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ON A BOLTZMANN MEAN FIELD MODEL FOR KNOWLEDGE GROWTH*

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Abstract. In this paper we analyze a Boltzmann-type mean field game model for knowledge growth, which was proposed by Lucas et al. [*J. Political Econ.*, 122 (2014), pp. 1–51]. We discuss the underlying mathematical model, which consists of a coupled system of a Boltzmann-type equation for the agent density and a Hamilton–Jacobi–Bellman equation for the optimal strategy. We study the analytic features of each equation separately and show local in time existence and uniqueness for the fully coupled system. Furthermore we focus on the construction and existence of special solutions, which relate to exponential growth in time—so-called balanced growth path solutions. Finally, we illustrate the behavior of solutions for the full system and the balanced growth path equations with numerical simulations.

Key words. mean field games, Boltzmann-type equations, Hamilton–Jacobi–Bellman equations

AMS subject classifications. 49J20, 49N90, 35Q20, 70H20, 35Q91

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1. Introduction. Endogenous growth theory is based on the assumption that human capital, innovation, and knowledge are significant factors for economic growth. Understanding how innovation and knowledge lead to long-term economic growth has attracted a lot of interest in the macroeconomic literature. Different models have been proposed to describe knowledge increase; most of them relate innovation or imitation to knowledge growth. In [11, 12] economic growth is initiated by imitation in random meetings. Köenig, Lorenz, and Zilibotti [13] proposed a decision-based model, in which firms decide between “imitation” or “innovation.” Luttmer [17, 16] claims that noise and imitation lead to growth and considered a model in which individuals are characterized by their productivity or knowledge level. Here the cumulative distribution function describes the distribution of knowledge, which evolves as individuals meet each other, compare ideas, and improve their own knowledge level. Individual meetings are modeled by “collisions” in a Boltzmann-type equation for the distribution function. The interaction frequency among individual agents is assumed to be given; agents do not decide how much time they invest in learning or working. Lucas and Moll [15] extended this approach by modeling agents as rational individuals, who decide between either option. Each agent’s decision is based on maximizing its

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individual earnings given the distribution of all other agents with respect to their knowledge level. The resulting system corresponds to a Boltzmann equation for the agent distribution that is coupled to an Hamilton–Jacobi–Bellman equation for the optimal strategy. This novel approach combines mathematical models developed in the field of kinetic equations as well as mean field games.

Mathematical tools and methods from statistical mechanics such as kinetic theory have become a popular and successful tool in economics and social sciences. The kinetic theory developed by Boltzmann studies the statistical behavior of a system not in equilibrium and has its origins in analyzing the thermodynamics of dilute gases. The Boltzmann equation describes the evolution of the probability distribution function of molecules due to microscopic interactions, namely, the collisions between particles. In socioeconomic applications these collisions correspond to trading events [3, 9], the exchange of opinion [20, 8, 18, 19, 2, 1], or noncooperative games [7].

In mean field models the effect of all other individuals on a single individual is replaced by an averaged effect. In mean field game theory the dynamics of a single individual are determined from stochastic optimal control problems, in which the optimal strategy is influenced by the knowledge of the distribution of all other players at all times. Mean field game models received an increasing interest in economics, e.g., for describing the strategic decision making processes in large player stochastic games; see, for example, [14, 10], in the last years. Their general structure corresponds to a coupled system of a Fokker–Planck equation describing the evolution of the macroscopic agent density (forward in time) and a Hamilton–Jacobi–Bellman equation for the optimal strategy (backward in time).

Degond, Liu, and Ringhofer [7] introduced a different mean-field kinetic model for rational agents that act in a game-theoretical framework. This framework, initially developed to describe herding, has been extended to model wealth evolution in [6] and further investigated with respect to model predictive control in [5].

In this paper we investigate the existence and uniqueness of solutions to a Boltzmann mean field game (BMFG) model for knowledge growth, with a particular focus on the construction of the so-called balanced growth path (BGP). We prove the existence of local in time solutions using a fixed point argument. The BMFG system exhibits interesting analytic features, such as mass accumulation of agents at the maximum initial knowledge level or BGP solutions. These special solutions correspond to an exponentially growing production function—a characteristic feature particularly interesting for economists. Identifying conditions which initiate sustained exponential growth gives important indications of which factors drive economic progress. In this work we prove the existence of BGP solutions in a special case and provide first analytic results, which ensure the existence for a suitable chosen optimal strategy. This result corresponds to an intermediate step in an elaborate fixed point argument which ensures existence in the general case. The detailed construction of this existence argument is given in [4].

This paper is organized as follows: in section 2 we present the BMFG model of Lucas and Moll and discuss specific modeling assumptions. Section 3 focuses on the separate analysis of the Boltzmann and the Hamilton–Jacobi–Bellman equation as well as the coupled system. In section 4 we discuss the existence of BGP solutions. We conclude by presenting various numerical examples to illustrate the behavior of the proposed model in section 5.

2. A Boltzmann-type model for knowledge growth. Lucas et al. [15] studied the following scenario: consider an economy or society with a constant number

of interacting agents, which are characterized by their knowledge level. Agents can decide how much time they devote to working (by producing goods with the knowledge they already have) and how much to learning (by exchanging ideas with others). Each agent is characterized by its knowledge level $z \in \mathcal{I}$ and the fraction of time $s = s(z, t) : \mathcal{I} \times [0, T] \rightarrow [0, 1]$ it devotes to learning. The interval \mathcal{I} may correspond to the positive real line, i.e., $\mathcal{I} = \mathbb{R}^+$, or the bounded interval with maximum knowledge level \bar{z} , i.e., $\mathcal{I} = [0, \bar{z}]$. The function $f = f(z, t)$ describes the distribution of the agents with respect to their knowledge z and time t . We assume that each agent has one unit of time, and hence the time devoted to working corresponds to $1 - s(z, t)$.

The evolution of the distribution $f = f(z, t)$ is modeled by a Boltzmann-type approach; individuals meet (i.e., collide) and exchange ideas and knowledge. Lucas and Moll proposed the following minimal interaction law to model knowledge increase. If two individuals with knowledge levels z and z' meet, their postcollision knowledge corresponds to

$$(2.1) \quad z = \max(z, z'),$$

i.e., the agent with the lower knowledge level matches its level with the other. Based on (2.1), the evolution of all agents $f = f(z, t)$ can be described by the following Boltzmann-type equation:

$$(2.2a) \quad \partial_t f(z, t) = -\alpha(s(z, t))f(z, t) \int_z^{\bar{z}} f(y, t)dy + f(z, t) \int_0^z \alpha(s(y, t))f(y, t)dy,$$

$$(2.2b) \quad f(z, 0) = f_0(z).$$

The function $\alpha = \alpha(s(z, t)) : [0, 1] \rightarrow [0, 1]$ denotes the learning function, i.e., the interaction probability of an individual with knowledge level z . We assume that the initial distribution of all agents $f_0 = f_0(z)$ is a probability distribution, i.e., it satisfies $\int_{\mathcal{I}} f_0(z) dz = 1$ and $f_0(z) \geq 0$ for all $z \in \mathcal{I}$.

Note that (2.2) can be written in terms of the Heaviside function $H = H(z)$, namely,

$$(2.3) \quad \partial_t f(z, t) = -\alpha(s(z, t))f(z, t)((1 - H) * f) + f(z, t)(H * (\alpha f)).$$

Another reformulation of (2.3) is based on the cumulative distribution function (CDF) $F(z, t) = \int_0^z f(x, t) dx$ and reads as

$$(2.4) \quad \partial_t F(z, t) = -[1 - F(z, t)] \int_0^z \alpha(s(x, t))f(x, t)dx.$$

We assume that the working and learning time is directly related to the individual benefit—more precisely that the earnings $y = y(z, t)$ of an agent with knowledge level z corresponds to the product of the time the individual spends on working, i.e., $1 - s(z, t)$, times its knowledge level z . Hence we have

$$(2.5) \quad y(z, t) = (1 - s(z, t))z.$$

The per-capita production illustrates the total earning in a society and is given by

$$(2.6) \quad Y(t) = \int_{\mathcal{I}} (1 - s(z, t))zf(z, t) dz.$$

Based on the per-capita production each agent wants to maximize its earnings (discounted by a given temporal discount factor $r \in \mathbb{R}^+$), by choosing an optimal partition of its working, respectively, learning, time. Then the optimal ratio of working to learning time (a quantity related to the work-life balance) is determined by the solution $s = s(z, t)$ of the optimal control problem

$$V(x, t') = \max_{s \in \mathcal{S}} \left[\int_{t'}^T \int_0^{\bar{z}} e^{-r(t-t')} (1 - s(z, t)) z \rho_x(z, t) dz dt \right],$$

subject to

$$\partial_t \rho_x(z, t) = -\alpha(s) \rho_x(z, t) \int_z^{\bar{z}} f(y, t) dy + f(z, t) \int_0^z \alpha(s) \rho_x(y, t) dy$$

with $\rho_x(z, t') = \delta_x$. Here \mathcal{S} denotes the set of admissible controls given by

$$(2.7) \quad \mathcal{S} = \{s : \mathcal{I} \times [0, T] \rightarrow [0, 1]\}.$$

Then the optimal strategy can be calculated via the Lagrange functional

$$\begin{aligned} \mathcal{L} = & \int_{t'}^T \int_0^{\bar{z}} e^{-r(t-t')} (1 - s(z, t)) z \rho_x(z, t) \\ & - \left[\partial_t \rho_x(z, t) + \alpha(s(z, t)) \rho_x(z, t) \int_z^{\bar{z}} f(y, t) dy \right. \\ & \left. - f(z, t) \int_0^z \alpha(s(y, t)) \rho_x(y, t) dy \right] e^{-r(t-t')} V(z, t) dz dt. \end{aligned}$$

The optimality condition with respect to f corresponds to the Hamilton–Jacobi–Bellman equation:

$$\begin{aligned} \partial_t V(z, t) - rV(z, t) + \max_{s \in \mathcal{S}} [(1 - s(z, t))z - \alpha(s(z, t))V(z, t)((1 - H) * f) \\ + \alpha(s(z, t))((1 - H) * (Vf))] = 0. \end{aligned}$$

The function $V = V(z, t)$ denotes the value function and represents the production level starting from knowledge level z at time t and controlling the system until a final or infinite time. Altogether we obtain a BMFG of the form

$$(2.8a) \quad \partial_t f(z, t) = -\alpha(S(z, t))f(z, t)((1 - H) * f) + f(z, t) \left(H * (\alpha(S(z, t))f) \right),$$

$$(2.8b) \quad \begin{aligned} \partial_t V(z, t) - rV(z, t) = - [(1 - S(z, t))z - \alpha(S(z, t))V(z, t)((1 - H) * f) \\ + \alpha(S(z, t))((1 - H) * (Vf))], \end{aligned}$$

$$(2.8c) \quad \begin{aligned} S(z, t) = \arg \max_{s \in \mathcal{S}} [(1 - s(z, t))z - \alpha(s(z, t))V(z, t)((1 - H) * f) \\ + \alpha(s(z, t))((1 - H) * (Vf))], \end{aligned}$$

$$(2.8d) \quad f(z, 0) = f_0(z),$$

$$(2.8e) \quad V(z, T) = 0.$$

Lucas and Moll refer to $\alpha = \alpha(s) : [0, 1] \rightarrow \mathbb{R}^+$ as the learning technology function, which may have the form

$$(2.9) \quad \alpha(s) = \alpha_0 s^n, \quad n \in [0, 1).$$

We conclude this section by stating three special modeling situations, which we shall discuss and analyze later.

2.1. Symmetric meetings. In the initial model only one party in the interaction/collisions gains knowledge; the other one has no benefit. To capture symmetric meetings Lucas and Moll propose a modified Boltzmann-type equation of the form

$$(2.10) \quad \begin{aligned} \partial_t f(z, t) = & - f(z, t) \int_z^\infty [\alpha(s(z, t)) + \beta\alpha(s(y, t))] f(y, t) dy \\ & + f(z, t) \int_0^z [\alpha(s(y, t)) + \beta\alpha(s(z, t))] f(y, t) dy, \end{aligned}$$

where $\beta \in [0, 1]$ encodes the probability to learn from each other. The case $\beta = 0$ corresponds to the original model, $\beta = 1$ to perfectly symmetric meetings.

2.2. Special case: $\alpha = \alpha_0 \in \mathbb{R}^+$. Let us consider the BMFG model (2.8) with a given constant learning function $\alpha = \alpha_0 \in \mathbb{R}$, i.e., $n = 0$ in (2.9). Note that in this case the value function $V = V(z, t)$ is positive by definition and that $f = f(z, t)$ is nonnegative for all times if $f(z, 0) \geq 0$. Then the maximum of

$$((1 - s(z, t))z + \alpha_0 (V(z, t)((1 - H) * f) + ((1 - H) * (Vf))))$$

is given by $s(z, t) = 0$. In this case system (2.8) decouples and the Boltzmann-type equation (2.8a) can be written in terms of the cumulative distribution function $F = F(z, t)$, i.e.,

$$(2.11) \quad \partial_t F(z, t) = -\alpha_0(1 - F(z, t))F(z, t).$$

Hence the optimal strategy $V = V(z, t)$ can be calculated independently, which also motivates the separate analysis of the Boltzmann and the Hamilton–Jacobi–Bellman equation in the next section.

2.3. Balanced growth path. Lucas and Moll postulated the existence of BGP solutions to system (2.8), for which the production rate (2.6) grows exponentially in time. BGPs correspond to solutions in the rescaled variables (ϕ, v, σ) :

$$(2.12) \quad f(z, t) = e^{-\gamma t} \phi(ze^{-\gamma t}), \quad V(z, t) = e^{\gamma t} v(ze^{-\gamma t}) \text{ and } s(z, t) = \sigma(ze^{-\gamma t})$$

assuming the existence of a constant $\gamma \in \mathbb{R}^+$. If this is the case the production (2.6) can be transformed to

$$Y(t) = e^{\gamma t} \int_0^\infty [1 - \sigma(x)] x \phi(x) dx.$$

The rescaled equations for $(\phi, v, \sigma) = (\phi(x), v(x), \sigma(x))$ read as

$$(2.13a) \quad -\gamma\phi(x) - \gamma\phi'(x)x = \phi(x) \int_0^x \alpha(\sigma(y))\phi(y) dy - \alpha(\sigma(x))\phi(x) \int_x^\infty \phi(y) dy,$$

$$(2.13b) \quad (r - \gamma)v(x) + \gamma v'(x)x = \max_{\sigma \in \Xi} \left\{ (1 - \sigma)x + \alpha(\sigma) \int_x^\infty [v(y) - v(x)]\phi(y) dy \right\},$$

where $\Xi = \{\sigma : \mathbb{R}^+ \rightarrow [0, 1]\}$ denotes the set of admissible controls.

3. Analysis of the Boltzmann mean field model. In this section we present a local existence result for the fully coupled Boltzmann mean field model (2.8). We start with the analysis of the Boltzmann equation (2.2) for a given interaction rate $\alpha = \alpha(z, t)$. Then we discuss the existence and uniqueness of solutions for the Hamilton–Jacobi equation and conclude the section by studying the fully coupled system.

In this section we will need the following assumptions on the initial datum $f_0 = f_0(z)$:

(A1) Let $f_0 \in L^\infty(\mathcal{I})$ be a probability density, i.e., $\int_{\mathcal{I}} f_0(z) dz = 1$ and $f_0(z) \geq 0$ for all $z \in \mathcal{I}$.

3.1. Analysis of the Boltzmann equation for a given learning function α . We start with the analysis of the Boltzmann-type equation (2.2) for a given learning function $\alpha = \alpha(z, t)$. Hence we consider

$$(3.1a) \quad \partial_t f(z, t) = -\alpha(z, t) f(z, t) \int_z^{\bar{z}} f(y, t) dy + f(z, t) \int_0^z \alpha(y, t) f(y, t) dy,$$

$$(3.1b) \quad f(z, 0) = f_0(z),$$

on the interval $\mathcal{I} = [0, \bar{z}]$.

Remark 3.1. Let us introduce the operators $G_s = \alpha(s(z, t)) \int_z^{\bar{z}} f(y, t) dy$ and $L_s = \int_0^z \alpha(s(y, t)) f(y, t) dy$. Then (3.1) can be written as

$$\partial_t f(z, t) = f(z, t) (L_s f(z, t) - G_s f(z, t)).$$

Note that G_s and L_s are adjoint operators, i.e., $L_s^* = G_s$ since

$$\begin{aligned} (L_s f, g) &= \int_0^{\bar{z}} \left[\int_0^z \alpha(s(y, t)) f(y, t) dy \right] g(z, t) dz \\ &= \int_0^{\bar{z}} \left[\int_y^{\bar{z}} g(z) dz \right] \alpha(s(y, t)) f(y, t) dy. \end{aligned}$$

First we present a global existence result for (3.1) in time.

THEOREM 3.2. *Let (A1) be satisfied and $\alpha = \alpha(z, t) \in L^1(\mathcal{I}) \times L^\infty([0, T])$. Then (3.1) has a global in time solution $f = f(z, t) \in L^1(\mathcal{I}) \times L^\infty([0, T])$.*

Proof. Let $T > 0$ be given, $\bar{\alpha} := \max \alpha$, and A the following closed subset:

$$A = \{g \in C([0, T], L^1(\mathcal{I})), g \geq 0, \|g(\cdot, t)\|_1 \leq a\},$$

where $a = c \int f_0(y) dy$. For each $g \in A$ we define the operator $\Phi(g)$ as the solution of

$$(3.2) \quad \partial_t f(z, t) = -\alpha(z, t) f(z, t) \int_z^{\bar{z}} g(y, t) dy + f(z, t) \int_0^z \alpha(y, t) g(y, t) dy$$

with initial data $f(z, 0) = f_0(z)$. Then the existence of a solution follows from Picard–Lindelöf by showing that the operator Φ

- (a) maps A onto itself,
- (b) is a contraction for T small.

A priori estimates are as follows: The change of the total mass can be estimated by

$$\frac{d}{dt} \int_{\mathcal{I}} f(y, t) dy \leq \bar{\alpha} \|g\|_1 \int_{\mathcal{I}} f(y, t) dy.$$

Based on this estimate we use Gronwall to deduce that

$$\int_{\mathcal{I}} f(y, t) dy \leq \exp\left(\bar{\alpha} \int_0^t \|g\|_1 ds\right) \int_{\mathcal{I}} f_0(y) dy.$$

Hence (a) is satisfied. To show that Φ is a contraction we consider (3.2) for two given functions g_1 and g_2 , $g_i \in A$, $i = 1, 2$. Then their difference satisfies

$$\begin{aligned} \partial_t(f_1 - f_2)(z, t) &= -\alpha(z, t)f_1(z, t) \int_z^{\bar{z}} g_1(y, t) dy + f_1(z, t) \int_0^z \alpha(y, t)g_1(y, t) dy \\ &\quad + \alpha(z, t)f_2(z, t) \int_z^{\bar{z}} g_2(y, t) dy - f_2(z, t) \int_0^z \alpha(y, t)g_2(y, t) dy. \end{aligned}$$

Using $\|f_2\|_1 \leq \|f_0\|_1 e^{a\bar{\alpha}T} =: K_{loc}$, for the L^1 -norm of the difference, we obtain the differential inequality

$$\partial_t \|f_1 - f_2\|_1 \leq 2a\bar{\alpha} \|f_1 - f_2\|_1 + 2K_{loc}\bar{\alpha} \|g_1 - g_2\|_1.$$

Since $f_1(z, 0) = f_2(z, 0)$ we deduce, using Gronwall, that Φ is a contraction for sufficiently small time T . Note that all constants in the local existence argument depend on the initial mass only. Since we have mass conservation, i.e.,

$$\int_{\mathcal{I}} f(z, t) dz = \int_{\mathcal{I}} f_0(z) dz \text{ for all } t > 0,$$

we can iterate the local existence argument at $T, 2T, \dots$ to obtain global existence. \square

Next we show that the support of f remains bounded if the initial datum f_0 has a compact support.

PROPOSITION 3.1. *Let $\alpha \in C([0, T] \times \mathcal{I})$ and $f = f(z, t)$ be a continuous solution to (3.1), i.e., $f \in C([0, T] \times \mathcal{I})$ with $\text{supp}f(\cdot, 0) \subset [0, M]$, $M < \bar{z}$. Then*

$$\text{supp}(f(\cdot, t)) \subset [0, M] \text{ for all times } t > 0.$$

Proof. The proof is based on the maximum principle. Let us assume there exists a point \hat{z} in $(M, \bar{z}]$ such that $f(\hat{z}, t) > 0$. Then $0 < \partial_t f(\hat{z}, t)$. But since $f(z, 0) = 0$ for all $z > M$, we deduce that $f(z, t) = 0$ for all $z > M$. \square

Note that Lemma 3.1 is only valid for continuous solutions. However we expect that solutions of (3.1) converge to Dirac deltas as time evolves. This can be explained by the fact that individuals with a lower knowledge level gain knowledge in each collision, but individuals with the greater knowledge level cannot improve. Hence we conjecture the formation of Dirac deltas at $z = M$ if $\text{supp}(f) \subset [0, M]$. This can be observed in the evolution of the first order moment, i.e.,

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{I}} z f(z, t) dz &= \int_{\mathcal{I}} \left[z f(z, t) \int_0^z \alpha(y, t) f(y, t) dy - f(z, t) \int_z^{\bar{z}} \alpha(z, t) f(y, t) z dy \right] dz \\ (3.3) \quad &= \int_{\mathcal{I}} f(z, t) \int_z^{\bar{z}} \alpha(y, t) f(y, t) \underbrace{(y - z)}_{\geq 0} dy dz, \end{aligned}$$

where we used the fact that L_s and G_s are adjoint operators (see Remark 3.1). Hence we deduce that the first order moment is increasing in time. Also the mass located in

the interval (z_0, \bar{z}) is increasing, because

$$\begin{aligned}
 \frac{d}{dt} \int_{z_0}^{\bar{z}} f(z, t) dz &= \int_{z_0}^{\bar{z}} \left[\int_0^z f(z, t) f(y, t) \alpha(y, t) dy - \int_z^{\bar{z}} f(z, t) \alpha(z, t) f(y, t) dy \right] dz \\
 &= \int_{z_0}^{\bar{z}} \left[\int_0^z f(z, t) f(y, t) \alpha(y, t) dy - \int_{z_0}^z f(z, t) f(y, t) \alpha(y, t) dy \right] dz \\
 (3.4) \quad &= \int_{z_0}^{\bar{z}} \int_0^{z_0} f(z, t) f(y, t) \alpha(y, t) dy dz \geq 0.
 \end{aligned}$$

From the previous estimates we deduce the following theorem.

THEOREM 3.3. *Let $\alpha(z, t) \geq \underline{\alpha} > 0$ and $\bar{z} \in \text{supp}(f)$, and then*

$$f(\cdot, t) \rightharpoonup^* \delta_{\bar{z}}.$$

Proof. Setting $z_0 = 0$ in (3.4) gives $\frac{d}{dt} \int_{z_0}^{\bar{z}} f(z, t) dz = 0$ and therefore implies mass conservation. From (3.4) we deduce that

$$-\frac{d}{dt} F(z_0, t) = \frac{d}{dt} (1 - F(z_0, t)) \geq \underline{\alpha} \int_{z_0}^{\bar{z}} f(z, t) dz F(z_0, t) = \underline{\alpha} (1 - F(z_0, t)) F(z_0, t).$$

This differential inequality implies that the CDF $F(z_0, t) \rightarrow 0$ for $z_0 < \bar{z}$ as time $t \rightarrow \infty$. Since we can choose z_0 close to \bar{z} we conclude that f converges to a Dirac mass. \square

Remark 3.4. Note that the formation of a Delta Dirac mass accumulates at $z = \bar{z}$, where $\bar{z} = \max_z \text{supp}(f)$ for compactly supported initial datum and $\bar{z} = \bar{z}$ for positive initial data on the bounded domain $\mathcal{I} = [0, \bar{z}]$. If $f_0(z) > 0$ for all $z \in \mathbb{R}^+$, the mass accumulates at $z = \infty$.

3.2. Analysis of the Hamilton–Jacobi–Bellman equation. Next we study the analytic behavior of the Hamilton–Jacobi–Bellman equation for a given $f \in C(0, T, L^1)$ on $\mathcal{I} = \mathbb{R}^+$:

$$\begin{aligned}
 \partial_t V(z, t) - rV(z, t) &= - \max_{s \in \mathcal{S}} [(1 - s(z, t))z - \alpha(s(z, t))V(z, t)((1 - H) * f) \\
 (3.5a) \quad &\quad + \alpha(s(z, t))((1 - H) * (Vf))],
 \end{aligned}$$

$$(3.5b) \quad V(z, T) = 0.$$

We shall need the following assumption for the terminal value function in the rest of the section:

(A2) Let the final data $V(\cdot, T)$ in (3.5) be nonnegative and nondecreasing.

To ensure the existence of a maximizer we need the following assumptions on the learning function $\alpha = \alpha(s)$:

(A3) Let $\alpha : [0, 1] \rightarrow \mathbb{R}^+$, $\alpha \in C^\infty([0, 1])$, $\alpha(0) = 0$, $\alpha'(0) = \infty$, $\alpha'' < 0$, and α monotone.

In the following we shall use the variable

$$B = -V(z, t)(1 - H) * f + (1 - H) * (Vf)$$

to enhance readability.

LEMMA 3.5. *Let assumption (A3) be satisfied, $z > 0$, and $B \in \mathbb{R}$. Then there exists a unique solution $S = S(B)$ of the optimization problem*

$$(3.6) \quad \max_{s \in \mathcal{S}} ((1 - s)z + \alpha(s)B)$$

with $S = \arg \max_{s \in \mathcal{S}} ((1 - s)z + \alpha(s)B)$.

Proof. Let $\beta := \alpha^{-1}$, and then the problem with $\zeta = \alpha(s)$ is equivalent to

$$\arg \max_{\zeta \in [0, \alpha(1)]} z(1 - \beta(\zeta)) + B\sigma.$$

Because of the strict concavity of $-\beta$, there exists a unique solution. □

LEMMA 3.6. *Let assumption (A3) be satisfied, $z > 0$, $B \in \mathbb{R}$, and $S = S(B)$ be the optimal control satisfying (3.6) for a given B . If*

$$\lim_{B \rightarrow 0} \alpha''(S(B))B^3 < 0,$$

then the maps $B \rightarrow S(B)$, $B \rightarrow \alpha(S(B))$, and $B \rightarrow \alpha(S(B))B$ are Lipschitz.

Proof. We distinguish between the following three cases:

Case 1: If $B < 0$, then $s = 0$.

Case 2: If $B > \frac{z}{\alpha'(1)}$ and since α is concave we deduce that

$$\begin{aligned} z(1 - s) + B\alpha(s) &\leq z(1 - s) + B\alpha(1) + B\alpha'(1)(s - 1) \\ &= B\alpha(1) + (1 - s)(z - B\alpha'(1)) < B\alpha(1). \end{aligned}$$

Hence the maximum is attained at $s = 1$.

Case 3: If $0 < B < \frac{z}{\alpha'(1)}$ let $\alpha'(s) = \frac{d}{ds}\alpha(s)$ and $\alpha''(s) = \frac{d^2}{ds^2}\alpha(s)$. In this case there exists a unique solution of $\frac{d}{ds}\alpha(s) = \frac{z}{B}$, which gives the maximum. Furthermore we have that

$$\frac{d^2}{ds^2}\alpha(s)s'(B) = -\frac{z}{B^2} \Rightarrow s'(B) = -\frac{z}{B^2\alpha''(s)}$$

and

$$(3.7) \quad \frac{d}{dB}\alpha(s) = \frac{d}{ds}\alpha(s)s'(B) = \alpha'(s)s'(B) = \frac{z}{B}s'(B) = -\frac{z^2}{B^3\alpha''} = -\frac{\alpha'^3}{z\alpha''}.$$

Because $\lim_{B \rightarrow 0} -B^3\alpha''(B) > 0$, we conclude that S is piecewise continuously differentiable. Since

$$\lim_{B \rightarrow 0} -B^2\alpha''(B) = \infty,$$

we deduce that $S'(B) \rightarrow 0$ as $B \rightarrow 0$. Hence we have continuity at $B = 0$ and $B = \frac{A}{\alpha'(1)}$, Lipschitz continuity follows for $S(B)$. The same is true for $\alpha(S)B$. □

LEMMA 3.7. *Let assumption (A2) be satisfied. Then the value function $V(\cdot, t)$ solving (3.5) is nonnegative and nondecreasing for all times $t \in [0, T]$.*

Proof. Equation (3.5) can be written as

$$(3.8) \quad \begin{aligned} \partial_t V(z, t) - rV(z, t) &= -(1 - S(z, t))z \\ &\quad - \alpha(S(z, t)) \left[\int_z^\infty V(y, t)f(y, t)dy - V(z, t) \int_z^\infty f(y, t)dy \right]. \end{aligned}$$

In a minimum point z_0 we have

$$\begin{aligned} \partial_t V(z_0, t) - rV(z_0, t) &\leq -(1 - S(z_0, t))z_0 \\ &\quad - \alpha \left[V(z_0, t) \int_{z_0}^{\infty} f(y, t) dy - V(z_0, t) \int_{z_0}^{\infty} f(y, t) dy \right]. \end{aligned}$$

Since this inequality is backward, it preserves nonnegativity.

Calculating the derivative of (3.8) with respect to z , we obtain

$$\begin{aligned} \partial_t V_z(z, t) - rV_z(z, t) &= S_z(z, t)z + S(z, t) - 1 \\ &\quad - \alpha'(S(z, t))S_z(z, t)B + \alpha(S(z, t))V_z(z, t) \int_z^{\infty} f(y, t) dy. \end{aligned}$$

Using $\alpha' = z/B$, it follows that

$$\partial_t V_z(z, t) - rV_z(z, t) = S(z, t) - 1 + \alpha(S(z, t))V_z(z, t) \int_z^{\infty} f(y, t) dy.$$

Since $S \leq 1$ we deduce that V_z stays nonnegative. \square

LEMMA 3.8. *Let assumption (A2) and (A3) be satisfied. Then $B(\cdot, t)$ is nonincreasing and the maximizer $S(\cdot, t)$ is nonincreasing for all times $t \in [0, T]$. Moreover the function $S(\cdot, t)$ is strictly decreasing on the interval where $0 < S(\cdot, t) < 1$, except in the degenerate case of $B(0, t) = 0$. Then $S(z, t) = 1$ for small z .*

Proof. We have already seen above that $B(z, t) \geq 0$ and

$$B_z = -V_z(z, t) \int_z^{\infty} f(y, t) dy \leq 0.$$

If $B(0, t) = 0$, then $S(z, t) = 1$ for all z . Otherwise $B(0, t) > 0$, so $B > \frac{z}{\alpha'(1)}$ and therefore the maximizer $S(z, t)$ is equal to 1 for small z . If $0 < B < \frac{z}{\alpha'(1)}$, we have $S(z, t) = (\alpha')^{-1}(\frac{z}{B})$. Since α is strictly concave, $S(z, t)$ is strictly decreasing. \square

The previous results lead to the following existence and uniqueness theorem for the Hamilton–Jacobi–Bellman equation.

THEOREM 3.9. *Let $f \in C(0, T, L^1)$ be given and $\alpha = \alpha(s, t)$ satisfy assumption (A3). Then there exists a unique solution $V \in C(0, T, L^\infty)$ of (3.5) with $V(z, T) = 0$. Moreover, let \tilde{V} be a solution of (3.5) with \tilde{f} , and then there exist constants m and D (independent of \tilde{V} and \tilde{f}) such that*

$$(3.9) \quad \|V - \tilde{V}\|_\infty \leq De^{mt} \|f - \tilde{f}\|_{C(0, T, L^1)} \|\tilde{V}\|_\infty.$$

Proof. The proof is based on the following statements:

- (i) $V \rightarrow B(V, f) = (1 - H) \star (Vf) - V(1 - H) \star f$ is Lipschitz on the spaces given.
- (ii) $B(V) \rightarrow S$ and $B(V) \rightarrow \alpha(S)B$ are Lipschitz because of Lemma 3.6.

Then (3.5) is of the form

$$\partial_t V = R(V)$$

with R Lipschitz, and we can conclude the proof with Picard–Lindelöf. For the difference $V - \tilde{V}$ we obtain

$$\partial_t(V - \tilde{V}) = R(V) - \tilde{R}(\tilde{V}) = R(V) - R(\tilde{V}) + R(\tilde{V}) - \tilde{R}(\tilde{V}).$$

Since the estimate $\|B(\tilde{V}, f) - B(\tilde{V}, \tilde{f})\|_\infty \leq \|\tilde{V}\|_\infty \|f - \tilde{f}\|_{C(0,T,L^1)}$ holds, we can show that

$$\|R(\tilde{V}) - \tilde{R}(\tilde{V})\| \leq K \|f - \tilde{f}\| \|\tilde{V}\|_\infty$$

with K independent of \tilde{V}, f, \tilde{f} . Therefore we conclude

$$\|V - \tilde{V}\|_\infty \leq De^{mt} \|f - \tilde{f}\|_{C(0,T,L^1)} \|\tilde{V}\|_\infty. \quad \square$$

3.3. Analysis of the fully coupled Boltzmann mean field game system.

We show existence and uniqueness of the fully coupled system using a fixed-point argument.

Because of the term $(1 - s)z$, we expect linear growth of V in z . Therefore we are looking for a solution in the space $C(0, T, L^\infty_{1+z}(\mathbb{R}^+))$, where

$$W^\infty := L^\infty_{1+z}(\mathbb{R}^+) := \{u = (1 + z)w \mid w \in L^\infty(\mathbb{R}^+)\} \text{ with } \|u\|_{L^\infty_{1+z}} := \text{ess sup } \frac{|u(z)|}{1 + z}.$$

To compensate we use a weighted L^1 -norm for f :

$$W^1 := L^1_{\frac{1}{1+z}}(\mathbb{R}^+) := \left\{ u = \frac{w}{1+z} \mid w \in L^1(\mathbb{R}^+) \right\} \text{ with } \|u\|_{L^1_{\frac{1}{1+z}}} := \int |u(z)|(1+z) dz.$$

First we state two lemmas, which provide the necessary estimates and bounds for the local existence proof.

LEMMA 3.10. *Let the initial datum $f_0 = f_0(z)$ satisfy assumption (A1) and the learning function $\alpha = \alpha(s)$ assumption (A3). Then every solution $f = f(z, t)$ to (2.3) has a bounded first order moment.*

Proof. We reiterate from (3.3) that

$$\frac{d}{dt} \int_{\mathcal{I}} z f(z, t) dz \leq \int_{\mathcal{I}} f(z) \int_z^\infty \alpha(y, t) f(y, t) y dy dz \leq \bar{\alpha} \int_{\mathcal{I}} f(y, t) y dy.$$

This gives an exponential bound for the first order moment. □

LEMMA 3.11. *Let $V = V(z, t)$ be a solution to (2.8b). Then V is bounded in $L^\infty_{1+z}(\mathbb{R}^+)$.*

Proof. Dividing (2.8b) by $1 + z$ and changing t to $-t$, we obtain

$$\partial_t \frac{V}{1+z} \leq r \frac{V}{1+z} + \frac{z}{1+z} + \frac{1}{1+z} \alpha B(V, f)$$

and

$$B(V, f) \leq \int_z^\infty (Vf)(x) dx \leq \|V\|_{W^\infty} \|f\|_{W^1} \leq C \|V\|_{W^\infty} e^{Kt}.$$

This implies

$$\partial_t \frac{V}{1+z} \leq r \frac{V}{1+z} + \frac{z}{1+z} + \frac{1}{1+z} \bar{\alpha} \|V\|_{W^\infty} C e^{Kt}. \quad \square$$

THEOREM 3.12. *Let assumptions (A1)–(A3) be satisfied. If $\lim_{s \rightarrow 0} \frac{(\alpha')^3}{\alpha''} < \infty$, then the fully coupled BMFG system (2.8) on $\mathcal{I} = \mathbb{R}^+$ has a unique local in time solution.*

Proof. For local existence, we take a $g \in C(0, T, L^1_{\frac{1}{1+z}}(\mathbb{R}^+))$ and solve

$$(3.10) \quad \partial_t V(z, t) - rV(z, t) = -\max_{s \in S} [(1 - s(z, t))z - \alpha(s(z, t))B(V, g)]$$

for V and S . Then we solve

$$(3.11) \quad \partial_t f(z, t) = -\alpha(S(z, t))f(z, t)((1 - H) * f) + f(z, t)(H * (\alpha(S(z, t))f))$$

for f . Let Ψ denote the operator mapping g to f . We show that Ψ is a contraction by considering two solutions g and \tilde{g} . Taking the difference of (3.10) for g and \tilde{g} gives

$$\partial_t(V - \tilde{V}) = r(V - \tilde{V}) + z(S - \tilde{S}) - (\alpha(S)B(V, g) - \alpha(\tilde{S})B(\tilde{V}, \tilde{g})).$$

To show that the second term on the right-hand side (RHS) is Lipschitz, we calculate $S'(B) = -\frac{z}{B^2\alpha''} = -\frac{(\alpha')^2}{z\alpha''}$. Similarly for the third term on the RHS we deduce that

$$\frac{d}{dB}(\alpha B) = \alpha'(S)S'(B)B + \alpha = zS' + \alpha = -\frac{(\alpha')^2}{\alpha''} + \alpha.$$

Since both derivatives are bounded we obtain

$$\partial_t(V - \tilde{V}) \leq r(V - \tilde{V}) + K|(B(V, g) - B(\tilde{V}, \tilde{g}))|.$$

Let us consider the last term only. Due to the structure of B , i.e., $B(V, g) = (1 - H) * (Vg) - V(1 - H) * g$, we work on the part $(1 - H) * (Vg)$ only. The other part can be estimated using similar arguments. We deduce that

$$(3.12) \quad \begin{aligned} (1 - H) * (Vg) - (1 - H) * (\tilde{V}\tilde{g}) &= (1 - H) * [(V - \tilde{V})g + \tilde{V}(g - \tilde{g})] \\ &= \int_z^\infty (V - \tilde{V})g \, dx + \int_z^\infty \tilde{V}(g - \tilde{g}) \, dx \\ &\leq \|V - \tilde{V}\|_{W^\infty} \|g\|_{W^1} + \|\tilde{V}\|_{W^\infty} \|g - \tilde{g}\|_{W^1}. \end{aligned}$$

This implies the following inequality:

$$\partial_t(V - \tilde{V}) \leq r(V - \tilde{V}) + 2K\|V - \tilde{V}\|_{W^\infty} \|g\|_{W^1} + \|\tilde{V}\|_{W^\infty} \|g - \tilde{g}\|_{W^1}.$$

So for $\int_0^T \|g - \tilde{g}\|_{W^1} \, dt$ small, $\|V - \tilde{V}\|_{W^\infty}$ is small. Moreover, according to Lemma 3.8, $S = \tilde{S} = 1$ for small z , so the Lipschitz constant coming from (3.7) is bounded. Therefore we obtain that also the term $|\alpha(S) - \alpha(\tilde{S})|$ is small. This implies for (3.10) that $\|f - \tilde{f}\|_{W^1} \leq a\|g - \tilde{g}\|_{W^1}$ with $a < 1$ for T small enough. Hence the operator Ψ is a contraction, which concludes the proof. \square

4. Balanced growth paths. In this section we discuss the existence of BGP solutions and the convergence behavior toward them. BGPs correspond to solutions (ϕ, v, σ) for which the production function (2.6) grows exponentially in time. We reiterate that the rescaled equations for the BMFG (2.8) in the variables $(\phi, v, \sigma) = (\phi(x), v(x), \sigma(x))$ with

$$(4.1) \quad f(z, t) = e^{-\gamma t} \phi(ze^{-\gamma t}), \quad V(z, t) = e^{\gamma t} v(ze^{-\gamma t}) \text{ and } s(z, t) = \sigma(ze^{-\gamma t})$$

read as

$$(4.2a) \quad -\gamma\phi(x) - \gamma\phi'(x)x = \phi(x) \int_0^x \alpha(\sigma(y))\phi(y) \, dy - \alpha(\sigma(x))\phi(x) \int_x^\infty \phi(y) \, dy$$

$$(4.2b) \quad (r - \gamma)v(x) + \gamma v'(x)x = \max_{\sigma \in \Xi} \left\{ (1 - \sigma)x + \alpha(\sigma) \int_x^\infty [v(y) - v(x)]\phi(y) \, dy \right\},$$

where $\Xi = \{\sigma : \mathbb{R}^+ \rightarrow [0, 1]\}$ denotes the set of admissible controls. A necessary prerequisite for the existence of BGP solutions is the assumption that the initial cumulative distribution function $F(z, 0)$ has a Pareto tail, which is given by the following:

(A4) The productivity function $F(z, 0) = \int_0^z f_0(y)dy$ has a Pareto tail, i.e., there exist constants $k, \theta \in \mathbb{R}^+$ such that

$$(4.3) \quad \lim_{z \rightarrow \infty} \frac{1 - F(z, 0)}{z^{-1/\theta}} = k.$$

Condition (4.3) in the rescaled variable ϕ reads as

$$(4.4) \quad \lim_{z \rightarrow \infty} \frac{\int_z^\infty \phi(y) dy}{z^{-1/\theta}} = k.$$

LEMMA 4.1. *Let assumption (A4) be satisfied. Then $F = F(z, t)$ has a Pareto tail with the same decay rate θ for all times $t \in [0, T]$.*

Proof. Note that we can rewrite (2.4) as

$$\partial_t[1 - F(z, t)] = [1 - F(z, t)]G(z, t) \quad \text{with } G(z, t) := \int_0^z \alpha(s(x, t))f(x, t)dx.$$

Then the solution is given by

$$1 - F(z, t) = [1 - F(z, 0)] \exp\left(\int_0^t G(z, s) ds\right).$$

Multiplication with $z^{-1/\theta}$ yields

$$[1 - F(z, t)]z^{1/\theta} = [1 - F(z, 0)]z^{1/\theta} \exp\left(\int_0^t G(z, s) ds\right).$$

For a fixed time s the function $G(z, s)$ is monotonically increasing in z and bounded. Hence we can pass to the limit $z \rightarrow \infty$ and deduce that the function $F(z, t)$ also has a Pareto tail for all time $t \in [0, T]$. \square

LEMMA 4.2. *Let assumption (A4) be satisfied. Then the growth parameter $\gamma \in \mathbb{R}$ is given by*

$$(4.5) \quad \gamma = \theta \int_0^\infty \alpha(\sigma(y))\phi(y)dy.$$

Proof. Recall that $F = \int_0^z f(y, t)dy$ satisfies (2.4). Using that $F(z, t) = \Phi(ze^{-\gamma t})$ we deduce that

$$(4.6) \quad \gamma\Phi'(x)x = [1 - \Phi(x)] \int_0^x \alpha(\sigma)\phi(y) dy.$$

Dividing this equation by $x^{-\theta}$, passing to the limit $x \rightarrow \infty$, and using assumption (A4), we obtain

$$-\gamma \frac{1}{\theta} k = -k \int_0^\infty \alpha(\sigma(y))\phi(y)dy$$

and subsequently formula (4.5). \square

4.1. Existence. Next we study the existence of BGP. In the special case of $\alpha = \alpha_0$ we can show the existence of solutions; for the fully coupled problem we shall provide first results on the existence of (γ, ϕ) for a given (v, σ) .

4.2. Special case: $\alpha = \alpha_0$. We reiterate that system (4.2) decouples in the case of a constant learning function $\alpha = \alpha_0 \in \mathbb{R}^+$. Hence we solve (4.2a) for $\phi = \phi(z)$ first. Since α is constant, we have $\sigma = 0$ and can solve the equation (4.2b) for $v = v(z)$, giving us a global solution.

THEOREM 4.3. *Let assumption (A4) be satisfied and $\alpha = \alpha_0$. Then there exists a unique BGP solution $(\Phi, v, 0)$ and a scaling constant γ to (4.2) given by*

$$\gamma = \alpha_0 \theta \int_{\mathcal{I}} f_0(z) dz, \quad \Phi(x) = \frac{1}{1 + kx^{-1/\theta}}.$$

Proof. Note that the initial datum f_0 uniquely determines the scaling constant γ for the BGP in the case of constant learning function $\alpha = \alpha_0$. Since

$$\int_{\mathcal{I}} f_0(z) dz = \int_{\mathcal{I}} f(z) dz = \int e^{-\gamma t} \phi(z e^{-\gamma t}) dz = \int \phi(y) dy,$$

we deduce for γ that

$$\gamma = \theta \alpha_0 \int \phi(y) dy = \theta \alpha_0 \int_{\mathcal{I}} f(z, t) dz.$$

For $\alpha = \alpha_0$ we can write (4.2a) as

$$(4.7) \quad -\gamma \Phi'(x)x = -\alpha_0 [1 - \Phi(x)]\Phi(x).$$

We define $\tilde{\Phi} := \frac{1}{1-\Phi}$ i.e., $\Phi := 1 - \frac{1}{\tilde{\Phi}}$ and obtain

$$x\tilde{\Phi}' = a(\tilde{\Phi} - 1)$$

with $a = \frac{\alpha_0}{\gamma}$. This equation has the solution

$$\tilde{\Phi} = bx^a + 1 \quad \text{and} \quad \Phi = \frac{bx^a}{bx^a + 1}.$$

Since we look for solutions which have a Pareto tail we can determine the constant b from the Pareto tail assumption (A4) and obtain

$$(4.8) \quad \Phi = \frac{1}{1 + kx^{-1/\theta}}.$$

In the case $\alpha = \alpha_0$, the maximum on the RHS of (4.2b) is attained at $\sigma = 0$. In order to solve (4.2b) with ϕ given and $\sigma = 0$, we consider $w := v\phi$ which satisfies

$$(4.9) \quad rw + \gamma w'x = \phi x + \alpha_0 \phi \int_x^\infty w dy - \alpha_0 w \int_0^x \phi dy.$$

We define $W := \int_x^\infty w dy$ and rewrite (4.9) as

$$rw + \gamma w'x = \Phi'x + \alpha_0 \Phi'W - \alpha_0 \Phi w.$$

This gives

$$(\gamma - r)W' - \gamma(W'x)' = \Phi'x + \alpha_0(\Phi W)'$$

With $H'(x) = x\Phi'(x)$, it follows that

$$(4.10) \quad (\gamma - r)W - \gamma W'x = H + \alpha_0\Phi W + K.$$

Equation (4.10) is a first order ODE with $W(\infty) = 0$, which can be solved by considering the solution to the homogeneous equation and then using the variation of constants method. Hence we obtain a unique solution w , and therefore $v = w/\phi$. \square

For constant α we are able to prove the following results about the stability of BGPs.

THEOREM 4.4. *Let assumption (A4) be satisfied, $\alpha = \alpha_0$, and f denote a solution to the original BMFG problem (2.8). Then the rescaled density $\psi = e^{\gamma t}f(xe^{\gamma t}, t)$ converges pointwise to the BGP solution ϕ , given by (4.1), as $t \rightarrow \infty$.*

Proof. The function $\psi(x, t)$ solves

$$\partial_t \psi(x, t) - \gamma \psi(x, t) - \gamma \partial_x \psi(x, t)x = \psi(x, t) \int_0^x \alpha_0 \psi(x, t) dy - \alpha_0 \psi(x, t) \int_x^\infty \psi(x, t) dy,$$

and therefore its primitive $\Psi(x, t) := \int_0^x \psi(y, t) dy$ satisfies

$$(4.11) \quad \partial_t \Psi(x, t) - \gamma \Psi(x, t) - \gamma x \partial_x \Psi(x, t) = -\alpha_0 [1 - \Psi(x, t)] \Psi(x, t).$$

We define $U(x^{-1/\theta}, t) := x^{1/\theta} \frac{1}{k} (\frac{1}{\Psi} - 1)$, i.e., $\Psi = \frac{1}{1+kx^{-1/\theta}U}$ and deduce

$$\begin{aligned} \partial_t \Psi &= -\Psi^2 k x^{-1/\theta} \partial_t U, \\ \partial_x \Psi &= \Psi^2 k x^{-1/\theta} \frac{1}{\theta} \left(x^{-1/\theta-1} U + x^{-2/\theta-1} \partial_y U \right), \\ \Psi(1 - \Psi) &= \Psi^2 k x^{-1/\theta} U. \end{aligned}$$

Hence we can rewrite (4.11) as

$$-\partial_t U - \frac{\gamma}{\theta} U - \frac{\gamma}{\theta} y \partial_y U = -\alpha_0 U.$$

Since φ has a Pareto-tail, U solves the equation

$$\partial_t U + \frac{\gamma}{\theta} y \partial_y U = 0 \text{ with } U_0(x) = U(x, 0) = 1.$$

This equation has the solution $U(y, t) = U_0(e^{-\gamma t}y)$, and hence $U(y, t)$ converges pointwise to 1 for $t \rightarrow \infty$. \square

4.3. Existence for the general model. We conclude the section by proving the existence of a solution (γ, ϕ) , which has a Pareto tail for a given (σ, v) . The existence of the fully coupled system is a challenging problem, which is solved in a subsequent paper by the authors, see [4].

THEOREM 4.5. *Let assumption (A3) hold and $\sigma \in C^1([0, \infty))$ denote a given function, which satisfies*

$$\sigma(z) = 1 \text{ for } z \in [0, z_0], \sigma'(z) \leq 0.$$

Then there exists a $\gamma \in \mathbb{R}^+$ and a solution $\phi \in L^1([0, \infty))$ to (4.2a), which has a Pareto tail.

Proof. The function σ is nonincreasing and equal to 1 in the interval $[0, x_0]$, and hence $\alpha(\sigma(z)) = \alpha(1)$ on $[0, x_0]$. We know from the previous subsection that in the case $\alpha = \alpha_0$ there exists a solution $\Phi = \Phi(x)$, $x \in [0, x_0]$ of the form

$$(4.12) \quad \Phi(z) = \frac{bx^a}{bx^a + 1},$$

where $a = \alpha(1)/\gamma$ on the interval $[0, x_0]$. That way we obtain $\Phi(x_0)$.

We rewrite (4.6) by integration by parts as

$$(4.13) \quad \gamma\Phi'(x)x = [1 - \Phi(x)] \left[\Phi(x)\alpha(\sigma(x)) - \int_0^x \Phi(y) \frac{d}{dy} \alpha(\sigma(y)) dy \right]$$

and can obtain a solution on $[x_0, \infty)$ starting from $\Phi(x_0)$ using Picard–Lindelöf. Next we check that the Pareto-tail condition is satisfied. To do so we change variables to $\tilde{x} := x^{-1/\theta}$ and define $\Phi(x) =: 1 - K(\tilde{x})\tilde{x}$. The existence of the limit

$$\lim_{x \rightarrow \infty} \frac{1 - \Phi(x)}{x^{-1/\theta}}$$

is equivalent to the existence of $\lim_{\tilde{x} \rightarrow 0} K(\tilde{x})$. In this new variable (4.6) reads as

$$K'\tilde{x} = -K \int_0^{\tilde{x}} \alpha(\tilde{\sigma})(K\xi)' d\xi,$$

where $\tilde{\sigma}$ is σ transformed in the new variable \tilde{x} . Since $(K\tilde{x})'$ is nonnegative we can estimate the integral on the RHS

$$K'\tilde{x} \geq -K \int_0^{\tilde{x}} \bar{\alpha}(K\xi)' d\xi = -K\bar{\alpha}K\tilde{x}.$$

Then it follows that

$$-\frac{K'}{K^2} \leq \bar{\alpha} \quad \text{and therefore} \quad K(0) \leq \frac{K(\tilde{x}_0)}{1 - \bar{\alpha}\tilde{x}_0K(\tilde{x}_0)}.$$

Since $\Phi(x) \rightarrow 1$ for $x \rightarrow \infty$, we have $\tilde{x}_0K(\tilde{x}_0) \rightarrow 0$ for $\tilde{x}_0 \rightarrow 0$. This means that the limit $\lim_{x \rightarrow \infty} \frac{1 - \Phi(x)}{x^{-1/\theta}}$ exists.

The proof follows by iterating between Φ and γ . For $\theta < 1$, the derivative Φ' is bounded. So we obtain a fixed point (γ, ϕ) satisfying (4.2a) and (4.5). \square

5. Numerical simulations. In this section we present an iterative scheme to solve (2.8) numerically. First we illustrate the formation of blow-up solutions for the Boltzmann-type model (2.8a). Then we compare the numerical solution of our scheme with the BGP calculated using the code provided by Lucas and Moll. For additional information on the construction of solutions and the numerical solver for the BGP we refer to [15]. Finally we study the stability of BGP solutions with respect to initial perturbations.

5.1. Numerical scheme. We consider a bounded domain $\mathcal{I} = [0, \bar{z}]$, where \bar{z} denotes to the maximum knowledge level. The spatial discretization corresponds to N logarithmically spaced intervals. The temporal domain is split into equidistant time steps of size Δt . Let z_i denote the i -th grid point and $t_k = k\Delta t$ the k -th time step. We set the initial agent distribution to $f(x, 0) = f_0(x)$ to

$$f_0(x) = \frac{k}{\theta} x^{-\frac{\theta+1}{\theta}} e^{-kx^{\frac{1}{\theta}}}.$$

Note that in this case the cumulative distribution function of $F_0(z) = \int_0^z f_0(y)dy$ has a Pareto tail, i.e., assumption (A4) is satisfied. Let $V(z, T) = 0$ be the terminal condition of the optimal strategy in the iterative solver. Then the numerical results of the full model are based on the following iterative procedure:

1. Calculate the solution $f = f(z, t)$ of the Boltzmann-type equation (2.8a) by updating the solution via an explicit in time discretization:

$$f(z_i, t_{k+1}) = f(z_i, t_k) + \Delta t \left[-\alpha(s(z_i, t_k))f(z_i, t_k) \int_{z_i}^{\bar{z}} f(y, t_k)dy + f(z_i, t_k) \int_0^{z_i} \alpha(s(y, t_k))f(y, t_k)dy \right],$$

where the integrals on the RHS are evaluated using the trapezoidal rule.

2. Solve the Hamilton–Jacobi–Bellman equation (2.8b), by first calculating the maximum via the optimality conditions. Then determine the new optimal velocity using the updated density distribution $f = f(z, t)$ and the function s .

In particular

$$\max_{s(z) \in \mathcal{S}} \left((1-s)z + \alpha(s) \int_z^{\bar{z}} (V(y, t) - V(z, t))f(y, t)dy \right)$$

is given by the following:

- (a) $z = 0$: Then $s = 1$ since $\alpha = \alpha(s)$ is a strictly monotone function.
 - (b) $z = \bar{z}$: Then the integral is equal to zero, and hence $s = 0$ gives the maximum.
 - (c) $z \in (0, \bar{z})$: Then $s = \min\left(\left(\frac{z}{\alpha_0 n B}\right)^{\frac{1}{n-1}}, 1\right)$, and hence we cut off s if it lies outside the interval $[0, 1]$.
3. Go to (1) until convergence.

5.2. Numerical simulations of the Boltzmann-type equation for a given α .

In our first example we study mass accumulation in the case of compactly supported initial data $f(z, 0)$; cf. Lemma 3.3. We assume that the maximum knowledge level is denoted by $\bar{z} = 1$ and choose an initial agent distribution of

$$f(z, 0) = \begin{cases} 2 & \text{for all } z \leq 0.5, \\ 0 & \text{otherwise.} \end{cases}$$

We set $\alpha = 1 - z$, i.e., individuals with the lowest knowledge devote all their time to “interactions,” those with the maximum knowledge do not interact at all. Figure 1 illustrates the formation of a Dirac at $z = 0.5$ for symmetric and nonsymmetric meetings. Note that the lines correspond to f at different discrete time steps. Furthermore we observe that the formation of the Dirac happens much faster in case of symmetric meetings, i.e., $\beta = 1$.

5.3. Numerical simulations of the full Boltzmann mean field game model. Next we compare our results with the numerical simulations of the BGP solutions by Lucas and Moll. We choose the same simulation parameters in (2.9) and

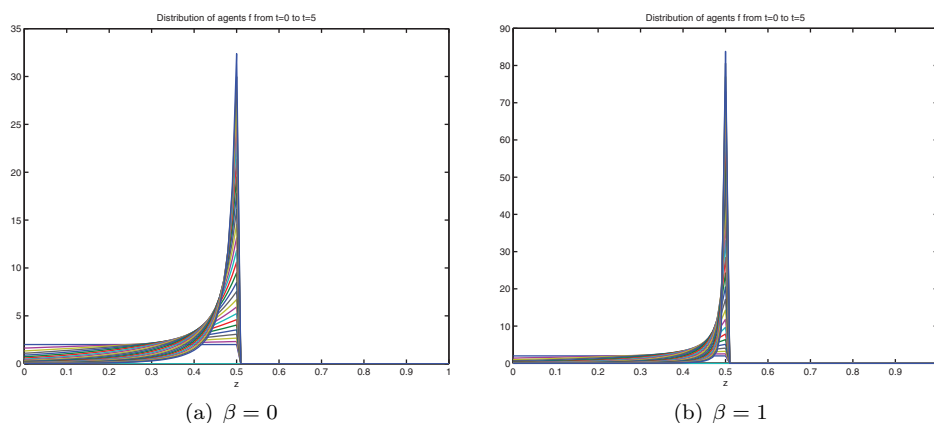


FIG. 1. Distribution of agents $f = f(z, t)$ at different times starting at $t = 0$ up to $t = 5$.

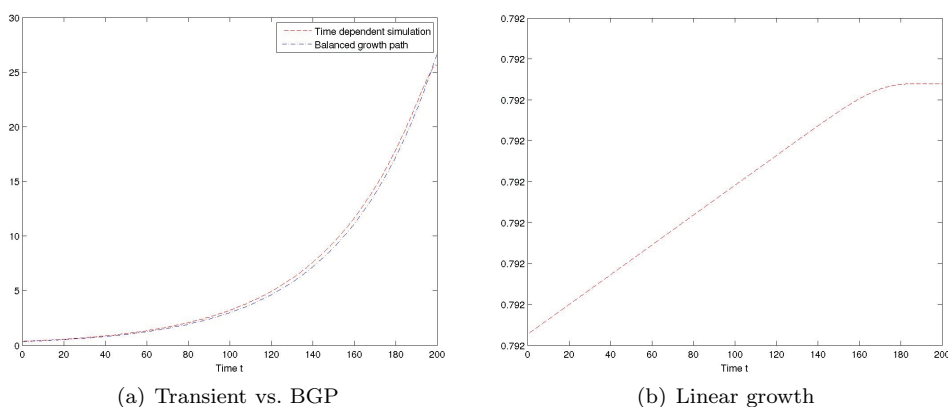


FIG. 2. Evolution of the production function $Y = Y(t)$ in time for different choices of n and θ

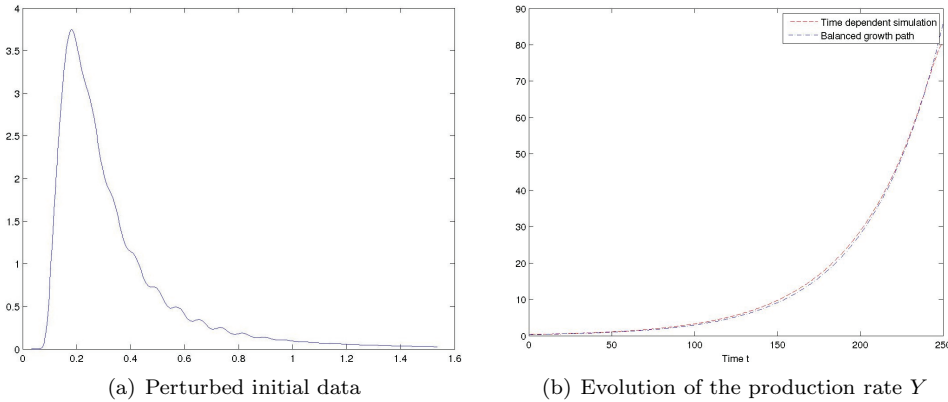
(4.3):

$$\alpha_0 = 0.0849, \quad n = 0.3, \quad \theta = 0.5, \quad k = 0.05, \quad \text{and} \quad r = 0.06.$$

To compare the results of the two numerical solvers we fix a sufficiently large final time T , i.e., $t \in [0, 200]$, and rescale the computational domain \mathcal{I} using the parameter γ determined from the BGP simulations of Lucas and Moll. The simulation results for $k = 400$ time steps and $N = 1001$ discretization points are depicted in Figure 2(a). Note that we can not expect the existence of BGP solutions in general. If we choose, for example,

$$\alpha_0 = 0.0849, \quad n = 0.9, \quad \theta = 0.1, \quad k = 0.05, \quad \text{and} \quad r = 0.06,$$

i.e., a larger value of n and a smaller value of θ , then the BGP solver of Lucas and Moll is not converging. The simulation results in Figure 2(a) reveal the reason why this is the case. Here the production function $Y = Y(t)$ is growing linear in time, and hence the ansatz proposed by Lucas and Moll is not satisfied.

FIG. 3. *Stability of BGP solutions.*

5.4. Stability of balanced growth path. In our final example we illustrate the stability of the BGP with respect to small perturbations of the initial data. We consider a perturbation of the initial distribution of the form

$$f_0^p(z) = f_0(z) + 0.1(1-z)\sin(25\pi z)\chi_{[0.1,1]},$$

which corresponds to a perturbation on the interval $[0.1, 1]$ that does not change the overall mass; see Figure 3(a). Note that the initial datum still has a Pareto tail. We set α_0 , n , k , θ , and r to the same values as in the previous example and solve the system on the time interval $t \in [0, 250]$ using 500 equidistant time steps. Figure 3 compares the evolution of BGP for the corresponding unperturbed initial datum, with the transient simulation. We observe that this perturbation does not change the long time behavior of the production rate. Note that the difference of the two solutions at $t = 250$ can be explained by the terminal condition $V(z, t = 250) = 0$ for the transient simulation.

6. Conclusion. In this paper we present first analytic results as well as numerical simulations for a novel BMFG model for knowledge growth proposed by Lucas and Moll [15]. In this model the distribution of individuals with respect to the knowledge level evolves according to a Boltzmann equation. Collisions correspond to knowledge exchange, the individual interaction rate is determined by maximizing the individual productivity (given the common knowledge of the distribution of all other agents). This gives to a coupled system of a Boltzmann equation and a Hamilton–Jacobi–Bellman equation. Knowledge growth is an inherent nature of the model, which is also reflected in the analytic results. The value function of the HJB equation is growing linearly in z , and hence we can only provide local in time existence. BGP solutions, which also correspond to the exponential growth of the production function in time, illustrate this nature as well—although we are not able to provide existence results of these special solutions in a general situation, we provide first results on their existence and stability in the case of a special interaction function.

This summary gives indications about several open analytic problems which shall be addressed in the near future, e.g., the existence and stability of BGP solutions for the fully coupled system. Another point of interest corresponds to the generalization of the model, for example, by considering a common noise via an additional diffusion

term, more general interaction laws for knowledge growth, or knowledge decay caused by forgetting information.

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