BLOCKS WITH TRANSITIVE FUSION SYSTEMS

LÁSZLÓ HÉTHELYI, RADHA KESSAR, BURKHARD KÜLSHAMMER, AND BENJAMIN SAMBALE

ABSTRACT. Suppose that all nontrivial subsections of a p-block B are conjugate (where p is a prime). By using the classification of the finite simple groups, we prove that the defect groups of B are either extraspecial of order p^3 with $p \in \{3, 5\}$ or elementary abelian.

1. INTRODUCTION

Let p be a prime, and let $\mathcal{F}$ be a saturated fusion system on a finite p-group P (cf. [1] and [8]). We call $\mathcal{F}$ *transitive* if any two nontrivial elements in P are $\mathcal{F}$-conjugate. In this case, P has exponent $\exp(P) \leq p$, and $\mathrm{Aut}_{\mathcal{F}}(P)$ acts transitively on $\mathrm{Z}(P) \setminus \{1\}$. This paper is motivated by the following:

Conjecture 1.1. *(cf. [23]) Let $\mathcal{F}$ be a transitive fusion system on a finite p-group P where p is a prime. Then P is either extraspecial of order p^3 or elementary abelian.*

Moreover, if P is extraspecial of order p^3 then results by Ruiz and Viruel [26] imply that $p \in \{3, 5, 7\}$. Note that the conjecture is trivially true for $p = 2$ since groups of exponent 2 are abelian. Thus Conjecture 1.1 is only of interest for $p > 2$. The aim of this paper is to prove the conjecture above for saturated fusion systems coming from blocks.

Theorem 1.2. *Let p be a prime, and let B be a p-block of a finite group G with defect group P. If the fusion system $\mathcal{F} = \mathcal{F}_P(B)$ of B on P is transitive then P is either extraspecial of order p^3 or elementary abelian.*

If P is extraspecial of order p^3 then the results in [26] and [20] imply that $p \in \{3, 5\}$. We call a block B with defect group P and transitive fusion system $\mathcal{F}_P(B)$ *fusion-transitive*. Whenever B has full defect then the theorem is a consequence of the results in [23]. In our proof of the theorem above, we will make use of the classification of the finite simple groups.

2010 *Mathematics Subject Classification.* Primary 20C20; Secondary 20D20.

Key words and phrases. Block, defect group, fusion system.

The first author gratefully acknowledges support by the Hungarian National Science Foundation Research No. 77476. The third author gratefully acknowledges support by the DFG (SPP 1388). The fourth author gratefully acknowledges support by the Carl Zeiss Foundation and the Daimler and Benz Foundation.

2. Saturated fusion systems

We begin with some results on arbitrary saturated fusion systems.

Proposition 2.1. *Let p be a prime, and let $\mathcal{F}$ be a transitive fusion system on a finite p-group P where $|P| \geq p^4$. Suppose that P contains an abelian subgroup of index p. Then P is abelian.*

Proof. We assume the contrary. Then $p > 2$.

Suppose first that P contains two distinct abelian subgroups A, B of index p. Then $AB = P$, $A \cap B \subseteq \mathrm{Z}(P)$ and $|P : A \cap B| = p^2$. Since P is nonabelian we conclude that $|P : \mathrm{Z}(P)| = p^2$. Thus $1 \neq P' \subseteq \mathrm{Z}(P)$. Since $\mathrm{Aut}_{\mathcal{F}}(P)$ acts transitively on $\mathrm{Z}(P) \setminus \{1\}$, we conclude that $P' = \mathrm{Z}(P)$. Hence there are $x, y \in P$ such that $P = \langle x, y \rangle$. Then $P' = \langle [x, y] \rangle$ (cf. III.1.11 in [17]); in particular, we have $|P'| = p$ and $|P| = p^3$, a contradiction.

It remains to consider the case where P contains a unique abelian subgroup A of index p. Let Z be a subgroup of order p in $\mathrm{Z}(P)$, and let B be an arbitrary subgroup of order p in A. By transitivity, there is an isomorphism $\phi : B \longrightarrow Z$ in $\mathcal{F}$. By definition, Z is fully $\mathcal{F}$-normalised. Thus, by Proposition 4.20 in [8], Z is also fully $\mathcal{F}$-automised and receptive. Hence ϕ extends to a morphism $\psi : N_\phi \longrightarrow P$ in $\mathcal{F}$. Since $|B| = p$ we have

$$A \subseteq \mathrm{N}_P(B) = \mathrm{C}_P(B) \subseteq N_\phi$$

(cf. p. 99 in [8]). Since $\psi(A)$ is also an abelian subgroup of index p in A we conclude that $\psi(A) = A$. Thus $\psi|A \in \mathrm{Aut}_{\mathcal{F}}(A)$, and $\psi|A$ maps B to Z. This shows that $\mathrm{Aut}_{\mathcal{F}}(A)$ acts transitively on the set of subgroups of order p in A.

In the following, we view A as a vector space over $\mathbb{F}_p$ and $G := \mathrm{Aut}_{\mathcal{F}}(A)$ as a subgroup of $\mathrm{GL}(A)$. If S denotes the group of scalar matrices in $\mathrm{GL}(A)$ then $H := GS$ is a transitive subgroup of $\mathrm{GL}(A)$. The transitive linear groups were classified by Hering (cf. [16] or Remark XII.7.5 in [18]). We are going to use the list in Theorem 15.1 of [27].

Before we do this, we observe the following. By the uniqueness of A, A is fully $\mathcal{F}$-automised, i.e. $P/A = \mathrm{N}_P(A)/\mathrm{C}_P(A) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(A))$. Thus $G = \mathrm{Aut}_{\mathcal{F}}(A)$ and $H = GS$ both have a Sylow p-subgroup of order p.

Now we write $|A| = p^n$ and go through the list in Theorem 15.1 of [27]:

(i) $H \subseteq \Gamma\mathrm{L}_1(p^n)$; in particular, $|H|$ divides $|\Gamma\mathrm{L}_1(p^n)| = n(p^n - 1)$.

In this case we can identify A with the finite field $L := \mathbb{F}_{p^n}$. Moreover, P is the semidirect product of L with $B = \langle \beta \rangle$ where β is a field automorphism of L. For $x \in L$, we have $x\beta \in P$ and

$$1 = (x\beta)^p = x\beta x\beta \ldots x\beta = x\beta(x)\beta^2(x) \ldots \beta^{p-1}(x) = \mathrm{N}_K^L(x)$$

where K is the fixed field of β. However, it is known that $\mathrm{N}_K^L(L) = K$, a contradiction.

(ii) $n = km$ where $k \geq 2$ and $\mathrm{SL}_k(p^m) \trianglelefteq H$.
Since the Sylow p-subgroups of H have order p, we conclude that $m = 1$ and $k = 2$. Then $n = 2$ and $|P| = p^3$, a contradiction.

(iii) $n = km$ where $k \geq 4$ is even and $\mathrm{Sp}_k(p^m)' \trianglelefteq H$.
Since $p > 2$ we have $\mathrm{Sp}_k(p^m)' = \mathrm{Sp}_k(p^m)$. Thus $\mathrm{Sp}_k(p^m)$ has a Sylow p-subgroup of order $p^{k^2/4} \geq p^4$, a contradiction.

(iv) $n = 6m$, $p = 2$ and $G_2(2^m)' \trianglelefteq H$.
This case is impossible as $p > 2$.

(v) $n = 2$ and $p \in \{5, 7, 11, 19, 23, 29, 59\}$.
Then $|P| = p^3$ which is again a contradiction.

(vi) $n = 4$, $p = 2$ and $H \cong \mathfrak{A}_7$.
This case is also impossible as $p > 2$.

(vii) $n = 4$, $p = 3$ and H is one of the groups in Table 15.1 of [27].
In this case we have $|P| = 3^5 = 243$. Then Proposition 15.12 in [27] leads to a contradiction.

(viii) $n = 6$, $p = 3$ and $H \cong \mathrm{SL}_2(13)$.
In this case we have $|P| = 3^7 = 2187$. However, one can check that P has exponent 9 in this case, a contradiction. □

Proposition 2.2. *Let P be a nonabelian p-group with a transitive fusion system. Then P is indecomposable (as a direct product).*

Proof. Let $P = N_1 \times \cdots \times N_k$ be a decomposition into indecomposable factors $N_i \neq 1$. Assume by way of contradiction that $k \geq 2$. Since P carries a transitive fusion system we have

$$\mathrm{Z}(N_1) \times \cdots \times \mathrm{Z}(N_k) = \mathrm{Z}(P) \subseteq P' = N_1' \times \cdots \times N_k'.$$

Let $1 \neq x \in \mathrm{Z}(N_1)$. By hypothesis there exists $\alpha \in \mathrm{Aut}(P)$ such that $\alpha(x) \in \mathrm{Z}(P) \setminus (\mathrm{Z}(N_1) \cup \ldots \cup \mathrm{Z}(N_k))$. By the Krull-Remak-Schmidt Theorem (see Satz I.12.5 in [17]) there is a normal automorphism β of P such that $\beta(N_i) = \alpha(N_1)$ for some $i \in \{1, \ldots, k\}$. In particular, there is $y \in \mathrm{Z}(N_i)$ such that $\beta(y) = \alpha(x)$. By Hilfssatz I.10.3 in [17], for every $g \in P$ there is a $z_g \in \mathrm{Z}(P)$ such that $\beta(g) = gz_g$. Obviously the map $P \longrightarrow \mathrm{Z}(P)$, $g \longmapsto z_g$, is a homomorphism. Since $\mathrm{Z}(N_i) \subseteq N_i'$, we obtain $z_y = 1$. This gives the contradiction $\alpha(x) = \beta(y) = y \in \mathrm{Z}(N_i)$. □

Proposition 2.3. *Let $P = \prod_{i=1}^{\infty} P_i^{a_i}$ where $P_i = C_{p^{r_i}} \wr C_p \wr \ldots \wr C_p$ (i factors in the wreath product) and $a_i \in \mathbb{N}_0$, $r_i \in \mathbb{N}$ for $i \in \mathbb{N}$. Moreover, let U be a normal subgroup of P such that P/U is cyclic, and let Z be a cyclic subgroup of $\mathrm{Z}(U)$. Suppose that $R := U/Z$ supports a transitive fusion system. Then R has order p^3 or is elementary abelian.*

Proof. We assume the contrary. Then $|R| \geq p^4$ and $p > 2$.

Suppose first that $r_j > 1$ for some $j > 1$. Since $p > 2$, P' contains a subgroup isomorphic to $C_{p^{r_j}} \times C_{p^{r_j}}$. Since $P' \subseteq U$ we conclude that $\exp(R) \geq p^2$, a contradiction.

Thus $r_j = 1$ for $j > 1$, and P_j is the iterated wreath product of j copies of C_p in this case.

Suppose next that $a_j > 0$ for some $j \geq 3$. Since $p > 2$, P' contains a subgroup isomorphic to $P_{j-1} \times P_{j-1}$. By Satz III.15.3 in [17], P_{j-1} has exponent $p^{j-1} \geq p^2$. Since $P' \subseteq U$ we conclude that $\exp(R) \geq p^2$, a contradiction again.

Thus $P = P_1^{a_1} \times P_2^{a_2}$ where $P_1 = C_{p^{r_1}}$ and $P_2 = C_p \wr C_p$. If $a_2 \leq 1$ then P and R contain abelian subgroups of index p. In this case Proposition 2.2 gives a contradiction.

Hence we may assume that $a_2 \geq 2$. Let $\pi : P \longrightarrow P_2^{a_2}$ be the relevant projection. Since $\exp(P_2) = p^2$ we cannot have $\pi(U) = P_2^{a_2}$. On the other hand, P_2/P_2' is elementary abelian. Since $P_2^{a_2}/\pi(U)$ is cyclic, $\pi(U)$ is a maximal subgroup of $P_2^{a_2}$. Let $\pi_1 : P_2^{a_2} \longrightarrow P_2^{a_2-1}$ be the projection onto the direct product of the first $a_2 - 1$ copies of P_2, and let $\pi_2 : P_2^{a_2} \longrightarrow P_2^{a_2-1}$ be the projection onto the direct product of the last $a_2 - 1$ copies of P_2.

Now suppose that $a_2 \geq 3$. Then an argument similar to the one above shows that $\pi_1(\pi(U))$ is a maximal subgroup of $P_2^{a_2-1} = \pi_1(P_2^{a_2})$. Thus $\mathrm{Ker}(\pi_1) \subseteq \pi(U)$ and, similarly, $\mathrm{Ker}(\pi_2) \subseteq \pi(U)$. Thus $\pi(U)$ contains a subgroup isomorphic to P_2^2. Hence $\exp(R) \geq p^2$, a contradiction.

We are left with the case $a_2 = 2$, i.e. $P = A \times P_2 \times P_2$ where $A = P_1^{a_1} \cong C_{p^{r_1}}^{a_1}$ is abelian. Since $\pi(U)$ is a maximal subgroup of $P_2 \times P_2$, we see that $A \times \pi(U)$ is a maximal subgroup of P. Let $x \in P$ such that $P = U\langle x \rangle$. Then $U\langle x^p \rangle \subseteq A \times \pi(U)$. Since $|P : U\langle x \rangle| \leq p$ we conclude that $U\langle x^p \rangle = A \times \pi(U)$. Note that $x^p \in \mho(P) \subseteq \mathrm{Z}(P)$.

Suppose that $\exp(A) > p$, and choose an element $a \in A$ of maximal order. We write $x = x_1x_2$ with $x_1 \in A$ and $x_2 \in P_2^2$, we write $a = ux^{pi}$ with $u \in U$ and $i \in \mathbb{Z}$, and we write $u = u_1u_2$ with $u_1 \in A$ and $u_2 \in P_2^2$. Then $a^p = u^p x^{p^2 i} = u_1^p x_1^{p^2 i} u_2^p x_2^{p^2 i} = u_1^p x_1^{p^2 i} u_2^p$. We conclude that $u_2^p = 1$ and $a^p = u_1^p x_1^{p^2 i}$. Thus $p < \exp(A) = |\langle a \rangle| = |\langle u_1 \rangle| = |\langle u \rangle|$, and $1 \neq u^p \in \mho(U) \cap A$.

By Aufgabe III.15.36 in [17], the elements of order 1 or p form a union of two maximal subgroups. Thus P_2^2 contains $p^{2p-2}(2p-1)^2 < p^{2p+1}$ elements of order 1 or p. Hence $\pi(U)$ contains elements of order p^2; in particular, $\mho(U)$ is noncyclic. Since $\mho(U) \subseteq Z$, this is a contradiction.

This contradiction shows that $\exp(A) \leq p$, i.e. $P = A \times P_2 \times P_2$ where A is elementary abelian. Hence P/P' is elementary abelian. Since P/U is cyclic we conclude that U is a maximal subgroup of P. Thus $U = A \times \pi(U)$ and $\mho(U) \subseteq \pi(U)$. Since $\pi(U)$ contains elements of order p^2, we have $1 \neq \mho(U) \subseteq Z$. On the other hand, Satz III.15.4 in [17] implies that $\mathrm{Z}(U)$ is elementary abelian. Thus $|Z| = p$ and $Z = \mho(U) \subseteq \pi(U)$. Since R supports a transitive fusion system we have

$$AZ/Z \subseteq \mathrm{Z}(U)/Z \subseteq \mathrm{Z}(R) \subseteq R' = U'Z/Z = \pi(U)'Z/Z \subseteq \pi(U)/Z.$$

Therefore $A = 1$, i.e. $P = P_2 \times P_2$. Recall that U is a maximal subgroup of P and that $\pi_1, \pi_2 : P \longrightarrow P_2$ denote the two projections. Without loss of generality we have $\pi_1(U) = P_2$. Since $\mho(U)$ is cyclic, $K_1 := \mathrm{Ker}(\pi_1)$ has order p^p and exponent p.

If $\pi_2(U) \neq P_2$ then $U = P_2 \times \pi_2(U)$ and $\exp(\pi_2(U)) = p$. Thus $Z = \mho(U) \subseteq P_2 \times 1$ and $R \cong P_2/Z \times \pi_2(U)$, a contradiction to Proposition 2.2.

Thus we must also have $\pi_2(U) = P_2$. Then also $K_2 := U \cap \mathrm{Ker}(\pi_2)$ has order p^p and exponent p. Moreover, we have $K_1 \times K_2 \subseteq U$.

We may choose elements $x, y \in U$ such that $\pi_1(x)$ and $\pi_2(x)$ have order p^2. Since $\langle x^p \rangle = Z = \langle y^p \rangle$ we see that $\pi_2(x)$ and $\pi_1(y)$ have order p^2. However, we may choose y such that yK_1 contains an element y' such that $\pi_2(y')$ has order p. Since $\pi_1(y) = \pi_1(y')$ still has order p^2, we have a final contradiction. □

3. Blocks

We now present the proof of Theorem 1.2.

Proof. Suppose that the result is false. Then P is nonabelian with $|P| \geq p^4$ and $p > 2$.

By [1, Proposition IV.6.3] we may assume that B is quasiprimitive. This means that, for any normal subgroup H of G, B covers a unique p-block of H.

Now let H be a normal subgroup of G, and let b be the unique p-block of H covered by B. Suppose that $P \cap H = 1$. (This is satisfied, for example, whenever H is a p'-subgroup.) Then b has defect zero. By Clifford theory, there exist a finite group G^*, a central p'-subgroup H^* of G^*, and a p-block B^* of G^* with defect group $P^* \cong P$ such that $\mathcal{F}_{P^*}(B^*)$ is equivalent to $\mathcal{F}$. Thus we may replace G by G^* and B by B^*.

Repeating the argument above we may therefore assume that every normal subgroup H of G with $P \cap H = 1$ is central. In particular, we have $\mathrm{O}_{p'}(G) \subseteq \mathrm{Z}(G)$.

It is well-known that $M := \mathrm{O}_p(G) \subseteq P$. Suppose first that $M \neq 1$. Since $\mathcal{F}$ is transitive this implies $M = P$. Then $\Phi(P)$ is a normal subgroup of G and properly contained in P. Since $\mathcal{F}$ is transitive, we must have $\Phi(P) = 1$. Thus P is elementary abelian in this case.

Hence, in the following, we may assume that $\mathrm{O}_p(G) = 1$. Then $\mathrm{F}(G) = \mathrm{O}_{p'}(G) = \mathrm{Z}(G)$. Moreover, the layer $\mathrm{E}(G)$ is nontrivial. Let b be the unique p-block of $\mathrm{E}(G)$ covered by B. Then b has defect group $P \cap \mathrm{E}(G) \neq 1$. Since B is transitive, this implies that $P \subseteq \mathrm{E}(G)$.

Let $L_1, \dots, L_n$ denote the components of G. Then $\mathrm{E}(G) = L_1 * \cdots * L_n$ is a central product. For $i = 1, \dots, n$, the unique p-block b_i of L_i covered by b has defect group $P_i := P \cap L_i \neq 1$. Moreover, we have $P = P_1 \times \cdots \times P_n$. Since $\mathcal{F}$ is transitive, this implies that $n = 1$. Thus $\mathrm{E}(G) = L_1 =: L$ is quasisimple, and $G/\mathrm{Z}(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(L)$.

If $|P| = p^4$ then Proposition 15.14 in [27] gives a contradiction. Thus we may assume that $|P| \geq p^5$; in particular, $|L|$ is divisible by p^5. If P is a Sylow p-subgroup of G then the results of [23] imply our theorem. Hence we may assume that $|G|$ is divisible by p^6.

We now make use of the classification of the finite simple groups and discuss the various possibilities for the simple group $\mathrm{F}^*(G)/\mathrm{Z}(G) \cong L/\mathrm{Z}(L)$. Since $\mathcal{F}$ is transitive we have $\mathrm{C}_L(u) \cong \mathrm{C}_L(v)$ for any $u, v \in P \setminus \{1\}$. This will be a very useful fact.

It can be checked with GAP [13] that $L/\mathrm{Z}(L)$ cannot be a sporadic simple group. Similarly, $L/\mathrm{Z}(L)$ cannot be a simple group with an exceptional Schur multiplier.

Suppose that $L = \mathfrak{A}_n$ is an alternating group. Then P is a defect group of a p-block of $\mathfrak{A}_n$. Hence P is also a defect group of a p-block of the symmetric group $\mathfrak{S}_n$. Thus P is a direct product of (iterated) wreath products of groups of order p. Since $C_p \wr C_p$ has exponent p^2 we conclude that P is a direct product of groups of order p, and the result follows in this case.

Suppose next that $L = \hat{\mathfrak{A}}_n$ is the 2-fold cover of $\mathfrak{A}_n$. We may assume that b is a faithful block of $\hat{\mathfrak{A}}_n$. In this case the defect groups of b have a similar structure as those in $\mathfrak{A}_n$ (cf. [24, Theorem 5.8.8]), so we are done here by the same argument.

Suppose now that $L/\mathrm{Z}(L)$ is a group of Lie type in characteristic p. Then the p-block b of L has full defect, i.e. P is a Sylow p-subgroup of L. Since $\mathcal{F}$ is transitive, every nontrivial element $u \in P$ is conjugate in G to an element $v \in \mathrm{Z}(P)$. Thus $|L : \mathrm{C}_L(u)| = |L : \mathrm{C}_L(v)|$ is not divisible by p. Therefore the results in [25] imply that P is abelian.

Finally suppose that $L/\mathrm{Z}(L)$ is a group of Lie type in characteristic $r \neq p$. First we deal with the exceptional groups of Lie type. Let $S \in \mathrm{Syl}_p(L)$. By §10.1 in [14], S contains an abelian normal subgroup N such that S/N is isomorphic to a subgroup of the Weyl group of $L/\mathrm{Z}(L)$. If $|S/N| \leq p$, then Proposition 2.1 gives a contradiction. This already implies the claim for $p \geq 7$. Now let $p = 5$. Then by the same argument we may assume that $L/\mathrm{Z}(L) \cong E_8(q)$ where $q \equiv \pm 1 \pmod 5$. This case will be handled in Section 6. Now let $p = 3$. Here we need to discuss the following groups: F_4, E_6, ${}^2E_6$, E_7 and E_8. For $L/\mathrm{Z}(L) \cong F_4(q)$ we have $|P| \leq p^6$ and the result follows by Proposition 15.13 in [27]. The remaining cases will be discussed in Section 6.

We may therefore assume that $L/\mathrm{Z}(L)$ is a classical group. In this case our theorem follows from the results of the next section. □

4. Classical Groups in non-describing characteristic

We keep the notation of the previous section. We suppose in this section that $L/\mathrm{Z}(L)$ is a simple group of Lie type in characteristic r, $r \neq p$. Let q be a power of r. Suppose that $L = \mathbf{L}^F/Z$, where $\mathbf{L}$ is a simple simply connected algebraic group defined over an algebraic closure $\bar{\mathbb{F}}_q$ of a field $\mathbb{F}_q$ of q elements, $F : \mathbf{L} \to \mathbf{L}$ a Frobenius morphism with respect to an $\mathbb{F}_q$-structure on $\mathbf{L}$ and Z is a central subgroup of $\mathbf{L}^F$. Note that by the classification of finite simple groups, we may assume that if q is a power of 2, then $\mathbf{L}$ is not of type C_n. Let $\tilde{b}$ be the block of $\mathbf{L}^F$ dominating b and $\tilde{P}$ be a defect group of $\tilde{b}$ such that $\tilde{P}Z/Z = P$.

We define groups $\mathbf{H}$ as follows. If $L/\mathrm{Z}(L) = B_n(q)$, then $\mathbf{H} = \mathrm{SO}_{2n+1}(\bar{\mathbb{F}}_q)$. If $L/\mathrm{Z}(L) = C_n(q)$, then $\mathbf{H} = \mathrm{Sp}_{2n}(\bar{\mathbb{F}}_q)$. If $L/\mathrm{Z}(L) = D_n^{\pm}(q)$, then $\mathbf{H} = \mathrm{SO}_{2n}(\bar{\mathbb{F}}_q)$. Here, if q is a power of 2, and $\mathbf{L}$ is of type B_n, then by $\mathrm{SO}_{2n}(\bar{\mathbb{F}}_q)$ we mean the adjoint simple group of type B_n. If q is a power of 2 and if $\mathbf{L}$ is of type D_n, then by $\mathrm{SO}_{2n}(\bar{\mathbb{F}}_q)$ we mean the simple algebraic group of type D_n corresponding to the root datum $(X, \Phi, Y, \Phi^\vee)$ for which the fundamental roots are $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n$ and $X = \{\sum_{i=1}^n a_i e_i : a_i \in \mathbb{Z}\}$ for an orthonormal basis, $e_1, e_2, \cdots, e_n$, of n-dimensional Euclidean space. We may and will assume that $\mathbf{H}$ is an F-stable quotient of $\mathbf{L}$.

Proposition 4.1. *Suppose that p is an odd prime and $L/\mathrm{Z}(L)$ is a classical group in non-describing characteristic different from triality D_4. Suppose that B is a fusion-transitive block with P of order at least p^5. Then P is abelian.*

Proof. Suppose that $L/\mathrm{Z}(L)$ is the projective special linear group $\mathrm{PSL}_n(q)$, so $\mathbf{L} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ and $L = \mathrm{SL}_n(q)$. Let D be a defect group of a block of $\mathrm{GL}_n(q)$ covering $\tilde{b}$ such that $\tilde{P} = D \cap \mathrm{SL}_n(q)$. By the results of Fong and Srinivasan on blocks of finite general linear groups [12, Theorem (3C)], D is isomorphic to the Sylow p-subgroup of a direct product of general linear groups over finite extensions of $\mathbb{F}_q$. Since $\mathrm{Z}(L)$ and $D/\tilde{P}$ are cyclic, the claim follows from Proposition 2.3. The case that $L/\mathrm{Z}(L)$ is the projective special unitary group can be handled similarly.

Now consider the case that $L/\mathrm{Z}(L)$ is of type B, C or D. Then $\tilde{P}$ is a defect group of $\mathbf{L}^F$. Let $1 \neq z \in \mathrm{Z}(\tilde{P})$. Since p is odd, $\mathrm{C}_{\mathbf{L}}(z)$ is a Levi subgroup of $\mathbf{L}$. For any subset A of $\mathbf{L}$, denote by $\overline{A}$ the image of A under the isogeny from $\mathbf{L}$ onto $\mathbf{H}$ and denote by U the kernel of the isogeny. Since U is a central 2-subgroup of $\mathbf{L}$, $\overline{\mathrm{C}_{\mathbf{L}}(z)} = \mathrm{C}_{\mathbf{H}}(\bar{z})$.

The group $\mathrm{C}_{\mathbf{H}}(\bar{z})$ is a direct product

$$\mathrm{C}_{\mathbf{H}}(\bar{z}) = \mathbf{H}_0 \times \cdots \times \mathbf{H}_r,$$

where $\mathbf{H}_0$ is either the identity or a classical group and for $i \geq 1$, $\mathbf{H}_i$ is a direct product of general linear groups with F transitively permuting the factors. This follows easily from the standard description of the root datum of $\mathbf{H}$. So,

$$\mathrm{C}_{\mathbf{H}}(\bar{z})^F = \mathbf{H}_0^F \times \cdots \times \mathbf{H}_r^F,$$

where $\mathbf{H}_i^F$ is a finite general linear or unitary group for $i \geq 1$ and $\mathbf{H}_0^F$ is a finite classical group (possibly the identity).

Let $\mathbf{L}_i$ be the inverse image in $\mathrm{C}_{\mathbf{L}}(z)$ of $\mathbf{H}_i$, $0 \leq i \leq r$. Then $\mathbf{L}_i$ is a normal F-stable subgroup of $\mathrm{C}_{\mathbf{L}}(z)$, $\mathrm{C}_{\mathbf{L}}(z) = \mathbf{L}_0 \cdots \mathbf{L}_r$ and

$$[\mathbf{L}_i, \mathbf{L}_0 \cdots \mathbf{L}_{i-1}\mathbf{L}_{i+1} \cdots \mathbf{L}_r] \leq \mathbf{L}_i \cap (\mathbf{L}_0 \cdots \mathbf{L}_{i-1}\mathbf{L}_{i+1} \cdots \mathbf{L}_r) = U.$$

We claim that $\overline{\mathbf{L}_i^F}$ is a normal subgroup of $\mathbf{H}_i^F$ of 2-power index. Indeed, let M be the inverse image in $\mathbf{L}_i$ of $\mathbf{H}_i^F$. Then M is F-stable since U is F-stable. Further, $[M, F] \leq U$. Since U is central in M, the map $M \to U$ defined by $x \to x^{-1}F(x)$ is a group homomorphism. The kernel of this map is $\mathbf{L}_i^F$ whence $\mathbf{L}_i^F$ is a normal subgroup of M and the index of $\mathbf{L}_i^F$ in M divides $|U|$. The claim follows since U is a 2-group.

The claim implies that $\mathbf{L}_0^F \cdots \mathbf{L}_r^F$ is a normal subgroup of 2-power index of $\mathrm{C}_{\mathbf{L}}(z)^F$. So, $\tilde{P}$ is a defect group of $\mathbf{L}_0^F \cdots \mathbf{L}_r^F$. The commutator relationship given above then implies that $\tilde{P}$ is a direct product $P_0 \cdots P_r$, where P_i is a defect group of $\mathbf{L}_i^F$, $0 \leq i \leq r$. By Proposition 2.2, $\tilde{P} = P_i$ for some i, $1 \leq i \leq r$. Since z is central in $\mathrm{C}_{\mathbf{L}}(z)$, $i \geq 1$ and $\mathbf{H}_i^F$ is a general linear or unitary group with a central p-element. Let $R = \tilde{P} \cap [\mathbf{L}_i, \mathbf{L}_i]^F$, a defect group of $[\mathbf{L}_i, \mathbf{L}_i]^F$. By suitably replacing $\tilde{P}$ by an $\mathbf{L}_i^F$-conjugate, we may assume that the relevant block of $[\mathbf{L}_i, \mathbf{L}_i]^F$ is $\tilde{P}$-stable and hence that $\tilde{P}$ is a defect group of $[\mathbf{L}_i, \mathbf{L}_i]^F \tilde{P}$.

The isogeny $\mathbf{L}_i \to \mathbf{H}_i$ restricts to an isogeny $[\mathbf{L}_i, \mathbf{L}_i] \to [\mathbf{H}_i, \mathbf{H}_i]$ with kernel $U \cap [\mathbf{L}_i, \mathbf{L}_i]$. However $[\mathbf{H}_i, \mathbf{H}_i]$ is a simply connected semisimple group, being the direct product of special linear groups. Thus, $U \cap [\mathbf{L}_i, \mathbf{L}_i] = 1$ and the restriction of the isogeny to $[\mathbf{L}_i, \mathbf{L}_i]$ is an abstract group isomorphism from $[\mathbf{L}_i, \mathbf{L}_i]$ to $[\mathbf{H}_i, \mathbf{H}_i]$ which commutes with F. Consequently, $[\mathbf{L}_i, \mathbf{L}_i]^F \cong [\mathbf{H}_i, \mathbf{H}_i]^F$. Also, $U \cap [\mathbf{L}_i, \mathbf{L}_i]\tilde{P} = 1$ and the induced map $[\mathbf{L}_i, \mathbf{L}_i]^F \tilde{P} \to \mathbf{H}_i^F$ is injective. Thus $\overline{\tilde{P}} \cong \tilde{P} \cong P$ is a defect group of $\overline{[\mathbf{L}_i, \mathbf{L}_i]^F \tilde{P}} \cong [\mathbf{H}_i, \mathbf{H}_i]^F \overline{\tilde{P}}$. Since $\mathbf{H}_i^F$ is a finite general linear or unitary group, the result now follows from [12, Theorem (3C)] and Proposition 2.3 in the same way as for the case that $L/\mathrm{Z}(L)$ is a projective special linear or unitary group. □

5. On A_{p-1}-components

Lemma 5.1. *Suppose that p is an odd prime and let G be a finite group isomorphic to one of the groups $\mathrm{SL}_p(q)$ or $\mathrm{SU}_p(q)$ for some prime power q not divisible by p. Let U be a non-abelian p-subgroup of G. Then U contains a normal abelian subgroup U_0 of index p such that any element of $U \setminus U_0$ has order p. If $|U| \geq p^{p+1}$, then U_0 contains an element of order p^2.*

Proof. First, consider the case that G is special linear or unitary. By replacing q if necessary by some power we may assume that $U \leq \mathrm{SL}_p(q)$ and p divides $q - 1$. Let S_0 be the Sylow p-subgroup of the group of diagonal matrices of $\mathrm{SL}_p(q)$ and let σ be a non-diagonal, monomial matrix in $\mathrm{SL}_p(q)$ of order p. Then $S := \langle S_0, \sigma \rangle$ is a Sylow p-subgroup of $\mathrm{SL}_p(q)$, S_0 is normal in S, abelian, of index p in S, rank $p - 1$ and any element of S not in S_0 has order p. Let $U_0 = U \cap S_0$. Then U_0 has index at most p in U. On the other hand, since U is non-abelian and S_0 is abelian, U is not contained in U_0. Thus U_0 has index p in U, proving the first assertion. Now suppose that U has exponent p. Then U_0 is elementary abelian. On the other hand, $U_0 \leq S_0$ and the p-rank of S_0 is $p - 1$. Hence, $|U| = p|U_0| \leq p^p$.

□

In the rest of this section, p will denote a fixed prime and $\mathbf{G}$ will denote a connected reductive group in characteristic $r \neq p$ with a Frobenius morphism F with respect to some $\mathbb{F}_{r'}$ structure for some power r' of r. In what follows, whenever we talk of a component of $\mathbf{G}$, we will mean a simple component of $[\mathbf{G}, \mathbf{G}]$.

We need a slight variation of the previous lemma.

Lemma 5.2. *Suppose that p is odd. If $[\mathbf{G}, \mathbf{G}] = \mathrm{SL}_p$, then any p-subgroup of $\mathbf{G}^F$ has an abelian subgroup of index p.*

Proof. Since $\mathbf{G} = \mathrm{Z}^\circ(\mathbf{G})[\mathbf{G}, \mathbf{G}]$ any element and hence any subgroup of $\mathbf{G}^F$ is contained in $\mathrm{Z}^\circ(\mathbf{G})^{F^d}[\mathbf{G}, \mathbf{G}]^{F^d}$ for some $d \geq 1$. This can be seen as follows. Since $\mathbf{G} = \mathrm{Z}^\circ(\mathbf{G})[\mathbf{G}, \mathbf{G}]$, any element u of $\mathbf{G}$ can be written in the form $u = xy$, where $x \in \mathrm{Z}^\circ(\mathbf{G})$ and $y \in [\mathbf{G}, \mathbf{G}]$. Let $\iota : \mathbf{G} \to \mathrm{GL}_n$ be an embedding. Then for some power, say F^t of F, some power, say s of r, and for all $g \in \mathbf{G}$, $F^t \circ \iota(g) = F_s(\iota(g))$ where F_s is the standard Frobenius morphism of GL_n raising every matrix entry to the s-th power. The claim follows since for any $h \in \mathrm{GL}_n$, $F_s^m(h) = h$ for some natural number m. Since any Sylow p-subgroup of $\mathrm{Z}^\circ(\mathbf{G})^{F^d}[\mathbf{G}, \mathbf{G}]^{F^d}$ is of the form R_1R_2, where R_1 is a Sylow p-subgroup of $\mathrm{Z}^\circ(\mathbf{G})^{F^d}$ and R_2 is a Sylow p-subgroup of $[\mathbf{G}, \mathbf{G}]^{F^d}$, the result follows from the previous Lemma and the fact that R_1 is central in R_1R_2. □

Lemma 5.3. *Suppose that p is odd. Let $\mathbf{X} = \mathrm{SL}_p$ be an F-stable component of $\mathbf{G}$ such that $\mathbf{X}^F$ has a central element of order p and let $\mathbf{Y}$ be the product of all other components of $\mathbf{G}$ and $\mathrm{Z}^\circ(\mathbf{G})$. Let P be a p-subgroup of $\mathbf{G}^F$ such that $P \cap \mathbf{X}^F$ is non-abelian of order at least p^p and P is not contained in $\mathbf{X}^F\mathbf{Y}^F$. Then there exists an element of order p^2 in P. Further, if Z is a central subgroup of $\mathbf{G}^F$ of order p such that P/Z has exponent p, then $Z \leq \mathbf{X}^F$.*

Proof. Let $\tilde{P}$ be the inverse image of P under the surjective group homomorphism $\mathbf{X} \times \mathbf{Y} \to \mathbf{G}$ induced by multiplication. The kernel of the multiplication map is isomorphic to $\mathbf{X} \cap \mathbf{Y} = \mathrm{Z}(\mathbf{X}) \cap \mathrm{Z}(\mathbf{Y})$. Since $\mathbf{X}$ is a simple group of type A_{p-1}, the kernel of the multiplication map is a group of order p and in particular, $\tilde{P}$ is a finite p-group. Let $P_1 \leq \mathbf{X}$ be the image of $\tilde{P}$ under the projection of $\mathbf{X} \times \mathbf{Y} \to \mathbf{X}$. Clearly P_1 contains $P \cap \mathbf{X}^F$. We claim that $P \cap \mathbf{X}^F$ is proper in P_1. Indeed, otherwise $\tilde{P} \leq (P \cap \mathbf{X}^F) \times \mathbf{Y}$, whence $P \leq (P \cap \mathbf{X}^F)\mathbf{Y}$. This implies that $P \leq (P \cap \mathbf{X}^F)(P \cap \mathbf{Y}^F) \leq P \cap \mathbf{X}^F\mathbf{Y}^F$, a contradiction. Since $P \cap \mathbf{X}^F$ is assumed to have order at least p^p, the claim implies that $|P_1| \geq p^{p+1}$.

Now P_1 is a finite subgroup of $\mathbf{X}$, thus of some finite special linear (or unitary) group. Hence, by Lemma 5.1, there exists an element $x \in P_1$ of order p^2. Let $y \in \mathbf{Y}$ be such that $w = xy \in P$. Since $P \cap \mathbf{X}^F$ is non-abelian again by Lemma 5.1, there exists $\sigma \in P \cap \mathbf{X}^F$ such that $x\sigma$ has order p. Then w and $w\sigma \in P$, $w^p = x^py^p$ and $(w\sigma)^p = y^p$. Then either $w^p \neq 1$ or $(w\sigma)^p \neq 1$, proving the first part of the result.

Suppose that P/Z has exponent p. Then, $w^p, (w\sigma)^p$ are in Z. Hence $x^p \in Z$. Since $1 \neq x^p$ and Z has order p the second assertion follows. □

Lemma 5.4. *Let $\mathcal{X}$ be an F-stable subset of components of $\mathbf{G}$. Let $\mathbf{X}$ be the product of all elements of $\mathcal{X}$ and let $\mathbf{Y}$ be the product of $\mathrm{Z}^\circ(\mathbf{G})$ and all the components of $[\mathbf{G}, \mathbf{G}]$ not in $\mathcal{X}$.*

(i) *Let P be a defect group of a block b of $\mathbf{G}^F$. Then $P \cap \mathbf{X}^F\mathbf{Y}^F$ is a defect group of a block of $\mathbf{X}^F\mathbf{Y}^F$ covered by b and is of the form P_1P_2, where P_1 is a defect group of a block of $\mathbf{X}^F$ covered by b and P_2 is a defect group of a block of $\mathbf{Y}^F$ covered by b. If $\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$ has p'-order, then $P = P_1P_2$ and the product is direct.*

(ii) *Let c be a p-block of $\mathbf{X}^F\mathbf{Y}^F$. Then the index of the stabiliser of c in $\mathbf{G}^F$ is prime to p. Suppose further that $\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$ is a p-group. Then c is $\mathbf{G}^F$-stable, c is covered by a unique block of $\mathbf{G}^F$ and if P is a defect group of the block of $\mathbf{G}^F$ covering c, then $P \cap \mathbf{X}^F\mathbf{Y}^F$ is a defect group of c and $P/(P \cap \mathbf{X}^F\mathbf{Y}^F) \cong \mathbf{G}^F/\mathbf{X}^F\mathbf{Y}^F$.*

Proof. The first statement of (i) follows from the theory of covering blocks as $\mathbf{X}^F\mathbf{Y}^F$ is a normal subgroup of $\mathbf{G}^F$, $\mathbf{X}^F$ and $\mathbf{Y}^F$ centralise each other and $\mathbf{X}^F \cap \mathbf{Y}^F = \mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F \subseteq \mathrm{Z}(\mathbf{G})^F$ is central in $\mathbf{X}^F\mathbf{Y}^F$. The second assertion of (i) follows from the first assertion, the fact that $|\mathbf{G}^F| = |\mathbf{X}^F||\mathbf{Y}^F|$ and $\mathbf{X}^F \cap \mathbf{Y}^F = \mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$.

We now prove (ii). Let $u \in \mathbf{G}^F$ be a p-element. Then $u = xy$, with $x \in \mathbf{X}$ and $y \in \mathbf{Y}$ such that $x^{-1}F(x) = yF(y^{-1})$ is an element of $\mathrm{Z}(\mathbf{X}) \cap \mathrm{Z}(\mathbf{Y})$. We may assume without loss of generality that x and y are p-elements. The block c of $\mathbf{X}^F\mathbf{Y}^F$ is a product c_1c_2 of blocks c_1 of $\mathbf{X}^F$ and c_2 of $\mathbf{Y}^F$. Thus, it suffices to prove that ${}^xc_1 = c_1$ and ${}^yc_2 = c_2$.

Now consider a regular embedding $\mathbf{X} \leq \tilde{\mathbf{X}}$, where $\tilde{\mathbf{X}}$ is a connected reductive group with connected centre containing $\mathbf{X}$ as a closed subgroup, such that $[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] = [\mathbf{X}, \mathbf{X}]$ and such that F extends to a Frobenius morphism of $\tilde{\mathbf{X}}$. Since $x^{-1}F(x) \in \mathrm{Z}(\mathbf{X}) \leq \mathrm{Z}^\circ(\tilde{\mathbf{X}})$, $x = x_1z$ for some $x_1 \in \tilde{\mathbf{X}}^F$, and $z \in \mathrm{Z}^\circ(\tilde{\mathbf{X}})$. We may assume also that x_1 is a p-element. Then ${}^xc_1 = {}^{x_1}c_1$. On the other hand, c_1 contains an ordinary irreducible character χ in a Lusztig series corresponding to a semisimple element of order prime to p in the dual group of $\mathbf{X}$, hence the index in $\tilde{\mathbf{X}}^F$ of the stabiliser in $\tilde{\mathbf{X}}^F$ of χ has order prime to p (see for instance [3, Corollaire 11.13]). This proves the first assertion. If $\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$ is a p-group, then $|\mathbf{G}^F/\mathbf{X}^F\mathbf{Y}^F| = |\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F|$ is a power of p. By the first assertion, c is $\mathbf{G}^F$-stable and by standard block theory, there is a unique block of $\mathbf{G}^F$ covering c. The second assertion of (ii) now follows from (i). □

Lemma 5.5. *Suppose that p is odd. Let $\mathbf{X}$ be an F-stable component of $\mathbf{G}$ of type A_{p-1} and let $\mathbf{Y}$ be the product of all other components of $\mathbf{G}$ and $\mathrm{Z}^\circ(\mathbf{G})$. Suppose that*

$\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F \neq 1$ *and that* P *is a defect group of* $\mathbf{G}^F$ *such that* $P \cap \mathbf{X}^F$ *is abelian. Then there exists an* F*-stable torus* $\mathbf{T}$ *of* $\mathbf{X}$ *such that* P *is a defect group of* $(\mathbf{YT})^F$.

Proof. In the proof, we will identify blocks with the corresponding central primitive idempotents. Let b be a block of $\mathbf{G}^F$ with P as defect group and let $P_0 := P \cap \mathbf{X}^F\mathbf{Y}^F$. The hypothesis implies that $|\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F| = p$. So, by Lemma 5.4, b is a block of $\mathbf{X}^F\mathbf{Y}^F$, P_0 is a defect group of b as block of $\mathbf{X}^F\mathbf{Y}^F$ and P/P_0 is isomorphic to $\mathbf{G}^F/\mathbf{X}^F\mathbf{Y}^F$. Let $b = b_1b_2$, where b_1 is the block of $\mathbf{X}^F$ covered by b and b_2 is the block of $\mathbf{Y}^F$ covered by b.

Let $u \in P$ generate P modulo P_0 and write $u = xy$, $x \in \mathbf{X}$, $y \in \mathbf{Y}$. Since u is a p-element, we may assume that both x and y are p-elements.

Now consider an F-compatible regular embedding of $\mathbf{X}$ in $\tilde{\mathbf{X}}$ such that $\tilde{\mathbf{X}}^F$ is a finite general linear (or unitary) group. Since $\mathrm{Z}(\tilde{\mathbf{X}})$ is connected, there exists $z \in \mathrm{Z}^\circ(\tilde{\mathbf{X}})$ such that $g := xz^{-1} \in \tilde{\mathbf{X}}^F$. Further, we may choose z such that g is a p-element. Since $u = xy$ normalises P_1, x normalises P_1 and therefore g normalises P_1. Therefore $S = \langle P_1, g\rangle \leq \tilde{X}^F$ is a p-group. Since u normalises b_1 it also follows that b_1 is S-stable.

We claim that there exists a block of $\tilde{\mathbf{X}}^F$ covering b_1 with a defect group D containing S. Indeed, in order to prove the claim, it suffices to prove that $\mathrm{Br}_S(b_1) \neq 0$. Since b_1 and b_2 are both $\mathbf{G}^F$-stable,

$$0 \neq \mathrm{Br}_P(b) = \mathrm{Br}_P(b_1)\mathrm{Br}_P(b_2)$$

and consequently $\mathrm{Br}_P(b_1) \neq 0 \neq \mathrm{Br}_P(b_2)$. Hence writing $b_1 = \sum_{v \in \mathbf{X}^F} \alpha_v v$ as an element of the modular group algebra of $\mathbf{X}^F$ there exists $v \in \mathbf{X}^F$ with α_v non-zero such that v centralises P and in particular v centralises P_1 and u. Since z is central, and y centralises $\mathbf{X}$, we have that v also commutes with g. Hence v centralises S and it follows that $\mathrm{Br}_S(b_1) \neq 0$, proving the claim.

By the block theory of finite general linear (or unitary) groups (see [12]; noting that p divides $q-1$ in the linear case and that p divides $q+1$ in the unitary case) D is a Sylow p-subgroup of the centraliser of some semisimple element of $\tilde{\mathbf{X}}^F$. Since by hypothesis $P_1 = D \cap \mathbf{X}^F$ is abelian, we have that D is abelian, hence D is the Sylow p-subgroup of $\tilde{\mathbf{T}}^F$ for some F-stable maximal torus $\tilde{\mathbf{T}}$ of $\tilde{\mathbf{X}}$. Set $\mathbf{T} = \mathbf{X} \cap \tilde{\mathbf{T}}$, an F-stable maximal torus of $\mathbf{X}$. Then $P_1 = D \cap \mathbf{X}^F$ is a Sylow p-subgroup of $\mathbf{T}^F$. Now $g = xz \in S \leq D \leq \tilde{\mathbf{T}}$, and $z \in \tilde{\mathbf{T}}$ (as z is central), hence $x = gz^{-1} \in \tilde{\mathbf{T}} \cap \mathbf{X} = \mathbf{T}$.

Set $\mathbf{G}_0 = \mathbf{TY}$. We have $u = xy \in \mathbf{G}_0^F$. Since $\mathbf{X} \cap \mathbf{Y} \leq \mathrm{Z}(\mathbf{X}) \leq \mathbf{T}$, we have that $\mathbf{G}_0^F \cap \mathbf{X}^F\mathbf{Y}^F = \mathbf{T}^F\mathbf{Y}^F$ and $\mathbf{G}_0^F/\mathbf{T}^F\mathbf{Y}^F$ is isomorphic to a subgroup of $\mathbf{G}^F/\mathbf{X}^F\mathbf{Y}^F$ and in particular has order p. Hence $\mathbf{G}_0^F = \langle \mathbf{T}^F\mathbf{Y}^F, u\rangle$. Let e be a block of $\mathbf{T}^F$ such that $eb_2 \neq 0$. Since $\mathbf{T}^F$ and $\mathbf{Y}^F$ commute, eb_2 is a block of $\mathbf{T}^F\mathbf{Y}^F$. Since $\mathbf{T}$ is central in $\mathbf{G}_0$, e is $\mathbf{G}_0^F$-stable. Further, b_2 is P-stable hence b_2 is $\mathbf{G}_0^F$-stable. So eb_2 is a $\mathbf{G}_0^F$-stable block of $\mathbf{T}^F\mathbf{Y}^F$ and therefore a block of $\mathbf{G}_0^F$. Since P_1 is the Sylow p-subgroup of $\mathbf{T}^F$ and $\mathbf{T}^F$ is abelian, P_1 is the defect group of e and P_2 is a defect group of b_2. Thus, P_1P_2 is a defect group of eb_2 as block of $\mathbf{T}^F\mathbf{Y}^F$. Since

$\mathrm{Br}_P(eb_2) = \mathrm{Br}_P(e)\mathrm{Br}_P(b_2)$ is non-zero, it follows by order considerations that P is a defect group of eb_2. □

6. The case $p = 3, 5$

In this section we handle the remaining exceptional groups of Lie type for $p \leq 5$.

Lemma 6.1. *Let G, H be finite groups, B a p-block of G and C a p-block of H such that B and C are Morita equivalent. Let P be a defect group of B, and Q a defect group of C. Suppose that P has exponent p. Then P is abelian if and only if Q is abelian. Further, P has an abelian subgroup of index p if and only if Q has an abelian subgroup of index p.*

Proof. By [21, Satz J], the exponent of defect groups is an invariant of Morita equivalence, hence Q has exponent p. In particular any abelian subgroup of P or of Q is elementary abelian. The remaining statements follow by the fact that Morita equivalence preserves the rank of the corresponding defect groups (see [2, Theorem 2.6]). □

Lemma 6.2. *Let $\mathbf{L}$ be connected reductive, with Frobenius morphism F, and let Z be a central p-subgroup of $\mathbf{L}^F$. Let b be a block of $\mathbf{L}^F$ and P a defect group of b. Suppose that P/Z is non-abelian, supports a transitive fusion system and $|P/Z| \geq p^4$. Let $\mathbf{H}$ be an F-stable Levi subgroup of $\mathbf{L}$, let c be a Bonnafé-Rouquier correspondent of b in $\mathbf{H}$ and let Q be a defect group of c. Then Q/Z has exponent p and Q/Z does not have an abelian subgroup of index p. In particular, a Sylow p-subgroup of $\mathbf{H}^F$ does not have an abelian subgroup of index p.*

Proof. Let $\bar{b}$ be the block of $\mathbf{L}^F/Z$ dominated by b and let $\bar{c}$ be the block of $\mathbf{H}^F/Z$ dominated by c. By [10, Prop. 4.1], $\bar{b}$ and $\bar{c}$ are Morita equivalent. Further, P/Z is a defect group of $\bar{b}$ and Q/Z is a defect group of $\bar{c}$. The result now follows from Lemma 2.1 and Lemma 6.1. □

Proposition 6.3. *Let $\mathbf{L}$ be connected reductive, in characteristic $r \neq p = 3$ with Frobenius morphism F, and suppose that $[\mathbf{L}, \mathbf{L}]$ is simply connected of type E_6 in characteristic $r \neq 3$. Let Z be a cyclic subgroup of $\mathrm{Z}(\mathbf{L}^F)$ of order 1 or 3 and let P be a defect group of $\mathbf{L}^F$. Suppose that P/Z supports a transitive fusion system and $|P/Z| \geq 3^7$. Suppose further that either $Z = 1$ or that $\mathbf{L}$ is simple. Then P/Z is abelian.*

Proof. Suppose that P/Z is non-abelian. Let $\mathbf{H}$ be an F-stable Levi subgroup of $\mathbf{L}$ and c a block of $\mathbf{H}^F$ such that c is quasi-isolated and b and c are Bonnafé-Rouquier correspondents. Let $s \in \mathbf{H}^*$ be a semisimple label of c (and b). Since b and c are Bonnafé-Rouquier correspondents, $\mathrm{C}_{\mathbf{L}^*}(s) = \mathrm{C}_{\mathbf{H}^*}(s)$. Let Q be a defect group of c. By Lemma 6.2, we may assume that Q/Z has exponent 3 and does not have an abelian subgroup of index 3. Note that all components of $\mathbf{L}$ and hence of $\mathbf{H}$ are simply connected.

If $\mathbf{H}^F$ has a component of type D_4 or D_5, then the only other possible components are of type A_1. We get a contradiction by Lemma 5.4(i), Lemma 6.2 and the fact that finite groups of type $D_4(q)$, $D_5(q)$, ${}^2D_4(q)$, ${}^2D_5(q)$ and ${}^3D_4(q)$ have a Sylow 3-subgroup with an abelian subgroup of index 3.

Thus, either all components of $\mathbf{H}$ are of type A or $\mathbf{H}$ has a component of type E_6. Let us first consider the case that all components of $\mathbf{H}$ are of type A. In particular, $\mathrm{C}^\circ_{\mathbf{H}^*}(s)$ is a Levi subgroup of $\mathbf{H}^*$ and since s has order prime to 3, $\mathrm{C}_{\mathbf{L}^*}(s) = \mathrm{C}_{\mathbf{H}^*}(s)$ is connected. It follows that s is central in $\mathbf{H}^*$, hence that Q is a defect group of a unipotent block of $\mathbf{H}^F$.

Suppose that $\mathbf{H}$ has a component $\mathbf{X}$ of type A_5. Then $\mathbf{X}$ is F-stable and is the only component of $\mathbf{H}$. If $\mathbf{X}^F$ does not contain a central element of order 3, then by Lemma 5.4(i), a Sylow 3-subgroup of $\mathbf{H}^F$ is a direct product of a Sylow 3-subgroup of $\mathbf{X}^F$ with the Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{H})^F$. Furthermore in this case a Sylow 3-subgroup of $\mathbf{X}^F$ has an abelian subgroup of index 3. If $\mathbf{X}^F$ contains a central element of order 3, then by [5, Prop. 3.3 and Theorem], the principal block is the only unipotent block of $\mathbf{X}^F$, and it follows that Q/Z has an element of order 9 since $\mathrm{PSL}_6(q)$ (respectively $\mathrm{PSU}_6(q)$) has elements of order 9 if $3 \mid q-1$ (respectively $3 \mid q+1$).

Suppose that $\mathbf{H}$ has a component of type A_4. Then the only other possible component is of type A_1 and it follows from Lemma 5.4(i) that a Sylow 3-subgroup of $\mathbf{H}^F$ has an abelian subgroup of index 3.

Suppose that $\mathbf{H}$ has a component $\mathbf{X}$ of type A_3. If all other components are of type A_1, then the above argument applies. If $\mathbf{H}$ has a component of type A_2, say $\mathbf{Y}$, then this is the only other component of $\mathbf{H}$. If the Sylow 3-subgroups of $\mathbf{X}^F$ are abelian, then Lemma 5.4(i) and Lemma 5.2 give the result. Thus, we may assume that the Sylow 3-subgroups of $\mathbf{X}^F$ are non-abelian. Thus, $\mathbf{X}^F$ is isomorphic to $\mathrm{SL}_4(q)$ (respectively $\mathrm{SU}_4(q)$) with $3 \mid q-1$ (respectively $3 \mid q+1$). Consequently, the principal block is the unique unipotent block of $\mathbf{X}^F$. In particular, Q contains a Sylow 3-subgroup of $\mathbf{X}^F$ and Q/Z has an element of order 9.

Thus, we may assume that all components of $\mathbf{H}$ are of type A_2 or A_1. By rank considerations, there can be at most two components of type A_2. By Lemma 5.4 (i) and Lemma 5.2 we may assume that there are two F-stable components $\mathbf{X}$ and $\mathbf{Y}$ of type A_2 such that both $\mathbf{X}^F$ and $\mathbf{Y}^F$ have central elements of order 3. Consequently, the principal block of $\mathbf{X}^F$ is the only unipotent block of $\mathbf{X}^F$ and similarly for $\mathbf{Y}^F$. The only other component of $\mathbf{H}$, if it exists is of type A_1, which also has a unique unipotent block. Hence Q is a Sylow 3-subgroup of $\mathbf{H}^F$.

Since $\mathbf{H}$ is a Levi subgroup of $\mathbf{L}$, there is surjective group homomorphism from $\mathrm{Z}(\mathbf{G})/\mathrm{Z}^\circ(\mathbf{G})$ to $\mathrm{Z}(\mathbf{H})/\mathrm{Z}^\circ(\mathbf{H})$ (see [3, Prop. 4.1]) and by hypothesis, $[\mathbf{L}, \mathbf{L}]$ is simple of type E_6. Hence $\mathrm{Z}(\mathbf{H})/\mathrm{Z}^\circ(\mathbf{H})$ is cyclic of order 1 or 3. Since $\mathbf{X}$ and $\mathbf{Y}$ are the only components of $\mathbf{H}$ with central elements of order 3, it follows that either $\mathrm{Z}(\mathbf{X})$ or $\mathrm{Z}(\mathbf{Y})$ covers $\mathrm{Z}(\mathbf{H})/\mathrm{Z}^\circ(\mathbf{H})$. Thus, either $\mathrm{Z}(\mathbf{X}) \le \mathrm{Z}(\mathbf{Y})\mathrm{Z}^\circ(\mathbf{H})$ or $\mathrm{Z}(\mathbf{Y}) \le \mathrm{Z}(\mathbf{X})\mathrm{Z}^\circ(\mathbf{H})$.

Assume that $\mathrm{Z}(\mathbf{X}) \leq \mathrm{Z}(\mathbf{Y})\mathrm{Z}^\circ(\mathbf{H})$. Let $\mathbf{U}$ be the product of all components of $\mathbf{H}$ other than $\mathbf{X}$ and $\mathrm{Z}^\circ(\mathbf{H})$. Then, $\mathrm{Z}(\mathbf{X})^F \leq (\mathrm{Z}(\mathbf{Y})\mathrm{Z}^\circ(\mathbf{H}))^F \leq \mathbf{U}^F$ and hence $3 \mid |\mathbf{X}^F \cap \mathbf{U}^F|$. Since Q is a Sylow 3-subgroup of $\mathbf{H}^F$ and $|\mathbf{H}^F| = |\mathbf{X}^F||\mathbf{U}^F|$, Q is not contained in $\mathbf{X}^F\mathbf{U}^F$. Further, $Q \cap \mathbf{X}^F$ is a Sylow 3-subgroup of $\mathbf{X}^F$ and in particular is non-abelian of order at least 3^3. By Lemma 6.2, Q/Z has exponent 3. So, by Lemma 5.3, $1 \neq Z \leq \mathrm{Z}(\mathbf{X})$ whence $Z = Z(\mathbf{X})$. Since $Z \neq 1$, $\mathbf{L}$ is simple by hypothesis. In particular, $Z = \mathrm{Z}(\mathbf{X})$ covers $\mathrm{Z}(\mathbf{G})/\mathrm{Z}^\circ(\mathbf{G})$. It follows that $\mathrm{Z}(\mathbf{Y}) \leq \mathrm{Z}(\mathbf{X})\mathrm{Z}^\circ(\mathbf{H})$. By the same argument as above with $\mathbf{Y}$ replacing $\mathbf{X}$, we get that $Z = \mathrm{Z}(\mathbf{Y})$. In particular $\mathrm{Z}(\mathbf{X}) = \mathrm{Z}(\mathbf{Y})$, a contradiction since $\mathbf{X} \cap \mathbf{Y} = 1$.

Finally, consider the case that $\mathbf{H}$ has a component of type E_6. Then $\mathbf{H} = \mathbf{L}$ and $b = c$. Let b_0 be a block of $[\mathbf{L},\mathbf{L}]^F$ covered by b and let $P_0 = P \cap [\mathbf{L},\mathbf{L}]^F$ be a defect group of b_0. Let R be the Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{L})^F$. By Lemma 5.4(i) applied with $\mathbf{X} = [\mathbf{L},\mathbf{L}]$ and $\mathbf{Y} = \mathrm{Z}^\circ(\mathbf{L})$, $P \cap [\mathbf{L},\mathbf{L}]^F\mathrm{Z}^\circ(\mathbf{L})^F = P_0R$. So, P/P_0R is a subgroup of $\mathbf{L}^F/([\mathbf{L},\mathbf{L}]^F\mathrm{Z}^\circ(\mathbf{L})^F)$. Since $\mathbf{L}^F/([\mathbf{L},\mathbf{L}]^F\mathrm{Z}^\circ(\mathbf{L})^F)$ is either trivial or has order 3, we have that P_0R has index at most 3 in P. If P_0 is abelian, then P and hence P/Z has an abelian subgroup of index 3. Thus, P_0 is non-abelian. We claim that $R \leq P_0$. Indeed, by hypothesis, either $Z = 1$ or $[\mathbf{L},\mathbf{L}] = \mathbf{L}$. If $\mathbf{L} = [\mathbf{L},\mathbf{L}]$, then $R = 1$ and the claim holds trivially. If $Z = 1$, then P supports a transitive fusion system. Hence $R \leq \mathrm{Z}(P) \leq [P,P] \leq [\mathbf{L},\mathbf{L}]^F$ and the claim is proved. Thus, $P_0 = PR$ has index at most 3 in P.

Assume first that b_0 is unipotent. The unipotent 3-blocks of exceptional groups have been described in [11]. If b_0 is the principal block, then P/Z has exponent greater than 3. So, b_0 is non-principal and P_0 is non-abelian. By [11] (last part of the proofs for Tableau I), P_0 is the extension of a homocyclic group, say T, of rank 2 by a group of order 3. If T is not elementary abelian, then TZ/Z has exponent at least 9 and hence so does P/Z. Thus, we may assume that T is elementary abelian. So, $|P_0| = 3^3$ and $|P| \leq 3^4$, a contradiction.

So, we may assume that b_0 is quasi-isolated but not unipotent. Here the blocks are described in [19, Section 4.3]. In particular, b_0 corresponds to one of lines 13, 14, or 15 of Table 4 of [19] (and the corresponding Ennola duals; see the last remark of Section 4 of [19]). If b_0 corresponds to line 15, then P_0 is abelian. If b_0 corresponds to line 14, then P_0 is the extension of a homocyclic group, say T, of rank 4 by a group of order 3. If T is not elementary abelian, then TZ/Z has exponent at least 9 and if T is elementary abelian, then $|P_0| \leq 3^5$, whence $|P| \leq 3^6$, a contradiction. If b_0 corresponds to line 13, then P_0 contains a subgroup isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_6(q)$ with $3 \mid q-1$. In particular, $\mho^1(P)$ is not cyclic. On the other hand, since P/Z has exponent 3, $\mho^1(P) \leq Z$. This is a contradiction as Z is cyclic. □

Proposition 6.4. *Suppose that either $p = 3$ and $\mathbf{L}$ is simple and simply connected of type E_7 or E_8 in characteristic $r \neq 3$ or that $p = 5$ and $\mathbf{L}$ is simple of type E_8 in characteristic $r \neq 5$. Let F be a Frobenius morphism on $\mathbf{L}$ and let P be a defect group* *of a p-block of $\mathbf{L}^F$. Suppose that P supports a transitive fusion system and $|P| \geq 3^7$ if $p = 3$. Then P is abelian.*

Proof. Suppose if possible that P is not abelian. As before P has exponent p, and is indecomposable and P does not have an abelian subgroup of index p. Let $z \in \mathrm{Z}(P)$. Since $\mathbf{L}$ is simply connected, $\mathbf{H} := \mathrm{C}_{\mathbf{L}}(z)$ is a connected reductive subgroup of $\mathbf{L}$ of maximal rank and of semisimple rank at most 8 and by [24, Chapter 5, Theorem 9.6], P is a defect group of $\mathbf{H}^F$. The possible components of $\mathbf{H}$ are of type A, D, E_6 or E_7.

Let $\mathcal{X}$ be an F-stable subset of components of $\mathbf{H}$ and let $\mathbf{X}$ be the product of the elements of $\mathcal{X}$. Suppose that $\mathbf{X}^F$ does not have a central element of order p. By Lemma 5.4(i), $P = (P \cap \mathbf{X}^F) \times (P \cap \mathbf{Y}^F)$ where $\mathbf{Y}$ is the product of $\mathrm{Z}^\circ(\mathbf{H})$ and all components of $\mathbf{H}$ other than those in $\mathcal{X}$. The indecomposability of P implies that either $P \leq \mathbf{X}^F$ or $P \leq \mathbf{Y}^F$. Since z is a central p-element of $\mathbf{H}^F$, and $\mathbf{X}^F$ does not have a central element of order p, it follows that $P \leq \mathbf{Y}^F$. By replacing $\mathbf{H}$ by $\mathbf{Y}$, we may assume that the fixed points of every F-orbit of components of $\mathbf{H}$ have central elements of order p ($\mathbf{Y}$ may have rank less than $\mathbf{H}$). Thus, if $p = 5$ the only possible components are of type A_4 and if $p = 3$, then the only possible components are of type A_2, A_5, A_8 or E_6.

Suppose that $\mathbf{H}$ has an F- stable component $\mathbf{X}$ of type A_{p-1}. Let $\mathbf{Y}$ be the product of all components of $\mathbf{H}$ other than those in $\mathbf{X}$ with $\mathrm{Z}^\circ(\mathbf{H})$. By Lemma 5.4(i) and the indecomposability of P, we may assume that $\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$ and hence $\mathbf{H}^F/\mathbf{X}^F\mathbf{Y}^F$ has order p. So, by Lemma 5.4(ii), P is not contained in $\mathbf{X}^F\mathbf{Y}^F$. By Lemma 5.5, we may assume that $P \cap \mathbf{X}^F$ is not abelian since otherwise we can replace $\mathbf{X}$ by a torus. Since $\mathbf{X}^F$ has a central element of order p, $\mathbf{X}^F$ is a special linear (respectively unitary) group. The only non-abelian defect groups of a finite special linear (or unitary) group of degree p in non-describing characteristic are Sylow p-subgroups and $P \cap \mathbf{X}^F$ is a non-abelian defect group of $\mathbf{X}^F$. Thus, $P \cap \mathbf{X}^F$ is a Sylow p-subgroup of $\mathbf{X}^F$ and consequently has order at least p^p. Since we have shown above that P is not contained in $\mathbf{X}^F\mathbf{Y}^F$, by Lemma 5.3, P has an element of order p^2, a contradiction. Thus, we may assume that any component of $\mathbf{H}$ of type A_{p-1} lies in an F-orbit of size at least 2.

If $p = 5$, the only case left to consider is that $\mathbf{H}$ has two components of type A_4 (and these are the only ones) transitively permuted by F. In this case, by rank considerations, $\mathrm{Z}^\circ(\mathbf{H})$ is trivial, and hence $\mathbf{H}^F$ is isomorphic to a special linear or unitary group. In particular the Sylow 5-subgroups of $\mathbf{H}^F$ have an abelian subgroup of index 5, a contradiction. This completes the proof for the case that $p = 5$.

Now assume that $p = 3$. Let us first consider the case that there is a component $\mathbf{X}$ of $\mathbf{H}$ of type A_8. Then $\mathbf{H} = \mathbf{X} = \mathrm{SL}_8$ and we may argue as in the first part of the proof of Proposition 4.1.

Let us next consider the case that there is a component $\mathbf{X}$ of $\mathbf{H}$ of type A_5. If $\mathbf{X}$ also has a component of type A_2, then by rank consideration this is the unique component of type A_2 and we have ruled out this situation above. Thus $\mathbf{X}$ is the unique component of $\mathbf{H}$. Let P_0 be a defect group of a covered block of $\mathbf{X}^F$. The Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{H})^F$ is contained in $\mathrm{Z}(P)$ and $\mathrm{Z}(P) \leq [P,P] \leq [\mathbf{X},\mathbf{X}] \cap \mathbf{H}^F \leq \mathbf{X}^F$, hence we have that the Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{H})^F$ is contained in $\mathbf{X}^F$ and in particular has order at most 3. Thus, P_0 has index at most 3 in P. In particular P_0 is non-abelian. Now $\mathbf{X} = \mathbf{M}/Z$, where $\mathbf{M}$ is a special linear group of degree 6 (with a compatible F-action) and Z is a central subgroup. Since $\mathrm{Z}(\mathbf{M})$ is cyclic of order 6 (or 3 if $r = 2$) and since $\mathbf{X}$ has a central element of order 3, Z is either trivial or of order 2, Z is F-stable and $Z^F = Z$. Further, $\mathbf{M}^F/Z$ is a normal subgroup of $\mathbf{X}^F = (\mathbf{M}/Z)^F$ of index $|Z|$. Thus P_0 is a defect group of $\mathbf{M}^F/Z$ and up to isomorphism a defect group of $\mathbf{M}^F$ and $\mathbf{M}^F = \mathrm{SL}_6(q)$ (respectively $\mathrm{SU}_6(q)$). Since $\mathbf{M}^F/Z$ has index prime to 3, $\mathbf{M}^F/Z$ contains the 3-part of the centre of $\mathbf{X}^F$, hence $\mathbf{M}^F$ has a central element of order 3. Thus, P_0 is the intersection with $\mathbf{X}^F$ of a Sylow 3-subgroup of the centraliser of a semisimple $3'$-element of $\mathrm{GL}_6(q)$ (or $\mathrm{GU}_6(q)$). Since P_0 has exponent 3 and is non-abelian, the possible structures of semisimple centralisers in $\mathrm{GL}_6(q)$ (or $\mathrm{GU}_6(q)$) force that the centraliser in $\mathrm{GL}_6(q)$ (respectively $\mathrm{GU}_6(q)$) has the form $\mathrm{GL}_3(q^2)$. Hence $|P_0| \leq p^3$ and $|P| \leq p^4$ a contradiction.

Suppose $\mathbf{H}$ has a component of type E_6. Arguing as in the previous case $\mathbf{H}$ has no components of type A_2 and hence the E_6-component is the unique component of $\mathbf{H}$. This component is of simply connected type since as explained in the beginning of the proof we may assume that the F-fixed point subgroup of every F-orbit of components of $\mathbf{H}$ has central elements of order 3 and we are done by Proposition 6.3 (note that we apply Proposition 6.3 here in the case that $Z = 1$).

The only case left to consider is that all components of $\mathbf{H}$ are of type A_2 and no component is F-stable. By rank considerations and the fact that groups of type E_8 do not have semisimple centralisers with component type A_2^4 (see the tables in [9]), we are left with two possibilities: either $\mathbf{H}$ has exactly three components, all of type A_2 and in a single F-orbit or $\mathbf{H}$ has exactly two components both of type A_2 and in a single F-orbit. In any case, $[\mathbf{H},\mathbf{H}]^F$ has a quotient or subgroup H_0 isomorphic to $\mathrm{PSL}_3(q)$ (respectively $\mathrm{PSU}_3(q)$) for some q such that $|[\mathbf{H},\mathbf{H}]^F|/|H_0|$ equals 1 or 3. Let $P_0 = P \cap [\mathbf{H},\mathbf{H}]$ and let P_0' be either the intersection of P_0 with H_0 or the image of P_0 in H_0. Then P_0' has exponent 3. Since any 3-subgroup of a finite projective special linear or unitary group of degree 3 has an abelian subgroup of index 3 and since the 3-rank of these groups is 2, it follows that $|P_0'| \leq 3^3$. Hence $|P_0| \leq 3^4$.

We claim that the index of P_0 in P is at most 3. Indeed, let R be the Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{H})^F$. Then $R \leq \mathrm{Z}(P) \leq [P,P] \leq [\mathbf{H},\mathbf{H}]$, that is $R \leq P_0$. On the other hand, $|P/P_0R|$ divides $|\mathrm{Z}([\mathbf{H},\mathbf{H}]^F)|_3$ and we have seen from the structure of $[\mathbf{H},\mathbf{H}]^F$ that $\mathrm{Z}([\mathbf{H},\mathbf{H}]^F)$ has order at most 3. This proves the claim. Hence $|P| \leq 3^5$, a contradiction. $\square$

7. Consequences

We note some consequences of Theorem 1.2.

Theorem 7.1. *Let B be a block of a finite group such that $k(B) - l(B) = 1$ (e. g. a block with multiplicity 1). Then B has elementary abelian defect groups.*

Proof. See proof of Theorem 3.6 in [23]. □

Corollary 7.2. *Let B be a block of a finite group such that $k(B) = 3$. Then B has elementary abelian defect groups.*

Proof. We have $l(B) \in \{1, 2\}$. In case $l(B) = 1$ it was shown by Külshammer [22] that the defect groups of B have order 3. The remaining case $l(B) = 2$ follows from Theorem 7.1. □

References

[1] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, Vol. 391, Cambridge University Press, Cambridge, 2011.
[2] C. Bessenrodt and W. Willems, *Relations between complexity and modular invariants and consequences for p-soluble groups*, J. Algebra **86** (1984), 445–456.
[3] C. Bonnafé, *Sur les caractères des groupes réductifs finis a centre non connexe: applications aux groupes spéciaux linéaires et unitaires*, Astérisque **306** (2006).
[4] C. Bonnafé, R. Rouquier, *Catégories dérivées et variétés de Deligne-Lusztig* , Publ. Math. Inst. Hautes Études Sci. **97** (2003), 1–59.
[5] M. Cabanes and M. Enguehard, *On unipotent blocks and their ordinary characters*, Invent. Math. **117** (1994), 149–164.
[6] M. Cabanes and M. Enguehard, *On blocks of finite reductive groups and twisted induction*, Advances Math. **145** (1999), 189–229.
[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.
[8] D. A. Craven, *The Theory of Fusion Systems*, Cambridge Studies in Advanced Mathematics, Vol. 131, Cambridge University Press, Cambridge, 2011.
[9] D. I. Deriziotis, *Centralisers of semisimple elements of the Chevalley groups E_7 and E_8*. Tokyo. J. Math **6** (1983), 191–216.
[10] C. Eaton, R. Kessar, B. Külshammer, B. Sambale, *2-blocks with abelian defect groups* Adv. Math. **254** (2014), 706–735.
[11] M. Enguehard, *Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais*, J. Algebra **230** (2000), 334–377.
[12] P. Fong and B. Srinivasan, *The blocks of finite general linear and unitary groups*, Inv. Math. **69** (1982), 101–153.
[13] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.6.4*, 2013, `http://www.gap.system.org`.
[14] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic 2 type*, Mem. Amer. Math. Soc. **42** (1983).
[15] D. Gorenstein, R. Lyons and R. Solomon, *The classification of finite simple groups 3*, Mathematical Surveys and Monographs **40**, American Mathematical Society (1998).

[16] C. Hering, *Transitive linear groups and linear groups which contain irreducible subgroups of prime order*, Geom. Dedic. **2** (1974), 425–460.
[17] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin 1967.
[18] B. Huppert and N. Blackburn, *Finite groups III*, Springer-Verlag, Berlin 1982.
[19] R. Kessar and G. Malle, *Quasi-isolated blocks and Brauer's height zero conjecture*, Ann. of Math. (2) **178** (2013) 321–384.
[20] R. Kessar and R. Stancu, *A reduction theorem for fusion systems of blocks*, J. Algebra **319** (2008), 806–823.
[21] B. Külshammer, *Bemerkungen über die Gruppenalgebra als symmetrische Algebra II*, J. Algebra **75** (1982), 59-69.
[22] B. Külshammer, *Symmetric local algebras and small blocks of finite groups*, J. Algebra **88** (1984), 190–195.
[23] B. Külshammer, G. Navarro, B. Sambale and P. H. Tiep, *Finite groups with two conjugacy classes of p-elements and related questions for p-blocks*, Bull. London Math. Soc. **46** (2014), 305–314
[24] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press Inc., Boston, MA, 1989.
[25] G. Navarro and P. H. Tiep, *Abelian Sylow subgroups in a finite group*, J. Algebra **398** (2014), 519-526.
[26] A. Ruiz and A. Viruel, *The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p*, Math. Z. **248** (2004), 45–65.
[27] B. Sambale, *Blocks of finite groups and their invariants*, Lecture Notes in Math. Vol. 2127, Springer-Verlag, Berlin, 2015.
[28] R. Steinberg, *Lectures on Chevalley Groups*, Notes by J. Faulkner and R. Wilson, Mimeographed notes, Yale University Mathematics Department (1968).

Department of Algebra, Budapest University of Technology and Economics, H-1521 Budapest, Hungary
E-mail address: hethelyi@math.bme.hu

Department of Mathematics, City University London, Northampton Square, London, EC1V 0HB, Great Britain
E-mail address: radha.kessar.1@city.ac.uk

Institut für Mathematik, Friedrich-Schiller-Universität, 07743 Jena, Germany
E-mail address: kuelshammer@uni-jena.de

Institut für Mathematik, Friedrich-Schiller-Universität, 07743 Jena, Germany
E-mail address: benjamin.sambale@uni-jena.de