Černý, A. (2009). Characterization of the oblique projector U(VU)V-dagger with application to constrained least squares. Linear Algebra and its Applications, 431(9), pp. 1564-1570. doi: 10.1016/j.laa.2009.05.025

# Characterization of the oblique projector $U(VU)^{\dagger}V$ with application to constrained least squares☆

Aleš Černý

*Cass Business School, City University London*

**Abstract**

We provide a full characterization of the oblique projector $U(VU)^{\dagger}V$ in the general case where the range of $U$ and the null space of $V$ are not complementary subspaces. We discuss the new result in the context of constrained least squares minimization which finds many applications in engineering and statistics.

*AMS classification:* 15A09, 15A04, 90C20

*Key words:* oblique projection, constrained least squares, Zlobec formula

## 1. Introduction

Let $E \in \mathbb{C}^{m\times m}$ be idempotent, $E^2 = E$. The null space and range of any idempotent matrix are complementary, cf. [1, Theorem 2.8],

$$R(E) + N(E) = \mathbb{C}^m, R(E) \cap N(E) = \{0\},$$

and we say that $E$ is an oblique projector onto $R(E)$ along $N(E)$. For any two complementary subspaces of $\mathbb{C}^m$ we denote the oblique projector onto $L$ along $M$ by $P_{L,M}$. The orthogonal projector onto $L$ is denoted by $P_L := P_{L,L^{\perp}}$, where $L^{\perp}$ is the orthogonal complement of $L$. Oblique projectors arise in numerous engineering and statistical applications, see [1, Chapter 8], [2] and references therein. Many of their properties follow from the general solution to the matrix equation $XAX = X$ studied in 1960-ies in the context of the various pseudoinverses, cf. [3]. This literature is mature, with excellent monographs such as [1]. In particular it is very well understood how to construct an oblique projector with a prescribed range and null space.

**Proposition 1.1.** *Let $L, M$ be complementary subspaces of $\mathbb{C}^m$. For any two matrices $U, V$ with $R(U) = L$ and $N(V) = M$ one has*

$$P_{L,M} = U(VU)^{\dagger}V,$$

*where the superscript "†" denotes the Moore-Penrose inverse. If $U$ and $V$ are in addition orthogonal projectors (i.e. they are Hermitian and idempotent) one obtains an even simpler form due to Greville [4, (3.1) and Theorem 2],*

$$P_{L,M} = P_L(P_{M^{\perp}}P_L)^{\dagger}P_{M^{\perp}} = (P_{M^{\perp}}P_L)^{\dagger}. \tag{1}$$

☆I would like to thank Adi Ben-Israel, Laura Rebollo-Neira, Götz Trenkler and two anonymous referees for helpful comments and pointers to references. Any errors in the manuscript are my responsibility.

*Preprint submitted to Linear Algebra and Its Applications* *May 26, 2009*

The converse problem of characterizing the range and null space of a given idempotent matrix has not received the same amount of attention. The motivation for studying idempotents of the form $U(VU)^{\dagger}V$ in the general case where $R(U) + N(V) \subsetneq \mathbb{C}^m$ and/or $R(U) \cap N(V) \neq \{0\}$ comes, among others, from constrained least squares optimization with a range of applications mentioned above. Briefly, the problem

$$\min_{x\in\mathbb{C}^n} \|A_1x - b_1\|^2, \text{ subject to } A_2x = b_2,$$

gives rise to the projector $D_2(A_1D_2)^{\dagger}A_1$ where $D_2$ is an arbitrary but fixed matrix with the property $R(D_2) = N(A_2)$. In this situation we typically have neither $R(D_2) + N(A_1) = \mathbb{C}^m$ nor $R(D_2) \cap N(A_1) = \{0\}$. Oblique projectors of the form $U(VU)^{\dagger}V$ with $R(U) + N(V) = \mathbb{C}^m$ and $R(U) \cap N(V) \neq \{0\}$ feature also in signal reconstruction, cf. [5].

Given that $U(VU)^{\dagger}V$ has a wide range of applications it is desirable to understand its geometric nature. One might conjecture that in general

$$U(VU)^{\dagger}V = P_{L,M}, \text{ where} \tag{2}$$
$$L = P_{R(U)}N(V)^{\perp} = R(U) \cap (R(U) \cap N(V))^{\perp}, \tag{3}$$
$$M = N(V) + (N(V) + R(U))^{\perp}, \tag{4}$$

but the behaviour of the projector is somewhat more intricate and cannot be described based on the knowledge of $R(U)$ and $N(V)$ alone. The conjecture (2)-(4) turns out to be true only when both $U$ and $V$ are orthogonal projectors. Surprisingly, the main tool in proving the general result is the Zlobec formula [6] in conjunction with Proposition 1.1.

The result presented here is different from the problem discussed by Rao and Yanai [7] in which projectors onto and along two given subspaces are considered under the assumption that the subspaces are not necessarily spanning the whole space. In such a situation, the projector no longer needs to be idempotent.

The paper is organized as follows. In section 2 we introduce required terminology and notation, we establish the main tools and prove Proposition 1.1. In section 3 we state and prove the main result. In section 4 we discuss application of the main result to constrained least squares minimization and the link to the minimal norm solution of Eldén [8].

## 2. Preliminaries

We use notation of [1]. $A^*$ denotes the conjugate transpose of matrix $A$. We write $r(A)$, $R(A)$, $N(A)$ for the rank, range and null space of $A$, respectively. Consider the following relations

$$AXA = A, \tag{I.1}$$
$$XAX = X, \tag{I.2}$$
$$AX = (AX)^*, \tag{I.3}$$
$$XA = (XA)^*. \tag{I.4}$$

We write $X \in A\{i, j, \ldots, k\}$, if $X$ satisfies conditions (I.$i$), (I.$j$), ..., (I.$k$). $A^{\dagger}$ denotes the Moore-Penrose inverse which is the unique element of $A\{1, 2, 3, 4\}$. The following theorem is our main tool.

**Theorem 2.1** ([1, Theorem 2.13])**.** *Let* $A \in \mathbb{C}^{m\times n}, \tilde{U} \in \mathbb{C}^{n\times s}, \tilde{V} \in \mathbb{C}^{t\times m}$ *and*

$$Z = \tilde{U}(\tilde{V}A\tilde{U})^{(1)}\tilde{V},$$

*where* $(\tilde{V}A\tilde{U})^{(1)}$ *is a fixed but arbitrary element of* $(\tilde{V}A\tilde{U})\{1\}$*. Then*

*a)* $Z \in A\{1\}$ *if and only if* $r(\tilde{V}A\tilde{U}) = r(A)$*;*

*b)* $Z \in A\{2\}$ *and* $R(Z) = R(\tilde{U})$ *if and only if* $r(\tilde{V}A\tilde{U}) = r(\tilde{U})$*;*

*c)* $Z \in A\{2\}$ *and* $N(Z) = N(\tilde{V})$ *if and only if* $r(\tilde{V}A\tilde{U}) = r(\tilde{V})$*;*

*d)* $Z = A^{(1,2)}_{R(\tilde{U}),N(\tilde{V})}$ *if and only if* $r(\tilde{U}) = r(\tilde{V}) = r(\tilde{V}A\tilde{U}) = r(A)$*, where* $A^{(1,2)}_{R(\tilde{U}),N(\tilde{V})}$ *is the unique element of* $A\{1,2\}$ *with range* $R(\tilde{U})$ *and null space* $N(\tilde{V})$*, also known as the oblique pseudoinverse (cf. [9]).*

**Corollary 2.2.** *The Zlobec formula [6],*

$$A^{\dagger} = A^{*}(A^{*}AA^{*})^{(1)}A^{*}, \tag{5}$$

*is now obtained by setting* $\tilde{U} = \tilde{V} = A^{*}$ *in part d) and arguing* $A^{(1,2)}_{R(A^{*}),N(A^{*})} = A^{\dagger}$*.*

The following is a pre-cursor to the main result in this note. The "if" part appears, for example, in [10, (3.51)].

**Corollary 2.3.** $\tilde{U}(\tilde{V}\tilde{U})^{(1)}\tilde{V} = P_{R(\tilde{U}),N(\tilde{V})}$ *if and only if* $r(\tilde{V}\tilde{U}) = r(\tilde{V}) = r(\tilde{U})$*.*

Next we show that the form $U(VU)^{\dagger}V$ covers all idempotent matrices.

**Lemma 2.4.** *Let* $U \in \mathbb{C}^{m\times p}, V \in \mathbb{C}^{q\times m}$*.* $R(U)$ *and* $N(V)$ *are complementary subspaces of* $\mathbb{C}^{m}$ *if and only if* $r(U) = r(V) = r(VU)$*.*

Proof. If: By Corollary 2.3 $U(VU)^{\dagger}V = P_{R(U),N(V)}$ which implies that $R(U), N(V)$ are complementary.

Only if: i) complementarity implies $\dim(R(U)) + \dim(N(V)) = m$. On rearranging we obtain $r(U) = m - \dim(N(V))$ and by the rank-nullity theorem $r(U) = r(V)$.

ii) Complementarity also implies $R(U) \cap N(V) = \{0\}$ which yields $N(VU) = N(U)$. By rank-nullity theorem we obtain $r(VU) = r(U)$. □

**Proposition 2.5.** *Matrix* $E \in \mathbb{C}^{m\times m}$ *is idempotent if and only if there are matrices* $U \in \mathbb{C}^{m\times p}, V \in \mathbb{C}^{q\times m}$ *such that*

$$E := U(VU)^{\dagger}V. \tag{6}$$

Proof. The 'if' statement follows easily from (6) and (I.2),

$$E^{2} = U(VU)^{\dagger}VU(VU)^{\dagger}V = E.$$

The 'only if' part: construct $U$ so that its columns form a basis of $R(E)$ and construct $V^{*}$ so that its columns form the basis of $N(E)^{\perp}$. This implies $R(U) = R(E), N(V) = N(E)$. Since $E$ is idempotent $R(U), N(V)$ are by construction complementary and from Lemma 2.4 we obtain $r(U) = r(V) = r(VU)$. By Corollary 2.3 $U(VU)^{\dagger}V = P_{R(E),N(E)} = E$. □

**Remark 2.6.** *A comprehensive characterization of projectors appears in [3]. Proposition 2.5 resembles a result of Mitra [11, Theorem 3a] who shows that all idempotent matrices are of the form* $\tilde{U}(\tilde{V}\tilde{U})^{(1,2)}\tilde{V}$ *where* $(\tilde{V}\tilde{U})^{(1,2)}$ *is an arbitrary element of* $\tilde{V}\tilde{U}\{1,2\}$*. This result is generalized further in [1, Theorem 2.13] to the form* $\tilde{U}(\tilde{V}\tilde{U})^{(1)}\tilde{V}$*, see Corollary 2.3. Proposition 2.5 goes in the opposite direction in order to avoid the ambiguity associated with* $\{1,2\}$*-inverses.*

To conclude we provide a proof of Proposition 1.1.

Proof (Proposition 1.1). The first statement follows from the 'only if' part in the proof of Proposition 2.5. The second part follows from identities $(P_{M^\perp}P_L)^\dagger = P_L(P_{M^\perp}P_L)^\dagger = (P_{M^\perp}P_L)^\dagger P_{M^\perp}$, see [1, Exercise 2.57]. □

## 3. Result

**Theorem 3.1.** *Given two arbitrary matrices $U \in \mathbb{C}^{m\times p}, V \in \mathbb{C}^{q\times m}$ the matrix $E = U(VU)^\dagger V$ is idempotent with range and null space given by*

$$R(E) = R(UU^*V^*) = R(UU^*V^*V) = R(U) \cap ((UU^*)^\dagger(R(U) \cap N(V)))^\perp, \tag{7}$$

$$N(E) = N(U^*V^*V) = N(UU^*V^*V) = N(V) \oplus (V^*V)^\dagger(R(U) + N(V))^\perp. \tag{8}$$

Proof. By Zlobec's formula (5) with $A = VU$ we obtain

$$E = UU^*V^*(U^*V^*VUU^*V^*)^{(1)}U^*V^*V.$$

Setting $\tilde{U} = UU^*V^*, \tilde{V} = U^*V^*V$ we claim $r(\tilde{U}) = r(\tilde{V}) = r(\tilde{V}\tilde{U}) = r(VU)$. Indeed,

$$r(VU) = r(VUU^*V^*) = r(VUU^*V^*VUU^*V^*) \leq r(U^*V^*VUU^*V^*) = r(\tilde{V}\tilde{U}), \tag{9}$$

$$r(\tilde{V}\tilde{U}) \leq r(\tilde{U}) = r(UU^*V^*) \leq r(U^*V^*) = r(VU), \tag{10}$$

$$r(\tilde{V}\tilde{U}) \leq r(\tilde{V}) = r(U^*V^*V) \leq r(U^*V^*) = r(VU). \tag{11}$$

Corollary 2.3 yields $R(E) = R(\tilde{U}), N(E) = N(\tilde{V})$. From

$$r(VU) = r(VUU^*V^*) = r(VUU^*V^*VUU^*V^*) \leq r(UU^*V^*V) \leq r(U^*V^*) = r(VU),$$

and from (9)-(11) we obtain $r(VU) = r(UU^*V^*) = r(UU^*V^*V)$ which implies $R(UU^*V^*) = R(UU^*V^*V)$. The proof of $N(U^*V^*V) = N(UU^*V^*V)$ proceeds similarly by showing $r(U^*V^*V) = r(UU^*V^*V)$.

To show the last equality in (8) we observe $\mathbb{C}^m = N(V) \oplus R(V^*)$. Since $N(V) \subseteq N(U^*V^*V)$ we have

$$N(U^*V^*V) = N(V) \oplus (R(V^*) \cap N(U^*V^*V)). \tag{12}$$

Continuing with the second term on the right hand side we obtain

$$\begin{aligned} y \in R(V^*) \cap N(U^*V^*V) &\iff (V^*Vy \in N(U^*) \cap R(V^*)) \wedge (y \in R(V^*)) \\ &\iff y \in (V^*V)^\dagger(N(U^*) \cap R(V^*)), \end{aligned}$$

which yields

$$R(V^*) \cap N(U^*V^*V) = (V^*V)^\dagger(R(U)^\perp \cap N(V)^\perp) = (V^*V)^\dagger(R(U) + N(V))^\perp. \tag{13}$$

On substituting (13) into (12) we obtain the desired result.

The last equality in (7) is obtained by writing $R(UU^*V^*) = N(VUU^*)^\perp$ and then evaluating $N(VUU^*)$ by exchanging the role of $U$ and $V^*$ in (12) and (13). □

**Remark 3.2.** *Special cases of Theorem 3.1 include situations covered by Corollary 2.3 in which $r(U) = r(V) = r(VU)$ and we have $R(E) = R(U), N(E) = N(V)$; the Langenhop form [12, Lemma 2.2] with $VU = I$ is a case in point. The Greville formula (1) also falls into this category. Hirabayashi and Unser [5, Lemma 3] encounter the case $R(U) + N(V) = \mathbb{C}^m$ and $R(U) \cap N(V) \neq \{0\}$, yielding $R(E) = R(UU^*V^*), N(E) = N(V)$.*

## 4. Application

**Proposition 4.1.** *Let* $A_1 \in \mathbb{C}^{m\times n}, b_1 \in \mathbb{C}^m, A_2 \in \mathbb{C}^{k\times n}, r(A_2) = k \geq 1, b_2 \in \mathbb{C}^k$. *Solutions of the problem*

$$\min_{x\in\mathbb{C}^n} \|A_1 x - b_1\|^2\,,\ \textit{subject to } A_2 x = b_2, \tag{14}$$

*lie in the set*

$$\Xi = \{D_2(A_1D_2)^\dagger A_1 A_1^\dagger b_1 + (I - D_2(A_1D_2)^\dagger A_1)(A_2^\dagger b_2 + z) : z \in N(A_2)\}, \tag{15}$$

*where* $D_2$ *is an arbitrary but fixed matrix with the property* $R(D_2) = N(A_2)$.

Proof. See [1, Exercise 3.10]. □

In general, the projector $D_2(A_1D_2)^\dagger A_1$ will depend on how $D_2$ is chosen. However, Theorem 3.1 shows that there is a special case when $D_2(A_1D_2)^\dagger A_1$ is actually invariant to the choice of $D_2$.

**Corollary 4.2.** *Using the notation of Proposition 4.1 assume further* $r(A_1) = n$. *Then*

$$D_2(A_1D_2)^\dagger A_1 = P_{N(A_2),(A_1^*A_1)^{-1}R(A_2^*)},$$

*and* $\Xi$ *is a singleton,*

$$\Xi = \{A_1^\dagger b_1 + (A_1^*A_1)^{-1}A_2^*(A_2(A_1^*A_1)^{-1}A_2^*)^{-1}(b_2 - A_2A_1^\dagger b_1)\}.$$

Proof. We have $N(A_1) = 0$ and by Theorem 3.1

$$R(D_2(A_1D_2)^\dagger A_1) = R(D_2) \cap (\{0\})^\perp = N(A_2),$$
$$N(D_2(A_1D_2)^\dagger A_1) = (A_1^*A_1)^{-1}R(D_2)^\perp = (A_1^*A_1)^{-1}R(A_2^*).$$

This implies $(I - D_2(A_1D_2)^\dagger A_1)z = 0$ for all $z \in N(A_2)$ and by Proposition 1.1

$$(I - D_2(A_1D_2)A_1) = (A_1^*A_1)^{-1}A_2^*(A_2(A_1^*A_1)^{-1}A_2^*)^{-1}A_2.$$

The rest follows from Proposition 4.1. □

Note that Corollary 4.2 is not covered by Corollary 2.3 since $n - k = r(D_2) = r(A_1D_2) < r(A_1) = n$. In situations where the choice of $D_2$ impacts on the projector $D_2(A_1D_2)^\dagger A_1$ Theorem 3.1 guides us to the convenient choice of $D_2$ which simplifies the geometry of the result and also helps to identify the element of $\Xi$ with minimal distance from a given reference point.

**Corollary 4.3.** *Using the notation of Proposition 4.1 the following statements hold:*

1. *The constrained least squares minimizer in (14) lies in the set*

$$\Xi = \{A_1^\dagger b_1 + P_{\mathcal{Y},\mathcal{X}}(A_2^\dagger b_2 - A_1^\dagger b_1) + z : z \in N(A_1) \cap N(A_2)\}, \tag{16}$$

*with*

$$P_{\mathcal{X},\mathcal{Y}} = I - P_{\mathcal{Y},\mathcal{X}} = (A_1(I - A_2^\dagger A_2))^\dagger A_1, \tag{17}$$
$$\mathcal{X} = P_{N(A_2)}R(A_1^*) = N(A_2) \cap (N(A_2) \cap N(A_1))^\perp, \tag{18}$$
$$\mathcal{Y} = N(A_1) \oplus (A_1^*A_1)^\dagger (N(A_1) + N(A_2))^\perp. \tag{19}$$

2. *The element of* $\Xi$ *with the smallest Euclidean norm is given by*

$$\xi := A_1^\dagger b_1 + P_{\mathcal{Y},\mathcal{X}}(A_2^\dagger b_2 - A_1^\dagger b_1).$$

3. *For any* $y \in \mathbb{C}^n$ *the solution of* $\min_{x\in\Xi} \|x - y\|$ *is given by*

$$\psi(y) := \xi + P_{N(A_1)\cap N(A_2)} y. \tag{20}$$

Proof. 1. On setting $D_2 = I - A_2^\dagger A_2 = P_{N(A_2)}$ Proposition 4.1 and Theorem 3.1 yield

$$\Xi = A_1^\dagger b_1 + P_{\mathcal{Y},\mathcal{X}}(A_2^\dagger b_2 - A_1^\dagger b_1 + N(A_2)), \tag{21}$$

with $P_{\mathcal{X},\mathcal{Y}}$, $\mathcal{X}$ and $\mathcal{Y}$ given in (17)-(19). From (18) we obtain $N(A_2) = \mathcal{X} \oplus (N(A_1) \cap N(A_2))$ which implies

$$P_{\mathcal{Y},\mathcal{X}} N(A_2) = P_{\mathcal{Y},\mathcal{X}}(N(A_1) \cap N(A_2)) = N(A_1) \cap N(A_2), \tag{22}$$

the last equality following from $N(A_1) \cap N(A_2) \subseteq \mathcal{Y}$. Substitution of (22) into (21) yields (16).

2. By (18) we have $\mathcal{X} \subseteq (N(A_2) \cap N(A_1))^\perp = R(A_1^*) + R(A_2^*)$. Consequently

$$P_{\mathcal{Y},\mathcal{X}}(R(A_1^*) + R(A_2^*)) = (I - P_{\mathcal{X},\mathcal{Y}})(R(A_1^*) + R(A_2^*)) \subseteq R(A_1^*) + R(A_2^*). \tag{23}$$

This implies

$$\xi \in R(A_1^*) + R(A_2^*) = (N(A_2) \cap N(A_1))^\perp. \tag{24}$$

By (16) $x - \xi \in N(A_1) \cap N(A_2)$ for any $x \in \Xi$ which together with (24) yields

$$\|x\|^2 = \|x - \xi + \xi\|^2 = \|x - \xi\|^2 + \|\xi\|^2 \text{ for all } x \in \Xi.$$

3. By (16), (20) and (24) we obtain $x - \psi(y) \in N(A_2) \cap N(A_1)$ and $\psi(y) - y \in (N(A_2) \cap N(A_1))^\perp$ which implies $\|x - y\|^2 = \|x - \psi(y) + \psi(y) - y\|^2 = \|x - \xi\|^2 + \|\xi - y\|^2$, for all $x \in \Xi$. □

**Remark 4.4.** *It is well known that vector* $A_1^\dagger b_1$ *has the smallest Euclidean norm among all solutions of the unconstrained least squares problem* $\min_{x\in\mathbb{C}^n} \|A_1 x - b_1\|$. *We have shown in part 2. of Corollary 4.3 that* $\xi = A_1^\dagger b_1 + P_{\mathcal{Y},\mathcal{X}}(A_2^\dagger b_2 - A_1^\dagger b_1)$ *is the shortest solution of the constrained least squares problem (14).*

*Eldén [8, Theorem 2.1] studied minimal norm solutions of constrained least squares. On setting*

$$h = b_2 - A_2 A_1^\dagger b_1, \quad f = x - A_1^\dagger b_1, \quad K = A_1, \quad L = A_2, \quad M = I,$$

*Eldén's solution yields that*

$$\zeta := A_1^\dagger b_1 + (I - P_{N(A_2)}(A_1 P_{N(A_2)})^\dagger A_1) A_2^\dagger (b_2 - A_2 A_1^\dagger b_1)$$

*minimizes the Euclidean distance* $\|x - A_1^\dagger b_1\|$ *among all constrained minimizers* $x \in \Xi$.

*With a little bit of work one finds* $\zeta = \xi - P_{\mathcal{Y},\mathcal{X}} P_{N(A_2)} A_1^\dagger b_1 = \xi$, *since* $P_{N(A_2)} A_1^\dagger \in \mathcal{X}$ *by virtue of (18). Thus part 3. of Corollary 4.3 simplifies and extends Eldén's result.*

## References


[1] A. Ben-Israel, T. N. E. Greville, Generalized inverses: Theory and applications, 2nd Edition, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 15, Springer-Verlag, New York, 2003.

[2] G. Corach, A. Maestripieri, Weighted generalized inverses, oblique projections, and least-squares problems, Numer. Funct. Anal. Optim. 26 (6) (2005) 659–673.

[3] G. Trenkler, Characterizations of oblique and orthogonal projectors, in: Proceedings of the International Conference on Linear Statistical Inference LINSTAT '93 (Poznań, 1993), Vol. 306 of Math. Appl., Kluwer Acad. Publ., Dordrecht, 1994, pp. 255–270.

[4] T. N. E. Greville, Solutions of the matrix equation $XAX = X$, and relations between oblique and orthogonal projectors, SIAM J. Appl. Math. 26 (1974) 828–832.

[5] A. Hirabayashi, M. Unser, Consistent sampling and signal recovery, IEEE Trans. Signal Process. 55 (8) (2007) 4104–4115.

[6] S. Zlobec, An explicit form of the Moore-Penrose inverse of an arbitrary complex matrix, SIAM Rev. 12 (1970) 132–134.

[7] C. R. Rao, H. Yanai, General definition and decomposition of projectors and some applications to statistical problems, J. Statist. Plann. Inference 3 (1) (1979) 1–17.

[8] L. Eldén, Perturbation theory for the least squares problem with linear equality constraints, SIAM J. Num. Anal. 17 (3) (1980) 338–350.

[9] R. D. Milne, An oblique matrix pseudoinverse, SIAM J. Appl. Math. 16 (1968) 931–944.

[10] D. S. G. Pollock, The algebra of econometrics, John Wiley & Sons Ltd., Chichester, 1979, Wiley Series in Probability and Mathematical Statistics.

[11] S. K. Mitra, On a generalised inverse of a matrix and applications, Sankhyā Ser. A 30 (1968) 107–114.

[12] C. E. Langenhop, On generalized inverses of matrices, SIAM J. Appl. Math. 15 (1967) 1239–1246.