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ON ISOTYPIES BETWEEN GALOIS CONJUGATE BLOCKS

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Abstract. We show that between any pair of Galois conjugate blocks of a finite group, there is an isotypy with all signs positive.

1. Introduction

Let p be a prime number, let k be an algebraic closure of the field of p elements and let G be a finite group. Let

$$\sigma : k \to k, \quad (\lambda \to \lambda^p, \quad \lambda \in k)$$

be the Frobenius homomorphism of k and we write also $\sigma : kG \to kG$ for the ring automorphism induced by σ. This is defined by

$$\sigma\Big(\sum_{g \in G} \alpha_g g\Big) = \sum_{g \in G} \alpha_g^p g.$$

The map σ is a ring automorphism but not a k-algebra automorphism. Since σ is a ring automorphism, σ induces a permutation of the blocks of kG. Here by a block of a ring A, we mean a primitive idempotent of the center of A. Blocks b and c of kG are said to be *Galois conjugate* if $c = \sigma^n(b)$ for some natural number n.

Let $(K, \mathcal{O}, k)$ be a p-modular system and let $^- : \mathcal{O} \to k$ be the canonical quotient mapping. Let $^- : \mathcal{O}G \to kG$ be the induced $\mathcal{O}$-algebra homomorphism of the group algebras. In particular, $b \to \bar{b}$ induces a bijection between the set of blocks of $\mathcal{O}G$ and the set of blocks of kG. Blocks b and c of $\mathcal{O}G$ are said to be *Galois conjugate* if $\bar{b}$ and $\bar{c}$ are Galois conjugate blocks of kG.

Our main result is the following.

Theorem 1.1. *With the notation above, suppose that K contains a primitive $|G|$-th root of unity. Then any two Galois conjugate blocks of $\mathcal{O}G$ are isotypic.*

As a corollary, we obtain a new proof of the following result of Cliff, Plesken and Weiss [7].

Corollary 1.2. *Let G be a finite group and b be a block of kG. Then the center $Z(kGb)$ of kGb has an $\mathbb{F}_p$-form.*

We recall that a finite dimensional k-algebra A is said to have an $\mathbb{F}_p$-form if there exists an $\mathbb{F}_p$-algebra A_0 such that $A \cong k \otimes_{\mathbb{F}_p} A_0$ as k-algebra.

The notion of isotypic blocks (see Definition 2.2) is due to Michel Broué [6, Definition 4.3]. Isotypies between blocks are interesting as they are often the character theoretic shadow of a derived or Morita equivalence between the k-linear module categories of the corresponding block algebras. In [2], it was shown that there exist pairs of Galois conjugate blocks which are not derived equivalent as k-algebras. Thus the blocks studied in [2] provide examples of pairs of blocks which are isotypic and isomorphic as rings but not derived equivalent as k-algebras. However, it is conjectured that the number of Morita equivalence classes of algebras in any set of Galois conjugate blocks is bounded by a number which depends only on the defect of the blocks and which is independent of G. This conjecture is related to the Donovan finiteness conjectures in block theory [9].

For a non-negative integer d and a finite dimensional commutative k-algebra A, we will say that A occurs as the center of a d-block if there exists a finite group H and a block c of kH with defect d such that $A \cong Z(kHc)$ as k-algebras. Combining Corollary 1.2 with a theorem of Brauer and Feit [4] gives a proof of the following finiteness result. The result itself is known to experts, but as far as I am aware has not appeared before in the literature.

Corollary 1.3. *Let d be a non-negative integer and set $m := \frac{1}{4}p^{2d} + 1$. Up to isomorphism there are at most p^{m^3} k-algebras that occur as centers of d-blocks.*

The paper is divided into three sections. In Section 2, we set up notation and recall the definitions of perfect isometries and isotypies. Section 3 contains the proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.3.

2. Notation and definitions

Throughout, G is a finite group, k is algebraically closed of characteristic p and $(K, \mathcal{O}, k)$ is a p-modular system such that K contains a primitive $|G|$-th root of unity. For an element g of G, we will denote by g_p the p-part of g and by $g_{p'}$-the p'-part of G. Denote by G_p the set of elements of G of p-power order and by $G_{p'}$ the set of elements of G of p'-order. For a natural number n, let n_p denote the p-part n and $n_{p'}$ denote the p'-part of n.

2.1. **Generalized decomposition maps.** Let b be a central idempotent of $\mathcal{O}G$. Denote by $\mathrm{CF}(G, K)$ the K-space of K-valued class functions on G. Identifying K-valued class functions on G with their canonical K-linear extensions to K-valued functions on KG, denote by $\mathrm{CF}(G, b, K)$ the K-subspace of class functions ϕ in $\mathrm{CF}(G, K)$ such that $\phi(gb) = \phi(g)$ for all $g \in G$. Denote by $\mathrm{CF}_{p'}(G, K)$ (respectively $\mathrm{CF}_{p'}(G, b, K)$) the K-subspaces of $\mathrm{CF}(G, K)$ (respectively $\mathrm{CF}(G, b, K)$) of class functions which vanish on the p-singular classes of G; $\mathrm{Irr}(G, K)$ (respectively $\mathrm{Irr}(G, b, K)$) the subset of $\mathrm{CF}(G, K)$ (respectively $\mathrm{CF}(G, b, K)$) of irreducible K-valued characters of G; by $\mathrm{IBr}(G,K)$ (respectively $\mathrm{IBr}(G,b,K)$) the set of irreducible Brauer characters of G (respectively irreducible Brauer characters of G in b). For $\phi \in \mathrm{IBr}(G,K)$, let $\phi_0 \in \mathrm{CF}_{p'}(G,K)$ be defined by $\phi_0(y) = \phi(y)$ if $y \in G_{p'}$. Denote by $\mathrm{IBr}(G,K)_0$ the set of class functions ϕ_0, for $\phi \in \mathrm{IBr}(G,K)$ and by $\mathrm{IBr}(G,b,K)_0$ the subset of $\mathrm{IBr}(G,K)_0$ consisting of those ϕ_0 for which $\phi \in \mathrm{IBr}(G,b,K)$

Let x be a p-element of G. The generalized decomposition map

$$d_G^x : \mathrm{CF}(G,K) \to \mathrm{CF}_{p'}(C_G(x),K)$$

is defined by $d_G^x(\alpha)(y) = \alpha(xy)$ for $\alpha \in \mathrm{CF}(G,K)$, $y \in C_G(x)_{p'}$. If $\chi \in \mathrm{Irr}(G,K)$, $x \in G_p$, then

$$d_G^x(\chi) = \sum_{\phi \in \mathrm{IBr}(C_G(x),K)} \delta_{(G,\chi)}^{(x,\phi)} \phi_0$$

for uniquely determined elements $\delta_{(G,\chi)}^{(x,\phi)}$ of K; $\delta_{(G,\chi)}^{(x,\phi)}$ is called the generalized decomposition number associated to the triple (χ,ϕ,x).

Let e be a central idempotent of $\mathcal{O}C_G(x)$. The map

$$d_G^{(x,e)} : \mathrm{CF}(G,K) \to \mathrm{CF}_{p'}(C_G(x),e,K)$$

is defined by $d_G^{(x,e)}(\alpha)(y) = \alpha(xey)$ for $\alpha \in \mathrm{CF}(G,K)$, $y \in C_G(x)_{p'}$.

2.2. **Local structure of blocks.** Let R denote one of the rings $\mathcal{O}$ or k. For Q a p-subgroup of G, let $(RG)^Q$ denote the R-subalgebra of RG consisting of the elements of RG which are fixed by the conjugation action of Q. The Brauer homomorphism $\mathrm{Br}_Q : (RG)^Q \to kC_G(Q)$ is the map defined by

$$\mathrm{Br}_Q(\sum_{g\in g} \alpha_g g) = \sum_{g \in kC_G(Q)} \bar{\alpha}_g g.$$

Here if $R = k$, then $\bar{\alpha}_g$ is to be interpreted as α_g. If b is a block of $\mathcal{O}G$, then a $\bar{b}$-Brauer pair is a pair $(Q,\bar{e})$, where Q is a p-subgroup of G and e is a block of $\mathcal{O}C_G(Q)$ such that $\mathrm{Br}_Q(b)\bar{e} = \bar{e}$. Let $(P,\bar{e}_P)$ be a maximal b-Brauer pair under the Alperin-Broué inclusion of Brauer pairs [1, Definition 1.1], and for each subgroup Q of P, let e_Q be the unique block of $\mathcal{O}C_G(Q)$ such that $(Q,\bar{e}_Q) \le (P,\bar{e}_P)$.

Let $\mathcal{F}_{(P,\bar{e}_P)}(G,\bar{b})$ denote the category whose objects are the subgroups of P and whose morphisms are defined as follows: For Q,R subgroups of P, the set of $\mathcal{F}_{(P,\bar{e}_P)}(G,\bar{b})$ morphisms from Q to R is the set of those group homomorphisms $\varphi : Q \to R$ for which there exists an element g of G such that $\varphi(x) = gxg^{-1}$ for $x \in Q$ and such that $({}^gP, {}^g\bar{e}_Q) \le (R,\bar{e}_R)$; composition of morphisms is the usual composition of functions.

2.3. **Perfect isometries and isotypies.** Let H be a finite group such that K contains a primitive $|H|$-th root of unity. Let b be a central idempotent of $\mathcal{O}G$ and c a central idempotent of $\mathcal{O}H$.

Definition 2.1. *A perfect isometry between b and c is a K-linear map*

$$I : \mathrm{CF}(G, b, K) \to \mathrm{CF}(H, c, K)$$

such that the following holds. For each $\chi \in \mathrm{Irr}(G, b, K)$, there exists an $\epsilon_\chi \in \{\pm 1\}$ such that the map $\chi \to \epsilon_\chi I(\chi)$ is a bijection between $\mathrm{Irr}(G, b, K)$ and $\mathrm{Irr}(H, c, K)$ and such that setting

$$\mu := \sum_{\chi \in \mathrm{Irr}(G,b,K)} \chi \times I(\chi),$$

the class function on $G \times H$ which sends an element (x, y) of $G \times H$ to the element $\sum_{\chi \in \mathrm{Irr}(G,b,K)} \chi(x)I(\chi)(y)$ of $\mathcal{O}$ the following holds:

(a) For each $x \in G$, $y \in H$, $\frac{\mu(x,y)}{|C_G(x)|} \in \mathcal{O}$.

(b) If $x \in G$, $y \in H$ are such that exactly one of x and y is p-singular, then $\mu(x, y) = 0$.

If I as in the above definition is a perfect isometry between b and c, then I induces by restriction a map

$$I_{p'} : \mathrm{CF}_{p'}(G, b, K) \to \mathrm{CF}_{p'}(H, c, K).$$

Definition 2.2. *Let b be a block of $\mathcal{O}G$ and c a block of $\mathcal{O}H$. Then b and c are isotypic if the following conditions hold:*

(a) There exists a p-group P and inclusions $P \hookrightarrow G$, $P \hookrightarrow H$ such that identifying P with its image in G and in H, there exists a block e_P of $\mathcal{O}C_G(P)$ such that $(P, \bar{e}_P)$ is a maximal $\bar{b}$-Brauer pair and a block f_P of $\mathcal{O}C_H(P)$ such that $(P, \bar{f}_P)$ is a maximal $\bar{c}$-Brauer pair and such that

$$\mathcal{F}_{(P,\bar{e}_P)}(G, \bar{b}) = \mathcal{F}_{(P,\bar{f}_P)}(H, \bar{c}).$$

(b) For each cyclic subgroup Q of P, there exists a perfect isometry

$$I^Q : \mathrm{CF}(C_G(Q), e_Q, K) \to \mathrm{CF}(C_H(Q), f_Q, K)$$

where e_Q (respectively f_Q) is the unique block of $\mathcal{O}C_G(Q)$ (respectively $\mathcal{O}C_H(Q)$) with $(Q, \bar{e}_Q) \le (P, \bar{e}_P)$ (respectively $(Q, \bar{f}_Q) \le (P, \bar{f}_P)$) such that for every generator x of Q, we have

$$I^Q_{p'} \circ d_G^{(x,e_Q)} = d_H^{(x,f_Q)} \circ I^{\{1\}}.$$

3. Proofs.

Keep the notation and hypothesis of Theorem 1.1. In addition, let $W(k)$ be the unique absolutely non-ramified complete discrete valuation ring having k as residue field, and identify $W(k)$ with its image under the canonical injective homomorphism $W(k) \to \mathcal{O}$ (see [12, Chapter 2 §5, Theorem 3 and Theorem 4]). There is a unique ring automorphism $\sigma_{W(k)} : W(k) \to W(k)$ such that $\overline{\sigma_{W(k)}(\eta)} = \sigma(\bar{\eta})$ for all $\eta \in W(k)$. Further, note that for any p'-root of unity η in K, $\eta \in W(k)$ and $\sigma_{W(k)}(\eta) = \eta^p$.

Let K_0 be the algebraic closure of $\mathbb{Q}$ in K. Choose a field automorphism $\sigma_{K_0} : K_0 \to K_0$ such that if $\eta \in K$ is any $|G|$-th root of unity, then $\sigma_{K_0}(\eta) =$

η^p if the order of η is relatively prime to p and $\sigma_{K_0}(\eta) = \eta$ if η is a power of p. Then, $K_0 \cap W(k)$ contains a primitive $|G|_{p'}$-root of unity and σ_{K_0} and $\sigma_{W(k)}$ coincide on $\mathbb{Q}[\eta] \cap W(k)$ for any $|G|_{p'}$-root of unity η. Note that we are not claiming that σ_0 extends to an automorphism of $\mathcal{O}$ which agrees with σ_{K_0} on restriction to $\mathbb{Q}[\eta] \cap \mathcal{O}$ for any $|G|$-th root of unity η.

Denote by $\sigma_{W(k)}$ (respectively σ_{K_0}) the natural extension of $\sigma_{W(k)}$ (respectively σ_{K_0}) to $W(k)G$ (respectively K_0G).

Recall from [11, Chapter 3, Theorem 6.22 (ii)] that any block of $\mathcal{O}H$, for H a finite group is an $\mathcal{O}$-linear combination of p'-elements of H.

Lemma 3.1. *Let η be a primitive $|G|_{p'}$ root of unity in K and let $H \leq G$. Let*

$$c = \sum_{g \in H_{p'}} \alpha_g g, \alpha_g \in \mathcal{O}$$

be a block of $\mathcal{O}H$. Then,

(i) $\alpha_g \in \mathbb{Q}[\eta] \cap W(k)$ for all $g \in H_{p'}$.

(ii) $\sigma_{W(k)}(c)$ is a block of $\mathcal{O}H$, $\sigma_{W(k)}(c) = \sigma_{K_0}(c)$ and

$$\overline{\sigma_{W(k)}(c)} = \sigma(\bar{c}).$$

Proof. (i) By idempotent lifting, the canonical quotient map $^- : \mathcal{O} \to k$ induces a bijection between the set of central idempotents of $\mathcal{O}H$ and kH. Similarly, the restriction of $^-$ to $W(k)$ induces a bijection between the set of central idempotents of $W(k)H$ and kH. Since a central idempotent of $W(k)H$ is a central idempotent of $\mathcal{O}H$, it follows that $\mathcal{O}H$ and $W(k)H$ have the same central idempotents. In particular, $c \in W(k)H$. So, $\alpha_g \in W(k)$ for all $g \in H_{p'}$. On the other hand, by [11, Chapter 3, Theorem 6.22], for $g \in H_{p'}$, α_g is a $\mathbb{Q}$-linear combination of $|g|$-th roots of unity whence $\alpha_g \in \mathbb{Q}[\eta]$. This proves (i).

(ii) As shown above, the set of blocks of $\mathcal{O}H$ is the same as the set of blocks of $W(k)H$ and $\sigma_{W(k)}$ is an automorphism of $W(k)H$. This proves the first assertion. The others are immediate from (i).

Definition 3.2. *For H a subgroup G, let*

$$I^H : \mathrm{CF}(H, K) \to \mathrm{CF}(H, K)$$

denote the K-linear map defined by

$$I^H(\phi)(x) = \phi(x_p x_{p'}^p), \quad \text{for } \phi \in \mathrm{CF}(H, K),\ x \in G.$$

If an element $\phi \in \mathrm{CF}(H, K)$ takes values in K_0, denote by $\sigma(\phi)$ the element of $\mathrm{CF}(H, K)$ which sends $g \in H$ to $\sigma_{K_0}(\phi(g))$. Similarly, if ϕ takes values in $W(k)$, then $\sigma_{W(k)}(\phi)$ will denote the element of $\mathrm{CF}(H, K)$ which sends $g \in H$ to $\sigma_{W(k)}(\phi(g))$. We use the same conventions on K-valued class functions defined on $H_{p'}$.

Lemma 3.3. *Let H be a subgroup of G and let c be a block of H.*

(i) For any $\chi \in \mathrm{Irr}(H,K)$,

$$I^H(\chi) = \sigma_{K_0}(\chi) \in \mathrm{Irr}(H,K).$$

(ii) The map

$$\chi \to \sigma_{K_0}(\chi)$$

is a bijection from $\mathrm{Irr}(H,c,K)$ to $\mathrm{Irr}(H,\sigma_{K_0}(c),K)$.

(iii) The restriction of I^H to $\mathrm{CF}(H,c,K)$ induces a perfect isometry between c and $\sigma_{K_0}(c)$.

(iv) For any $\phi \in \mathrm{IBr}(H,K)$ and any $\chi \in \mathrm{Irr}(G,K)$,

$$I^H(\phi_0) = \sigma_{K_0}(\phi)_0 \in \mathrm{IBr}(H,K)_0$$

and

$$\delta^{(\{1\},\sigma_{K_0}(\phi))}_{(H,\sigma_{K_0}(\chi))} = \delta^{(\{1\},\phi)}_{(H,\chi)}.$$

Proof. (i) Let $\chi \in \mathrm{Irr}(H,K)$, and let $\rho : H \to GL_n(K_0)$ be a representation affording χ. Such a ρ exists by Brauer's splitting field theorem (see for example [11, Chapter 3, Theorem 4.11]). Denoting also by σ_{K_0} the automorphism (as abstract group) of $GL_n(K_0)$ induced by σ_{K_0}, it follows that $\sigma_{K_0} \circ \rho$ is an irreducible representation of H with character $\sigma_{K_0}(\chi)$. We show that $I^H(\chi) = \sigma(\chi)$. Let $h = h_p h_{p'} \in H$. Since K_0 contains all the eigen values of $\rho(h)$, by replacing ρ with an equivalent representation if necessary, we may assume that $\rho(h)$ is a diagonal matrix. Since h_p and $h_{p'}$ are powers of h, it follows that $\rho(h_p)$ is a diagonal matrix $diag(\zeta_1, \cdots, \zeta_n)$, $\rho(h_{p'})$ is a diagonal matrix $diag(\eta_1, \cdots, \eta_n)$, where each ζ_i is a $|H|_p$ root of unity and each η_j is an $|H|_{p'}$ root of unity, $\rho(h)$is the diagonal matrix $diag(\zeta_1\eta_1, \cdots, \zeta_n\eta_n)$ and $\rho(h_p h_{p'}^p)$ is the diagonal matrix $diag(\zeta_1\eta_1^p, \cdots, \zeta_n\eta_n^p)$. Thus,

$$I^H(\chi)(h) = \sum_{1\le i\le n} \zeta_i\eta_i^p = \sigma_{K_0}(\sum_{1\le i\le n} \zeta_i\eta_i) = \sigma(\chi)(h).$$

(ii) Let $\chi \in \mathrm{Irr}(H,K)$ and let

$$e_\chi = \frac{\chi(1)}{|H|}\sum_{h\in H}\chi(h)h^{-1}$$

be the primitive central idempotent of KH corresponding to χ. Then,

$$e_{\sigma_{K_0}(\chi)} = \frac{\chi(1)}{|H|}\sum_{h\in H}\sigma_{K_0}(\chi(h))h^{-1} = \sigma_{K_0}(e_\chi).$$

From this it is immediate that if $\chi \in \mathrm{Irr}(H,c,K)$ then $\sigma_{K_0}(\chi) \in \mathrm{Irr}(H,\sigma_{K_0}(c),K)$.

(iii) By (i) and (ii) it suffices to prove that the function

$$\mu := \sum_{\chi\in\mathrm{Irr}(H,c,K)} \chi \times I^H(\chi),$$

satisfies conditions (a) and (b) of Definition 1.1.

Set

$$\iota := \sum_{\chi \in \mathrm{Irr}(H,K,c)} \chi \times \chi$$

and let $x, y \in H$.

Then

$$\mu(x,y) = \iota(x, y_p y_{p'}^p).$$

Since the identity map on $\mathrm{CF}(H,c,K)$ is a perfect isometry with ι the corresponding (virtual) character of $H \times H$, we see that $\mu(x,y)$ is divisible in $\mathcal{O}$ by both $|C_G(x)|$ and by $|C_G(y_p y_{p'}^p)|$ and that $\mu(x,y)$ is 0 if exactly one of x and $y_p y_{p'}^p$ is p-singular. But, clearly $C_G(y_p y_{p'}^p) = C_G(y)$ and $y_p y_{p'}^p$ is p-singular if and only if y is p-singular. This proves (iii).

(iv) Let $\phi \in \mathrm{IBr}(H,K)$ and let $\tau : H \to GL_n(k)$ be an irreducible representation of H affording the Brauer character ϕ. Then $\sigma \circ \tau$ is an irreducible representation of H, where again we denote by σ the induced automorphism of $GL_n(k)$. Let ϕ' be the Brauer character associated to $\sigma \circ \tau$ and let $h \in H_{p'}$. Let Λ be the multiset of eigen values of $\tau(h)$. The multiset of eigen values of $\sigma \circ \tau(h)$ is $\{\lambda^p : \lambda \in \Lambda\}$, hence $\phi'(h) = \sigma_{W(k)}(h)$. Further, since $\{\lambda^p : \lambda \in \Lambda\}$ is also the multiset of eigen values of $\tau(h^p)$, $I^H(\phi_0)(h) = \phi(h^p) = \sigma_{W(k)}(h)$ for all $h \in H_{p'}$. Thus,

$$I^H(\phi_0) = \sigma_{W(k)}(\phi)_0 = \phi' \in \mathrm{IBr}(H,K).$$

Since ϕ takes values in $\mathbb{Z}[\eta]$, with η a primitive $|H|_{p'}$-root of unity, $\sigma_{W(k)}(\phi)_0 = \sigma_{K_0}(\phi)_0$ whence $I^H(\phi_0) = \sigma_{K_0}(\phi)_0 \in \mathrm{IBr}(H,K)$.

The compatibility of decomposition numbers is clear from the fact that for any $\chi \in \mathrm{Irr}(H,K)$,

$$\mathrm{Res}|_{H_{p'}} \chi = \sum_{\phi \in \mathrm{IBr}(H,K)} \delta_{H,\chi}^{1,\phi} \phi$$

and that $\delta_{H,\chi}^{1,\phi} \in \mathbb{Z}$ for all $\phi \in \mathrm{IBr}(H,K)$.

Proof of Theorem 1.1. Let b be a block of $\mathcal{O}G$. By Lemma 3.1 it suffices to show that b and $\sigma_{K_0}(b)$ are isotypic. Let P a p-subgroup of G and e_P be a block of $\mathcal{O}C_G(P)$ such that $(P, \bar{e}_P)$ is a maximal $\bar{b}$-Brauer pair. For each $Q \leq P$, let e_Q be the unique block of $\mathcal{O}C_G(Q)$ such that $(Q, \bar{e}_Q) \leq (P, \bar{e}_P)$. For any p-subgroup Q of G and any $a \in (kG)^Q$, $\sigma(\mathrm{Br}_P(a)) = \mathrm{Br}_P(\sigma(a))$. So, the map $(Q,f) \to (Q, \sigma(f))$ is an isomorphism from the G-poset of $\bar{b}$-Brauer pairs to the G-poset of $\sigma(\bar{b})$-Brauer pairs. By Lemma 3.1, $\sigma(\bar{b}) = \overline{\sigma_{K_0}(b)}$ and $\sigma(\bar{f}) = \overline{\sigma_{K_0}(f)}$ for any block f of $kC_G(Q)$, Q a p-subgroup of G. Thus, $(P, \overline{\sigma_{K_0}(e_P)})$ is a maximal $\overline{\sigma_{K_0}(b)}$-Brauer pair; for every subgroup Q of P, $\sigma_{K_0}(e_Q)$ is the unique block of $\mathcal{O}C_G(Q)$ such that $(Q, \overline{\sigma_{K_0}(e_Q)}) \leq (P, \overline{\sigma_{K_0}(e_P)})$, and $\mathcal{F}_{(P,\overline{\sigma_{K_0}(e_P)})}(G, \overline{\sigma_{K_0}(b)}) = \mathcal{F}_{(P,\bar{e}_P)}(G, \bar{b})$.

We will use the maps $I^{C_G(Q)}$ of Definition 3.2 to produce an isotypy between b and $\sigma_{K_0}(b)$. For $Q \leq P$, let I^Q be the restriction of $I^{C_G(Q)}$ to

$\mathrm{CF}(C_G(Q), e_Q, K)$. We will show that the maps I^Q, as Q runs over the cyclic subgroups of P defines an isotypy between b and c. So, let $Q \leq P$ be a cyclic group. By Lemma 3.3,

$$I^Q : \mathrm{CF}(C_G(Q), e_Q, K) \to \mathrm{CF}(C_G(Q), \sigma_{K_0}(e_Q), K)$$

is a perfect isometry. It remains only to check the compatibility condition (b) of Definition 2.2.

Set $H = C_G(Q)$ and let $Q = \langle x \rangle$. We claim that

$$\delta_{(G,\chi)}^{(x,\phi)} = \delta_{(G,\sigma_{K_0}(\chi))}^{(x,\sigma_{K_0}(\phi))}$$

for all $\chi \in \mathrm{Irr}(G, K)$ and all $\phi \in \mathrm{IBr}(H, K)$.

Indeed, for $\tau \in \mathrm{Irr}(H, K)$, let $\zeta_{(\tau,x)} \in K$ be such that $\rho(x)$ is the scalar matrix $(\zeta_{(\tau,x)}, \cdots, \zeta_{(\tau,x)})$ in any representation of H affording τ. Since the order of $\zeta_{(\tau,x)}$ is a divisor of $|G|_p$,

$$\zeta_{(\sigma_{K_0}(\tau),x)} = \sigma_{K_0}(\zeta_{(\tau,x)}) = \zeta_{(\tau,x)}.$$

Also, for $\chi \in \mathrm{Irr}(G, K)$, $\phi \in \mathrm{IBr}(H, K)$,

$$\delta_{(G,\chi)}^{(x,\phi)} = \sum_{\tau \in \mathrm{Irr}(H,K)} \langle \mathrm{Res}|_H \chi, \tau \rangle \zeta_{(\tau,x)} \delta_{(H,\tau)}^{(\{1\},\phi)}.$$

The claim follows since for all $\tau \in \mathrm{Irr}(H, K)$ and all $\phi \in \mathrm{IBr}(H, K)$, $\zeta_{(\sigma_{K_0}(\tau),x)} = \zeta_{(\tau,x)}$, $\delta_{(H,\sigma_{K_0}\tau))}^{(\{1\},\sigma_{K_0}(\phi))} = \delta_{(H,\tau)}^{(\{1\},\phi)}$ by Lemma 3.3 (iv) and $\langle \mathrm{Res}|_H \chi, \tau \rangle \zeta_{(\tau,x)} \in \mathbb{Z}$.

For $\chi \in \mathrm{Irr}(G, K)$, $\phi \in \mathrm{IBr}(H, K)$,

$$d_G^{(x,e_Q)}(\chi) = \sum_{\phi \in \mathrm{IBr}(H,e_Q,K)} \delta_{(G,\chi)}^{(x,\phi)} \phi_0,$$

The compatibility condition (b) of Definition 2.2 is easily seen to follow from the linearity of the maps I^Q, the claim and Lemma 3.3

$$\begin{aligned}
I_{p'}^Q \circ d_G^{(x,e_Q)}(\chi) &= \sum_{\phi \in \mathrm{IBr}(H,e_Q,K)} \delta_{(G,\chi)}^{(x,\phi)} \sigma_{K_0} I^Q(\phi_0) \\
&= \sum_{\phi \in \mathrm{IBr}(H,e_Q,K)} \delta_{(G,\sigma_{K_0}(\chi))}^{(x,\sigma_{K_0}(\phi))} I^Q(\phi_0) \\
&= \sum_{\phi \in \mathrm{IBr}(H,e_Q,K)} \delta_{(G,\sigma_{K_0}(\chi))}^{(x,\sigma_{K_0}(\phi))} \sigma_{K_0}(\phi) \\
&= \sum_{\phi' \in \mathrm{IBr}(H,\sigma_{K_0}(e_Q),K)} \delta_{(G,\sigma_{K_0}(\chi))}^{(x,\phi')} \phi_0' \\
&= d_G^{(x,\sigma_{K_0}(e_Q))} I^{\{1\}}(\chi).
\end{aligned}$$

□

Proof of Corollary 1.2. Let $(K, \mathcal{O}, k)$ be a p-modular system such that k is an algebraic closure of $\mathbb{F}_p$ and K contains a primitive $|G|$-th root of unity. By the theorem, there is a perfect isometry between $\mathcal{O}Gb$ and $\mathcal{O}G\sigma_{K_0}(b)$. Hence, by [5, Théorème 1.4], there is an $\mathcal{O}$-algebra isomorphism $f : Z(\mathcal{O}G\sigma_{K_0}(b)) \to Z(\mathcal{O}Gb)$. This induces a k-algebra isomorphism $f : Z(kG\sigma(\bar{b})) \to Z(kG\bar{b})$. On the other hand, σ induces a ring isomorphism $\sigma : Z(kG\bar{b}) \to Z(kG\sigma(\bar{b}))$ such that for all $a \in Z(kG\bar{b})$ and all $\lambda \in k$, $\sigma(\lambda a) = \lambda^p \sigma(a)$. Thus, $\sigma \circ f : Z(kG\sigma(b)) \to Z(kG\sigma(b))$ is a ring automorphism which satisfies $\sigma \circ f(\lambda a) = \lambda^p \sigma \circ f(a)$ for all $a \in Z(kG\sigma(\bar{b}))$ and all $\lambda \in k$. By [9, Lemma 2.1], the fixed points $(Z(kG\sigma(\bar{b})))^{\sigma\circ f}$ of $Z(kG\sigma(\bar{b}))$ under $\sigma \circ f$ are an $\mathbb{F}_p$-subspace of $Z(kG\sigma(\bar{b}))$ such that $Z(kG\sigma(\bar{b})) \cong k \otimes_{\mathbb{F}_p} (Z(kG\sigma(\bar{b})))^{\sigma\circ f}$ as k-vector spaces. Since $\sigma \circ f$ is a homomorphism of rings, $(Z(kG\sigma(\bar{b})))^{\sigma\circ f}$ is an $\mathbb{F}_p$-algebra. Thus $Z(kG\sigma(\bar{b}))$ and hence $Z(kGb)$ has an $\mathbb{F}_p$-form. □

Proof of Corollary 1.3. Let b be a block of kG with defect d. Since $\dim_k(Z(kGb)) = \dim_K(Z(KGb)) = |\mathrm{Irr}(G, b, K)|$, by [4, Theorem 1] the k-dimension of $Z(kGb)$ is bounded by m. By Corollary 1.2, $Z(kGb)$ has a k-basis such that the multiplicative constants of $Z(kGb)$ with respect to this basis are all in $\mathbb{F}_p$. Thus there are at most p^{m^3} possibilities for the isomorphism type of $Z(kGb)$. □

Remarks 3.4. *(i)* The proof of Theorem 1.1 can be readily adapted to prove that between any pair of Galois conjugate blocks there is a global isotypy in the sense of [3, 1.9]. It is not known whether there is a p-permutation equivalence (cf. [3, Definiton 1.3], [10, Definiton 1.3]) between any pair of Galois conjugate blocks.

(ii) By Lemma 3.3, the isometries between various blocks in Theorem 1.1 all appear without signs, which seems to render even more surprising the fact that Galois conjugate blocks need not be Morita equivalent [2].

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