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Approximate Solutions of the Determinantal Assignment Problem and Distance Problems

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Abstract: The paper introduces the formulation of an exact algebrogeometric problem, the study of the Determinantal Assignment Problem (DAP) in the set up of design, where approximate solutions of the algebraic problem are sought. Integral part of the solution of the Approximate DAP is the computation of distance of a multivector from the Grassmann variety of a projective space. We examine the special case of the calculation of the minimum distance of a multivector in $\wedge^2(\mathbb{R}^5)$ from the Grassmann variety $G_2(\mathbb{R}^5)$. This problem is closely related to the problem of decomposing the multivector and finding its best decomposable approximation. We establish the existence of the best decomposition in a closed form and link the problem of distance to the decomposition of multivectors. The uniqueness of this decomposition is then examined and several new alternative decompositions are presented that solve our minimization problem based on the structure of the problem.

1. INTRODUCTION

Systems and Control provide a paradigm that introduces many open problems of mathematical nature. The Determinantal Assignment Problem (DAP) has emerged as the abstract problem to which the study of pole, zero assignment of linear systems may be reduced (Karcanias and Giannakopoulos [1984]), (Hodge and Pedoe [1994]), (Giannakopoulos, Kalogeropoulos and Karcanias [1983]), (Giannakopoulos and Karcanias [1989]), (Leventides and Karcanias [1995]). This approach unifies the study of frequency assignment problems (pole, zero) of multivariable systems under constant, dynamic centralised, or decentralised control structure, has been developed. The Determinantal Assignment Problem (DAP) demonstrates the significance of exterior algebra and classical algebraic geometry for control problems. The importance of tools and techniques of algebraic geometry for control theory problems has been demonstrated by the work of (Brockett and Byrnes [1981]) etc. The approach adopted from (Karcanias and Giannakopoulos [1984]), (Giannakopoulos, Kalogeropoulos and Karcanias [1983]) (Giannakopoulos and Karcanias [1989]) differs from that of (Brockett and Byrnes [1981]) in the sense that the problem is studied in a projective, rather than an affine space setting; the former approach relies on exterior algebra and on the explicit description of the Grassmann variety (Hodge and Pedoe [1994]), in terms of the QPRs, and has the advantage of being computational.

The multilinear nature of DAP suggests that the natural framework for its study is that of exterior algebra Marcus, M., 1973. The study of DAP (Karcanias and Giannakopoulos [1984]) may be reduced to a linear problem of zero assignment of polynomial combinants and a standard problem of multilinear algebra, the decomposability of multivectors (Marcus [1973]). The solution of the linear subproblem, whenever it exists, defines a linear space in a projective space $\mathbb{P}^t$ whereas decomposability is characterized by the set of Quadratic Plucker Relations (QPR), which define the Grassmann variety of $\mathbb{P}^t$ (Hodge and Pedoe [1994]). Thus, solvability of DAP is reduced to a problem of finding real intersections between the linear variety and the Grassmann variety of $\mathbb{P}^t$. This novel Exterior Algebra-Algebraic Geometry method, has provided new invariants (Plucker Matrices and the Grassmann vectors) for the characterisation of rational vector spaces, solvability of control problems, ability to discuss both generic and non generic cases and it is flexible as far as handling dynamic schemes, as well as structurally constrained compensation schemes. The additional advantage of the new framework is that it provides a unifying computational framework for finding the solutions, when such solutions exist. The multilinear nature of DAP has been recently handled by a "blow up" type methodology, using the notion of degenerate solution and known as "Global Linearisation" (Leventides [1993]), (Leventides and Karcanias [1995]). Under certain conditions, this methodology allows the computation of solutions of the DAP problem.

The approach defined by the DAP formulation is based on polynomial matrix theory, exterior algebra and properties of the Grassmann variety of projective space. The aim of this paper is to introduce the basics that can turn this methodology from an exact algebra based approach to one that can handle issues of design. This may be achieved by developing an analytic dimension based on distance problems and optimization tools. In fact what we propose here is the development of approximate solutions to purely algebraic problems and thus expand the potential of the existing algebraic framework by developing its analytic dimension. The development of the "approximate" dimension of DAP involves a number of aspects that can transform the existence results and general computational schemes to tools for control design. There are many challenging issues in the development of the DAP framework and amongst them are its ability to provide solutions even for non-generic cases, as well as providing approximate solutions to the cases where generically there is no solution of the exact problem. The paper extends the formulation of the approximate DAP developed in (Karcanias and Leventides [2007]) by considering a more complex case involving the computation of the distance of a point in $\wedge^2(\mathbb{R}^5)$ from the Grassmann variety $G_2(\mathbb{R}^5)$.

2. THE APPROXIMATE DAP: BACKGROUND RESULTS.

The development of the approximate DAP requires a framework for approximation (provided by distance problems) which is equivalent to the formulation of an appropriate constrained optimization problem. The solvability of the exact problem is equivalent to finding real intersections between the Grassmann variety and a linear variety depending on the polynomial to be assigned. The need for "approximate" DAP solutions emerges when when the exact problem has no solutions , or there is model uncertainty that leads to a family of linear varieties. The key problem we need to address in the development of the "approximate DAP" is defining the distance between a linear variety and the Grassmann variety of a projective space. Central to this investigation is studying the problem of "approximate decomposability" of multivectors that has been considered for the first time in this context in (Karcanias and Leventides [2007]).

A new framework for searching for approximate solutions has been recently proposed based on the notion of "approximate decomposability" of multi-vectors (Karcanias and Leventides [2007]), which deals with the special case of the $G_2(\mathbb{R}^4)$ Grassmann variety . This study is in its early stages of formulation and it is based on the characterization of decomposability by the properties of a new family of matrices known as Grassmann Matrices (Karcanias and Giannakopoulos [1984]) which has been introduced as an alternative criterion to the standard description of the Grassmann variety provided by the QPRs (Hodge and Pedoe [1994]). The problem of decomposability of the multivector $\underline{z} \in \wedge^m \mathfrak{U}$ where $\mathfrak{U}$ is a vector space , is equivalent to the solvability of the exterior equation

$$\underline{v}_1 \wedge \underline{v}_2 \wedge ... \wedge \underline{v}_m = \underline{z},\ \underline{v}_i \in \mathfrak{U} \tag{2.1}$$

The formulation of a distance problem emerges here as the main task. The problem of approximate decomposability is a very difficult problem of multi-linear algebra that has not been completely solved (De Silva and Lim [2006]). Here we consider the case $G_2(\mathbb{R}^5)$ which is the next step to the previous one and we will develop a closed form solution to the problem.

In its general form the distance problem is related to several important general problems of multi-linear algebra, such as:
a) Low rank tensor approximation (De Silva and Lim [2006]) b) Multi-linear singular value decomposition (De Lathauwer, De Moor and Vaulewalle [2000]),(De Silva and Lim [2006]) c) Determination of the tensor rank (De Silva and Lim [2006]). The main theme of these problems is to decompose a tensor X as a sum of rank one or low rank tensors , i.e

$$X = \sum_{i=1}^{r} X_i \tag{2.2}$$

where X_i is a rank one or low rank tensor. The least r of this decomposition is the rank of the tensor.

Our work deals with skew symmetric tensors, i.e. multilinear tensors X that arise from determinantal problems and we will try to approximate them by decomposable vectors, i.e. to find vectors $a_1, a_2, ..., a_r$ such that the norm $\|X - a_1 \wedge a_2 \wedge ... \wedge a_r\|$ is minimized. This problem can be viewed into two ways, either as a low rank approximation of skew symmetric tensors, or as a distance problem from the Grassmann variety embeded in a projective space by the Plucker embedding. Here, we consider the case of $G_2(\mathbb{R}^5)$ and thus we deal with skew symmetric tensors $x_1 \in \wedge^2(\mathbb{R}^5)$ or points x_2 of the projective space $\mathbb{P}^9(\mathbb{R})$. We aim to find decomposable vectors $\underline{a}_1 \wedge \underline{a}_2$ that approximate x_1 in the Euclidean metric or x_2 in the gap metric. In this form we use the spectral decomposition of T_{x_1} corresponding to the skew symmetric tensor x_1 and we prove that x_1 can be written as a sum of two perpendicular decomposable vectors, one of which is the best decomposable approximation. The main result in this paper is the calculation of the closest decomposable vector $\underline{x}$ to a given vector $\underline{z}$ where $\underline{z}, \underline{x} \in \wedge^2(\mathbb{R}^5)$ and the least distance of $\underline{z}$ from the set of all decomposable vectors and the Grassmann variety $G_2(\mathbb{R}^5)$ - the latter with the use of the Gap metric. Also, we present two alternative ways for the above calculations; one with the use of the Grassmann matrix and the second via the Hodge- star duality properties.

3. THE BEST DECOMPOSABLE APPROXIMATION OF A MULTI-VECTOR IN $\wedge^2(\mathbb{R}^5)$.

In this section we will examine the minimization problem defined as follows:
Problem: Given a multi-vector $\underline{z} \in \wedge^2(\mathbb{R}^5) \simeq \mathbb{R}^{10}$, find the nearest decomposable vector to $\underline{z}$.
In other words, for $\underline{z} \in \mathbb{R}^{10}$ solve the minimization problem:

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \ \text{ when } \ \underline{x} \in \wedge^2(\mathbb{R}^5) \text{ is decomposable} \tag{3.1}$$

3.1 Decomposition via spectral analysis.

Let $\underline{z} \in \wedge^2(\mathbb{R}^5) \simeq \mathbb{R}^{10}$. We now examine the approximation of $\underline{z}$ by a decomposable vector, i.e. find $\underline{a} \wedge \underline{b}$ such that the distance $\|\underline{z} - \underline{a} \wedge \underline{b}\|$ is minimum. If we write $\underline{a} \wedge \underline{b} = \lambda \cdot \underline{w}_1 \wedge \underline{w}_2$ where $\|\underline{w}_1\| = \|\underline{w}_2\| = 1,\ <\underline{w}_1, \underline{w}_2>= 0$, then

$$\|\underline{z} - \underline{a} \wedge \underline{b}\|^2 = \|\underline{z}\|^2 - 2\lambda < \underline{z}, \underline{w}_1 \wedge \underline{w}_2 > + \lambda^2$$

which is minimized for $\lambda =< \underline{z}, \underline{w}_1 \wedge \underline{w}_2 >$. In this case

$$\|\underline{z} - \underline{w}_1 \wedge \underline{w}_2\|^2 = \|\underline{z}\|\|^2 - < \underline{z}, \underline{w}_1 \wedge \underline{w}_2 >^2$$

Therefore, we are looking to find orthonormal vectors $\underline{w}_1, \underline{w}_2$ such that the inner product $< \underline{z}, \underline{w}_1 \wedge \underline{w}_2 >^2$ is maximum. To achieve this, we need the following results.

Lemma 3.1. If

$$T \equiv T_{\underline{z}} = \begin{pmatrix} 0 & z_1 & z_2 & z_3 & z_4 \\ -z_1 & 0 & z_5 & z_6 & z_7 \\ -z_2 & -z_5 & 0 & z_8 & z_9 \\ -z_3 & -z_6 & -z_8 & 0 & z_{10} \\ -z_4 & -z_7 & -z_9 & -z_{10} & 0 \end{pmatrix}$$

is the skew symmetric matrix representing the multi-vector $\underline{z} = (z_1, z_2, z_3, ..., z_9, z_{10})$, then

$$< \underline{z}, \underline{w}_1 \wedge \underline{w}_2 >= \underline{w}_1^t T \underline{w}_2, \forall \underline{w}_1, \underline{w}_2 \in \mathbb{R}^5.$$

Proof. This is evident, from the form of the matrix T and that if $\underline{w}_1 = (w_1, w_2, w_3, w_4, w_5)$ and $\underline{w}_2 = (w_1', w_2', w_3', w_4', w_5')$, then $\underline{w}_1 \wedge \underline{w}_2$ is equal to a vector with coordinates the 2nd order minors of the matrix

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 \\ w_1' & w_2' & w_3' & w_4' & w_5' \end{pmatrix} \qquad \square$$

Lemma 3.2. The skew symmetric matrix T of the multi-vector $\underline{z}$ can be written in the form

$$(\underline{r}, \underline{b}_1, \underline{b}_2, \underline{a}_1, \underline{a}_2) \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_4 & 0 & 0 \\ 0 & -\sigma_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_2 \\ 0 & 0 & 0 & -\sigma_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{r}^t \\ \underline{b}_1^t \\ \underline{b}_2^t \\ \underline{a}_a^t \\ \underline{a}_2^t \end{pmatrix}$$

where, $\sigma_2 \leq \sigma_4$ are the real parts of the respective imaginary eigenvalues $\{0, i\sigma_4, -i\sigma_4, i\sigma_2, -i\sigma_2\}$ of T and $\{\underline{r}, \underline{b}_1 + i\underline{b}_2, \underline{b}_1 - i\underline{b}_2, \underline{a}_1 + i\underline{a}_2, \underline{a}_1 - i\underline{a}_2\}$ their orthonormal eigenvectors respectively.

Proof. This can be proved by the spectral analysis of the skew symmetric matrix T (Bellmann [1996]). $\square$

Theorem 3.1. For any $\underline{z} \in \mathbb{R}^{10}$ the following holds true :

$$\underline{z} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 \tag{3.2}$$

Proof. We take $\{\underline{r}, \underline{b}_1, \underline{b}_2, \underline{a}_1, \underline{a}_2\} \equiv \{\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4, \underline{e}_5\}$ be an orthonormal basis of $\mathbb{R}^5$. Then the set

$$B = \{\underline{e}_i \wedge \underline{e}_j,\ 1 \leq i \leq j \leq 5\}$$

is an orthonormal basis of $\wedge^2(\mathbb{R}^5) \simeq \mathbb{R}^{10}$. Therefore $\underline{z}$ can be written in the form

$$\underline{z} = \sum_{1 \leq i < j \leq 5} \lambda_{ij} \underline{e}_i \wedge \underline{e}_j$$

Since the basis is orthonormal, in order to calculate the coefficients λ_{ij} we need to calculate the products $< \underline{z}, \underline{e}_i \wedge \underline{e}_j >$. Now, from Lemma 3.1 we get:

$$\begin{aligned} &< \underline{z}, \underline{e}_4 \wedge \underline{e}_5 >= \underline{e}_4^t T \underline{e}_5 = \sigma_2 \underline{e}_4^t \underline{e}_4 = \sigma_2, \\ &< \underline{z}, \underline{e}_2 \wedge \underline{e}_3 >= \underline{e}_2^t T \underline{e}_3 = \sigma_4 \underline{e}_2^t \underline{e}_2 = \sigma_4, \\ &< \underline{z}, \underline{e}_1 \wedge \underline{e}_i >= 0,\ \forall i = 2, ..., 5 \text{ and } < \underline{z}, \underline{e}_2 \wedge \underline{e}_4 >= \\ &=< \underline{z}, \underline{e}_2 \wedge \underline{e}_5 >=< \underline{z}, \underline{e}_3 \wedge \underline{e}_4 >=< \underline{z}, \underline{e}_3 \wedge \underline{e}_5 >= 0. \end{aligned}$$

Hence, $\underline{z} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$. $\square$

The decomposition defined by (3.2) will be referred as *the prime decomposition* of $\underline{z}$.

Remark 3.1. It is obvious that every decomposable multi-vector $\underline{x}$ can be written in the form

$$\underline{x} = \|\underline{x}\| \underline{w}_1 \wedge \underline{w}_2$$

where $\underline{w}_1$, $\underline{w}_2$ are orthonormal by matching an orthonormal selection $\{\underline{w}_1, \underline{w}_2\}$ of a basis of the subspace corresponding to $\underline{x}$. $\square$

Theorem 3.2. For every decomposable multi-vector $\underline{x} \in \wedge^2(\mathbb{R}^5)$ and $\underline{z} \in \mathbb{R}^{10}$ we have that:

$$|< \underline{z}, \underline{x} >| \leq \sigma_4 \cdot \|\underline{x}\|$$

Proof. From Theorem 3.1 and the previous Remark, we have

$$\underline{z} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2,\ \underline{x} = \|\underline{x}\| \underline{w}_1 \wedge \underline{w}_2$$

where $\underline{w}_1$, $\underline{w}_2$ are orthonormal. Therefore,

$$| < \underline{z}, \underline{x} > | =< \underline{z}, \|\underline{x}\| \underline{w}_1 \wedge \underline{w}_2 > | = \|\underline{x}\| < |\underline{z}, \underline{w}_1 \wedge \underline{w}_2 > | \tag{3.3}$$

Also, for $\underline{w}_1$, $\underline{w}_2$ orthonormal we have that

$$\begin{aligned} &\max | < |\underline{z}, \underline{w}_1 \wedge \underline{w}_2 > | = \max < \underline{w}_1^t T, \underline{w}_2 >= \\ &= \max_{\|\underline{w}_1\|=1} \left\langle \underline{w}_1^t T, \frac{\underline{w}_1^t T}{\|\underline{w}_1^t T\|} \right\rangle = \max_{\|\underline{w}_1\|=1} \|\underline{w}_1^t T\| = \sigma_4 \end{aligned} \tag{3.4}$$

From the equations (3.3), (3.4) the result is established. $\square$

Theorem 3.3. Let $\underline{z}$ be a fixed multi-vector in $\wedge^2(\mathbb{R}^5)$. Then, the decomposable vector

$$\underline{x}_0 = \sigma_4 \underline{b}_1 \wedge \underline{b}_2 \tag{3.5}$$

solves the minimization problem $\min \|\underline{x} - \underline{z}\|$ where $\underline{x}$ is decomposable. Furthermore, the distance from the set of all decomposable vectors is

$$\|\underline{z} - \underline{x}_0\| = \sigma_2 \tag{3.6}$$

Proof. Let $\underline{x} = \underline{w}_1 \wedge \underline{w}_2$ be any decomposable vector. Then

$$\begin{aligned} \|\underline{x} - \underline{z}\|^2 &= \|\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 - \underline{x}\|^2 = \\ &= \sigma_2^2 + \sigma_4^2 + \|\underline{x}\|^2 - 2 < \underline{z}, \underline{x} > \end{aligned}$$

But from Theorem 3.2 we have

$$| < \underline{z}, \underline{x} > | \leq \sigma_4 \cdot \|\underline{x}\|$$

and as Remark 3.1 states we can consider $\underline{w}_1 \perp \underline{w}_2$, therefore

$$\begin{aligned} \|\underline{x} - \underline{z}\|^2 &\geq \sigma_2^2 + \sigma_4^2 + \|\underline{x}\|^2 - 2\sigma_4 \cdot \|\underline{x}\| \geq \sigma_2^2 + (\sigma_4 - \|\underline{x})\|)^2 \\ &\geq \sigma_2^2 \end{aligned}$$

for any decomposable multi-vector $\underline{x}$. Thus

$$\|\underline{z} - \underline{x}_0\| = \|\sigma_2 \underline{a}_1 \wedge \underline{a}_2\| = \sigma_2. \square$$

3.2 Minimization of the gap metric.

In the previous section, we have calculated the best decomposable approximation of a multivector. One can solve the same approximation problem in a different space, i.e. $\underline{z} \in \mathbb{P}^9(\mathbb{R})$ where the set of decomposable vectors is identified by the Grassmann variety $G_2(\mathbb{R}^5)$. In the projective space $\mathbb{P}^9(\mathbb{R})$ a natural metric is the gap metric g and the previous approximation problem becomes $\min_{\underline{x} \in G_2(\mathbb{R}^5)} g(\underline{x}, \underline{z})$.

Definition 3.1. (El-Sakkary [1985]), (Georgiou [1988]) The gap metric $g(\underline{z}, \underline{x})$ between two multi-vectors $\underline{z}, \underline{x}$ is the absolute value of the sine of the angle they form, i.e. $g(\underline{z}, \underline{x}) = |\sin(\underline{z} \hat{,} \underline{x})|$. $\square$

An equivalent definition is the following:

Definition 3.2. (El-Sakkary [1985]), (Georgiou [1988]) The gap metric $g(\underline{z}, \underline{x})$ between two lines $\text{span}\{\underline{z}\}, \text{span}\{\underline{x}\}$ in the projective space $P^9(\mathbb{R})$ is a function $g : P^9(\mathbb{R}) \times P^9(\mathbb{R}) \to \mathbb{R}$ such that:

$$g(\underline{z}, \underline{x}) = \min_{\lambda} \left\| \frac{\underline{z}}{\|\underline{z}\|} - \frac{\underline{x}}{\|\underline{x}\|} \cdot \lambda \right\| \qquad \square$$

Lemma 3.3. For every $\underline{z} \in \mathbb{R}^{10}$ and any orthonormal decomposition $\underline{z} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$, we have that

$$\sigma_2^2 + \sigma_4^2 = \|\underline{z}\|^2, \ \|\underline{z} \wedge \underline{z}\| = 2\sigma_2\sigma_4$$

Proof. Since the decomposition (3.2) is orthonormal (Bellmann [1996]) we have

$$(\underline{a}_1 \wedge \underline{a}_2) \perp (\underline{b}_1 \wedge \underline{b}_2)$$

Thus, from Theorem 3.1 we have

$$\|\underline{z}\|^2 = \|\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2\|^2 = \sigma_2^2 + \sigma_4^2$$

and

$$\|\underline{z} \wedge \underline{z}\| = \|2\sigma_2\sigma_4 \underline{a}_1 \wedge \underline{a}_2 \wedge \underline{b}_1 \wedge \underline{b}_2\| = 2\sigma_2\sigma_4$$

$\square$

Theorem 3.4. The gap metric between $\underline{z}$ and $\underline{x}_0$ is given by

$$g(\underline{z}, \underline{x}_0) = \frac{\sigma_2}{\|\underline{z}\|}$$

Proof. From Definitions 3.1 and 3.2 we have

$$\sin^2(\underline{z}, \hat{\underline{x}}_0) = \min_{\lambda} \left\| \frac{\underline{z}}{\|\underline{z}\|} - \frac{\underline{x}_0}{\|\underline{x}_0\|} \cdot \lambda \right\| =$$

$$= \min_{\lambda} \left(1 + \lambda^2 - 2 \cdot \frac{<\underline{z}, \underline{x}_0>}{\|\underline{z}\| \cdot \|\underline{x}_0\|} \cdot \lambda \right)$$

This is a second order degree polynomial in terms of λ that is minimized for

$$\lambda = \frac{<\underline{z}, \underline{x}_0>}{\|\underline{z}\| \cdot \|\underline{x}_0\|}$$

Thus, the distance is

$$\sin^2(\underline{z}, \hat{\underline{x}}_0) = 1 - \frac{<\underline{z}, \underline{x}_0>^2}{\|\underline{z}\|^2 \cdot \|\underline{x}_0\|^2}$$

Now, from Definition 3.1 and for $\underline{x}_0 = \sigma_4 \underline{b}_1 \wedge \underline{b}_2$ and $\underline{z} = \sigma_2 \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \underline{b}_1 \wedge \underline{b}_2$ we have

$$\sin^2(\underline{z}, \hat{\underline{x}}_0) = \frac{\|\underline{z}\|^2 \cdot \|\underline{x}_0\|^2 - <\underline{z}, \underline{x}_0>^2}{\|\underline{z}\|^2 \cdot \|\underline{x}_0\|^2} =$$

$$= \frac{\|\underline{z}\|^2 \cdot \sigma_4^2 - \sigma_4^4}{\|\underline{z}\|^2 \cdot \sigma_4^2} = \frac{\sigma_2^2}{\|\underline{z}\|^2}$$

due to Lemma 3.3. This proves the result. $\square$

Corollary 3.1. The multi-vector $\underline{x}_0 = \sigma_4 \underline{b}_1 \wedge \underline{b}_2$ minimizes the gap between $\underline{z}$ and any decomposable vector $\underline{x}_1$.

Proof. This is evident due to the previous theorem. $\square$

3.3 Uniqueness Criteria.

In this section we examine the conditions under which (3.2) is unique. It is obvious that under no assumptions, (3.2) is not unique since we can get an infinite number of decompositions for $\underline{z}$, i.e.

$$\underline{z} = \underbrace{\sigma_4(\underline{a}_1 + \underline{b}_1) \wedge (\underline{b}_2 + \underline{b}_1)}_{\underline{x}_1} + \underbrace{\underline{a}_1 \wedge (\sigma_2 \underline{a}_2 - \sigma_4 \underline{b}_1 - \sigma_4 \underline{b}_2)}_{\underline{x}_2}$$

or

$$\underline{z} = \underbrace{(\underline{a}_1 + 3\underline{b}_1) \wedge \left(\frac{\underline{a}_2 - \underline{b}_1}{2} \right)}_{\underline{x}_1} + \underbrace{(\underline{a}_1 - 3\underline{b}_2) \wedge \left(\frac{\underline{a}_2 + \underline{b}_1}{2} \right)}_{\underline{x}_2}$$

Definition 3.3. Let

$$\underline{z} = \underline{x}_1 + \underline{x}_2 \tag{3.7}$$

a (rank-2) decomposition of $\underline{z}$. If $\underline{x}_1 = \underline{a}_1 \wedge \underline{a}_2$, $\underline{x}_2 = \underline{b}_1 \wedge \underline{b}_2$ where $\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2$ are orthonormal, i.e. if $U := [\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2]$ then $U \cdot U^t = \mathbb{I}_5$, then the decomposition (3.7) is called orthonormal. $\square$

Definition 3.4. We say that two orthonormal decompositions

$$\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2, \quad \sigma_2' \cdot \underline{a}_1' \wedge \underline{a}_2' + \sigma_4' \cdot \underline{b}_1' \wedge \underline{b}_2'$$

are equivalent if

$$\sigma_4 = \sigma_4', \quad \sigma_2 = \sigma_2', \quad \underline{b}_1 \wedge \underline{b}_2 = \underline{b}_1' \wedge \underline{b}_2', \quad \underline{a}_1 \wedge \underline{a}_2 = \underline{a}_1' \wedge \underline{a}_2'$$

for $\sigma_4 \geq \sigma_2 \geq 0$, $\sigma_4' \geq \sigma_2' \geq 0$. $\square$

Theorem 3.5. The decomposition $\underline{z} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$ is unique if $\sigma_4 > \sigma_2 > 0$.

Proof. The existence of the decomposition $\underline{z} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$ has been proved in Theorem 3.1. Assume there is another decomposition,

$$\underline{z} = \sigma_2' \cdot \underline{a}_1' \wedge \underline{a}_2' + \sigma_4' \cdot \underline{b}_1' \wedge \underline{b}_2'.$$

Then from Lemma 3.3 we have that

$$\|\underline{z} \wedge \underline{z}\| = 2\sigma_2\sigma_4 = 2\sigma_2'\sigma_4'$$

and

$$\|\underline{z}\|^2 = \sigma_2^2 + \sigma_4^2 = \sigma_2'^2 + \sigma_4'^2$$

Therefore,

$$\sigma_2 = \sigma_2', \ \sigma_4 = \sigma_4'$$

Also, we have that

$$colspan[\underline{b}_1, \underline{b}_2, \underline{a}_1, \underline{a}_2] = colspan[\underline{b}_1', \underline{b}_2', \underline{a}_1', \underline{a}_2']$$

since they both are perpendicular to the same vector $(\underline{z} \wedge \underline{z})^*$. We now consider matrix U such that

$$[B, A] \cdot U = [B', A'], \ U = [U_1, U_2],$$

where $B = [\underline{b}_1, \underline{b}_2]$, $A = [\underline{a}_1, \underline{a}_2]$, $B' = [\underline{b}_1', \underline{b}_2']$, $A' = [\underline{a}_1', \underline{a}_2'.]$ Then by taking compounds we have that

$$\underline{b}_1' \wedge \underline{b}_2' = C_2[B, A] \cdot C_2[U_1], \ \underline{a}_1' \wedge \underline{a}_2' = C_2[B, A] \cdot C_2[U_2],$$

where $C_2[U_1], C_2[U_2] \in \wedge(\mathbb{R}^4)$ and if $\underline{x} := C_2[U_1]$, then $\underline{x}^* := C_2[U_2]$. Therefore,

$$\sigma_2' \cdot \underline{a}_1' \wedge \underline{a}_2' + \sigma_4' \cdot \underline{b}_1' \wedge \underline{b}_2' = C_2[B, A] \left(\sigma_4 \underline{x} + \sigma_2 \underline{x}^* \right)$$

and

$$\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 = C_2[B, A] \left(\sigma_4 \underline{e}_1 + \sigma_2 \underline{e}_6 \right)$$

Thus if,

$$C_2[B, A] \left(\sigma_4 \underline{x} + \sigma_2 \underline{x}^* \right) = C_2[B, A] \left(\sigma_4 \underline{e}_1 + \sigma_2 \underline{e}_6 \right),$$

by taking the left inverse matrix of $C_2[B, A]$ we have that

$$\sigma_4 \underline{x} + \sigma_2 \underline{x}^* = \sigma_4 \underline{e}_1 + \sigma_2 \underline{e}_6$$

Therefore, by applying the Hodge star operator we obtain

$$\sigma_4 \underline{x}^* + \sigma_2 \underline{x} = \sigma_4 \underline{e}_6 + \sigma_2 \underline{e}_1$$

Hence,

$$(\underline{x}, \underline{x}^*) \cdot \begin{pmatrix} \sigma_4 & \sigma_2 \\ \sigma_2 & \sigma_4 \end{pmatrix} = (\underline{e}_1, \underline{e}_6) \cdot \begin{pmatrix} \sigma_4 & \sigma_2 \\ \sigma_2 & \sigma_4 \end{pmatrix}$$

Since $\sigma_4 \neq \sigma_2$ the matrix $\begin{pmatrix} \sigma_4 & \sigma_2 \\ \sigma_2 & \sigma_4 \end{pmatrix}$ is invertible, therefore $\underline{x} = \underline{e}_1, \underline{x}^* = \underline{e}_6$. Hence, the representation is unique. $\square$

4. ALTERNATIVE DECOMPOSITIONS.

Having developed the prime decomposition of a multi-vector, studied the optimization problem (3.1) and investigating the uniqueness of the prime decomposition, we are now concerned with deriving alternative methods for the computation of the decomposition, without the use of skew-symmetric matrices or SVDs as in the original prime decomposition.

4.1 Minimum distance from $G_2(\mathbb{R}^5)$ via the Grassmann matrix.

The Grassmann matrix (Karcanias and Giannakopoulos [1984]) for the Grassmann variety $G_2(\mathbb{R}^5)$ has been defined as:

$$\mathbf{\Phi} = \begin{pmatrix} a_{23} & -a_{13} & a_{12} & 0 & 0 \\ a_{24} & -a_{14} & 0 & a_{12} & 0 \\ a_{25} & -a_{15} & 0 & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} & 0 \\ a_{35} & 0 & -a_{15} & 0 & a_{13} \\ a_{45} & 0 & 0 & -a_{15} & a_{14} \\ 0 & a_{34} & -a_{24} & a_{23} & 0 \\ 0 & a_{35} & -a_{25} & 0 & a_{23} \\ 0 & a_{45} & 0 & -a_{25} & a_{24} \\ 0 & 0 & a_{45} & -a_{35} & a_{34} \end{pmatrix}$$

If

$$\underline{z} = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}) \equiv \equiv (a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}, a_{35}, a_{45})$$

then the singular values of Φ can be calculated as the eigenvalues of $\Phi^t\Phi$.

Lemma 4.1. (Karcanias and Giannakopoulos [1984]) For every $\underline{z} \in \mathbb{R}^n, \underline{w} \in \mathbb{R}^m$ where $n > m$, $n, m \in \mathbb{N}$, the Grassmann matrix $\Phi \in \mathbb{R}^{n\times m}$ satisfies the equation:

$$\Phi \cdot \underline{w} = \underline{z} \wedge \underline{w}$$

i.e, matrix Φ is a matrix representation of the operation $\underline{z} \wedge (\cdot)$. □

Theorem 4.1. The singular values of Φ for a multi-vector $\underline{z}$ are

$$\|\underline{z}\| = \sqrt{\sigma_2^2 + \sigma_4^2}, \sigma_4, \sigma_4, \sigma_2, \sigma_2$$

and the right singular vectors are $\underline{r}, \underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2$ as these were established in Lemma 3.2. Thus, the minimum distance of $\underline{z}$ from $G_2(\mathbb{R}^5)$ coincides with the smallest singular value of Φ.

Proof. From Lemma 4.1 we have:

$$\underline{w}^t\Phi^t\Phi\underline{w} = \|\underline{z} \wedge \underline{w}\|^2 \tag{4.1}$$

We can now set:

$$\underline{w} = \kappa_1 \cdot \underline{a}_1 + \kappa_2 \cdot \underline{a}_2 + \kappa_3 \cdot \underline{b}_1 + \kappa_4 \cdot \underline{b}_2 + \kappa_5 \cdot \underline{r}$$

and $\underline{z} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$. Then

$$\underline{z} \wedge \underline{w} = \sigma_2(\kappa_3\underline{a}_1 \wedge \underline{a}_2 \wedge \underline{b}_1 + \kappa_4\underline{a}_1 \wedge \underline{a}_2 \wedge \underline{b}_2 + + \kappa_5\underline{a}_1 \wedge \underline{a}_2 \wedge \underline{r})\sigma_4(\kappa_1\underline{b}_1 \wedge \underline{b}_2 \wedge \underline{a}_1 + + \kappa_2\underline{b}_1 \wedge \underline{b}_2 \wedge \underline{a}_2 + \kappa_5\underline{b}_1 \wedge \underline{b}_2 \wedge \underline{r})$$

Therefore,

$$\|\underline{z} \wedge \underline{w}\|^2 = \sigma_2^2(\kappa_3^2 + \kappa_4^2 + \kappa_5^2) + \sigma_4^2(\kappa_1^2 + \kappa_2^2 + \kappa_5^2) = \underline{u}\Sigma\underline{u}^t$$

where $\underline{u} = (\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$ and

$$\Sigma = \begin{pmatrix} \sigma_2^2 + \sigma_4^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_4^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_4^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_2^2 \end{pmatrix}$$

Thus, from (4.1) the proof of the theorem is complete.
□

4.2 Decomposition via Duality Properties.

In this section we will see how the Hogde star duality properties help us simplify the prime decomposition.

Proposition 4.1. If $\underline{z} \wedge \underline{z} \neq 0$ and $\underline{r}_{\underline{z}} := \left((\underline{z} \wedge \underline{z})^*\right) / \|\underline{z} \wedge \underline{z}\|$ then $\left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^* = \sigma_4 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$

Proof. We have that

$$\begin{aligned}\left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^* &= \left((\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2) \wedge \underline{r}_{\underline{z}}\right)^* = \\ &= \sigma_4 \left(\underline{b}_1 \wedge \underline{b}_2 \wedge \underline{r}_{\underline{z}}\right)^* + \sigma_2 \left(\underline{a}_1 \wedge \underline{a}_2 \wedge \underline{r}_{\underline{z}}\right)^* = \\ &= \sigma_4 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2\end{aligned}$$

□

The above leads to the following result.

Theorem 4.2. If $\underline{z} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$, $\sigma_4 > \sigma_2 > 0$ then

$$\sigma_4 = \frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} + \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2},$$

$$\sigma_2 = \frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} - \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2},$$

$$\underline{b}_1 \wedge \underline{b}_2 = \frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} + \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right),$$

$$\underline{a}_1 \wedge \underline{a}_2 = \frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} - \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right)$$

Proof. 1) As we have already proved: $\sigma_2^2 + \sigma_4^2 = \|\underline{z}\|^2$, $\|\underline{z} \wedge \underline{z}\| = 2\sigma_2\sigma_4$. Therefore,

$$\frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} + \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2} = \sigma_4$$

since $\sigma_4 > \sigma_2 > 0$. Similarly,

$$\frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} - \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2} = \sigma_2$$

2) In Proposition 4.1 we saw that

$$\left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^* = \sigma_4 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$$

Thus, using the prime decomposition (3.2) and the fact that $\sigma_4 > \sigma_2 > 0$ again, we have that

$$\begin{aligned}&\frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} + \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right) = \\ &= \frac{\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 + \sigma_4 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2}{2\sigma_4 + 2\sigma_2} + \\ &+ \frac{\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 - \sigma_4 \cdot \underline{a}_1 \wedge \underline{a}_2 - \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2}{2\sigma_4 - 2\sigma_2} = \\ &= \frac{\underline{b}_1 \wedge \underline{b}_2}{2} + \frac{\underline{b}_1 \wedge \underline{b}_2}{2} = \underline{b}_1 \wedge \underline{b}_2\end{aligned}$$

Similarly, we obtain that

$$\frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} - \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right) = \underline{a}_1 \wedge \underline{a}_2$$

Remark 4.1. The formulae derived in the previous theorem give us the terms of the prime decomposition (3.2) without applying any SVD or spectral analysis of the skew-symmetric matrix of $\underline{z}$. Every term is a function of $\underline{z}$, which means that the calculations for the best approximation of $\underline{z}$, i.e. the minimum distance from the Grassmann variety, are easier and faster. □

5. CONCLUSIONS AND FUTURE WORK.

In this paper we have considered a key problem in the development of the "approximate" DAP, which is the computation of the distance of a point in a projective space from the Grassmann variety. This is equivalent to the problem of decomposing and approximating a multi-vector by a decomposable one. Equivalently, we found a vector's minimum distance from the set of all decomposable vectors and from the Grassmann variety $G_2(\mathbb{R}^5)$. This was achieved by the spectral analysis of the related skew- symmetric matrix of the multi-vector. The existence of the prime decomposition (3.2) also helped us examine the uniqueness of the decomposition and have provided alternative methods for defining the prime decomposition and calculating the minimum distance. This was achieved via the Grassmann matrix and the calculation of the terms of (3.2) as functions of the coordinates of the given point, thus avoiding the spectral analysis of the respective skew symmetric matrix or the singular value decomposition method which have been used up until now in this field of research. Finally, the current research was based on the special case of the Grassmann variety $G_2(\mathbb{R}^5)$. The interest of this case is due to the fact that closed form solutions may be derived. The extension to the $G_2(\mathbb{R}^n)$ case is the next challenge and it is an issue under investigation. Of special interest is the derivation of closed form solutions rather than the numerical approach that is an alternative way of looking at the general case. The current research provides the basics required for finding solutions to the central problem of the approximate DAP which is determining the distance of parameter dependent linear varieties from the Grassmann variety of the projective space.

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