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The Approximate Determinantal Assignment Problem

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Abstract

The Determinantal Assignment Problem (DAP) has been introduced as the unifying description of all frequency assignment problems in linear systems and it is studied in a projective space setting. This is a multi-linear nature problem and its solution is equivalent to finding real intersections between a linear space, associated with the polynomials to be assigned, and the Grassmann variety of the projective space. This paper introduces a new relaxed version of the problem where the computation of the approximate solution, referred to as the approximate DAP, is reduced to a distance problem between a point in the projective space from the Grassmann variety $G_m(\mathbb{R}^n)$. The cases $G_2(\mathbb{R}^n)$ and its Hodge-dual $G_{n-2}(\mathbb{R}^n)$ are examined and a closed form solution to the distance problem is given based on the skew-symmetric matrix description of multivectors via the gap metric. A new algorithm for the calculation of the approximate solution is given and stability radius results are used to investigate the acceptability of the resulting perturbed solutions.

Key words. Algebraic Control Theory, Frequency Assignment Problems, Exterior Algebra, Approximation.

1 Introduction

The Determinantal Assignment Problem (DAP) belongs to the family of algebraic synthesis methods and has emerged as the abstract problem formulation of pole, zero assignment of linear systems [Kar. 1]. This approach has unified the study of frequency assignment problems (pole, zero) of multivariable systems under

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constant, dynamic centralized, or decentralized control structure, [Kar. 2]. This problem is equivalent to finding solutions to an inherently non-linear problem which has a determinantal character demonstrating the significance of exterior algebra and classical algebraic geometry for this family of control problems.

The multilinear nature of DAP has suggested, [Kar. 1], an Exterior Algebra framework for its study. Specifically, in [Kar. 1] it has been proved that DAP may be reduced to a linear problem of zero assignment of polynomial combinants, described by the solutions of the linear system $P\underline{h} = \underline{a}$ where P is the corresponding Plücker matrix [Kar. 1] and a standard problem of multi-linear algebra expressed by the decomposability of the multivector $\underline{h}$, [Mar. 1]. The solution of the linear sub-problem, defines a linear space in a projective space whereas decomposability is characterized by the set of *Quadratic Plücker Relations* (QPR), which define the Grassmann variety of a related projective space [Hod. 1]. Thus, the solvability of DAP is reduced to a problem of finding real intersections between the linear variety and the Grassmann variety. This novel Exterior Algebra-Algebraic Geometry method, has provided, [Kar. 1], a number of invariants, such as the Plücker Matrices and Grassmann vectors, suitable for the characterization of rational vector spaces and the solvability of control problems, in both generic and non-generic cases, and it is flexible as far as handling dynamic schemes, as well as structurally constrained compensation schemes, [Gia. 1]. An additional advantage of this framework is that it provides a unifying computational framework for finding the solutions, when such solutions exist.

The above approach for the study of DAP in a projective, rather than an affine space setting, as in [Bro. 1], [Mart. 1], among others, provides a computational approach that relies on exterior algebra and on the explicit description of the Grassmann variety in terms of the QPR, which allows its formulation as a distance problem between varieties in the (real) projective space. This may transform the problem of exact intersection to a problem of "approximate intersection", i.e., small distance -via a suitable metric- between varieties, thus transform the exact DAP synthesis method to a DAP design methodology, where approximate solutions to the exact problem are sought. This enables the derivation of solutions, even for non-generic cases and handles problems of model uncertainty, as well as approximate solutions to the cases where generically there is no solution of the exact problem.

We propose the distance problem

$$\hat{H}_\epsilon := \{\underline{\hat{h}} \in G_q(\mathbb{R}^n) : \mathrm{dist}(\underline{\hat{h}}, \mathbb{K}) \leq \epsilon\} \tag{1.1}$$

where $\mathbb{K}$ is described by the solutions of a linear system and $G_q(\mathbb{R}^n)$ the Grassmann variety described by the set of QPR, as a relaxation of the exact intersection problem, which is referred to as the *approximate DAP*. This extension makes the investigation relevant to problems where there are no real intersections and thus approximate solutions are sought. Note that a solution to the approximate problem produces an approximate polynomial that will be assigned and this requires studying the stability properties of this perturbed polynomial, which are very important for the perturbed solutions to be acceptable.

In this paper, we consider the solution of problem (1.1) and the stability properties of the "approximate" polynomial $\hat{a}(s)$ that corresponds to an approximation $\underline{\hat{h}}$. Our approach views the problem as a minimization problem between a solution $\underline{z}(\underline{x})$ of a linear problem and the Grassmann variety $G_2(\mathbb{R}^n)$, i.e.,

$$\min_{\underline{x}} g(\underline{z}(\underline{x}), G_2(\mathbb{R}^n)) \tag{1.2}$$

where g is the gap, [Wey. 1], between the parameterized multivector $\underline{z}(\underline{x})$ in the projective space and the Grassmann variety with $\underline{x}$ being the vector of free parameters that describe the linear variety $\mathbb{K}$. It is shown that the solution of (1.2) is implied by the solution of $\min_{\underline{z}} g(\underline{z}, G_2(\mathbb{R}^n))$ when $\underline{z}$ is constant, i.e., least distance of a *fixed point* from the Grassmann variety.

The methodology used for the computation of the above minimization is based on the *best decomposable approximation of multivectors*, [Delav. 1], [Kar. 3], [Yok. 1], in the vicinity of a parametric multivector in the projective space. All results are given in closed-form formulae and further connection with similar optimization-approximation problems and techniques is provided. Simplifications of the results are also given in case of specific Grassmann varieties. Our suggestion can be used independently of generic or exact solvability conditions.

The paper is organized in the following way: In section 2, we present some preliminary definitions and results concerning DAP and how these imply the approximate DAP. In section 3 it is shown how the approximate DAP is formulated and solved for a constant and fixed point $\underline{z} \in \wedge^2(\mathbb{R}^n)$. Specifically, in 3.1 we use some primary results with regard to antisymmetric matrices and multi-vector decomposability in order to derive a specific formula for our problem, the so called prime decomposition of 2-vectors. In section 3.2, we apply the previous results to the related projective space and new upper-lower bounds are obtained for the gap between a point in the projective and the Grassmann variety. In section 3.3 the same problem is studied when $\underline{z} \in \wedge^{n-2}(\mathbb{R}^n)$. In Section 4, solutions are given for parameterized multivectors: In section 4.1, the approximate DAP is solved for $\underline{z}(\underline{x})$ and for special cases it is shown that the solution is reduced to the optimization of a 4th order polynomial constrained to the unit sphere. The stability of the "approximate" polynomial $\hat{a}(s)$, implied by $\underline{\hat{h}}$ is examined in section 4.2 via stability radius type results. Finally, an algorithm for direct calculations and applications on the above results is presented in section 4.3.

Throughout this paper the following notation is adopted: Scalars are denoted by lower-case letters, e.g., a, b, etc. Vectors and q-vectors (multivectors) are written as lower underline case letters, e.g., $\underline{x}, \underline{y}$, etc. All q-vectors in this article are considered elements of the set $\wedge^q(\mathbb{R}^n)$ or its Hodge-dual, [Mar. 1], $\wedge^{n-q}(\mathbb{R}^n)$ where $q \leq n$. The respective Hodge-star operator for an n-dimensional oriented vector space $\mathbb{V}$ will be denoted as $(\cdot)^*$. The *wedge* or *exterior* product between multivectors $\underline{a}, \underline{b}$ is denoted as $\underline{a} \wedge \underline{b}$. Matrices will be denoted in upper case letters, e.g., X, Y, etc. The right and the left kernel a matrix A, are denoted respectively by $\mathcal{N}_r(A)$, $\mathcal{N}_\ell(A)$. To denote the *compound matrix* of a matrix A, i.e., all the $q \times q$ minors of A we use the notation $C_q(A)$. A q-vector in $\wedge^q(\mathbb{R}^n)$ written as $\underline{a}_1 \wedge \underline{a}_2 \wedge \cdots \wedge \underline{a}_q$, with $\underline{a}_i$ in $\mathbb{R}^n$, $i = 1, ..., n$, will be called decomposable [Mar. 1], and equivalently $\underline{a}_1 \wedge \underline{a}_2 \wedge \cdots \wedge \underline{a}_q = C_q(A)$, $A := [\underline{a}_1, \underline{a}_2, ..., \underline{a}_q]$. The Grassmann variety of a real projective space $\mathbb{P}^{\binom{n}{q}-1}(\mathbb{R})$, i.e., the variety of the decomposable vectors of $\mathbb{P}^{\binom{n}{q}-1}(\mathbb{R})$, is denoted as $G_q(\mathbb{R}^n)$.

2 Background Results

The *constant Determinantal Assignment Problem* has been defined, [Kar. 1], as the problem of the derivation of a matrix $H \in \mathbb{H}$, $\mathbb{H} = \{H \in \mathbb{R}^{q \times n},\ q \leq n\}$ with rank$H = q$, such that

$$\det\left(H \cdot M(s)\right) = a(s) \tag{2.1}$$

where $M(s)$ is a given polynomial matrix in $\mathbb{R}^{n \times q}[s]$ with rank$\{M(s)\} = q$ and $a(s)$ is an arbitrary polynomial polynomial of an appropriate degree d. If H is a polynomial matrix, problem (2.1) is usually called *Dynamical DAP*. However, since all dynamics can be shifted from $H(s)$ to M(s), (2.1) is usually referred to as *the Determinantal Assignment Problem* (DAP) and its difficulty is mainly due to the multi-linear nature of the problem as this is described by its determinantal character.

Remark 2.1. *The degree of the polynomial $a(s)$ depends firstly on the order of $M(s)$, [For. 1], and secondly, on the structure of H. However, in most of our problems the degree of $a(s)$ is equal to the order of $M(s)$.*

Remark 2.2. *For an n-state, k-input, m-output linear system described by the $S(A, B, C, D)$ state space description, or the transfer function $G(s)$ represented by the coprime Matrix Fractional Description (MFD) $G(s) = D_L^{-1}(s)N_L(s) = D_R^{-1}(s)N_R(s)$, DAP takes the following forms:*

i) Pole Assignment by State Feedback. If $F \in \mathbb{R}^{n \times k}$ is a state feedback, then the closed-loop characteristic polynomial is

$$a_F(s) = \det(sI_n - A - BF) = \det(M(s)\tilde{F})$$

where $M(s) = [sI_n - A, -B]$, $\tilde{F} = [I_n, F^t]^t$.

ii) Pole assignment by constant output feedback. If $K \in \mathbb{R}^{m \times k}$ is an output feedback, then the closed-loop characteristic polynomial is

$$a_K(s) = \det(D_L(s) + N_L(s)K) = \det(D_R(s) + KN_R(s)) =$$

$$= \det(\tilde{K}_R T_R(s)) = \det(T_L(s)\tilde{K}_L)$$

where

$$T_L(s) = [D_L(s), N_L(s)],\ T_R(s) = \begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix},\ \tilde{K}_L = [I_m, K^t]^t,\ \tilde{K}_R = [I_k, K]$$

More forms, such as the design of an n-state observer and the zero assignment by squaring down may be found in [Kar. 1].

Algebraic techniques have provided both *exact solvability conditions*, [Lev.1], [Lev. 2] and *generic solvability conditions*, [Byr. 1], [Lev. 2], [Mart. 1], whereas geometric methodologies have used Schubert calculus/Enumerative Geometry, [Bro. 1] and projective geometry techniques, [Kar. 1]. The projective methodology in [Lev.1], [Lev. 2] has provided for the first time a systematic procedure for finding solutions to such nonlinear problems using a blow-up methodology, known as *Global Linearization* which has the significant advantage over the affine space approach [Byr. 1], [Mart. 1] that it does not just interpret DAP as an intersection problem between a linear and the Grassmann variety of a projective space, but also provides a computational framework that is suitable for computing solutions. From that aspect, eqn.(2.1) is reduced to:

i) *Linear subproblem.* Suppose that $\underline{h}$ is free. Find the conditions under which vectors $\underline{h} \in \mathbb{R}^{1\times\binom{n}{q}}$ exist such that

$$< \underline{h}, \underline{m}(s) >= a(s) \Leftrightarrow \underline{h}^t \cdot P = \underline{a}^t \tag{2.2}$$

where $P \in \mathbb{R}^{\binom{n}{q}\times(d+1)}$ is the Plücker matrix of the vector space col-span $\{M(s)\}$, i.e., the matrix whose i-th row is formed by the coefficients of the polynomials in the i-th coordinate of $\underline{m}(s)$ and d is the order of $M(s)$.

ii) *Multilinear subproblem.* Assume that the above linear subproblem is solvable and $\mathbb{K} \subseteq \mathbb{P}^{\binom{n}{q}-1}(\mathbb{R})$ be the family of the solution vectors $\underline{h}$ of (2.2). Then find whether there exists $\underline{h} \in \mathbb{K}$ such that $\underline{h}$ is decomposable. If such vector exists, determine the matrix $H \in \mathbb{R}^{q\times n}$ such that $C_q(H) = c\underline{h}^t,\ c \in \mathbb{R}^*$.

Where $\underline{h}^t := \underline{h}_1^t \wedge \cdots \wedge \underline{h}_q^t \in \mathbb{R}^{1\times\binom{n}{q}}$, $\underline{m}(s) := \underline{m}_1(s) \wedge \cdots \wedge \underline{m}_q(s) \in \mathbb{R}^{\binom{n}{q}\times 1}$ and

$$H = \begin{bmatrix} \underline{h}_1^t \\ \underline{h}_2^t \\ \vdots \\ \underline{h}_q^t \end{bmatrix},\ M(s) = [\underline{m}_1(s), \underline{m}_2(s), ..., \underline{m}_q(s)]$$

Therefore, if $\tilde{H}$ is the set of solutions of (2.1), then

$$\tilde{H} = \mathbb{K} \cap G_q(\mathbb{R}^n) \tag{2.3}$$

and $\tilde{H}$, may be approximated by the following set

$$\hat{H}_\epsilon := \{\hat{\underline{h}} \in G_q(\mathbb{R}^n) : \mathrm{dist}(\hat{\underline{h}}, \mathbb{K}) \leq \epsilon\} \tag{2.4}$$

where $\mathbb{K}$ is described by the solutions of the linear system (2.2) and $G_q(\mathbb{R}^n)$ by the set of QPR. Then

$$\tilde{H} = \bigcap_{\epsilon \geq 0} \hat{H}_\epsilon \tag{2.5}$$

Defining $\hat{H}_\epsilon$ represents a relaxation of the exact intersection problem, and it is referred to as *approximate DAP*.

3 Distances between a Constant Point and Decomposable Multivectors

In this section, we start with the study of the distance of a point from the Grassmann variety, where the point is considered as fixed. The results of this analysis will be very helpful when afterwards the point will be considered as being a member of the linear variety $\mathbb{K}$.

3.1 The Prime Decomposition of a 2-vector

Most optimization problems that concern Grassmann manifolds are resolved into multilinear SVD problems, [Delat. 1], [Delav. 1], [Kol. 1], [Sav. 1], [Yok. 1]. A first step is to decompose the multi-vector or the tensor into a sum of lower-rank tensors. In this section we prove a very useful formula for a 2-vector decomposition along with some new properties. We will need the exact matrix form of a 2-vector first.

Definition 3.1. *Let $\underline{z} \in \wedge^2(\mathbb{R}^n)$, $\underline{y}_1$, $\underline{y}_2 \in \mathbb{R}^n$ and $f(\underline{y}_1, \underline{y}_2) = <\underline{z}, \underline{y}_1 \wedge \underline{y}_2>$ be a bilinear form, from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ with respect to $\underline{y}_1$, $\underline{y}_2$. We define as $T_{\underline{z}} \in \mathbb{R}^{n\times n}$ the matrix representation of f, i.e.,*

$$< \underline{z}, \underline{y}_1 \wedge \underline{y}_2 >= \underline{y}_1^t \cdot T_{\underline{z}} \cdot \underline{y}_2 \tag{3.1}$$

The skew-symmetric matrix $T_{\underline{z}}$ now easily follows.

Lemma 3.1. *For every multivector $\underline{z} \in \wedge^2(\mathbb{R}^n)$, $T_{\underline{z}}$ is given by*

$$T_{\underline{z}} = \begin{pmatrix} 0 & z_{12} & z_{13} & \cdots & z_{1,n} \\ -z_{12} & 0 & z_{23} & \cdots & z_{2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -z_{1,n-1} & \cdots & -z_{n-2,n-1} & 0 & z_{n-1,n} \\ -z_{1,n} & -z_{2,n} & \cdots & -z_{n-1,n} & 0 \end{pmatrix} \tag{3.2}$$

Proof. If $\{\underline{e}_i \wedge \underline{e}_j\}_{1\le i<j\le n}$ an orthonormal basis of $\wedge^2(\mathbb{R}^n)$ then we have that $\underline{z} = \sum_{1\le i<j\le n} z_{ij}\underline{e}_i \wedge \underline{e}_j$, where z_{ij} are the components of $\underline{z}$ known as *Plücker coordinates*, [Hod. 1]. Therefore,

$$f(\underline{e}_i, \underline{e}_j) =< \underline{z}, \underline{e}_i \wedge \underline{e}_j >= z_{ij},\ f(\underline{e}_j, \underline{e}_i) =< \underline{z}, \underline{e}_j \wedge \underline{e}_i >= -z_{ij}$$

and $f(\underline{e}_i, \underline{e}_i) = 0$. Hence, $T_{\underline{z}}$ readily follows. □

Lemma 3.2. *[Gan. 1] Every skew-symmetric matrix has $k := [n/2]$ imaginary eigenvalues, $\pm i\sigma_1, ..., \pm i\sigma_k$ with $\sigma_k \ge \sigma_{k-1} \ge \cdots \ge \sigma_1 \ge 0$, corresponding to the complex eigenvectors $\underline{e}_{2k} \pm i\underline{e}_{2k-1}, ..., \underline{e}_2 \pm i\underline{e}_1$ when $n = 2k$ and $0, \pm i\sigma_1, ..., \pm i\sigma_k$, $\underline{e}_{2k+1}, \underline{e}_{2k} \pm i\underline{e}_{2k-1}, ..., \underline{e}_2 \pm i\underline{e}_1$ when n is odd, where $\{\underline{e}_j\}_{j=1}^{2k}$, $\{\underline{e}_j\}_{j=1}^{2k+1}$ are orthonormal bases for $\mathbb{R}^n$ when $n = 2k$, $n = 2k+1$, respectively.*

For the rest of the paper, σ_i will denote the imaginary part of the eigenvalues of $T_{\underline{z}}$ and $\underline{e}_i$ the corresponding vectors obtained by the eigenvectors of $T_{\underline{z}}$.

Theorem 3.1. *Let $\underline{z} \in \wedge^2(\mathbb{R}^n)$. Then $\underline{z}$ is expressed as:*

$$\underline{z} = \sigma_k \underline{x}_k + \sigma_{k-1}\underline{x}_{k-1} + ... + \sigma_1 \underline{x}_1 \tag{3.3}$$

where $\underline{x}_k := \underline{e}_{2k} \wedge \underline{e}_{2k-1}, ..., \underline{x}_1 := \underline{e}_2 \wedge \underline{e}_1$.

Proof. Following [Gan. 1], $T_{\underline{z}}$, is written as

$$\left[\underline{e}_{2k}, \underline{e}_{2k-1}, \ldots, \underline{e}_2, \underline{e}_1\right] \cdot \text{block-diag}\left\{\begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}\right\}_{i=1}^{k} \cdot \begin{bmatrix} \underline{e}_{2k}^t \\ \underline{e}_{2k-1}^t \\ \vdots \\ \underline{e}_2^t \\ \underline{e}_1^t \end{bmatrix}$$

if $n = 2k$, or

$$\left[\underline{e}_{2k+1}, \underline{e}_{2k}, \ldots, \underline{e}_2, \underline{e}_1\right] \cdot \text{block-diag}\left\{0, \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}, ..., \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix}\right\} \cdot \begin{bmatrix} \underline{e}_{2k+1}^t \\ \underline{e}_{2k}^t \\ \vdots \\ \underline{e}_2^t \\ \underline{e}_1^t \end{bmatrix}$$

if $n = 2k+1$. When $n = 2k$, we obtain:

$$\begin{aligned} T_{\underline{z}} &= \sigma_k(\underline{e}_{2k}\underline{e}_{2k-1}^t - \underline{e}_{2k-1}\underline{e}_{2k}^t) + \cdots + \sigma_1(\underline{e}_2\underline{e}_1^t - \underline{e}_1\underline{e}_2^t) = \\ &= \sigma_k T_{\underline{e}_{2k}\wedge\underline{e}_{2k-1}} + \cdots + \sigma_1 T_{\underline{e}_2\wedge\underline{e}_1} \end{aligned}$$

Hence, $\underline{z} = \sigma_k\underline{x}_k + \sigma_{k-1}\underline{x}_{k-1} + ... + \sigma_1\underline{x}_1$. The proof is similar for the extra zero eigenvalue, when $n = 2k+1$. □

The decomposition defined by (3.3) will be referred as *the prime decomposition* of $\underline{z}$.

Corollary 3.1. *(Properties of the Prime Decomposition) Let* $\underline{z} \in \wedge^2(\mathbb{R}^n)$ *and let us denote*

$$\underbrace{\underline{z} \wedge \underline{z} \wedge \cdots \wedge \underline{z}}_{k-times} \equiv \underline{z}^{\wedge k} \tag{3.4}$$

1) If $\sigma_1, \sigma_2, ..., \sigma_k$ *are the imaginary parts of the eigenvalues of* $T_{\underline{z}}$*, then*

$$\underline{z} = \sum_{i=1}^{k} \sigma_i \underline{x}_i, \ \underline{z} \wedge \underline{z} = 2! \sum_{j>i} \sigma_i \sigma_j \underline{x}_i \wedge \underline{x}_j, ..., \tag{3.5}$$

$$\underline{z}^{\wedge \mu} = \mu! \sum_{1 \leq i_1 < ... < i_\mu \leq k} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_\mu} \underline{x}_{i_1} \wedge \underline{x}_{i_2} \wedge ... \wedge \underline{x}_{i_\mu}, \ 2 \leq \mu \leq k, ..., \tag{3.6}$$

$$\underline{z}^{\wedge k} = k! \sigma_1 \sigma_2 \cdots \sigma_k \underline{x}_1 \wedge \underline{x}_2 \wedge ... \wedge \underline{x}_k \tag{3.7}$$

2) The characteristic polynomial of $T_{\underline{z}}$ *is given by*

$$\varphi(\lambda) = \begin{cases} \lambda^n + \|\underline{z}\|^2 \lambda^{n-2} + \dfrac{\|\underline{z} \wedge \underline{z}\|^2}{(2!)^2} \lambda^{n-4} + \cdots \dfrac{\|\underline{z}^{\wedge k}\|^2}{(k!)^2}, & n = 2k \\ \lambda^n + \|\underline{z}\|^2 \lambda^{n-2} + \dfrac{\|\underline{z} \wedge \underline{z}\|^2}{(2!)^2} \lambda^{n-4} + \cdots \dfrac{\|\underline{z}^{\wedge k}\|^2}{(k!)^2} \lambda, & n = 2k+1 \end{cases} \tag{3.8}$$

3) The best decomposable approximation of $\underline{z}$ *is given by* $\hat{\underline{z}} = \sigma_k \underline{x}_k$ *with*

$$\|\underline{z} - \hat{\underline{z}}\| = \sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2} \tag{3.9}$$

4) $\underline{z}$ *is decomposable if and only if* $\underline{z} \wedge \underline{z} = \underline{0} \in \wedge^4(\mathbb{R}^n)$.

Proof. 1) For k=2 and with the use of the prime decomposition we have that

$$\begin{aligned} \underline{z} \wedge \underline{z} = & \left(\sum_{i=1}^{k} \sigma_i \underline{x}_i \right) \wedge \left(\sum_{i=1}^{k} \sigma_i \underline{x}_i \right) = \\ & = \sigma_k \sigma_{k-1} \underline{x}_k \wedge \underline{x}_{k-1} + ... + \sigma_k \sigma_1 \underline{x}_k \wedge \underline{x}_1 + \\ & + \sigma_{k-1} \sigma_k \underline{x}_{k-1} \wedge \underline{x}_k + ... + \sigma_{k-1} \sigma_1 \underline{x}_{k-1} \wedge \underline{x}_1 + ... \\ & + \sigma_2 \sigma_k \underline{x}_2 \wedge \underline{x}_k + ... + \sigma_2 \sigma_1 \underline{x}_2 \wedge \underline{x}_1 + \\ & + \sigma_1 \sigma_k \underline{x}_1 \wedge \underline{x}_k + ... + \sigma_1 \sigma_2 \underline{x}_1 \wedge \underline{x}_2 = \\ & = 2! \sum_{j>i} \sigma_i \sigma_j \underline{x}_i \wedge \underline{x}_j \end{aligned}$$

Suppose that (3.6) holds true. Then

$$\underline{z}^{\wedge(\mu+1)} = \underline{z}^{\wedge\mu} \wedge \underline{z} = \left(\mu! \sum_{1\leq i_1<\ldots<i_\mu\leq k} \sigma_{i_1}\cdots\sigma_{i_\mu}\underline{x}_{i_1}\wedge\ldots\wedge\underline{x}_{i_\mu}\right)\wedge\sum_{i=1}^{k}\sigma_i\underline{x}_i =$$
$$= (\mu+1)\mu! \sum_{1\leq i_1<\ldots<i_{\mu+1}\leq k} \sigma_{i_1}\cdots\sigma_{i_{\mu+1}}\underline{x}_{i_1}\wedge\ldots\wedge\underline{x}_{i_{\mu+1}}$$

which proves the result.

2) If $n = 2k$, from the spectral decomposition of $T_{\underline{z}}$ we have that

$$\varphi(\lambda) = (\lambda^2+\sigma_1^2)(\lambda^2+\sigma_2^2)\cdots(\lambda^2+\sigma_k^2) =$$
$$= \lambda^n + (\sigma_1^2+\cdots+\sigma_k^2)\lambda^{n-1}+\cdots+(\sigma_1^2\cdots\sigma_k^2)$$

Similarly, if n is odd:

$$\varphi(\lambda) = (\lambda^2+\sigma_1^2)(\lambda^2+\sigma_2^2)\cdots(\lambda^2+\sigma_k^2)\lambda =$$
$$= \lambda^n + (\sigma_1^2+\cdots+\sigma_k^2)\lambda^{n-1}+\cdots+(\sigma_1^2\cdots\sigma_k^2)\lambda$$

The result now follows, due to the equations (3.5)-(3.7).

3) Let $D_{\wedge^2(\mathbb{R}^n)}$ denote the subsets of decomposable vectors in $\wedge^2(\mathbb{R}^n)$ and $\underline{x} \in D_{\wedge^2(\mathbb{R}^n)}$. Then,

$$\min_{\underline{x}\in D_{\wedge^2(\mathbb{R}^n)}} \|\underline{x}-\underline{z}\|^2 = \min_{\underline{x}\in D_{\wedge^2(\mathbb{R}^n)}} \left\{\|\underline{z}\|^2+\|\underline{x}\|^2 - 2 < \underline{z},\underline{x} >\right\} \tag{3.10}$$

Thus (3.10) is minimized at $\underline{x}_1 =< \underline{z},\underline{x} > \underline{x}/\|\underline{x}\|^2$ for some decomposable vector $\underline{x}$. Hence,

$$\min_{\underline{x}\in D_{\wedge^2(\mathbb{R}^n)}} \|\underline{x}-\underline{z}\|^2 = \min_{\underline{x}\in D_{\wedge^2(\mathbb{R}^n)},\|\underline{x}\|=1} \left\{\|\underline{z}\|^2 - < \underline{z},\underline{x} >^2\right\} =$$
$$= \|\underline{z}\|^2 - \max_{\underline{x}\in D_{\wedge^2(\mathbb{R}^n)},\|\underline{x}\|=1} < \underline{z},\underline{x} >^2$$

We are therefore aiming to maximize $< \underline{z},\underline{x} >$ when $\underline{x} = \underline{y}_1 \wedge \underline{y}_2$ where $\underline{y}_1$, $\underline{y}_2$ are orthonormal. Thus,

$$\max_{\underline{y}_1,\underline{y}_2} | < \underline{z},\underline{y}_1\wedge\underline{y}_2 > | = \max_{\underline{y}_1,\underline{y}_2} < \underline{y}_1^t T_{\underline{z}},\underline{y}_2 >= \max_{\|\underline{y}_1\|=1} \left\langle \underline{y}_1^t T_{\underline{z}}, \frac{\underline{y}_1^t T_{\underline{z}}}{\|\underline{y}_1^t T_{\underline{z}}\|}\right\rangle =$$
$$= \max_{\|\underline{y}_1\|=1} \|\underline{y}_1^t T_{\underline{z}}\| = \sigma_k$$

Thus, (3.10) implies

$$\|\underline{x}-\underline{z}\|^2 \geq \sigma_k^2+\sigma_{k-1}^2+\cdots+\sigma_1^2+\|\underline{x}\|^2 - 2\sigma_k\cdot\|\underline{x}\| \geq$$

$$\geq \sigma_{k-1}^2 + \sigma_{k-2}^2 + \cdots + \sigma_1^2 + (\sigma_k - \|\underline{x})\|)^2 \geq$$
$$\geq \sigma_{k-1}^2 + \sigma_{k-2}^2 + \cdots + \sigma_1^2$$

Hence, $\underline{\hat{z}} = \sigma_k \underline{x}_k$ realizes the least distance from all decomposable multi-vectors, which is given by eqn.(3.9).

4) ($\Rightarrow$) If $\underline{z}$ is decomposable then $\underline{z} = \underline{a} \wedge \underline{b}$, $\underline{a}$, $\underline{b} \in \mathbb{R}^n$. Hence, $\underline{z} \wedge \underline{z} = \underline{a} \wedge \underline{b} \wedge \underline{a} \wedge \underline{b} = \underline{0}$.

($\Leftarrow$) If $\underline{z} \wedge \underline{z} = 0$, then from the prime decomposition we have that $\sigma_i \sigma_j = 0$, for all $(i, j), j > i$ pairs. This means that $k-1$ the number σ_i have to be zero. Due to $0 \leq \sigma_1 \leq \sigma_2 \leq ... \leq \sigma_k$, we have that $\sigma_1, ..., \sigma_{k-1} = 0$. Therefore, $\underline{z} = \sigma_k \cdot \underline{x}_k$.

□

Remark 3.1. *A similar problem to Corollary 3.1-(3) is addressed at [Eck. 1] with the use of the SVD for lower matrix-rank approximations. Also, a different proof of Corollary 3.1-(4) may be found in [Hit. 1]. Clearly, eqn.*(3.3) *implies significant simplifications to both of these problems.*

Furthermore, Corollary 3.1-(3) implies the derivation of the QPR in the $\wedge^2(\mathbb{R}^n)$ case in a rather straight-forward way, contrary to the classical formulae [Hod. 1]. We may illustrate this by the following example.

Example 3.1. *We examine the non-trivial case* $n = 5$. *Then*

$$\underline{z} = z_1 e_1 \wedge e_2 + z_2 e_1 \wedge e_3 + z_3 e_1 \wedge e_4 + z_4 e_1 \wedge e_5 + z_5 e_2 \wedge e_3 + \\ + z_6 e_2 \wedge e_4 + z_7 e_2 \wedge e_5 + z_8 e_3 \wedge e_4 + z_9 e_3 \wedge e_5 + z_{10} e_4 \wedge e_5$$

Hence,

$$\underline{z} \wedge \underline{z} = \underline{0} \Leftrightarrow (z_1 z_8 - z_2 z_6 + z_3 z_5) e_1 \wedge e_2 \wedge e_3 \wedge e_4 + (z_1 z_9 - z_2 z_7 + z_4 z_5) e_1 \wedge \\ \wedge e_2 \wedge e_3 \wedge e_5 + (z_1 z_{10} - z_3 z_7 + z_4 z_6) e_1 \wedge e_2 \wedge e_4 \wedge e_5 + (z_2 z_{10} - z_3 z_9 + \\ + z_4 z_8) e_1 \wedge e_3 \wedge e_4 \wedge e_5 + (z_5 z_{10} - z_6 z_9 + z_7 z_8) e_2 \wedge e_3 \wedge e_4 \wedge e_5 = \underline{0}$$

The second order homogeneous equations

$$z_1 z_8 - z_2 z_6 + z_3 z_5 = 0, z_1 z_9 - z_2 z_7 + z_4 z_5 = 0, z_1 z_{10} - z_3 z_7 + z_4 z_6 = 0, \\ z_2 z_{10} - z_3 z_9 + z_4 z_8 = 0, z_5 z_{10} - z_6 z_9 + z_7 z_8 = 0$$

are the Quadratic Plucker Relations (QPR) that define the Grassmann variety of the projective space $P^9(\mathbb{R})$.

3.2 Optimization in the Projective Space

If g denotes the gap between a point and the Grassmann variety in the projective space $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$ and gap$(\cdot,\cdot)$ the gap metric between two points in $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$, we will calculate $g\left(\underline{z}, G_2(\mathbb{R}^n)\right)$, i.e., the best decomposable approximation representative $\hat{\underline{z}}$ of a given 2-vector representative $\underline{z}$ for the respective equivalence classes and proceed to further simplifications of the result for specific Grassmann varieties. In the following, for notational simplicity, we identify $\underline{z} \equiv \text{span}\{\underline{z}\}$. The unitary properties of distances in a projective space, [Wey. 1], imply the following definition.

Definition 3.2. *The gap metric* gap$(\underline{z}, \underline{x})$ *between two multi-vectors* $\underline{z}, \underline{x}$ *is given by*

$$\text{gap}(\underline{z}, \underline{x}) = |\sin(\underline{z}\hat{,}\underline{x})| = \min_{\lambda} \left\| \frac{\underline{z}}{\|\underline{z}\|} - \frac{\underline{x}}{\|\underline{x}\|} \cdot \lambda \right\| \tag{3.11}$$

Theorem 3.2. *The gap g between $\underline{z}$ and $G_2(\mathbb{R}^n)$ is equal to*

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} \tag{3.12}$$

Proof. We have that

$$\begin{aligned} g\left(\underline{z}, G_2(\mathbb{R}^n)\right) :&= \min_{\underline{x} \in G_2(\mathbb{R}^n)} \text{gap}(\underline{z}, \underline{x}) = \min_{\underline{x} \in G_2(\mathbb{R}^n)} \sqrt{1 - \frac{<\underline{z}, \underline{x}>^2}{\|\underline{x}\|^2 \|\underline{z}\|^2}} = \\ &= \left(1 - \frac{1}{\|\underline{z}\|^2} \max_{\underline{x} \in G_2(\mathbb{R}^n)} \frac{<\underline{z}, \underline{x}>^2}{\|\underline{x}\|^2}\right)^{\frac{1}{2}} = \sqrt{1 - \frac{\sigma_k^2}{\|\underline{z}\|^2}} = \\ &= \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} \end{aligned}$$

due to Corollary 3.1-(3). □

Theorem 3.3. *The maximum possible gap of a multi-vector $\underline{z} \in \wedge^2(\mathbb{R}^n)$ from $G_2(\mathbb{R}^n)$ is equal to*

$$\max_{\underline{z}} g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \sqrt{1 - \frac{1}{k}} \tag{3.13}$$

Proof. Using the result of Theorem 3.2 we have that

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\sqrt{\sigma_k^2 + \sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}} = \frac{1}{\sqrt{1 + \frac{\sigma_k^2}{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}} \leq$$

$$\leq \frac{1}{\sqrt{1+\frac{\sigma_k^2}{\sigma_k^2+\sigma_k^2+\sigma_k^2+\ldots+\sigma_k^2}}} = \frac{1}{\sqrt{1+\frac{1}{k-1}}} = \sqrt{1-\frac{1}{k}}$$

□

Remark 3.2. *Clearly, the minimization of g and consequently the multilinear subproblem of DAP are equivalent to the maximization of the largest eigenvalue of* $T_{\underline{z}}$*, since* $g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \sqrt{1-\sigma_{\max}^2/\|\underline{z}\|^2}$*. Note, that eigenvalue optimization problems of this form are usually addressed algorithmically, e,g., [Lew. 1]. Hence, Theorem 3.2 implies a closed-form solution for the optimization of the largest eigenvalue of a skew-symmetric matrix as well.*

Now, formula (3.12) can be further simplified if the set of QPR that describe the respective Grassmann varieties is given. Next theorem concerns this case for $n = 5$.

Theorem 3.4. *The minimum gap between a 2-vector* $\underline{z} \in \wedge^2(\mathbb{R}^5)$ *and* $G_2(\mathbb{R}^5)$ *is given by*

$$g_\wedge(\underline{z}, G_2(\mathbb{R}^5)) = \frac{\sqrt{\sum QPR^2(\underline{z})}}{\sum_{i=1}^{10} z_i^2} \tag{3.14}$$

Proof. The minimum gap is $g(\underline{z}, \hat{\underline{z}}) = \sigma_1/\|\underline{z}\| \equiv g(\underline{z})$, with $\hat{\underline{z}} = \sigma_2 \underline{e}_1 \wedge \underline{e}_2,\ \sigma_2 \geq \sigma_1$. From the prime decomposition of $\underline{z}$ we have that $\|\underline{z} \wedge \underline{z}\| = \|2\sigma_1\sigma_2 \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \wedge \underline{e}_4\|$. Hence, if

$$g_\wedge := \frac{\|\underline{z} \wedge \underline{z}\|}{\|\underline{z}\|^2} \tag{3.15}$$

we have that

$$g_\wedge = \frac{2\sigma_1\sigma_2}{\|\underline{z}\|^2} = \frac{2\sqrt{\|\underline{z}\|^2 - \sigma_1^2}\sigma_1}{\|\underline{z}\|^2} = 2\sqrt{1-\left(\frac{\sigma_1}{\|\underline{z}\|}\right)^2}\frac{\sigma_1}{\|\underline{z}\|} =$$
$$= 2\sqrt{1-g^2(\underline{z})}g(\underline{z}) = 2\sqrt{g^2(\underline{z})(1-g^2(\underline{z}))}$$

Since the function $f(x) = \sqrt{x^2(1-x^2)}$ is increasing at $[0, \sqrt{2}/2]$ we have that the minimization of g is equivalent to the minimization of $g_\wedge$. In other words, the norm of the QPR over the norm of $\underline{z}$ can be used for minimization in $\wedge^2(\mathbb{R}^5)$, i.e.,

$$g_\wedge(\underline{z}, G_2(\mathbb{R}^5)) = \frac{\|\underline{z} \wedge \underline{z}\|}{\|\underline{z}\|^2} = \frac{\sqrt{\sum QPR^2(\underline{z})}}{\sum_{i=1}^{10} z_i^2} \tag{3.16}$$

□

Theorem 3.4 implies the first important result; if a specific Grassmann variety is given, all distance computations are expressed in terms of $\underline{z}$, which is a remarkable simplification for Grassman optimization problems.

3.3 Approximation in $\wedge^{n-2}(\mathbb{R}^n)$

The results of the previous section may now be extended to the dual case. In fact, the previous results imply calculations over $G_{n-2}(\mathbb{R}^n)$ and rely on the Hodge $\star$-operator, [Mar. 1]. This operator was firstly defined in order to generalize the notion of the Laplacian on Riemannian manifolds.

Definition 3.3. *The Hodge $\star$-operator, for an oriented n-dimensional vector space V equipped with an inner product $< \cdot, \cdot >$, is an operator defined as*

$$* : \wedge^m(V) \to \wedge^{n-m}(V) \tag{3.17}$$

such that

$$\underline{a} \wedge (\underline{b}^*) =< \underline{a}, \underline{b} > \underline{w} \tag{3.18}$$

where $\underline{a}, \underline{b} \in \wedge^m(V)$, $\underline{w} \in \wedge^n(V)$ defines the orientation on V and $n > m$.

We first consider some background results, [Mar. 1].

Lemma 3.3. *The Hodge $\star$-operator is linear, one to one, onto and an isometry.*

Lemma 3.4. *Let $D_{\wedge^2(\mathbb{R}^n)}$, $D_{\wedge^{n-2}(\mathbb{R}^n)}$ denote the subsets of decomposable vectors in $\wedge^2(\mathbb{R}^n)$, $\wedge^{n-2}(\mathbb{R}^n)$ respectively. Then $\star : D_{\wedge^2(\mathbb{R}^n)} \to D_{\wedge^{n-2}(\mathbb{R}^n)}$ is also one to one and onto.*

Next we use conjugacy in general vector spaces to study the minimization problem for $G_{n-2}(\mathbb{R}^n)$.

Proposition 3.1. *Let $\mathbb{U}$, $\mathbb{V}$ be two finite dimensional vector spaces and $T : \mathbb{U} \to \mathbb{V}$ a linear, "1-1", onto isometry. If $\mathbb{U}_1$, $\mathbb{V}_1$ are two isometric subsets of $\mathbb{U}$, $\mathbb{V}$ respectively through T and*

$$f(\underline{u}) := \arg \min_{\underline{u}_1 \in \mathbb{U}_1} \|\underline{u} - \underline{u}_1\|, \; g(\underline{u}) := \arg \min_{\underline{v}_1 \in \mathbb{V}_1} \|\underline{u} - \underline{v}_1\| \tag{3.19}$$

then

$$f(\underline{u}) = T^{-1}(g(T(\underline{u}))) \tag{3.20}$$

Proof. Following the standard properties of operators and duality in Banach spaces, e.g., [Roc. 1], we have that

$$f(\underline{u}) = \arg \min_{\underline{u}_1 \in \mathbb{U}_1} \|T(\underline{u}) - T(\underline{u}_1)\| = T^{-1} \left(\arg \min_{T(\underline{u}_1) \in \mathbb{V}_1} \|T(\underline{u}) - T(\underline{u}_1)\| \right) =$$
$$= T^{-1}(\arg \min_{\underline{v}_1 \in \mathbb{V}_1} \|T(\underline{u}) - \underline{v}_1\|) = T^{-1}(g(T(\underline{u})))$$

□

The above result may be described by the commutative diagram:

$$\begin{array}{ccc} & T & \\ \mathbb{U} & \longrightarrow & \mathbb{V} \\ \downarrow f & & g\downarrow \\ \mathbb{U}_1 & \longrightarrow & \mathbb{V}_1 \\ & T & \end{array}$$

In our case, $\mathbb{U},\ \mathbb{V}$ are represented by $\wedge^2(\mathbb{R}^n),\ \wedge^{n-2}(\mathbb{R}^n)$ and $\mathbb{U}_1,\ \mathbb{V}_1$ by $D_{\wedge^2(\mathbb{R}^n)}$ and $D_{\wedge^{n-2}(\mathbb{R}^n)}$, respectively. Then the diagram above, shows how the minima for $D_{\wedge^{n-2}(\mathbb{R}^n)}$ and $D_{\wedge^2(\mathbb{R}^n)}$ in Proposition 3.1 may be derived from each other. Hence, if $T \equiv \star$ the following result is established.

Corollary 3.2. *For every* $\underline{z} \in \wedge^{n-2}(\mathbb{R}^n)$ *the following equality holds:*

$$\min_{\underline{a}_1,\dots,\underline{a}_{n-2}\in\mathbb{R}^n} \|\underline{z} - \underline{a}_1 \wedge \dots \wedge \underline{a}_{n-2}\| = \min_{\underline{b}_1,\underline{b}_2\in\mathbb{R}^n} \|\underline{z}^* - \underline{b}_1 \wedge \underline{b}_2\| \tag{3.21}$$

The above may be illustrated by the following example:

Example 3.2. *Let* $\underline{z} = (6, 1, 7, -3, -11, 0, -5, 1, 8, 2)^t \in \wedge^3(\mathbb{R}^5) \simeq \mathbb{R}^{10}$. *Then* $\underline{z}^* = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6)^t \in \wedge^2(\mathbb{R}^5) \simeq \mathbb{R}^{10}$. *Hence,*

$$T_{\underline{z}^*} = \begin{pmatrix} 0 & 2 & -8 & 1 & 5 \\ -2 & 0 & 0 & 11 & -3 \\ 8 & 0 & 0 & 7 & -1 \\ -1 & -11 & -7 & 0 & 6 \\ -5 & 3 & 1 & -6 & 0 \end{pmatrix}$$

From the canonical form of $T_{\underline{z}^*}$*, we have*

$\underline{r} = (-0.0785093, 0.094211, 0.549565, 0.204124, 0.800795)^t$,
$\underline{b}_1 = (0.395816, -0.0925693, -0.0410099, 0.900056, -0.151586)^t$,
$\underline{b}_2 = (-0.0182749, -0.613107, -0.616617, 0, 0.493507)^t$,
$\underline{a}_1 = (0.236471, 0.760887, -0.527125, 0, 0.29542)^t$,
$\underline{a}_2 = (-0.883693, 0.166455, -0.195497, 0.38501, -0.0701948)^t$
and $\sigma_2 = 8.16558, \sigma_4 = 15.5988$. *Therefore,*

$$\begin{aligned}\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 = \ & (5.81187, -4.18115, 0.743424, 1.99617, -0.49817, \\ & 2.3921, -0.83766, -1.65719, 0.77373, -0.928749)^t\end{aligned}$$

and

$$\begin{aligned}\sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 = \ & (-3.81187, -3.81885, 0.256576, 3.00383, 0.49817, \\ & 8.6079, -2.16234, 8.65719, -1.77373, 6.92875)^t\end{aligned}$$

Hence,

$$\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6)^t = \underline{z}^*$$

and $\underline{x} = \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$ *is the best decomposable approximation of* $\underline{z}^*$*. Thus*

$$\begin{aligned}(\sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2)^* = \sigma_4 \cdot \underline{a}_1 \wedge \underline{a}_2 \wedge \underline{r} = (&6.92874, 1.77373, 8.65719, -2.16234, \\ &-8.60789, 0.498171, -3.00383, 0.256576, 3.81885, -3.81187)^t\end{aligned}$$

is the best decomposable approximation of $\underline{z}$*.*

4 Solutions on the Approximate DAP

In this section, $\underline{z}$ is obtained as a function $\underline{z}(\underline{x})$ of the degrees of freedom $\underline{x}$ that describe the linear variety $\mathbb{K}$ of the linear system (2.2). Note that such linear varieties are functions of the coefficients of the polynomial that is to be assigned. The optimization problem (1.2) is solved and the solution leads to a perturbed solution of the exact assignment problem associated with a new assignable polynomial. A stability criterion is given to characterize the approximate polynomial that corresponds to $\hat{\underline{z}}$.

4.1 Least Distance between $\mathbb{K}$ and $G_2(\mathbb{R}^n)$

Here, the generalization of $\underline{z}$ into $\underline{z}(\underline{x})$ is considered and then used in order to calculate the gap between the linear variety $\mathbb{K}$ and the Grassmann variety $G_2(\mathbb{R}^n)$.

Proposition 4.1. *Let* $\underline{x} = (x_1, ..., x_d)$ *be the vector of the d- free parameters that define the linear variety* $\mathbb{K}$*,* $d \in \mathbb{N}$*. Then the least distance between* $\mathbb{K}$ *and* $G_2(\mathbb{R}^n)$ *is given by*

$$\min_{\underline{x}} \sqrt{\sigma_{k-1}^2(\underline{x}) + \sigma_{k-2}^2(\underline{x}) + ... + \sigma_1^2(\underline{x})} \quad \text{subject to} \sum_{i=1}^{k} \sigma_i^2(\underline{x}) = 1 \tag{4.1}$$

where $k := [n/2]$*,* $\sigma_i(\underline{x})$ *are the real parts of the* $i-th$ *eigenvalue of* $T_{\underline{z}}$ *and* $\underline{z}$ *is the parametric form of the linear variety* $\mathbb{K}$*.*

Proof. Let the determinantal assignment problem $\underline{h}^t P = \underline{a}^t$. Since the poles of the system remain the same under scalar multiplication, we are interested for the general solution of $\underline{h}^t P = \lambda \underline{a}^t,\ \lambda \in \mathbb{R}$. Therefore, if $\underline{h}_0^t$ is a particular solution of DAP then

$$\underline{h}^t = \lambda \underline{h}_0^t + \underline{\kappa}^t V = [\lambda, \underline{\kappa}^t] \begin{bmatrix} \underline{h}_0^t \\ V \end{bmatrix} \tag{4.2}$$

where $V \in \mathbb{R}^{(r-1)\times\binom{n}{2}}$ is the matrix representation of $\mathcal{N}_\ell(P)$ and $\underline{\kappa}^t \in \mathbb{R}^{r-1}$ for $r = \dim\left(\text{row-span}\left\{\begin{bmatrix} \underline{h}_0^t \\ V \end{bmatrix}\right\}\right)$. Hence, if $(\underline{v}_1, \underline{v}_2, ..., \underline{v}_r)$ is an orthonormal basis of the row-span$\left\{\begin{bmatrix} \underline{h}_0^t \\ V \end{bmatrix}\right\}$, then

$$\underline{z} = x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_r\underline{v}_r \equiv \underline{z}(\underline{x})$$

Thus, $\|\underline{z}\| = \|\underline{x}\| = 1$ and the result follows from Theorem 3.2. □

Similarly to Theorem 3.2, eqn.(4.1) can be further simplified if specific Grassmann varieties are given. The following theorem is one of the main results of this article, which characterizes the way the distance between the linear variety K and specific Grassmann varieties may be computed.

Theorem 4.1. *The least distance problem between the linear variety $\mathbb{K}$ and $G_2(\mathbb{R}^5)$ is equivalent to the minimization of a 4th order homogeneous polynomial $F(\underline{x})$, constrained to the unit sphere.*

Proof. We saw that the case $n = 5$ has implied that the minimization of $g_\wedge = \|\underline{z}\wedge\underline{z}\|/\|\underline{z}\|^2$, which may be used instead of the minimization of the gap g. Hence, problem (4.1) can be transformed into

$$\min_{\underline{z}} \|\underline{z} \wedge \underline{z}\|^2 \quad \text{subject to } \|\underline{z}\| = 1 \tag{4.3}$$

Therefore, due to eqn.(4.1) we obtain

$$\begin{aligned} \|\underline{z} \wedge \underline{z}\|^2 &= \left\| \left(\sum_{i=1}^{r} x_i\underline{v}_i\right) \wedge \left(\sum_{i=1}^{r} x_i\underline{v}_i\right) \right\|^2 = \left\| \sum_{i,j=1}^{r} x_i x_j \underline{v}_i \wedge \underline{v}_j \right\|^2 = \\ &= \left\langle \sum_{i,j=1}^{r} x_i x_j \underline{v}_i \wedge \underline{v}_j, \sum_{\varrho,\mu=1}^{r} x_\varrho x_\mu \underline{v}_\varrho \wedge \underline{v}_\mu \right\rangle = \\ &= \sum_{1\le i,j,\varrho,\mu\le r} x_i x_j x_\varrho x_\mu \langle \underline{v}_i \wedge \underline{v}_j, \underline{v}_\varrho \wedge \underline{v}_\mu \rangle \end{aligned} \tag{4.4}$$

Eqn.(4.4) is a 4th order homogeneous polynomial $F(\underline{x})$ in terms of $x_i,\ x_j,\ x_\varrho,\ x_\mu$. Thus (4.1) is written as

$$\min_{\underline{x}} F(\underline{x}) \quad \text{subject to } \|\underline{x}\| = 1 \tag{4.5}$$

□

If $\underline{z} = \underline{z}(\underline{x})$ implied by the solution of (4.1) or (4.5) is decomposable, then DAP is solved precisely. If $\underline{z}$ is not decomposable, its prime decomposition is implemented in order to obtain the best decomposable approximation $\hat{\underline{z}}$.

4.2 Stability Criteria for the Approximate DAP

In this section we assume that the solution $\underline{z}(\underline{x})$ is not a decomposable multi-vector, i.e., the gap between $\underline{z}(\underline{x})$ and the Grassmann variety in not zero. Then a perturbed polynomial $\hat{a}(s)$ is derived such that $\underline{\hat{z}}^t P = \underline{\hat{a}}^t$, with respect to the initial problem. We examine the stability properties of the resulting polynomial $\hat{a}(s)$ and its distance from the nominal polynomial $a(s)$ in eqn.(2.1), which we intended to assign. For this study we are using the results on the stability radius [Hin. 1].

Definition 4.1. *If*

$$a(\underline{\alpha}, s) = s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0,\ \underline{\alpha} = (1, a_{n-1}, ..., a_o) \tag{4.6}$$

is a Hurwitz polynomial written as, $a(\underline{\alpha}, s) = a_1(-s^2) + s a_2(-s^2)$ *where* $a_j(-s^2),\ j = 1, 2$, *are real polynomials in* $-s^2$, *then the stability radius* $r_{\underline{a}}$ *is given by*

$$r_{\underline{a}} = \min\left\{a_0, \left(\max_{\omega^2 \in \mathbb{R}^+} f(\omega^2)\right)^{-1/2}\right\} \tag{4.7}$$

where

$$f(\omega^2) = \frac{1 + \omega^4 + \cdots + \omega^{2n-4}}{a_1^2(\omega^2) + a_2^2(\omega^2)}, \quad \text{if}\, n = 2k$$

or

$$f(\omega^2) = \frac{(1 + \omega^4 + \cdots + \omega^{2n-6})(1 + \omega^4 + \cdots \omega^{2n-2})}{a_1^2(\omega^2)(1 + \omega^4 + \cdots + \omega^{2n-6}) + a_2^2(\omega^2)(1 + \omega^4 + \cdots + \omega^{2n-2})}, \quad \text{if}\, n = 2k+1$$

From (4.7), we easily verify the following result.

Lemma 4.1. *Let* $\|\cdot\|$ *be the Euclidean norm in* $\mathbb{R}^n$ *where* n *is the degree of a stable polynomial* $a(s)$. *If* $\hat{a}(s)$ *is the perturbed polynomial with respect to the coefficients* $\underline{\alpha}$ *of* a *and* $\|\underline{a} - \underline{\hat{a}}\| \leq r_{\underline{\alpha}}$, *where* $\underline{a}$, $\underline{\hat{a}}$ *their coefficient-vectors respectively, then* $\hat{a}(s)$ *is also stable.*

The following criterion is now obtained, linking the decomposability of DAP to its stability.

Theorem 4.2. *Let* $\underline{z} \in \wedge^2(\mathbb{R}^n)$ *be a 2-vector and* $\underline{\hat{z}}$ *be its best decomposable approximation. If* $\underline{a}$, $\underline{\hat{a}}$ *are the coefficient-vectors of* $a(s)$, $\underline{\hat{a}}(s)$ *respectively, and* $\|\underline{z} - \underline{\hat{z}}\| \leq r_{\underline{\alpha}}/\sigma_P$, *where* σ_P *is the largest singular value of the Plücker matrix* P *and* $a(s)$ *is a stable polynomial, then* $\hat{a}$ *is also stable.*

Proof. Let the spectral norm $\|A\|_2 = \max\{\lambda : \lambda \in \sigma(A^t A)\}$. From (4.7), the stability radius $r_{\underline{\alpha}}$ is computable and due to the forms of the initial and the approximate system, we immediately imply:

$$\|\underline{a} - \underline{\hat{a}}\| = \|(\underline{z} - \underline{\hat{z}})P\| \leq \|P\|_2 \frac{r_{\underline{\alpha}}}{\sigma_P} = r_{\underline{\alpha}}$$

The result now follows from Lemma 4.1. □

Theorem 4.2 does not only provide the means to test whether the perturbed solutions are at an acceptable distance from the original stable polynomial we had to assign, but also constitutes a criterion for the stability of the perturbed polynomial, *without* the calculation of its roots or their properties.

4.3 An Approximate DAP algorithm

We now consider the problem of approximate DAP that covers all related frequency assignment, i.e. solve the system $\det(H \cdot M(s)) = a(s)$ in terms of H when $a(s)$ is Hurwitz. Let $s_1, s_2, ..., s_n,\ n > 6$ be the roots of $a(s)$. If $\underline{z} := C_m(H)$ and V is the matrix representation of $\mathcal{N}_\ell(P)$, then DAP is transformed into $\underline{z}^t = \lambda \underline{h}_0^t + \underline{\kappa}^t V = [\lambda, \underline{\kappa}^t] \begin{bmatrix} \underline{h}_0^t \\ V \end{bmatrix}$. If this is not possible, one has to find $\underline{z}$ such that the gap between $\underline{z}$ and $G_m(\mathbb{R}^n)$ the least possible. Therefore, we have:

An Algorithm for the Approximate DAP

- Select s_i such that $Re(s_i) < 0$.

- Calculate an orthonormal matrix-basis $[\underline{v}_1, ..., \underline{v}_r]$ for the linear problem $\underline{h}^t P = \underline{a}^t$.

  Set $\underline{f}$ the vector of the corresponding degrees of freedom.

- Solve in terms of $\underline{f}$ the optimization problem $\min_{\underline{f}} g\left(\underline{f}^t \cdot [\underline{v}_1, ..., \underline{v}_r], G_m(\mathbb{R}^n)\right)$ and let $\underline{z}_0 = \underline{f}_0[\underline{v}_1, ..., \underline{v}_r]$ the optimal vector.

- If $\underline{z}_0$ is decomposable then obtain H by $\underline{z}_0 := C_m(H)$. Else, decompose $\underline{z}_0$ via the prime decomposition. A candidate solution is the decomposable vector $\sigma_k \cdot \underline{e}_{2k} \wedge \underline{e}_{2k-1}$.

Example 4.1. *Let*

$$M(s) = \begin{pmatrix} (1+s)^4 & 0 \\ -2+s^2 & s^3 \\ 1+s^3 & s^2 \\ 2s & -2+s \\ 1 & 1 \end{pmatrix}$$

We select the stable polynomial $a(s) = 9.80179 + 50.0464s + 109.122s^2 + 131.717s^3 + 95.06s^4 + 41.02s^5 + 9.8s^6 + s^7$ *whose roots are*

$$s_1 = -1.7, s_2 = -1.6, s_3 = -1.5, s_4 = -1.4, s_5 = -1.3, s_6 = -1.2, s_7 = -1.1$$

If P is the Plücker matrix, then the matrix representation of an orthonormal basis *of* $\underline{h}^t P = \underline{a}^t$ *is*

$$[\underline{v}_1, \underline{v}_2, \underline{v}_3] = \begin{bmatrix} 0.0212483 & 0.0031971 & 0.0198759 \\ 0.14123 & 0.0008332 & 0.107242 \\ 0.179195 & 0.108632 & 0.267088 \\ 0.686415 & 0.311853 & 0.54408 \\ 0.017989 & -0.0177103 & -0.00803 \\ 0.058825 & 0.247088 & -0.202414 \\ -0.480937 & 0.505633 & 0.293198 \\ 0.142346 & 0.268904 & -0.308593 \\ -0.127068 & 0.700893 & -0.219786 \\ -0.452585 & -0.101711 & 0.591784 \end{bmatrix}$$

The substitution of $\underline{z} := (f_1, f_2, f_3) \cdot [\underline{v}_1, \underline{v}_2, \underline{v}_3]$ *to the gap* g *(formula* (3.14)*) implies the 4th order homogeneous polynomial*

$$\begin{aligned} F(f_1, f_2, f_3) = \; & 0.0276749 f_1^4 - 0.0445024 f_1^3 f_2 + 0.050691 f_1^2 f_2^2 + 0.0127932 f_1 f_2^3 + \\ & + 0.0018031 f_2^4 - 0.0348223 f_1^3 f_3 - 0.0986078 f_1^2 f_2 f_3 - 0.0735073 f_1 f_2^2 f_3 + \\ & + 0.013395 f_2^3 f_3 + 0.0052206 f_1^2 f_3^2 + 0.0119637 f_1 f_2 f_3^2 + 0.041775 f_2^2 f_3^2 - \\ & - 0.0414718 f_3^3 + 0.0414718 f_2 f_3^3 + 0.054624 f_3^4 \end{aligned}$$

Hence, we have

$$\min F(f_1, f_2, f_3) \text{ subject to } f_1^2 + f_2^2 + f_3^2 = 1$$

The least gap is achieved at

$$(f_1 = -0.711111, f_2 = -0.348945, f_3 = 0.610376)$$

where we get

$$\begin{aligned} \underline{z}_0 = (& -0.0283575, -0.166179, -0.328359, -0.92903, -0.00117062, \\ & -0.00450339, -0.013399, -0.00669876, -0.0200615, -0.0038811) \end{aligned}$$

Vector $\underline{z}$ *is not decomposable since*

$$\begin{aligned} (\underline{z} \wedge \underline{z})/2 = (& -0.000174025, -0.000570197, \\ & -0.00010584, 0.000280932, 0) \neq \underline{0} \end{aligned}$$

Therefore, we proceed to the calculation of its best decomposable approximation. The spectral analysis of

$$T_{\underline{z}} = \begin{pmatrix} 0 & -0.0283575 & -0.166179 & -0.328359 & -0.92903 \\ 0.0283575 & 0 & -0.00117062 & -0.00450339 & -0.013399 \\ 0.166179 & 0.00117062 & 0 & -0.00669876 & -0.0200615 \\ 0.328359 & 0.00450339 & 0.00669876 & 0 & -0.0038811 \\ 0.92903 & 0.013399 & 0.0200615 & 0.0038811 & 0 \end{pmatrix}$$

will give

$$\underline{z} = \underline{z}_1 + \underline{z}_2$$

where

$$\underline{z}_1 = \sigma_4 \underline{b}_1 \wedge \underline{b}_2 = (-0.0283592, -0.166177, -0.328363, -0.929028, -0.00177318, \\ -0.00460241, -0.0133529, -0.00643763, -0.0201559, -0.00383742)$$

and

$$\underline{z}_2 = \sigma_2 \underline{a}_1 \wedge \underline{a}_2 = (1.86224 \cdot 10^{-6}, -2.04133 \cdot 10^{-6}, 4.24203 \cdot 10^{-6}, -1.19104 \cdot 10^{-6}, \\ -0.0000669825, 0.0000990208, -0.0000469799, -0.000261124, 0.0000943383, \\ -0.0000436849)$$

and $\sigma_2 = 0.00667496 < 1 = \sigma_4$. *Hence, the closest decomposable vector to* $\underline{z}$ *is* $\underline{z}_1$*, which can be re-written dy division by the first coordinate as:*

$$\underline{v} = (1, 5.85971, 11.5787, 32.7594, 0.0625258, \\ 0.16229, 0.470849, 0.227004, 0.710735, 0.135315)$$

This corresponds to the 2×5 *matrix* $\hat{H}$

$$\hat{H} = \begin{pmatrix} 1 & 0 & -0.0625258 & -0.16229 & -0.470849 \\ 0 & 1 & 5.85971 & 11.5787 & 32.7594 \end{pmatrix}$$

The approximate matrix $\hat{H}$ *implies the perturbed polynomial*

$$\hat{a}(s) = \hat{H} \cdot M(s) = -9.83676 - 28.2452s - -22.4508s^2 + \\ + 13.8753s^3 + 35.4504s^4 + 24.9495s^5 + 7.99053s^6 + s^7$$

whose roots are

$$(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (-2.27804 - 0.576901i, -2.27804 + 0.576901i, -1.34073- \\ - 0.809347i, -1.34073 + 0.809347i, -0.903325, -0.82816- \\ - 0.343749i, -0.82816 + 0.343749i)$$

Hence, $\hat{a}(s)$ *is stable. Furthermore,*

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 \\ 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 \\ -2 & -7 & -8 & -2 & 2 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 & -1 & 0 \\ 4 & -2 & -2 & 1 & -2 & 0 & 0 & 0 \\ -2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & -4 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\sigma_P = 14.0414$. *The stability radius for* $p(s) = (s+1.7)(s+1.6)(s+1.5)(s+1.4)(s+1.3)(s+1.2)(s+1.1)$ *from* (4.7) *is* $r_{\underline{\alpha}} = 7.3246$. *Therefore,*

$$\|\underline{z} - \hat{\underline{z}}\| = 0.000673828 < 0.522663 = \frac{r_{\underline{\alpha}}}{\sigma_P}$$

Hence the approximate polynomial $\hat{a}(s)$ *is stable, which verifies Theorem 4.2.*

5 Conclusions

The approximate determinantal assignment problem has been defined and solved as a distance problem between the Grassmann variety and a linear variety defined by the properties of a desirable polynomial. The study of the problem was split to three basic problems: (i) The distance problem of a point of the projective space from the Grassmann variety; (ii) The extension of the above to the case where we have the distance of a linear variety from the Grassmann variety; (iii) The characterization of acceptability of the optimal distance solutions as far as the nature of the resulting assigned polynomial. The key minimization problem implied by the second problem (formulation of the (1.2) type), was addressed and a closed form solution was derived, which is similar in nature to the first optimization problem for a constant point in $\wedge^2(\mathbb{R}^n)$. The results of this approach have been demonstrated via a DAP algorithm and can be easily specialized to different types of frequency assignment, such as the output feedback. The polynomials corresponding to the approximate solutions are at some distance from the nominal polynomial and for solutions to be acceptable we need the resulting polynomials to be stable. We have used the stability radius results [Hin. 1] to derive a condition that can be used to check stability without root calculations.

The above results are based on the prime decomposition (3.3) of 2-vectors which has implied significant simplifications, such as the formulation of DAP into a 4th order polynomial minimization problem, constrained to the unit sphere for the $G_2(\mathbb{R}^5)$ case. Moreover, this analysis has connected the approximate DAP with similar optimization problems, such as eigenvalue maximization.

This approach has not been used before for general frequency assignment problems and it may be implemented as a new pole placement method, that uses no generic or exact solvability conditions. Future work will examine the challenging case of generalization to $G_q(\mathbb{R}^n)$ and derive if possible similar closed-form formulae, since higher order Grassmann-optimization problems are usually solved numerically via algebraic geometry toolboxes [Eis. 1]. The difficulty in q-decompositions, $q \geq 3$, lies in the fact that the matrices which in our case provide the representation of the points of the projective space, become q-tensors and thus the multi-linear subproblem should be naturally examined via tensor decomposition algorithms, [Kol. 1], [Sav. 1], which is the area of future research.

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