s \to \infty$, with

$$R_o \sim c(e^{-s} - \frac{c}{8} e^{-3s}) \qquad (6.3.28)$$

where a, b, c are arbitrary constants. Solutions (6.3.28) occur at a discrete set of values of the parameter $a_1 = R_o'(0)$ and the first three such values, and corresponding eigenfunctions, are shown in figures 6.5 - 6.7. Each eigensolution is characterized by the number of zeros it possesses for $0 < s < \infty$.

Note that, in practice, the shooting method used to obtain the solutions in figures 6.5 - 6.7 inevitably leads to the eventual divergence into the form (6.3.27) although in each case the region of exponential decay, (6.3.28), is clearly visible.

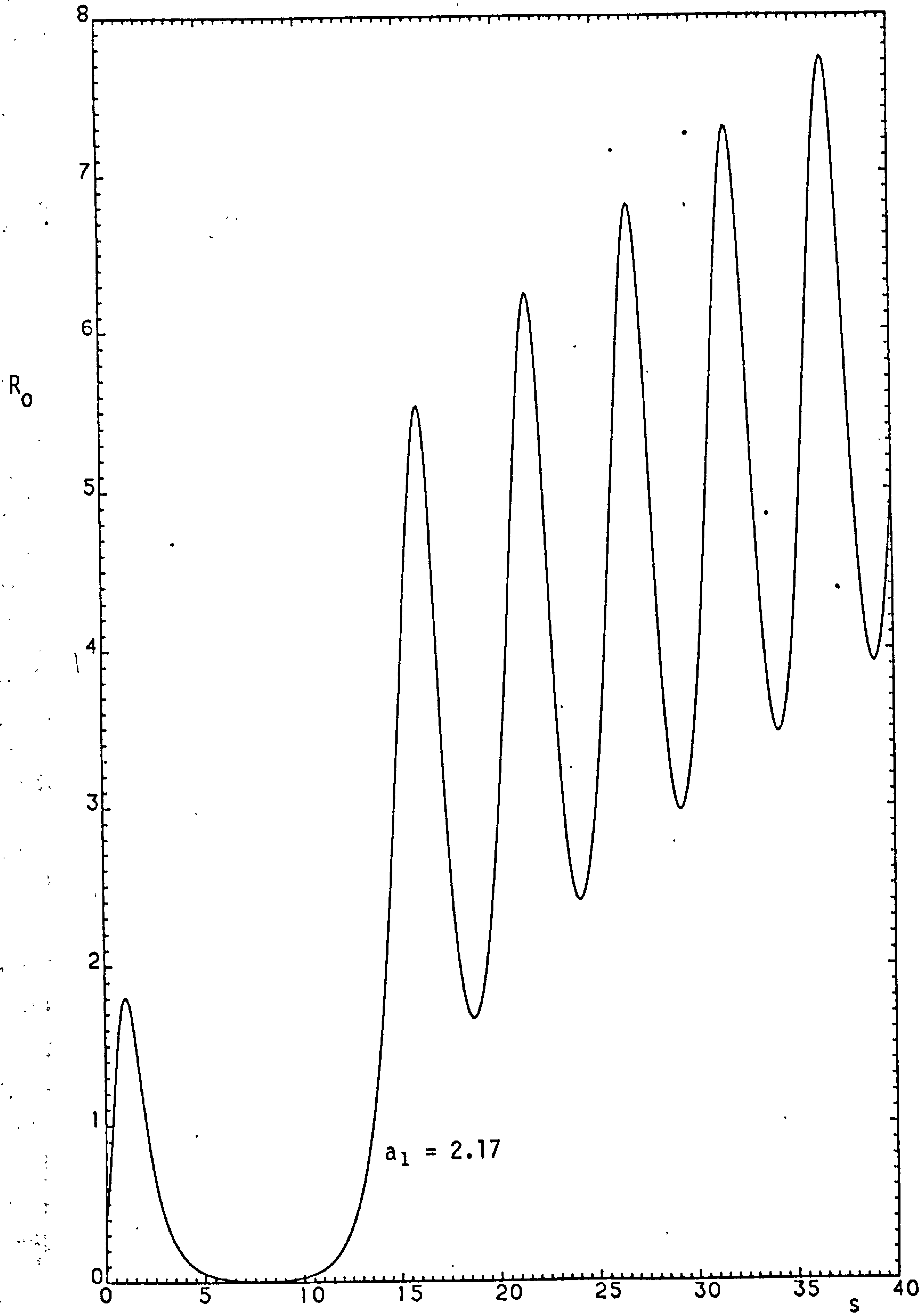

Figure 6.5 : Profile of R_o (Subcritical solution for $\bar{\xi} < 0$).

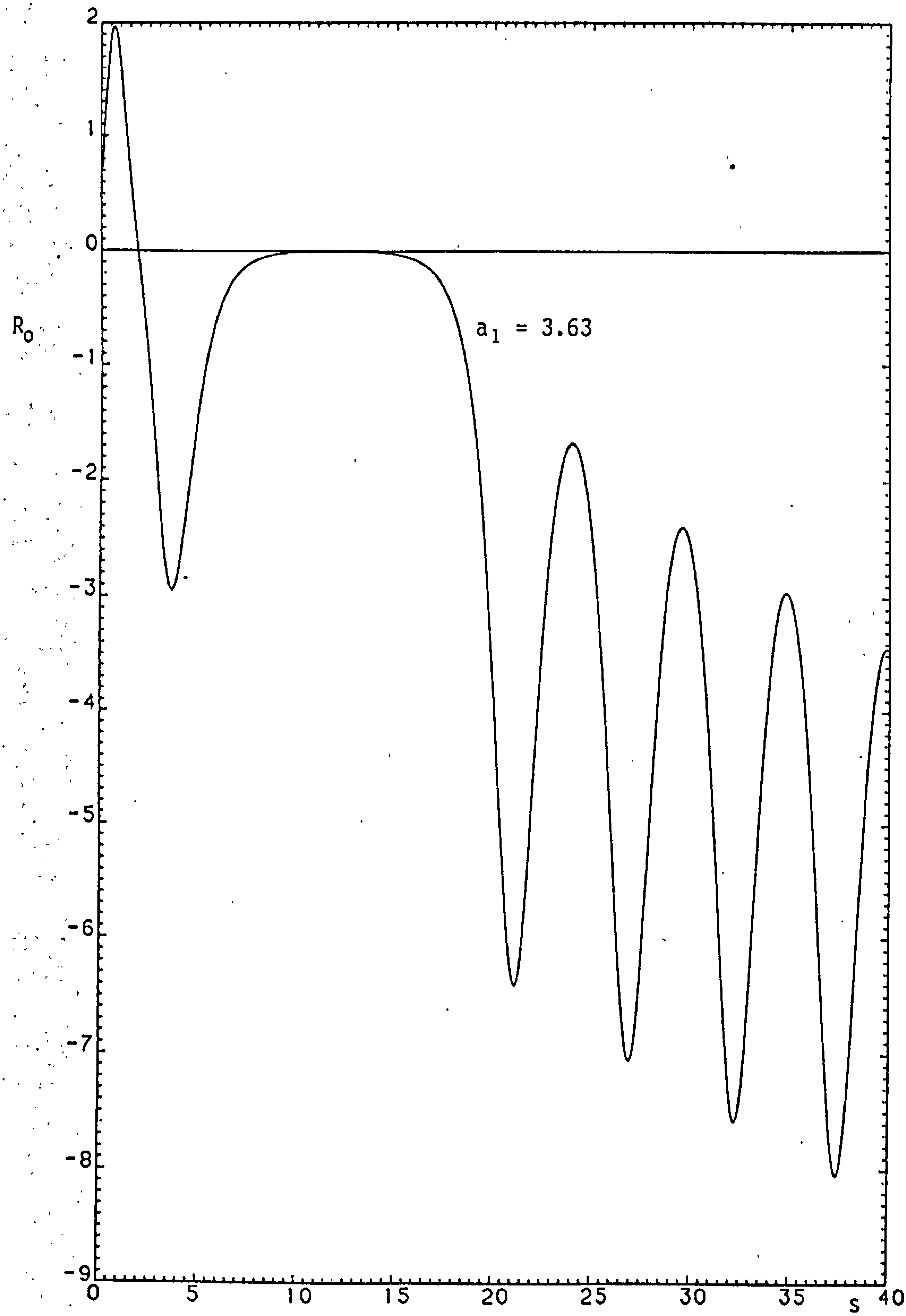

Figure 6.6: Profile of R_o (Subcritical Solution for $\bar{\xi} < 0$)

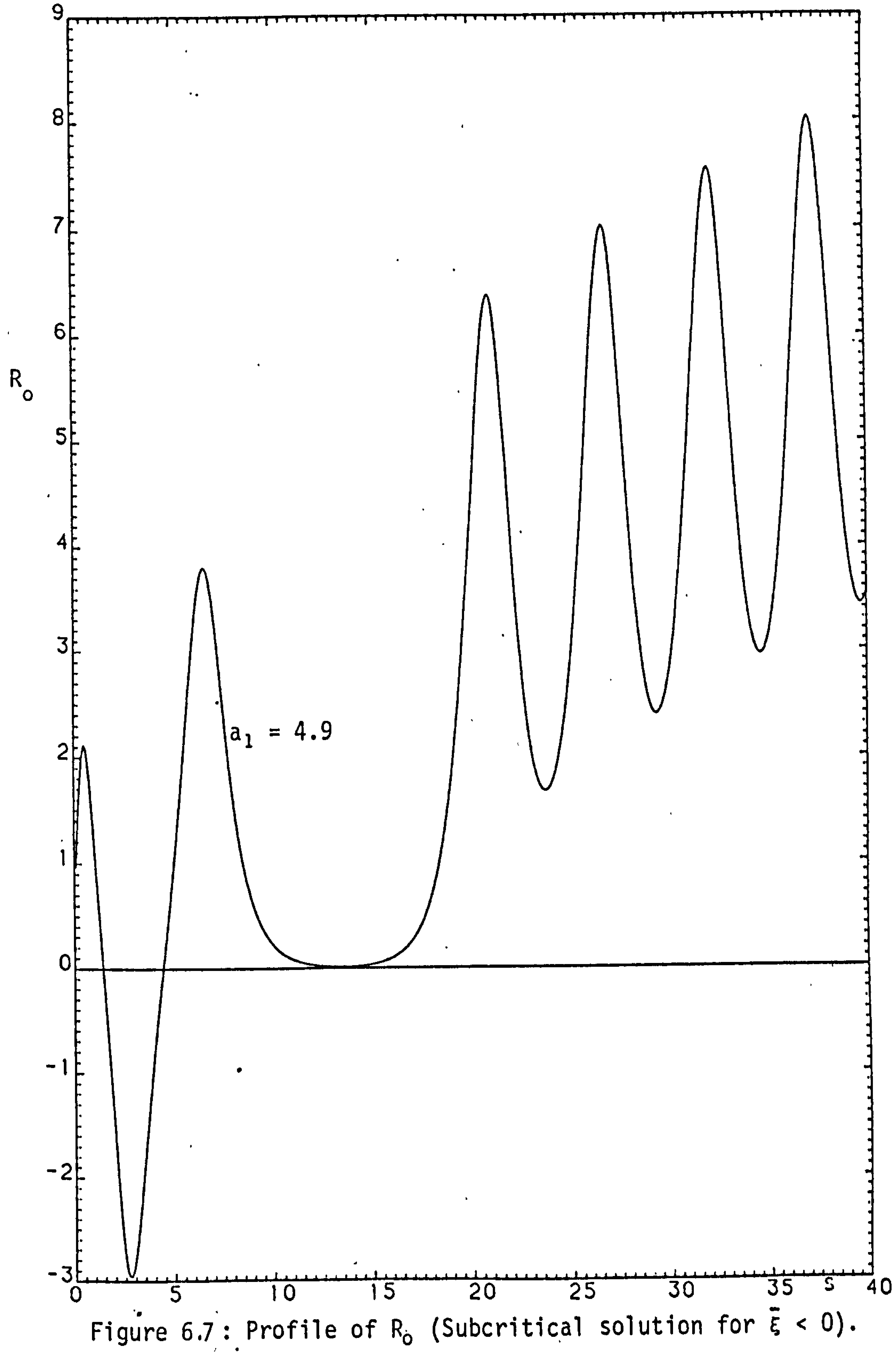

Figure 6.7 : Profile of R_0 (Subcritical solution for $\bar{\xi} < 0$).

6.4. Roll Pattern

In view of (6.2.6) - (6.2.8) the amplitude function, A, may be expressed as

$$A \sim R_0 e^{i(\gamma + \bar{\varepsilon}\, \theta_0)} \tag{6.4.1}$$

where R_0 and θ_0 satisfy equations (6.3.1) and (6.2.11) respectively and $s = \varepsilon r$ is O(1). Here the second order term θ_0, in the phase of A is included in order to indicate any possible change in wavelength of the roll pattern.

From (5.1.6), (5.2.6) and (7.4.1), the stream function $\psi \sim \varepsilon \int_0^z u_\perp \, dz$, is given by

$$\psi \sim \frac{2\varepsilon R_0 f(z)}{s^{\frac{1}{2}} |\bar{\xi}|^{\frac{1}{2}}} \cos \{\alpha_c r - \frac{\pi}{4} + \gamma + \bar{\varepsilon}\, c_1 \int \frac{ds}{R_0^2}\} \tag{6.4.2}$$

where the numerical solution of f(z) is given in § 4.1, and c_1 is the phase winding parameter related to γ by (6.2.24), and γ itself is an arbitrary constant.

The discussion in § 6.3 indicates that when $\mathrm{sgn}(\bar{\xi}) = 1$ and for supercritical Rayleigh numbers there is a stable solution of the system in which a_1, the slope of R_0 at the origin is given by (6.3.6). Using the corresponding solution for R_0, the roll patterns obtained from (7.4.2) for various values of the parameter γ and for given values of ε, T, σ, are shown in figures 6.8, 6.11.

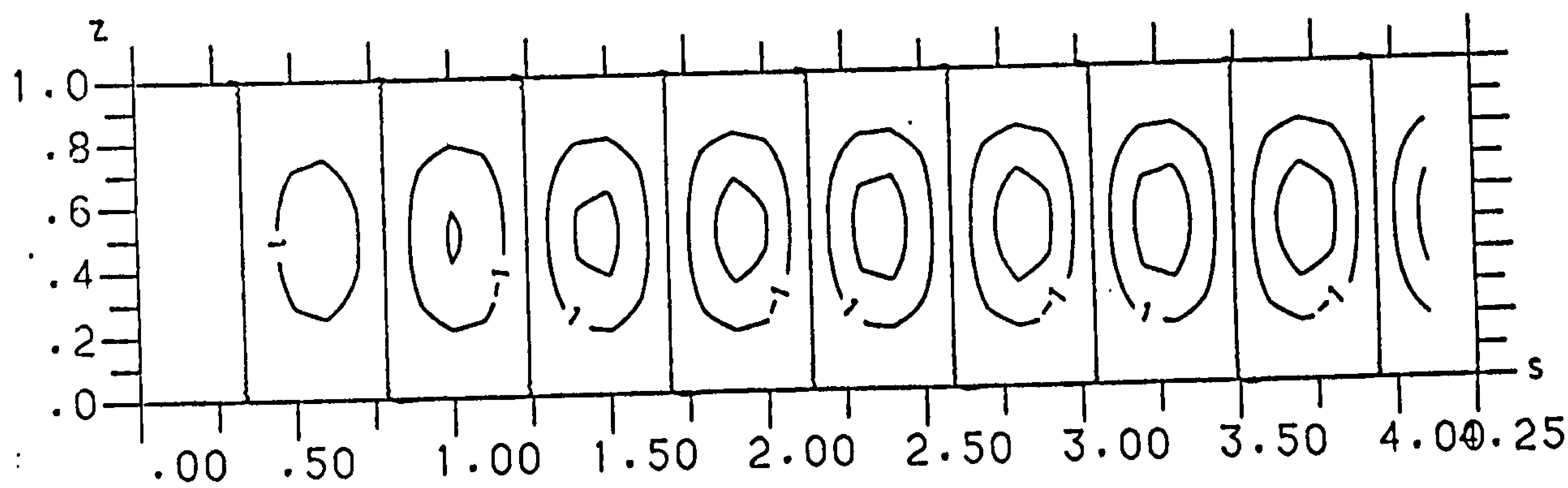

Figure 6.8: Roll Pattern for
$T = 10,\ \varepsilon = 0.1,\ \gamma = 0$

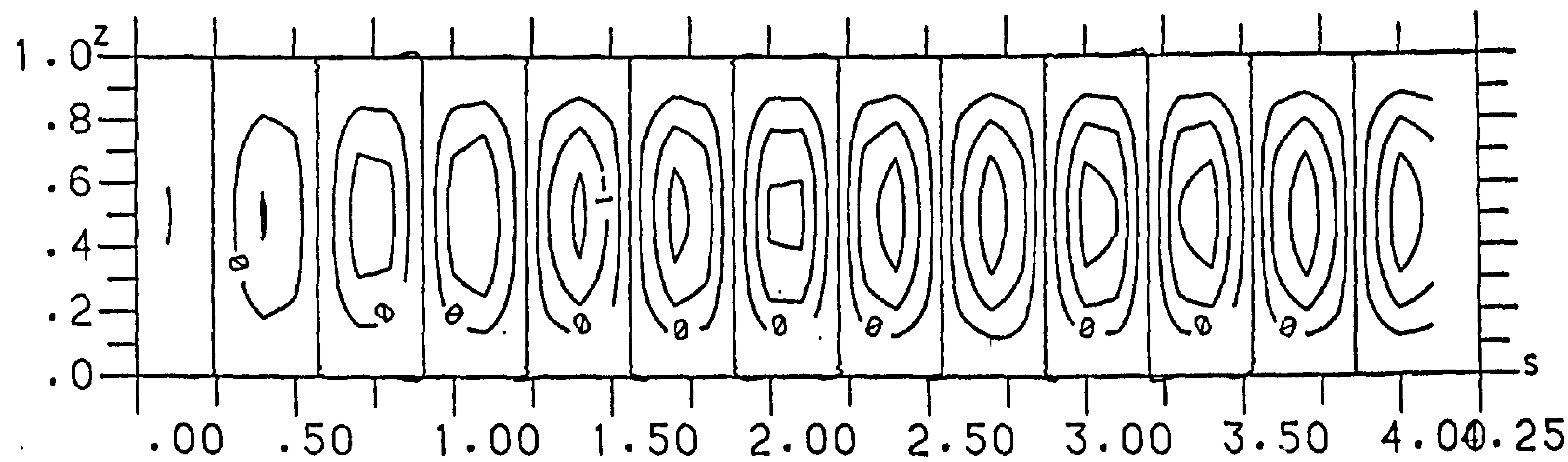

Figure 6.9: Roll pattern for $T = 10^4$, $\varepsilon = 0.1$, $\gamma = 0.0$

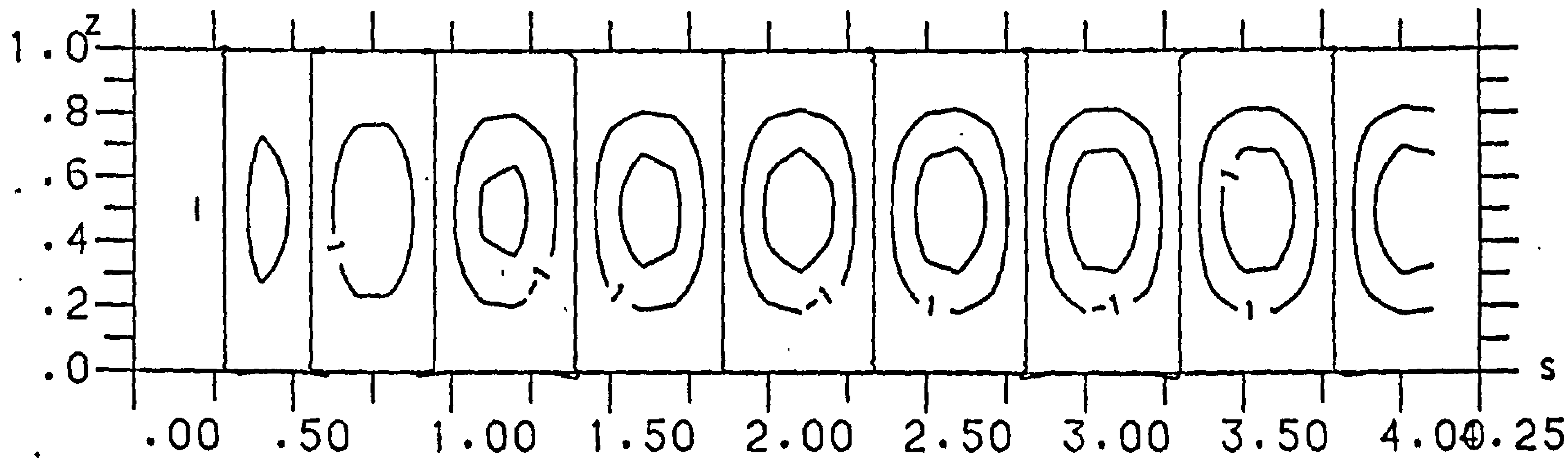

Figure 6.10 : Roll pattern for
$T = 10,\ \varepsilon = 0.1,\ \gamma = \frac{\pi}{3}$

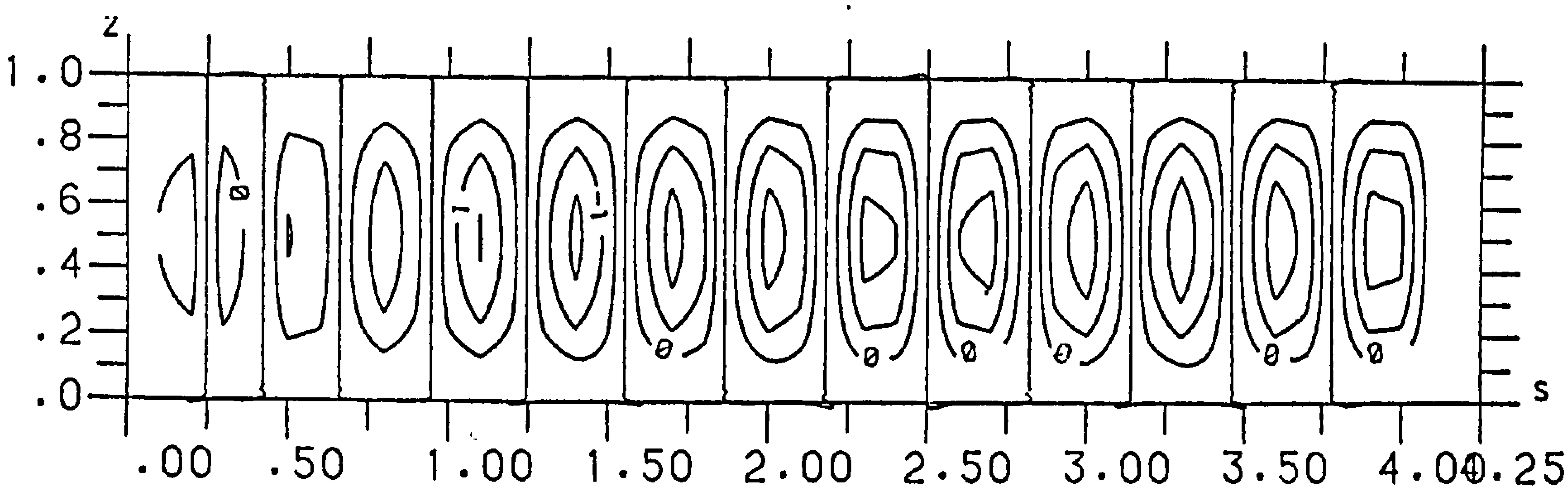

Figure 6.11 Roll Pattern for
$T = 10^4$, $\varepsilon=0.1$, $\gamma = \frac{\pi}{3}$

Note the wavelength decreases as T increases and that there is a minor change in the pattern caused by phase-winding effect characterised by the value of the γ.

CHAPTER 7

Non-linear Rotating Benard Convection:

Overstability

In this chapter we restrict attention to stress-free boundaries. As the Rayleigh number is increased in the rotating case, the system is initially prone to overstability for certain ranges of the Prandtl number and speed of rotation (Chandrasekhar 1953). The motion that develops at the critical Rayleigh number is oscillatory in time and the stationary structure of the type described in chapters 5 and 6 is no longer relevant. In the present chapter, we consider the overstable situation and discuss the formation of an oscillatory axisymmetric finite amplitude equilibrium state.

Following the approach of Brown and Stewartson (1978, 1979) it is found that the flow domain can be subdivided into a central region near the axis of rotation where a linearised solution can be constructed in terms of Bessēl functions (7.1.2) and the remainder of the flow where non-linear effects are significant and can be incorporated in a pair of amplitude functions for A_1 and A_2. Amplitude equations for the overstable case incorporating spatial modulation were derived by Daniels (1978) in order to discuss a finite amplitude motion in terms of two-dimensional rolls in a rotating system bounded by distant sidewalls. Here these equations are further modified by the influence of curvature effects.

The expansions are based on a small parameter ε where

$$R - R_c = O(\varepsilon), \quad (7.1)$$

R is the Rayleigh number and R_c is the critical Rayleigh number at which overstability first occurs. The radial co-ordinate, r, non-dimensionalised with respect to the height of the layer, is then subdivided into the central zone (Figure 7.1) where

$$r = O(1) \quad (7.2)$$

and the outer zone where

$$s = \varepsilon r = O(1). \quad (7.3)$$

The additional subdivisions of the outer zone are then defined by an inner region where

$$s = \varepsilon^X, \quad 0 < X < 1 \quad (7.4)$$

and an outer region where

$$s \sim \frac{\tilde{\mu} + 1}{2(1-\tilde{\mu})} \ln|\ln\varepsilon| \quad (7.5)$$

and $\tilde{\mu}$ is a parameter in the range $-1 < \tilde{\mu} < 1$ to be defined in (7.2.6) below. The inner region (7.4) provides a vital link between the central zone (7.2) and the main outer zone (7.3) and the rather unusual definition of the local co-ordinate X is a novel feature of the asymptotic structure. The amplitude of the disturbance at the centre of the layer is found to be unusually large,

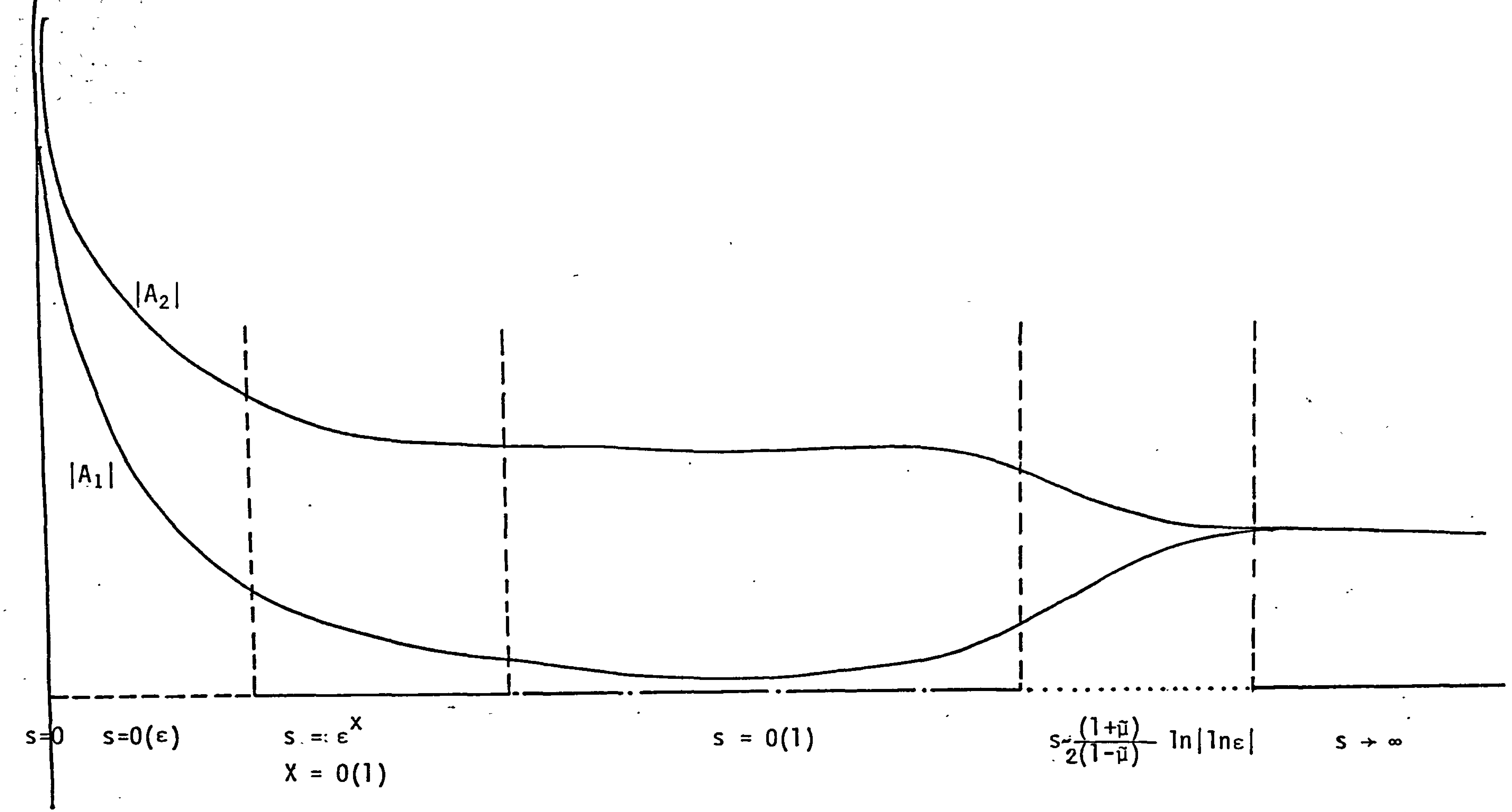

Figure 7.1. Structure of solution.

$$O(|\ln\varepsilon|^{-\frac{1}{2}}). \tag{7.6}$$

This is considerably larger than even the unexpectedly large value found by Brown & Stewartson (1978) in the exchange case. Away from the centre, the disturbance has the more familiar size of $O(\varepsilon^{\frac{1}{2}})$.

The results are discussed in §7.5.

7.1 Linearised Solution

The equations that govern the axisymmetric motion relative to the rotating frame and in which the coupling of the density variation with the centrifugal acceleration is neglected are given in (4.1) - (4.5). The horizontal surfaces at z = 0 and z = 1 are assumed to be stress-free.

The boundary conditions at these surfaces are then

$$\theta = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \text{ at } z = 0 \text{ and } z = 1. \tag{7.1.1}$$

The basic state is one of rigid body rotation with a uniform vertical temperature gradient.

The linearised versions of the equations (4.1) - (4.5) and the boundary conditions (7.1.1) are satisfied by normal mode solutions of the form

$$(\theta,w) = (a,\ b)\ J_0\ (\alpha r)e^{i\omega t}\sin n\pi z + \text{c.c.}$$
$$(u,v) = (c,\ d)\ J_0'\ (\alpha r)e^{i\omega t}\cos n\pi z + \text{c.c.} \tag{7.1.2}$$

where

$$b = a(i\omega+\alpha^2+n^2\pi^2)/R,\ c=n\pi b/\alpha,\ d=-c\sigma T^{\frac{1}{2}}/(i\omega+\sigma\{\alpha^2+n^2\pi^2\}), \quad (7.1.3)$$

and the frequency ω and wavenumber α are related by the equation

$$(i\omega+\sigma\{\alpha^2+n^2\pi^2\})^2(i\omega+\alpha^2+n^2\pi^2)(\alpha^2+n^2\pi^2)+(i\omega+\alpha^2+n^2\pi^2)$$
$$n^2\pi^2\sigma^2T-(i\omega+\sigma\{\alpha^2+n^2\pi^2\})\sigma\alpha^2R = 0. \quad (7.1.4)$$

The corresponding solution for $\omega = 0$ was first given by Muller (1965). Equation (7.1.4) is the same characteristic equation as that first derived by Chandrasekhar (1953) using rectangular Cartesian co-ordinates and thus the results of their analysis are immediately applicable. For certain values of the Prandtl number, σ, and Taylor number, T, solutions of (7.1.4) for integer n, real α and $Im(\omega) = 0$ first occur as the Rayleigh number is increased from zero with $Re(\omega) \neq 0$ and overstability is said to occur. The lowest value of R corresponds to $n = 1$ and we define this critical value, and the corresponding values of α and ω as

$$R = R_c,\ \alpha = \alpha_c,\ \omega = \omega_c. \quad (7.1.5)$$

Thus at $R = R_c$ a non-zero oscillatory motion with

$$\theta = aJ_0(\alpha_c r)e^{i\omega_c t}\sin\pi z + c.c. \quad (7.1.6)$$

is possible and in the following sections we seek to determine, by the inclusion of the non-linear terms in the basic equations (4.1) - (4.5), the finite amplitude form of the motion for Rayleigh numbers slightly larger than R_c.

7.2 The Amplitude Equations

At large distance from the central axis the local radius is much greater than the wavelength of the assumed axisymmetric disturbance that evolves near $R = R_c$. Thus curvature effects are of secondary importance and the wavelength has the almost constant value of π/α_c. It follows that the solution in this region can usefully be expressed in the form

$$\theta=\varepsilon^{\frac{1}{2}}\{\bar{A}_1(s,\bar{\tau})e^{i\alpha_c r+i\omega_c t}+\bar{A}_2(s,\bar{\tau})e^{-i\alpha_c r+i\omega_c t}+c.c.\}\sin\pi z+\varepsilon\theta_1+\varepsilon^{\frac{3}{2}}\theta_2+\ldots, \quad (7.2.1)$$

$$u=\varepsilon^{\frac{1}{2}}\{\frac{i\pi(i\omega_c+\alpha_c^2+\pi^2)}{\alpha_c R_c}\{\bar{A}_1\varepsilon^{i\alpha_c r+i\omega_c t}-\bar{A}_2 e^{-i\alpha_c r+i\omega_c t}\}+c.c.\}\cos\pi z+\varepsilon u_1+\varepsilon^{\frac{3}{2}}u_2+\ldots, \quad (7.2.2)$$

where

$$s=\varepsilon r,\ \bar{\tau}=\varepsilon t,\ R=R_c+\varepsilon\delta \quad (7.2.3)$$

It is assumed that $\varepsilon \ll 1$ and δ is a constant factor introduced for convenience. In the subsequent analysis it will be necessary to further structure the solution by assuming that $\bar{A}_1$ and $\bar{A}_2$ can themselves be expended in powers of $|\ln\varepsilon|$, but the required adjustments can be made within the framework of a single pair of amplitude equations for $\bar{A}_1$ and $\bar{A}_2$. These equations arise from the compatability conditions that ensure bounded solutions for the terms u_2, θ_2, ... at order $\varepsilon^{\frac{3}{2}}$. The appropriate equations for two-dimensional rolls in Cartesian co-ordinates have already been derived by Daniels (1978) and we may immediately infer that in cylindrical co-ordinates the corresponding equations have the form

$$\mu_{or}\frac{\partial\bar{A}_1}{\partial\bar{\tau}} + \frac{\partial\bar{A}_1}{\partial s} + \tfrac{1}{2}s^{-1}\bar{A}_1 = \bar{A}_1\{(\mu_{1r}+i\mu_{1i})\delta-(\mu_{2r}+i\mu_{2i})|\bar{A}_1|^2-(\mu_{3r}+i\mu_{3i})|\bar{A}_2|^2\}, \qquad (7.2.4)$$

$$\mu_{or}\frac{\partial\bar{A}_2}{\partial\bar{\tau}} - \frac{\partial\bar{A}_2}{\partial s} - \tfrac{1}{2}s^{-1}\bar{A}_2 = \bar{A}_2\{(\mu_{1r}+i\mu_{1i})\delta-(\mu_{2r}+i\mu_{2i})|\bar{A}_2|^2-(\mu_{3r}+i\mu_{3i})|\bar{A}_1|^2\}, \qquad (7.2.5)$$

where μ_{or}, μ_{nr}, μ_{ni} $(n = 1, 2, 3)$ are real coefficients which have complicated dependencies upon σ and T and whose values are discussed in more detail in the earlier study (Daniels 1978). In fact the only difference between (7.2.4), (7.2.5) and the equations given in this earlier work is the appearance of the curvature terms $\bar{s}^{-1}\bar{A}_1$ and $\bar{s}^{-1}\bar{A}_2$. The coefficients of these terms can be deduced from the reasoning that the linear forms of (7.2.4), (7.2.5) must be consistent with the asymptotic form of the Bessel function solution (7.1.2) as $r \to \infty$; this is only possible if the linear forms of (7.2.4), (7.2.5) are satisfied by $\bar{A}_{1,2} = \bar{s}^{-\frac{1}{2}}$ when $\delta = 0$ and it then follows that the coefficients of the terms $\bar{s}^{-1}\bar{A}_1$, and $\bar{s}^{-1}\bar{A}_2$ must be $\pm\frac{1}{2}$. These values were obtained by direct substitution of (7.2.1), (7.2.2) into the governing equations (4.1) - (4.5).

To simplify the systems (7.2.4), (7.2.5) we set

$$\bar{\tau} = \mu_{or}\tau,\ \bar{A}_{1,2} = \mu_{2r}^{-\frac{1}{2}}A_{1,2},\ \tilde{\mu} = \mu_{3r}/\mu_{2r},\ \delta = \mu_{1r}^{-1}, \qquad (7.2.6)$$

the analysis of Daniels (1978) suggesting that for the relevant values of σ and T, $\mu_{nr}>0$ $(n = 0, 1, 2)$. Then the equations for the complex amplitude functions A_1 and A_2 are

$$\frac{\partial A_1}{\partial\tau} + \frac{\partial A_1}{\partial s} + \tfrac{1}{2}s^{-1}A_1 = A_1\{(1+i\Theta_1)-(1+i\Theta_2)|A_1|^2-(\tilde{\mu}+i\Theta_3)|A_2|^2\}, \qquad (7.2.7)$$

$$\frac{\partial A_2}{\partial\tau} - \frac{\partial A_2}{\partial s} - \tfrac{1}{2}s^{-1}A_2 = A_2\{(1+i\Theta_1)-(1+i\Theta_2)|A_2|^2-(\tilde{\mu}+i\Theta_3)|A_1|^2\} \qquad (7.2.8)$$

and if we write

$$A_{1,2} = R_{1,2}\ (s,\tau)e^{i\phi_{1,2}(s,\tau)} \tag{7.2.9}$$

then the amplitude moduli R_1 and R_2 satisfy the pair of simultaneous equations

$$\frac{\partial R_1}{\partial \tau} + \frac{\partial R_1}{\partial s} + \tfrac{1}{2}\, s^{-1}\, R_1 \;=\; R_1(1-R_1^2 - \tilde{\mu} R_2^2\ ), \tag{7.2.10}$$

$$\frac{\partial R_2}{\partial \tau} - \frac{\partial R_2}{\partial s} - \tfrac{1}{2}\, s^{-1}\, R_2 \;=\; R_2(1-R_2^2 - \tilde{\mu} R_1^2\ ), \tag{7.2.11}$$

where it is believed that the parameter $\tilde{\mu}$ lies in the range

$$-1 < \tilde{\mu} < 1 \tag{7.2.12}$$

for all relevant values of σ and T (cf. Daniels 1978); this will be assumed in the subsequent analysis.

The phase functions ϕ_1 and ϕ_2 satisfy the equations

$$\frac{\partial \phi_1}{\partial \tau} + \frac{\partial \phi_1}{\partial s} = \Theta_1 - \Theta_2 R_1^2 - \Theta_3 R_2^2 \tag{7.2.13}$$

$$\frac{\partial \phi_2}{\partial \tau} - \frac{\partial \phi_2}{\partial s} = \Theta_1 - \Theta_2 R_2^2 - \Theta_3 R_1^2 \tag{7.2.14}$$

where the coefficients $\Theta_{1,2,3}$ are functions of σ and T. In § 7.3 we will briefly consider solutions of (7.2.10) - (7.2.14) in the form of travelling waves, but the main purpose of this chapter is to look for constant frequency equilibrium solutions, equivalent to assuming that R_1 and R_2 are independent of τ, and with

$$\phi_{1,2} = \Omega_{1,2}\tau + \bar{\phi}_{1,2}(s). \tag{7.2.15}$$

Then, if $\Omega_1 = \Omega_2 = \tilde{\Omega}$, say, the quantity

$$\omega_c + \varepsilon\mu_{or}^{-1}\tilde{\Omega} \qquad (\varepsilon \ll 1), \tag{7.2.16}$$

is the corrected frequency of the disturbance.

Following the method of Brown & Stewartson (1978) for the non-rotating problem we now attempt a complete description of the non-linear finite amplitude equilibrium state that evolves for $R > R_c$ by finding a solution of (7.2.10)-(7.2.14) that has acceptable behaviour at large distances ($s \to \infty$) and that matches with a valid solution of the governing equations in the neighbourhood of the central axis, where $r = O(1)$. Since $R = R_c + O(\varepsilon)$ it may be assumed that the amplitude of the motion in the latter region is small and so the leading order term in the expansion of the solution there as $\varepsilon \to 0$ is given by the linearised form (7.1.6):

$$\theta \sim \{a(\tau)e^{i\omega_c t} + \text{c.c.}\} J_0(\alpha_c r)\sin\pi z, \tag{7.2.17}$$

where $a(\tau) \ll 1$ is an unknown function of τ to be determined through matching with the solution of (7.2.10)-(7.2.14). Other solutions of the linearised equations which have exponential growth as $r \to \infty$ must be discarded. It should be noted that our expansion procedure is really based on the assumption that non-linear effects first become important outside the central zone, $r = O(1)$, an assumption which is justified a posteriori. An alternative strategy would be to assume that the nonlinear terms in the central zone solution themselves lead to a local compatability condition which fixes the value of a. However, there appears to be no obvious condition which can be applied directly as $r \to \infty$ because of the decay of the Bessel function solution (7.2.17).

As a first step we consider the situation where $\tilde{\mu} = 0$ since in this case the equations (7.2.10), (7.2.11) are uncoupled and thus, explicit solutions can be obtained for the outer zone in simple form.

7.3 Solution for $\tilde{\mu} = 0$

When $\tilde{\mu} = 0$ the general steady solutions of (7.2.10) and (7.2.11) are

$$R_1 = s^{-\frac{1}{2}} e^{s} \{2 \int_1^s e^{2s'} s'^{-1} \, ds' + C_1\}^{-\frac{1}{2}}, \qquad (7.3.1)$$

and

$$R_2 = s^{-\frac{1}{2}} e^{-s} \{2 \int_s^\infty e^{-2s'} s'^{-1} ds' + C_2\}^{-\frac{1}{2}}, \qquad (7.3.2)$$

where C_1 and C_2 are arbitrary constants. As $s \to \infty$ the solution for R_1 automatically approaches 1, irrespective of the value of C_1. The solution for R_2 approaches 1 only if $C_2 = 0$ and otherwise R_2 decays exponentially as $s \to \infty$.

We shall restrict attention to solutions with $C_2 = 0$ since it seems likely that all those solutions with C_2 non-zero will be unstable. This is suggested by a stability analysis of the time-dependent versions of (7.2.10), (7.2.11).

$$\frac{dR_1}{d\tau} = R_1(1-R_1^2 - \tilde{\mu}R_2^2)$$

$$\frac{dR_2}{d\tau} = R_2(1-R_2^2 - \tilde{\mu}R_1^2) \qquad (7.3.3)$$

in which the spatial dependence is ignored. If the equilibrium solution in which

$$R_1 = R_2 = (1 + \tilde{\mu})^{-\frac{1}{2}} \qquad (7.3.4)$$

is subjected to a disturbance of the form

$$R_{1,2} = (1 + \tilde{\mu})^{-\frac{1}{2}} + k_{1,2}e^{\beta\tau}, \qquad (k_{1,2} \ll 1), \qquad (7.3.5)$$

the growth rates are

$$\beta = \frac{2(\tilde{\mu} - 1)}{(\tilde{\mu} + 1)}, \ (k_2 = -k_1) \text{ and } \beta = -2, \ (k_2 = k_1), \qquad (7.3.6)$$

and so, from (7.2.12) are both negative. Thus the solution with $R_{1,2}$ both non-zero (and equivalent to the choice $C_2 = 0$ in (7.3.2) above) has the possibility of being stable. The other non-zero equilibrium solutions of (7.3.3),

$$R_1 = 0, \ R_2 = 1 \qquad \text{and} \qquad R_1 = 1, \ R_2 = 0 \qquad (7.3.7)$$

are definitely unstable. For example, in the second case (equivalent to the choice of non-zero C_2 in (7.3.2)), if we set

$$R_1 = 1 + k_1e^{\beta_1\tau}, \ R_2 = k_2e^{\beta_2\tau} \qquad (k_{1,2} \ll 1), \qquad (7.3.8)$$

we find that

$$\beta_1 = -2, \qquad \beta_2 = 1 - \tilde{\mu} \tag{7.3.9}$$

so that $\beta_2 > 0$.

From (7.3.1) it can be seen that the solution for R_1 exists only for $s > s_0$ where

$$2\int_1^{s_0} e^{2s'} s'^{-1}\, ds' + C_1 = 0 \tag{7.3.10}$$

and, if $s_0 \ll 1$,

$$C_1 \sim -2\ln s_0. \tag{7.3.11}$$

Thus the solution for R_1 exists close to the central zone ($s \to 0$) only if $C_1 \gg 1$ and then, from (7.3.1), $R_1 \ll 1$ for all $s = O(1)$. The precise size of C_1 is fixed by matching the outer zone solution (7.2.1) as $s \to 0$ to the central zone solution (7.2.17) as $r \to \infty$. Setting $C_2 = 0$ we have, from (7.2.1), (7.2.6), (7.3.1) and (7.3.2):

$$\theta \sim \varepsilon^{\frac{1}{2}}\mu_2 r^{-\frac{1}{2}}\sin\pi z\{(C_1+2\ln s)^{-\frac{1}{2}}s^{-\frac{1}{2}}e^{i\phi_1+i\alpha_c r+i\omega_c t}$$
$$+(-2\ln s)^{-\frac{1}{2}}s^{-\frac{1}{2}}e^{i\phi_2-i\alpha_c r+i\omega_c t} + C.C.\} \tag{7.3.12}$$

where ϕ_1 and ϕ_2 take appropriate limiting forms which may be deduced from (7.2.13), (7.2.14) and will be discussed in greater detail in § 7.4. We now set $s = \varepsilon r$ and in view of the appearance of terms $(C_1 + 2\ln s)$ we also write

$$C_1 = c_1|\ln\varepsilon| + \ldots., \quad (\varepsilon \to 0), \tag{7.3.13}$$

where c_1 is a finite constant. We then have

$$\theta \sim |\ln\varepsilon|^{-\frac{1}{2}} r^{-\frac{1}{2}} \mu_{2r}^{-\frac{1}{2}} \sin\pi z\{(c_1 - 2 - \frac{2\ln r}{\ln\varepsilon})^{-\frac{1}{2}} e^{i\phi_1 + i\alpha_c r + i\omega_c t}$$

$$+ (2 + \frac{2\ln r}{\ln\varepsilon})^{-\frac{1}{2}} e^{i\phi_2 - i\alpha_c r + i\omega_c t} + \text{c.c.}\}. \qquad (7.3.14)$$

For $s \ll 1$ we have $r \ll \varepsilon^{-1}$ so that $\ln r \ll |\ln\varepsilon|$ and we can express (7.3.14) in the form

$$\theta \sim |\ln\varepsilon|^{-\frac{1}{2}} r^{-\frac{1}{2}} \mu_{2r}^{-\frac{1}{2}} \sin\pi z\{(c_1 - 2)^{-\frac{1}{2}} e^{i\phi_1 + i\alpha_c r + i\omega_c t} + 2^{-\frac{1}{2}} e^{i\phi_2}$$

$$^{-i\alpha_c r + i\omega_c t} + \text{c.c}\} + O(|\ln\varepsilon|^{-\frac{3}{2}} r^{-\frac{1}{2}} \ln r). \qquad (7.3.15)$$

It is important here that variations in the limiting forms of ϕ_1 and ϕ_2 with r are $O(|\ln\varepsilon|^{-1})$, and so do not influence the matching procedure at leading order.

From (7.2.17), as $r \to \infty$,

$$\theta \sim r^{-\frac{1}{2}} e^{i\omega_c t} \sin\pi z \cos(\alpha_c r - \frac{\pi}{4}) a(\tau) \sqrt{\frac{2}{\pi\alpha_c}} + \text{c.c.} \qquad (7.3.16)$$

so that comparison with (7.3.15) gives

$$a(\tau) \sqrt{\frac{2}{\pi\alpha_c}} \, \frac{1}{2} e^{\frac{-i\pi}{4}} = (c_1 - 2)^{-\frac{1}{2}} \mu_{2r}^{-\frac{1}{2}} e^{i\phi_1} |\ln\varepsilon|^{-\frac{1}{2}}, \qquad (7.3.17)$$

and

$$a(\tau) \sqrt{\frac{2}{\pi\alpha_c}} \, \frac{1}{2} e^{\frac{i\pi}{4}} = 2^{-\frac{1}{2}} \mu_{2r}^{-\frac{1}{2}} e^{i\phi_2} |\ln\varepsilon|^{-\frac{1}{2}} \qquad (7.3.18)$$

Taking the modulus of each side we have

$$(c_1 - 2)^{-\frac{1}{2}} = 2^{-\frac{1}{2}} \qquad (7.3.19)$$

and so

$$c_1 = 4. \qquad (7.3.20)$$

A measure of the amplitude of the motion at the central axis is

$$|a| = \sqrt{\frac{\pi\alpha_c}{\mu_{2r}}} \quad |\ln\varepsilon|^{-\frac{1}{2}}. \qquad (7.3.21)$$

This indicates an extremely rapid growth in amplitude at the centre as the Rayleigh number increases above R_c and substantially larger than the non-linear motion in the outer zone. It is also significantly larger than the motion at the centre when stationary convection occurs (Brown and Stewartson 1978, chapter 6). Figure 7.2 shows the solutions (7.3.1), 7.3.2) for R_1 and R_2 as functions of s, for ε=0.1.

The above results are obtained from the asymptotic limit of a full solution of the amplitude equations (7.2.10), (7.2.11), available when $\tilde{\mu} = 0$. In order to find a solution for general values of $\tilde{\mu}$ it is instructive to analyse the asymptotic structure in some deatil. The central zone is seen to provide a 'reflection condition' which fixes the amplitude of the 'reflected wave' R_1 through the value of C_1 given by (7.3.13) and (7.3.20). A consistent match is only possible if C_1 is asymptotically large as $\varepsilon \to 0$ so that R_1, given by (7.3.10), is asymptotically small in the outer zone where s = 0(1):

$$R_1 \sim \frac{s^{-\frac{1}{2}} e^{+s}}{2|\ln\varepsilon|^{\frac{1}{2}}}. \qquad (7.3.22)$$

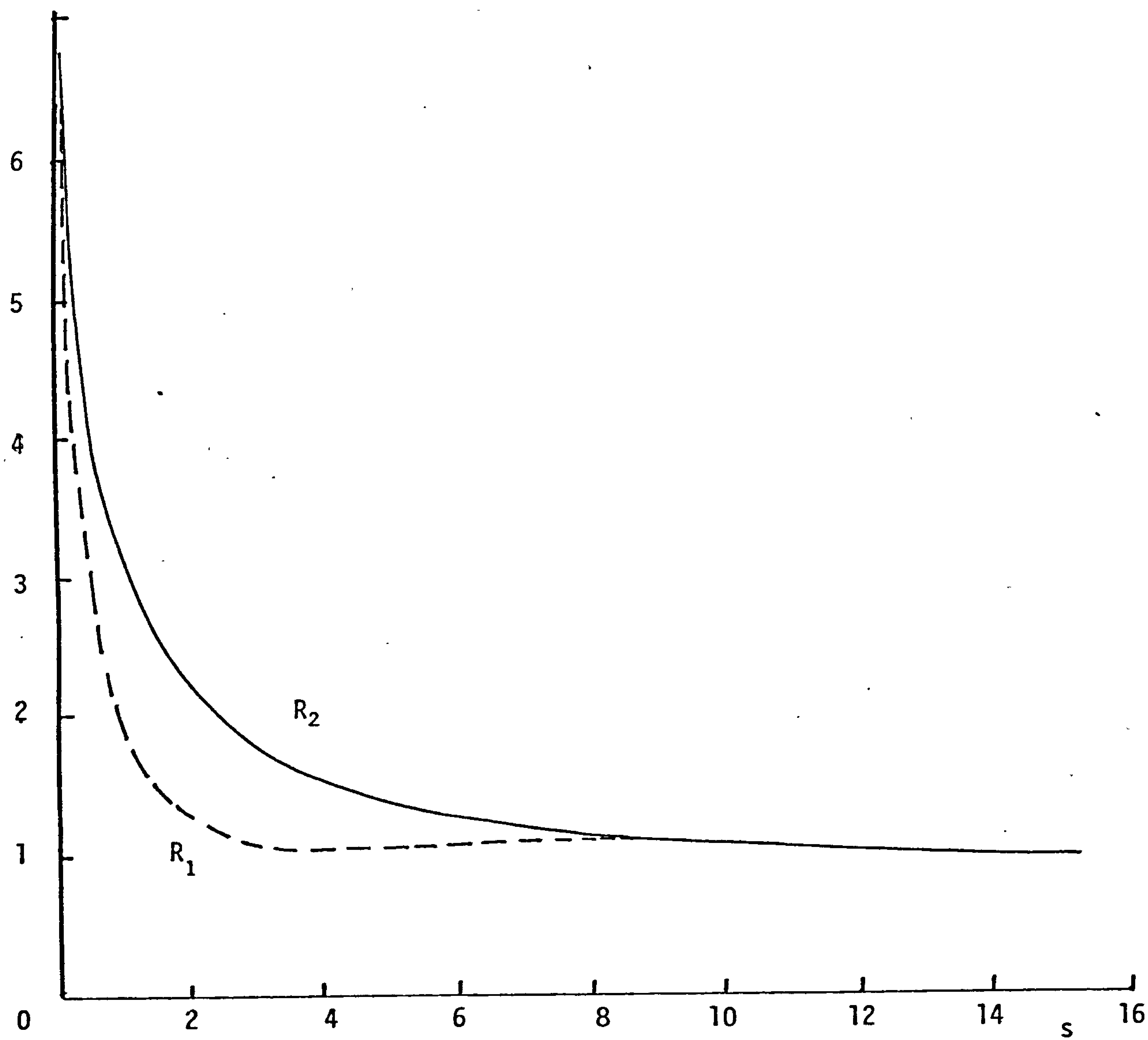

Figure 7.2: Profiles of R_1 and R_2 for $\varepsilon = 0.1$, $\mu = 0$.

The solution (7.3.22) is in fact only valid for $s<\frac{1}{2}\ln|\ln\varepsilon|$ since the integral component of (7.3.1) eventually dominates C_1 as s increases, and R_1 then grows to achieve the same amplitude as R_2 at infinity.

The reflection condition at the centre is perhaps more readily understood in terms of travelling-wave solutions of (7.2.10), (7.2.11). With $\tilde{\mu} = 0$, interactions between a wave component R_2 travelling towards the centre and a reflected component R_1 travelling away from the centre can be ignored. If we assume that an incident component is generated in the region $s_1<s<s_2$ at $\tau = 0$ by an initial disturbance in which

$$R_2(s,0) = D(s),\ (s_1<s<s_2),\ R_2(s,0) = 0\ (s<s_1, s>s_2), \tag{7.3.23}$$

then an incident wave solution of (7.2.11) of the form

$$R_2=s^{-\frac{1}{2}}e^{-s}\{2\int_s^{s+\tau} e^{-2s'}s'^{-1}ds' +(s+\tau)^{-1}e^{-2(s+\tau)}\{D(s+\tau)\}^{-2}\}^{-\frac{1}{2}} \tag{7.3.24}$$

is produced. This represents a wave which travels towards the centre, being confined to the region $s_1-\tau<s<s_2-\tau$ and being reflected at the centre over the period $s_1<\tau<s_2$. As $s\to 0$ the effect of the initial profile D diminishes and the integral part of (7.3.24) dominates the matching with the central zone where a non-zero $a(\tau)$ is generated for $s_1<\tau<s_2$. The details are thus similar to those of the constant frequency case already described, with the result that a reflected wave solution of (7.2.10) of the form

$$R_1 \sim s^{-\frac{1}{2}}e^{s}\{2\int_1^{s} e^{2s'}s'^{-1}ds' + 4|\ln\varepsilon|\}^{-\frac{1}{2}} \tag{7.3.25}$$

is generated for $\tau - s_2 < s < \tau - s_1$ and $\tau > s_1$, R_1 being zero for other values of s. We see that for s = O(1) the reflection damps the wave amplitude considerably, but sufficient remains to provoke the eventual return of the wave to finite amplitude as $s \to \infty$.

The constant frequency equilibrium state can be regarded as that produced by the limit of (7.3.24) and (7.3.25) for large times, τ, and for an initial disturbance D(s) extending to infinity ($s_2 \to \infty$).

In the next section we show that the asymptotic structure found in this section for the case $\tilde{\mu} = 0$ can be used to construct the constant frequency equilibrium solution for general values of $\tilde{\mu}$.

7.4 Solution for general $\tilde{\mu}$

(i) Outer zone $s = O(1)$:

It can be shown that as $s \to 0$ the only possible steady non-zero solution of the full pair of nonlinear equations (7.2.10), (7.2.11) has the form

$$R_1 = 0 \ , \ R_2 \sim (2s|\ln s|)^{-\frac{1}{2}}. \qquad (7.4.1)$$

It follows that in the region where $s = O(1)$ the only acceptable structure consists of a solution in which R_2 is fully nonlinear but R_1 is small and thus, at leading order, satisfies the linearised form of (7.2.10) in which R_2 is regarded as a known function of s. We therefore set

$$R_2 = R_{20}(s) + \ldots + |\ln\varepsilon|^{-2\gamma} R_{22} + \ldots, \qquad (7.4.2)$$

$$(\varepsilon \ll 1),$$

$$R_1 = |\ln\varepsilon|^{-\gamma} R_{11} + \ldots, \qquad (7.4.3)$$

where the value of γ remains to be determined.

The equations for R_{20} and R_{11} are

$$\frac{dR_{20}}{ds} + \frac{R_{20}}{2s} = R_{20}^3 - R_{20} \ , \qquad (7.4.4)$$

$$\frac{dR_{11}}{ds} + \frac{R_{11}}{2s} = R_{11} - \tilde{\mu} R_{20}^2 R_{11}. \qquad (7.4.5)$$

The solution of (7.4.4) which satisfies the condition $R_{2o} \to 1$ as $s \to \infty$ is

$$R_{2o} = e^{-s}(2s)^{-\frac{1}{2}}\{\int_s^\infty e^{-2s'} s'^{-1} ds'\}^{-\frac{1}{2}}, \tag{7.4.6}$$

and this has the behaviour

$$R_{2o} \sim (2s|\ln s|)^{-\frac{1}{2}} \text{ as } s \to 0. \tag{7.4.7}$$

The solution for R_{11} is,

$$R_{11} = c_{11}|\ln s|^{\frac{1}{2}\bar{\mu}} s^{-\frac{1}{2}} e^{s-\bar{\mu}I},$$

$$I = \int_0^s (R_{2o}^2 + \frac{1}{2s\ln s})\, ds, \tag{7.4.8}$$

where c_{11} is an arbitrary constant, and this has the behaviour

$$R_{11} \sim c_{11}\, s^{-\frac{1}{2}}|\ln s|^{\frac{1}{2}\bar{\mu}} \quad \text{as } s \to 0 \tag{7.4.9}$$

Since $\bar{\mu} > -1$, R_{11} grows more rapidly, as $s \to 0$, than does R_{2o}, leading to the possibility of a consistent match with the central zone solution where the inward and outward propagating components are of the same order of magnitude. In fact, this suggests that the value of γ in (7.4.2), (7.4.3) can be determined by setting $s = \varepsilon$ in (7.4.7), (7.4.8) and requiring that R_1 and R_2 are of equal magnitude. This gives

$$|\ln\varepsilon|^{-\frac{1}{2}} = |\ln\varepsilon|^{-\gamma}\, |\ln\varepsilon|^{\frac{1}{2}\bar{\mu}} \tag{7.4.10}$$

and thus

$$\gamma = \tfrac{1}{2}(1 + \tilde{\mu}). \tag{7.4.11}$$

(ii) Transitional zone $s = O(\varepsilon^X)$, $0<X<1$:

In fact it emerges that the solution (7.4.2), (7.4.3) where $s = O(1)$ does not match directly to the central zone solution (7.2.17). As $s\to 0$, (7.4.2), (7.4.3) take the forms

$$R_2 \sim (2s|\ln s|)^{-\frac{1}{2}} - |\ln\varepsilon|^{-(1+\tilde{\mu})} \frac{\tilde{\mu} c_{11}^2 \; s^{-\frac{1}{2}}}{\sqrt{2}(2+\tilde{\mu})} |\ln s|^{\frac{1}{2}+\tilde{\mu}} + \ldots.., \tag{7.4.12}$$

$$R_1 \sim |\ln\varepsilon|^{-\frac{1}{2}(1+\tilde{\mu})} c_{11} s^{-\frac{1}{2}} |\ln s|^{\frac{1}{2}\tilde{\mu}} + \ldots.., \quad (s\to 0, \varepsilon \ll 1) \tag{7.4.13}$$

where the dominant term in R_{22}, which arises from a particular solution generated by a term $\tilde{\mu} R_{20} R_{11}^2$ in the equation for R_{22}, has also been included. The expansions fail where

$$\left|\frac{\ln s}{\ln \varepsilon}\right| = O(1) \tag{7.4.14}$$

or, equivalently, where

$$s = \varepsilon^X, \quad X = O(1) \tag{7.4.15}$$

The region where X = O(1) merges into the outer zone as $X \to 0$, and where the solution must take the form (7.4.13), and into the central zone as $X \to 1$, where the solution must match with the Bessel function form (7.2.17). This transitional zone thus occupies the region

$$0 < X < 1. \qquad (7.4.16)$$

Locally, the amplitude functions R_1 and R_2 can be written in the form

$$R_{1,2} = \varepsilon^{-\frac{1}{2}X}|\ln\varepsilon|^{-\frac{1}{2}} X^{-\frac{1}{2}}F_{1,2}(\xi) + \ldots\ldots, \quad (\varepsilon \ll 1), \qquad (7.4.17)$$

where

$$\xi = X^{\frac{1}{2}(1+\bar{\mu})} \quad (0 < \xi < 1), \qquad (7.4.18)$$

and, from substitution into (7.2.10), (7.2.11), the functions F_1 and F_2 satisfy the coupled pair of nonlinear equations

$$\tfrac{1}{2}(1+\bar{\mu})\xi \frac{dF_1}{d\xi} = \tfrac{1}{2}F_1 + F_1^3 + \bar{\mu}F_1F_2^2, \qquad (7.4.19)$$

$$\tfrac{1}{2}(1+\bar{\mu})\xi \frac{dF_2}{d\xi} = \tfrac{1}{2}F_2 - F_2^2 - \bar{\mu}F_2F_1^2. \qquad (7.4.20)$$

One boundary condition arises from matching with (7.4.12) as $\xi \to 0$. This requires

$$F_2 = \sqrt{\tfrac{1}{2}} + O(\xi^2) \text{ as } \xi \to 0. \qquad (7.4.21)$$

A second boundary condition arises from matching with the central zone solution (7.2.17) as $\xi \to 1$. We can expect

$$F_{1,2} \sim a_{1,2} + O(1 - \xi) \text{ as } \xi \to 1 \qquad (7.4.22)$$

so that, writing $s = \varepsilon r$, we obtain

$$R_{1,2} \sim r^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}|\ln\varepsilon|^{-\frac{1}{2}} a_{1,2} + O(r^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}|\ln\varepsilon|^{-\frac{3}{2}}\ln r). \qquad (7.4.23)$$

Then,following an argument similar to that of (7.3.12) - (7.3.18) above, we arrive at the fomulae equivalent to (7.3.17) - (7.3.18):

$$a\sqrt{\frac{2}{\pi\alpha_c}}\ \tfrac{1}{2}e^{-\frac{i\pi}{4}} = a_1\ \mu_2 r^{-\frac{1}{2}}\ e^{i\phi_1}|\ln\varepsilon|^{-\frac{1}{2}}, \qquad (7.2.24)$$

$$a\sqrt{\frac{2}{\pi\alpha_c}}\ \tfrac{1}{2}e^{\frac{i\pi}{4}} = a_2\ \mu_2 r^{-\frac{1}{2}}\ e^{i\phi_2}\ |\ln\varepsilon|^{-\frac{1}{2}}. \qquad (7.4.25)$$

Thus $a_1 = a_2$, a reflection condition which supplies the second boundary condition for the solution of (7.4.19), (7.4.20):

$$F_1 = F_2 \qquad \text{at} \quad \xi = 1. \qquad (7.4.26)$$

Note that, as in the solution for $\tilde{\mu} = 0$, the amplitude of the motion at the centre is relatively large, with

$$|a(\tau)| = \sqrt{\frac{2\pi\alpha_c}{\mu_{2r}}}\,|\ln\varepsilon|^{-\frac{1}{2}}\,F_1(1), \tag{7.4.27}$$

where the value of $F_1(1)$ is to be determined by the solution of (7.4.19) - (7.4.21) and (7.4.26). Cubic non-linearities in the central zone solution can be expected to generate terms there of $O(|\ln\varepsilon|^{-\frac{3}{2}})$ which have the asymptotic behaviour $e^{\pm i\alpha_c r}\,r^{-\frac{1}{2}}\ln r$, $(r\to\infty)$, required to match with the correction term in (7.4.22).

The values of ϕ_1 and ϕ_2 in (7.4.24), (7.4.25) are the limiting forms, as $X\to 1$, of the solutions

$$\left.\begin{aligned}\phi_1 &\sim -\int_X^1 (\Theta_2 X^{-1} F_1^{\,2} + \Theta_3 X^{-1} F_2^{\,2})\,dX + q_1 + \Omega_1\,\tau, \\ \phi_2 &\sim \int_X^1 (\Theta_2 X^{-1} F_2^{\,2} + \Theta_3 X^{-1} F_1^{\,2})\,dX + q_2 + \Omega_2\,\tau,\end{aligned}\right\}\quad (0<X<1) \tag{7.4.28}$$

where q_1 and q_2 are arbitrary constants. As $X\to 1$,

$$\left.\begin{aligned}\phi_1 &= q_1 + \Omega_1\,\tau + O(1 - X), \\ \phi_2 &= q_2 + \Omega_2\,\tau + O(1 - X).\end{aligned}\right\} \tag{7.4.29}$$

The matching conditions (7.4.24), (7.4.25) give

$$q_2 = q_1 + \frac{\pi}{2} \tag{7.4.30}$$

and

$$\Omega_1 = \Omega_2 = \tilde{\Omega}, \text{ say,} \tag{7.4.31}$$

so that the phase difference between the two components of the disturbance is fixed, but not the absolute phase, which, along with the frequency correction, $\tilde{\Omega}$, remains arbitrary in the infinite layer, resulting in an infinite set of non-linear solutions. Note that, in (7.4.29), the correction terms are $O(1-X) = O(\frac{\ln r}{|\ln\varepsilon|})$ so that variations of ϕ_1 and ϕ_2 with r do not affect the matching procedure with the Bessel function solution at leading order.

As $X \to 0$, we have from (7.4.28),

$$\left.\begin{aligned} \phi_1 &\sim \tfrac{1}{2}\Theta_3 \ln X + q_1 + \tilde{\Omega}\tau + O(1)\;) \\ \phi_2 &\sim -\tfrac{1}{2}\Theta_2 \ln X + q_2 + \tilde{\Omega}\tau + O(1)\;). \end{aligned}\right. \tag{7.4.32}$$

In the outer zone appropriate forms for ϕ_1 and ϕ_2 are

$$\left.\begin{aligned} \phi_1 &\approx \tilde{\Omega}\tau + (\Theta_1 - \tilde{\Omega})s - \Theta_3 \int_1^s R_{2o}^2 \, ds + \tilde{q}_1, \\ \phi_2 &\approx \tilde{\Omega}\tau + (\Theta_1 + \tilde{\Omega})s + \Theta_2 \int_1^s R_{2o}^2 \, ds + \tilde{q}_2, \end{aligned}\right. \tag{7.4.33}$$

and matching with (7.4.32) implies that

$$\left.\begin{aligned} \tilde{q}_1 + \tfrac{1}{2}\,\Theta_3 \ln|\ln\varepsilon| &= q_1, \\ \tilde{\tilde{q}}_2 - \tfrac{1}{2}\,\Theta_2 \ln|\ln\varepsilon| &= q_2. \end{aligned}\right. \tag{7.4.37}$$

Thus, from (7.4.30), the phase difference in the outer zone is determined to leading order by

$$\tilde{q}_1 - \tilde{q}_2 \simeq -\tfrac{1}{2}(\Theta_2+\Theta_3)\ \ln|\ln\varepsilon|, \quad (\varepsilon \ll 1). \tag{7.4.35}$$

The solution of the transitional zone problem posed by (7.4.19) - (7.4.21) and (7.4.26) was solved numerically for various values of $\tilde{\mu}$ by a Runge-Kutta scheme. The solution was started at $\xi = 0$ using the Taylor series expansions for F_1 and F_2 for the first few steps. These expansions begin

$$F_1 = c\,\xi + O(\xi^3),$$

$$F_2 = \sqrt{\tfrac{1}{2}} - \frac{\tilde{\mu}c^2}{\sqrt{2}(2+\tilde{\mu})}\,\xi^2 + O(\xi^4), \tag{7.4.36}$$

where c is an arbirtary constant to be determined from the numerical solution. Matching with (7.4.13) then gives

$$c_{11} = c \tag{7.4.37}$$

and so completesthe leading order solution for R_1 in the outer zone. The value of c was guessed initially and the solution computed from $\xi = 0$ to $\xi = 1$ where, in general, $F_1 \neq F_2$; appropriate adjustment of c was then incorporated in an iterative scheme and the computations continued until $F_1 = F_2$ at $\xi=1$ to within a prescribed tolerance. An initial estimate for c was provided by its known value when $\tilde{\mu}=0$, for the solution of (7.4.19) - (7.4.21) and (7.4.26) is then

$$F_1 = \tfrac{1}{2}\,\xi(1 - \tfrac{1}{2}\xi^2)^{-\frac{1}{2}}, \quad F_2 = \tfrac{1}{\sqrt{2}}\,, \tag{7.4.38}$$

so that

$$c = F_1'(0) = \tfrac{1}{2}. \tag{7.4.39}$$

The result (7.4.39) can be extended by a perturbation expansion about $\tilde{\mu} = 0$ and this yields the result

$$c = \tfrac{1}{2} + \tfrac{1}{4}\,\tilde{\mu}(1 - \ln 2) + \; \ldots\ldots, \quad (\tilde{\mu} \ll 1), \tag{7.4.40}$$

which was found to be in good agreement with the numerical computations given in Table 7.1. Solutions for F_1 and F_2 for various values of $\tilde{\mu}$ are also shown in figure 7.3. Note also that the results (7.4.38), (7.4.39) are consistent with the solution of §7.3 since when $\tilde{\mu} = 0$, $X = \xi^2$ and from (7.4.38), (7.4.17) and (7.4.15) we recover the forms of (7.3.1) and (7.3.2) for small s:

$$R_1 \simeq s^{-\frac{1}{2}}(2\ln s - 4\ln\varepsilon)^{-\frac{1}{2}}, \; R_2 \simeq s^{-\frac{1}{2}}|2\ln s|^{-\frac{1}{2}}. \tag{7.4.41}$$

One feature of the results worthy of comment is the general trend for an increase in amplitude with a decrease in the value of $\tilde{\mu}$, leading to a singular form of the present structure as $\tilde{\mu} \to -1$. Evidence for this is found in the structure at infinity (see (7.4.64), (7.4.65) below), in the near-zero value of γ in (7.4.11), and also in the trend of the transitional zone solutions of figure 7.3. These suggest that the values of the amplitude functions at the

$\tilde{\mu}$	c	$F_1(1)$
0.0	0.5	0.7071
0.1	0.5073	0.6943
0.2	0.5142	0.6832
0.3	0.5205	0.6733
0.4	0.5265	0.6645
0.5	0.5321	0.6566
0.6	0.5373	0.6495
0.7	0.5422	0.6430
0.8	0.5468	0.6371
0.9	0.5512	0.6316
0.99	0.5549	0.6271
1	0.5553	0.6266
-0.1	0.4920	0.7218
-0.2	0.4833	0.7391
-0.3	0.4738	0.7595
-0.4	0.4633	0.7848
-0.5	0.4518	0.8164
-0.6	0.4388	0.8581
-0.7	0.4240	0.9164
-0.8	0.4067	1.007
-0.9	0.3856	1.191

Table 7.1 : Numerical results for different values of $\tilde{\mu}$.

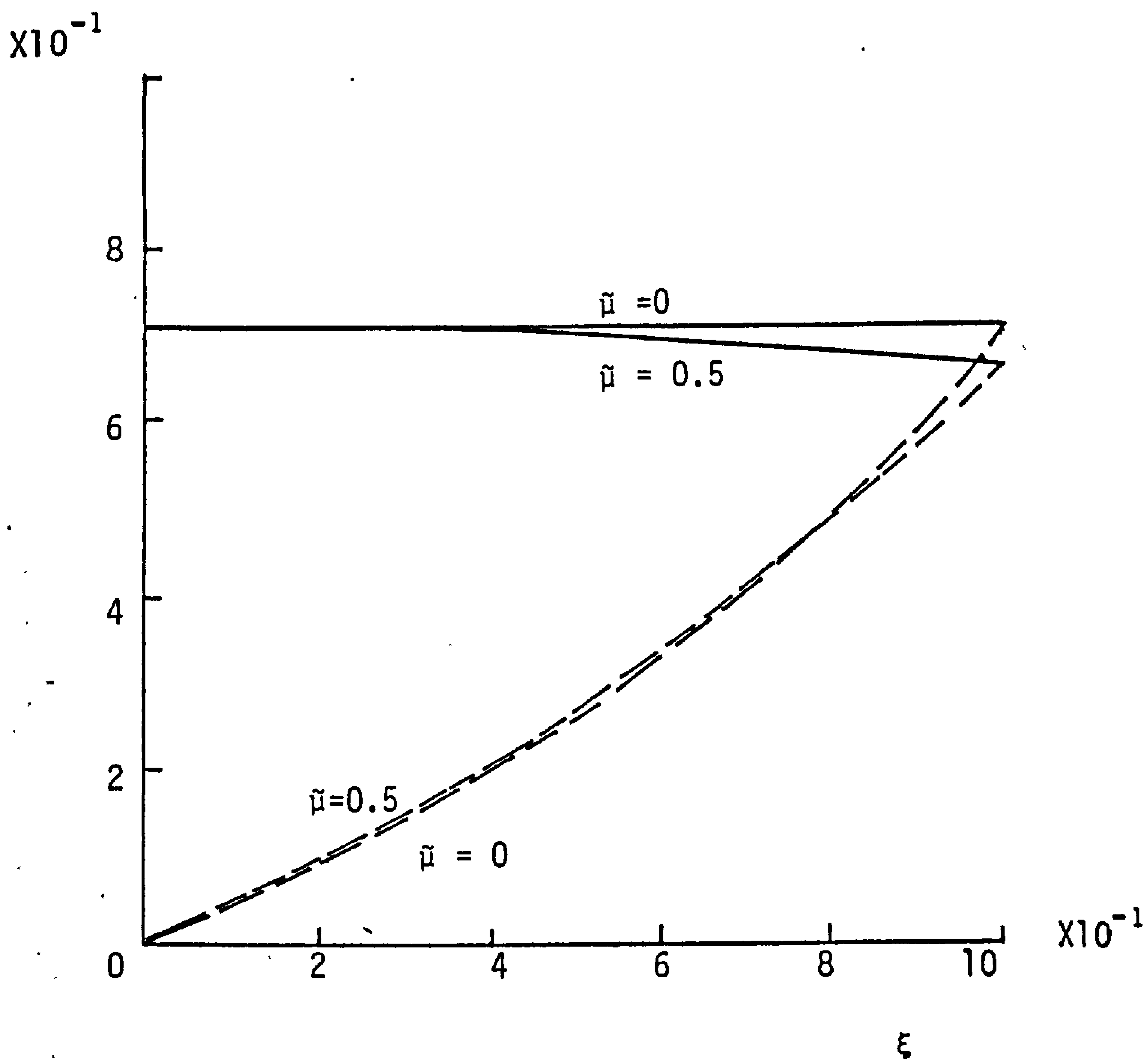

Figure 7.3: F_1 and F_2 profiles based on numerical results.

———————— F_1

- - - - - - - - - F_2

centre, $F_1(1) = F_2(1)$, become infinitely large as $\tilde{\mu} \to -1$ and this is confirmed by the following description of the limiting structure.

We set

$$\tilde{\mu} = -1 + \Delta \quad \text{with} \quad \Delta \ll 1 \tag{7.4.42}$$

and define new variables

$$Y(\xi) = F_1^{\,2}, \quad Z(\xi) = F_1^{\,2} - F_2^{\,2} + \tfrac{1}{2}. \tag{7.4.43}$$

Then the equations (7.4.19), (7.4.20) become

$$YZ = \Delta \{\tfrac{1}{4} \xi Y' - Y^2 + ZY - \tfrac{1}{2}Y\}$$

$$Z(Y - Z + \tfrac{1}{2}) = \Delta\{\tfrac{1}{4} \xi Y' - \tfrac{1}{4} \xi Z' + Y^2 - ZY + \tfrac{1}{2}Y\}, \tag{7.4.44}$$

and, from the boundary conditions (7.4.21), (7.4.26),

$$Y = Z = 0 \quad \text{at} \quad \xi = 0, \tag{7.4.45}$$

and

$$Z = \tfrac{1}{2} \quad \text{at} \quad \xi = 1. \tag{7.4.46}$$

The solution divides into a core region $0 < \xi < 1$ where, from terms of $O(\Delta)$ in (7.4.44) it may be shown that a solution satisfying (7.4.45) is

$$Y = \{-\tfrac{1}{4} + \tfrac{1}{4}(1-C\xi^2)^{-\frac{1}{2}}\} + O(\Delta),\ Z = O(\Delta), \qquad (7.4.47)$$

where C is an arbitrary constant. It emerges that $C = 1$ in order to match with a consistent solution in a boundary layer near $\xi = 1$, needed to adjust the value of Z in (7.4.47) to the value given by (7.4.46). We define a local co-ordinate

$$\xi_1 = \Delta^{-1}(1-\xi) \qquad (7.4.48)$$

and set

$$Y = \Delta^{-\frac{1}{2}} Y_o(\xi_1) + O(1),\ Z = Z_o(\xi_1) + O(\Delta^{\frac{1}{2}}). \qquad (7.4.49)$$

From terms of $O(\Delta^{-\frac{1}{2}})$ in (7.4.44) we obtain

$$Z_o Y_o = -\tfrac{1}{4} Y_o', \qquad (7.4.50)$$

and from terms of $O(1)$,

$$Z_o^2 - \tfrac{1}{2} Z_o = -2Y_o^2 - \tfrac{1}{4} Z_o'. \qquad (7.4.51)$$

These two equations must be solved subject to

$$Z_o = \tfrac{1}{2} \text{ at } \xi_1 = 0 \quad \text{and} \quad Z_o, Y_o \to 0 \text{ as } \xi_1 \to \infty \qquad (7.4.52)$$

From (5.50)

$$Y_0 = a_0 \exp\{-4\int_0^{\xi_1} Z_0 d\xi_1\}, \tag{7.4.53}$$

and then (7.4.51) becomes

$$\frac{1}{4} Z_0' + Z_0^2 - \tfrac{1}{2}Z_0 = -2a_0^2\exp\{-8\int_0^{\xi_1} Z_0 d\xi_1\} \tag{7.4.54}$$

to be solved subject to

$$Z_0 = \tfrac{1}{2} \text{ at } \xi_1 = 0 \quad \text{and } Z_0 = 0(\xi_1^{-1}) \text{ as } \xi_1 \to \infty. \tag{7.4.55}$$

The value of a_0 must be chosen in order that Z_0 decays as $\xi_1 \to \infty$. If a_0 is too large Z_0 reaches zero at a finite value of ξ_1 while if a_0 is too small, $Z_0 \to \frac{1}{2}$ as $\xi_1 \to \infty$. The required solution was thus achieved in a numerical solution of (7.4.54), (7.4.55) by a shooting method combined with a continual readjustment of the value of a_0. This leads to the final converged state in which

$$Z_0 \sim \tfrac{1}{8}\xi_1^{-1} \quad , \quad Y_0 \sim \frac{1}{4\sqrt{2}} \xi_1^{-\frac{1}{2}} \text{ as } \xi_1 \to \infty, \tag{7.4.56}$$

which matches correctly with (7.4.47) provided that we set $C = 1$.

The numerical solutions for Z_0 and Y_0 are shown in Figure 7.4 and the value of a_0 was found to be $a_0 = \frac{e^{.919}}{4\sqrt{2}}$.

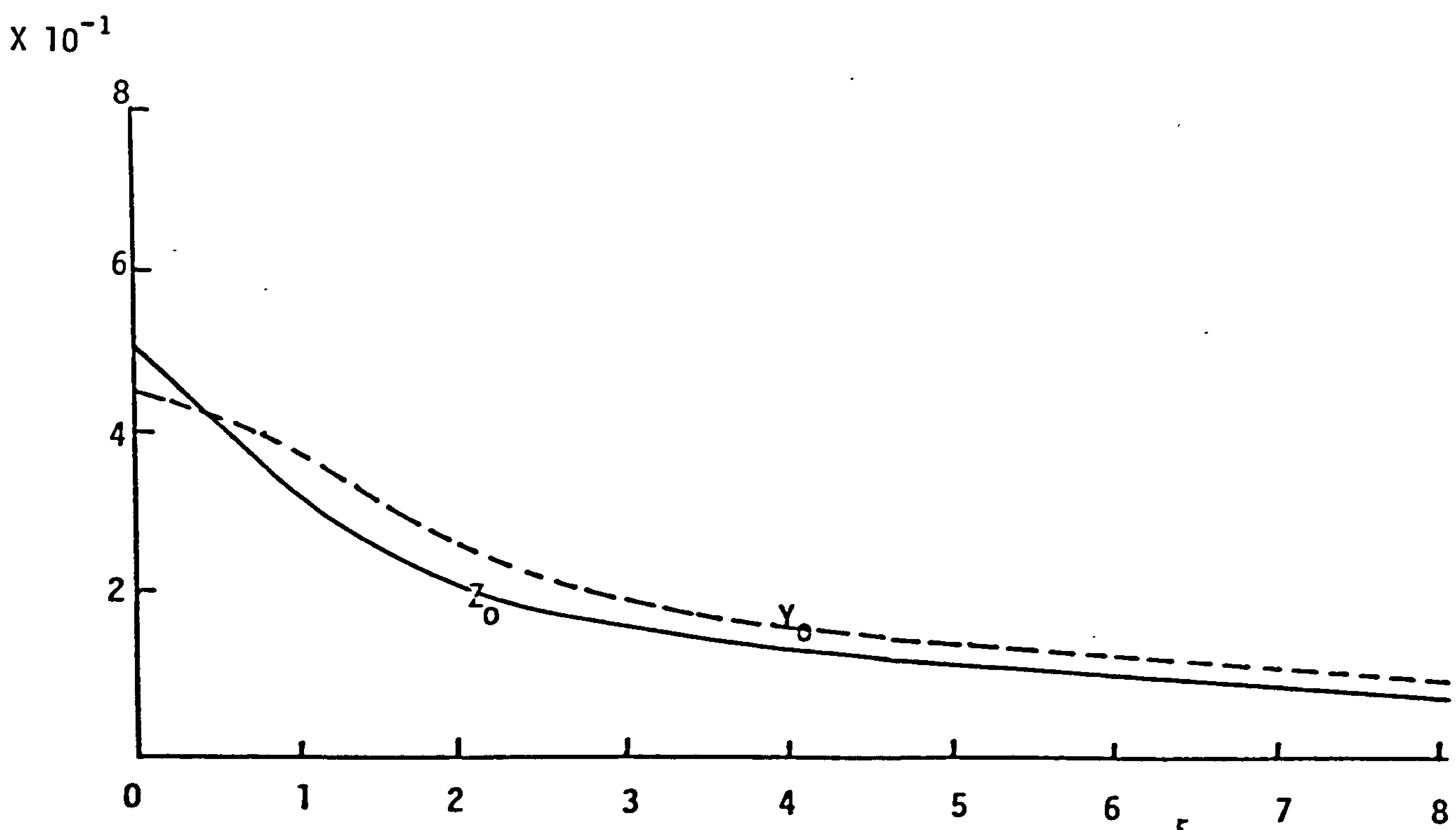

Figure 7.4: Profiles of Z_o and Y_o.

From (7.4.43) and (7.4.47) the value of c at $\tilde{\mu} = -1$ is

$$c = \frac{1}{2\sqrt{2}} = 0.3535 \ldots\ldots \qquad (7.4.57)$$

and, from (7.4.49) and (7.4.53) the values of F_1 and F_2 at $\xi = 1$ are

$$\sim a_0^{\frac{1}{2}} (\tilde{\mu} + 1)^{-\frac{1}{2}}, \qquad (\tilde{\mu} \to -1). \qquad (7.4.58)$$

These predictions both compare well with the numerical results given in table 7.1.

(iii) $s \to \infty$:

It now remains only to consider the behaviour of the amplitude functions for large s where, according to the results (7.4.2), (7.4.3), (7.4.6) and (7.4.8) we have

$$R_2 \sim \{1 + \tfrac{1}{4} s^{-1} + \ldots\ldots\} + \ldots\ldots,$$

$$(s \to \infty),$$

$$R_1 \sim |\ln\varepsilon|^{-\frac{1}{2}(1+\tilde{\mu})} r_{11} e^{(1-\tilde{\mu})s - \tilde{\mu} I_0} s^{-\frac{1}{2}(1+\tilde{\mu})} + \ldots, \qquad (7.4.59)$$

where I_0 is a numerical constant given by the finite part of I as $s \to \infty$. Thus R_1 increases in size and eventually, in the neighbourhood of $s = s_c$ where

$$s_c = \tfrac{1}{2}\left(\frac{1+\tilde{\mu}}{1-\tilde{\mu}}\right) \{\ln|\ln\varepsilon| + \ln(\ln|\ln\varepsilon|)\}, \qquad (7.4.60)$$

can no longer be neglected in comparison to R_2 in (7.2.11). We define a local variable S by

$$s = s_c + S \qquad (7.4.61)$$

and expand R_1 and R_2 as

$$R_1 = \tilde{R}_1(S) + \ldots, \quad R_2 = \tilde{R}_2(S) + \ldots \quad . \qquad (7.4.62)$$

The functions $\tilde{R}_1$ and $\tilde{R}_2$ satisfy the full nonlinear equations (7.2.10), (7.2.11) except that, in view of (7.4.60), the curvature terms can be neglected, giving

$$\frac{d\tilde{R}_1}{dS} = \tilde{R}_1 \left(1 - \tilde{R}_1^{\,2} - \tilde{\mu}\tilde{R}_2^{\,2}\right), \; \frac{d\tilde{R}_2}{dS} = -\tilde{R}_2^{\,2}\left(1-\tilde{R}_2^{\,2}-\tilde{\mu}\tilde{R}_1^{\,2}\right). \qquad (7.4.63)$$

Following the stability arguments given in §7.3 we restrict attention to solutions with

$$\tilde{R}_{1,2} \to (1 + \tilde{\mu})^{-\frac{1}{2}} \quad \text{as} \quad S \to \infty, \qquad (7.4.64)$$

so that at infinity the inward and outward propagating disturbances are of equal sizes. The required solution of (7.4.63) can then be computed backwards from $S = \infty$ by applying a small negative increment to the value of $\tilde{R}_1$ in (7.4.64). Precise forms for $\tilde{R}_1$ and $\tilde{R}_2$ as $S \to \infty$, used to start the numerical solution, are

$$\tilde{R}_{1,2} \sim (1+\tilde{\mu})^{-\frac{1}{2}} + \ell_{1,2}\exp\{-2(\tfrac{1-\tilde{\mu}}{1+\tilde{\mu}})^{\frac{1}{2}}S\}\ , \tag{7.4.65}$$

where

$$\ell_2 = -\tilde{\mu}\,(1+\{1-\tilde{\mu}^2\}^{\frac{1}{2}})^{-1}\ell_1 \tag{7.4.66}$$

and ℓ_1 is an arbitrary constant, equivalent to an origin shift in S.

For solutions with $\ell_1 < 0$, $\tilde{R}_1$ decreases monotonically to zero and $\tilde{R}_2$ automatically approaches one as $S \to \infty$. Asymptotic forms are

$$\begin{aligned} \tilde{R}_1 &\sim \exp\{(1-\tilde{\mu})(S+S_0)\}, \\ \tilde{R}_2 &\sim 1 - \tfrac{1}{2}\exp\{2(1-\tilde{\mu})(S+S_0)\} + \ell\, e^{2(s+S_0)}, \end{aligned} \qquad (S \to -\infty), \tag{7.4.67}$$

where the value of the constant ℓ can be determined from the numerical solution. The value of S_0 is fixed by the requirement that the forms (7.4.67)must match with those, (7.4.59), as $s \to \infty$. This implies that

$$S_0 = (1-\tilde{\mu})^{-1}(\ln c - \tilde{\mu}I_0) - \tfrac{1}{2}(\tfrac{1+\tilde{\mu}}{1-\tilde{\mu}})\ln\{\tfrac{1}{2}(\tfrac{1+\tilde{\mu}}{1-\tilde{\mu}})\}, \tag{7.4.68}$$

and fixes the origin shift in the solutions for $\tilde{R}_1$ and $\tilde{R}_2$. It seems likely that in general ℓ will be non-zero and so will generate a further term in the zone where $s = O(1)$

which, for $\tilde{\mu} < 0$ is larger than the term R_{22} in(7.4.2) (which matches with the second term in the expansion for $\tilde{R}_2$ in (7.4.67)). However, such a term remains uniformly smaller than R_{2o}, even as $s \to 0$, and therefore should not influence the matching procedure at the centre already determined.

When $\tilde{\mu} = 0$ the solutions for $\tilde{R}_1$ and $\tilde{R}_2$ are

$$\tilde{R}_1 = \sqrt{\tfrac{1}{2}}e^{s}(1 + \tfrac{1}{2}e^{2s})^{-\frac{1}{2}}, \quad \tilde{R}_2 = 1. \tag{7.4.69}$$

In this case $c = \frac{1}{2}$, $S_0 = -\frac{1}{2}\ln 2$, $\ell = \frac{1}{2}$ and $R_{22} = 0$. It can also be verified that the solutions (7.4.69) are equivalent to the appropriate limiting forms of (7.3.1) and (7.3.2). Some numerical solutions for other values of $\tilde{\mu}$ are shown in Figure 7.5.

Finally, we consider the behaviour of the phase functions ϕ_1 and ϕ_2 for large values of s. As $s \to \infty$, we have from (7.4.33)

$$\begin{aligned} \phi_1 &\sim \tilde{\Omega}\tau + (\Theta_1 - \Theta_3 - \tilde{\Omega})s + \tilde{q}_1, \\ & \qquad\qquad\qquad\qquad (s < s_c), \\ \phi_2 &\sim \tilde{\Omega}\tau + (\Theta_1 + \Theta_2 + \tilde{\Omega})s + \tilde{q}_2, \end{aligned} \tag{7.4.70}$$

with $\tilde{q}_1 - \tilde{q}_2$ given by (7.4.35). However, these forms are adjusted in the neighbourhood of s_c, so that for $s > s_c$,

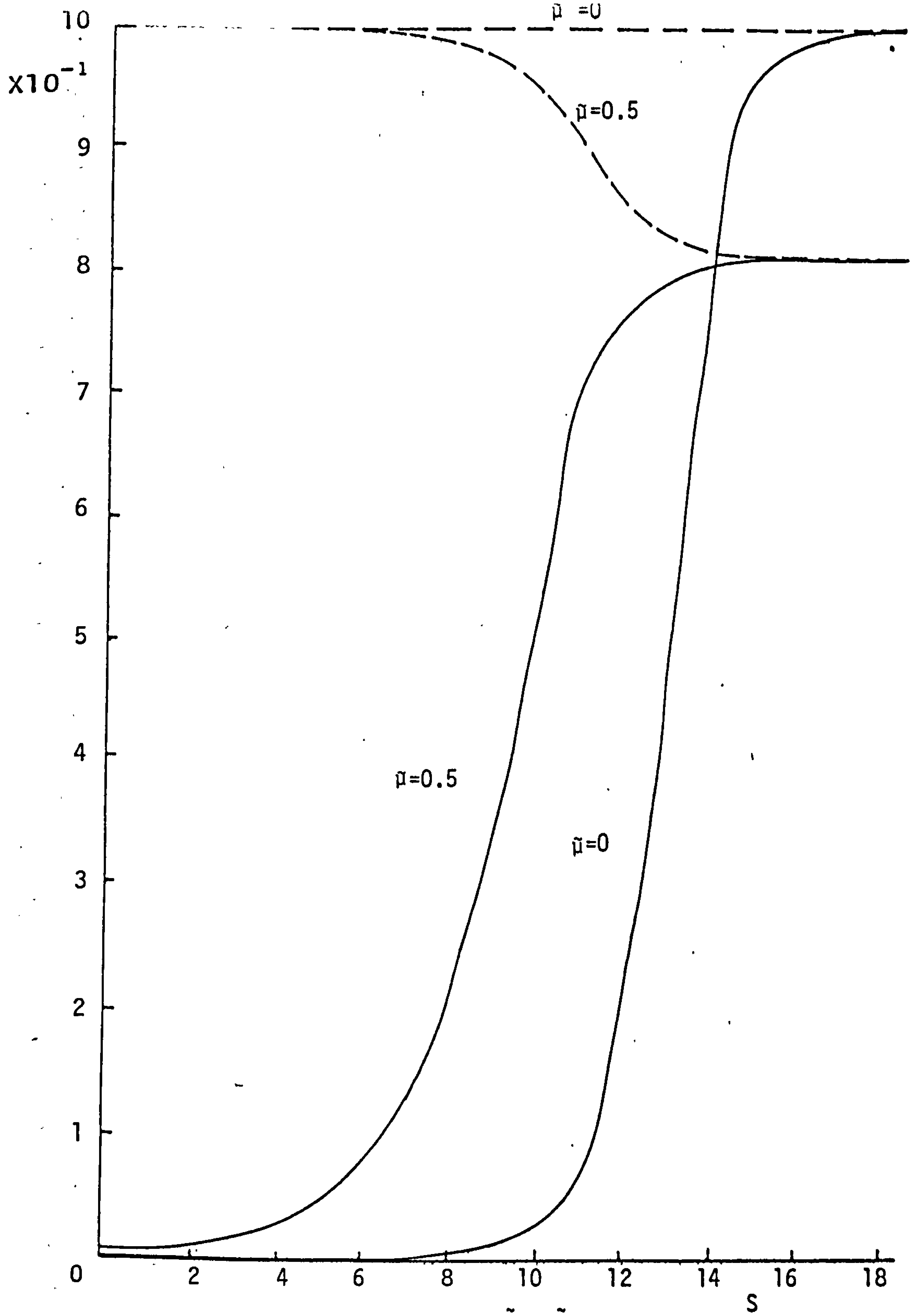

Figure 7.5 : Profiles of $\tilde{R}_1$, $\tilde{R}_2$ for $\tilde{\mu}$= 0,0.5

--------- $\tilde{R}_2$

———— $\tilde{R}_1$

$$\frac{\partial\phi_1}{\partial s} \sim - \frac{\partial\phi_2}{\partial s} \sim \{\Theta_1 - \tilde{\Omega} - (\Theta_2+\Theta_3)(1+\tilde{\mu})^{-1}\} \tag{7.4.71}$$

This represents a corrected disturbance wavelength equal to

$$\frac{\pi}{\alpha_c}\{1 - \frac{\varepsilon}{\alpha_c}\{\Theta_1 - \tilde{\Omega} - (\Theta_2+\Theta_3)(1+\mu)^{-1}\}\}, \qquad (\varepsilon \ll 1) \tag{7.4.72}$$

at infinity.

7.5. Discussion

The structure of an axisymmetric finite amplitude equilibrium state for a rotating system when the Rayleigh number exceeds its critical value for overstability by a small amount, $O(\varepsilon)$, has been described. At infinity, the solution is taken to be that for which the two components of the disturbance, A_1 and A_2, are of equal magnitude and the solution is seen to consist of several distinct regions in the limit as $\varepsilon \to 0$. Near $r = \frac{1}{2}\left(\frac{1+\bar{\mu}}{1-\bar{\mu}}\right)\varepsilon^{-1} \ln|\ln\varepsilon|$ there is a region where, as r decreases, the two components are adjusted so that A_1 is damped almost to zero, and, where $r = O(\varepsilon^{-1})$, is reduced to $O(\varepsilon^{\frac{1}{2}}|\ln\varepsilon|^{-\frac{1}{2}(1+\bar{\mu})})$. This region is dominated by the component A_2 of $O(\varepsilon^{\frac{1}{2}})$ because the central region of the flow acts as a poor reflector. The central region itself consists of two zones, first a transitional region where $r = O(\varepsilon^{X-1})$, $(0<X<1)$, and the two components rise to comparable amplitudes again, of $O(\varepsilon^{\frac{1}{2}(1-X)}|\ln\varepsilon|^{-\frac{1}{2}})$. These amplitudes become equal as $X \to 1$ and this permits the solution to match with the Bessel function form in the innermost zone where $r = O(1)$, and the amplitude reaches a peak value of $O(|\ln\varepsilon|^{-\frac{1}{2}})$. This indicates an intense motion near the axis of rotation as R increases above R_c and although a similar effect occurs in the exchange case, there the corresponding amplitude is considerably weaker, of $O(\varepsilon^{\frac{1}{4}}|\ln\varepsilon|^{-\frac{1}{3}})$.

The present work does not address the question of stability in a comprehensive fashion, although the indications are that the equilibrium state described is a stable one. Although the region where A_1 is small might at first sight be thought to be susceptible to the type of disturbance discussed in (7.3.7) - (7.3.9), it should be remembered that any natural disturbance to the equilibrium state will have to obey a reflection condition at the centre similar in form to that of the basic solution. Thus the component of the disturbance which gives rise to the positive growth rate of $1 - \tilde{\mu}$ in (7.3.9) will itself be significantly restricted and may therefore not destabilise the solution.

APPENDIX I

In (5.2.2) the functions $\bar{X}_1$, $\bar{X}_2$ and $\bar{X}_3$ can be expressed as follows:

$$\bar{X}_1=h_1\frac{\partial A_o}{\partial\tilde{\tau}}-x_1A_o|A_o|^2+\alpha f_1A_o-(2\alpha H+h_1)\left(\frac{\partial^2 A_o}{\partial s^2}+\frac{1}{s}\frac{\partial A_o}{\partial s}\right)+\frac{A_o}{2s^2}(\alpha H),$$

$$\bar{X}_2=\sigma^{-1}k_1\frac{\partial A_o}{\partial\tilde{\tau}}+\sigma^{-1}x_2A_o|A_o|^2+\frac{A_o}{s^2}(k_1-\alpha T_{12})+\frac{\alpha}{s}\frac{\partial A_o}{\partial s}\left\{T_{11}-\frac{k_1}{\alpha}+2T_{12}-T^{\frac{1}{2}}D\left(\frac{2f_1}{\alpha}+T_{22}+T_{21}\right)\right\}.$$

$$+\left\{2\alpha T_{11}-k_1-T^{\frac{1}{2}}D\left(\frac{f_1}{\alpha^2}+\frac{T_{21}}{\alpha}\right)\right\}\frac{\partial^2 A_o}{\partial s^2},$$

$$\bar{X}_3=\sigma^{-1}(D^2f_1-\alpha^2f_1)\frac{\partial A_o}{\partial\tilde{\tau}}+\sigma^{-1}x_3A_o|A_o|^2+\left\{\frac{D^4(f_1+\alpha T_{21})}{\alpha^2}+(2\alpha D^2-3\alpha^2)T_{21}+3\alpha^2f_1-T_{31}\right\}\frac{\partial^2 A_o}{\partial s^2}$$

$$+\left\{\frac{D^4(2f_1+\alpha T_{22}+\alpha T_{21})}{\alpha^2}+(2\alpha D^2-3\alpha^3)T_{22}+2\alpha^2f_1-\alpha^3T_{21}-T_{32}\right\}\frac{\partial A_o}{\partial s}+\frac{A_o}{s}\{-\alpha^2f_1+T_{32}+$$

$$(2\alpha^3-2\alpha D^2)T_{22}\},$$

where $D=\frac{d}{dz}$ and $\alpha=\alpha_c$, $R=R_c$.

The functions x_1, x_2, x_3 are given by

$$x_1=\alpha(2h_{21}Df_1+h_1Df_{21}+f_1Dh_{21}-f_1Dh_{22}+2f_{21}Dh_1),$$

$$x_2=\alpha(k_1Df_{21}-2k_{21}Df_1-f_1DK_{21}+2f_{21}Dk_1),$$

$$x_3=\alpha(2D^2f_1D^2f_{21}-f_1D^3f_{21}+2f_{21}D^3f_1+6\alpha^2f_{21}Df_1+3\alpha^2f_1Df_{21}-Df_{21}D^2f_1-2Df_1D^2f_{21}).$$

Also the functions $\bar{D}_i$ ($i=1, 2, \ldots., 6$) in (5.2.4) are found to be

$$\bar{D}_1 = \bar{R}^1 h_1^2 - \bar{\sigma}^1\{k_1^2 + f_1(D^2 f_1 - \alpha^2 f_1)\}\ ,$$

$$\bar{D}_2 = \bar{R}^1 h_1 x_1 + \sigma^{-1}(k_1 x_2 + x_3 f_1)\ ,$$

$$\bar{D}_3 = k_1^2 - \alpha k_1 T_{12} - \frac{\alpha h_1 H}{2R} + f_1(2\alpha^3 T_{22} - \alpha^2 f_1 + T_{32} - 2\alpha D^2 T_{22}),$$

$$\bar{D}_4 = \bar{R}^1(h_1^2 + 2\alpha h_1 H) + \frac{f_1 D^4(f_1 + \alpha T_{21})}{\alpha^2} - \alpha f_1(3\alpha T_{21} + 3\alpha f_1 + 2D^2 T_{21}) - T_{31} f_1 + 2\alpha k_1 T_{11}$$

$$-k_1^2 - \frac{T^{\frac{1}{2}} k_1 D}{\alpha}\left(\frac{f_1}{\alpha} + T_{21}\right),$$

$$\bar{D}_5 = h_1^2 + 2\alpha h_1 H + k_1(\alpha T_{11} - k_1 + 2\alpha T_{12}) - \frac{T^{\frac{1}{2}} k_1 D}{\alpha}\left(\frac{2f_1}{\alpha} + T_{22} + T_{21}\right) + \frac{f_1 D^4}{\alpha^2}(2f_1 + \alpha T_{22} + T_{21})$$

$$+ f_1(-3\alpha^3 T_{22} + 2\alpha^2 f_1 - \alpha^3 T_{21} - T_{32} + 2\alpha D^2 T_{22}),$$

$$\bar{D}_6 = \bar{R}^1 h_1 x_1 + \bar{\sigma}^1(k_1 x_2 + f_1 x_3)$$

where the numerical solutions of all functions involved in these expressions, are given in § 4.2, § 4.3 and § 5.1.

APPENDIX II

Matrix Formulation of the Rotating Linear Problem

The linearised equations (4.1) - (4.5) with (5.1.2) for leading order terms can be expressed (cf. (3.3)) in the matrix form

$$\frac{\partial \underline{U}}{\partial z} = \underline{A} \cdot \underline{U} \,, \qquad \text{(II.1)}$$

where $\underline{U}$ is a vector functions of $s, \tilde{\tau}, r, z$

and

$$\underline{U} = \{ \tilde{u} \;\; \frac{\partial p}{\partial r} \;\; \frac{\partial \tilde{\theta}}{\partial r} \;\; \tilde{v} \;\; u \;\; \frac{\partial w}{\partial r} \;\; \frac{\partial \theta}{\partial r} \;\; v \}^{tr},$$

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & -L & 0 & 0 & T^{\frac{1}{2}} \\ L & 0 & 0 & 0 & 0 & L & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -R & -L & 0 \\ 0 & 0 & 0 & 0 & T^{\frac{1}{2}} & 0 & 0 & -L \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(II.2)}$$

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \qquad \text{(II.3)}$$

We can write the matrix $\underline{A}$ in the form

$$\underline{A} = L \, \underline{A}_1 + R \, \underline{A}_2 + T^{\frac{1}{2}} \underline{A}_3 + \underline{A}_4 \qquad \text{(II.4)}$$

where

$$
\underline{A}_1=\begin{bmatrix}
0\;0\;0 & -1\;0\;0\;0\\
1\;0\;0 & 0\;1\;0\;0\\
0\;0\;0 & 0\;0\;-1\;0\\
0\;0\;0 & 0\;0\;0\;-1\\
 & 0\;0\;0\;0\\
 & 0\;0\;0\;0\\
\underline{0} & -1\;0\;0\;0\\
 & 0\;0\;0\;0
\end{bmatrix},
\underline{A}_2=\begin{bmatrix}
\underline{0} & 0\;0\;0\\
 & 0\;0\;0\\
 & -1\;0\;0\\
 & \\
 & \\
 & \\
\underline{0} & \underline{0}\\
 &
\end{bmatrix},
\underline{A}_3=\begin{bmatrix}
\underline{0} & 0\;0\;0\;-1\\
 & 0\;0\;0\;0\\
 & 0\;0\;0\;0\\
 & 1\;0\;0\;0\\
 & \\
 & \\
\underline{0} & \underline{0}\\
 &
\end{bmatrix},
\underline{A}_4=\begin{bmatrix}
0\;1\;0\;0\;0\;0\;0\;0\\
0\;0\;0\;0\;0\;0\;1\;0\\
0\;0\;0\;0\;0\;0\;0\;0\\
0\;0\;0\;0\;0\;0\;0\;0\\
1\;0\;0\;0 & \\
0\;0\;0\;0 & \\
0\;0\;1\;0 & \underline{0}\\
0\;0\;0\;1 &
\end{bmatrix}.
$$

In (II.I) the vector $\underline{U}$ may be expanded as

$$U = e^{i\alpha_c r}\{\varepsilon \underline{F}_0 + \varepsilon^2 \underline{F}_1 + \varepsilon^3 \underline{F}_3 + \ldots\ldots\}, \qquad \text{(II.5)}$$

where each $\underline{F}_i (i=0,2, \ldots)$ is a function of z, s, $\tilde{\tau}$.

Substitution of (5.1.1) and (II.4), (II.5) into equation (II.1) and from leading order terms and higher order terms, we find that

$$\mathcal{L}\, \underline{F}_0 = 0; \quad \gamma_1 \qquad \text{(II.6)}$$

$$\mathcal{L}\, \underline{F}_1 = i\alpha_c\, \underline{A}_1\left(2\frac{\partial \underline{F}_0}{\partial s} + \frac{\underline{F}_0}{s}\right);\gamma_1 \qquad \text{(II.7)}$$

$$\mathcal{L}\, \underline{F}_2 = \underline{A}_1\left(\frac{\partial^2 \underline{F}_0}{\partial s^2} + \frac{1}{s}\frac{\partial \underline{F}_0}{\partial s} - \frac{\underline{F}_0}{s^2}\right) + i\alpha_c \cdot \left(2\frac{\partial \underline{F}_1}{\partial s} + \frac{\underline{F}_1}{s}\right) + \beta\, \underline{A}_2\, \underline{F}_0\, ;\ \gamma_1 \qquad \text{(II.8)}$$

where

$$\mathcal{L} = \frac{\partial}{\partial z} - (T^{\frac{1}{2}}\, \underline{A}_3 + \underline{A}_4 + R_c\, \underline{A}_2 - \alpha_c^2\, \underline{A}_1), \qquad \text{(II.9)}$$

is a linear operator and the boundary conditions, γ_1, are defined in (3.3.22).

We define

$$\mathcal{L}^* = \frac{\partial}{\partial z} + (T^{\frac{1}{2}}\, \underline{A}_3 + \underline{A}_4 + R_c \quad \underline{A}_2 - \alpha_c^2\, \underline{A}_1)^{tr}$$

as an adjoint operator (cf. (3.5)) and find solutions of equations (II.6) - (II.8).

From (II.6)

$$\underline{F}_0 = \underline{h}_0\ (z)\ A_0\ (s,\ \tau), \qquad \text{(II.10)}$$

where the numerical solution of $\underline{h}_0$ is given in § 4.2, and A_0 denotes the amplitude of disturbance. To solve the equation (II.7), first we apply the adjoint condition (cf. (3.5)) which implies that

$$(2\, \frac{\partial A_0}{\partial s} + \frac{A_0}{s}) \int_0^1 \underline{A}_1\ \underline{f}^{tr,a}\ \underline{h}_0\ dz = 0 \qquad \text{(II.11)}$$

where $\underline{f}^{tr,a}$ is the adjoint eigen function (cf. (3.5)). We remind the reader that this condition is equivalent to (4.3.5); therefore from (II.7),

$$\underline{F}_1 = i\alpha_c(2\, \frac{\partial A_0}{\partial s} + \frac{A_0}{s})\underline{h}_1 + A_1\ (s,\tau)\ \underline{h}_0, \qquad \text{(II.12)}$$

where the numerical solution of $\underline{h}_1$ is given in § 4.3.

From substitution of (II.10) and (II.12), into equation (II.8) we find that

$$\mathcal{L}\,\underline{F}_2=\{\underline{A}_1 L^*(A_o)+i\alpha_c(2\frac{\partial}{\partial s}+\frac{1}{s})A_1\}\underline{h}_o-4\underline{A}_1\{(L^*-\frac{3}{4s^2})A_o\}\underline{h}_1+\beta\underline{A}_2\,\underline{F}_o, \quad \text{(II.13)}$$

where β is an arbitrary constant which is defined in (5.1.1) and L^* is given by

$$L^* = \frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}-\frac{1}{s^2}.$$

Now from the adjoint condition we find that

$$\frac{\partial^2 A_o}{\partial s^2}+\frac{1}{s}\frac{\partial A_o}{\partial s}-\frac{A_o}{4s^2}-\frac{\beta}{4}\frac{\int_0^1 \underline{A}_2\,\underline{f}^{a,tr}\,\underline{h}_o\,dz}{\int_0^1 \underline{A}_1\,\underline{f}^{a,tr}\,\underline{h}_1\,dz}A_o = 0, \quad \text{(II.14)}$$

or

$$\frac{\partial^2 A_o}{\partial s^2}+\frac{1}{s}\frac{\partial A_o}{\partial s}-\frac{A_o}{4s^2}+A_o = 0 \quad \text{(II.15)}$$

where we set

$$\beta = \frac{-4\int_0^1 \underline{A}_1\,\underline{f}^{a,tr}\,\underline{h}_1\,dz}{\int_0^1 \underline{A}_2\,\underline{f}^{a,tr}\,\underline{h}_o\,dz}. \quad \text{(II.16)}$$

Equation (II.15) is the linear form of the amplitude equation for the rotating layer and it can be seen that in (5.2.3)

$$a_4 = a_5, \; a_4 = -4a_3 \qquad \text{(II.17)}$$

It should be noted that the numerical value of (II.16) is given in (5.2.8).

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