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QUASI-ISOLATED BLOCKS AND BRAUER'S HEIGHT ZERO CONJECTURE

RADHA KESSAR AND GUNTER MALLE

Abstract. This paper has two main results. Firstly, we complete the parametrisation of all p-blocks of finite quasi-simple groups by finding the so-called quasi-isolated blocks of exceptional groups of Lie type for bad primes. This relies on the explicit decomposition of Lusztig induction from suitable Levi subgroups. Our second major result is the proof of one direction of Brauer's long-standing height zero conjecture on blocks of finite groups, using the reduction by Berger and Knörr to the quasi-simple situation. We also use our result on blocks to verify a conjecture of Malle and Navarro on nilpotent blocks for all quasi-simple groups.

1. Main results

Brauer's famous height zero conjecture [9] from 1955 states that a p-block B of a finite group has an abelian defect group if and only if every ordinary irreducible character in B has height zero.

Here we are concerned with one direction of this conjecture:

(HZC1) If a p-block B of a finite group has abelian defect groups, then every ordinary irreducible character of B has height zero.

One of the main aims of this paper is the proof of the following result:

Theorem 1.1. *The 'if part' (HZC1) of Brauer's height zero conjecture holds for all finite groups.*

Our proof relies on the crucial paper of Berger and Knörr [3] where they show that this direction of the conjecture holds for all groups, provided that it holds for all quasi-simple groups. An alternative proof of this reduction was later given by Murai [41].

Many particular cases of (HZC1) had been considered before. Olsson [44] showed the claim for the covering groups of alternating groups. The case of unipotent blocks of groups of Lie type was treated by Broué–Malle–Michel [10] and Broué–Michel [11]. Cabanes–Enguehard [12] then showed (HZC1) for most blocks of finite reductive groups. In addition to these results we use the theorem of Blau–Ellers [4] that this direction of the conjecture holds for all central quotients of special linear and special unitary groups.

Let us mention a few other important partial results: Gluck and Wolf [25] proved both directions of the height zero conjecture for p-solvable groups. Fong–Harris showed

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(HZC1) for principal 2-blocks, Navarro–Tiep [43] recently proved both directions for 2-blocks of maximal defect and Kessar–Koshitani–Linckelmann [31] proved (HZC1) for 2-blocks whose defect groups are elementary abelian of order 8. Our paper is independent of the latter results.

As our second main result and as a crucial ingredient to the proof of Theorem 1.1 we complete the parametrisation of the ℓ-blocks of finite quasi-simple groups, where ℓ is a prime number. (See Remark 6.12 for historic comments on this problem.) The only case that remains to be considered is the one of quasi-isolated blocks of exceptional groups of Lie type when ℓ is bad, that is, non-unipotent blocks parametrised by non-identity semisimple elements whose centraliser in the dual group is not contained in any proper Levi subgroup. This is the case which we solve here.

Although our determination of quasi-isolated blocks proceeds in a case-by-case manner, the result on blocks and their defect groups can be phrased in the following general, generic form, which also appeared for the blocks considered in the earlier work of Cabanes and Enguehard. Throughout this introduction, G denotes a simple simply connected algebraic group over an algebraic closure of a finite field $\mathbb{F}_p$ with Steinberg endomorphism $F : G \to G$. See Sections 2–6 for further notation and the proofs.

Theorem 1.2 (Parametrization of Blocks). *Assume that G is simple simply connected of exceptional Lie type in characteristic p and $\ell \neq p$ a bad prime for G. Let $1 \neq s \in G^{*F}$ be a quasi-isolated ℓ'-element.*

(a) *There is a natural bijection*

$$b_{G^F}(L, \lambda) \longleftrightarrow (L, \lambda)$$

between ℓ-blocks of G^F in $\mathcal{E}_\ell(G^F, s)$ and pairs (L, λ) up to G^F-conjugation, where

(1) *L is an e-split Levi subgroup of G, with $e = e_\ell(q)$,*

(2) *$\lambda \in \mathcal{E}(L^F, s)$ is e-cuspidal, and*

(3) *λ is of quasi-central ℓ-defect.*

(b) *There is a defect group $P \leq N_G(L, \lambda)^F$ of $b_{G^F}(L, \lambda)$ with a normal series*

$$Z(L)^F_\ell \trianglelefteq D := C_P(Z(L)^F_\ell) \trianglelefteq P,$$

with quotients P/D isomorphic to a Sylow ℓ-subgroup of $W_{G^F}(L, \lambda)$ and $D/Z(L)^F_\ell$ isomorphic to a Sylow ℓ-subgroup of $L^F/Z(L)^F_\ell[L, L]^F$.

(c) *Here, $b_{G^F}(L, \lambda)$ has abelian defect if and only if $W_{G^F}(L, \lambda)$ is an ℓ'-group.*

(d) *Further, when $\ell \neq 2$ then $D = Z(L)^F_\ell$ in* (b) *and P is a Sylow ℓ-subgroup of the extension of $Z(L)^F_\ell$ by $W_{G^F}(L, \lambda)$.*

In [8], Bonnafé and Rouquier proved that every ℓ-block of a finite reductive group in characteristic different from ℓ is Morita equivalent, via Lusztig induction, to a quasi-isolated block of some Levi subgroup. This comparison theorem provides a crucial reduction in the proof of Theorem 1.1. But note that it is not known in general whether Morita equivalences preserve abelianess of defect groups. In our context, relying on previous results, mainly of Cabanes–Enguehard, we prove the following result:

Theorem 1.3 (Preservation of Abelian Defect Groups). *Let G be simple, simply connected in characteristic p and $\ell \neq p$ a prime. Let M be an F-stable Levi subgroup of G, and let b and c be Bonnafé–Rouquier corresponding ℓ-blocks of G^F and M^F respectively (see*

Definition 7.7). Let Z be a central ℓ-subgroup of G^F and let $\bar{b}$ and $\bar{c}$ be the images of b and c in G^F/Z and M^F/Z respectively. If either $\bar{b}$ or $\bar{c}$ has abelian defect groups, then the defect groups of $\bar{b}$ and $\bar{c}$ are isomorphic.

The above result should ideally follow from general properties of the bimodules inducing Bonnafé–Rouquier Morita equivalences, but our proof is different. In fact, one expects that if b and c are Bonnafé–Rouquier correspondents, then any defect group of c is a defect group of b — this is known to hold in many cases.

In order to prove Theorem 1.2 we apply a criterion of Cabanes and Enguehard (see Proposition 2.12 below) which allows one to determine the blocks if Lusztig induction from suitable Levi subgroups can be shown to satisfy a generalised Harish-Chandra theory. The following result is not only a crucial ingredient for our proofs but of independent interest:

Theorem 1.4 (e-Harish-Chandra Theory)**.** *Assume that G is simple simply-connected of exceptional Lie type in characteristic p and $\ell \neq p$ a bad prime for G. Let $s \in G^{*F}$ be a quasi-isolated ℓ'-element. Then with $e = e_\ell(q)$ we have:*

(a) *The sets $\mathcal{E}(G^F, (L, \lambda))$, where (L, λ) runs over a set of representatives of the G^F-conjugacy classes of e-cuspidal pairs of G below $\mathcal{E}(G^F, s)$, partition $\mathcal{E}(G^F, s)$.*

(b) *G^F satisfies an e-Harish-Chandra theory above each e-cuspidal pair (L, λ) of G^F below $\mathcal{E}(G^F, s)$ (see Definition 2.9 below).*

The case when $s = 1$, that is, the case of unipotent characters, was the main result in [10, Thm. 3.2] (where there was no restriction on the type of G, but ℓ was assumed to be large enough).

Finally, we use the previous results to characterise blocks of quasi-simple groups all of whose height zero characters have the same degree, thus completing a programme begun by Malle–Navarro [37], and continued by Gramain [26] for the case of spin-blocks of alternating groups:

Theorem 1.5 (Characterization of Nilpotent Blocks)**.** *Let S be a finite quasi-simple group and p a prime. Assume that B is a p-block of S all of whose height zero characters have the same degree. Then the defect group of B is abelian and thus B is nilpotent.*

The paper is organised as follows. In Section 2 we collect various results on groups of Lie type, Lusztig induction, blocks and Brauer pairs and we state our main criteria for block distribution and the structure of defect groups. In Sections 3–6 we determine the decomposition of Lusztig induction from suitable Levi subgroups in the Lusztig series belonging to quasi-isolated elements of exceptional groups of rank at least 4 and the block distribution in these series. Section 7 is devoted to showing Theorem 1.3. The remaining steps of the proof of (HZC1) are given in Section 8, see Theorem 8.9. Finally, in Section 9 we prove Theorem 1.5.

2. Background results and methods

Throughout this paper, ℓ denotes a prime number.

2.1. **Blocks and Brauer pairs.** Let G be a finite group and let $(K, \mathcal{O}, k)$ be a splitting modular system for G, i.e., $\mathcal{O}$ is a complete discrete valuation ring with residue field k of characteristic ℓ and field of fractions K such that k and K are splitting fields for all groups involved in G. Let $\mathrm{CF}(G, K)$ denote the set of K-valued class functions on G and let $\mathrm{Irr}(G)$ denote the subset of $\mathrm{CF}(G, K)$ consisting of irreducible characters of G. Let $\langle\ ,\ \rangle_G$ denote the standard inner product on $\mathrm{CF}(G, K)$.

By an ℓ-block of G we will mean a primitive idempotent of $Z(kG)$. By idempotent lifting, the canonical surjection of $\mathcal{O}$ onto k induces a bijection between the set of primitive idempotents of $Z(\mathcal{O}G)$ and primitive idempotents of $Z(kG)$, and this induces an orthogonal decomposition of the set of K-valued class functions $\mathrm{CF}(G, K)$ on G with respect to the standard inner product. For $f \in \mathrm{CF}(G, K)$ and b an ℓ-block of G, the projection of f onto the component of b in $\mathrm{CF}(G, K)$ is denoted by $b.f$, and we write $b = b_G(f)$ if $f = b.f$. This defines a partition $\mathrm{Irr}(G) = \coprod_b \mathrm{Irr}(b)$, where $\mathrm{Irr}(b) = \{\chi \in \mathrm{Irr}(G) \mid b.\chi = \chi\}$.

A *Brauer pair* of G (or *G-Brauer pair*) with respect to ℓ is a pair (Q, c), such that Q is an ℓ-subgroup of G and c is an ℓ-block of $C_G(Q)$. The set of G-Brauer pairs has a structure of a G-poset such that the following properties hold: If (Q, c) and (R, d) are Brauer pairs with $(R, d) \subseteq (Q, c)$, then $R \leq Q$, and for any Brauer pair (Q, c) and any subgroup R of Q, there is a unique Brauer pair (R, d) such that $(R, d) \subseteq (Q, c)$. In particular, for each Brauer pair (Q, c), there exists a unique ℓ-block, say b of G such that $(\{1\}, b) \subseteq (Q, c)$, and in this case we say that (Q, c) is a *b-Brauer pair* or that (Q, c) is *associated to b*. A Brauer pair (Q, c) is a b-Brauer pair if and only if $\mathrm{Br}_Q(b)c = c$, if and only if $\mathrm{Br}_Q(b)c \neq 0$, where $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$ denotes the Brauer homomorphism.

For an ℓ-block b of G, the subset of the set of Brauer pairs of G associated to b is closed under inclusion and under the action of G. For any Brauer pair (Q, c), $Z(Q)$ is contained in every defect group of c and (Q, c) is said to be *centric* (or *self-centralising*) if $Z(Q)$ is a defect group of c. A Brauer pair (Q, c) is maximal if and only if (Q, c) is centric and $N_G(Q, c)/QC_G(Q)$ is an ℓ'-group, where $N_G(Q, c)$ denotes the stabiliser in G of (Q, c). Further, (Q, c) is maximal if and only if Q is a defect group of the unique ℓ-block of G to which (Q, c) is associated. G acts transitively on the subset of maximal b-Brauer pairs.

If (Q, c) and (R, d) are Brauer pairs with $(R, d) \subseteq (Q, c)$, and such that R is normal in Q, then we write $(R, d) \trianglelefteq (Q, c)$.

For a more detailed exposition on Brauer pairs, we refer the reader to the monographs [47, §40], [2, Part IV], or to the original article of Alperin and Broué [1] — in the latter reference Brauer pairs are referred to as subpairs. Here we recall a few stray facts which will be used in the sequel.

Let R be an ℓ-subgroup of G and let H be a subgroup of G such that $RC_G(R) \leq H \leq N_G(R)$. Every central idempotent of kH is in $kC_G(R) = kC_H(R)$ (see [2, Part IV, Lemma 3.17]). Now let (R, d) be a G-Brauer pair and suppose that $RC_G(R) \leq H \leq N_G(R, d)$. Then, d is an ℓ-block of H. Further, for any subgroup Q of H containing R, $C_G(Q) = C_H(Q)$, the H-Brauer pairs with first component Q are the G-Brauer pairs with first component Q and for any block c of $C_H(Q) = C_G(Q)$, $(\{1\}, d) \subseteq (Q, c)$ as H-Brauer pairs if and only if $(R, d) \subseteq (Q, c)$ as H-Brauer pairs, if and only if $(R, d) \subseteq (Q, c)$ as G-Brauer pairs (see [2, Part IV, Lemma 3.18]). We will use these facts without further comment.

We will need a few facts about covering blocks. For $\tilde{G}$ a finite group containing G as normal subgroup, $\tilde{b}$ an ℓ-block of $\tilde{G}$ and b an ℓ-block of G, $\tilde{b}$ is said to cover b if $\tilde{b}b \neq 0$.

Lemma 2.1. *Let b be an ℓ-block of G and let $(A,u) \subseteq (D,v) \subseteq (P,w)$ be b-Brauer pairs such that D is maximal with respect to $D \leq AC_G(A)$ and P is maximal with respect to $P \leq N_G(A,u)$. Let $\tilde{G}$ be a finite group with $G \trianglelefteq \tilde{G}$. Then:*

(a) *D is a defect group of the block u of $AC_G(A)$ and P is a defect group of the block u of $N_G(A,u)$. Further, $D = P \cap AC_G(A)$ and P/D is isomorphic to a Sylow ℓ-subgroup of $N_G(A,u)/AC_G(A)$.*

(b) *Let $\tilde{b}$ be an ℓ-block of $\tilde{G}$ and $(A,\tilde{u})$ a $\tilde{b}$-Brauer pair. If $\tilde{u}$ covers u, then $\tilde{b}$ covers b.*

(c) *There exists an ℓ-block $\tilde{b}$ of $\tilde{G}$ covering b, and $\tilde{b}$-Brauer pairs $(A,\tilde{u}) \trianglelefteq (\tilde{P},y)$ such that $\tilde{P} \leq N_{\tilde{G}}(A,u)$, $\tilde{u}$ covers u, $\tilde{P} \cap G = P$ and $\tilde{P}/P$ is isomorphic to a Sylow ℓ-subgroup of $N_{\tilde{G}}(A,u)/N_G(A,u)$.*

Proof. By [2, Part IV, Lemma 3.18], (D,v) is a maximal $AC_G(A)$-Brauer pair, and is associated to u so D is a defect group of u. Similarly, P is a defect group of u as block of $N_G(A,u)$. Consider the normal inclusion $AC_G(A) \trianglelefteq N_G(A,u)$. As u is the only block of $kN_G(A,u)$ covering the block u of $kC_G(A)$, by covering block theory, $D = P \cap AC_G(A)$ and P/D is isomorphic to a Sylow ℓ-subgroup of $N_G(A,u)/AC_G(A)$ (see [42, Ch. 5, Thm. 5.16]). This proves (a).

Let $\tilde{b}$ and $\tilde{u}$ be as in (b) and suppose that $\tilde{u}$ covers u. Then, $\mathrm{Br}_A(\tilde{b})\tilde{u} = \tilde{u}$, $\mathrm{Br}_A(b)u = u$ and $\tilde{u}u \neq 0$. Since Br_A is an algebra homomorphism, and $\tilde{u}$ is central in $C_{\tilde{G}}(A)$, $\mathrm{Br}_A(\tilde{b}b)\tilde{u}u = \tilde{u}u \neq 0$, and it follows that $\mathrm{Br}_A(\tilde{b}b) \neq 0$, whence $\tilde{b}b \neq 0$, proving (b).

For (c), consider the normal inclusion $N_G(A,u) \trianglelefteq N_{\tilde{G}}(A,u)$. By (a), u is a block of $N_G(A,u)$ with defect group P. So, again by [42, Ch. 5, Thm. 5.16], there exists a block u' of $N_{\tilde{G}}(A,u)$ covering u such that u' has a defect group $\tilde{P} \leq N_{\tilde{G}}(A,u)$ with $\tilde{P} \cap G = \tilde{P} \cap N_G(A,u) = P$ and $\tilde{P}/P$ isomorphic to a Sylow ℓ-subgroup of $N_{\tilde{G}}(A,u)/N_G(A,u)$. Now, $\tilde{P}$ being a defect group of u' implies that $\mathrm{Br}_{\tilde{P}}(u') \neq 0$. Also, u is the unique block of $N_G(A,u)$ covered by u', hence $u'u = u'$. So,

$$\mathrm{Br}_{\tilde{P}}(u')\mathrm{Br}_{\tilde{P}}(u) = \mathrm{Br}_{\tilde{P}}(u') \neq 0,$$

whence $\mathrm{Br}_{\tilde{P}}(u) \neq 0$.

Now consider the normal inclusion $C_G(A) \trianglelefteq C_{\tilde{G}}(A)$. Let $\mathcal{U}$ be the set of ℓ-blocks of $C_{\tilde{G}}(A)$ covering u. Since $\tilde{P}$ normalises $C_{\tilde{G}}(A)$ and stabilises u, $\tilde{P}$ acts by conjugation on $\mathcal{U}$. In particular, $\sum_{f\in\mathcal{U}} f \in (k\tilde{G})^{\tilde{P}}$. Also, $u(\sum_{f\in\mathcal{U}} f) = u$. So,

$$\mathrm{Br}_{\tilde{P}}(u)\mathrm{Br}_{\tilde{P}}(\sum_{f\in\mathcal{U}} f) = \mathrm{Br}_{\tilde{P}}(u) \neq 0.$$

Since $\tilde{P}$ permutes the elements of $\mathcal{U}$, the above equation yields that there is an element say $\tilde{u}$, of $\mathcal{U}$ such that $\tilde{u} \in (k\tilde{G})^{\tilde{P}}$ and $\mathrm{Br}_{\tilde{P}}(\tilde{u}) \neq 0$. Consequently, there exists a $\tilde{G}$-Brauer pair $(\tilde{P},y)$ such that $(A,\tilde{u}) \leq (\tilde{P},y)$. Let $\tilde{b}$ be the unique ℓ-block of $\tilde{G}$ such that $(A,\tilde{u})$ is a $\tilde{b}$-Brauer pair. Since $\tilde{u}$ covers u, (b) gives that $\tilde{b}$ covers b. This proves (c). ☐

Let $\chi \in \mathrm{Irr}(G)$ and let θ be a linear character of G. Then $\theta\otimes\chi$ is an irreducible character of G and the map $\chi \mapsto \theta \otimes \chi$ is a permutation on $\mathrm{Irr}(G)$ which respects ℓ-blocks: for any ℓ-block b of G, the set $\{\theta \otimes \chi \mid \chi \in \mathrm{Irr}(b)\}$ is the set of irreducible characters of an ℓ-block of G, which we will denote by $\theta \otimes b$. Denoting also by θ the restriction of θ to any subgroup of G, the map $(Q, f) \mapsto (Q, \theta \otimes f)$ is a G-poset isomorphism between the set of b-Brauer pairs and the set of $\theta \otimes b$-Brauer pairs.

Lemma 2.2. *Let $\tilde{G}$ be a finite group such that $G \trianglelefteq \tilde{G}$, b an ℓ-block of G and $\tilde{b}$ an ℓ-block of $\tilde{G}$ covering b. Suppose that $\tilde{G}/G$ is abelian. Then:*

(a) *Any ℓ-block of $\tilde{G}$ covering b is of the form $\theta \otimes \tilde{b}$, where θ is a linear character of $\tilde{G}/G$.*

(b) *Assume that b has a defect group $Z \leq Z(\tilde{G})$. Suppose that the unique character $\chi \in \mathrm{Irr}(b)$ containing Z in its kernel extends to its stabiliser I in $\tilde{G}$. Then, $\tilde{b}$ is nilpotent, and if D is a defect group of $\tilde{b}$, then $D \leq I$, $D \cap G = Z$ and D/Z is isomorphic to the Sylow ℓ-subgroup of I/G. Moreover, there are $|I : G|_{\ell'}$ ℓ-blocks of $\tilde{G}$ covering b.*

Proof. Let b' be an ℓ-block of $\tilde{G}$ covering b and let $\chi \in \mathrm{Irr}(b)$. Then there exists $\eta \in \mathrm{Irr}(\tilde{b})$ and $\eta' \in \mathrm{Irr}(b')$ such that η and η' both cover b (see [42, Ch. 5, Lemma 5.8(ii)]). But since $\tilde{G}/G$ is abelian, $\eta' = \theta \otimes \eta$ for some linear character θ of $\tilde{G}/G$. This proves (a).

Suppose that the hypotheses of (b) hold. Then I is the stabiliser in $\tilde{G}$ of b. Induction induces a bijection between the set of ℓ-blocks of I covering b and the set of ℓ-blocks of $\tilde{G}$ covering b; corresponding blocks under this bijection are source algebra equivalent, (see for instance [30, Sec. 2]) and in particular the correspondence preserves the nilpotency of blocks and corresponding blocks have common defect groups. Hence, we may assume that $I = \tilde{G}$.

Since Z is a central subgroup of $\tilde{G}$, the canonical surjection of $\tilde{G}$ onto $\tilde{G}/Z$ induces a bijection between the ℓ-blocks of $\tilde{G}$ and the ℓ-blocks of $\tilde{G}/Z$ (see [42, Ch. 5, Thm. 8.11]), and also between the ℓ-blocks of G and the ℓ-blocks of G/Z; for any block d of $\tilde{G}$ (respectively G) denote by $\bar{d}$ the corresponding block of $\tilde{G}/Z$ (respectively G/Z). Then a block b' of $\tilde{G}$ covers b if and only if $\bar{b}'$ covers $\bar{b}$, b' is nilpotent if and only if $\bar{b}'$ is nilpotent, and the defect groups of $\bar{b}'$ are of the form D/Z, where D is a defect group of b'. Further, χ extends to an irreducible character of $\tilde{G}$, so χ considered as an element of $\tilde{G}/Z$ extends to an irreducible character of $\tilde{G}/Z$.

Thus, we may assume that $Z = 1$. If $\tilde{G}/G$ is an ℓ'-group, the claim is immediate. Thus we may also assume that $\tilde{G}/G$ has ℓ-power order. Then there is a unique block $\tilde{b}$ lying above b. By assumption, χ extends to $\tilde{G}$, so a defect group D of $\tilde{b}$ is isomorphic to $\tilde{G}/G$ and satisfies $D \cap G = 1$. In particular, $\tilde{b}$ is nilpotent. □

2.2. **Lusztig series and ℓ-blocks.** We set up the following notation. Let G be a connected reductive algebraic group over an algebraic closure of a finite field $\mathbb{F}_p$ with a Steinberg endomorphism $F : G \to G$, and G^F the finite group of fixed points. We are interested in the ℓ-blocks of G^F, where ℓ is a prime number different from the defining characteristic p of G. We first recall several useful results.

Let T be an F-stable maximal torus of G, and G^* a group in duality with G with respect to T, with corresponding Steinberg endomorphism again denoted by F (see [19, 13.10]). We denote by q the absolute value of the eigenvalues of F on the character group of T, an integral power of $\sqrt{p}$. By the fundamental results of Lusztig, the set of complex irreducible characters of G^F is a disjoint union of rational Lusztig series $\mathcal{E}(G^F, s)$, where

s runs over semisimple elements of G^{*F} up to conjugation. Lusztig series are compatible with block theory in the following sense (see [15, Thm. 9.12]):

Theorem 2.3 (Broué–Michel, Hiss). *Let $s \in G^{*F}$ be a semisimple ℓ'-element. Then:*
(a) *The set*
$$\mathcal{E}_\ell(G^F, s) := \bigcup_{t \in C_{G^*}(s)^F_\ell} \mathcal{E}(G^F, st)$$
is a union of ℓ-blocks (where t runs over the ℓ-elements in $C_{G^*}(s)^F$ up to conjugation).
(b) *Any ℓ-block in $\mathcal{E}_\ell(G^F, s)$ contains a character from $\mathcal{E}(G^F, s)$.*

Thus, to parametrise the ℓ-blocks of G^F, it suffices to decompose $\mathcal{E}(G^F, s)$ into ℓ-blocks, for all ℓ'-elements $s \in G^{*F}$.

We'll also use the following notation for the union of Lusztig series corresponding to ℓ'-elements:
$$\mathcal{E}(G^F, \ell') := \bigcup_{\ell'\text{-elements } s \in G^{*F}} \mathcal{E}(G^F, s).$$

2.3. **Quasi-central defect and defect groups.** In this subsection we develop some results which will allow us to identify the defect groups of blocks.

Definition 2.4. Let $\zeta \in \mathcal{E}(G^F, \ell')$. We say that ζ is *of central ℓ-defect* if $|G^F|_\ell = \zeta(1)_\ell |Z(G)^F|_\ell$ and that ζ is *of quasi-central ℓ-defect* if some (and hence any) character of $[G, G]^F$ covered by ζ is of central ℓ-defect.

The above definition makes sense since if $\zeta \in \mathcal{E}(G^F, \ell')$, then any character of $[G, G]^F$ covered by ζ is in $\mathcal{E}([G, G]^F, \ell')$. The following are some properties of characters of quasi-central and central ℓ-defect. They rely on Lusztig's result [34, Prop. 10] on the restriction of irreducible characters under regular embeddings being multiplicity free.

Proposition 2.5. *Let $\zeta \in \mathcal{E}(G^F, \ell')$, $A = Z(G)^F_\ell$, and $A_0 = Z([G, G])^F_\ell$. Then:*
(a) *ζ is of quasi-central ℓ-defect if and only if $|[G, G]^F|_\ell = \zeta(1)_\ell |Z([G, G])^F|_\ell$.*
(b) *If ζ is of central ℓ-defect, then ζ is of quasi-central ℓ-defect.*
(c) *ζ is of central ℓ-defect if and only if A is a defect group of $b_{G^F}(\zeta)$.*
Suppose that ζ is of quasi-central ℓ-defect. Then:
(d) *$b_{G^F}(\zeta)$ is nilpotent.*
(e) *Any defect group D of $b_{G^F}(\zeta)$ contains A with D/A isomorphic to a Sylow ℓ-subgroup of $G^F/A[G, G]^F$ and $D \cap [G, G]^F = A_0$.*
(f) *$\mathcal{E}(G^F, \ell') \cap \mathrm{Irr}(b_{G^F}(\zeta)) = \{\zeta\}$.*
(g) *ζ is of central ℓ-defect if and only if $G^F/A[G, G]^F$ is an ℓ'-group.*

Proof. Let ζ_0 be an irreducible constituent of the restriction of ζ to $[G, G]^F$ and let I be the stabiliser in G^F of ζ_0. Since $\zeta_0 \in \mathcal{E}([G, G]^F, \ell')$, the index of I in G^F is prime to ℓ. On the other hand, by [34, Prop. 10], ζ_0 extends to an irreducible character of I. Thus, $\zeta_0(1)_\ell = \zeta(1)_\ell$, proving (a) and (b). Since A is a central ℓ-subgroup of G and $\zeta \in \mathcal{E}(G^F, \ell')$, A is in the kernel of ζ, from which (c) is immediate.

Assume till the end of the proof that ζ is of quasi-central ℓ-defect. By (c), $b_{[G,G]^F}(\zeta_0)$ has defect group A_0. Further, since $G = Z(G)[G, G]$ and $A_0 \le Z([G, G])^F$, A_0 is central in G^F. So the hypotheses of Lemma 2.2(b) are satisfied for the normal subgroup $[G, G]^F$

of G^F and the blocks $b_{[G,G]^F}(\zeta_0)$ and $b_{G^F}(\zeta)$, proving (d) and (e). The index of I in G^F is prime to ℓ, so again by Lemma 2.2(b) there are $|I : [G,G]^F|_{\ell'}$ ℓ-blocks of G^F covering $b_{[G,G]^F}(\zeta_0)$. Also, there are $|I : [G,G]^F|_{\ell'}$ elements of $\mathcal{E}(G^F, \ell')$ covering ζ_0. Now (f) follows from Theorem 2.3 and (g) follows from (c) and (e). □

The next results will be our main tools for the identification of defect groups. We first derive an easy upper bound for the orders of defect groups:

Lemma 2.6. *Let $s \in G^{*F}$ be a semisimple ℓ'-element.*

(a) *The defect groups of any ℓ-block in $\mathcal{E}_\ell(G^F, s)$ have order at most $|C_{G^{*F}}(s)|_\ell$.*
(b) *There exists an ℓ-block in $\mathcal{E}_\ell(G^F, s)$ whose defect groups have order $|C_{G^{*F}}(s)|_\ell$.*

In particular, if $\mathcal{E}_\ell(G^F, s)$ is a single ℓ-block, then the defect groups of this block have order $|C_{G^{*F}}(s)|_\ell$.

Proof. Let t be an ℓ-element of $C_{G^{*F}}(s)$ and let $\chi \in \mathcal{E}(G^F, st)$. By the Jordan decomposition of characters, there exists an irreducible (unipotent) character ψ of $C_{G^{*F}}(st)$ such that

$$\frac{|G^F|_\ell}{\chi(1)_\ell} = \frac{|C_{G^{*F}}(st)|_\ell}{\psi(1)_\ell}.$$

In particular,

$$\frac{|G^F|_\ell}{\chi(1)_\ell} \le |C_{G^{*F}}(st)|_\ell \le |C_{G^{*F}}(s)|_\ell.$$

This proves the first part. If $\chi \in \mathcal{E}(G^F, s)$ corresponds to the trivial character of $C_{G^{*F}}(s)$, then by the above formula, the ℓ-defect of χ is $|C_{G^{*F}}(s)|_\ell$, hence the block containing χ has defect at least $|C_{G^{*F}}(s)|_\ell$. This proves (b). □

Proposition 2.7. *Let $L \le G$ be an F-stable Levi subgroup and $A = Z(L)^F_\ell$, $A_0 = Z([L,L])^F_\ell$. Suppose that*

$$L = C^\circ_G(A), \qquad L^F = C_{G^F}(A).$$

For $\lambda \in \mathcal{E}(L^F, \ell')$ of quasi-central ℓ-defect let $u = b_{L^F}(\lambda)$ and let b be the block of G^F such that (A, u) is a b-Brauer pair. Then

(a) *$N_{G^F}(A) = N_{G^F}(L)$, $N_{G^F}(A, u) = N_{G^F}(L, \lambda)$, and $N_{G^F}(A, u)/C_{G^F}(A) = W_{G^F}(L, \lambda)$, where $W_{G^F}(L, \lambda) := N_{G^F}(L, \lambda)/L^F$.*

Let $(A, u) \subseteq (D, v) \subseteq (P, w)$ be b-Brauer pairs such that D is maximal with respect to $D \le C_{G^F}(A)$ and P is maximal with respect to $P \le N_{G^F}(L, \lambda)$. Then:

(b) *D/A is isomorphic to a Sylow ℓ-subgroup of $L^F/A[L,L]^F$, $P \cap L^F = D$ and P/D is isomorphic to a Sylow ℓ-subgroup of $W_{G^F}(L, \lambda)$.*

Let $s \in G^{*F}$ be an ℓ'-element such that $\mathrm{Irr}(b) \subseteq \mathcal{E}_\ell(G^F, s)$. *Then:*

(c) *If*

$$|C_{G^{*F}}(s)|_\ell = |Z^\circ(L)^F_\ell| \cdot |A_0| \cdot |W_{G^F}(L, \lambda)|_\ell,$$

then P is a defect group of b.

(d) *If A is characteristic in P, then P is a defect group of b.*
(e) *If the defect groups of b are abelian, then ℓ does not divide $|W_{G^F}(L, \lambda)|$.*
(f) *If $A = D$ and ℓ does not divide $|W_{G^F}(L, \lambda)|$, then A is a defect group of b.*
(g) *If $Z^\circ(L)^F \cap [L,L]^F$ is an ℓ'-group, then $A = D$.*

Proof. Since $L = C_G^\circ(A)$, $N_{G^F}(A) \le N_{G^F}(L)$ and since $A = Z(L)_\ell^F$, $N_{G^F}(L) \le N_{G^F}(A)$. Thus, $N_{G^F}(A) = N_{G^F}(L)$. By Proposition 2.5(f), λ is the unique element of $\mathcal{E}(L^F, \ell') \cap \mathrm{Irr}(u)$. Since conjugation by elements of $N_{G^F}(L)$ stabilises $\mathcal{E}(L^F, \ell')$, we get that

$$N_{G^F}(A, u) = N_{G^F}(L, u) = N_{G^F}(L, \lambda).$$

This proves (a). From this, (b) follows by Lemma 2.1 and Proposition 2.5(e).

By (b) we have

$$|P| = \frac{|A| \cdot |L^F|_\ell \cdot |W_{G^F}(L, \lambda)|_\ell}{|A[L, L]^F|_\ell}.$$

Now, $|L^F| = |Z^\circ(L)^F| \cdot |[L, L]^F|$ and as pointed out in the proof of Proposition 2.5, $A_0 = A \cap [L, L]^F$. Hence

$$|P| = |Z^\circ(L)_\ell^F| \cdot |A_0| \cdot |W_{G^F}(L, \lambda)|_\ell$$

and (c) follows from Lemma 2.6(a).

Suppose that A is characteristic in P. Let $(P, w) \subseteq (R, f) \subseteq (S, j)$ be b-Brauer pairs with (S, j) maximal and $R = N_S(P)$. Since R normalises A, $P \le R \le N_{G^F}(A, u) = N_{G^F}(L, \lambda)$. So, by maximality of P, $R = P$, whence $S = P$, proving (d).

Part (e) is immediate from part (b).

If $A = D$ and $W_{G^F}(L, \lambda)$ is an ℓ'-group, then $P = A$, which means that if (S, j) is any maximal Brauer pair containing (A, u), then $N_S(A) = A$. But this implies that (A, u) is maximal, proving (f).

If $Z^\circ(L)^F \cap [L, L]^F$ is an ℓ'-group, then from the equality $|L^F| = |Z^\circ(L)^F| \cdot |[L, L]^F|$ it follows that $L^F/Z^\circ(L)^F[L, L]^F$ and hence $L^F/A[L, L]^F$ is an ℓ'-group. But by part (a), D/A is isomorphic to a Sylow ℓ-subgroup of $L^F/A[L, L]^F$. This proves (g). $\square$

2.4. **Lusztig induction and e-Harish-Chandra theory.** It is known that the ℓ-blocks of G^F are in close relation with Lusztig induction. For any F-stable Levi subgroup L of a (not necessarily F-stable) parabolic subgroup P of G Lusztig defines linear maps

$$R_{L\subset P}^G : \mathbb{Z}\mathrm{Irr}(L^F) \longrightarrow \mathbb{Z}\mathrm{Irr}(G^F),$$
$${}^*R_{L\subset P}^G : \mathbb{Z}\mathrm{Irr}(G^F) \longrightarrow \mathbb{Z}\mathrm{Irr}(L^F),$$

adjoint to each other with respect to the standard scalar product on complex characters. This Lusztig induction enjoys the following important properties:

Theorem 2.8. *Let L be an F-stable Levi subgroup of a parabolic subgroup P of G.*

(a) *If $M \le L$ is an F-stable Levi subgroup of a parabolic subgroup Q of P then*

$$R_{M\subset Q}^G = R_{L\subset P}^G \circ R_{M\subset Q\cap L}^L.$$

(b) *Let L^* be an F-stable Levi subgroup of G^* in duality with L. For any semisimple element $s \in L^*$, $R_{L\subset P}^G$ restricts to a linear map*

$$R_{L\subset P}^G : \mathbb{Z}\mathcal{E}(L^F, s) \longrightarrow \mathbb{Z}\mathcal{E}(G^F, s).$$

(c) *If P is F-stable, then*

$$R_{L\subset P}^G = \mathrm{Ind}_{P^F}^{G^F} \circ \mathrm{Infl}_{L^F}^{P^F}.$$

(d) *The Mackey formula holds for $R_{L\subset P}^G$ except possibly if G^F has a simple component ${}^2E_6(2)$, $E_7(2)$ or $E_8(2)$.*

Proof. See [19, 11.5] for (a), part (b) is immediate from this and the definition of the Lusztig series, for (c) see [19, §11], and the recent paper of Bonnafé–Michel [7] for (d). □

Note that, as a formal consequence of the validity of the Mackey formula, Lusztig induction is independent of the choice of parabolic subgroup P containing L (except possibly in the groups excluded in Theorem 2.8(d)). We will henceforth just write R_L^G.

An F-stable torus T of G is called an *e-torus* if it splits completely over $\mathbb{F}_{q^e}$ but does not split over any smaller field. Equivalently, there is $a \geq 0$ such that $|T^{F^k}| = \Phi_e(q^k)^a$ for all $k \geq 1$, where Φ_e denotes the eth cyclotomic polynomial. The centralisers of e-tori of G are called *e-split Levi subgroups.* (Note that these are indeed Levi subgroups, which are F-stable.) A character $\chi \in \mathrm{Irr}(G^F)$ is called *e-cuspidal* if ${}^*R_L^G(\chi) = 0$ for every e-split proper Levi subgroup L of G. A pair (L, λ) consisting of an e-split Levi subgroup L and an e-cuspidal character $\lambda \in \mathrm{Irr}(L^F)$ is then called an *e-cuspidal pair.* Given an e-cuspidal pair (L, λ), we write

$$\mathcal{E}(G^F, (L, \lambda)) := \{\chi \in \mathrm{Irr}(G^F) \mid \langle {}^*R_L^G(\chi), \lambda\rangle \neq 0\}$$

for the set of constituents of $R_L^G(\lambda)$. This is called the *e-Harish-Chandra series of G^F above* (L, λ).

Definition 2.9. We say that R_L^G *satisfies an e-Harish-Chandra theory above the e-cuspidal pair* (L, λ) if there exists a collection of isometries

$$I_{(L,\lambda)}^M : \mathbb{Z}\mathrm{Irr}(W_{M^F}(L, \lambda)) \to \mathbb{Z}\mathcal{E}(M^F, (L, \lambda))\,,$$

where M runs over the set of all e-split Levi subgroups of G containing L, such that

(1) for all M we have

$$R_M^G \circ I_{(L,\lambda)}^M = I_{(L,\lambda)}^G \circ \mathrm{Ind}_{W_{M^F}(L,\lambda)}^{W_{G^F}(L,\lambda)}\,;$$

(2) the collection $\left(I_{(L,\lambda)}^M\right)_{M,(L,\lambda)}$ is stable under the conjugation action by W_{G^F}; and

(3) $I_{(L,\lambda)}^L$ maps the trivial character of the trivial group $W_{L^F}(L, \lambda)$ to λ.

The following is shown in [10, Prop. 3.15 and Thm. 3.11]:

Proposition 2.10. *Assume that R_L^G satisfies an e-Harish-Chandra theory above (L, λ). Then for any e-split Levi subgroup $L \leq H \leq G$ and any $\chi \in \mathrm{Irr}(H^F)$ with $\langle R_L^H(\lambda), \chi\rangle \neq 0$ we have:*

(a)

$${}^*R_L^H(\chi) = \langle {}^*R_L^H(\chi), \lambda\rangle_{L^F} \sum_{g \in N_{H^F}(L)/N_{H^F}(L,\lambda)} {}^g\lambda.$$

(b) *Every constituent ψ of $R_H^G(\chi)$ is a constituent of $R_L^G(\lambda)$.*

2.5. **ℓ-adapted Levi subgroups and Cabanes' criterion.** The results in this section are adaptations and extensions of a powerful criterion of Cabanes, formulated in [20, Prop. 3], which provides a strong relation between the explicit decomposition of the Lusztig induction functor R_L^G for suitable Levi subgroups L of G and the subdivision of $\mathcal{E}_\ell(G^F, s)$ into ℓ-blocks through the inclusion of Brauer pairs.

For an ℓ-element z of G^F, we write d^{z,G^F} for the generalised decomposition map which sends a K-valued class function f of G^F to the class function $d^{z,G^F}(f)$ on $C_{G^F}(z)$ by the rule $d^{z,G^F}(f)(zy) = f(zy)$ if $y \in C_{G^F}(z)$ has order prime to ℓ and $d^{z,G^F}(f)(zy) = 0$ if the order of $y \in C_{G^F}(z)$ is divisible by ℓ. The map d^{1,G^F} is the usual decomposition map. Note that if A is an abelian ℓ-subgroup of G^F contained in a maximal torus of G, then $C_G(A)/C_G^\circ(A)$ is an ℓ-group (see [38, Prop. 14.20], [13, Prop. 2.1(i)]). So, $C_{G^F}(A)/C_G^\circ(A)^F$ is an ℓ-group, and hence, each ℓ-block of $C_G^\circ(A)^F$ is covered by a unique block of $C_{G^F}(A)$. We will use this fact without further comment.

Lemma 2.11. *Let L be an F-stable Levi subgroup of G, let $\lambda \in \mathcal{E}(L^F,\ell')$ and $\chi \in \mathcal{E}(G^F,\ell')$. Suppose that $\langle \chi, R_L^G(\lambda)\rangle \neq 0$ and $\langle {}^*R_L^G(\chi), d^{1,L^F}(\lambda)\rangle \neq 0$. Then, for any $z \in Z(L)_\ell^F$, there exists an irreducible constituent ϕ of $R_L^H(\lambda)$, where $H := C_G^\circ(z)$, such that denoting by $\tilde{b}$ the unique block of $C_{G^F}(z)$ covering the block containing ϕ, $(\langle z\rangle, \tilde{b})$ is a $b_{G^F}(\chi)$-Brauer pair.*

Proof. We have

$$d^{1,L^F}({}^*R_L^G(\chi)) = d^{z,L^F}({}^*R_L^G(\chi)) = {}^*R_L^H(d^{z,G^F}(\chi)),$$

the first equality holding since z is a central ℓ-element of L^F whereas χ is in an ℓ'-series, and the second because of the commutation of Lusztig restriction and generalised decomposition maps (see [15, Thm. 21.4]). It follows that

$$\langle d^{z,G^F}(\chi), R_L^H(\lambda)\rangle \neq 0.$$

Now the result follows by Brauer's second main theorem and the fact that the index of $H = C_G^\circ(z)$ in $C_G(z)$ is a power of ℓ. □

Proposition 2.12. *Let L be an F-stable Levi subgroup of G and let $\lambda \in \mathcal{E}(L^F,\ell')$. Let Z be a subgroup of $Z(L)_\ell^F$ and $\{z_1,\ldots,z_m\}$ a generating set for Z. Set $H_i = C_G^\circ(z_1,\ldots,z_i)$, $1 \leq i \leq m$, and $H_0 = G$. Suppose the following:*

(1) *For any i, $0 \leq i \leq m-1$, and any character $\chi \in \mathrm{Irr}(H_i^F)$ with $\langle R_L^{H_i}(\lambda), \chi\rangle_{H_i^F} \neq 0$ we have $\langle d^{1,L^F}(\lambda), {}^*R_L^{H_i}(\chi)\rangle_{L^F} \neq 0$.*
(2) *The irreducible constituents of $R_L^{H_m}(\lambda)$ lie in a single ℓ-block of H_m^F.*

Then, for all i, $0 \leq i \leq m$, there exists a unique block, say b_i of H_i^F such that all constituents of $R_L^{H_i}(\lambda)$ lie in b_i. Further, letting $\tilde{b}_i$ be the unique block of $C_G(z_1,\ldots,z_i)^F$ covering b_i for $1 \leq i \leq m$, we have inclusions of Brauer pairs

$$(\{1\}, b_0) \subseteq (\langle z_1\rangle, \tilde{b}_1) \subseteq \ldots \subseteq (Z, \tilde{b}_m).$$

Proof. Proceed by induction on m. Suppose first that $m = 1$, and let b_1 be the block of H_1^F in which all constituents of $R_L^{H_1}(\lambda)$ lie. By (1) and Lemma 2.11, for any irreducible constituent χ of $R_L^G(\lambda)$, we have an inclusion of Brauer pairs

$$(\{1\}, b_{G^F}(\chi)) \subseteq (\langle z_1\rangle, \tilde{b}_1) = (Z, \tilde{b}_1).$$

Now by the uniqueness of inclusion of Brauer pairs it follows that $b_{G^F}(\chi) = b_{G^F}(\chi')$ for any irreducible constituents χ, χ' of $R_L^G(\lambda)$.

Now suppose $m>1$. Since $H_m=C^\circ_{H_{m-1}}(z_m)$, by the previous argument there exists a unique block b_{m-1} of H_{m-1}^F such that all constituents of $R_L^{H_{m-1}}(\lambda)$ lie in b_{m-1} and there is an inclusion of H_{m-1}^F-Brauer pairs

$$(\{1\},b_{m-1})\subseteq(\langle z_m\rangle,b'_m),$$

where b'_m is the unique block of $C_{H_{m-1}^F}(z_m)$ covering the block b_m of H_m^F (note that $C_{H_{m-1}}(z_m)$ may be a proper subgroup of $C_G(z_1,\ldots,z_m)$). This yields an inclusion of $C_G(z_1,\ldots,z_{m-1})^F$-Brauer pairs

$$(\{1\},\tilde{b}_{m-1})\subseteq(\langle z_m\rangle,\tilde{b}_m),$$

and hence we have the inclusion of G^F-Brauer pairs

$$(\langle z_1,\ldots,z_{m-1}\rangle,\tilde{b}_{m-1})\subseteq(Z,\tilde{b}_m).$$

The result now follows by induction since we have shown above that all constituents of $R_L^{H_{m-1}}(\lambda)$ lie in the same block. ☐

The following gives sufficient criteria for condition (2) of Proposition 2.12 to hold.

Proposition 2.13. *In the notation of Proposition 2.12 condition (2) is satisfied for any $\lambda\in\mathcal{E}(L^F,\ell')$ if one of the following holds:*
(1) $L=C^\circ_G(Z)$; or
(2) $\ell=2$ and the simple components of $C^\circ_G(Z)$ are of classical type B, C or D.

Proof. The assertion is obvious in the first case since then $H_m=L$. In the second case, the assertion follows by [21, Prop. 1.5(b)]. ☐

We will make mostly use of condition (1) above which has been checked in many cases by Enguehard [20] for the choice $Z=Z(L)^F_\ell$.

Now we develop sufficient criteria for condition (1) of Proposition 2.12 to hold.

Definition 2.14. Let $L\le G$ be an e-split Levi subgroup. We say that L is *(e,ℓ)-adapted*, if there exist generators $Z(L)^F_\ell=\langle z_1,\ldots,z_m\rangle$ such that $C^\circ_G(z_1,\ldots,z_i)$ is an e-split Levi subgroup of G for all $1\le i\le m$.

Proposition 2.15. *Let $e\ge1$ and let (L,λ) be an e-cuspidal pair such that $\lambda\in\mathcal{E}(L^F,\ell')$. Assume that R_L^G satisfies an e-Harish-Chandra theory above λ. Then for any e-split Levi subgroup $L\le H\le G$ and any $\chi\in\mathrm{Irr}(H^F)$ such that $\langle R_L^H(\lambda),\chi\rangle\ne0$, we have*

$$\left\langle d^{1,L^F}({}^*R_L^H(\chi)),{}^*R_L^H(\chi)\right\rangle_{L^F}\ne0.$$

Further, if L is (e,ℓ)-adapted in G with respect to the generating set $\{z_1,\ldots,z_m\}$ of $Z(L)^F_\ell$, then condition (1) of Proposition 2.12 is satisfied with respect to $z_1,\ldots,z_m$.

Proof. Let $L\le H\le G$ be e-split and $\chi\in\mathrm{Irr}(H^F)$ such that $\langle R_L^H(\lambda),\chi\rangle\ne0$. By Proposition 2.10 we have

$${}^*R_L^H(\chi)=a\sum_{g\in N_{H^F}(L)/N_{H^F}(L,\lambda)}{}^g\lambda$$

with $a:=\langle\lambda,{}^*R_L^H(\chi)\rangle=\langle R_L^H(\lambda),\chi\rangle\ne0$, whence we see that ${}^*R_L^H(\chi)(1)\ne0$, and thus

$$\left\langle d^{1,L^F}({}^*R_L^H(\chi)),{}^*R_L^H(\chi)\right\rangle_{L^F}\ne0.$$

But,

$$\begin{aligned}\left\langle d^{1,L^F}({}^*R_L^H(\chi)), {}^*R_L^H(\chi)\right\rangle_{L^F} &= a \sum_{g\in N_{H^F}(L)/N_{H^F}(L,\lambda)} \left\langle d^{1,L^F}({}^g\lambda), {}^*R_L^H(\chi)\right\rangle_{L^F} \\ &= a|N_{H^F}(L) : N_{H^F}(L,\lambda)|\left\langle d^{1,L^F}(\lambda), {}^*R_L^H(\chi)\right\rangle_{L^F}.\end{aligned}$$

This proves the first assertion. The second part follows by repeatedly applying the first assertion to the cases $H = G$, respectively $H = C_G^\circ(z_1, \ldots, z_i)$, $1 \le i \le m$. □

The next result contains further useful criteria for condition (1) of Proposition 2.12.

Proposition 2.16. *Let L be an F-stable Levi subgroup of G, let $\lambda \in \mathcal{E}(L^F, \ell')$ and let b be the ℓ-block of L^F containing λ. Suppose that one of the following holds:*
(1) *L is a torus;*
(2) *ℓ is good for L and the ℓ-block of L^F containing λ is nilpotent;*
(3) *λ is of quasi-central ℓ-defect; or*
(4) $\mathrm{Irr}(b) \cap \mathcal{E}(L^F, \ell') = \{\lambda\}$.
Then for any character $\chi \in \mathrm{Irr}(G^F)$ with $\langle R_L^G(\lambda), \chi\rangle_{G^F} \neq 0$ we have

$$\langle d^{1,L^F}(\lambda), {}^*R_L^G(\chi)\rangle_{L^F} \neq 0.$$

Consequently, condition (1) of Proposition 2.12 holds for any subgroup Z of $Z(L)_\ell^F$ and any generating set $\{z_1, \ldots, z_m\}$ of Z.

Proof. (1) is a special case of (2) and of (3), and by Proposition 2.5(f), (3) is a special case of (4). Also, the second assertion follows by applying the first part with G replaced by H_i, $1 \le i \le m$. For any irreducible character χ of G^F, we have that

$$\left\langle d^{1,L^F}(\lambda), {}^*R_L^G(\chi)\right\rangle_{L^F} = \left\langle d^{1,L^F}(\lambda), b\cdot {}^*R_L^G(\chi)\right\rangle_{L^F} = \left\langle d^{1,L^F}(\lambda), d^{1,L^F}(b.{}^*R_L^G(\chi))\right\rangle_{L^F}.$$

Hence, in order to prove the proposition it suffices to show that if either (2) or (4) of the statement hold, then

$$\left\langle d^{1,L^F}(\lambda), d^{1,L^F}(b.{}^*R_L^G(\chi))\right\rangle_{L^F} \neq 0$$

for any $\chi \in \mathrm{Irr}(G^F)$ such that $\langle R_L^G(\lambda), \chi\rangle_{G^F} \neq 0$. Indeed, for such χ we have by adjunction

$${}^*R_L^G(\chi) = a\lambda + \sum_{\phi\in I} a_\phi \phi$$

for suitable $a \neq 0$, $a_\phi \in \mathbb{Z}$, where I is a subset of $\mathcal{E}(L^F, \ell') \setminus \{\lambda\}$. So,

$$b.{}^*R_L^G(\chi) = a\lambda + \sum_{\phi\in I'} a_\phi \phi$$

where $I' = I \cap \mathrm{Irr}(b)$.

Suppose first that (2) holds. Since ℓ is good for L, by [14, Thm. 1.7] the restriction of the right hand side of the above equation to the ℓ'-elements of L^F is non-zero. On the other hand, since b is nilpotent, $\{d^{1,L^F}(\lambda)\}$ is an ℓ-basic set for b. Hence $d^{1,L^F}(b.{}^*R_L^G(\chi)) = md^{1,L^F}(\lambda)$ for some non-zero m. The result follows since $\lambda(1) \neq 0$.

Now suppose (4) holds. Then again since $I' \subseteq \mathrm{Irr}(b) \cap \mathcal{E}(L^F, \ell') \setminus \{\lambda\}$, the hypothesis implies that $I' = \emptyset$. The result follows since $\lambda(1) \neq 0$. □

The previous results combine to give the following criterion which will be crucial for the proof of Theorem 1.2. Here, (L,λ) is said to *lie below* $\mathcal{E}(G^F,s)$ if the constituents of $R_L^G(\lambda)$ lie in $\mathcal{E}(G^F,s)$, or equivalently, if $\lambda\in\mathcal{E}(L^F,s)$.

Proposition 2.17. *Let $e\geq 1$ and let $s\in G^{*F}$ be a semisimple ℓ'-element. Suppose the following.*

(1) *The assertions of Theorem 1.4 hold for the set of e-cuspidal pairs below $\mathcal{E}(G^F,s)$.*
(2) *For any e-cuspidal pair (L,λ) below $\mathcal{E}(G^F,s)$ we have $L=C_G^\circ(Z(L)_\ell^F)$, and $L^F=C_{G^F}(Z(L)_\ell^F)$, and L is (e,ℓ)-adapted.*

Then for any e-split Levi subgroup H of G such that $H^F=C_{G^F}(Z(H)_\ell^F)$ the following holds:

For any ℓ-block b of H^F such that $\mathrm{Irr}(b)\cap\mathcal{E}(H^F,s)\neq\emptyset$, there exists a unique ℓ-block c of G^F such that for any $\chi\in\mathrm{Irr}(b)\cap\mathcal{E}(H^F,s)$ all constituents of $R_H^G(\chi)$ lie in c. Moreover, $(Z(H)_\ell^F,b)$ is a c-Brauer pair.

Proof. Let H be as in the statement and let (L,λ) be an e-cuspidal pair of G such that $L\leq H$. We claim that $L=C_H^\circ(Z(L)_\ell^F)$, $L^F=C_{H^F}(Z(L)_\ell^F)$, L is (e,ℓ)-adapted in H, and R_L^H satisfies an e-Harish-Chandra theory above λ. The first two assertions of our claim follow from

$$L\leq C_H^\circ(Z(L)_\ell^F)\leq C_G^\circ(Z(L)_\ell^F)\cap H=L\cap H=L.$$

Next, we show that L is (e,ℓ)-adapted in H. Let $Z(L)_\ell^F=\langle z_1,\ldots,z_m\rangle$ be a system of generators such that $L_i:=C_G^\circ(z_1,\ldots,z_i)$ is e-split, and for $1\leq i\leq m$ let T_i be the Sylow e-torus of $Z(L_i)$ and T the Sylow e-torus of $Z(H)$, so that $L_i=C_G(T_i)$ and $H=C_G(T)$. Since T is central in H, and $L\leq H$ is a Levi subgroup, we have $T\leq Z(L_i)$, so $T\leq T_i$ and

$$C_H^\circ(z_1,\ldots,z_i)\leq C_G(T_i)\cap C_H(T)=C_H(T_i)=H\cap L_i=C_H^\circ(z_1,\ldots,z_i)$$

for $1\leq i\leq m$. Finally, since any e-split Levi subgroup of H is an e-split Levi subgroup of G, R_L^H satisfies an e-Harish-Chandra theory over λ by condition (1), proving the claim.

Now let b be as in the statement and let $\chi\in\mathrm{Irr}(b)\cap\mathcal{E}(H^F,s)$. Let (L,λ) be an e-cuspidal pair of G such that $L\leq H$, and χ is a constituent of $R_L^H(\lambda)$. By Proposition 2.10(b) every constituent ψ of $R_H^G(\chi)$ is a constituent of $R_L^G(\lambda)$. As R_L^G satisfies an e-Harish-Chandra theory above λ, condition (1) of Proposition 2.12 holds for $\{z_1,\ldots,z_m\}$ by Proposition 2.15. Further, condition (2) holds by hypothesis and by Proposition 2.13. Hence we have an inclusion of G^F-Brauer pairs

$$(\{1\},b_{G^F}(\psi))\subseteq(Z(L)_\ell^F,b_{L^F}(\lambda)).$$

On the other hand, by using the claim one sees that the arguments in the preceding section all apply to H also, hence we have an inclusion of H^F-Brauer pairs

$$(\{1\},b)\subseteq(Z(L)_\ell^F,b_{L^F}(\lambda)).$$

Since by hypothesis $H^F=C_{G^F}(Z(H)_\ell^F)$, this also yields an inclusion of G^F-Brauer pairs

$$\big(Z(H)_\ell^F,b\big)\subseteq(Z(L)_\ell^F,b_{L^F}(\lambda)).$$

Let c be the unique block of G^F such that we have an inclusion of G^F-Brauer pairs

$$(\{1\},c)\subseteq\big(Z(H)_\ell^F,b\big)\,.$$

By transitivity and uniqueness of inclusion of Brauer pairs, we get that $c = b_{G^F}(\psi)$. This proves the result. □

3. The quasi-isolated blocks in $F_4(q)$

In this section we prove Theorems 1.2 and 1.4 on quasi-isolated blocks of simple groups of type F_4.

For this recall that a semisimple element s of a connected reductive group G is called *quasi-isolated* if its centraliser $C_G(s)$ is not contained in any proper Levi subgroup of G. Correspondingly, a *quasi-isolated ℓ-block* is a block lying in the Lusztig series parametrised by a quasi-isolated ℓ'-elements of the dual group.

By earlier results on blocks (see Remark 6.12) the decomposition of $\mathcal{E}_\ell(G^F, s)$ into ℓ-blocks of G^F is known except when ℓ is a bad prime for G and $s \neq 1$ is a quasi-isolated ℓ'-element of G^*, an exceptional group of adjoint type. The various (ℓ, s) will be treated case-by-case in Sections 3–6, so to start we need to recall the classification of quasi-isolated elements in exceptional groups of adjoint type from [5, Prop. 4.9 and Table 3].

Proposition 3.1 (Bonnafé)**.** *Let G be a simple exceptional algebraic group of adjoint type and of rank at least 4. Then the conjugacy classes of quasi-isolated elements s whose order is not divisible by all bad primes for G, the root system of their centraliser $C_G(s)$, the group of components $A(s) := C_G(s)/C_G^\circ(s)$ and the automiser $A(C) := N_G(C_G(s))/C_G^\circ(s)$ are given in Table 1.*

Table 1. Quasi-isolated elements in exceptional groups

G	$o(s)$	$C_G^\circ(s)$	$A(s)$	$A(C)$
F_4	2	$C_3{+}A_1,\ B_4$	1	1
	3	$A_2{+}\tilde{A}_2$	1	2
	4	$A_3{+}\tilde{A}_1$	1	2
E_6	2	$A_5{+}A_1$	1	1
	3	$A_2{+}A_2{+}A_2,\ D_4$	3	$\mathfrak{S}_3$
E_7	2	$D_6{+}A_1$	1	1
	2	$A_7,\ E_6$	2	2
	3	$A_5{+}A_2$	1	2
	4	$A_3{+}A_3{+}A_1,\ D_4{+}A_1{+}A_1$	2	4
E_8	2	$D_8,\ E_7{+}A_1$	1	1
	4	$D_5{+}A_3, A_7{+}A_1$	1	2
	3	$A_8,\ E_6{+}A_2$	1	2
	5	$A_4{+}A_4$	1	4

In Table 1, in the last two columns, n stands for a cyclic group of order n. Furthermore, $\tilde{A}_k$ denotes a component of $C_G^\circ(s)$ of type A_k generated by short root subgroups.

From now on let G be simple of type F_4, with Steinberg endomorphism $F : G \to G$, so $G^F = F_4(q)$, and let $\ell \in \{2, 3\}$ be one of the two bad primes for G. According to Proposition 3.1 there exist four different types of centralisers of quasi-isolated elements

$1 \neq s \in G^{*F}$. In Table 2 we have collected various information on their centralisers and the corresponding Lusztig series in $\mathrm{Irr}(G^F)$ as follows. Firstly, in the second column we list the possible rational structures of centralisers of quasi-isolated elements. Here, a quasi-isolated element of order 4 with centraliser structure $A_3(q)\tilde{A}_1(q)$ exists when $q \equiv 1 \pmod 4$, while there is one with centraliser structure ${}^2A_3(q)\tilde{A}_1(q)$ when $q \equiv 3 \pmod 4$. Similarly, a quasi-isolated 3-element with centraliser structure $A_2(q)\tilde{A}_2(q)$ exists when $q \equiv 1 \pmod 3$, while there is one with centraliser structure ${}^2A_2(q){}^2\tilde{A}_2(q)$ when $q \equiv 2 \pmod 3$.

In each case there is a unique bad prime ℓ not dividing $o(s)$. The third column contains one of the two possibilities for

$$e = e_\ell(q) := \text{order of } q \text{ modulo} \begin{cases} \ell & \text{if } \ell > 2, \\ 4 & \text{if } \ell = 2. \end{cases}$$

More precisely, in order to avoid duplication of arguments, we assume that $e = 1$, that is, $q \equiv 1 \pmod 4$ when $\ell = 2$, and $q \equiv 1 \pmod 3$ when $\ell = 3$, respectively. The cases where $e = 2$, that is, where $q \equiv 1 \pmod 4$ for $\ell = 2$, respectively $q \equiv 2 \pmod 3$ for $\ell = 3$, can be obtained from the former by formally exchanging q by $-q$ in all arguments to come (the operation of Ennola duality, see [10, (3A)]). Note that G^F itself is its own Ennola dual.

TABLE 2. Quasi-isolated blocks in $F_4(q)$

No.	$C_{G^*}(s)^F$	(ℓ, e)	L^F	$C_{L^*}(s)^F$	λ	$W_{G^F}(L,\lambda)$
1	$A_2(q)\,\tilde{A}_2(q)$	$(2,1)$	Φ_1^4	L^{*F}	1	$A_2 \times A_2$
2	${}^2A_2(q)\,{}^2\tilde{A}_2(q)$	$(2,1)$	$\Phi_1^2.A_1(q)^2$	$\Phi_1^2\Phi_2^2$	1	$A_1 \times A_1$
			$\Phi_1.B_3(q)$	$\Phi_1\Phi_2.{}^2A_2(q)$	ϕ_{21}	A_1
			$\Phi_1.C_3(q)$	$\Phi_1\Phi_2.{}^2\tilde{A}_2(q)$	$\tilde{\phi}_{21}$	A_1
			G^F	$C_{G^*}(s)^F$	$\phi_{21} \otimes \tilde{\phi}_{21}$	1
3	$B_4(q)$	$(3,1)$	Φ_1^4	L^{*F}	1	B_4
4			$\Phi_1^2.B_2(q)$	L^{*F}	$B_2[1]$	B_2
5	$C_3(q)\,A_1(q)$	$(3,1)$	Φ_1^4	L^{*F}	1	$C_3 \times A_1$
6			$\Phi_1^2.B_2(q)$	L^{*F}	$B_2[1]$	$A_1 \times A_1$
7	$A_3(q)\,\tilde{A}_1(q)$	$(3,1)$	Φ_1^4	L^{*F}	1	$A_3 \times A_1$
8	${}^2A_3(q)\,\tilde{A}_1(q)$	$(3,1)$	$\Phi_1^3.\tilde{A}_1(q)$	$\Phi_1^3\Phi_2$	1	$C_2 \times A_1$
$2b$		$(2,2)$	Φ_2^4	L^{*F}	1	$A_2 \times A_2$

For each type of centraliser occurring in the table we have also listed in Table 2 all e-cuspidal pairs (L, λ) in G (up to G^F-conjugacy) such that $\lambda \in \mathcal{E}(L^F, s)$, together with their relative Weyl groups. More precisely, we denote λ by the standard name of its unipotent correspondent under Lusztig's Jordan decomposition of characters; for example ϕ_{21} denotes the unipotent character of $\mathrm{SL}_3(q)$ parametrised by the partition 21 of 3. Thus, in particular, if $\lambda \in \mathcal{E}(L, s)$ corresponds to $\rho \in \mathcal{E}(C_{L^*}(s), 1)$, then $\lambda(1) = |L^* : C_{L^*}(s)|_{p'}\, \rho(1)$.

The relative Weyl groups $W_{G^F}(L,\lambda) = N_{G^F}(L,\lambda)/L^F$ can be computed using the GAP-package Chevie [39], see also the paper of Howlett [28]; they are Coxeter groups of the indicated type.

The last line 2b will be needed in one of the arguments below.

Proposition 3.2. *Let $s \neq 1$ be a quasi-isolated ℓ'-element of $G^{*F} = F_4(q)$, and assume that $e = e_\ell(q) = 1$. Then we have:*

(a) *$\mathcal{E}(G^F, s)$ is the disjoint union of the e-Harish-Chandra series listed in the rows of Table 2.*

(b) *The assertion of Theorem 1.4 holds for G of type F_4.*

Proof. We first determine the decomposition of R_L^G in the relevant cases. If L is 1-split, this is given by the usual 1-Harish-Chandra theory. Secondly, if L is a maximal torus, or if λ is uniform, this was determined by Lusztig [33, Thm. 4.23]. Thus, the decomposition of R_L^G is known in all cases listed in Table 2, and also for their Ennola duals unless $\ell = 2$, $e = 2$, and L is the Ennnola twist of lines 2 or 3 in case 2, or $\ell = 3$, $e = 2$ and $L^F = \Phi_2^2.B_2(q)$ is the Ennola twist of case 4 or 6. In the second situation, by the Mackey formula in Theorem 2.8(d), $R_L^H(\lambda)$, with $H \geq L$ an e-split Levi subgroup of type B_3 or C_3 has norm 2, while $R_H^G(\mu)$, for μ a constituent of $R_L^H(\lambda)$, has norm 3. So in both cases the decomposition can be recovered from the uniform projections, for which the decomposition is known by Lusztig's work. Similarly, in the case that $\ell = 2$ we use that $R_L^G(\lambda)$ has norm 3 to determine its decomposition.

It turns out that all decompositions are independent of q. Both (a) and (b) can now be checked from these decompositions. □

We now verify the assumptions for Proposition 2.17.

Lemma 3.3. *Let L and ℓ be as in Table 2, with $e = e_\ell(q) = 1$. Then:*

(a) *in Cases 1–8, $L = C_G(Z(L)_\ell^F)$ and L is (e,ℓ)-adapted;*

(b) *λ is of quasi-central ℓ-defect precisely in the numbered lines of the table; and*

(c) *in Case 2b, there is $z \in Z(L)_2^F$ with $C_G(z)$ of type B_4.*

Proof. This is easy to check using Chevie or by hand calculations in the root system of type F_4. □

In fact, in all numbered lines except 2, λ is even of central ℓ-defect.

Corollary 3.4. *For each quasi-isolated ℓ'-element $1 \neq s \in G^{*F}$ the e-Harish-Chandra series above any e-cuspidal pair (L,λ) below $\mathcal{E}(G^F, s)$ is contained in a unique ℓ-block of G^F.*

Proof. By Proposition 3.2(b) and Lemma 3.3 the assumptions of Proposition 2.17 are satisfied, so each e-Harish-Chandra series in $\mathcal{E}(G^F, s)$ lies in a unique ℓ-block. □

We're now ready to determine the quasi-isolated ℓ-blocks and their defect groups:

Proposition 3.5. *Assume that $e_\ell(q) = 1$. For any quasi-isolated ℓ'-element $1 \neq s \in G^{*F} = F_4(q)$ the block distribution of $\mathcal{E}(G^F, s)$ is as indicated by the horizontal lines in Table 2.*

For each ℓ-block corresponding to one of the cases 1–8 in the table, there is a defect group $P \leq N_{G^F}(L,\lambda)$ with the structure described in Theorem 1.2. In particular, the defect groups are abelian precisely in cases 4, 6 and 8.

Proof. In cases 1, 7 and 8 in particular, $\mathcal{E}(G^F, s)$ is a single 1-Harish-Chandra series. Then $\mathcal{E}_\ell(G^F, s)$ must be an ℓ-block by Theorem 2.3. In case 2b, by Lemma 3.3(c) there is $z \in Z(L)_2^F$ with centraliser C of type B_4. But by [21, Prop. 1.5] each $\mathcal{E}_2(C^F, s)$ is a single 2-block. So by Proposition 2.12 all constituents of $R_L^G(\lambda)$ lie in a unique 2-block. Since this 2-Harish-Chandra series contains all four 1-Harish-Chandra series under line 2, they all must lie in the same 2-block. In order to complete the proof of the first assertion, it remains to show that the blocks in lines 3 and 4 correspond to distinct blocks as well as the blocks in lines 5 and 6. We will do this after determining the defect groups.

By Lemma 3.3, the assumptions on (L,λ) of Proposition 2.7 are satisfied. Let P be as in Proposition 2.7. We show that P is a defect group of the corresponding block B. In lines 1, 2, 3, 5, 7, and 8 one checks the equality

$$|C_{G^*}(s)^F|_\ell = |Z^\circ(L)_\ell^F| \cdot |Z([L,L])_\ell^F| \cdot |W_{G^F}(L,\lambda)|_\ell,$$

whence by Proposition 2.7(c), P is a defect group of B. Further, in cases 1, 2, 3, 5, and 7, $W_{G^F}(L,\lambda)$ is not an ℓ'-group, so by Proposition 2.7(e), P is not abelian.

In cases 4, 6 and 8, $Z^\circ(L)^F \cap [L,L]^F$ and $W_{G^F}(L,\lambda)$ are both ℓ'-groups, hence by Proposition 2.7(f),(g), $Z(L)_\ell^F = D = P$ is a defect group of B.

Finally, since the block corresponding to line 3 has non-abelian defect groups whereas the one corresponding to line 4 has abelian defect groups, these lines correspond to different blocks. Similarly, lines 5 and 6 correspond to different blocks. □

This completes the proof of Theorem 1.2 for type F_4.

4. The quasi-isolated blocks in $E_6(q)$ and ${}^2E_6(q)$

Here we prove Theorems 1.2 and 1.4 for G a simple simply connected group of type E_6. Let's first assume that $G^F = E_6(q)_{\mathrm{sc}}$. The situation here is more complicated than for type F_4 since the dual group G^* of adjoint type contains semisimple elements with disconnected centralisers. In Table 3 we have collected the six possible types of quasi-isolated elements $1 \neq s \in G^{*F}$ and their centralisers according to Proposition 3.1. Note that, whether $\ell = 2$ or $\ell = 3$, we may have $e = e_\ell(q) = 1$ or 2, which explains the fact that each centraliser occurs twice in the table.

Again, for each element s we have listed all e-cuspidal pairs (L,λ) below $\mathcal{E}(G^F, s)$ up to G^F-conjugation. (If L is a proper Levi subgroup of G, the e-cuspidality of the given character λ is known by induction; when $L = G$ it will be a consequence of the explicit decomposition of Lusztig induction.) We denote the characters λ as explained for F_4. Moreover, ϕ, ϕ', ϕ'' denote the three extensions of the unique 2-cuspidal unipotent character of $D_4(q)$ to its extension by the graph automorphism of order 3.

The column headed $W_{G^F}(L,\lambda)$ describes the relative Weyl group for the given e-cuspidal pairs as a Coxeter group, possibly extended by a cyclic group of order 3 if $C_{G^*}(s)$ is disconnected.

We now proceed as in the case of F_4 and first discuss the decomposition of R_L^G for each line in Table 3:

TABLE 3. Quasi-isolated blocks in $E_6(q)$

No.	$C_{G^*}(s)^F$	(ℓ, e)	L^F	$C_{L^*}(s)^F$	λ	$W_{G^F}(L,\lambda)$
1	$A_2(q)^3.3$	$(2,1)$	Φ_1^6	L^{*F}	1	$A_2 \wr 3$
2	$A_2(q^3).3$	$(2,1)$	$\Phi_1^2.A_2(q)^2$	$\Phi_1^2\Phi_3^2.3$	1	A_2
3	$\Phi_1^2.D_4(q).3$	$(2,1)$	Φ_1^6	L^{*F}	1	$D_4.3$
			$\Phi_1^2.D_4(q)$	L^{*F}	$D_4[1]$	3
4	$\Phi_1\Phi_2.{}^2D_4(q)$	$(2,1)$	$\Phi_1^4.A_1(q)^2$	$\Phi_1^4\Phi_2^2$	1	B_3
5	$\Phi_3.{}^3D_4(q).3$	$(2,1)$	$\Phi_1^2.A_2(q)^2$	$\Phi_1^2\Phi_3^2.3$	1	G_2
			G^F	$C_{G^*}(s)^F$	${}^3D_4[\pm 1]$	1
6	$A_2(q^2).{}^2A_2(q)$	$(2,1)$	$\Phi_1^3.A_1(q)^3$	$\Phi_1^3\Phi_2^3$	1	$A_2 \times A_1$
			$\Phi_1^2.D_4(q)$	$\Phi_1^2\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	A_2
7	$A_2(q)^3.3$	$(2,2)$	$\Phi_1^2\Phi_2^3.A_1(q)$	$\Phi_1^3\Phi_2^3$	1	$A_1 \wr 3$
			$\Phi_1\Phi_2^2.A_3(q)$	$\Phi_1^2\Phi_2^2.A_2(q)$	ϕ_{21}	$A_1 \times A_1$
			$\Phi_2.A_5(q)$	$\Phi_1\Phi_2.A_2(q)^2$	$\phi_{21} \otimes \phi_{21}$	A_1
			G^F	$C_{G^*}(s)^F$	$\phi_{21}^{\otimes 3}$	1
8	$A_2(q^3).3$	$(2,2)$	$\Phi_2.A_2(q^2)A_1(q)$	$\Phi_1\Phi_2\Phi_3\Phi_6.3$	1	A_1
			G^F	$C_{G^*}(s)^F$	ϕ_{21}	1
9	$\Phi_1^2.D_4(q).3$	$(2,2)$	$\Phi_1^2\Phi_2^4$	L^{*F}	1	$D_4.3$
			G^F	$C_{G^*}(s)^F$	ϕ, ϕ', ϕ''	1
10	$\Phi_1\Phi_2.{}^2D_4(q)$	$(2,2)$	$\Phi_1^2\Phi_2^4$	L^{*F}	1	B_3
11	$\Phi_3.{}^3D_4(q).3$	$(2,2)$	$\Phi_2^2.A_2(q^2)$	$\Phi_2^2\Phi_3\Phi_6.3$	1	G_2
			G^F	$C_{G^*}(s)^F$	$\phi_{2,1}, \phi_{2,2}$	1
12	$A_2(q^2).{}^2A_2(q)$	$(2,2)$	$\Phi_1^2\Phi_2^4$	L^{*F}	1	$A_2 \times A_2$
13	$A_5(q)A_1(q)$	$(3,1)$	Φ_1^6	L^{*F}	1	$A_5 \times A_1$
14	$A_5(q)A_1(q)$	$(3,2)$	$\Phi_1^2\Phi_2^4$	L^{*F}	1	$C_3 \times A_1$
15			$\Phi_2.A_5(q)$	L^{*F}	ϕ_{321}	A_1

Proposition 4.1. *Let $1 \neq s \in G^{*F} = E_6(q)_{\mathrm{ad}}$ be a quasi-isolated ℓ'-element, and $e = e_\ell(q)$. Then we have:*

(a) *$\mathcal{E}(G^F, s)$ is the disjoint union of the e-Harish-Chandra series listed in Table 3.*

(b) *The assertion of Theorem 1.4 holds for G of type E_6.*

Proof. The characters of all proper Levi subgroups in cases 1, 4, 6, 7, 10, and 12–15 are uniform, so the decomposition of Lusztig induction can be reduced to the known decomposition of $R_T^G(\theta)$ for suitable maximal tori T. The same is true for the first line in cases 3 and 9. Whenever $L = G$, there is nothing to do. For each of the two Levi subgroups L of type A_2^2 (cases 2, 5 and 11) there are three $N_{G^F}(L)$-orbits of characters of degree $\frac{1}{3}\Phi_1^4\Phi_2^2$, their sum being uniform. Since L only involves factors of type A, Lusztig induction of this sum can be decomposed. In the second line in case 3, $R_L^G(\lambda)$ has norm 3 by Theorem 2.8(d), and from its known degree one concludes that it equals the sum of the three remaining characters of $\mathcal{E}(G^F, s)$ not occurring in the e-Harish-Chandra series in line 3 of the table. The same considerations apply to case 8.

It follows from the explicit decompositions that both (a) and (b) hold. □

The following is easily checked by explicit computation:

Lemma 4.2. *Let L and ℓ be as in Table 3, with $e = e_\ell(q)$. Then:*
(a) *$L = C_G(Z(L)^F_\ell)$ and L is (e, ℓ)-adapted; and*
(b) *in the table, λ is of quasi-central ℓ-defect precisely in the numbered lines.*

In fact, in all numbered lines except 6–8, λ is even of central ℓ-defect.

By Proposition 4.1(b) and Lemma 4.2, the assumptions of Proposition 2.17 are satisfied, so again each e-Harish-Chandra series in Table 3 is contained in a unique ℓ-block of G^F.

Proposition 4.3. *Let $e = e_\ell(q)$. For any quasi-isolated ℓ'-element $1 \neq s \in G^{*F} = E_6(q)_{\rm ad}$ the block distribution of $\mathcal{E}(G^F, s)$ is as indicated by the horizontal lines in Table 3.*

For each ℓ-block corresponding to one of the cases 1–15 in the table there is a defect group $P \leq N_{G^F}(L, \lambda)$ with the structure described in Theorem 1.2. In particular, the defect groups are abelian precisely in case 15.

Proof. In cases 1, 2, 4, 10, 12 and 13, $\mathcal{E}(G^F, s)$ is a single e-Harish-Chandra series, hence $\mathcal{E}_\ell(G^F, s)$ is an ℓ-block by Theorem 2.3. The Levi subgroup in the second line of case 3 contains the one in the first line of case 3. In the second line of case 3, the irreducible characters in $\mathcal{E}(L^F, s)$ are products of a fixed linear character of $Z(L)^F$ of order 2 with unipotent characters of the derived group $[L, L]^F$ of type D_4. Now by [21, Prop. 1.5] all unipotent characters of this derived group are contained in a single 2-block, hence all elements of $\mathcal{E}(L^F, s)$ are in the same 2-block so the two Harish-Chandra series of G^F lie above a single 2-block of L^F, and hence lie in a single 2-block of G^F by Proposition 2.17. The same argument applies to case 6, using again that $\mathcal{E}_2(L^F, s)$ forms a single 2-block and that the Levi subgroup corresponding to the second line contains the one corresponding to the first.

In case 7 we also use 1-Harish-Chandra theory from the 1-cuspidal pair (L, λ) in line 1. It turns out that all assertions of Lemma 4.2(a) are also satisfied there when $q \equiv 3 \pmod 4$. Then, by Proposition 2.17, all constituents of $R_L^G(\lambda)$ lie in a single 2-block. Since this 1-Harish-Chandra series contains all 2-Harish-Chandra series below 7, the latter must form a single 2-block. The same argument applies to line 9, using line 3.

For case 8, we verify that the 1-cuspidal pair (L, λ) in line 2, with $q \equiv 3 \pmod 4$, satisfies $L = C_G^\circ(Z(L)^F_\ell)$ and that λ is of central ℓ-defect. We may conclude by Proposition 2.16 that the 1-Harish-Chandra series in 2 lies in a unique 2-block. Since this contains both 2-Harish-Chandra series below 8, these lie in a single 2-block.

In case 5 all character of $\mathcal{E}(G^F, s)$ but three (corresponding to the cuspidal unipotent character of D_4) lie in the same 1-Harish-Chandra series, hence in the same 2-block. Now we also consider Lusztig induction from the 2-split Levi subgroup with fixed point group $\Phi_2^2.A_2(q^2)$ (line 11 with $q \equiv 1 \pmod 4$). Then again all characters in $\mathcal{E}(G^F, s)$ but three different ones lie in the same 2-Harish-Chandra series, hence in the same 2-block. The same argument applies to case 11, using line 5.

In order to complete the proof of the distribution of blocks, it remains only to show that lines 14 and 15 correspond to different blocks, and this will be done after the determination of defect groups.

By Lemma 4.2, the assumptions on (L, λ) of Proposition 2.7 are satisfied. Let P be as in Proposition 2.7. In lines 1–14 one checks the assumption of Proposition 2.7(c), whence P is a defect group of B. Further, in all these cases $W_{G^F}(L, \lambda)$ is not an ℓ'-group, so by Proposition 2.7(e), P is not abelian. In case 15, $Z^\circ(L)^F \cap [L, L]^F$ and $W_{G^F}(L, \lambda)$ are both ℓ'-groups, hence by Proposition 2.7(f) and (g), $Z(L)^F_\ell = D = P$ is a defect group of B.

Finally, since the block corresponding to line 14 has non-abelian defect groups whereas the one corresponding to line 15 has abelian defect groups, these lines correspond to different blocks. □

This completes the proof of Theorem 1.2 for $G = E_6(q)$.

The Lusztig series to consider in ${}^2E_6(q)$ are Ennola duals of those in $E_6(q)$, and thus precisely the same arguments as for the latter case apply. We obtain ℓ-blocks as in Table 3, with the cases $(\ell, 1)$ and $(\ell, 2)$ interchanged, and the Levi subgroups replaced by their Ennola-duals.

5. The quasi-isolated blocks in $E_7(q)$

We now prove Theorems 1.2 and 1.4 for G a simple simply connected group of type E_7, so $G^F = E_7(q)_{\mathrm{sc}}$. The relevant non-central quasi-isolated elements $s \in G^{*F}$ and their centralisers when $q \equiv 1 \pmod 4$ (for the first two entries) respectively $q \equiv 1 \pmod 3$ (for the remaining entries) are given in Table 4 according to Proposition 3.1. Thus, we have $e = e_\ell(q) = 1$ for the cases listed in the table, and hence $\ell|(q-1)$. The cases where $q \equiv 3 \pmod 4$ and $\ell = 2$ (respectively $q \equiv 2 \pmod 3$ and $\ell = 3$) are obtained from these by Ennola duality. Note that cases 12, 15, 16 and 19 only occur for $q \equiv 1 \pmod 4$, and cases 13, 17, 18 and 20 only for $q \equiv 3 \pmod 4$.

As for F_4 and E_6, in each case we give all relevant 1-cuspidal pairs (L, λ) (up to G^F-conjugation) lying below characters from $\mathcal{E}(G^F, s)$ and their relative Weyl groups. Case 2b, with $e = 2$, case 10b, with $e = 3$, will be used to further investigate the ℓ-blocks in cases 2 and 10.

In order to fit the table on the page, we've adopted the following notation for the Levi subgroups L, except in lines 2b and 10b: we just give the Dynkin type of the derived subgroup $[L, L]$, with the understanding that L contains a maximally split torus (since $e = 1$).

(We remark that the conjugacy class of parabolic subgroups of type A_1^3 of $W(E_7)$ with normaliser quotient F_4, denoted by $(A_1^3)'$ in the above table, seems to have been overseen in [28].)

Proposition 5.1. *Let $1 \neq s \in G^{*F} = E_7(q)_{\mathrm{ad}}$ be a quasi-isolated ℓ'-element, and assume that $e = e_\ell(q) = 1$. Then we have:*

(a) *$\mathcal{E}(G^F, s)$ is the disjoint union of the e-Harish-Chandra series listed in the upper part of Table 4.*

(b) *The assertion of Theorem 1.4 holds for G of type E_7.*

Proof. Whenever λ is uniform, the decomposition of R_L^G is obtained from the known decomposition of R_T^G for various maximal tori T of G. Secondly, whenever the relative Weyl group is of order 2, $R_L^G(\lambda)$ is of norm 2 by the Mackey formula, and its constituents are easily determined from the uniform projection. Furthermore, in all cases the induction

Table 4. Quasi-isolated blocks in $E_7(q)$

No.	$C_{G^*}(s)^F$	(ℓ, e)	L^F	$C_{L^*}(s)^F$	λ	$W_{G^F}(L,\lambda)$
1	$A_5(q)A_2(q)$	$(2,1)$	$\emptyset$	L^{*F}	1	$A_5 \times A_2$
2	${}^2A_5(q){}^2A_2(q)$	$(2,1)$	A_1^3	$\Phi_1^4\Phi_2^3$	1	$C_3 \times A_1$
			D_4	$\Phi_1^3\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	C_3
			D_6	$\Phi_1\Phi_2.{}^2A_5(q)$	ϕ_{321}	A_1
			E_7	$C_{G^*}(s)^F$	$\phi_{321} \otimes \phi_{21}$	1
3	$D_6(q)A_1(q)$	$(3,1)$	$\emptyset$	L^{*F}	1	$D_6 \times A_1$
4			D_4	L^{*F}	$D_4[1]$	$B_2 \times A_1$
5	$A_7(q).2$	$(3,1)$	$\emptyset$	L^{*F}	1	$A_7.2$
6	${}^2A_7(q).2$	$(3,1)$	$(A_1^3)'$	$\Phi_1^4\Phi_2^3.2$	1	C_4
7			D_6	$\Phi_1\Phi_2.{}^2A_5(q).2$	ϕ_{321}	A_1
8	$\Phi_1.E_6(q).2$	$(3,1)$	$\emptyset$	L^{*F}	1	$E_6.2$
9			D_4	L^{*F}	$D_4[1]$	$A_2.2$
			E_6	L^{*F}	$E_6[\theta^{\pm 1}]$	2
10	$\Phi_2.{}^2E_6(q).2$	$(3,1)$	$(A_1^3)'$	$\Phi_1^4\Phi_2^3.2$	1	F_4
			E_7	$C_{G^*}(s)^F$	${}^2E_6[\theta^{\pm 1}], {}^2E_6[1]$	1
11			D_6	$\Phi_1\Phi_2.{}^2A_5(q).2$	ϕ_{321}	A_1
12	$A_3(q){}^2A_1(q).2$	$(3,1)$	$\emptyset$	L^{*F}	1	$A_3 \wr 2 \times A_1$
13	${}^2A_3(q){}^2A_1(q).2$	$(3,1)$	A_1^2	$\Phi_1^5\Phi_2^2$	1	$B_2 \wr 2 \times A_1$
14	$A_3(q^2)A_1(q).2$	$(3,1)$	$(A_1^3)'$	$\Phi_1^4\Phi_2^3.2$	1	$A_3 \times A_1$
15	$\Phi_1.D_4(q)A_1(q)^2.2$	$(3,1)$	$\emptyset$	L^{*F}	1	$(D_4 \times A_1^2).2$
16			D_4	L^{*F}	$D_4[1]$	$A_1 \wr 2$
17	$\Phi_2.D_4(q)A_1(q)^2.2$	$(3,1)$	A_1	$\Phi_1^6\Phi_2$	1	$(D_4 \times A_1^2).2$
18			$D_4 \cdot A_1$	$\Phi_1^2\Phi_2 D_4(q)$	$D_4[1]$	$A_1 \wr 2$
19	$\Phi_1.{}^2D_4(q)A_1(q^2).2$	$(3,1)$	A_1^2	$\Phi_1^5\Phi_2^2$	1	$(B_3 \times A_1).2$
20	$\Phi_2.{}^2D_4(q)A_1(q^2).2$	$(3,1)$	$(A_1^3)'$	$\Phi_1^4\Phi_2^3.2$	1	$B_3 \times A_1$
$2b$		$(2,2)$	Φ_2^7	L^{*F}	1	$A_5 \times A_2$
$10b$		$(3,3)$	$\Phi_3^2.A_1(q^3)$	$\Phi_2\Phi_3^2\Phi_6.2$	1	G_5

to a Levi subgroup of type E_6 respectively ${}^2E_6$ is known by the results of the previous section. The norm of characters induced from these Levi subgroups is small enough to again determine them uniquely from their uniform projections. $\square$

The conditions on L and on λ can be checked as in the previous cases:

Lemma 5.2. *Let L and ℓ be as in cases 1–20 in Table 4 (and recall that $e = e_\ell(q) = 1$). Then:*
(a) *$L = C_G(Z(L)_\ell^F)$, and L is (e,ℓ)-adapted; and*
(b) *in the table, λ is of quasi-central ℓ-defect precisely in the numbered lines.*
Additionally, in cases 2b and 10b we have $L = C_G^\circ(Z(L)_\ell^F)$.

In fact, in all numbered lines except 2, λ is even of central ℓ-defect.

Proposition 5.3. *Assume that $e_\ell(q) = 1$. For any quasi-isolated ℓ'-element $1 \neq s \in G^{*F} = E_7(q)_{\mathrm{ad}}$ the block distribution of $\mathcal{E}(G^F, s)$ is as indicated by the horizontal lines in Table 4.*

For each ℓ-block corresponding to one of the cases 1–20 in the table, there is a defect group $P \leq N_{G^F}(L, \lambda)$ with the structure described in Theorem 1.2. In particular, the defect groups are abelian precisely in cases 4, 7, 11, 13, 16 and 18.

Proof. By Proposition 2.17, each e-Harish-Chandra series in Table 4 is contained in a unique ℓ-block of G^F. In cases 1, 5, 12–14, 19 and 20, $\mathcal{E}(G^F, s)$ is a single ℓ-block by Theorem 2.3. By [21, Tables for $E_6(q)$] all unipotent characters of positive 3-defect of the Levi subgroup of type E_6 lie in the same 3-block, so by Proposition 2.17 the Harish-Chandra series in line 9 and the following line belong to the same 3-block. Here note that the Levi subgroup in the second line in each case contains the one in the first line.

In case 2, we claim that all four Harish-Chandra series lie in the same 2-block. For this note that the 2-split Levi subgroup L in case 2b satisfies $L = C^\circ_G(Z(L)^F_\ell)$, and then the claim follows from Proposition 2.16 applied to the 2-Harish-Chandra series in case 2b, which contains all Harish-Chandra series from case 2. By the same arguments, the 1-cuspidal characters $\lambda = E_6[\theta^{\pm 1}]$ in the second line of case 10 lie in the same block as line 10, since these lie in the same 3-Harish-Chandra series as in case 10b. We will show that different numbered lines corresponding to the same quasi-isolated element lie in different blocks after the determination of the defect groups.

Now let B be an ℓ-block in $\mathcal{E}_\ell(G^F, s)$ and let P be as in Proposition 2.7. In all numbered lines which are in the first line of the part of the table corresponding to s we have that P is a defect group of B by Proposition 2.7(c). Further, in all of these cases, except line 13, $W_{G^F}(L, \lambda)$ is not an ℓ'-group, so by Proposition 2.7(e), P is not abelian.

For lines 4, 7, 11, 13, 16 and 18, $Z^\circ(L)^F \cap [L, L]^F$ and $W_{G^F}(L, \lambda)$ are both ℓ'-groups, hence by Proposition 2.7(f),(g), $Z(L)^F_\ell = D = P$ is a defect group of B.

For case 9, we note that by Proposition 2.7(b),(g) there is a subgroup P of a defect group of B of the required type and with $D = A = Z(L)^F_3$. Further, $Z(L)^F_3 = E^3$, where E is a cyclic group of order $(q-1)_3$, $Z(L)^F_3$ has index 3 in P and if $\sigma \in P \setminus Z(L)^F_3$, then σ acts on E^3 by cyclically permuting the factors. Thus $C_P(\sigma)$ has order $3(q-1)_3$ whereas $C_P(\tau)$ for any $\tau \in Z(L)^F_3$ has order at least $3(q-1)^3_3$. So, $Z(L)^F_3$ is characteristic in Q, and it follows from Proposition 2.7(d) that P is a defect group of B.

The defect groups in cases 8 and 9 have different orders, hence they correspond to different blocks. The defect groups in cases 3, 6, 10, 15 and 17 are non-abelian whereas those in cases 4, 7, 11, 16 and 18 are abelian, hence correspond to different blocks. □

6. The quasi-isolated blocks in $E_8(q)$

Throughout this section, G is a simple group of type E_8, so $G^F = E_8(q)$. The situation is yet more complicated since now there are three bad primes $\ell = 2, 3, 5$ to deal with, which we'll do one at a time. Until Section 6.4 we assume that $q \neq 2$.

6.1. **Quasi-isolated 2-blocks of $E_8(q)$.** We begin by considering the case when $\ell = 2$. Table 5 contains the possible rational types of centralisers of quasi-isolated 3- and 5-elements $1 \neq s \in G^{*F}$, all e-cuspidal pairs (L, λ) with $s \in L^{*F}$ and their relative Weyl groups for the case $q \equiv 1 \pmod 4$. Here, a quasi-isolated 3-element as in cases 1 and 3 occurs when $q \equiv 1 \pmod 3$, as in cases 2 and 5 when $q \equiv 2 \pmod 3$; and a quasi-isolated 5-element as in case 7 occurs when $q \equiv 1 \pmod 5$, as in case 8 when $q \equiv 2, 3 \pmod 5$ and as in case 9 when $q \equiv -1 \pmod 5$. The notation for Levi subgroups and for the cuspidal characters is as in Table 4 above.

The cases where $q \equiv 3 \pmod 4$ can be obtained from the former by Ennola duality.

TABLE 5. Quasi-isolated 2-blocks in $E_8(q)$, $q \equiv 1 \pmod 4$

No.	$C_{G^*}(s)^F$	e	L^F	$C_{L^*}(s)^F$	λ	$W_{G^F}(L,\lambda)$
1	$A_8(q)$	1	$\emptyset$	L^{*F}	1	A_8
2	${}^2A_8(q)$	1	A_1^4	$\Phi_1^4\Phi_2^4$	1	B_4
			$D_4{\cdot}A_1$	$\Phi_1^3\Phi_2^3.{}^2A_2(q)$	ϕ_{21}	B_3
3	$E_6(q).A_2(q)$	1	$\emptyset$	L^{*F}	1	$E_6 \times A_2$
			D_4	L^{*F}	$D_4[1]$	$G_2 \times A_2$
4			E_6	L^{*F}	$E_6[\theta^{\pm1}]$	A_2
5	${}^2E_6(q).{}^2A_2(q)$	1	A_1^3	$\Phi_1^5\Phi_2^3$	1	$F_4 \times A_1$
			D_4	$\Phi_1^4\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	F_4
			D_6	$\Phi_1^2\Phi_2.{}^2A_5(q)$	ϕ_{321}	$A_1 \times A_1$
			E_7	$\Phi_1.{}^2A_5(q){}^2A_2(q)$	$\phi_{321} \otimes \phi_{21}$	A_1
			E_7	$\Phi_1\Phi_2.{}^2E_6(q)$	${}^2E_6[1]$	A_1
			E_8	$C_{G^*}(s)^F$	${}^2E_6[1] \otimes \phi_{21}$	1
6			E_7	$\Phi_1\Phi_2.{}^2E_6(q)$	${}^2E_6[\theta^{\pm1}]$	A_1
			E_8	$C_{G^*}(s)^F$	${}^2E_6[\theta^{\pm1}] \otimes \phi_{21}$	1
7	$A_4(q)^2$	1	$\emptyset$	L^{*F}	1	A_4^2
8	${}^2A_4(q^2)$	1	A_3^2	$\Phi_1^2\Phi_2^2\Phi_4^2$	1	B_2
			D_7	$\Phi_1\Phi_2\Phi_4.{}^2A_2(q^2)$	ϕ_{21}	A_1
9	${}^2A_4(q)^2$	1	A_1^4	$\Phi_1^4\Phi_2^4$	1	B_2^2
			$D_4{\cdot}A_1$	$\Phi_1^3\Phi_2^3.{}^2A_2(q)$	ϕ_{21} (2×)	$B_2 \times A_1$
			D_6	$\Phi_1^2\Phi_2^2.{}^2A_2(q)^2$	$\phi_{21} \otimes \phi_{21}$	A_1^2
$5b$	${}^2E_6(q).{}^2A_2(q)$	2	Φ_2^8	L^{*F}	1	$E_6 \times A_2$
$6b$			$\Phi_2^2.{}^2E_6(q)$	L^{*F}	${}^2E_6[\theta^{\pm1}]$	A_2

Lets' point out one particularity here. Since E_7 has two non-conjugate Levi subgroups of type A_1^3 (see the remark before Proposition 5.1), the quasi-isolated involution in case 5 embeds in two different ways into a 1-split Levi subgroup of type E_7, with non-isomorphic centralizers (see rows 4 and 5 in case 5).

Proposition 6.1. *Let $1 \neq s \in G^{*F} = E_8(q)$ be a quasi-isolated $2'$-element and assume that $e = e_\ell(q) = 1$. Then we have:*

(a) *$\mathcal{E}(G^F, s)$ is the disjoint union of the e-Harish-Chandra series listed in the upper part of Table 5.*

(b) *The assertion of Theorem 1.4 holds for G of type E_8 with $\ell = 2$.*

Proof. We determine the decomposition of $R_L^G(\lambda)$ for the e-Harish-Chandra series occurring in Table 5 as in the previous proofs, using mainly the Mackey formula and transitivity. □

Lemma 6.2. *Let L be as in cases 1–9 of Table 5, and recall that $q \equiv 1 \pmod 4$. Then $L = C_G(Z(L)_2^F)$, and L is $(1,2)$-adapted. In each numbered line of the table, and no other, λ is of quasi-central 2-defect. It is of central ℓ-defect in the lines 1, 3, 4, 7, 5b and 6b.*

Moreover, in Cases 5b and 6b we have $L = C_G^\circ(Z(L)_2^F)$.

Note that for $q \equiv 3 \pmod 4$ this is no longer true; there are many cases for which $L < C_G^\circ(Z(L)_2^F)$:

Example 6.3. Assume that $q \equiv 3 \pmod 4$ and let L^F be of type $\Phi_1^2.A_3(q)^2$. Then $C_G^\circ(Z(L)_2^F)$ is of type $D_4(q)^2$. Similarly, for L^F of type $\Phi_1^4.A_1(q)^4$ we have $C_G^\circ(Z(L)_2^F)$ is of type $A_1(q)^8$.

But as explained above, for that congruence we choose the Ennola duals of the above Levi subgroups, and for those the analogue of Lemma 6.2 continues to hold.

Proposition 6.4. *Suppose that $q \equiv 1 \pmod 4$. For any quasi-isolated $2'$-element $1 \neq s \in G^{*F}$ the block distribution of $\mathcal{E}(G^F, s)$ is as indicated by the horizontal lines in the upper part of Table 5.*

For each 2-block corresponding to one of the cases 1–9 in the table, there is a defect group $P \leq N_{G^F}(L, \lambda)$ with the structure described in Theorem 1.2. In particular, the defect groups are non-abelian.

Proof. We first prove part of the block distribution. Again, each e-Harish-Chandra series in Table 5 is contained in a unique 2-block of G^F. The lines 1 and 7 in Table 5 both correspond to a unique block by Theorem 2.3. The unnumbered Harish-Chandra series below cases 2, 3, 8 and 9 lie in the same 2-block as the respective numbered line by Proposition 2.17 since all characters of $\mathcal{E}(L^F, s)$ lie in the same 2-block by [21, Prop. 1.5].

Similarly, all characters in each of the two Lusztig series $\mathcal{E}(L^F, s)$, for L a Levi of type E_7 in rows 4 and 5 of case 5, lie in a single 2-block, except for those denoted ${}^2E_6[\theta^{\pm 1}]$, hence so do the characters in $\mathcal{E}(G^F, s)$ above them. To see that the cuspidal character in the line before case 6 belongs to the block in case 5, we use the alternative 2-Harish-Chandra series above (L, λ) given in cases 5b, which by Lemma 6.2 still satisfies $L = C_G^\circ(Z(L)_2^F)$. Thus Proposition 2.16 applies. The 1-Harish-Chandra series in line 6 and the subsequent line are both contained in the 2-Harish-Chandra series above (L, λ) in line 6b. As λ is of central defect, an application of Proposition 2.16 shows that both 1-Harish-Chandra series lie in the same 2-block.

Again, we defer the question of different numbered lines corresponding to different blocks to after the discussion on defect groups.

For any ℓ-block B in $\mathcal{E}(G^F, s)$, let $(D, v) \leq (P, w)$ be B-Brauer pairs as in Proposition 2.7. In all numbered cases, $W_{G^F}(L, \lambda)$ is not a $2'$-group, so by Proposition 2.7(e), P is not abelian. In all numbered lines which are at the top of the part of the table corresponding to a particular s, we conclude by Proposition 2.7(c) that P is a defect group of B.

In case 4, by Proposition 2.7(g), $D = A = Z(L)_2^F$. Further, $Z(L)_2^F = E^2$, where E is a cyclic group of order $(q-1)_2$. The Levi subgroup of type E_6 is contained in the maximal rank subgroup of type $E_6 + A_2$, and $Z(L)_2^F.W_{G^F}(L, \lambda)$ is contained in the normaliser of the maximal torus of the A_2-factor. Any 2-element $\sigma \in P \setminus Z(L)_2^F$ interchanges two cyclic subgroups of $Z(L)_2^F$ of order at least 4, so $Z(L)_2^F$ is the only abelian subgroup of P properly containing $Z(P)[P,P]$. Further, since $[P,P] \not\subseteq Z(P)$ and $Z(L)_2^F$ is of index 2 in P, $Z(L)_2^F = C_P([P,P])$. Since any subgroup of index 2 of P contains $[P,P]$, it follows that $Z(L)_2^F$ is the unique abelian subgroup of index 2 of P. In particular, $Z(L)_2^F$ is characteristic in P and it follows from Proposition 2.7(d) that P is a defect group of B.

The Levi subgroup of type E_7 in case 6 lies in a maximal rank subgroup of type $E_7 + A_1$, and L is a central product $E_7 \circ T$, where T is a split torus of A_1 and where the involution of the centre of E_7 is identified with the involution of T. Thus $Z(L)_2^F = |T^F|_2$ is cyclic of order $(q-1)_2$ and by Proposition 2.7(b), $Z(L)_2^F$ has index 2 in D. By considering the projection of D into T, one sees that D is cyclic of order $2(q-1)_2$. Further, if $\sigma \in P \setminus D$, then σ acts by inversion on A. Since D is cyclic of order at least 8, and A is of index 2 in D, it follows that D is the unique cyclic subgroup of index 2 in P. Thus, D and hence A is characteristic in P. Hence by Proposition 2.7(d), P is a defect group of B.

Since the defect groups in cases 3 and 4 have different order as do the defect groups in cases 5 and 6, we see that these lines correspond to distinct blocks. In case 4, as shown above $Z(L)_2^F$ is the unique abelian subgroup of P of index 2. So, if the Brauer pairs corresponding to the two choices of λ in case 4 correspond to the same block, then they are G^F-conjugate, and hence by Lemma 6.2 the corresponding e-cuspidal pairs are G^F-conjugate, which is not the case. Thus, the two entries of case 4 correspond to different blocks. A similar argument applies in case 6. The subgroup D is the unique cyclic subgroup of P of index 2, and the group $Z(L)_2^F$ is the unique subgroup of index 2 in D. $\square$

6.2. **Quasi-isolated 3-blocks of $E_8(q)$.** Now let $\ell = 3$. In Table 6 we present the centralisers of quasi-isolated 2- and 5-elements together with data for the relevant cuspidal pairs in the case where $q \equiv 1 \pmod 3$. Again those for $q \equiv 2 \pmod 3$ are obtained by Ennola duality. As in the case when $\ell = 2$, there occurs just one type of quasi-isolated 5-elements, depending on $q \pmod 5$. The quasi-isolated 4-elements in cases 6 and 9 occur when $q \equiv 1 \pmod 4$, those in cases 8 and 10 when $q \equiv 3 \pmod 4$.

Proposition 6.5. *Let $1 \neq s \in G^{*F} = E_8(q)$ be a quasi-isolated $3'$-element and recall that $e = e_\ell(q) = 1$. Then we have:*
(a) *$\mathcal{E}(G^F, s)$ is the disjoint union of the e-Harish-Chandra series listed in Table 6.*
(b) *The assertion of Theorem 1.4 holds for G of type E_8, $q \neq 2$, and $\ell = 3$.*

Proof. The decomposition of $R_L^G(\lambda)$ for the e-Harish-Chandra series 12–17 has already been computed in the proof of Proposition 6.1. For the remaining Harish-Chandra series, the usual arguments yield the claim. $\square$

Lemma 6.6. *Let L be as in Table 6 and recall that $q \equiv 1 \pmod 3$. Then $L = C_G(Z(L)_3^F)$, and L is $(1,3)$-adapted. Moreover, in each numbered line of the table, and only in those, λ is of central 3-defect.*

We obtain:

Table 6. Quasi-isolated 3-blocks in $E_8(q)$, $q \equiv 1 \pmod 3$

No.	$C_{G^*}(s)^F$	L	$C_{L^*}(s)^F$	λ	$W_{G^F}(L,\lambda)$
1	$D_8(q)$	$\emptyset$	L^{*F}	1	D_8
2		D_4	L^{*F}	$D_4[1]$	B_4
3	$E_7(q)A_1(q)$	$\emptyset$	L^{*F}	1	$E_7 \times A_1$
4		D_4	L^{*F}	$D_4[1]$	$C_3 \times A_1$
		E_6	L^{*F}	$E_6[\theta^{\pm1}]$	$A_1 \times A_1$
5		E_7	L^{*F}	$E_7[\pm\xi]$	A_1
6	$D_5(q)A_3(q)$	$\emptyset$	L^{*F}	1	$D_5 \times A_3$
7		D_4	L^{*F}	$D_4[1]$	$A_3 \times A_1$
8	${}^2D_5(q).{}^2A_3(q)$	A_1^2	$\Phi_1^6\Phi_2^2$	1	$B_4 \times C_2$
9	$A_7(q)A_1(q)$	$\emptyset$	L^{*F}	1	$A_7 \times A_1$
10	${}^2A_7(q)A_1(q)$	A_1^3	$\Phi_1^5\Phi_2^3$	1	$C_4 \times A_1$
11		D_6	$\Phi_1^2\Phi_2.{}^2A_5(q)$	ϕ_{321}	A_1^2
12	$A_4(q)^2$	$\emptyset$	L^{*F}	1	A_4^2
13	${}^2A_4(q^2)$	A_3^2	$\Phi_1^2\Phi_2^2\Phi_4^2$	1	B_2
14		D_7	$\Phi_1\Phi_2\Phi_4.{}^2A_2(q^2)$	ϕ_{21}	A_1
15	${}^2A_4(q)^2$	A_1^4	$\Phi_1^4\Phi_2^4$	1	B_2^2
16		$D_4{\cdot}A_1$	$\Phi_1^3\Phi_2^3.{}^2A_2(q)$	ϕ_{21} (2×)	$B_2 \times A_1$
17		D_6	$\Phi_1^2\Phi_2^2.{}^2A_2(q)^2$	$\phi_{21} \otimes \phi_{21}$	A_1^2

Proposition 6.7. *Suppose that $q \equiv 1 \pmod 3$. For any quasi-isolated $3'$-element $1 \neq s \in G^{*F}$ the block distribution of $\mathcal{E}(G^F, s)$ as indicated by the horizontal lines in Table 6.*

For each 3-block B corresponding to one of the cases 1–17 in the table, there is a defect group $P \leq N_{G^F}(L,\lambda)$ with the structure described in Theorem 1.2.

In particular, the defect groups of B are abelian precisely in the cases 5, 11 and 13–17, and then $Z(L)_3^F$ is a defect group of B.

Proof. Each e-Harish-Chandra series in Table 6 is contained in a unique 3-block of G^F. Next, note that lines 8, 9 and 12 correspond to a single 3-block each. The two 1-cuspidal unipotent characters $E_6[\theta^{\pm1}]$ of the derived subgroup of the Levi subgroup of type E_6 below line 4 lie in the same 3-block of $E_6(q)$ as those above $D_4[1]$ by [20], so by Proposition 2.17 their Harish-Chandra series are contained in the 3-block from case 4. All other separations of blocks will be argued once we've determined defect groups.

Concerning the structure of the defect groups, in all numbered lines which are at the top of the part of the table corresponding to a particular s, we conclude as usual by Proposition 2.7(c). Further, for all of these except cases 13 and 15, $W_{G^F}(L,\lambda)$ is not a $3'$-group, so P is non-abelian by Proposition 2.7(e).

In cases 5, 11 and 13–17, $Z^\circ(L)^F \cap [L,L]^F$ and $W_{G^F}(L,\lambda)$ are both $3'$-groups, hence by Proposition 2.7(f),(g), $Z(L)_3^F = D = P$ is a defect group of B.

In cases 2, 4, 7, by embedding L in a maximal rank subgroup of type $D_4{+}D_4$, we see that L is a central product of D_4 with a split maximal torus T of type D_4 and $Z(L)_3^F = (T)_3^F$. Since $Z^\circ(L)^F \cap [L,L]^F$ is a $3'$-group, by Proposition 2.7(g), $D = Z(L)_3^F$. The action of

$\sigma \in P \setminus D$ on $D = (T)_3^F$ can be determined through the action of the Weyl group of type D_4 on T. We have $D = \langle z_1, z_2, z_3, z_4\rangle$ with σ cyclically permuting the z_1, z_2, z_3 and fixing z_4. So, $Z(P)[P,P] = \langle z_1z_2z_3, z_4, z_1z_2^{-1}, z_2z_3^{-1}\rangle$ is a subgroup of index 3 in D and it follows that D is the only abelian subgroup of Q properly containing $Z(P)[P,P]$. Thus, D is characteristic in P, and it follows by Proposition 2.7(d) that P is a defect group of B.

In all cases, except the two represented by cases 5, respectively 16, one sees that different numbered lines correspond to different blocks by comparing orders of the defect group or noting that one of the lines corresponds to abelian defect while the other doesn't. To see that the two blocks represented by case 16 are different, note that each has a maximal Brauer pair of the form $(Z(L)_3^F, \lambda)$ and that by Lemma 6.6, $L = C_G(Z(L)_3^F)$. Since the pairs (L, λ) are not G^F-conjugate, neither are the corresponding maximal Brauer pairs. Similarly, the two blocks represented by case 5 are different. □

6.3. **Quasi-isolated 5-blocks of $E_8(q)$.** Finally, let $\ell = 5$. Here, we distinguish two cases according to whether $q \equiv \pm1 \pmod 5$ or $q \equiv \pm2 \pmod 5$. The cuspidal pairs for the case $e = 1$, are collected in Table 7; here the decomposition of R_L^G was already determined in the previous two subsections. The case $e = 2$ is obtained from this by Ennola duality. Table 8 contains the relevant information in the case $e = 4$. Here, the relative Weyl groups are, in general, no longer true Weyl groups, but various types of complex reflection groups occur.

Proposition 6.8. *Let $1 \neq s \in G^{*F} = E_8(q)$ be a quasi-isolated $5'$-element. Then we have:*

(a) *$\mathcal{E}(G^F, s)$ is the disjoint union of the e-Harish-Chandra series listed in Tables 7 and 8.*
(b) *The assertion of Theorem 1.4 holds for G of type E_8, $q \neq 2$, and $\ell = 5$.*

Proof. The decomposition of $R_L^G(\lambda)$ for the e-cuspidal pairs (L, λ) in Table 7 was already determined in Propositions 6.1 and 6.5. As for Table 8, λ is always uniform except in case 2, or when $L = G$ (in which case $R_L^G(\lambda) = \lambda$). □

Lemma 6.9. *Let L be as in Table 7 or 8. Then $L = C_G(Z(L)_5^F)$ and L is $(e,5)$-adapted. Moreover, each character λ in the tables is of central 5-defect.*

Proposition 6.10. *Suppose that $2 \neq q \equiv 1, 2, 3 \pmod 5$. For any quasi-isolated $5'$-element $1 \neq s \in G^{*F}$ the block distribution of $\mathcal{E}(G^F, s)$ is as given in Tables 7 and 8 for the respective congruences of $q \pmod 5$.*

For each 5-block B corresponding to one of the cases in Table 7 or 8 there is a defect group $P \leq N_{G^F}(L, \lambda)$ with the structure described in Theorem 1.2.

In particular, B has abelian defect groups precisely when the order $W_{G^F}(L, \lambda)$ is not divisible by 5, in which case $Z(L)_5^F$ is a defect group of B.

Proof. Again, each e-Harish-Chandra series in the tables is contained in a unique 5-block of G^F. In all numbered lines which are at the top of the part of the table corresponding to a particular s we conclude by Proposition 2.7(c). In all cases, $Z^\circ(L)^F \cap [L, L]^F$ is a $5'$-group, so $D = A$, and in all lines which are not at the top of the part of the table corresponding to a particular s, $W_{G^F}(L, \lambda)$ is a $5'$-group. The assertion on the defect groups follows by Proposition 2.7(f),(g). We see that different numbered lines correspond to different blocks by comparing orders of defect groups, or differentiating on the basis

TABLE 7. Quasi-isolated 5-blocks in $E_8(q)$, $q \equiv 1 \pmod 5$

No.	$C_{G^*}(s)^F$	L	$C_{L^*}(s)^F$	λ	$W_{G^F}(L,\lambda)$
1	$D_8(q)$	$\emptyset$	L^{*F}	1	D_8
2		D_4	L^{*F}	$D_4[1]$	B_4
3	$E_7(q)A_1(q)$	$\emptyset$	L^{*F}	1	$E_7 \times A_1$
4		D_4	L^{*F}	$D_4[1]$	$C_3 \times A_1$
5		E_6	L^{*F}	$E_6[\theta^{\pm 1}]$	$A_1 \times A_1$
6		E_7	L^{*F}	$E_7[\pm\xi]$	A_1
7	$D_5(q)A_3(q)$	$\emptyset$	L^{*F}	1	$D_5 \times A_3$
8		D_4	L^{*F}	$D_4[1]$	$A_1 \times A_3$
9	${}^2D_5(q).{}^2A_3(q)$	A_1^2	$\Phi_1^6\Phi_2^2$	1	$B_4 \times C_2$
10	$A_7(q)A_1(q)$	$\emptyset$	L^{*F}	1	$A_7 \times A_1$
11	${}^2A_7(q)A_1(q)$	A_1^3	$\Phi_1^5\Phi_2^3$	1	$C_4 \times A_1$
12		D_6	$\Phi_1^2\Phi_2.{}^2A_5(q)$	ϕ_{321}	A_1^2
13	$A_8(q)$	$\emptyset$	L^{*F}	1	A_8
14	${}^2A_8(q)$	A_1^4	$\Phi_1^4\Phi_2^4$	1	B_4
15		$D_4{\cdot}A_1$	$\Phi_1^3\Phi_2^3.{}^2A_2(q)$	ϕ_{21}	B_3
16	$E_6(q).A_2(q)$	$\emptyset$	L^{*F}	1	$E_6 \times A_2$
17		D_4	L^{*F}	$D_4[1]$	$G_2 \times A_2$
18		E_6	L^{*F}	$E_6[\theta^{\pm 1}]$	A_2
19	${}^2E_6(q).{}^2A_2(q)$	A_1^3	$\Phi_1^5\Phi_2^3$	1	$F_4 \times A_1$
20		D_4	$\Phi_1^4\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	F_4
21		D_6	$\Phi_1^2\Phi_2.{}^2A_5(q)$	ϕ_{321}	$A_1 \times A_1$
22		E_7	$\Phi_1\Phi_2.{}^2E_6(q)$	${}^2E_6[1], {}^2E_6[\theta^{\pm 1}]$	A_1
23		E_7	$\Phi_1.{}^2A_5(q){}^2A_2(q)$	$\phi_{321} \otimes \phi_{21}$	A_1
24		E_8	$C_{G^*}(s)^F$	${}^2E_6[1] \otimes \phi_{21}$, ${}^2E_6[\theta^{\pm 1}] \otimes \phi_{21}$	1

of whether the defect groups are abelian or not. For the cases where one numbered line corresponds to several cuspidal pairs, eg. cases $5, 6, \ldots$, one notes that the maximal Brauer pairs of the two blocks are not conjugate (see the argument for the two blocks represented by line 16 of Table 6). □

6.4. **The group $E_8(2)$.** The general Mackey formula has not (yet) been proved for $E_8(2)$. Since this group has three bad primes 2, 3, and 5, the 3-blocks for quasi-isolated 5-elements and the 5-blocks for quasi-isolated 3-elements are not covered by previous results.

Proposition 6.11. *The results on 3-blocks and 5-blocks of $E_8(q)$, $q > 2$, stated in Sections 6.2 and 6.3 above continue to hold for $q = 2$.*

Proof. Let first $\ell = 3$, so we are in the situation of the Ennola dual of Table 6. Here, only quasi-isolated 5-elements need to be considered, that is, the Ennola duals of lines 12–17 in that table. Now all relevant centralisers $C_{L^*}(s)$ are of type A, so their cuspidal characters λ are uniform. In this case, the decomposition of $R_L^G(\lambda)$ is known by the results of Lusztig,

TABLE 8. Quasi-isolated 5-blocks in $E_8(q)$, $q \equiv \pm 2 \pmod 5$

No.	$C_{G^*}(s)^F$	L^F	$C_{L^*}(s)^F$	λ	$W_{G^F}(L,\lambda)$
25	$D_8(q)$	Φ_4^4	L^{*F}	1	$G(4,2,4)$
26		$\Phi_4^2.D_4(q)$	L^{*F}	4 chars	$G(4,1,2)$
27		G^F	$C_{G^*}(s)^F$	4 chars	1
28	$E_7(q)A_1(q)$	$\Phi_4^2.D_4(q)$	$\Phi_4^2.A_1(q)^4$	4 chars	G_8
29		$\Phi_4^2.D_4(q)$	$\Phi_4^2.A_1(q)^4$	4 chars	$G(4,1,2)$
30		G^F	$C_{G^*}(s)^F$	32 chars	1
31	$D_5(q)A_3(q)$	$\Phi_4^3.A_1(q^2)$	$\Phi_1\Phi_2\Phi_4^3$	1	$G(4,1,2)\times Z_4$
32		$\Phi_4^2.D_4(q)$	$\Phi_1\Phi_4^2.A_3(q)$	ϕ_{22}	$G(4,1,2)$
33		$\Phi_4^2.D_4(q)$	$\Phi_1\Phi_4^2.{}^2A_3(q)$	ϕ_{22}	$Z_4\times Z_4$
34		$\Phi_4.{}^2D_6(q)$	$\Phi_1.{}^2A_3(q)A_3(q)$	$\phi_{22}\otimes\phi_{22}$	Z_4
35		G^F	$C_{G^*}(s)^F$	4 chars	1
36	$A_7(q)A_1(q)$	$\Phi_4^2.D_4(q)$	$\Phi_1\Phi_2^2\Phi_4^2.A_1(q)$	$1,\phi_{11}$	$G(4,1,2)$
37		$\Phi_4.{}^2D_6(q)$	$\Phi_1\Phi_2\Phi_4.A_3(q)A_1(q)$	$\phi_{22}\otimes 1,\phi_{11}$	Z_4
38		G^F	$C_{G^*}(s)^F$	8 chars	1
39	$A_8(q)$	$\Phi_4^2.A_1(q^2)^2$	$\Phi_1^2\Phi_2^2\Phi_4^2$	1	$G(4,1,2)$
40		$\Phi_4.{}^2D_6(q)$	$\Phi_1\Phi_2\Phi_4.A_4(q)$	$\phi_{41},\phi_{311},\phi_{2111}$	Z_4
41		G^F	$C_{G^*}(s)^F$	4 chars	1
42	$E_6(q).A_2(q)$	$\Phi_4^2.D_4(q)$	$\Phi_1^2\Phi_4^2.A_2(q)$	3 chars	G_8
43		$\Phi_4.{}^2D_6(q)$	$\Phi_1\Phi_4.{}^2A_3(q)A_2(q)$	3 chars	Z_4
44		G^F	$C_{G^*}(s)^F$	30 chars	1

and the Mackey formula is not needed. We thus obtain the same Harish-Chandra series as in the case $q > 2$, and the results there continue to hold.

Similarly for $\ell = 5$, since $q = 2$, we are in the situation of Table 8 and we only need to consider Lusztig-series for quasi-isolated 3-elements. Thus, only cases 39–44 in that table matter. But note that again either λ is uniform (cases 39,40,42,43) in which case the Mackey formula is not needed for the determination of $R_L^G(\lambda)$, or $L = G$ and λ is of 5-defect zero, so lies in a block of defect zero. □

Remark 6.12. This completes the parametrization of ℓ-blocks of the finite quasi-simple groups. Indeed, the ℓ-blocks of the covering groups of alternating groups were found by Brauer and Robinson, and by Cabanes and Humphreys (see e.g [44]). The case of groups of Lie type in their defining characteristic was solved a long time ago by Humphreys; here, the non-trivial block are in bijection with irreducible characters of the centre, and they all have full defect.

This leaves the case of groups of Lie type where ℓ is different from the defining characteristic. The first general results on block distribution for classical type groups were obtained in the landmark papers of Fong and Srinivasan [22, 23], which also introduced some of the fundamental methods. For exceptional type groups, the case of unipotent blocks, that is, blocks containing some unipotent character, was first considered by Schewe [46]; complete results for some groups of low rank were obtained by Hiss, Deriziotis and

Michler, Malle in [27, 18, 35]. The parametrisation of all unipotent blocks for large primes ℓ was obtained in [10] in terms of e-Harish-Chandra series. Cabanes and Enguehard [14] determined all ℓ-blocks whenever ℓ is a good prime. Bonnafé and Rouquier [8] showed that ℓ-blocks parametrised by semisimple elements of the dual group whose centraliser lies in a proper Levi subgroup are Morita equivalent via Lusztig induction to unipotent blocks of smaller groups. The unipotent blocks for small ℓ and the quasi-isolated blocks of classical groups were found by Enguehard [20, 21].

6.5. **Quasi-isolated blocks for G_2 and ${}^3D_4$.** For later use we also record the following easy observations on quasi-isolated blocks for small exceptional type groups:

Lemma 6.13. *Let $G^F = G_2(q)$ or $G^F = {}^3D_4(q)$, $p \neq \ell \in \{2, 3\}$ and $s \in G^{*F}$ a quasi-isolated ℓ'-element. Then for $e = e_\ell(q)$, the e-Harish-Chandra series in $\mathcal{E}(G^F, s)$ satisfy Theorem 1.4 and $\mathcal{E}_\ell(G^F, s)$ is a single ℓ-block. Moreover in each numbered line in Table 9, $L = T$ is a torus with $T = C_G(T_\ell^F)$. For each ℓ-block corresponding to one of the numbered lines in the table there is a defect group $P \leq N_{G^F}(L, \lambda)$ with the structure described in Theorem 1.2. In particular, the defect groups are abelian precisely in case 3 when $G^F = G_2(q)$, and in case 1 when $G^F = {}^3D_4(q)$.*

Proof. In Table 9 we give the information on the 1-Harish-Chandra series in $\mathcal{E}(G^F, s)$ with the same conventions as earlier. The decomposition of R_L^G was determined by Lusztig, and from that it is easy to check Theorem 1.4 in this case. The situation for $e = 2$ is completely analogous. The assertion on the block and defect group structure can be deduced as previously; it was also already obtained in [27] for $G_2(q)$ and in [18] for ${}^3D_4(q)$. □

Table 9. Quasi-isolated blocks in $G_2(q)$ and ${}^3D_4(q)$

G^F	No.	$C_{G^*}(s)^F$	(ℓ, e)	L^F	λ	$W_{G^F}(L, \lambda)$
G_2	1	$A_2(q)$	$(2, 1)$	Φ_1^2	1	A_2
	2	${}^2A_2(q)$	$(2, 1)$	$\Phi_1\Phi_2$	1	A_1
				G^F	ϕ_{21}	1
	3	$A_1(q)A_1(q)$	$(3, 1)$	Φ_1^2	1	$A_1 \times A_1$
${}^3D_4$	1	$A_1(q)A_1(q^3)$	$(3, 1)$	$\Phi_1^2\Phi_3$	1	$A_1 \times A_1$

This concludes and completes the proof of Theorem 1.2 on the parametrization of quasi-isolated blocks for exceptional type groups and bad primes.

7. Defect groups and Bonnafé–Rouquier equivalences

The aim of this section is the proof of Theorem 1.3 which shows that abelian defect groups are preserved under Bonnafé–Rouquier Morita equivalences. Throughout, G will denote a connected reductive algebraic group over the algebraic closure of a finite field, and $F : G \to G$ a Steinberg endomorphism.

7.1. **Miscellany.** We start by proving some auxiliary statements.

Lemma 7.1. *Let G be connected reductive with derived subgroup of simply connected type, L an F-stable Levi subgroup of G and ℓ a prime. Let $G_1, \ldots, G_r$ be a set of representatives for the F-orbits on the set of simple components of $[G,G]$, and $L_i := G_i \cap L$. Suppose that*

$$C_{G_i}(Z(L_i)_\ell^{F^{d_i}}) = L_i \qquad \text{for } i = 1, \ldots, r,$$

where d_i denotes the length of the F-orbit of G_i. Then $C_G(Z(L)_\ell^F) = L$.

Proof. Let $H_1, H_2, \ldots, H_d$ denote an F-orbit on the set of simple components of $[G,G]$, $H := H_1 \cdots H_d$, and set $M_j := L \cap H_j$ for $1 \le j \le d$, $M := M_1 \cdots M_d = L \cap H$. Then $M^F \cong M_1^{F^d}$ (see e.g. [38, Ex. 30.2]), and similarly $C_H(Z(M)_\ell^F) \cong C_{H_1}(Z(M_1)_\ell^{F^d}) = M_1$ by assumption.

Now $G' = [G,G]$ is the direct product of F-orbits as before, and hence

$$C_{G'}(Z(L \cap G')_\ell^F) = C_{G_1}(Z(L_1)_\ell^{F^{d_1}}) \cdots C_{G_r}(Z(L_r)_\ell^{F^{d_r}}) = L_1 \cdots L_r = L \cap G'.$$

Finally, $G = G'T$ for a central torus T, whence the claim follows. □

Lemma 7.2. *Suppose that G has simply connected derived subgroup and ℓ is a good prime for G. For any finite abelian ℓ-subgroup A of G, $C_G(A)$ is a Levi subgroup of G.*

Proof. Let $A = \langle z_1, \ldots, z_r \rangle$ be a generating system for A. Since $[G,G]$ is simply connected, $C := C_G(z_1)$ is connected, and it is a Levi subgroup of G by [15, Prop. 13.16] since ℓ is a good prime for G. By [38, Prop. 12.14], $[C,C]$ is simply connected. We may now replace G by C and apply induction to conclude. □

Proposition 7.3. *Assume that G has simply connected derived subgroup over a field of odd characteristic. Let $T \le G$ be an F-stable maximal torus of G containing a Sylow e-torus, where $e = e_2(q)$, and $A = T_2^F$ the Sylow 2-subgroup of T^F. Then:*

(a) *$N_G(T)^F$ contains a Sylow 2-subgroup of G^F.*

(b) *$N_{G^F}(T)/T^F$ acts faithfully on A.*

(c) *$C_G(A) = T$.*

Proof. Since $G = Z(G)[G,G]$, we may argue in $[G,G]$, which is a direct product of simple groups, with F permuting the factors. So after possible extension of scalars we are reduced to G being simple.

By [36, Prop. 5.20] for example, $N_G(T)^F$ contains a Sylow 2-subgroup of G^F. For G of exceptional type, or for G^F of type ${}^3D_4$, the assertion in (b) can be checked using [39]. Otherwise T is the centralizer of a Sylow 1- or 2-torus and $W_G(T)^F$ is a Coxeter group of type A_l, B_l or D_l, with l suitable. Let's then write $W_l := W_G(T)^F$. The cases where $l \le 4$ can again be checked by computer, so now assume $l \ge 5$. Then it is easy to see that all non-trivial normal subgroups N of W_l have non-trivial intersection with its parabolic subgroup W_{l-1}, and thus act non-trivially on T^F by induction, except for $N = \langle w_0 \rangle$ generated by the longest element w_0 in types B_l and D_l. But the longest element acts by inversion on T^F, hence also non-trivially as T^F contains elements of order 4. So W_l acts faithfully in all cases.

For (c) let $M := \langle T^g \mid g \in G,\ A^g = A \rangle$ be generated by the maximal tori of G containing A. Then M is connected (see e.g. [38, Prop. 1.16]), F-stable, $W_{G^F}(T)$-invariant, and

$T \leq M \leq C_G(A)$. Let X denote its unipotent radical. If $[M/X, M/X] \neq 1$ then we obtain a non-trivial 2-element in the Weyl group $N_{M^F}(T)$, centralizing A but not lying in A, contradicting (b). Thus M is solvable. Let B denote a Borel subgroup of G containing M, with unipotent radical U, so $B = U.T$ with $X \leq U$. Let $w_0 \in W_{G^F}(T)$ be the longest element. If $u \in M$ is unipotent, then

$$u^{w_0} \in X^{w_0} \cap U^{w_0} = X \cap U^{w_0} \subseteq U \cap U^{w_0} = 1$$

(see [38, Cor. 11.18]), so $X = 1$ and $M = T$. This show that $C_G(A) \leq N_G(A) \leq N_G(T)$, but $W_{G^F}(T)$ acts faithfully on A by (b), whence $C_G(A) = T$. □

Lemma 7.4. *Assume that G has simply connected derived subgroup with all simple factors of type A, over a field of odd characteristic. Let $T \leq G$ denote an F-stable maximal torus such that $N_G(T)^F$ contains a Sylow 2-subgroup P of G^F and $C_G(T_2^F) = T$, and set $A = T_2^F$. Let $Z \leq Z(G^F)$ be a central subgroup of order 2. Then:*

(a) *If P centralises A/Z, then either G is a torus and $P = A$ or the components of $[G, G]$ are of type A_1, form a single F-orbit and the index of A in P is 2.*
(b) *Suppose that $Z = Z(P)$ and P/Z is abelian. Then P is quaternion of order 8.*

Proof. For (a), suppose that G is not a torus. Let I be the set of F-orbits on the simple components of $[G, G]$, and for each $i \in I$, let H_i denote the product of the simple components in i. So $[G, G]$ is a direct product of the H_i's and by the above, T is a product

$$Z^\circ(G)(\prod_{i \in I} T_i),$$

where T_i is an F-stable maximal torus of H_i such that $N_{H_i^F}(T_i)$ contains a Sylow 2-subgroup, say P_i of H_i^F, $C_{H_i}((T_i)_2^F) = T_i$ and P contains $\prod_{i \in I} P_i$. Since T is a maximal torus of G, T^F covers $G^F/[G, G]^F$ and hence $P = A(\prod_{i \in I} P_i)$.

Set $A_i = (T_i)_2^F = A \cap T_i$. For each i, P_i centralises $A_i Z/Z$. We claim that $|I| = 1$. Else, since Z is cyclic and the product of the H_i's is direct, $H_i \cap Z = 1$ for some $i \in I$, whence P_i centralises $A_i \cong A_i Z/Z$. But this is impossible as the Sylow 2-subgroups of H_i^F are non-abelian. So, $|I| = 1$, and either $[G, G]^F \cong \mathrm{SL}_n(q^d)$ or $[G, G]^F \cong \mathrm{SU}_n(q^d)$. If $n \geq 3$, then P_i does not centralise A_i/U, for a central subgroup U of order 2 of $[G, G]^F$. So, $n = 2$ and $[P_1 : A_1] = 2$. Since $P = AP_1$, it follows that $[P : A] = 2$.

We prove (c). Since P_1 is a Sylow 2-subgroup of a special linear or unitary group of degree 2, P_1 is quaternion. Also, P_1/Z is abelian, hence P_1 has order 8.

Since $Z = Z(P)$, $Z(G)_2^F \leq Z$, and hence the natural surjection of G onto $G/Z(G)$ induces an injection of P/Z into $(G/Z(G))^F \cong \mathrm{PGL}_2(q^d)$ (or $\mathrm{PGU}_2(q^d)$). The Sylow 2-subgroups of $(G/Z(G))^F$ are non-abelian of order 8, hence $|P/Z| \leq 4$. So $P = P_1$ is quaternion of order 8. □

For M an F-stable Levi subgroup M of G and s a semi-simple element of M^{*F}, we will be interested in the condition $C_{G^*}(s) \leq M^*$. The following translates this into the corresponding condition on G and M.

Lemma 7.5. *Let M be an F-stable-Levi subgroup of G and $s \in M^{*F}$ a semi-simple element. The following are equivalent.*

(i) $C_{G^*}(s) \leq M^*$.

(ii) *For $i = 1, 2$, let $T_i \leq M$ be F-stable maximal tori of M and $\theta_i \in \mathrm{Irr}(T_i^F)$ such that the M-geometric conjugacy class of (T_i, θ_i) both correspond via duality to the class of s. Then any $g \in G$ which geometrically conjugates (T_1, θ_1) to (T_2, θ_2) is in M.*

Proof. The implication (i) $\Rightarrow$ (ii) is in [19, Lemma 13.26(i)] and the reverse implication follows from reversing the argument of [19, Lemma 13.26(i)], and noting that for any F-stable torus T^* of M^* containing s, if $N_{G^*}(T^*) \cap C_{G^*}(s) \subseteq M^*$, then $C_{G^*}(s) \subseteq M^*$ (since $N_{G^*}(T^*) \cap C^\circ_{G^*}(s)$ is a Levi-subgroup of $C^\circ_{G^*}(s)$). $\square$

Lemma 7.6. *With the notation of Lemma 7.5, suppose that $C_{G^*}(s) \leq M^*$. Let L be an F-stable Levi subgroup of M and $\lambda \in \mathrm{Irr}(L^F)$ such that all constituents of $R_L^M(\lambda)$ lie in $\mathcal{E}(M^F, s)$. Let $z \in Z(L)^F$ and set $G_0 = C^\circ_G(z)$ and $M_0 = C^\circ_M(z)$. Let $s' \in {M_0^*}^F$ be a semisimple element such that all constituents of $R_L^{M_0}(\lambda)$ lie in $\mathcal{E}(M_0^F, s')$. Then $C_{G_0^*}(s') \leq M_0^*$.*

Proof. Let T be an F-stable maximal torus of L and θ an irreducible character of T^F such that λ is a constituent of $R_T^L(\theta)$. Since Lusztig induction preserves Lusztig series, the M-geometric conjugacy class of (T, θ) corresponds to the M^*-class of s and the M_0-geometric conjugacy class of (T, θ) corresponds to the G_0^*-class of s'. The assertion follows from Lemma 7.5 — here, note that $M_0 = G_0 \cap M$. $\square$

7.2. **Bonnafé–Rouquier correspondents.** In this subsection, M will denote an F-stable Levi subgroup of G and $s \in {M^*}^F$ a semi-simple ℓ'-element. We let c be an ℓ-block of M^F contained in $\mathcal{E}_\ell(M^F, s)$ and let b be an ℓ-block of G^F contained in $\mathcal{E}_\ell(G^F, s)$.

Recall that if $C_{G^*}(s) \leq M^*$, then for any semisimple ℓ-element $t \in C_{G^*}(s)$, $\epsilon_M \epsilon_G R_M^G$ induces a bijection between $\mathcal{E}(M^F, st)$ and $\mathcal{E}(G^F, st)$. This bijection is independent of choice of parabolic containing M (see [19, Rem. 13.28]) and it induces a bijection between ℓ-blocks in $\mathcal{E}_\ell(M^F, s)$ and in $\mathcal{E}_\ell(G^F, s)$. Further, by [8, Thm. B'] there is a Morita equivalence over $\mathcal{O}$ between pairs of corresponding blocks which induces the bijection $\chi \mapsto \epsilon_M \epsilon_G R_M^G(\chi)$ on ordinary irreducible characters.

Definition 7.7. We say that blocks b and c are *Bonnafé–Rouquier correspondents* if $C_{G^*}(s) \subseteq M^*$ and for some (and hence any) $\chi \in \mathcal{E}(M^F, s) \cap \mathrm{Irr}(c)$ we have $\epsilon_M \epsilon_G R_M^G(\chi) \in \mathrm{Irr}(b)$.

Lemma 7.8. *Suppose that b and c are Bonnafé–Rouquier correspondents. Let L be an F-stable Levi subgroup of M, let $\lambda \in \mathcal{E}(L^F, \ell')$, and suppose that every constituent of $R_L^M(\lambda)$ is contained in $\mathrm{Irr}(c)$. Then every constituent of $R_L^G(\lambda)$ is contained in $\mathrm{Irr}(b)$, and for every $\chi_0 \in \mathrm{Irr}(c)$, $\langle \chi_0, R_L^M(\lambda) \rangle \neq 0$ if and only if $\langle \chi, R_L^G(\lambda) \rangle \neq 0$, where $\chi = \epsilon_G \epsilon_M R_M^G(\chi_0)$ denotes the corresponding element of $\mathrm{Irr}(b)$. Further, $\langle {}^*R_L^M(\chi_0), d^{1,M^F}(\lambda) \rangle \neq 0$ if and only if $\langle {}^*R_L^G(\chi), d^{1,G^F}(\lambda) \rangle \neq 0$.*

Proof. The first two assertions follow from transitivity of Lusztig induction (see Theorem 2.8(a)). For the third claim, let $\chi_0 \in \mathrm{Irr}(c)$ and write

$$d^{1,M}(\chi_0) = \alpha_1 \phi_0^1 + \cdots + \alpha_t \phi_0^r$$

with α_i non-zero for all i and the ϕ_0^i pairwise distinct irreducible characters in c. So,

$$d^{1,G}(\chi) = \alpha_1 \phi^1 + \cdots + \alpha_r \phi^r.$$

Then,

$$\langle d^{1,M^F}(\chi_0), R_L^M(\lambda)\rangle \neq 0$$

if and only if $\phi_0^j = \chi_0^i$ for some i,j, if and only if $\phi^j = \chi^i$ for some i,j, if and only if $\langle d^{1,G^F}(\chi), R_L^G(\lambda)\rangle \neq 0$. Now the result follows since

$$\langle {}^*R_L^M(\chi_0), d^{1,M^F}(\lambda)\rangle = \langle d^{1,M^F}(\chi_0), R_L^M(\lambda)\rangle$$

and

$$\langle {}^*R_L^G(\chi), d^{1,G^F}(\lambda)\rangle = \langle d^{1,G^F}(\chi), R_L^G(\lambda)\rangle.$$

□

Proposition 7.9. *Suppose that b and c are Bonnafé–Rouquier correspondents. Let L be an F-stable Levi subgroup of M and let $\lambda \in \mathcal{E}(L^F, \ell')$ be such that $R_L^M(\lambda)$ has a constituent in the block c. Let $A = \langle z_1, \ldots, z_m\rangle$ be a subgroup of $Z(L)_\ell^F$ and set $M_i = C_M^\circ(z_1, \ldots, z_i)$, $G_i = C_G^\circ(z_1, \ldots, z_i)$, $1 \leq i \leq m$, $M_0 = M$, $G_0 = G$. Suppose the following:*

(1) *For any i, $1 \leq i \leq m$, and any character $\chi \in \mathrm{Irr}(M_i^F)$ with $\langle R_L^{M_i}(\lambda), \chi\rangle_{M_i^F} \neq 0$ we have $\langle d^{1,L^F}(\lambda), {}^*R_L^{M_i}(\chi)\rangle_{L^F} \neq 0$.*

(2) *The irreducible constituents of $R_L^{M_m}(\lambda)$ lie in a single block.*

Then, (1) and (2) hold with M_i replaced by G_i for $1 \leq i \leq m$. Consequently, for any i there exists a unique block, say b_i of G_i^F containing the constituents of $R_L^{G_i}(\lambda)$, and a unique block say c_i of M_i^F containing all constituents of $R_L^{M_i}(\lambda)$. Further, the following holds.

(a) *$b_0 = b$, $c_0 = c$, and for all i, $0 \leq i \leq m$, b_i and c_i are Bonnafé–Rouquier correspondents.*

(b) *Let $\tilde{c}_m$ denote the unique ℓ-block of $C_{M^F}(A)$ covering c_m and $\tilde{b}_m$ denote the unique ℓ-block of $C_{G^F}(A)$ covering b_m. Then $(A, \tilde{c}_m)$ is a c-Brauer pair and $(A, \tilde{b}_m)$ is a b-Brauer pair. Moreover,*

(c) *$N_{M^F}(A, \tilde{c}_m) \leq N_{G^F}(A, \tilde{b}_m)$, and hence*

$$N_{M^F}(A, \tilde{c}_m)/C_{M^F}(A) \leq N_{G^F}(A, \tilde{b}_m)/C_{G^F}(A)$$

under the inclusion of M in G.

Proof. By Proposition 2.12, for all i, $0 \leq i \leq m$, the irreducible constituents of $R_L^{M_i}(\lambda)$ lie in a unique block, c_i of M_i^F. Further, by Lemma 7.6, c_i has a Bonnafé–Rouquier correspondent, say b_i in G_i^F. The first assertion and (a) holds by Lemma 7.8, applied to c_i and b_i, $0 \leq i \leq m$. (b) follows from Proposition 2.12 applied to both M and G.

For (c), let $g \in N_{M^F}(A, \tilde{c}_m)$. Then, ${}^g c_m$ is covered by $\tilde{c}_m$, whence ${}^g c_m = {}^h c_m$ for some $h \in C_{M^F}(A)$. Let $\chi_0 \in \mathrm{Irr}(c_m)$. Then ${}^{h^{-1}g}\chi_0$ is an irreducible character of c_m. Since c_m and b_m are Bonnafé–Rouquier correspondents, $R_{M_m}^{G_m}({}^{h^{-1}g}\chi_0)$ and $R_{M_m}^{G_m}(\chi_0)$ are irreducible characters in b_m. So, noting the independence from the choice of parabolic subgroup of G_m containing M_m as pointed out before Definition 7.7,

$$R_{M_m}^{G_m}({}^{h^{-1}g}\chi_0) = {}^{h^{-1}g}R_{M_m}^{G_m}(\chi_0),$$

and ${}^{h^{-1}g}R_{M_m}^{G_m}(\chi_0)$ is in ${}^{h^{-1}g}\, b_m$. Hence, ${}^g b_m = {}^h b_m$ and it follows that ${}^g\tilde{b}_m = \tilde{b}_m$. □

7.3. **Good pairs.**

Definition 7.10. Let G be connected reductive with Steinberg endomorphism $F : G \to G$. Let ℓ be a prime and let b be an ℓ-block of G^F. A pair (L, λ) consisting of an F-stable Levi subgroup of G and $\lambda \in \mathcal{E}(L^F, \ell')$ is called a *good pair for* b if the following holds:

(1) $L = C_G(Z(L)^F_\ell)$,
(2) λ is of quasi-central ℓ-defect,
(3) $(Z(L)^F_\ell, b_{L^F}(\lambda))$ is a b-Brauer pair, and
(4) there is a maximal G^F-Brauer pair (P, f) such that $(Z(L)^F_\ell, b_{L^F}(\lambda)) \trianglelefteq (P, f)$.

Note that when (L, λ) satisfies (1)–(3) then by Propositions 2.5, 2.16(4), 2.13, and 2.12 all irreducible constituents of $R_L^G(\lambda)$ lie in b.

The notion of good pairs is related to that of e-cuspidal pairs in that many ℓ-blocks of G^F have good pairs which are also e-cuspidal pairs of G where $e = e_\ell(q)$ (see Theorem 7.12 below). However, the two notions are not identical, and it will be easier to track the structure of defect groups through Bonnafé–Rouquier Morita equivalences using good pairs (see Proposition 7.13 below).

Lemma 7.11. *Let $\tilde{G}$ be connected reductive with Steinberg endomorphism $F : \tilde{G} \to \tilde{G}$, containing G as an F-stable closed subgroup with $[\tilde{G}, \tilde{G}] \le G$ and let $\tilde{\mathbf{Z}} = Z^\circ(\tilde{G})$. Let b be a block of G^F and $\tilde{b}$ a block of $\tilde{G}^F$ covering b. Let L be an F-stable Levi subgroup of G and $\tilde{L} = \tilde{\mathbf{Z}}L$, a Levi subgroup of $\tilde{G}$. Set $A = Z(L)^F_\ell$ and $\tilde{A} = Z(\tilde{L})^F_\ell$. Suppose that $\lambda \in \mathcal{E}(L^F, \ell')$ is such that (L, λ) is a good pair for b, and let (P, f) be a maximal b-Brauer pair such that $(A, b_{L^F}(\lambda)) \trianglelefteq (P, f)$. Then:*

(a) *There exists $\tilde{\lambda} \in \mathcal{E}(\tilde{L}^F, \ell')$ covering λ and a maximal $\tilde{b}$-Brauer pair $(\tilde{P}, d)$ such that $(\tilde{L}, \tilde{\lambda})$ is a good pair for $\tilde{b}$, $(\tilde{A}, b_{\tilde{L}^F}(\tilde{\lambda})) \trianglelefteq (\tilde{P}, d)$ and $P \le \tilde{P} \le N_{\tilde{G}^F}(A, b_{\tilde{L}^F}(\lambda))$.*
(b) *Further, if b and $\tilde{b}$ are unipotent and (L, λ) is an e-cuspidal pair for G, then $(\tilde{L}, \tilde{\lambda})$ is a unipotent e-cuspidal pair of $\tilde{G}$.*
(c) *Suppose that $\tilde{\mathbf{Z}} \cap G$ contains no non-trivial ℓ-element. If λ is of central ℓ-defect, then so is $\tilde{\lambda}$.*

Proof. We have $\tilde{L} \le C_{\tilde{G}}(\tilde{A}) \le C_{\tilde{G}}(A) = \tilde{\mathbf{Z}}L = \tilde{L}$, hence $\tilde{L} = C_{\tilde{G}}(A) = C_{\tilde{G}}(\tilde{A})$. Also, note that the inclusion of $\tilde{L}$ in $\tilde{G}$ induces an isomorphism between $\tilde{L}^F/L^F$ and $\tilde{G}^F/G^F$.

By Lemma 2.1, there exists an ℓ-block b' of $\tilde{G}^F$ covering b and b'-Brauer pairs (A, v) and $(\tilde{P}, d)$ such that $(A, v) \trianglelefteq (\tilde{P}, d)$, $\tilde{P} \le N_{\tilde{G}^F}(A, b_{L^F}(\lambda))$, v covers $b_{L^F}(\lambda)$, $\tilde{P} \cap G^F = P$ and $\tilde{P}/P$ is isomorphic to a Sylow ℓ-subgroup of $N_{\tilde{G}^F}(A, b_{L^F}(\lambda))/N_{G^F}(A, b_{L^F}(\lambda))$. Since $\tilde{G}^F/G^F$ is abelian, by Lemma 2.2, $\tilde{b} = \theta \otimes b'$, for some linear character θ of $\tilde{G}^F/G^F$. So, $(A, \theta \otimes v)$ and $(\tilde{P}, \theta \otimes d)$ are $\tilde{b}$-Brauer pairs and since θ contains L^F in its kernel, $\theta \otimes v$ covers $\theta \otimes b_{L^F}(\lambda) = b_{L^F}(\lambda)$. Thus, replacing $\theta \otimes d$ by d and $\theta \otimes v$ by v, we may assume that $b' = \tilde{b}$. Since $\tilde{A}$ is central in $\tilde{L}^F$, we also get that $(\tilde{A}, v)$ is a $\tilde{b}$-Brauer pair and $(\tilde{A}, v) \trianglelefteq (\tilde{P}, d)$.

We claim that $\tilde{P}$ is a defect group of $\tilde{b}$. Indeed, since P is a defect group of b, and $\tilde{b}$ covers b, it suffices to prove that $|\tilde{P} : P| \ge |\tilde{G}^F : G^F|_\ell$. But,

$$\begin{aligned}|\tilde{P} : P| &= |N_{\tilde{G}^F}(A, b_{L^F}(\lambda)) : N_{G^F}(A, b_{L^F}(\lambda))|_\ell \\ &\ge |N_{\tilde{L}^F}(A, b_{L^F}(\lambda)) : L^F|_\ell = |\tilde{L}^F : L^F|_\ell = |\tilde{G}^F : G^F|_\ell.\end{aligned}$$

Here, for the second equality, note that the index of $N_{\tilde{L}^F}(A, b_{L^F}(\lambda))$ in $\tilde{L}^F$ is prime to ℓ. This proves the claim.

Now let $\tilde{\lambda} \in \mathcal{E}(\tilde{L}^F, \ell') \cap \mathrm{Irr}(v)$. Then $\tilde{\lambda}$ covers an element of $\mathcal{E}(L^F, \ell') \cap \mathrm{Irr}(b_{L^F}(\lambda))$ (see [6, Prop. 11.7(b)]). Since $\mathrm{Irr}(b_{L^F}(\lambda)) \cap \mathcal{E}(L^F, \ell') = \{\lambda\}$ by Proposition 2.5, $\tilde{\lambda}$ covers λ. Further, $\tilde{\lambda}$ and λ cover a common character of $[\tilde{L}, \tilde{L}]^F = [L, L]^F$, so $\tilde{\lambda}$ is of quasi-central ℓ-defect. This proves (a).

(b) follows from (a) and the fact that restriction induces a bijection between $\mathcal{E}(\tilde{G}^F, 1)$ and $\mathcal{E}(G^F, 1)$ which commutes with Lusztig induction and restriction.

It remains to prove (c). Since $L \leq G$, $\tilde{\mathbf{Z}} \cap L$ contains no non-trivial ℓ-element. So, any Sylow ℓ-subgroup of $\tilde{L}^F$ is a direct product of the Sylow ℓ-subgroup of $\tilde{\mathbf{Z}}^F$ and a Sylow ℓ-subgroup of L^F, and similarly, the Sylow ℓ-subgroup of $Z(\tilde{L}^F)$ is a direct product of the Sylow ℓ-subgroup of $\tilde{\mathbf{Z}}^F$ and the Sylow ℓ-subgroup of $Z(L)^F$. The result follows as $|\tilde{\lambda}(1)|_\ell = |\lambda(1)|_\ell$. □

The next result shows in particular that all quasi-isolated blocks have good pairs.

Theorem 7.12. *Suppose that $[G, G]$ is simply connected. Let b be an ℓ-block of G^F with $\mathrm{Irr}(b) \subseteq \mathcal{E}_\ell(G^F, s)$ and let $e = e_\ell(q)$.*

(a) *Suppose that ℓ is odd, good for G and $\ell \neq 3$ if ${}^3D_4(q)$ is involved in G^F. Then b has a good pair (L, λ) and a maximal b-Brauer pair (P, f) with $(Z(L)^F_\ell, b_{L^F}(\lambda)) \trianglelefteq (P, f)$, such that λ is of central ℓ-defect, the extension*
$$1 \to Z(L)^F_\ell \to P \to P/Z(L)^F_\ell \to 1$$
is split and $Z(L)^F_\ell$ is the unique maximal normal abelian subgroup of P. If s is central, then (L, λ) can be chosen to be e-cuspidal, and in that case (L, λ) is unique up to G^F-conjugacy.

(b) *Suppose that $\ell = 2$ and all components of G are of type A. Then b has a good pair (L, λ) and a maximal b-Brauer pair (P, f) with $(Z(L)^F_2, b_{L^F}(\lambda)) \trianglelefteq (P, f)$, such that $Z(L)^F_2 = T^F_2$, λ is of central 2-defect and $\mathrm{Aut}_P(T^F_2) = \mathrm{Aut}_{P'}(T^F_2)$. Here, T is an F-stable maximal torus of G such that $C_{G_1}(T^F_2) = T$ for a Levi subgroup G_1 of G in duality with $C^\circ_{G^*}(s)$, and such that $N_{G_1^F}(T)$ contains a Sylow 2-subgroup P' of G_1^F.*

(c) *Suppose that $\ell = 2$, G is simple, of classical type different from type A and s is quasi-isolated in G^*. Then b has a good pair (L, λ), where L is an F-stable maximal torus of G containing a Sylow e-torus of G.*

(d) *Suppose that s is quasi-isolated and either G is simple of exceptional type and ℓ is bad for G, or G is of rational type ${}^3D_4$ and $\ell = 2, 3$. Then b has a good pair (L, λ) which is e-split cuspidal. Further, if ℓ is odd, λ is of central ℓ-defect.*

In particular, if b is quasi-isolated then b has a good pair.

Proof. Suppose the assumptions of (a) hold. Then there exists a pair (L, λ) (denoted (M, ζ_M) in [14]) such that $C^\circ_G(Z(L)^F_\ell) = L$ ([14, Lemma 4.8]), λ is of central ℓ-defect ([14, Lemma 4.11]), and letting $\hat{u}$ denote the unique ℓ-block of $C_{G^F}(Z(L)^F_\ell)$ covering $b_{L^F}(\lambda)$, $(Z(L)^F_\ell, \hat{u})$ is a b-Brauer pair ([14, Lemma 4.10]). Further, there exists a maximal G^F-Brauer pair (P, f) with $(Z(L)^F_\ell, \hat{u}) \trianglelefteq (P, f)$ and such that the short exact sequence above has the required properties ([14, Lemma 4.16]). Thus, in order to prove that (L, λ) is a good pair for b, we need only show that $L = C_G(Z(L)^F_\ell)$. But since $C^\circ_G(Z(L)^F_\ell) = L$ this follows from Lemma 7.2. The final assertion follows from [13, Thm. 1.1, Lemma 4.5].

Suppose that the assumptions of (b) hold and note that T as in the statement exists by Proposition 7.4 applied to G_1. Let $G \to \tilde G$ be a regular embedding. For $\tilde G_1$ an F-stable Levi subgroup of $\tilde G$ with $\tilde G_1 \cap G = G_1$ we let $\tilde T$ be an F-stable maximal torus of $\tilde G_1$ with $\tilde T \cap G_1 = T$. Set $A = T_2^F$ and $\tilde A = \tilde T_2^F$. By Lemma 7.2, $\tilde L := C_{\tilde G}(A)$ is a Levi subgroup of $\tilde G$ and $L := C_G(A)$ is a Levi subgroup of G.

Let $\tilde s \in \tilde G^{*F}$ be an element of odd order lifting s such that $C_{\tilde G^*}(s) = \tilde G_1^*$, where $\tilde G_1^*$ is the dual of $\tilde G_1$ in $\tilde G^*$ and let θ be the linear character of $\tilde G_1^F$ in duality with $\tilde s$ (see [19, Prop. 13.30]). By [21, Prop. 1.5], $\mathcal{E}_2(\tilde G_1^F, \tilde s)$ is a single 2-block, say $\tilde c$ and $\mathcal{E}_2(\tilde G^F, \tilde s)$ is a single 2-block, say $\tilde b$. In particular, $\tilde b$ covers b. Moreover, $R^{\tilde G}_{\tilde G_1}$ induces a Morita equivalence between $\tilde c$ and $\tilde b$. Since $C_{G_1}(A) = T$, $C_{\tilde G_1}(A) = \tilde T$, and hence by Proposition 7.9, applied with $M = \tilde G_1$, $G = \tilde G$, and $L = \tilde T$, $R^{\tilde L}_{\tilde T}(\theta)$ is (up to sign) an irreducible character, say $\tilde\chi$ of $\tilde L^F = C_{\tilde G}(A)^F$ and $(A, b_{\tilde L^F}(\tilde\chi))$ is a $\tilde b$-Brauer pair.

Let $P' \le N_{G_1}(T)$ be a Sylow 2-subgroup of G_1^F. Then $P' \le N_{\tilde G_1}(A, \theta)$, so by Proposition 7.9, $P' \le N_{\tilde G}(A, b_{\tilde L^F}(\tilde\chi))$. In particular, P' acts on the blocks of L^F covered by $b_{\tilde L^F}(\tilde\chi)$. There is an odd number of such blocks, so there exists a block f of L^F covered by $b_{\tilde L^F}(\tilde\chi)$ which is P'-stable. Let $\chi' \in \mathrm{Irr}(f) \cap \mathcal{E}(L^F, \ell')$ be covered by $\tilde\chi$ and let b' be the block of G^F such that (A, f) is a b'-Brauer pair. Then b' is covered by $\tilde b$. Since $\pm R^{\tilde L}_{\tilde T}(\theta) \in \mathrm{Irr}(\tilde L^F, \tilde s)$, and $\tilde s$ is an odd order element,

$$\chi'(1)_2 = \tilde\chi(1)_2 = \frac{|\tilde L^F|_2}{|\tilde T^F|_2} \ge \frac{|L^F|_2}{|T^F|_2} = \frac{|L^F|_2}{|A|}.$$

Since A is central in L^F, $A \le \ker(\chi')$. From the above displayed equation it follows that $A = Z(L)_2^F$ and χ' is of central 2-defect. Let (P, d) be a b'-Brauer pair, maximal with respect to $(A, f) \trianglelefteq (P, d)$. Then $P \cap L^F = A$ and by Lemma 2.1(a), P/A is a Sylow ℓ-subgroup of $N_{G^F}(A, f)/L^F$. Since $P' \le N_{G^F}(A, f)$ and $P' \cap C_{G^F}(A) = A$, it follows that $|P| \ge |P'|$. On the other hand, since b' being covered by b means that $\mathrm{Irr}(b') \subseteq \mathcal{E}_\ell(G^F, s)$, by [21, Prop. 1.5], any Sylow 2-subgroup of G_1^F is a defect group of b'. Hence, (P, d) is a maximal b'-pair. Since L is a Levi subgroup of G, (A, χ') is a good pair for b'. Now b and b' are both covered by $\tilde b$ hence replacing (L, χ') by a suitable $\tilde G^F$-conjugate gives the desired result.

Now suppose that the assumptions of (c) hold. So G is simple of type B, C or D. Then $s = 1$ is the only odd order quasi-isolated element of G^*. By [21, Prop. 1.5] G^F has a unique unipotent 2-block, hence by Proposition 7.3 $(L, 1)$ is a good pair for b for any F-stable maximal torus L of G containing a Sylow e-torus of G.

Suppose the assumptions of (d) hold. If s is non-central in G^*, then the result follows from Theorem 1.2 and its proof. If s is central in G^*, then (d) follows from [20]. Note that Enguehard does not state the equality $L = C_G(Z(L)^F_\ell)$ for all unipotent e-cuspidal pairs (L, λ) but this can be checked — the Levis occurring for central quasi-isolated elements also occur in our tables, except for the 1-split Levis of type D_4 in E_6, and of type E_6 in E_7 and their Ennola duals, and these cases can be easily checked also.

Finally, suppose that b is a quasi-isolated block of G^F. Then any block of $[G,G]^F$ covered by b is quasi-isolated, so by Lemma 7.11, we may assume that $G = [G,G]$. Since G is a direct product of simple simply connected groups and the component of b in the fixed points of each F-orbit is quasi-isolated, by Lemma 7.1 we may assume that G is simple. Now the result follows from parts (a)–(d). □

Proposition 7.13. *Suppose that b and c are Bonnafé–Rouquier correspondents and that c has a good pair (L,λ). Let $A = Z(L)_\ell^F$, let $u = b_{L^F}(\lambda)$ and let (P,f) be a maximal c-Brauer pair with $(A,u) \trianglelefteq (P,f)$. Then:*

(a) *Let v be the ℓ-block of $C_G^\circ(A)^F$ containing the constituents of $R_L^{C_G^\circ(A)}(\lambda)$ and let $\tilde{v}$ be the block of $C_G(A)^F$ covering v. Then there is a maximal G^F-Brauer pair (Q,d) such that $(A,\tilde{v}) \trianglelefteq (Q,d)$, $C_Q(A) \cong C_P(A)$ and $\mathrm{Aut}_Q(A) = \mathrm{Aut}_P(A)$.*
(b) *b has abelian defect groups if and only if c has abelian defect groups. If this is the case, then the defect groups of b and c are isomorphic.*
(c) *Let Z be a central ℓ-subgroup of G^F and let $\bar{c}$ (respectively $\bar{b}$) be the image of c (respectively b) in M^F/Z (respectively G^F/Z). If either $\bar{b}$ or $\bar{c}$ has abelian defect groups, then P centralises A/Z and Q centralises A/Z.*
(d) *If λ is of central ℓ-defect and c has abelian defect groups, then A is a defect group of both b and c.*

Proof. Let U be a subgroup of A. Since $C_M(A) = L$, $C_{C_M(U)}(A) = L$. So, since λ is of quasi-central ℓ-defect, by Proposition 2.16, the conditions of Proposition 7.9 hold for any choice of generators $\langle z_1,\ldots,z_m\rangle$ of A and the statement of (a) makes sense. In particular, u and v are Bonnafé–Rouquier correspondents.

Let $(A,\tilde{v}) \trianglelefteq (Q,d)$ where Q is maximal with respect to the property that $Q \leq N_{G^F}(A,\tilde{v})$. Then $QC_{G^F}(A)/C_{G^F}(A)$ is a Sylow ℓ-subgroup of $N_{G^F}(A,\tilde{v})/C_{G^F}(A)$ (see Lemma 2.1). So by Proposition 7.9(c), and by replacing if necessary (Q,d) by an $N_{G^F}(A,\tilde{v})$-conjugate, $QC_{G^F}(A)/C_{G^F}(A)$ contains $PC_{M^F}(A)/C_{M^F}(A)$. In particular, $P/C_P(A)$ is isomorphic to a subgroup of $Q/C_Q(A)$.

Now $C_P(A)$ is a defect group of the block u, u is nilpotent and is Morita equivalent to v (over $\mathcal{O}$). By a result of Puig a Morita equivalence over $\mathcal{O}$ between a nilpotent block and a block preserves nilpotency and isomorphism type of defect groups (see [45, Thm. 8.4 and Cor. 7.3]), so v is nilpotent and a defect group of v is isomorphic to $C_P(A)$. Since $C_Q(A)$ contains a defect group of $\tilde{v}$, $C_Q(A)$ contains a defect group of v, and hence $|C_Q(A)| \geq |C_P(A)|$. We have shown above that $P/C_P(A)$ is isomorphic to a subgroup of $Q/C_Q(A)$, hence $|Q| \geq |P|$. On the other hand, $|Q| \leq |P|$ as Q is contained in a defect group of b, and P is a defect group of c, and b and c are Morita equivalent. Thus Q is a defect group of b, $C_Q(A) \cong C_P(A)$ is a defect group of v and $PC_{G^F}(A)/C_{G^F}(A) = QC_{G^F}(A)/C_{G^F}(A)$. This proves (a).

Part (b) follows from (a) as Q is abelian if and only if $Q = C_Q(A)$ and $C_Q(A)$ is abelian, and similarly P is abelian if and only if $P = C_P(A)$ and $C_P(A)$ is abelian. Part (c) follows from (a) and (b) on observing that P/Z is a defect group of $\bar{c}$ and Q/Z is a defect group of $\bar{b}$. Finally, suppose that λ is of central ℓ-defect and c has abelian defect groups. Then $P = A = Q$. □

7.4. **Proof of Theorem 1.3.** In this subsection, M will denote an F-stable Levi subgroup of G, and b and c will be ℓ-blocks of G^F and M^F respectively. For large ℓ, Theorem 1.3 follows from the work of Cabanes–Enguehard and Enguehard.

Proposition 7.14. *Suppose that b and c are Bonnafé–Rouquier correspondents. Let Z be a central ℓ-subgroup of G^F and let $\bar{b}$ and $\bar{c}$ be the images of b and c in G^F/Z and M^F/Z respectively.*

(a) *If $[G,G]$ is simply connected, ℓ is odd, good for G and $\ell \neq 3$ if ${}^3D_4(q)$ is involved in G^F, then $\bar{b}$ and $\bar{c}$ have isomorphic defect groups.*

(b) *If $\ell = 2$, and all components of G are classical, then $\bar{b}$ and $\bar{c}$ have a common defect group.*

Proof. In the situation of (a), c has a good pair (L,λ) by Theorem 7.12(a), and $A = Z(L)_\ell^F$ has a complement in P and $C_P(A) = A$, with (P,f) a maximal c-Brauer pair with $(A,u) \trianglelefteq (P,f)$. If (Q,d) is a maximal b-Brauer pair as in Proposition 7.13(a), then $C_Q(A) = A$, whence A is a maximal normal abelian subgroup of Q. But by the structure of the defect groups of b as given in [14, Lemma 4.16], Q has a unique maximal normal abelian subgroup and this subgroup has a complement in Q. So, A has a complement in Q. The result follows from Proposition 7.13(a).

In (b) let $G_1 \leq M$ be a Levi subgroup of G in duality with $C_{G^*}(s) \leq M^*$. Then, by [21, Prop. 1.5], any Sylow 2-subgroup of G_1^F is a defect group of both b and c. □

In fact, as pointed out to us by Marc Cabanes, it can be deduced from [14, Lemma 4.16] that the two blocks in the situation of Proposition 7.14(a) have a common defect group. The next result will be needed to deal with E_6 at $\ell = 3$.

Lemma 7.15. *Suppose that G is simply connected in characteristic not 3 and all components of G are of type A. Let b be a unipotent 3-block of G^F, (L,λ) an $e_3(q)$-cuspidal unipotent pair which is a good pair for b as in Theorem 7.12(a), $A = Z(L)_3^F$ and let (P,u) be a maximal b-Brauer pair with $(A, b_{L^F}(\lambda)) \trianglelefteq (P,u)$. Let Z be a central subgroup of order 3 of G^F. Suppose that P is non-abelian and that P acts trivially on A/Z. Then:*

(a) *There is an F-orbit of irreducible components of $[G,G]$ of type A_2 whose group of F-fixed points contains Z, and this is the only F-orbit of irreducible components of $[G,G]$ whose fixed points contain a central subgroup of order 3. Further, P/A is cyclic.*

(b) *If moreover $Z(P) = Z$ and P/Z is abelian, then P is extra-special of order 3^3.*

Proof. Since restriction induces a bijection between the unipotent characters of G^F and $[G,G]^F$, there is a unique block, say b_0 of $[G,G]^F$ covered by b and it is unipotent. Let I be the set of F-orbits on the simple components of $[G,G]$, and for each $i \in I$, let H_i denote the product of the simple components in i. So $[G,G]$ is a direct product of the H_i's and $[G,G]^F$ is a direct product of the H_i^F's. For $i \in I$ let b_i be the block of H_i^F covered by b_0, let (L_i,λ_i) be an e-cuspidal unipotent good pair for b_i and let $(L_0,\lambda_0) = (\prod_{i\in I} L_i, \prod_{i\in I} \lambda_i)$ be the corresponding pair for b_0. Further, let P_i be the first component of a maximal b_i-Brauer pair normalising $(A, b_{L_i}^F(\lambda_i))$. Then, up to replacing (L,λ) and then (P,u) by a G^F-conjugate, (L,λ) is an extension of (L_0,λ_0) as in Lemma 7.11, $A \cap L_i^F = Z(L_i)_3^F$ and $P_i = P \cap L_i^F$ for all $i \in I$.

For each $i \in I$, $[P_i, A_i] \leq [P, A] \leq Z$, and Z is cyclic, hence there exists at most one $i \in I$ with $[P_i, A_i] \neq 1$, say $i = j$. Since, by Theorem 7.12(a), λ_i is of central 3-defect, we have $A_i = P_i$ for all $i \neq j$.

Let $i \in I$ and suppose that the rational type of H_i^F is $(A_n, \epsilon q^m)$. The group H_i^F contains a central element of order 3 if and only if $3|d_i := \gcd(q^m - \epsilon, n + 1)$. Further, if $3|d_i$, then by [13, Prop. 3.3], b_i is the principal block of H_i^F, and P_i is a Sylow 3-subgroup of H_i^F. Consequently, if $3|d_i$, then P_i is non-abelian. So for all $j \neq i \in I$, $3 \not| d_i$ and in particular H_i^F does not contain a central element of order 3.

By Theorem 7.12(a), λ is of central defect, hence $C_P(A) = A$. Since P is non-abelian, $Z \leq [P, P] \leq [G, G]^F$. Hence, H_j^F contains a central element of order 3, thus $3|d_j$. Suppose the rational type of H_j^F is $(A_n, \epsilon q^m)$. If $n \geq 5$, or $3^2|(q^m - \epsilon)$ then $[P_j, A_j] \leq Z$ has order at least 9. Thus, $n = 2$, $3||(q^m - \epsilon)$ and P_j is extra-special of order 3^3.

Let $H' = Z(G)H_j$, let b' be the (unique) block of H'^F covered by b and $P' = P \cap H'^F$, a defect group of b'. Then P' is a Sylow 3-subgroup of H'^F and $P = P' \times \prod_{i \in I, j \neq i} A_i$. Since $Z(G) \cap H_j^F$ has order at most 3, $Z(G)_3^F \leq A$, and A_j has index 3 in P_j, it follows that A has index 3 in P. This proves (a).

Now suppose that the hypothesis of (b) hold. We have shown above that P_j has order 3^3. Since $Z = Z(P)$, $P = P'$ and $Z(H')_3^F \leq Z$. Thus the surjection of H' onto $(H'/Z(H'))$ induces an injection of P/Z into $(H'/Z(H'))^F$. But $(H'/Z(H'))^F$ has non-abelian Sylow 3-subgroups of order $|P_j|$, hence $|P/Z| < |P_j|$ which means that $P = P_j$ is extra-special of order 3^3. $\square$

Theorem 7.16. *Suppose that G is simple, simply connected and that b and c are Bonnafé–Rouquier correspondents. Let Z be a central ℓ-subgroup of G^F and let $\bar{b}$ and $\bar{c}$ be the images of b and c in G^F/Z and M^F/Z respectively. If either $\bar{b}$ or $\bar{c}$ has abelian defect groups, then the defect groups of $\bar{b}$ and $\bar{c}$ are isomorphic.*

Proof. By Proposition 7.14, we may assume that either G is of exceptional type and ℓ is a bad prime for G, or $\ell = 3$ and $G^F = {}^3D_4(q)$. If $Z = 1$, the statement is immediate from Proposition 7.13(b). So, we may assume that $Z \neq 1$, whence either $\ell = 3$ and G is of type E_6, or $\ell = 2$ and G is of type E_7.

We first consider the case that c is quasi-isolated. By Theorem 7.12, c has a good pair, say (L, λ). Set $A = Z(L)_\ell^F$, let (P, f) be a maximal c-Brauer pair with $(A, b_{L^F}(\lambda)) \trianglelefteq (P, f)$ and let (Q, d) be a maximal b-Brauer pair as in Proposition 7.13(b). By Proposition 7.13(c), $[P, A] \leq Z$.

We make some reductions. Suppose that λ is of central defect. Then $C_P(A) = A$, hence either $P = A$ or $[P, A] = Z$ and $Z(P) \leq A$. If $P = A$, then $P/Z = Q/Z = A/Z$ and there is nothing to prove. Also, $[P, P]$ is contained in $[C_G(z_1), C_G(z_1)]$ and in $[C_M(z_1), C_M(z_1)]$ for $z_1 \in Z(P)$. Thus, we may assume the following: If λ is of central defect, then $Z = [P, A]$, $Z(P) \leq A$ and for any $z_1 \in Z(P)$, $[C_G(z_1), C_G(z_1)]$ and $[C_M(z_1), C_M(z_1)]$ contain non-trivial central ℓ-elements.

Next, let $z_1 \in A$, $M_1 = C_M(z_1)$, $G_1 = C_G(z_1)$. By Proposition 7.9, there exist blocks c_1 and b_1 of M_1^F and G_1^F respectively, which are Bonnafé–Rouquier correspondents, and such that $(\langle z_1 \rangle, c_1)$ is a c-Brauer pair, and $(\langle z_1 \rangle, b_1)$ is a b-Brauer pair. Note that since G and M are simply connected, G_1 and M_1 are connected. If $z_1 \in Z(P)$, then $P \leq M_1^F$ is a defect group of c_1, and also $Q \leq G_1^F$ is a defect group of b_1. Thus, by Proposition 7.14, applied to the blocks b_1 and c_1 we may assume the following. If G is of type E_6, $\ell = 3$ and $z_1 \in (Z(P) \cap A) \setminus Z$ then $C_G(z_1)$ contains a component of type D_4 and if G is of type E_7, $\ell = 2$ and $z_1 \in (Z(P) \cap A) \setminus Z$, then $C_G(z_1)$ contains a component of type E_6.

Suppose that G is of type E_6 and $\ell = 3$. Then M is classical, so 3 is good for M and by Theorem 7.12, λ_0 is of central 3-defect. Suppose first that $[M, M]$ has a component of type D_4 or D_5. By rank considerations $[M, M]$ does not contain a central element of order 3, so λ is of central defect by Lemma 7.11(c), whence by the first reduction above $Z \leq [P, P]$. But $[P, P] \leq [M, M]$, a contradiction.

So, we may assume that all components of M are of type A. By Theorem 7.12, λ is of central 3-defect. By rank consideration, if $z \in P$ is such that $[C_G(z), C_G(z)]$ contains a component of type D_4 or D_5, then $[C_G(z), C_G(z)]$ does not contain a central element of order 3, hence by the first reduction $z \notin Z(P)$. By the second reduction, we may assume that $Z(P) = Z$.

Now $C_{M^*}(s)/C^\circ_{M^*}(s)$ is isomorphic to a subgroup of $Z(M)/Z^\circ(M)$, hence to a subgroup of $Z(G)/Z^\circ(G)$, the latter being of order 3. On the other hand, the exponent of $C_{M^*}(s)/C^\circ_{M^*}(s)$ divides the order of s, which is prime to 3. Thus, $C_{M^*}(s)$ is connected, whence s is isolated in M^*. But all components of M^* are of type A, hence M^* has no non-central isolated elements. Thus s is central in M^* and $c = \theta \otimes c'$, where c' is a unipotent block of M^F and θ is a linear character of M^F in duality with s. In particular, P is a defect group of a unipotent block of M^F. By Lemma 7.15(a), P/A (and hence Q/A) is cyclic of order 3, thus P/Z is abelian if and only if Q/Z is abelian. So, we may assume that P/Z is abelian. We have shown above that $Z = Z(P)$. By Lemma 7.15(b), P is extra-special of order 3^3. Thus, Q is extra-special of order 3^3, so P/Z and Q/Z are elementary abelian of order 3^2, and in particular isomorphic.

Suppose that G is of type E_7 and $\ell = 2$. Let Z be the centre of G of order 2. Suppose first that M has a component of type E_6. Then $[M, M]$ is simple of type E_6. Consequently $[M, M]$ does not contain a central element of order 2, and it follows that P is abelian. By Proposition 4.3, $[M, M]^F$ does not contain a non-unipotent, quasi-isolated 2-block with abelian defect groups, so c covers a unipotent block of $[M, M]^F$. By the tables for $E_6(q)$ and ${}^2E_6(q)$ in [20], c_0 is of defect 0. Since $M^F/[M, M]^F$ has cyclic Sylow 2-subgroups, P and hence Q are cyclic and so are P/Z and Q/Z.

Thus, we may assume that M is classical. Suppose that M has a simple component, say H_1 of type D_n, $n \geq 4$. The principal 2-block of H_1^F is the only quasi-isolated 2-block of H_1^F, hence P contains a Sylow 2-subgroup of H_1^F and by Theorem 7.12(c), we may assume that this Sylow 2-subgroup normalises T_2^F where T is an F-stable maximal torus containing a Sylow e-torus and such that the commutator of the Sylow subgroup with T_2^F is contained in a cyclic group of order 2. But this is not the case.

So, we may assume that all components of M are of type A. Then by Theorem 7.12, λ is of central 2-defect. By the same argument as in the E_6-case above we conclude that $z \notin Z(P)$ and $Z(P) = Z$.

Let $G_1 \leq M$ be a Levi subgroup of G in duality with $C^\circ_{G^*}(s) = C^\circ_{M^*}(s)$. Let $P' \leq N_{G_1}(A)$ be a Sylow 2-subgroup of G_1^F as in Theorem 7.12(b). Since $[P, A] \leq Z$, $[P', A] \leq Z$. So, by Lemma 7.4(b), the index of A in P' is 2. Hence the index of A in P and Q is also 2 and P/Z is abelian if and only if Q/Z is abelian. So, we may assume that P/Z is abelian, and hence that P'/Z is abelian.

Since $Z(P) = Z$, $Z(P') = Z$, and by Lemma 7.4(c), P' is quaternion of order 8, hence both P and Q are non-abelian of order 8, and P/Z and Q/Z are elementary abelian of order 4.

Now suppose that c is not quasi-isolated in M. Then, replacing M by an F-stable Levi subgroup whose dual contains $C_{M^*}(s)$ and in which s is quasi-isolated, and G by M, the above argument again gives the desired result (note that above we do not use that G is of type E_6 or E_7, but only that Z has order 2 or 3 and that the rank of G is at most 7). □

8. Brauer's height zero conjecture

In this section we give the arguments which are necessary to combine our results and those obtained previously by various authors to prove (HZC1), that is, Theorem 1.1.

8.1. **Groups not of Lie type.**

Proposition 8.1. *Let S be a perfect central extension of a sporadic simple group or the Tits simple group ${}^2F_4(2)'$. Then (HZC1) holds for S.*

Proof. It is well known that a Sylow p-subgroup of a covering group of a sporadic simple group of order at least p^3 is non-abelian unless $S = J_1$ and $p = 2$, or $S = ON$ and $p = 3$. Since the block distribution of ordinary characters as well as the size of the respective defect groups can easily be obtained using GAP, this deals with most blocks in question. For the remaining blocks (i.e., non-principal blocks with defect group of order at least p^3) which are only in characteristic 2 or 3, either the structure of the defect group is given by Landrock [32], or it can easily be shown to be of extra-special type (see Müller [40]). □

Proposition 8.2. *Let S be an exceptional covering group of a finite simple group of Lie type, or of $\mathfrak{A}_7$. Then (HZC1) holds for S.*

Proof. From the ordinary character tables in [16] it follows that all p-blocks of the groups in question fall into three categories: either all characters in the block are of height zero, or the block is principal and the Sylow p-subgroups are non-abelian, or $p = 2$, $S = 3.\mathrm{O}_7(3)$ or $6.\mathrm{O}_7(3)$.

Let $S = 6.\mathrm{O}_7(3)$, let b be a 2-block of S and denote by $\bar{b}$ the corresponding 2-block of $\bar{S} := 3.\mathrm{O}_7(3)$. By the modular atlas [17], the defect groups of b have order 2^{10}, 16, 4 or 2. In the first case, the defect groups of b (respectively $\bar{b}$) are Sylow 2-subgroups of S (respectively $\bar{S}$) and hence non-abelian. If the defect groups of b are cyclic or Klein 4-groups, then all characters in b and $\bar{b}$ are of height zero. So assume that the defect groups of b have order 16, and hence that the defect groups of $\bar{b}$ have order 8. From ordinary character tables it follows that there exists an irreducible character in $\bar{b}$ which does not vanish on an element of $\bar{S}$ of order 4. Thus, the defect groups of $\bar{b}$ are not elementary abelian. On the other hand, by [17], $\bar{b}$ has two modular irreducible characters. Since blocks with defect groups isomorphic to $C_4 \times C_2$ or to C_8 have a unique modular irreducible character and since blocks with defect groups isomorphic to Q_8 have either one or three modular irreducible characters, it follows that the defect groups of $\bar{b}$ are dihedral. In particular, the defect groups of $\bar{b}$ and of b are non-abelian. □

8.2. **Bad primes for exceptional type groups.** We will need the following result of Enguehard [20, §3.2], respectively Ward [48] and Malle [35]. Let G be connected reductive with a Steinberg endomorphism $F : G \to G$ and let ℓ be a prime number different from the defining characteristic of G. By [20, Thm. A], the assertions of Theorem 1.2 hold for G^F and ℓ for the case $s = 1$, and with the "quasi-central ℓ-defect" condition in (a3) of Theorem 1.2 replaced by "central ℓ-defect". For a unipotent e-cuspidal pair (L, λ) of G such that λ is of central ℓ-defect and $S = G^F/Z$ for some central subgroup Z of G^F we denote by $b_S(L, \lambda)$ the ℓ-block of S corresponding to (L, λ), respectively its image in G^F/Z, and by $W_{G^F}(L, \lambda)$ the relative Weyl group $N_{G^F}(L, \lambda)/L^F$.

Proposition 8.3. *Suppose that G is simple, simply connected of exceptional type and that ℓ is a bad prime for G. Let $S = G^F/Z$ for some central subgroup Z of G^F and $B = b_S(L, \lambda)$ a unipotent ℓ-block of S with non-trivial abelian defect groups. Then B is as in Table 10 or Ennola dual to an entry there. Moreover, $W_{G^F}(L, \lambda)$ is an ℓ'-group, except for the listed entries for $E_6(q)$ and $E_7(q)$ and their Ennola duals.*

TABLE 10. Unipotent ℓ-blocks of quasi-simple exceptional groups with non-trivial abelian defect group, ℓ bad

S	(ℓ, e)	L^F	λ	conditions
${}^2G_2(q^2)$	$(2, 1)$	$\Phi_1\Phi_2$	1	
$F_4(q)$	$(3, 1)$	$\Phi_1^2.B_2(q)$	$B_2[1]$	
$E_6(q)$	$(3, 1)$	$\Phi_1^2.D_4(q)$	$D_4[1]$	$3\|\|q-1,\ Z(S) = 1$
${}^2E_6(q)$	$(3, 1)$	$\Phi_1.{}^2A_5(q)$	ϕ_{321}	
$E_7(q)$	$(2, 1)$	$\Phi_1.E_6(q)$	$E_6[\theta^{\pm 1}]$	$4\|\|q-1,\ Z(S) = 1$
$E_8(q)$	$(3, 1)$	$\Phi_1.E_7(q)$	$E_7[\pm\xi]$	
	$(5, 1)$	$\Phi_1^4.D_4(q)$	$D_4[1]$	
	$(5, 1)$	$\Phi_1^2.E_6(q)$	$E_6[\theta^{\pm 1}]$	
	$(5, 1)$	$\Phi_1.E_7(q)$	$E_7[\pm\xi]$	
	$(5, 4)$	$\Phi_4^2.D_4(q)$	$\xi_1, \ldots, \xi_4$	

For a prime ℓ, by a minimal counterexample to (HZC1) we will mean a pair (χ, S), such that S is a finite group, $\chi \in \mathrm{Irr}(S)$ is an irreducible character lying in an ℓ-block of S with abelian defect such that χ has positive height and such that $(\chi(1), |S|)$ is minimal with respect to the lexicographical ordering on such pairs.

Proposition 8.4. *Suppose that G is simple, simply connected of exceptional type and that ℓ is a bad prime for G. Let $S = G^F/Z$ for some central subgroup Z of G^F and $B = b_S(L, \lambda)$ a unipotent ℓ-block of S. Then (χ, S) is not a minimal counterexample to (HZC1) for any $\chi \in \mathrm{Irr}(B)$.*

Proof. For ${}^2G_2(q^2)$ the validity of (HZC1) follows from the results in [48]. Note that it is also known to hold for blocks with cyclic defect group. So now let $B = b_S(L, \lambda)$, for (L, λ) a unipotent e-cuspidal pair of central ℓ-defect and suppose that B has non-cyclic abelian defect groups.

Assume that G is of type F_4 or E_8. Then $S = G^F$. Further, by Proposition 8.3, $W_{G^F}(L,\lambda)$ is an ℓ'-group. We claim that $Z(L)_\ell^F$ is a Sylow ℓ-subgroup of $C_G^\circ([L,L])^F$. Indeed, since $L = C_G(Z^\circ(L)_{\Phi_e})$ and since by [13, Prop. 1.7(ii)], $Z^\circ(L)_{\Phi_e}$ is a Sylow e-torus of $C_G^\circ([L,L])$, by the argument before [13, Lemma 4.5], $N_{G^F}(L)$ contains a Sylow ℓ-subgroup, say D of $C_G^\circ([L,L])^F$. Since D centralises $[L,L]^F$, and λ is determined by its restriction to $[L,L]^F$, $D \leq N_{G^F}(L,\lambda)$. But $W_{G^F}(L,\lambda)$ being an ℓ'-group means that $D \leq L$, and hence $D \leq Z(L)$.

Since G is self-dual, we may and will identify G^* with G in such a way that the resulting correspondence between unipotent e-cuspidal pairs of G and G^* is the correspondence of [20, Prop. 15]. Let $t \in G^*$ be an ℓ-element such that $\chi \in \mathcal{E}(G^F,t)$. Let $H = C_G(t)$ and $\psi \in \mathcal{E}(H^F,1)$ be the Jordan correspondent of χ in $C_{G^*}(t)$. Since G has trivial centre, by [20, Thm. B, Prop. 17], there is a unipotent e-cuspidal pair (L_t,λ_t) for H, with central ℓ-defect such that $([L,L],\mathrm{Res}^{L^F}_{[L,L]^F}\lambda)$ and $([L_t,L_t],\mathrm{Res}^{L_t^F}_{[L_t,L_t]^F}\lambda_t)$ are G^F-conjugate, and such that ψ is in the block $b_{H^F}(L_t,\lambda_t)$. Further, (L_t,λ_t) is uniquely determined up to H^F-conjugacy.

Since t commutes with $[L_t,L_t]$, some G^F-conjugate of t commutes with $[L,L]$, and so by the claim above, we may assume that $t \in Z(L)_\ell^F$, and hence that $(L_t,\lambda_t) = (L,\lambda)$. By [20, Prop. 8, 8.bis], $L = C_G^\circ(Z(L)_\ell^F)$ and $L^F = C_{G^F}(Z(L)_\ell^F)$ (in fact, $L = C_G(Z(L)_\ell^F)$), hence, also $L = C_H^\circ(Z(L)_\ell^F)$ and $L^F = C_{H^F}(Z(L)_\ell^F)$. Since $W_{H^F}(L,\lambda) \leq W_{G^F}(L,\lambda)$ are ℓ'-groups, $Z(L)_\ell^F$ is a defect group of $b_{H^F}(L_t,\lambda_t)$ and of B by Proposition 2.7.

Now by the degree formula for Jordan decomposition, ψ and χ have the same defect and thus the same height, whence (ψ,H^F) is a counterexample to (HZC1). Since $\psi(1) \leq \chi(1)$, (χ,S) is a minimal counterexample only if t is central, so 1, and hence only if χ is a unipotent character. But it is easy to check from the decomposition of Lusztig induction of the relevant unipotent e-cuspidal pairs that all unipotent characters in B are of zero height.

Now suppose that G is of type E_6. If S is as in line 4 of the table, then the defect groups of B are cyclic. So, we assume that S is as in line 5 of the table. By Proposition 8.3, $\ell = 3$, $3||(q-1)$, $Z \neq 1$, and S may be assumed to be the commutator subgroup of $\hat{G}^F$, where $\hat{G} = (E_6)_{\mathrm{ad}}$. Denote by $\hat{B}$ the unipotent block of $\hat{G}^F$ covering B. Let $\hat{\chi} \in \mathrm{Irr}(\hat{B})$ cover χ and let $t \in \hat{G}^{*F}$ be such that $\hat{\chi} \in \mathcal{E}(\hat{G}^F,t)$. Since $\hat{G}$ has connected centre, again by [20, Thm. B, Prop. 17], (and using the canonical correspondences between unipotent e-cuspidal pairs for groups of the same type), $C_{\hat{G}^*}(t)$ contains a unipotent e-cuspidal pair (L_t,λ_t) such that $[L_t,L_t] \cong [L,L]$. There are only three classes of centralisers of semisimple elements of $\hat{G}^{*F} = E_6(q)_{\mathrm{sc}}$ containing Levi subgroups of type D_4: one of type $\Phi_1^2.D_4(q)$, one of type $\Phi_1.D_5(q)$, and $\hat{G}^{*F}$ itself. For $t \in Z(\hat{G}^*)$, the elements of $\mathcal{E}(\hat{G}^F,t)$ have the same restrictions to S as the unipotent characters. Since 3 divides $q-1$ precisely once, there are no 3-elements with centraliser $\Phi_1.D_5(q)$. Finally, there is exactly one class of elements of order 3 with centraliser $H = \Phi_1^2.D_4(q)$. The Jordan correspondent of $\hat{\chi}$ is therefore the unique (cuspidal) character $D_4[1]$ in $\mathcal{E}(H,1)$, $\hat{\chi}$ is the only possible non-unipotent character in $\hat{B}$, and it has height 1. Since the image of t in the adjoint type group $\hat{G}^*/Z(\hat{G}^*)$ has disconnected centraliser, the restriction of $\hat{\chi}$ to S has three irreducible constituents, which are thus of height zero. In particular, (HZC1) is satisfied for $\hat{B}$. Exactly the same reasoning applies to the Ennola dual case, and a slight variation is valid for G of type E_7. □

Next we show that no other quasi-isolated block provides a minimal counterexample to (HCZ1).

Lemma 8.5. *Let G be connected reductive such that G and G^* have connected centre. Let $s \in G^{*F}$ be a semisimple ℓ'-element such that G and s satisfy the assertions of Theorem 1.4 and that all e-cuspidal pairs (L, λ) of G below $\mathcal{E}(G^F, s)$ satisfy*

$$C_G^\circ(Z(L)_\ell^F) = L, \quad C_{G^F}(Z(L)_\ell^F) = L^F,$$

and λ is of central ℓ-defect. Let $t \in G^{*F}$ be an ℓ-element commuting with s and suppose that there exists a proper F-stable Levi subgroup M of G such that the following holds.
(1) *$C_{G^*}(st) \leq M$.*
(2) *M is e-split.*
(3) *One of the following holds.*
  (a) *For all e-cuspidal pairs (L, λ) of G below $\mathcal{E}(G^F, s)$, $W_{G^F}(L, \lambda)$ is an ℓ'-group and there exists an F-stable Levi subgroup M_0 of M such that $C_{M^*}(s) \leq M_0$ and such that ℓ is good for M_0.*
  (b) *$\ell \geq 5$ and ℓ is good for M.*

Then (χ, G^F) is not a minimal counterexample to (HZC1) for any $\chi \in \mathcal{E}(G^F, st)$.

Proof. The hypotheses on G (and our results in Section 2) imply that for any e-split Levi subgroup H of G and for any H^F-conjugacy class of e-cuspidal pairs (L, λ) of G below $\mathcal{E}(H^F, s)$ there is an ℓ-block $b_{H^F}(L, \lambda)$ of H^F in $\mathcal{E}_\ell(H^F, s)$ such that all constituents of $R_L^H(\lambda)$ lie in $\mathrm{Irr}(b_{H^F}(L, \lambda) \cap \mathcal{E}(H^F, s)$, and such that $(Z(L)^F, b_{L^F}(\lambda))$ is a centric $b_{H^F}(L, \lambda)$-Brauer pair. Moreover, any ℓ-block of H^F in $\mathcal{E}_\ell(H^F, s)$ is of the form $b_{H^F}(L, \lambda)$ for some e-cuspidal pair (L, λ) of G below $\mathcal{E}(H^F, s)$.

Let $\chi \in \mathcal{E}(G^F, st)$. By (1), there exists $\phi \in \mathcal{E}(M^F, st)$ such that $\chi = \pm R_M^G(\phi)$. Let $b := b_{M^F}(L, \lambda)$ be the ℓ-block of M^F containing ϕ and set $c := b_{G^F}(L, \lambda)$. Since $W_{M^F}(L, \lambda)$ is a subgroup of $W_{G^F}(L, \lambda)$, by Proposition 2.7(e),(f) if c has abelian defect groups, then b has abelian defect groups, and $Z(L)^F$ is a defect group of both b and c. So, since $\phi(1) < \chi(1)$ and ϕ and χ have the same ℓ-defect, it suffices to prove that c contains χ, or equivalently that $d^{1,G^F}(\chi) \in c$.

We claim that for any $\psi \in \mathrm{Irr}(b) \cap \mathcal{E}(M^F, s)$, all constituents of $R_M^G(\psi)$ lie in c. Indeed, note that by Proposition 2.10 in order to prove the claim, it suffices to prove that $\mathrm{Irr}(b) \cap \mathcal{E}(M^F, s)$ is precisely the set of constituents of $R_L^M(\lambda)$. If (3b) holds then this follows from the main theorem of [14]. Suppose that (3a) holds. Then for any e-split Levi subgroup H of G and any e-cuspidal pair $(\tilde{L}, \tilde{\lambda})$ of G below $\mathcal{E}(H^F, s)$, $(Z(\tilde{L}^F), b_{\tilde{L}^F}(\tilde{\lambda}))$ is a maximal $b_{H^F}(\tilde{L}, \tilde{\lambda})$-Brauer pair. Consequently, the map

$$(\tilde{L}, \tilde{\lambda}) \to b_{H^F}(\tilde{L}, \tilde{\lambda})$$

induces a bijection between the H^F-conjugacy classes of e-cuspidal pairs of G below $\mathcal{E}(H^F, s)$ and the set of ℓ-blocks in $\mathcal{E}(H^F, s)$, and $\mathrm{Irr}(b_{H^F}(\tilde{L}, \tilde{\lambda})) \cap \mathcal{E}(H^F, s)$ is precisely the set of constituents of $R_L^H(\lambda)$.

Now,

$$d^{1,G^F}(\chi) = \pm d^{1,G^F}(R_M^G(\phi)) = R_M^G(d^{1,M^F}(\phi)).$$

Hence, by the claim above, it suffices to prove that $\mathrm{Irr}(b) \cap \mathcal{E}(M^F, s)$ is an ℓ-basic set for b. Suppose first that (3a) holds. Since G has connected centre, so does M and by hypothesis, ℓ is good for M. So, by [24, Thm. A], $\mathrm{Irr}(b) \cap \mathcal{E}(M^F, s)$ is an ℓ-basic set for b. In case (3b), let b_0 be the Bonnafé–Rouquier correspondent of b in M_0^F. By the previous argument, applied to M_0 instead of M, $\mathrm{Irr}(b_0) \cap \mathcal{E}(M_0^F, s)$ is an ℓ-basic set for b_0. The result follows as the Bonnafé–Rouquier Morita equivalence preserves basic sets. □

The next two results will allow us to verify the conditions of the previous Lemma for certain situations in E_8.

Proposition 8.6. *Let H be connected reductive with Steinberg endomorphism F. Let ℓ be a prime different from the defining characteristic of H, good for H, not dividing $|Z(H)/Z^\circ(H)|$, and not a torsion prime, and set $e = e_\ell(q)$. Assume that one of the following holds:*

(1) *e is the unique integer such that $\ell|\Phi_e(q)$ and $\Phi_e(q)\big||H^F|$; or*

(2) *$e \in \{1, 2\}$ and a Sylow e-torus of H is a maximal torus.*

Then the centraliser of any ℓ-element $1 \neq t \in H^F$ lies in the centraliser of a non-trivial e-torus.

Proof. Clearly it suffices to prove the assertion for t of order ℓ. Let t be of order ℓ and set $C := C_H^\circ(t)$. Then C is a Levi subgroup of H, and $t \in C$ (see [38, Prop. 14.1]). Moreover, $t \in Z(C)$, and as $|Z(C)/Z^\circ(C)|$ divides $|Z(H)/Z^\circ(H)|$ by [21, Prop. 1.1.2(b)], we even have $t \in Z^\circ(C)$. Thus $Z^\circ(C)$ is a torus with $|Z^\circ(C)^F|$ divisible by ℓ. Under assumption (1) this implies that $Z^\circ(C)$ contains a non-trivial e-torus T, and thus $C \leq C_H(T)$.

In case (2), let T denote a Sylow e-torus of H. Then t is H-conjugate to some element of T, by [38, Cor. 6.11]. As $\ell|\Phi_e(q)$ we see that all elements of order ℓ of T are F-stable, so lie in T^F. But the centraliser of t is connected by [38, Ex. 20.16], hence t is even H^F-conjugate to an element of T^F by [38, Thm. 26.7]. We may assume that actually $t \in T^F$. Then in particular C contains the maximal torus T, whence $Z^\circ(C) \leq T$ is a non-trivial e-torus. □

Lemma 8.7. *Assume that $G^F = E_8(q)$ with $q \equiv 1 \pmod 3$. Let $s \in G^F$ be a quasi-isolated 5-element such that F induces a nonsplit Steinberg endomorphism on $H := C_G(s)$. Then for any non-trivial 3-element $t \in H^F$, $C_G(st)$ is contained in a Levi subgroup M_0 of G of classical type, which itself lies in a proper 1-split Levi subgroup M of G.*

Proof. According to Table 1 (or Table 6) we have H^F of type either ${}^2A_4(q^2)$ or ${}^2A_4(q)^2$. It is easy (using Jordan normal forms) to work out the types of 3-elements in H^F and the isomorphism types of their centralisers; the result is given in Table 11.

Clearly, $M_0 := C_G(Z^\circ(C_H(t)))$ and $M := C_G(Z^\circ(C_H(t))_{\Phi_1})$ are Levi subgroups of G containing $C_G(st)$, $M_0 \leq M$, and M is 1-split and proper. Further, M_0 has semisimple rank at most $8 - \dim(Z^\circ(C_H(t)))$, hence it is of classical type unless $\dim(Z^\circ(C_H(t)) \leq 2$, which happens precisely for the last two centralisers for $H = {}^2A_4(q)^2$. But there, $C_H(t)$ is of type $A_4 + A_2$, $A_4 + 2A_1$ respectively, and these do not embed into a group of type E_6, by the Borel–de Siebenthal algorithm (see [38, Thm. 13.12]). Hence in these cases as well, M_0 is of classical type. □

Table 11. Centralisers of 3-elements in H^F

$H^F = {}^2A_4(q)^2$		$H^F = {}^2A_4(q^2)$	
$C_{H^F}(t)$	$Z^\circ(C_{H^F}(t))$	$C_{H^F}(t)$	$Z^\circ(C_{H^F}(t))$
$\mathrm{GL}_2(q^2)^2$	$\Phi_1^2\Phi_2^2$	$\mathrm{GL}_2(q^4)$	$\Phi_1\Phi_2\Phi_4$
$\Phi_1.\mathrm{GL}_2(q^2)\mathrm{GU}_3(q)$	$\Phi_1^2\Phi_2^2$	$\Phi_1\Phi_2.\mathrm{GU}_3(q^2)$	$\Phi_1\Phi_2\Phi_4$
$\Phi_1^2.\mathrm{GU}_3(q)^2$	$\Phi_1^2\Phi_2^2$		
$\mathrm{GL}_2(q^2)\mathrm{SU}_5(q)$	Φ_1^2		
$\Phi_1.\mathrm{GU}_3(q)\mathrm{SU}_5(q)$	$\Phi_1\Phi_2$		

Proposition 8.8. *Suppose that G is simple, simply connected of exceptional type F_4, E_6, E_7 or E_8 and that ℓ is a bad prime for G. Let $S = G^F/Z$ for Z a central subgroup of G^F and let B be an ℓ-block of S such that the block of G^F lifting B is a quasi-isolated, non-unipotent block of abelian defect as in Tables 2–8 or their Ennola duals. Then (χ, S) is not a minimal counterexample to (HZC1) for any $\chi \in \mathrm{Irr}(B)$.*

Proof. For $F_4(q)$, by Proposition 3.5, B is one of the blocks numbered 3, 5 or 7 or their Ennola duals. Now note that in all three cases, the ℓ-power in the degrees of characters in $\mathrm{Irr}(B)\cap\mathcal{E}(G^F, s)$ is maximal among all elements of $\mathcal{E}_\ell(G^F, s)$, while on the other hand, they are of height zero in B. Hence, no character in $\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(G^F, s)$ can have positive height.

If S is of type $E_6(q)$, then we note by Table 3 that B has abelian defect groups only if the block of G^F lifting B has abelian defect groups (note that in Lines 13 and 14 of the table the action of the relative Weyl group does not become trivial on passing to $G^F/Z(G^F)$). Hence, only the block numbered 15 has to be considered. Here, either we can apply the same argument as for $F_4(q)$, or alternatively observe that the defect groups are cyclic. The same arguments apply to the unique quasi-isolated block with abelian defect group of ${}^2E_6(q)$.

According to Proposition 5.3 and Table 4, for $E_7(q)$ again the defect groups of B are abelian if and only if the defect groups of the block of G^F lifting B are abelian and this occurs only for the blocks 4, 7, 11, 13, 16 and 18. In all cases, the characters in $\mathrm{Irr}(B) \cap \mathcal{E}_\ell(G^F, s)$ have the maximal possible ℓ-part in their degree, so we conclude as before.

For $E_8(q)$ the blocks with abelian defect were described in Propositions 6.4, 6.7 and 6.10. In particular, there are no cases when $\ell = 2$. For $\ell = 3$, we need to treat blocks 5, 11, and 13–17. Here, the cases 5 and 11 follow by the standard argument on maximal ℓ-power in the degrees. In the remaining cases, that is, when s is a quasi-isolated 5-element with non-split centraliser, the conditions (1)–(3a) of Lemma 8.5 were shown to hold in Lemma 8.7, whence the claim.

Finally assume that $\ell = 5$. We conclude as before when $\mathcal{E}_5(G^F, s)$ is a single 5-block. It is straightforward to check that all other centralisers H of quasi-isolated elements s in Table 8 satisfy condition (1) of Proposition 8.6, and those in Table 7 satisfy either (1) or (2). Thus there is a non-trivial e-torus T contained in the centre of $C_{G^{*F}}(st)$, and we let $M \leq G$ be a Levi subgroup in duality with $C_{G^{*F}}(T)$. So M is proper in G and (1), (2) and (3b) of Lemma 8.5 hold, which gives the claim. This completes the proof. □

8.3. **Proof of (HZC1).**

Theorem 8.9. *The 'if part' (HCZ1) of Brauer's height zero conjecture holds.*

Proof. We investigate minimal counterexamples to the assertion. For this let S be a finite group, p a prime, B a p-block of S with abelian defect group D and $\chi \in \mathrm{Irr}(B)$ an irreducible character of B of positive height, such that $(\chi(1), |S|)$ is minimal with respect to the lexicographic ordering on such pairs.

Then by the principal result of [3], S is quasi-simple, with simple central factor group X, say. We may assume that X is not alternating, by Olsson's result [44], respectively by Proposition 8.2 for the exceptional covering groups of $\mathfrak{A}_6$ and $\mathfrak{A}_7$. (Note that for the double cover of $\mathfrak{A}_n$ Olsson only treats the odd primes but the abelian defect groups of 2-blocks of alternating groups are of order at most 4, and (HCZ1) is known for such blocks by results of Brauer.) Furthermore, X is not sporadic by Proposition 8.1, nor a special linear or unitary group by the theorem of Blau and Ellers [4, Thm. 5], respectively by Proposition 8.2 for their exceptional covering groups. Thus by the classification of finite simple groups, X is a simple group of Lie type not of type A_n or ${}^2A_n$.

There is a simple algebraic group G of simply connected type with a Steinberg endomorphism $F : G \to G$ such that $X \cong G^F/Z(G)^F$ (recall that we consider ${}^2F_4(2)'$ as a sporadic group). Moreover, by Proposition 8.2 we may assume that $S = G^F/Z$ for some central subgroup $Z \leq Z(G)^F$. Now first assume that p is the defining prime for X. Then $\gcd(p, |Z|) = 1$, so any p-block of S is also a p-block of G^F, with the same defect group. By the theorem of Humphreys [29] the p-blocks of G^F are either of defect zero or of full defect. But the Sylow p-subgroup of G^F is non-abelian unless $G = \mathrm{SL}_2$, which was excluded before.

Thus, p is not the defining prime for G. Let G^* be a group in duality with G, thus of adjoint type, with compatible Steinberg morphism $F : G^* \to G^*$. Let B_0 be the p-block of G^F containing the lift, say χ_0, of χ to G^F and let $s \in G^{*F}$ be a semisimple p'-element such that $B_0 \subseteq \mathcal{E}_p(G, s)$.

Now assume first that p is a good prime for G, and $p \neq 3$ if F is a triality automorphism. Since G is not of type A_n and p is good for G, p does not divide $|Z(G^F)|$, hence B_0 and B have isomorphic defect groups. So, χ and χ_0 have equal degree and equal height. In particular, χ_0 is not of zero height. Again since p is good for G and $p \neq 3$ if F is a triality automorphism, by the theorem of Enguehard [21, Thm. 1.6], there is a reductive algebraic group $G(s)$ with a Steinberg endomorphism $F : G(s) \to G(s)$, such that $G(s)^\circ$ is in duality with $C^\circ_{G^*}(s)$, a p-block b of $G(s)^F$ with isomorphic defect group D and a height preserving bijection $\Xi : \mathrm{Irr}(B_0) \to \mathrm{Irr}(b)$. Moreover, $\Xi(\psi)(1)|\psi(1)$ for all $\psi \in \mathrm{Irr}(B_0)$. Thus, if $|G(s)^F| < |S|$, then $(\Xi(\chi), G(s)^F)$ cannot be a counterexample, so neither can (χ, S). Hence, in this case we must have $|G(s)^F| \geq |S|$, so $s \in Z(G^*) = 1$ (since G^* is adjoint). So B_0 is unipotent.

Since G is not of type A_n, p is good and $p \neq 3$ when F induces triality, p is even (G, F)-excellent in the sense of Broué–Michel [11, Def. 1.11]. Thus, by [11, Thm. 3.1] there is an isotypie between the unipotent block B_0 and a block b of the normaliser of some non-trivial p-subgroup; in particular Brauer's height zero conjecture holds for B_0 and hence for B, contradicting our choice.

Thus, either p is a bad prime for G, or $p = 3$ and F induces triality. Let M be an F-stable Levi-subgroup of G such that $C_{G^*}(s) \leq M^*$ and s is quasi-isolated in M^*. Let C_0 be the Bonnafé–Rouquier correspondent of B_0 in M^F. Then, $\psi \mapsto \epsilon_G \epsilon_M R_M^G(\psi)$ is a height preserving bijection between $\mathrm{Irr}(C_0)$ and $\mathrm{Irr}(B_0)$. For $\psi \in \mathrm{Irr}(C_0)$ and $z \in G^F$, $z \in \ker(\psi)$ if and only if $z \in \ker(\epsilon_G \epsilon_M R_M^G(\psi))$. Hence, the character of C_0 corresponding to χ_0 via the above bijection is the lift of a character, say τ, of M^F/Z to M^F. Let C be the ℓ-block of M^F/Z containing τ. By Theorem 7.16 the defect groups of C are abelian and of the same size as the defect groups of B (note that Theorem 7.16 applies to the Sylow p-subgroup of Z and defect groups remain unchanged on passing to quotients by p'-groups). Now τ and χ have equal heights, and $\tau(1) \leq \chi(1)$. So, $M^F/Z = G^F/Z$, $M = G$ and s is quasi-isolated in G^*.

Assume first that $p = 2$ and G^F is of classical type different from A_n or ${}^2A_n$. The quasi-isolated elements of G^* are 2-elements, hence $s = 1$. But then by [21, Prop. 1.5], B is the principal block of S, and in particular has non-abelian defect groups. Thus, S is of exceptional type. For the groups ${}^2B_2(q^2)$ the only bad prime is the defining one, so this case does not occur here. The height zero conjecture for the groups ${}^2G_2(q^2)$, $G_2(q)$, ${}^3D_4(q)$ and ${}^2F_4(q^2)$ has been checked by Ward [48], Hiß [27], Deriziotis–Michler [18] and Malle [35] respectively. So s is quasi-isolated in a quasi-simple exceptional group of Lie type of rank at least 4 and p is a bad prime. In this case the claim is contained in Propositions 8.4 and 8.8.

This completes the proof of Theorem 8.9. □

9. Blocks with equal height zero degrees

Here, we complete the proof of Theorem 1.5. According to the result in [37, Thm. 6.1] the only blocks left to consider are spin blocks (i.e., faithful blocks) of the double cover of alternating groups, and quasi-isolated blocks of exceptional groups of Lie type of rank at least 4. The validity of Theorem 1.5 for spin blocks of alternating groups has recently been shown by Gramain [26, Thm. 4.1 and Cor. 4.2].

So we may assume that S is quasi-simple of exceptional Lie type in characteristic p, and B is a quasi-isolated ℓ-block of S with $\ell \neq p$. It is immediate from our explicit description of such blocks in Sections 3–6 and the degree formula resulting from Lusztig's Jordan decomposition of characters that the only quasi-isolated ℓ-blocks with all height zero characters of the same degree are those consisting of a single cuspidal character. In those cases, the defect groups are central, and in particular abelian. Together with the criterion in [37, Thm. 4.1], the claim follows.

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Institute of Mathematics, University of Aberdeen, Fraser Noble Building, Aberdeen, AB243UE, UK
E-mail address: `r.kessar@abdn.ac.uk`

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
E-mail address: `malle@mathematik.uni-kl.de`