# Limit Structures and Property Testing

by

**Tereza Klimošová**

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# Declarations

The thesis contains results which were obtained together with collaborators and published as follows.

The results from Chapter 2 were obtained together with R. Glebov, A. Grzesik and D. Král'. The corresponding paper *R. Glebov, A. Grzesik, T. Klimošová, and D. Král': Finitely forcible graphons and permutons* was published in Journal of Combinatorial Theory, Series B **110** (2015), 112–135 and is also available as arXiv:1307.2444. The results from this chapter are also a part of the PhD thesis of A. Grzesik.

The results from Chapter 3 were obtained together with R. Glebov and D. Král'. The corresponding paper *R. Glebov, T. Klimošová, and D. Král': Infinite dimensional finitely forcible graphon* is available as arXiv:1404.2743 and has been submitted to a journal.

The results from Chapters 4 and 5 were obtained together with R. Glebov, C. Hoppen, Y. Kohayakawa, D. Král', and H. Liu. The corresponding paper *R. Glebov, C. Hoppen, T. Klimošová, Y. Kohayakawa, D. Král', and H. Liu: Large permutations and parameter testing* is available as arXiv:1412.5622 and has been submitted to a journal.

The results from Chapter 6 were obtained together with D. Král'. The corresponding paper *T. Klimošová and D. Král': Hereditary properties of permutations are strongly testable* was published in Proceedings of SODA 2014, SIAM, Philadelphia, PA, 1164–1173 and is also available as arXiv:1208.2624.

With the exception of Chapter 2, as indicated above, none of the results appeared in any other thesis.

All the collaborators have agreed with the inclusion of our joint work into this thesis.

# Abstract

In the thesis, we study properties of large combinatorial objects. We analyze these objects from two different points of view.

The first aspect is analytic—we study properties of limit objects of combinatorial structures. We investigate when graphons (limits of graphs) and permutons (limits of permutations) are finitely forcible, i.e., when they are uniquely determined by finitely many densities of their substructures. We give examples of families of permutons that are finitely forcible but the associated graphons are not and we disprove a conjecture of Lovász and Szegedy on the dimension of the space of typical vertices of a finitely forcible graphon. In particular, we show that there exists a finitely forcible graphon $W$ such that the topological spaces $T(W)$ and $\overline{T}(W)$ have infinite Lebesgue covering dimension.

We also study the dependence between densities of substructures. We prove a permutation analogue of the classical theorem of Erdős, Lovász and Spencer on the densities of connected subgraphs in large graphs.

The second aspect of large combinatorial objects we concentrate on is algorithmic—we study property testing and parameter testing. We show that there exists a bounded testable permutation parameter that is not finitely forcible and that every hereditary permutation property is testable (in constant time) with respect to the Kendall's tau distance, resolving a conjecture of Kohayakawa.

# Notation and Preliminaries

In this section we survey basic notation and terminology for graphs and permutations that is used throughout the thesis.

We use $\mathbb{N}^*$ for $\mathbb{N} \cup \{\infty\}$ and $[n]$ for $\{1, \ldots, n\}$. We also set $[\infty] = \mathbb{N}$. If $a$ and $b$ are integers, then $a$ mod $b$ is equal to the integer $x \in [b]$ with the same remainder as $a$ after division by $b$. An *interval* $I$ in $[n]$ is a set of integers of the form $\{k \mid a \leq k \leq b\}$ for some $a, b \in [n]$. If $a < b$ and $I \neq [n]$ we say that $I$ is *proper*.

A collection of sets $\mathcal{S} = \{S_1, \ldots, S_\ell\}$ is a *partition of a set $S$ of order $\ell$* if $S = \cup_{i \in [\ell]} S_i$ and $S_i \cap S_j = \emptyset$ for every $i \neq j$, $i, j \in [\ell]$. We denote the order of a partition $\mathcal{S}$ by $|\mathcal{S}|$.

We use $\lambda_k$ for the $k$-dimensional Lebesgue measure and $\upsilon_k$ for its restriction to the $\sigma$-algebra of Borel sets. (In different parts of the thesis, we need to specifically consider either one or the other.) In other words, $\upsilon_k$ is a uniform measure on the $\sigma$-algebra of Borel sets. We omit the subscript if the dimension is clear from the context.

For a non-trivial convex polygon $A \subseteq [0,1]^2$, i.e., a convex polygon different from a point (but it can be a segment), we define $\Upsilon_A$ to be the unique probability measure on the $\sigma$-algebra of Borel sets of $[0,1]^2$ with support $A$ and mass uniformly distributed inside $A$. That is, for every Borel set $S \subseteq [0,1]^2$, $\Upsilon_A(S) = \upsilon_2(A \cap S)/\upsilon_2(A)$ for $A$ with $\upsilon_2(A) > 0$, and $\Upsilon_A(S) = \upsilon_1(A \cap S)/\upsilon_1(A)$ if $\upsilon_2(A) = 0$, in which case $A$ must be a segment and $\upsilon_1$ is the uniform measure on the line containing the segment $A$. In particular, $\Upsilon_{[0,1]^2}$ coincides with $\upsilon_2$ on $[0,1]^2$. We set $\Upsilon = \Upsilon_{[0,1]^2}$ to simplify the notation.

A *graph* is a pair $(V, E)$ where $E \subseteq \binom{V}{2}$. The elements of $V$ are called *vertices* and the elements of $E$ are called *edges*. The *order* of a graph $G$ is the number of its vertices and it is denoted by $|G|$.

If $G$ and $G'$ are graphs, then $G \cup G'$ is the disjoint union of $G$ and $G'$ and $G + G'$ is the graph obtained from $G \cup G'$ by adding all edges between $G$ and $G'$. If $G$ is a graph and $U$ is a subset of its vertices, then $G \setminus U$ is the graph obtained from $G$ by removing the vertices of $U$ and all edges containing at least one vertex from $U$.

The *density* $t(H, G)$ of a graph $H$ in a graph $G$ is the probability that $|H|$ distinct vertices of $G$ chosen uniformly at random induce a subgraph isomorphic to $H$. If $|H| > |G|$, we set $t(H, G) = 0$.

A *permutation of order* $n$ is a bijective mapping from $[n]$ to $[n]$. The order of a permutation $\pi$ is also denoted by $|\pi|$. We will call a permutation non-trivial if it has order greater than 1. The set of all permutations is denoted by $\mathfrak{S}$ and the set all permutations of order $n$ by $S_n$. In what follows, we identify a sequence of $n$ distinct integers $a_1 \dots a_n$ between 1 and $n$ with a permutation $\pi$ by setting $\pi(i) = a_i$. For example, the identity permutation of order 4 is denoted by 1234. An *inversion* in a permutation $\sigma$ is a pair $(i, j)$, $i, j \in [|\sigma|]$, such that $i < j$ and $\sigma(i) > \sigma(j)$.

Let $\sigma$ be a permutation of order $n$ and $X = \{x_1, \dots, x_k\} \subseteq [n]$ such that $x_1 < \cdots < x_k$. A *subpermutation* induced by $X$ in $\sigma$ denoted by $\pi = \sigma \upharpoonright X$ is a permutation of order $k$ such that $\pi(j) < \pi(j')$ if and only if $\sigma(x_j) < \sigma(x_{j'})$. For example, the subpermutation of 7126354 induced by $3, 4, 6$ is 132. We say that $\sigma$ contains $\pi$ as a subpermutation if there exists $X \subseteq [n]$ such that $\pi = \sigma \upharpoonright X$. In some literature, subpermutations are referred to as *patterns*. However, we follow the terminology from previous papers related to testing permutation properties and to permutation limits, which also makes the terminology closer to the case of graphs.

The *density* $t(\pi, \sigma)$ of a permutation $\pi$ of order $k$ in a permutation $\sigma$ of order $n$ is the probability that a (uniform) random subset of $[n]$ of size $k$ induces a subpermutation $\pi$ in $\sigma$. We set $t(\pi, \sigma) = 0$ if $k > n$.

# Chapter 1

# Introduction

In the thesis, we study properties of large combinatorial objects. We analyze these objects from two different points of view.

The first aspect is analytic—we study properties of limit objects of combinatorial structures. We investigate when graphons (limits of graphs) and permutons (limits of permutations) are finitely forcible, i.e., when they are uniquely determined by finitely many densities of their substructures. In Chapter 2, we give examples of families of permutons that are finitely forcible but the associated graphons are not.

Our efforts in studying finite forcibility culminate in Chapter 3, where we disprove a conjecture of Lovász and Szegedy on the dimension of the space of typical vertices of a finitely forcible graphon. In particular, we show that there exists a finitely forcible graphon $W$ such that the topological spaces $T(W)$ and $\overline{T}(W)$ have infinite Lebesgue covering dimension.

In Chapter 4, we study the dependence between densities of substructures. We prove a permutation analogue of the classical theorem of Erdős, Lovász and Spencer on the densities of connected subgraphs in large graphs.

The second aspect of large combinatorial objects we concentrate on is algorithmic—we study property testing and parameter testing algorithms, i.e., probabilistic algorithms for determining properties and parameters of large input in sublinear time. In fact, this topic is related to combinatorial limits in a closer way than it might seem at the first sight. Several results on property testing have been proved or reproved using limit structures (see, e.g., [54, 69]). In Chapter 5, we use limits of permutations to obtain a result on testing permutation parameters. In particular, we give a positive answer to a question of Hoppen, Kohayakawa, Moreira, and Sampaio [54, Question 5.5] whether there exists a bounded testable permutation parameter that is not finitely forcible.

In Chapter 6, we show that every hereditary permutation property is testable (in constant time) with respect to the Kendall's tau distance, resolving a conjecture of Kohayakawa [58].

In the remainder of this chapter, we survey definitions and put the results contained in the thesis into the context of previous work.

## 1.1 Limit objects and finite forcibility

Research on analytic objects associated with convergent series of combinatorial objects was initiated by the theory of limits of dense graphs [20–22,66], followed by limits of sparse graphs [18,34], permutations [52,53], partial orders [56] and others. This theory provides an analytic view of many standard concepts, e.g., the regularity method [68], and led to results in many areas of mathematics and computer science, in particular in extremal combinatorics [9–12,49–51,59, 60,71,72,74–76] and property testing [54,69].

In the thesis, we focus on limits of dense graphs, which are called *graphons* and limits of permutations, which are called *permutons*. We start our exposition with the slightly simpler notion of permutation limits.

### 1.1.1 Limits of permutations

The theory of permutation limits was initiated by Hoppen, Kohayakawa, Moreira, Ráth and Sampaio in [52, 53]. Here, we follow the analytic view on the limit as used in [61], which also appeared in an earlier work of Presutti and Stromquist [73].

An infinite sequence $(\pi_i)_{i\in\mathbb{N}}$ of permutations with $|\pi_i| \to \infty$ is *convergent* if the sequence $(t(\sigma, \pi_i))_{i\in\mathbb{N}}$ converges for every permutation $\sigma$. With every convergent sequence of permutations, one can associate the following analytic object: a *permuton* is a probability measure $\Phi$ on the $\sigma$-algebra $\mathcal{A}$ of Borel sets of the unit square $[0,1]^2$ such that $\Phi$ has *uniform marginals*, i.e., $\Phi([\alpha,\beta]\times[0,1]) = \Phi([0,1]\times[\alpha,\beta]) = \beta - \alpha$ for every $0 \le \alpha \le \beta \le 1$. We denote the set of all permutons by $\mathfrak{P}$.

We now describe the relation between convergent sequences of permutations and permutons. Let $\Phi$ be a permuton. For an integer $n$, let $(x_1, y_1), \ldots, (x_n, y_n)$ be points in $[0,1]^2$ sampled independently according to the distribution $\Phi$. Because $\Phi$ has uniform marginals, the $x$-coordinates of all these points are mutually different with probability one. The same holds for the $y$-coordinates. Assume that this is indeed the case. One can then define a permutation $\pi$ of order $n$ based on the $n$ points $(x_1, y_1), \ldots, (x_n, y_n)$ as follows: let $i_1, \ldots, i_n \in [n]$ be such that $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$ and define $\pi$ to be the

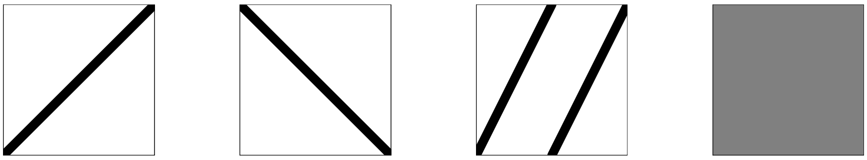

Figure 1.1: The limits of sequences $\left(\pi_i^1\right)_{i\in\mathbb{N}}$, $\left(\pi_i^2\right)_{i\in\mathbb{N}}$, $\left(\pi_i^3\right)_{i\in\mathbb{N}}$ and $\left(\pi_i^4\right)_{i\in\mathbb{N}}$.

unique bijective mapping from $[n]$ to $[n]$ satisfying $\pi(j) < \pi(j')$ if and only if $y_{i_j} < y_{i_{j'}}$. We say that a permutation $\pi$ of order $n$ obtained in the just described way is a $\Phi$-*random permutation* of order $n$. A *uniformly random permutation* is a $\Upsilon$-random permutation (note that $\Upsilon = \Upsilon_{[0,1]^2}$ is a permuton), i.e., each permutation of order $n$ is chosen with probability $(n!)^{-1}$.

If $\Phi$ is a permuton and $\sigma$ is a permutation of order $n$, then $t(\sigma, \Phi)$ is the probability that a $\Phi$-random permutation of order $n$ is $\sigma$. We now recall the core results from [52, 53]. For every convergent sequence $(\pi_i)_{i\in\mathbb{N}}$ of permutations, there exists a unique permuton $\Phi$ such that $t(\sigma, \Phi) = \lim_{i\to\infty} t(\sigma, \pi_i)$ for every permutation $\sigma$. This permuton is the *limit* of the sequence $(\pi_i)_{i\in\mathbb{N}}$. On the other hand, let $\Phi$ be a permuton and $(\pi_i)_{i\in\mathbb{N}}$ a sequence such that $\pi_i$ is a $\Phi$-random permutation of order $i$. With probability one, this sequence is convergent and its limit is $\Phi$.

We now give four examples of the notions we have just defined (the corresponding permutons are depicted in Figure 1.1). Let us consider a sequence $\left(\pi_i^1\right)_{i\in\mathbb{N}}$ such that $\pi_i^1$ is the identity permutation of order $i$, i.e., $\pi_i^1(k) = k$ for $k \in [i]$. This sequence is convergent and its limit is the the permuton $I = \Upsilon_A$ where $A = \{(x, x), x \in [0, 1]\}$. Similarly, the limit of a sequence $\left(\pi_i^2\right)_{i\in\mathbb{N}}$, where $\pi_i^2$ is the permutation of order $i$ defined as $\pi_i^2(k) = i + 1 - k$ for $k \in [i]$, is the permuton $\Omega = \Upsilon_B$ where $B = \{(x, 1 - x), x \in [0, 1]\}$. A little bit more complicated example is the following: the sequence $\left(\pi_i^3\right)_{i\in\mathbb{N}}$, where $\pi_i^3$ is the permutation of order $2i$ defined as

$$\pi_i^3(k) = \begin{cases} 2k - 1 & \text{if } k \in [i], \\ 2(k - i) & \text{otherwise} \end{cases}$$

is convergent and the limit of the sequence is the measure $\frac{1}{2}\Upsilon_C + \frac{1}{2}\Upsilon_D$, where $C = \{(x/2, x), x \in [0, 1]\}$ and $D = \{((x + 1)/2, x), x \in [0, 1]\}$. Next, consider a sequence $(\pi_i^4)_{i\in\mathbb{N}}$ such that $\pi_i^4$ is a uniformly random permutation of order $i$. This sequence is convergent with probability one and its limit is the measure $\Upsilon$ with probability one.

### 1.1.2 Limits of dense graphs

The other limit structures we consider are limits of dense graphs. We now survey basic results related to the theory of dense graph limits as developed in [20–22, 66]. A sequence of graphs $(G_i)_{i\in\mathbb{N}}$ with $|G| \to \infty$ is *convergent* if the sequence $(t(H, G_i))_{i\in\mathbb{N}}$ converges for every $H$. The associated limit object is called a *graphon*: it is a symmetric $\lambda$-measurable function from $[0,1]^2$ to $[0,1]$. Here, symmetric stands for the property that $W(x,y) = W(y,x)$ for every $x, y \in [0,1]$. If $W$ is a graphon, then a *$W$-random graph* of order $k$ is obtained by sampling $k$ random points $x_1, \dots, x_k \in [0,1]$ uniformly and independently and joining the $i$-th and the $j$-th vertex by an edge with probability $W(x_i, x_j)$. As in the case of permutations, we write $t(H, W)$ for the probability that a $W$-random graph of order $|H|$ is isomorphic to $H$.

The densities of graphs in a graphon $W$ can be expressed as integrals. If $W$ is a graphon and $H$ is a graph of order $k$ with vertices $v_1, \dots, v_k$ and edge set $E$, then

$$t(H,W) = \frac{k!}{|\mathrm{Aut}(H)|} \int\limits_{[0,1]^k} \prod_{v_i v_j \in E} W(x_i, x_j) \prod_{v_i v_j \notin E} (1 - W(x_i, x_j)) \mathrm{d}x_1 \dots \mathrm{d}x_k$$

where $\mathrm{Aut}(H)$ is the automorphism group of $H$.

For every convergent sequence $(G_i)_{i\in\mathbb{N}}$ of graphs, there exists a graphon $W$ such that $t(H,W) = \lim_{i\to\infty} t(H, G_i)$ for every graph $H$ [66]. We call such a graphon $W$ a *limit* of $(G_i)_{i\in\mathbb{N}}$. On the other hand, for a graphon $W$, the sequence $(G_i)_{i\in\mathbb{N}}$ where $G_i$ is a $W$-random graph of order $i$ is convergent with probability one and its limit is $W$ with probability one.

Unlike in the case of permutations, the limit of a convergent sequence of graphs is not unique. For a graphon $W$ and a measure preserving transformation $\varphi : [0,1] \to [0,1]$, let $W^\varphi = W(\varphi(x), \varphi(y))$. Then, if $W$ is a limit of $(G_i)_{i\in\mathbb{N}}$, $W^\varphi$ is also a limit of $(G_i)_{i\in\mathbb{N}}$. Let us introduce the following definition of equivalence of graphons: two graphons $W$ and $W'$ are *weakly isomorphic* if $t(H,W) = t(H,W')$ for every graph $H$. The following equivalent characterization of weak isomorphism was given in [19].

**Theorem 1.** *Two graphons $U$ and $W$ are weakly isomorphic if and only if there exist measure preserving maps $\varphi, \psi : [0,1] \to [0,1]$ such that $U^\varphi = W^\psi$ almost everywhere.*

### 1.1.3 Finite forcibility

In this section we introduce a notion of finite forcibility for graphons and permutons. A graphon $W$ is *finitely forcible* if there exists a finite set of graphs $\mathcal{H}$

such that every graphon $W'$ satisfying $t(H, W) = t(H, W')$ for every $H \in \mathcal{H}$ is weakly isomorphic to $W$. Similarly, a permuton $\Phi$ is *finitely forcible* if there exists a finite set $S$ of permutations such that every permuton $\Phi'$ satisfying $t(\sigma, \Phi) = t(\sigma, \Phi')$ for every $\sigma \in S$ is equal to $\Phi$. In other words, a graphon or a permuton is finitely forcible, if it can be uniquely determined (up to weak isomorphism in the case of graphons) by finitely many densities of substructures.

The question whether a graphon or a permuton is finitely forcible is particularly interesting from the point of view of extremal combinatorics, since every finitely forcible object corresponds to the unique solution of some extremal problem. Problems of this kind are also related to quasirandomness and they were studied well before the theory of limits of combinatorial objects emerged. For example, the results on quasirandom graphs from the work of Chung, Graham and Wilson [24], Rödl [78] and Thomason [84,85] imply that the homomorphic densities of $K_2$ and $C_4$ guarantee that densities of all subgraphs behave as in the random graph $G_{n,1/2}$. In the language of graphons, this result asserts that the graphon identically equal to $1/2$ is finitely forcible by densities of 4-vertex subgraphs. A similar result for permutations, which was originally raised as a question by Graham, was proven by Král' and Pikhurko [61] who exploited the analytic view on permutation limits to show that the random permuton is finitely forced by densities of permutations of order 4.

The results on graphs were generalized for stepwise graphons in [64]. Another example of a finitely forcible graphon is due to Diaconis, Homes, and Janson [32]. Their result is equivalent to the statement that the half-graphon $W_\triangle(x, y)$ defined as $W_\triangle(x, y) = 1$ if $x+y \geq 1$, and $W_\triangle = 0$, otherwise, is finitely forcible. These results were further extended by Lovász and Szegedy [65] who also gave several conditions when a graphon is not finitely forcible. In Chapter 2, we provide a generalization of these results in the realm of permutons.

### 1.1.4 Statements of our results

In Chapter 2, we focus on the interplay between finite forcibility of permutons and graphons. In [64], Lovász and Sós proved a result for more complex quasirandom graphs, which can be restated in the language of graphons as a statement that any stepwise graphon[1] is finitely forcible. We prove the following analogue of this result for permutons.

[1]A graphon $W$ is *stepwise* if there exists a partition of $[0, 1]$ into finitely many measurable sets $S_1, \dots, S_k$, such that $W$ is constant on $S_i \times S_j$ for every $i, j \in [k]$.

Figure 1.2: The graphons associated with the first three permutons depicted in Figure 1.1, where the point (0,0) is in the bottom left corner.

**Theorem 2.** *If $\Phi$ is a permuton satisfying $\Phi = \sum_{i\in[k]} \alpha_i \Upsilon_{A_i}$ for some non-negative reals $\alpha_1, \ldots, \alpha_k$ and some non-trivial polygons $A_1, \ldots, A_k \subseteq [0,1]^2$, then $\Phi$ is finitely forcible.*

A permutation $\pi$ of order $k$ can be associated with a graph $G_\pi$ of order $k$ as follows. The vertices of $G_\pi$ are the integers between 1 and $k$ and $ij$ is an edge of $G$ if and only if either $i < j$ and $\pi(i) > \pi(j)$, or $i > j$ and $\pi(i) < \pi(j)$, i.e., $i$ and $j$ form an inversion. If $(\pi_i)_{i\in\mathbb{N}}$ is a convergent sequence of permutations, then the sequence of graphs $(G_{\pi_i})_{i\in\mathbb{N}}$ is also convergent. Moreover, if two convergent sequences of permutations have the same limit, then the graphons associated with the two corresponding (convergent) sequences of graphs are weakly isomorphic. In this way, we may associate each permuton $\Phi$ with a graphon $W_\Phi$, which is unique up to a weak isomorphism (see Figure 1.2 for examples). We will provide examples of classes of permutons that are finitely forcible, while the associated graphons are not.

For $k \in \mathbb{N}^*$, let $(W)_{i\in[k]}$ be a sequence of graphons and $(p_i)_{i\in[k]} \in \mathbb{R}_+^k$ be a sequence of reals such that $\sum_{i\in[k]} p_i = 1$. We define a *direct sum of graphons* $W_i$ with weights $p_i$ denoted by $W = \bigoplus_{i\in[k]} p_i W_i$ as follows.

$$W(x,y) = \begin{cases} W_i(\varphi_i(x), \varphi_i(y)) & \text{if } x, y \in J_i \text{ for some } i \in [k], \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$J_i = \left( \sum_{j=1}^{i-1} p_j, \sum_{j=1}^{i} p_j \right] \text{ and}$$

$$\varphi_i(x) = \frac{x - \sum_{j=1}^{i-1} p_j}{p_i}$$

for every $i \in [k]$.

We now define the direct sum of permutons with weights in an analogous way. For $k \in \mathbb{N}^*$, a sequence of permutons $(\Phi_i)_{i\in[k]}$ and $(p_i)_{i\in[k]} \in \mathbb{R}_+^k$ such that $\sum_{i\in[k]} p_i = 1$, the *direct sum of permutons* $\Phi_i$ with weights $p_i$ is denoted by $\Phi = \bigoplus_{i\in[k]} p_i \Phi_i$ and is defined as follows;

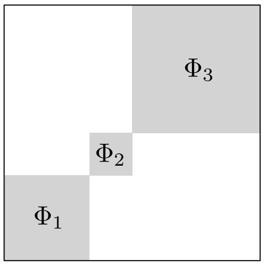


Figure 1.3: The permuton $\frac{1}{3}\Phi_1 \oplus \frac{1}{6}\Phi_2 \oplus \frac{1}{2}\Phi_3$.

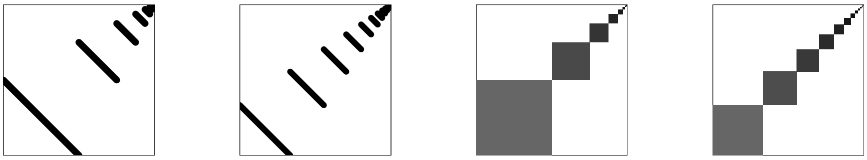

Figure 1.4: The permutons $\Upsilon_{\to 1/2}$, $\Upsilon_{\to 2/3}$, $\Omega_{\to 1/2}$, and $\Omega_{\to 2/3}$.

$$\Phi(S) = \sum_{i \in [k]} p_i \Phi_i(\theta_i(S \cap C_i))$$

for every Borel set $S$, where $C_i = J_i \times J_i$ and $\theta_i$ is a map from $C_i$ to $[0,1]^2$ defined as $\theta_i((x,y)) = (\varphi_i(x), \varphi_i(y))$ for every $i \in [k]$. See Figure 1.3 for an example.

For a graphon $W$ and $\alpha \in (0,1)$, we define a graphon

$$W_{\to\alpha} = \bigoplus_{i=1}^{\infty} (1-\alpha)\alpha^{i-1} W.$$

Similarly, for a permuton $\Phi$ and $\alpha \in (0,1)$, we define

$$\Phi_{\to\alpha} = \bigoplus_{i=1}^{\infty} (1-\alpha)\alpha^{i-1} \Phi.$$

Later, we prove finite forcibility of permutons $\Omega_{\to\alpha}$ and $\Upsilon_{\to\alpha}$. (Recall that $\Omega$ denotes the unique permuton with support consisting of the segment between $(0,1)$ and $(1,0)$.) Examples of these permutons can be found in Figure 1.4.

**Theorem 3.** *For every $\alpha \in (0,1)$, the permutons $\Omega_{\to\alpha}$ and $\Upsilon_{\to\alpha}$ are finitely forcible.*

Next, we turn our attention to graphons with similar recursive structure. We show that graphons of this kind are not finitely forcible unless they are equal to zero almost everywhere.

**Theorem 4.** *For every $\alpha \in (0,1)$ and every graphon $W$, if the graphon $W_{\to\alpha}$ is finitely forcible, then $W$ is equal to zero almost everywhere.*

Consequently, $W_{\to\alpha}$ is finitely forcible only if $W$ zero almost everywhere. Observe that $W_{\Phi_{\to\alpha}}$ and $(W_\Phi)_{\to\alpha}$ are weakly isomorphic. It follows that the graphons $W_{\Omega_{\to\alpha}}$ and $W_{\Upsilon_{\to\alpha}}$ associated with the permutons $\Omega_{\to\alpha}$ and $\Upsilon_{\to\alpha}$ are weakly isomorphic to $(W_\Omega)_{\to\alpha}$ and $(W_\Upsilon)_{\to\alpha}$, respectively and therefore not finitely forcible.

In Chapter 3, we study a relation between finite forcibility and properties of *typical vertices* of a graphon. Every graphon can be assigned a topological space associated with its typical vertices as follows [67]. For a graphon $W$ and $x \in [0,1]$, we define a function

$$f_x^W(y) = W(x,y).$$

Since almost every function $f_x^W$ belongs to $L^1([0,1])$, the graphon $W$ naturally defines a probability measure $\mu$ on $L^1([0,1])$. Let $T(W)$ be the set formed by the functions $f \in L^1([0,1])$ such that every neighborhood of $f$ in $L^1([0,1])$ has positive measure with respect to $\mu$. The set $T(W)$ with the topology inherited from $L^1([0,1])$ is called the *space of typical vertices* of $W$. The vertices $x$ of $W$ with $f_x^W \in T(W)$ are called *typical vertices* of a graphon $W$. A coarser topology on $T(W)$ can be defined using the similarity distance $d_W$ between $f,g \in L^1([0,1])$ defined as

$$d_W(f,g) = \int\limits_{[0,1]} \left| \int\limits_{[0,1]} W(x,y)(f(y)-g(y))\mathrm{d}y \right| \mathrm{d}x.$$

The space with this topology is denoted by $\overline{T}(W)$. The topological space $\overline{T}(W)$ is always compact [63, Chapter 13] and its structure is related to weak regular partitions of $W$ [68].

Unlike $\overline{T}(W)$, $T(W)$ does not need to be compact even if $W$ is finitely forcible [42]. Lovász and Szegedy [65, Conjecture 10] led by examples of known finitely forcible graphons proposed the following:

**Conjecture 1.** *If $W$ is a finitely forcible graphon, then $T(W)$ is finite dimensional.*

They said that they intentionally do not specify which notion of dimension is meant here—a result concerning any variant would be interesting. The Rademacher graphon $W_R$ constructed in [42] is finitely forcible and the space $T(W_R)$ has infinite Minkowski dimension but its Lebesgue dimension is one and $\overline{T}(W_R)$ has both Minkowski and Lebesgue dimension one.

We construct a graphon $W_\boxplus$, which we call a *hypercubical graphon*, such that $W_\boxplus$ is finitely forcible and both $T(W_\boxplus)$ and $\overline{T}(W_\boxplus)$ contain subspaces homeomorphic to $[0,1]^\mathbb{N}$.

**Theorem 5.** *The hypercubical graphon $W_\boxplus$ is finitely forcible and the topological spaces $T(W_\boxplus)$ and $\overline{T}(W_\boxplus)$ contain subspaces homeomorphic to $[0,1]^\mathbb{N}$.*

The proof of Theorem 5 extends the methods from [42] and [70]. In particular, Norine [70] constructed finitely forcible graphons with the space of typical vertices of arbitrarily large (but finite) Lebesgue dimension. In his construction, both $T(W)$ and $\overline{T}(W)$ contain a subspace homeomorphic to $[0,1]^d$. We show how the techniques from [42] and [70] can be refined to force a subspace homeomorphic to $[0,1]^\mathbb{N}$, which turned out to be quite technically challenging.

In Chapter 4, we use limit objects of permutations to prove a permutation analogue of a classical result of Erdős, Lovász and Spencer about subgraph densities in a graph. In the case of graphs, Erdős, Lovász and Spencer [35] considered three notions of substructure densities: the subgraph density, the induced subgraph density and the homomorphism density. They showed that these types of densities are strongly related and that the densities of connected graphs are independent. The result has a natural formulation in the language of graphons: the body of possible densities of any $k$ connected graphs in graphons, which is a subset of $[0,1]^k$, has a non-empty interior (in particular, it is full dimensional).

Our result asserts that the analogous statement is also true for permutations. As in the case of graphs, it is natural to cast the result in terms of limit objects—permutons. In particular, results of Chapter 4 (Theorem 32 and Lemma 31) say that the body of possible densities of any $k$ indecomposable permutations in permutons has a non-empty interior and is full dimensional. We use the notion of an *indecomposable* permutation, which is an analogue of graph connectivity in the sense that an indecomposable permutation cannot be split into *independent parts.* Specifically, a permutation $\sigma$ of order $n$ is *indecomposable* if there is no $m < n$ such that $\sigma([m]) = [m]$.

## 1.2 Property and parameter testing

In Chapters 5 and 6 we focus on algorithmic aspects of large combinatorial structures, in particular on property and parameter testing.

A *property tester* is an algorithm that decides whether a large input has the considered property by querying only a small sample of it. Since the tester is presented only with a part of the input, it is necessary to allow an error based on the robustness of the tested property. Following [43, 45, 48, 79–81], we say that a property $\mathcal{P}$ of a class of structures (e.g., functions, graphs) is testable if for every $\varepsilon > 0$, there exists a randomized algorithm (a tester) $\mathcal{A}$ such that the number of queries made by $\mathcal{A}$ is bounded by a function of $\varepsilon$ independent of the input and such that if the input has the property $\mathcal{P}$, then $\mathcal{A}$ accepts with probability at least $1 - \varepsilon$, and if the input is $\varepsilon$-far from $\mathcal{P}$, then $\mathcal{A}$ rejects with probability at least $1 - \varepsilon$. The exact notion depends on the studied class $\mathcal{C}$ of combinatorial structures, the considered property $\mathcal{P}$ and the chosen metric on $\mathcal{C}$. There are also some variants of this notion. For example, one can allow only a one-sided error, i.e., $\mathcal{A}$ is required to accept whenever the input has the property $\mathcal{P}$, or the size of the sample may also depend (in a sublinear way) on the input size (for example as in testing monotonicity of functions [1,33,38,44]).

A well-investigated area of property testing is testing properties of dense graphs, i.e., graphs with quadratically many edges. One of the most significant results in this area is that of Alon and Shapira [6] asserting that every hereditary graph property, that is, a property preserved by taking induced subgraphs, is testable with respect to the edit distance[2]. This extends several earlier results [7, 45, 77]. A characterization of testable graph properties can be found in [3]. A logic perspective of graph property testing was addressed in [2,37] and the connection to graph limits was explored in [69].

Besides the dense case, a property testing in sparse graphs has also attracted substantial attention. The bounded degree graph case was introduced in [47]. Unlike in the dense case, not all hereditary properties are testable [17] though many properties can be tested [14,27,28,30,46], also see surveys [29,43]. Testing properties of other objects have also been intensively studied. For example, results on testing properties of strings can be found in [4, 62], results related to constraint satisfaction problems in [5], and to more algebraically oriented properties in [13, 15, 16, 81–83].

[2]The edit distance of a graph $G$ from a graph property $\mathcal{P}$ is the minimum number of edges that need to be modified (added or removed) in $G$ to obtain a graph with property $\mathcal{P}$, divided by $|G|^2$.

In the thesis, we focus on testing of permutation properties. A *permutation property* $\mathcal{P}$ is a set of permutations. If $\pi \in \mathcal{P}$, we say that a permutation $\pi$ has the property $\mathcal{P}$. We often refer to permutation properties just as properties. We focus on properties which are *hereditary*, that is, closed under taking subpermutations. In other words, if $\pi$ has a hereditary property $\mathcal{P}$, then any subpermutation of $\pi$ has the property $\mathcal{P}$. An example of a hereditary property is the set of all permutations not containing a fixed permutation as a subpermutation.

In Chapter 6, we study testing properties of permutations in a property testing model analogous to the dense graph setting and we fill a gap related to testing hereditary properties with respect to the counterpart of the edit distance.

We consider testing permutation properties *through subpermutations*, where the tester is presented with a random subpermutation of the input permutation (the size of the subpermutation depends on the tested property and the required error). In particular, if an input permutation $\pi$ has order $n$, then a random subset $X \subseteq [n]$ is chosen and the tester is presented with $\pi \upharpoonright X$.

For a distance $d$ between permutations of the same order, we define distance of a permutation $\pi$ from a property $\mathcal{P}$ as follows;

$$d(\pi, \mathcal{P}) = \min_{\sigma \in \mathcal{P} \cap S_{|\pi|}} d(\pi, \sigma).$$

In particular, $d(\pi, \mathcal{P}) = \infty$ if $\mathcal{P}$ does not contain any permutation of order $|\pi|$. We say that a property $\mathcal{P}$ is *testable with respect to a distance* $d$ if for every $\varepsilon > 0$, there exist $M_\varepsilon$ and a tester $\mathcal{A}_\varepsilon$ which, based on a random subpermutation of size $M_\varepsilon$, accepts an input permutation $\pi \in \mathcal{P}$ with probability at least $1 - \varepsilon$ and rejects an input permutation $\pi$ such that $d(\pi, \mathcal{P}) > \varepsilon$ with probability at least $1 - \varepsilon$.

There are several notions of distance between permutations, see [31]. The rectangular distance and the Kendall's tau distance will be of most interest to us. Let $\pi$ and $\sigma$ be two permutations of the same order $n$. The *rectangular distance* of $\pi$ and $\sigma$, which is denoted by $\mathrm{dist}_\square(\pi, \sigma)$, is defined as

$$\max_{S,T} \frac{\big|\, |\pi(S) \cap T| - |\sigma(S) \cap T| \,\big|}{n}$$

where the maximum is taken over all subintervals $S$ and $T$ of $[n]$.

The *Kendall's tau distance* $\mathrm{dist}_K(\pi, \sigma)$ is defined as

$$\frac{|\{(i,j) \mid \pi(i) < \pi(j), \sigma(i) > \sigma(j), i, j \in [n]\}|}{\binom{n}{2}}.$$

Alternatively, the Kendall's tau distance of two permutations is the minimum number of swaps of pairs of the elements with values differing by one needed for transforming $\pi$ to $\sigma$, normalized by $\binom{n}{2}$.

It can be shown that if two permutations are close in the Kendall's tau distance, then they are close in the rectangular distance. The converse is not true: the rectangular distance of two random permutations is concentrated around zero but their Kendall's tau distance is concentrated around $1/2$.

Hence, testing permutation properties with respect to the Kendall's tau distance is more difficult than with respect to the rectangular distance (at least in the sense that every tester designed for testing with respect to the Kendall's tau distance also works for testing with respect to the rectangular distance but not vice versa in general). The Kendall's tau distance is considered to correspond to the edit distance of graphs which appears in the hereditary graph property testing, while the rectangular distance is considered to correspond to the cut norm appearing in the theory of graphs limits, see [66]. The latter is demonstrated in the notion of regularity decompositions of permutations developed by Cooper [25, 26] and permutation limits introduced by Hoppen et al. [52, 53] (also see [25, 61] for relation to quasirandom permutations).

Another notion of distance between permutations, the minimum number of insertions and deletions to transform one permutation to another normalized by the size of permutations, was considered [36]. This distance is "finer" than the Kendall's tau distance, that is, if two permutation are close in it, they are close in the Kendall's tau distance too but not vice versa. One of the results in [36] implies that the hereditary properties of permutations are not testable with constant sample size with respect to this distance. In particular, it was shown that monotonicity of a permutation is testable with $O(\log n/\varepsilon)$ queries and a logarithmic number of queries is needed.

Testing hereditary permutation properties with respect to the rectangular distance was addressed by Hoppen, Kohayakawa, Moreira and Sampaio [54, 55]. The main result of [54] is the following.

**Theorem 6.** *Let $\mathcal{P}$ be a hereditary property. For every positive real $\varepsilon$, there exists $M$ such that if $\pi$ is a permutation of order at least $M$ with* $\mathrm{dist}_\square(\pi, \mathcal{P}) \geq \varepsilon$*, then a random subpermutation of $\pi$ of order $M$ has the property $\mathcal{P}$ with probability at most $\varepsilon$.*

Theorem 6 implies that hereditary properties are testable through subpermutations with respect to the rectangular distance with one-sided error: the tester accepts if and only if the random subpermutation has the property $\mathcal{P}$ and thus the tester always accepts permutations having the property $\mathcal{P}$.

Kohayakawa [58] asked whether hereditary properties of permutations are also testable through subpermutations with respect to the Kendall's tau distance, which he refers to as *strong testability*. We resolve this question in the positive way. In particular, we prove the following analogue of Theorem 6.

**Theorem 7.** *Let $\mathcal{P}$ be a hereditary property. For every positive real $\varepsilon_0$, there exists $M_0$ such that if $\pi$ is a permutation of order at least $M_0$ with $\operatorname{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$, then a random subpermutation of $\pi$ of order $M_0$ has the property $\mathcal{P}$ with probability at most $\varepsilon_0$.*

Hence, we establish that hereditary properties are testable through subpermutations with respect to the Kendall's tau distance with one-sided error. Since the Kendall's tau distance is the counterpart of the edit distance for graphs, our result was proposed in [54] as a possible permutation analogue of the result of Alon and Shapira [6]. It is also worth noting that our arguments are purely combinatorial and are not based on regularity decompositions or on the analysis of limit structures.

*Parameter testing* is a notion related to property testing which was introduced by Borgs et al. [20, 21]. Here, the goal is to estimate some numerical parameter of the input structure with high probability by querying only a small sample of the input structure. For instance, Fisher and Newman [39] proved, that the edit distance from a property $\mathcal{P}$ is a testable graph parameter for every testable graph property $\mathcal{P}$. That is, there is an algorithm which with high probability estimates the edit distance of an input graph from a testable property up to an additive constant, using only constant number of queries.

In Chapter 5, we are concerned with testing permutation parameters. A *permutation parameter* $f$ is a function from $\mathfrak{S}$ to $\mathbb{R}$. We say that $f$ is *bounded* if for some constant $K$, $|f(\pi)| \leq K$ for every permutation $\pi$. A permutation parameter $f$ is *testable* if for every $\varepsilon > 0$ there exist an integer $n_0$ and $\tilde{f} : S_{n_0} \to \mathbb{R}$ such that for every permutation $\sigma$ of order at least $n_0$, a randomly chosen subpermutation $\pi$ of $\sigma$ of size $n_0$ satisfies $|f(\sigma) - \tilde{f}(\pi)| < \varepsilon$ with probability at least $1 - \varepsilon$.

Parameter testing for permutations was considered in [54]. The authors introduced the related notions of *finite approximability* and *finite forcibility* of permutation parameters.

A parameter $f$ is *finitely forcible* if there exists a finite family of permutations $\mathcal{A}$ such that for every $\varepsilon > 0$ there exist an integer $n_0$ and a real $\delta > 0$ such that if $\sigma$ and $\pi$ are permutations of order at least $n_0$ satisfying $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}$, then $|f(\sigma) - f(\pi)| < \varepsilon$. The set $\mathcal{A}$ is referred to as a *forcing family for $f$*.

A permutation parameter $f$ is *finitely approximable* if for every $\varepsilon > 0$ there exist $\delta > 0$, an integer $n_0$ and a finite family of permutations $\mathcal{A}_\varepsilon$ such that, if $\sigma$ and $\pi$ are permutations of order at least $n_0$ satisfying $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}_\varepsilon$, then $|f(\sigma) - f(\pi)| < \varepsilon$.

In [54], it was proved that a bounded permutation parameter is testable if and only if it is finitely approximable and the authors asked, whether such a parameter is also finitely forcible. In Chapter 5, we show that this is not the case.

**Theorem 8.** *There exists a bounded permutation parameter $f$ that is finitely approximable but not finitely forcible.*

Informally speaking, we utilize the proof methods used in Chapter 4 and we construct a permutation parameter that oscillates but the level of oscillation is bounded and the parameter is still testable (though it fails to be finitely forcible).

# Chapter 2

# Finitely forcible graphons and permutons

In this chapter, we discuss finite forcibility of graphons and permutons. We start with proving Theorem 2, which is an analogue of the result of Lovász and Sós [64] for permutons. We then focus on finite forcibility of graphons and permutons with infinite recursive structure. We show that there exist finitely forcible permutons such that the associated graphons are not finitely forcible. In Section 2.3, we prove finite forcibility of permutons $\Omega_{\to\alpha}$ and $\Upsilon_{\to\alpha}$ for $\alpha \in (0,1)$. In Section 2.4, we show that all graphons with analogous recursive structure, including the graphons associated with $\Omega_{\to\alpha}$ and $\Upsilon_{\to\alpha}$, are not finitely forcible.

## 2.1 Cumulative distribution function

For a permuton $\Phi$, let $F_\Phi$ be the function from $[0,1]^2$ to $[0,1]$ defined as $F_\Phi(x,y) = \Phi([0,x] \times [0,y])$. In other words, if we view $\Phi$ as a probability measure, $F_\Phi$ is its joint cumulative distribution function. For example, for $\Phi = \Upsilon$, we have $F_\Phi(x,y) = xy$. Observe that $F_\Phi$ is always a continuous function satisfying $F_\Phi(x,1) = F_\Phi(1,x) = x$ for every $x \in [0,1]$. Furthermore, notice that $\Phi \neq \Phi'$ implies $F_\Phi \neq F_{\Phi'}$, that is, the function $F_\Phi$ determines the permuton $\Phi$.

The next theorem was implicitly proven in [61]. We include its proof for completeness.

**Theorem 9.** *Let $p(x,y)$ be a polynomial and $k$ a non-negative integer. There exist a finite set $S$ of permutations and coefficients $\gamma_\sigma$, $\sigma \in S$, such that*

$$\int_{[0,1]^2} p(x,y) F_\Phi^k(x,y) \mathrm{d}\lambda = \sum_{\sigma \in S} \gamma_\sigma t(\sigma, \Phi) \tag{2.1}$$

*for every permuton $\Phi$.*

*Proof.* By additivity, it is sufficient to consider the case $p(x,y) = x^\alpha y^\beta$ for non-negative integers $\alpha$ and $\beta$. Fix a permuton $\Phi$. Since $\Phi$ has uniform marginals, the product $x^\alpha y^\beta F^k_\Phi(x,y)$ for $(x,y) \in [0,1]^2$ is equal to the probability that out of $\alpha+\beta+k$ points are chosen randomly independently based on $\Phi$, the first $\alpha$ points belong to $[0,x]\times[0,1]$, the next $\beta$ points belong to $[0,1]\times[0,y]$, and the last $k$ points belong to $[0,x]\times[0,y]$. So, the integral in (2.1) is equal to the probability that the above holds for a uniform choice of a point $(x,y)$ in $[0,1]^2$.

Since $\Phi$ is a measure with uniform marginals, a point $(x,y)$ uniformly distributed in $[0,1]^2$ can be obtained by sampling two points randomly independently based on $\Phi$ and setting $x$ to be the first coordinate of the first of these two points and $y$ to be the second coordinate of the second point. Thus, we can consider the following random event. Let us choose $\alpha+\beta+k+2$ points independently at random based on $\Phi$ and denote by $x$ the first coordinate of the last but one point, and by $y$ is the second coordinate of the last point. Then the integral on the left hand side of (2.1) is equal to the probability that the first $\alpha$ points belong to $[0,x]\times[0,1]$, the next $\beta$ points belong to $[0,1]\times[0,y]$, and the following $k$ points belong to $[0,x]\times[0,y]$. We conclude that the equation (2.1) holds with $S = S_{\alpha+\beta+k+2}$ and $\gamma_\sigma$ equal to the probability that the following holds for a random permutation $\pi$ of order $\alpha+\beta+k+2$: $\pi(i) \le \pi(\alpha+\beta+k+1)$ for $i \le \alpha$ and for $\alpha+\beta+1 \le i \le \alpha+\beta+k$, and $\sigma(\pi(i)) \le \sigma(\pi(\alpha+\beta+k+2))$ for $\alpha+1 \le i \le \alpha+\beta+k$. □

Instead of sampling two additional points to get a random point with respect to the uniform measure on $[0,1]^2$, we can also sample just a single point, which is a random point with respect to $\Phi$. This gives the following.

**Theorem 10.** *Let $p(x,y)$ be a polynomial and $k$ a non-negative integer. There exist a finite set $S$ of permutations and coefficients $\gamma_\sigma$, $\sigma \in S$, such that*

$$\int\limits_{[0,1]^2} p(x,y) F^k_\Phi(x,y) \mathrm{d}\Phi = \sum_{\sigma\in S} \gamma_\sigma t(\sigma,\Phi) \tag{2.2}$$

*for every permuton $\Phi$.*

Let now $\overline{S}_k$ be the set of permutations of order $k$ with one distinguished element; we call such permutations *rooted*. To denote rooted permutations, we add a bar above the distinguished element: e.g., if the second element of the permutation 2341 is distinguished, we write $2\overline{3}41$. Note that $\left|\overline{S}_k\right| = k!\cdot k$. For $\sigma \in \overline{S}_k$, let $F^\sigma_\Phi(x,y)$ be the probability that the point $(x,y)$ and $k-1$ points randomly independently chosen based on $\Phi$ induce the permutation $\sigma$ with the distinguished element corresponding to the point $(x,y)$. Observe that $F_\Phi(x,y) = F^{1\overline{2}}_\Phi(x,y)$, $F^{1\overline{2}}_\Phi(x,y) + F^{2\overline{1}}_\Phi(x,y) = x$ and $F^{1\overline{2}}_\Phi(x,y) + F^{\overline{2}1}_\Phi(x,y) = y$.

A reader familiar with the concept of flag algebras developed by Razborov [74] might recognize the notion of 1-labelled flags in the just introduced notation.

Similarly to Theorem 10, the following is true. Since the proof is completely analogous to that of Theorem 9, we decided to state the theorem without giving its proof.

**Theorem 11.** *Let $\Sigma$ be a multiset of rooted permutations. There exist a finite set $S$ of permutations and coefficients $\gamma_\sigma$, $\sigma \in S$, such that*

$$\int\limits_{[0,1]^2} \prod_{\sigma\in\Sigma} F^\sigma_\Phi(x,y)\mathrm{d}\Phi = \sum_{\sigma\in S} \gamma_\sigma t(\sigma,\Phi) \tag{2.3}$$

*for every permuton $\Phi$.*

## 2.2 Permutons with finite structure

In this section, we give a sufficient condition on a permuton to be finitely forcible. A function $f : [0,1]^2 \to \mathbb{R}$ is called piecewise polynomial if there exist finitely many polynomials $p_1, \dots, p_k$ such that $f(x,y) \in \{p_1(x,y), \dots, p_k(x,y)\}$ for every $(x,y) \in [0,1]^2$.

**Theorem 12.** *Every permuton $\Phi$ such that $F_\Phi$ is piecewise polynomial is finitely forcible.*

*Proof.* Let $\Phi$ be a permuton such that $F_\Phi$ is piecewise polynomial, that is, there exist polynomials $p_1, \dots, p_k$ such that $F_\Phi(x,y) \in \{p_1(x,y), \dots, p_k(x,y)\}$ for every $(x,y) \in [0,1]^2$. Let $\mathcal{F}$ be the set of all continuous functions $f$ on $[0,1]^2$ such that $f(x,y) \in \{p_1(x,y), \dots, p_k(x,y)\}$ for every $(x,y) \in [0,1]^2$. The set $\mathcal{F}$ is finite. Indeed, let

$$q(x,y) = \prod_{1\le i<j\le k} (p_j(x,y) - p_i(x,y))$$

and let $Q$ be the set of all points $(x,y) \in \mathbb{R}^2$ such that $q(x,y) = 0$. By Harnack's curve theorem, the set $Q$ has finitely many connected components. Bézout's theorem implies that the number of branching points in each of these components is finite and these points have finite degrees. Consequently, $\mathbb{R}^2 \setminus Q$ has finitely many components. If $A_1, \dots, A_\ell$ are all the connected components of $[0,1]^2 \setminus Q$, then each function $f \in \mathcal{F}$ coincides with one of the $k$ polynomials $p_1, \dots, p_k$ on every $A_i$. So, $|\mathcal{F}| \le k^\ell$.

Observe that the function $F_\Phi(x,y)$ is continuous since the measure $\Phi$ has uniform marginals. By the Stone-Weierstrass theorem, there exist a polynomial $p(x,y)$ and $\varepsilon > 0$ such that

$$\int\limits_{[0,1]^2} (F_\Phi(x,y) - p(x,y))^2 \,\mathrm{d}\lambda \quad < \quad \varepsilon \text{ , and} \tag{2.4}$$

$$\int\limits_{[0,1]^2} (f(x,y) - p(x,y))^2 \,\mathrm{d}\lambda \quad > \quad \varepsilon \text{ for every } f \in \mathcal{F},\ f \neq F_\Phi. \tag{2.5}$$

Let $\varepsilon_0$ be the value of the left hand side of (2.4). We claim that the unique permuton $\Phi'$ satisfying

$$\int\limits_{[0,1]^2} \prod_{i=1}^{k} (F_{\Phi'}(x,y) - p_i(x,y))^2 \,\mathrm{d}\lambda \quad = \quad 0 \text{ , and} \tag{2.6}$$

$$\int\limits_{[0,1]^2} (F_{\Phi'}(x,y) - p(x,y))^2 \,\mathrm{d}\lambda \quad = \quad \varepsilon_0 \tag{2.7}$$

is $\Phi$. Assume that $\Phi'$ is a permuton satisfying both (2.6) and (2.7). The equation (2.6) implies that $F_{\Phi'} \in \mathcal{F}$. Next, (2.5), (2.7), and (2.4) yield that $F_{\Phi'} \neq f$ for every $f \in \mathcal{F}$, $f \neq F_\Phi$. We conclude that $F_{\Phi'} = F_\Phi$ and thus $\Phi' = \Phi$.

By Theorem 9, the left hand sides of (2.6) and (2.7) can be expressed as finite linear combinations of densities $t(\sigma, \Phi)$. Let $S$ be the set of all permutations appearing in these linear combinations. Any permuton $\Phi'$ with $t(\sigma, \Phi') = t(\sigma, \Phi)$ for every $\sigma \in S$ satisfies both (2.6) and (2.7) and thus it must be equal to $\Phi$. This shows that $\Phi$ is finitely forcible. $\square$

We immediately obtain the following Theorem 2, which we restate below.

**Theorem 2.** *If $\Phi$ is a permuton satisfying $\Phi = \sum_{i\in[k]} \alpha_i \Upsilon_{A_i}$ for some non-negative reals $\alpha_1, \ldots, \alpha_k$ and some non-trivial polygons $A_1, \ldots, A_k \subseteq [0,1]^2$, then $\Phi$ is finitely forcible.*

*Proof.* Let $F_i$, $i \in [k]$, be the function from $[0,1]^2$ to $[0,1]$ defined as $F_i(x,y) = \Upsilon_{A_i}([0,x] \times [0,y])$. Clearly, each function $F_i$ is piecewise polynomial. Since $F_\Phi = \sum_i^k \alpha_i F_i$, the finite forcibility of $\Phi$ follows from Theorem 12. $\square$

A particular case of permutons that are finitely forcible by Theorem 2 is the following. If $k$ is an integer, $z_1, \ldots, z_k \in [0,1]$ are reals summing to one and $M$ is a square matrix of order $k$ with entries being non-negative reals summing

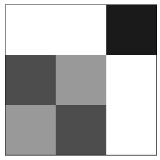

Figure 2.1: The permuton $\Phi_M$ constructed as an example at the end of Section 2.2. The gray area in the picture is the support of the measure and different shades correspond to the density of the measure.

to $z_i$ in the $i$-th row and in the $i$-th column, we can define a permuton $\Phi_M$ as

$$\Phi_M = \sum_{i,j=1}^{k} M_{ij} \Upsilon_{A_{ij}} ,$$

where $A_{ij} = [s_{i-1}, s_i] \times [s_{j-1}, s_j]$, $i, j \in [k]$ and $s_i = z_1 + \cdots + z_i$ (in particular, $s_0 = 0$ and $s_k = 1$). For instance, if $z_1 = z_2 = z_3 = 1/3$ and

$$M = \begin{pmatrix} 0 & 0 & 1/3 \\ 2/9 & 1/9 & 0 \\ 1/9 & 2/9 & 0 \end{pmatrix} ,$$

we get the permuton depicted in Figure 2.1.

## 2.3 Permutons with infinite structure

In this section, we prove Theorem 3 which asserts that permutons with infinite structure $\Omega_{\to\alpha}$ and $\Upsilon_{\to\alpha}$ are finitely forcible for every $\alpha \in (0, 1)$. We start with proving finite forcibility of $\Omega_{\to\alpha}$. We prove finite forcibility of $\Upsilon_{\to\alpha}$ later in this section as Theorem 16.

**Theorem 13.** *For every $\alpha \in (0, 1)$, the permuton $\Omega_{\to\alpha}$ is finitely forcible.*

*Proof.* We claim that any permuton $\Phi$ satisfying

$$t(231, \Phi) + t(312, \Phi) = 0 , \tag{2.8}$$

$$t(21, \Phi) = (1 - \alpha)^2 \sum_{i=0}^{\infty} \alpha^{2i} , \text{ and} \tag{2.9}$$

$$\int_{[0,1]^2} \left(1 - x - y + F_\Phi(x, y) - \tfrac{\alpha}{1-\alpha} \left(x + y - 2F_\Phi(x, y)\right)\right)^2 \mathrm{d}\Phi = 0 \tag{2.10}$$

is equal to $\Omega_{\to\alpha}$. This would prove the finite forcibility of $\Omega_{\to\alpha}$ by Theorem 10. Note that the permuton $\Omega_{\to\alpha}$ satisfies (2.8), (2.9), and (2.10).

Assume that a permuton $\Phi$ satisfies (2.8), (2.9), and (2.10). Let $X$ be the support of $\Phi$ and consider the binary relation $R$ defined on the support of $\Phi$ such that $(x, y)R(x', y')$ if

- $x = x'$ and $y = y'$, or
- $x < x'$ and $y > y'$, or
- $x > x'$ and $y < y'$.

The relation $R$ is an equivalence relation. Indeed, the reflexivity and symmetry is clear. To prove transitivity, consider three points $(x, y)$, $(x', y')$ and $(x'', y'')$ such that $(x, y)R(x', y')$ and $(x', y')R(x'', y'')$ but it does not hold that $(x, y)R(x'', y'')$. By the definition of $R$, either $x < x'$ and $x'' < x'$, or $x > x'$ and $x'' > x'$. If $x < x'$ and $x'' < x'$, then we obtain that $t(231, \Phi) > 0$ unless $x = x''$ (recall that $R$ is defined on the support of $\Phi$). We can now assume that $x = x''$ and $y < y''$. Since $\Phi$ has uniform marginals, the support of $\Phi$ intersects at least one of the open rectangles $(0, x) \times (y, y'')$, $(x, x') \times (y, y'')$ and $(x', 1) \times (y, y'')$. However, this yields that $t(231, \Phi) > 0$ in the first two cases and $t(312, \Phi) > 0$ in the last case. The case $x > x'$ and $x'' > x'$ is handled in an analogous way.

Let $\mathcal{R}$ be the set of equivalence classes of $R$. If $A \in \mathcal{R}$, let $A_x$ and $A_y$ be the projections of $A$ on the $x$ and $y$ axes. It is not hard to show that $A_x$ is a closed interval for each $A \in \mathcal{R}$ and these intervals are internally disjoint for different choices of $A \in \mathcal{R}$. The same holds for the projections on the $y$ axis. Since $\Phi$ has uniform marginals, the intervals $A_x$ and $A_y$ must have the same length for every $A \in \mathcal{R}$. Moreover, the definition of $R$ implies that if $A_x$ precedes $A'_x$, then $A_y$ also precedes $A'_y$ for any $A, A' \in \mathcal{R}$. We conclude that there exists a set $\mathcal{I}$ of internally disjoint closed intervals such that

$$\bigcup_{[z,z']\in\mathcal{I}} \left[z, z'\right] = [0, 1] \text{ and}$$

the support of $\Phi$ is equal to (because the density of subpermutations 231 and 312 is zero and the measure $\Phi$ has uniform marginals)

$$\bigcup_{[z,z']\in\mathcal{I}} \left\{(x, z' - x + z), x \in \left[z, z'\right]\right\} .$$

Note that some intervals contained in $\mathcal{I}$ may be formed by single points. Let $\mathcal{I}_0$ be the subset of $\mathcal{I}$ containing the intervals of positive length.

Let $[z, z'] \in \mathcal{I}_0$ and let $I = \{(x, z' - x + z), x \in [z, z']\}$. Since $\Phi([0, x] \times [0, y]) = \Phi([0, z] \times [0, z])$ and the measure $\Phi$ has uniform marginals, it follows that $F_\Phi(x, y) = z$. The equality (2.10) implies that the (continuous) function integrated in (2.10) is zero for every $(x, y) \in I$. Substituting $x + y = z + z'$ and $F_\Phi(x, y) = z$ into this function implies

$$z' = z + (1 - \alpha)(1 - z) . \tag{2.11}$$

Let $Z$ be the set formed by the left end points of intervals in $\mathcal{I}_0$. Define $z_1$ to be the minimum element of $Z$, and in general $z_i$ to be the minimum element of $Z \setminus \bigcup_{j<i} \{z_j\}$. The existence of these elements follows from (2.11) and the fact that the intervals in $\mathcal{I}_0$ are internally disjoint. If $Z$ is finite, we set $z_k = 1$ for $k > |Z|$. We derive from the definition of $Z$ and from (2.11) that

$$\mathcal{I}_0 \quad = \quad \left\{[z_i, z_i + (1 - \alpha)(1 - z_i)],\ i \in \mathbb{N}^+\right\} \setminus \{[1, 1]\} \ .$$

Consequently, we obtain

$$t(21, \Phi) = \sum_{i=1}^{\infty} (1 - \alpha)^2 (1 - z_i)^2 = (1 - \alpha)^2 \sum_{i=1}^{\infty} (1 - z_i)^2 \ . \tag{2.12}$$

For $j \in \mathbb{N}$, we define $\beta_j \in [0, 1]$ as follows:

$$\beta_j = \begin{cases} 1 - z_1 & \text{for } j = 1, \\ \frac{1-z_j}{\alpha(1-z_{j-1})} & \text{if } z_j \neq 1 \text{ and } j > 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The equation (2.12) can now be rewritten as

$$t(21, \Phi) = (1 - \alpha)^2 \sum_{i=1}^{\infty} \alpha^{2(i-1)} \prod_{j=1}^{i} \beta_j^2 \ . \tag{2.13}$$

Hence, the equality (2.9) can hold only if $\beta_j = 1$ for every $j$ which implies that $z_i = 1 - \alpha^{i-1}$. Consequently, the permutons $\Phi$ and $\Omega_{\to\alpha}$ are identical. $\square$

We remark that any permuton $\Phi$ obeying the constraints (2.8) and (2.10) must also be equal to $\Omega_{\to\alpha}$. However, we decided to include the additional constraint (2.9) to make the presented arguments more straightforward.

Next, we show finite forcibility of the permuton $\Upsilon_{\to\alpha}$ for every $\alpha \in (0, 1)$. Its structure is similar to that of $\Omega_{\to\alpha}$. The proof proceeds along similar lines as the proof of Theorem 13 but we have to overcome several new technical difficulties.

We start by proving an auxiliary lemma.

**Lemma 14.** *There exist a finite set $S$ of permutations and reals $\gamma_\sigma$, $\sigma \in S$, such that the following is equivalent for every permuton $\Phi$:*

- $\sum\limits_{\sigma \in S} \gamma_\sigma t(\sigma, \Phi) = 0$,
- *$\Phi$ restricted to $[x_1, x_2] \times [y_2, y_1]$ is a (possibly zero) multiple of $\Upsilon_{[x_1,x_2]\times[y_2,y_1]}$ for any two points $(x_1, y_1)$ and $(x_2, y_2)$ of the support of $\Phi$ with $x_1 < x_2$ and $y_1 > y_2$.*

*Proof.* The proof technique is similar to that used in [61]. For intervals $I, J \subseteq [0, 1]$ and $A = I \times J$, let $\upsilon_A(X) = \upsilon(X \cap A)$ for every Borel set $X \subseteq [0, 1]^2$. Equivalently, $\upsilon_A(X) = \upsilon(A) \cdot \Upsilon_A(X)$. Let $(x_1, y_1)$ and $(x_2, y_2)$ be two points of the support of $\Phi$ with $x_1 < x_2$ and $y_1 > y_2$. By Cauchy-Schwarz inequality, the measure $\Phi$ restricted $[x_1, x_2] \times [y_2, y_1]$ is a multiple of $\upsilon_{[x_1,x_2]\times[y_2,y_1]}$ if and only if it holds that

$$\left(\int\limits_{(x,y)} \Phi([x_1, x] \times [y_2, y])^2 \, \mathrm{d}\upsilon_{[x_1,x_2]\times[y_2,y_1]}\right) \times$$
$$\left(\int\limits_{(x,y)} (x - x_1)^2 (y - y_2)^2 \, \mathrm{d}\upsilon_{[x_1,x_2]\times[y_2,y_1]}\right) -$$
$$\left(\int\limits_{(x,y)} (x - x_1)(y - y_2)\Phi([x_1, x] \times [y_2, y]) \, \mathrm{d}\upsilon_{[x_1,x_2]\times[y_2,y_1]}\right)^2 = 0 \qquad (2.14)$$

Since the left hand side of (2.14) cannot be negative, we obtain that the second statement in the lemma is equivalent to

$$\int\limits_{(x_1,y_1)} \int\limits_{(x_2,y_2)} \int\limits_{(x,y)} \int\limits_{(x',y')} (x' - x_1)^2 (y' - y_2)^2 \cdot \Phi\left([x_1, x] \times [y_2, y]\right)^2 -$$
$$(x - x_1)(y - y_2) \cdot \Phi\left([x_1, x] \times [y_2, y]\right) \cdot (x' - x_1)(y' - y_2) \cdot \Phi([x_1, y_2] \times [x', y'])$$
$$\mathrm{d}\upsilon_{[x_1,x_2]\times[y_2,y_1]} \, \mathrm{d}\upsilon_{[x_1,x_2]\times[y_2,y_1]} \, \mathrm{d}\Phi \, \mathrm{d}\Phi \;=\; 0 \quad (2.15)$$

In the rest of the proof, we show that the left hand side of (2.15) can be expressed as a linear combination of finitely many subpermutation densities. Since this argument follows the lines of the proofs of Theorems 9–11, we only briefly explain the main steps.

The left hand side of (2.15) is equal to the expected value of the integrated function in (2.15) for two points $(x_1, y_1)$ and $(x_2, y_2)$ randomly chosen in $[0, 1]^2$ based on $\Phi$ and two points $(x, y)$ and $(x', y')$ randomly chosen in $[0, 1]^2$ based on $\Upsilon$ when treating the value of the integrated function to be zero if

$x_1 \geq x_2$, $y_1 \geq y_2$, $x \notin [x_1, x_2]$, $x' \notin [x_1, x_2]$, $y \notin [y_1, y_2]$, or $y' \notin [y_1, y_2]$. Such points $(x_1, y_1)$, $(x_2, y_2)$, $(x, y)$, and $(x', y')$ can be obtained by sampling six random points from $[0, 1]^2$ based on $\Phi$ since $\Phi$ has uniform marginals (see the proof of Theorem 9 for more details). When the four points $(x_1, y_1)$, $(x_2, y_2)$, $(x, y)$, and $(x', y')$ are sampled, any of the quantities $x_1$, $y_2$, $x$, $y$, $x'$, $y'$, $\Phi([x_1, y_2] \times [x, y])$, and $\Phi([x_1, y_2] \times [x', y'])$ appearing in the product is equal to the probability that a point randomly chosen in $[0, 1]^2$ based on $\Phi$ has a certain property in a permutation determined by the sampled points. Since we need to sample six additional points to be able to determine each of the products appearing in (2.15),the left hand side of (2.15) is equal to a linear combination of densities of 12-element permutations with appropriate coefficients. We conclude that the lemma holds with $S = S_{12}$. $\square$

Analogously, one can prove the following lemma. Since the proof follows the lines of the proof of Lemma 14, we omit further details.

**Lemma 15.** *There exist a finite set $S$ of permutations and reals $\gamma_\sigma$, $\sigma \in S$ such that the following is equivalent for every permuton $\Phi$:*

- $\sum_{\sigma \in S} \gamma_\sigma t(\sigma, \Phi) = 0$,
- *if $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are three points of the support of $\Phi$ with $x_1 < x_2 < x_3$ and $y_2 < y_3 < y_1$, then $\Phi$ restricted to $[x_2, x_3] \times [y_2, y_3]$ is a (possibly zero) multiple of $\Upsilon_{[x_2,x_3]\times[y_2,y_3]}$.*

We are now ready to show that each permuton $\Upsilon_{\to\alpha}$, $\alpha \in (0, 1)$, is finitely forcible.

**Theorem 16.** *For every $\alpha \in (0, 1)$, the permuton $\Upsilon_{\to\alpha}$ is finitely forcible.*

*Proof.* Let $S$ be the union of the two sets of permutations from Lemma 14 and Lemma 15. Next, consider the following eight functions:

$$\begin{aligned}
F_\Phi^{\nwarrow}(x, y) &= F_\Phi^{2\bar{1}}(x, y) , & f_\Phi^{\nwarrow}(x, y) &= F_\Phi^{23\bar{1}}(x, y) + F_\Phi^{32\bar{1}}(x, y) , \\
F_\Phi^{\nearrow}(x, y) &= F_\Phi^{\bar{1}2}(x, y) , & f_\Phi^{\nearrow}(x, y) &= F_\Phi^{\bar{2}31}(x, y) , \\
F_\Phi^{\swarrow}(x, y) &= F_\Phi^{1\bar{2}}(x, y) , & f_\Phi^{\swarrow}(x, y) &= F_\Phi^{31\bar{2}}(x, y) , \\
F_\Phi^{\searrow}(x, y) &= F_\Phi^{\bar{2}1}(x, y) , & f_\Phi^{\searrow}(x, y) &= F_\Phi^{\bar{3}12}(x, y) + F_\Phi^{\bar{3}21}(x, y) .
\end{aligned}$$

To save space in what follows, we often omit parameters when no confusion can arise, e.g., we write $F_\Phi^{\searrow}$ for the value $F_\Phi^{\searrow}(x, y)$ if $x$ and $y$ are clear from the context.

We claim that any permuton satisfying the following three conditions is equal to $\Upsilon_{\to\alpha}$:

$$t(\sigma,\Phi) = t(\sigma,\Upsilon_{\to\alpha}) \text{ for every } \sigma \in S, \tag{2.16}$$

$$\begin{aligned} \int\limits_{[0,1]^2} \Big( (1-\alpha)\left(F_\Phi^{\nearrow} f_\Phi^{\searrow} - F_\Phi^{\searrow} f_\Phi^{\nearrow}\right) f_\Phi^{\nwarrow} & \\ - \alpha\big(F_\Phi^{\nwarrow} f_\Phi^{\nwarrow} f_\Phi^{\searrow} + F_\Phi^{\searrow} f_\Phi^{\nwarrow} f_\Phi^{\searrow} + F_\Phi^{\nwarrow} f_\Phi^{\swarrow} f_\Phi^{\searrow} + F_\Phi^{\searrow} f_\Phi^{\nwarrow} f_\Phi^{\nearrow}\big)\Big)^2 \,\mathrm{d}\Phi &= 0\,, \end{aligned} \tag{2.17}$$

and

$$t(21,\Phi) = \frac{(1-\alpha)^2}{2}\sum_{i=0}^{\infty}\alpha^{2i}\,. \tag{2.18}$$

This would prove the finite forcibility of $\Upsilon_{\to\alpha}$ by Theorem 11.

Suppose that a permuton $\Phi$ satisfies (2.16), (2.17), and (2.18). Let $X$ be the support of $\Phi$ and consider the binary relation $R$ defined on the support of $\Phi$ such that $(x,y)R(x',y')$ if

- $x = x'$ and $y = y'$,
- $x < x'$ and $y > y'$, or
- $x > x'$ and $y < y'$.

Unlike in the proof of Theorem 13, the relation $R$ need not be an equivalence relation. Instead, we consider the transitive closure $R_0$ of $R$ and let $\mathcal{R}$ be the set of the equivalence classes of $R_0$.

We define $\rho((x,y),(x',y'))$, where $(x,y)$ and $(x',y')$ are two points of the support of $\Phi$ such that $(x,y)R(x',y')$, as follows

$$\rho\big((x,y),(x',y')\big) = \begin{cases} \frac{\Phi([x,x']\times[y',y])}{(x'-x)(y-y')} & \text{if } x < x' \text{ and } y > y', \\ \frac{\Phi([x',x]\times[y,y'])}{(x-x')(y'-y)} & \text{if } x > x' \text{ and } y < y', \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Phi$ satisfies (2.16), Lemma 14 implies that any three points $(x,y)$, $(x',y')$ and $(x'',y'')$ of the support of $\Phi$ such that $(x,y)R(x',y')$ and $(x',y')R(x'',y'')$ satisfy $\rho((x,y),(x',y')) = \rho((x',y'),(x'',y''))$. In particular, $\rho((x,y),(x',y'))$ has the same value for all pairs of points $(x,y)$ and $(x',y')$ with $(x,y)R(x',y')$ lying in the same equivalence class of $R_0$. So, we may define $\rho(A)$ to be this common value for each equivalence class $A \in \mathcal{R}$ or for a closure of such class.

As in the proof of Theorem 13, we define $A_x$ and $A_y$ to be the projections of an equivalence class $A \in \mathcal{R}$ on the $x$ and $y$ axes. The definition of $R$ yields that $A_x$ and $A_y$ are closed intervals for all $A \in \mathcal{R}$ and these intervals are internally disjoint for different choices of $A \in \mathcal{R}$. Since $\Phi$ has uniform marginals, the intervals $A_x$ and $A_y$ must have the same length for every $A \in \mathcal{R}$. As in the proof of Theorem 13, we conclude that there exists a set $\mathcal{I}$ of internally disjoint closed intervals such that

$$\bigcup_{[z,z']\in\mathcal{I}} [z,z'] = [0,1] \ ,$$

the support of $\Phi$ is a subset of

$$\bigcup_{[z,z']\in\mathcal{I}} [z,z'] \times [z,z'] \ ,$$

and the interior of each of these squares intersects at most one class $A \in \mathcal{R}$. Since some intervals contained in $\mathcal{I}$ may be formed by single points, we define $\mathcal{I}_0$ to be the subset of $\mathcal{I}$ containing the intervals of positive length.

Let $[z,z'] \in \mathcal{I}_0$ and let $A$ be the closure of the corresponding equivalence class from $\mathcal{R}$. Let $f(x)$, $x \in [z,z']$, be the minimum $y$ such that $(x,y)$ belongs to $A$; similarly, $g(x)$ denotes the maximum such $y$.

Assume first that $\rho(A) > 0$. Since $\Phi$ has uniform marginals, it holds that $g(x) - f(x) = \rho(A)^{-1}$ for every $x \in (z,z')$. From (2.16) and Lemma 14 we see that the functions $f$ and $g$ are non-decreasing, and similarly (2.16) and Lemma 15 imply that $f$ and $g$ are non-increasing. We conclude that $A = ([z,z'] \times [z,z'])$ and $\rho(A) = (z'-z)^{-1}$.

Assume now that $\rho(A) = 0$. Lemma 15 and (2.16) imply that if $(x,y) \in (z,z') \times (z,z')$ belongs to the support of $\Phi$, then $\Phi([x,z'] \times [z,y]) = 0$ (otherwise, $\rho(A) > 0$). But then $(x,y)$ cannot be in relation $R$ with another point of the support of $\Phi$. So, we conclude that the case $\rho(A) = 0$ cannot appear.

The just presented arguments show the support of the measure $\Phi$ is equal to

$$\bigcup_{[z,z']\in\mathcal{I}} [z,z'] \times [z,z']$$

and the measure is uniformly distributed inside each square $[z,z'] \times [z,z']$, $[z,z'] \in \mathcal{I}_0$.

Let $[z,z']$ be one of the intervals from $\mathcal{I}_0$. Recall that we have argued that

$$\Phi\big([0,z] \times [0,z] \cup [z,z'] \times [z,z'] \cup [z',1] \times [z',1]\big) = 1$$

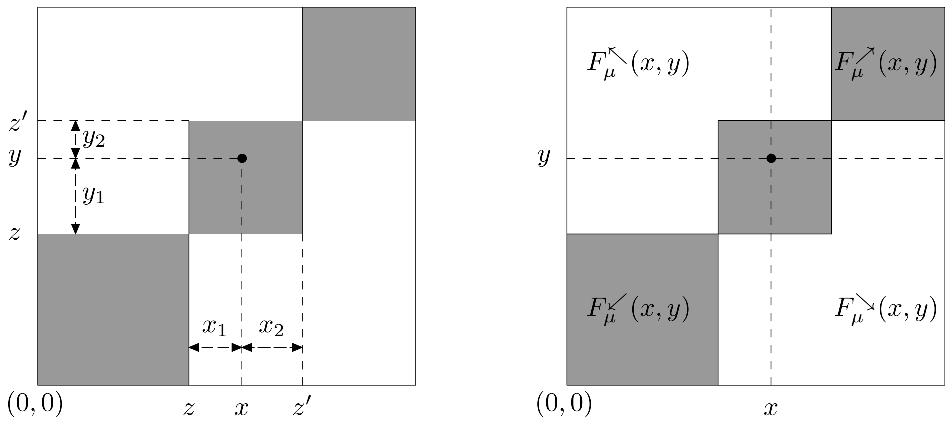


Figure 2.2: Notation used in equalities (2.20) and (2.21). Areas that can contain the support of $\Phi$ are drawn in grey.

and the measure $\Phi$ is uniform inside the square $[z,z'] \times [z,z']$ (see Figure 2.2). By (2.17), the following holds for almost every point of the support of $\Phi$:

$$(1-\alpha)\left(F_\Phi^{\nearrow} f_\Phi^{\searrow} - F_\Phi^{\searrow} f_\Phi^{\nearrow}\right) f_\Phi^{\nwarrow} = \alpha\big(F_\Phi^{\nwarrow} f_\Phi^{\nwarrow} f_\Phi^{\searrow} + F_\Phi^{\searrow} f_\Phi^{\nwarrow} f_\Phi^{\searrow} + F_\Phi^{\nwarrow} f_\Phi^{\swarrow} f_\Phi^{\searrow} + F_\Phi^{\searrow} f_\Phi^{\nwarrow} f_\Phi^{\nearrow}\big) . \quad (2.19)$$

In particular, this holds for all points in $[z,z'] \times [z,z']$ since the functions appearing in (2.19) are continuous.

Let $(x,y)$ be a point from $(z,z') \times (z,z')$. Let $x_1 = x - z$, $x_2 = z' - x$, $y_1 = y - z$, and $y_2 = z' - y$ (see Figure 2.2). Since all the quantities appearing in (2.19) are positive, we may rewrite (2.19) as

$$(1-\alpha)\left(F_\Phi^{\nearrow} - F_\Phi^{\searrow}\frac{f_\Phi^{\nearrow}}{f_\Phi^{\searrow}}\right) = \alpha\left(F_\Phi^{\nwarrow} + F_\Phi^{\searrow} + F_\Phi^{\nwarrow}\frac{f_\Phi^{\swarrow}}{f_\Phi^{\nwarrow}} + F_\Phi^{\searrow}\frac{f_\Phi^{\nearrow}}{f_\Phi^{\searrow}}\right) . \quad (2.20)$$

Observe that $F_\Phi^{\nearrow}(x,y) = \Phi([x,1] \times [y,1])$, $F_\Phi^{\nwarrow}(x,y) = \Phi([z,x] \times [y,z']) = \frac{x_1 y_2}{z'-z}$, and $F_\Phi^{\searrow}(x,y) = \Phi([x,z'] \times [z,y]) = \frac{x_2 y_1}{z'-z}$. Further observe that

$$\frac{f_\Phi^{\nearrow}(x,y)}{f_\Phi^{\searrow}(x,y)} = \frac{\frac{2x_2^2 y_1 y_2}{2(z'-z)^2}}{\frac{x_2^2 y_1^2}{(z'-z)^2}} = \frac{y_2}{y_1} \quad \text{and} \quad \frac{f_\Phi^{\swarrow}(x,y)}{f_\Phi^{\nwarrow}(x,y)} = \frac{\frac{2x_1^2 y_1 y_2}{2(z'-z)^2}}{\frac{x_1^2 y_2^2}{(z'-z)^2}} = \frac{y_1}{y_2} .$$

Plugging these observations in (2.20), we obtain that

$$(1-\alpha)\left(\Phi([x,1] \times [y,1]) - \frac{x_2 y_2}{z'-z}\right) = \alpha\frac{x_1 y_2 + x_2 y_1 + x_1 y_1 + x_2 y_2}{z'-z} . \quad (2.21)$$

Since $x_1 + x_2 = y_1 + y_2 = z' - z$ and $\frac{x_2 y_2}{z'-z} = \Phi([x, z'] \times [y, z'])$, we obtain from (2.21) that

$$(1 - \alpha)\Phi([z', 1] \times [z', 1]) = \alpha \frac{(z' - z)^2}{z' - z} = \alpha(z' - z) \; . \tag{2.22}$$

Finally, we substitute $1 - z'$ for $\Phi([z', 1] \times [z', 1])$ in (2.22) and get the following:

$$z' = z + (1 - \alpha)(1 - z) \; . \tag{2.23}$$

So, we conclude that the right end point of every interval in $\mathcal{I}_0$ is uniquely determined by its left end point.

Let $Z$ be the set formed by the left end points of intervals in $\mathcal{I}_0$. As in the proof of Theorem 13, for a positive integer $i$, let $z_i$ be the $i$-th smallest element of $Z$. Notice that the existence of minimum elements follows from (2.23). If $Z$ is finite, we set $z_k = 1$ for $k > |Z|$.

We derive from the definition of $Z$ and from (2.23) that

$$\mathcal{I}_0 \quad = \quad \left\{ [z_i, z_i + (1 - \alpha)(1 - z_i)], \; i \in \mathbb{N}^+ \right\} \setminus \{[1, 1]\} \; .$$

Consequently, we obtain

$$t(21, \Phi) = \sum_{i=1}^{\infty} \frac{(1 - \alpha)^2 (1 - z_i)^2}{2} = \frac{(1 - \alpha)^2}{2} \sum_{i=1}^{\infty} (1 - z_i)^2 \; . \tag{2.24}$$

Analogously to the proof of Theorem 13, for $j \in \mathbb{N}$, we define $\beta_j \in [0, 1]$ as follows:

$$\beta_j = \begin{cases} 1 - z_1 & \text{for } j = 1, \\ \frac{1 - z_j}{\alpha(1 - z_{j-1})} & \text{if } z_j \neq 1 \text{ and } j > 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The equation (2.24) can be rewritten as

$$t(21, \Phi) = \frac{(1 - \alpha)^2}{2} \sum_{i=1}^{\infty} \alpha^{2(i-1)} \prod_{j=1}^{i} \beta_j^2 \; . \tag{2.25}$$

Hence, the equality (2.18) can hold only if $\beta_j = 1$ for every $j$, i.e., $z_i = 1 - \alpha^{i-1}$. This implies that the permutons $\Phi$ and $\Upsilon_{\to\alpha}$ are identical. $\square$

## 2.4 Graphons with infinite structure

In this section, we show that graphons associated with the finitely forcible permutons from Section 2.3 are not finitely forcible. We start with graphons $W_{\Omega_{\to\alpha}}$ associated with permutons $\Omega_{\to\alpha}$, $\alpha \in (0,1)$.

### 2.4.1 Union of complete graphs

We now focus on graphons with $t(P_3, W) = 0$ where $P_3$ is the path on three vertices. We start with the following lemma, which seems to be of independent interest. Informally, the lemma asserts that any finitely forcible graphon with zero density of $P_3$ can be forced by finitely many densities of complete graphs.

**Lemma 17.** *If $W_0$ is a finitely forcible graphon and $t(P_3, W_0) = 0$, then there exists an integer $\ell_0$ such that any graphon $W$ with $t(P_3, W) = 0$ and $t(K_\ell, W) = t(K_\ell, W_0)$ for $\ell \le \ell_0$ is weakly isomorphic to $W_0$.*

*Proof.* To prove the statement of the lemma, it is enough to show the following claim: the density of any $n$-vertex graph $G$ in a graphon $W$ with $t(P_3, W) = 0$ can be expressed as a combination of densities of $K_1, \ldots, K_n$ in $W$. We proceed by induction on $n+k$ where $n$ and $k$ are the numbers of vertices and components of $G$ respectively. If $n = k = 1$, there exists only a single one-vertex graph $K_1$ and the claim holds.

Assume now that $n + k > 2$. If $G$ is not a union of complete graphs, then $t(G, W) = 0$ since $t(P_3, W) = 0$. So, we assume that $G$ is a union of $k$ complete graphs $G_1, \ldots, G_k$, i.e., $G = G_1 \cup \cdots \cup G_k$. If $k = 1$, then $G = K_n$ and the claim clearly holds. So, we assume $k > 1$.

For $2 \le i \le k$, we denote

$$H_i = (G_1 + G_i) \cup \bigcup_{j \in [k] \setminus \{1,i\}} G_j .$$

Observe that the following holds:

$$t(G_1, W) \cdot t(G_2 \cup \cdots \cup G_k, W) = p_1 \cdot t(G, W) + \sum_{i=2}^{k} p_i \cdot t(H_i, W) \tag{2.26}$$

where $p_1$ is the probability that a set $V$ of randomly chosen $|G_1|$ vertices of the graph $G$ induces a complete graph and the graph $G \setminus V$ is isomorphic to $G_2 \cup \cdots \cup G_k$, and $p_i$, $i > 1$, is the probability that a set $V$ of randomly chosen $|G_1|$ vertices of $H_i$ induces a complete graph and the graph $H_i \setminus V$ is isomorphic to $G_2 \cup \cdots \cup G_k$. To see (2.26), observe that $t(G_1, W) \cdot t(G_2 \cup \cdots \cup G_k, W)$

is equal to the product of the probability that a $W$-random graph of order $|G_1|$ is isomorphic to $G_1$ and the probability that a $W$-random graph of order $|G_2|+\cdots+|G_k|$ is isomorphic to $G_2\cup\cdots\cup G_k$. This is equal to the probability that randomly chosen $|G_1|$ vertices of a $W$-random graph of order $n$ induce a subgraph isomorphic to $G_1$ and the remaining vertices induce a subgraph isomorphic to $G_2\cup\cdots\cup G_k$. This probability is equal to the right hand side of (2.26).

By induction, $t(G_2\cup\cdots\cup G_k,W)$ and $t\left(H_i,W\right)$, $2\le i\le k$, can be expressed as combinations of densities of complete graphs of order at most $n$ in $W$. Rearranging the terms of (2.26), we obtain that $t(G,W)$ is equal to a combination of densities of complete graphs of order at most $n$ in $W$. $\square$

Let $U^\rho$ be a graphon identically equal to $\rho\in[0,1]$. The main result of this subsection asserts that unlike the permuton $\Omega_{\to\alpha}$, the associated graphon $W_{\Omega_{\to\alpha}}=U^1_{\to\alpha}$ is not finitely forcible. Although it immediately follows as a corollary of the more general Theorem 4, we give its proof here to increase the readability.

**Theorem 18.** *For every $\alpha\in(0,1)$, the graphon $U^1_{\to\alpha}$ is not finitely forcible.*

*Proof.* Observe that $d\left(P_3,U^1_{\to\alpha}\right)=0$. By Lemma 17, it is enough to show that $U^1_{\to\alpha}$ is not finitely forcible with $S=\{P_3,K_1,\dots,K_n\}$ for any $n\in\mathbb{N}$, i.e., by setting the densities of $P_3$ and the complete graphs of orders $1,\dots,n$. Suppose for the sake of contradiction that for some $n\in\mathbb{N}$ the graphon $U^1_{\to\alpha}$ is uniquely determined by the densities of $P_3$ and $K_1,\dots,K_n$. Let $a_i=(1-\alpha)\alpha^{i-1}$. Then, $U^1_{\to\alpha}=\bigoplus_{i\in\mathbb{N}}a_iU^1$. We define functions $F_i(x_1,\dots,x_{n+1}):\mathbb{R}^{n+1}\to\mathbb{R}$ for $i\in[n]$ as follows:

$$F_i(x_1,\dots,x_{n+1})=\sum_{j=1}^{n+1}\left(x_j^i-a_j^i\right)\ . \tag{2.27}$$

Observe that if it holds that $x_1+\cdots+x_{n+1}=a_1+\cdots+a_{n+1}$, which is equivalent to $F_1(x_1,\dots,x_{n+1})=0$, then it also holds that

$$t(K_i,\bigoplus_{i\in\mathbb{N}}b_iU^1)=t(K_i,U^1_{\to\alpha})+F_i(x_1,\dots,x_{n+1})\text{ for }i\in[n], \tag{2.28}$$

where $b_i=x_i$ for $i\le n+1$ and $b_i=a_i$ for $i>n+1$. Hence, to obtain the desired contradiction, it suffices to prove that there exist functions $g_j(x_{n+1})$, $j\in[n]$, on some open neighborhood of $a_{n+1}$ such that

$$F_i(g_1(x_{n+1}),\dots,g_n(x_{n+1}),x_{n+1})=0\text{ for every }i\in[n]. \tag{2.29}$$

Indeed, if such functions $g_j(x_{n+1})$, $j \in [n]$, exist, then (2.28) yields that the densities of $K_1, \dots, K_n$ in the graphon $\bigoplus_{i \in \mathbb{N}} b_i U^1$ with $b_i = g_i(x_{n+1})$ for $i \le n$, $b_{n+1} = x_{n+1}$ and $b_i = a_i$ for $i > n+1$ equal their densities in the graphon $U^1_{\to\alpha}$. This implies that $U^1_{\to\alpha}$ is not forced by the densities of $P_3$ and $K_1, \dots, K_n$.

We now establish the existence of functions $g_1, \dots, g_n$ satisfying (2.29) on some open neighborhood of $a_{n+1}$. Observe that

$$\frac{\partial F_i}{\partial x_j}(x_1, \dots, x_{n+1}) = i \cdot x_j^{i-1} \ .$$

We consider the Jacobian matrix of the functions $F_1, \dots, F_n$ with respect to $x_1, \dots, x_n$. The determinant of the Jacobian matrix is equal to

$$n! \prod_{1 \le j < j' \le n} \left(x_{j'} - x_j\right) \ . \tag{2.30}$$

Substituting $x_j = a_j$ for $j \in [n]$, we obtain that the Jacobian matrix has non-zero determinant. In particular, the Jacobian is non-zero. The Implicit Function Theorem now implies the existence of the functions $g_1, \dots, g_n$ satisfying (2.29). This concludes the proof. □

### 2.4.2 Union of random graphs

The graphons considered in the previous subsection were associated with the permutons $\Omega_{\to\alpha}$. We now focus on finite forcibility of graphons related to the permutons $\Upsilon_{\to\alpha}$.

In fact, we prove a more general result. Theorem 4, which we restate below, asserts that a graphon $W_{\to\alpha}$ is not finitely forcible unless $W$ is weakly isomorphic to $U^0$.

**Theorem 4.** *For every $\alpha \in (0,1)$ and every graphon $W$, if the graphon $W_{\to\alpha}$ is finitely forcible, then $W$ is equal to zero almost everywhere.*

*Proof.* It is enough to show that for every $n$, there exists a sequence $(b_i)_{i \in \mathbb{N}}$ different from $(a_i)_{i \in \mathbb{N}}$, $a_i = (1-\alpha)\alpha^{i-1}$, such that

$$t\left(G, \bigoplus_{i \in \mathbb{N}} b_i W\right) = t\left(G, W_{\to\alpha}\right) \text{ for every graph } G \text{ with } |G| \le n. \tag{2.31}$$

The proof of Theorem 18 yields that for every $n$, there exists such $(b_i)_{i \in \mathbb{N}}$ different from $(a_i)_{i \in \mathbb{N}}$, satisfying

$$t\left(G, \bigoplus_{i \in \mathbb{N}} b_i U^1\right) = t\left(G, U^1_\alpha\right) \text{ for every graph } G \text{ with } |G| \le n. \tag{2.32}$$

We claim that this $(b_i)_{i\in\mathbb{N}}$ also satisfies (2.31). Also note that (2.32) is non-zero only for graphs $G$ that are disjoint union of cliques.

Let $G$ be a graph with $n$ vertices and let $G_1,\ldots,G_k$ be the connected components of $G$. Furthermore, let $\mathcal{F}=\{I_1,\ldots,I_\ell\}$ be the partition of $[k]$ according to the isomorphism classes of the graphs $G_1,\ldots,G_k$, i.e., for every $i,j\in[\ell]$ with $i\neq j$ and every $a_1,a_2\in I_i$ and $a_3\in I_j$, the graphs $G_{a_1}$ and $G_{a_2}$ are isomorphic, and the graphs $G_{a_1}$ and $G_{a_3}$ are not isomorphic.

Observe that

$$t\left(G,\bigoplus_{i\in\mathbb{N}}b_iW\right)=\sum_{f:[k]\to\mathbb{N}}c(f)\left(\prod_{i=1}^{\infty}t\left(\bigcup_{j\in f^{-1}(i)}G_j,W\right)b_i^{\left|\bigcup_{j\in f^{-1}(i)}G_j\right|}\right)\tag{2.33}$$

with the normalizing factor

$$c(f)=\prod_{m\in[\ell]}\frac{\prod_{i=1}^{\infty}\left|f^{-1}(i)\cap I_m\right|!}{|I_m|!},$$

where we set $0!=1$ and the density $t(\emptyset,W)$ of the empty graph in the graphon $W$ to 1.

We consider partitions of the set of connected components of $G$. If $Q=\{Q_1,\ldots,Q_k\}$ is such a partition, we slightly abuse the notation and identify $Q_i$ with the subgraph of $G$ induced by the components of $Q_i$. In particular, $|Q_i|$ denotes the number of vertices in this subgraph.. Furthermore, we always view a partition $Q$ as a multiset, and also allow some of the $Q_i$'s to be empty. Let $\mathcal{Q}$ be the set of all such partitions. The identity (2.33) can now be rewritten as follows:

$$t\left(G,\bigoplus_{i\in\mathbb{N}}b_iW\right)=\sum_{Q\in\mathcal{Q}}\prod_{i\in[k]}t(Q_i,W)t\left(\bigcup_{i\in[k]}K_{|Q_i|},\bigoplus_{i\in\mathbb{N}}b_iU^1\right),\tag{2.34}$$

where $K_0$ is the empty graph. Since $(b_i)_{i\in\mathbb{N}}$ satisfies (2.32), we obtain that it satisfies (2.34), and therefore also (2.31). $\square$

We immediately obtain the following two corollaries.

**Corollary 19.** *For every $\alpha\in(0,1)$ and every $\rho\in(0,1]$, the graphon $U^\rho_{\to\alpha}$ is not finitely forcible.*

**Corollary 20.** *For every $\alpha\in(0,1)$, the graphon $W_{\Upsilon_{\to\alpha}}=(W_\Upsilon)_{\to\alpha}$, which is associated with the permuton $\Upsilon_{\to\alpha}$, is not finitely forcible.*

# Chapter 3

# Infinitely dimensional finitely forcible graphon

In this chapter we are concerned with the relation between finite forcibility and dimension of a space of typical vertices of a graphon. We prove Theorem 5 by constructing a graphon $W_{\boxplus}$, which we call a hypercubical graphon, such that $W_{\boxplus}$ is finitely forcible and both $T(W_{\boxplus})$ and $\overline{T}(W_{\boxplus})$ contain subspaces homeomorphic to $[0,1]^{\mathbb{N}}$.

Informally speaking, our approach is the following. The constructed hypercubical graphon $W_{\boxplus}$ has several parts (see Figure 3.1), which are determined by degrees of the vertices they contain. The parts $A_i$ serve to further partition the parts $B_i$ into infinitely many smaller parts and the part $C$ serves to introduce coordinate systems on the parts $B_i$ and $D$. Having this structure on the parts $B_i$ in place, we can force subspaces homeomorphic $[0,1]^d$ for every $d \in \mathbb{N}$ (corresponding to the parts $B_{1,d}$). Their structure is forced in an iterative (induction like) way, increasing dimension by one at each step. The proof is concluded by forcing the parts $B_{1,d}$ to be "projections" of another part of the graphon, the part $D$. The subspace homeomorphic to the part $D$ must then be homeomorphic to $[0,1]^{\mathbb{N}}$.

## 3.1 Finite forcibility

Following the framework from [42], when proving finite forcibility of a graphon, we give a set of *constraints* that uniquely determines $W$ instead of specifying the finitely many subgraphs and their densities that uniquely determine $W$. A *constraint* is an equality between two density expressions where a *density expression* is recursively defined as follows: a real number or a graph $H$ are density expressions, and if $D_1$ and $D_2$ are two density expression, then the sum

$D_1 + D_2$ and the product $D_1 \cdot D_2$ are also density expressions. The value of the density expression is the value obtained by substituting for every subgraph $H$ its density in the graphon. As observed in [42], if $W$ is a unique (up to weak isomorphism) graphon that satisfies a finite set $\mathcal{C}$ of constraints, then it is finitely forcible. In particular, W is the unique (up to weak isomorphism) graphon with densities of graphs appearing in $\mathcal{C}$ equal to their densities in $W$.

In [42], it was also observed that a more general form of constraints, called *rooted constraints*, can be used in finite forcibility. A subgraph is rooted if it has $m$ distinguished vertices labeled with numbers $1, \ldots, m$. These vertices are referred to as roots while the other vertices are non-roots. Two rooted graphs are *compatible* if the subgraphs induced by their roots are isomorphic through an isomorphism mapping the roots with the same label to each other. Similarly, two rooted graphs are isomorphic if there exists an isomorphism mapping the $i$-th root of one of them to the $i$-th root of the other.

A *rooted density* expression is a density expression such that all graphs that appear in it are mutually compatible rooted graphs. The meaning of a rooted density expression is defined in the next paragraphs.

Fix a rooted graph $H$. Let $H_0$ be the graph induced by the roots of $H$, and let $m = |H_0|$. For a graphon $W$ with $t(H_0, W) > 0$, we let the auxiliary function $c : [0,1]^m \to [0,1]$ denote the probability that an $m$-tuple $(x_1, \ldots, x_m) \in [0,1]$ induces a copy of $H_0$ in $W$ respecting the labeling of vertices of $H_0$:

$$c(x_1, \ldots, x_m) = \left( \prod_{(i,j) \in E(H_0)} W(x_i, x_j) \right) \cdot \left( \prod_{(i,j) \notin E(H_0)} (1 - W(x_i, x_j)) \right) .$$

We next define a probability measure $\mu$ on $[0,1]$. If $A \subseteq [0,1]^m$ is a Borel set, then:

$$\mu(A) = \frac{\int_A c(x_1, \ldots, x_m) \mathrm{d}\lambda_m}{\int_{[0,1]^m} c(x_1, \ldots, x_m) \mathrm{d}\lambda_m} .$$

When $x_1, \ldots, x_m \in [0,1]$ are fixed, then the density of a graph $H$ with root vertices $x_1, \ldots, x_m$ is the probability that a random sample of non-roots yields a copy of $H$ conditioned on the roots inducing $H_0$. Noticing that an automorphism of a rooted graph has all roots as fixed vertices, we obtain that this is equal to

$$\frac{(|H| - m)!}{|\mathrm{Aut}(H)|} \int_{[0,1]^{|H-m|}} \prod_{(i,j) \in E(H) \setminus E(H_0)} W(x_i, x_j) \prod_{(i,j) \notin E(H) \cup \binom{H_0}{2}} (1 - W(x_i, x_j)) \, \mathrm{d}\lambda_{|H|-m}.$$

We now consider a constraint such that both left and right hand sides $D$ and $D'$ are compatible rooted density expressions. Such a constraint represents that $D - D' = 0$ holds with probability one with respect to the choice of roots. At several occasions, we write a fraction of two rooted density expressions $D/D'$. A constraint containing such a fraction, say $D_1/D'_1 = D_2/D'_2$ should be interpreted as $D_1 \cdot D'_2 = D_2 \cdot D'_1$. It can be shown that for every rooted constraint $D = D'$, there exists a non-rooted constraint $C = C'$ such that $D = D'$ with probability one if and only if $C = C'$ holds [42].

A *degree* $\deg^W x$ of a vertex $x \in [0,1]$ in a graphon $W$ is equal to

$$\int_{[0,1]} W(x,y)\mathrm{d}y \ .$$

Note that the degree is well-defined for almost every vertex of $W$. We omit the superscript $W$ whenever the graphon is clear from the context.

Let $A$ be a measurable non-null subset of $[0,1]$. A *relative degree* $\deg_A^W x$ of a vertex $x \in [0,1]$ of a graphon $W$ in $A$ is equal to

$$\frac{\int_A W(x,y)\mathrm{d}y}{\lambda(A)} \ .$$

Fix a graphon $W$ and let $x, x' \in [0,1]$ and $Y \subseteq [0,1]$. Then $N_Y(x)$ denotes the set of $y \in Y$ such that $W(x,y) > 0$ and

$$N_Y(x \setminus x') = \{y \in Y | W(x,y) > 0 \text{ and } W(x',y) < 1\}.$$

If $Y$ is measurable, then $N_Y(x)$ is measurable for almost every $x$.

A graphon $W$ is *partitioned* if there exist $k \in \mathbb{N}$, positive reals $a_1, \dots, a_k$ summing to one and distinct reals $d_1, \dots, d_k$ between zero and one such that the set of vertices of $W$ with degree $d_i$ has measure $a_i$. If $W$ is a partitioned graphon, we write $A_i$ for the set of vertices of degree $d_i$ for $i \in [k]$.

A graph $H$ is *decorated* if its vertices are labeled with parts $A_1, \dots, A_k$. The density of a decorated graph $H$ is the probability that randomly chosen $|H|$ vertices induce a subgraph isomorphic to $H$ with its vertices contained in the parts corresponding to the labels, conditioned by the event that the sampled vertices are in the parts corresponding to the labels. For example, if $H$ is an edge with vertices labeled with parts $A_1$ and $A_2$, then the density of $H$ is the density of edges between $A_1$ and $A_2$, i.e.,

$$t(H,W) = \frac{1}{a_1 a_2} \int_{A_1} \int_{A_2} W(x,y) \, \mathrm{d}x \, \mathrm{d}y \ .$$

Similarly as in the case of non-decorated graphs, we can define rooted decorated subgraphs and use them in constraints. A constraint that uses (rooted or non-rooted) decorated subgraphs is referred to as decorated. In [42], it is shown that decorated constraints can be used in forcing (also see Lemma 22 below).

We depict roots of decorated graphs by squares, and non-root vertices by circles, labeled by the name of the respective part of a graphon. The full lines connecting vertices correspond edges, dashed lines to non-edges. No connection between two vertices means that both edge or non-edge are allowed between the vertices, i.e., the corresponding density is the sum of the densities of the graphs with and without such edge(s).

We omit the distinguishing labels of the roots. To avoid possible ambiguity, a drawing of the graph on the roots is identical for all occurrences of the graph in a constraint to make clear which roots correspond to each other.

We conclude this section by lemmas from [42], used in our proof and by introducing additional terminology for graphons. The first lemma guarantees the existence of a set of constraints that force a graphon satisfying these constraints to be a partitioned graphon with a given partition and given degrees between the parts.

**Lemma 21.** *Let $k \in \mathbb{N}$, let $a_1, \dots, a_k$ be positive reals summing to one and let $d_1, \dots, d_k$ be distinct reals between zero and one. There exists a finite set of constraints $\mathcal{C}$ such that a graphon $W$ satisfies $\mathcal{C}$ if and only if it is a partitioned graphon with parts of sizes $a_1, \dots, a_k$ and degrees $d_1, \dots, d_k$.*

The following lemma allows us to use decorated constraints by stating that they are equivalent to some non-decorated constraints.

**Lemma 22.** *Let $k \in \mathbb{N}$, let $a_1, \dots, a_k$ be positive reals summing to one and let $d_1, \dots, d_k$ be distinct reals between zero and one. If $W$ is a partitioned graphon with $k$ parts formed by vertices of degree $d_i$ and measure $a_i$ each, then any decorated (rooted or non-rooted) constraint can be expressed as a non-decorated non-rooted constraint, i.e., $W$ satisfies the decorated constraint if and only if it satisfies the non-decorated non-rooted constraint.*

Note that our definition of density of a decorated graph $H$ differs from the definition in [42]. However, density of a decorated graph $H$ in our sense is density of a decorated graph $H$ in the sense of [42] multiplied by the appropriate constant depending on measures of the parts of the graphon. Therefore, decorated constraints in our sense and in the sense of [42] are in one-to-one correspondence. Thus, Lemma 22 holds with our definition of decorated constraints, too.

Mimicking the terminology for graphs, we call a graphon $W$ restricted to $S \times T$ for $S, T \in [0,1]$, a *subgraphon* on $S \times T$ and we denote it by $W[S \times T]$. The *density* between $S$ and $T$ (or on $S \times T$) of a graphon $W$, is

$$\frac{\int_S \int_T W(x,y)\mathrm{d}x\mathrm{d}y}{\lambda(S)\lambda(T)} .$$

We say that $W[S \times S]$ for $S \subseteq [0,1]$ is a *clique* in a graphon $W$, if $W$ equals one almost everywhere on $S \times S$. A subgraphon $W[S \times T]$ of $W$ is a *complete bipartite subgraphon with sides $S$ and $T$* for some $S, T \subseteq [0,1]$, $S \cap T = \emptyset$ if $W$ equals one almost everywhere on $S \times T$. Similarly $W[S \times T]$ is a *pseudorandom bipartite subgraphon of density $p$* for $p \in [0,1]$ if $W$ equals $p$ almost everywhere on $S \times T$.

The following lemma from [42] states that we can force a pseudorandom bipartite subgraphon between any two parts of a partitioned graphon.

**Lemma 23.** *For every choice of $k \in \mathbb{N}$, positive reals $a_1, \dots, a_k$ summing to one, distinct reals $d_1, \dots, d_k$ between zero and one, $l, l' \leq k$, $l \neq l'$, and $p \in [0,1]$, there exists a finite set of constraints $\mathcal{C}$ such that every graphon $W$ that is a partitioned graphon with $k$ parts $A_1, \dots, A_k$ of measures $a_1, \dots, a_k$ and degrees $d_1, \dots, d_k$, respectively satisfies $\mathcal{C}$ if and only if it satisfies that $W(x,y) = p$ for almost every $x \in A_l$ and $y \in A_{l'}$.*

## 3.2 The hypercubical graphon

In this section, we describe a graphon $W_\boxplus$ which we call the *hypercubical graphon*. For readability, we include a sketch of the structure of $W_\boxplus$ in Figure 3.1.

The *hypercubical graphon* $W_\boxplus$ is a partitioned graphon with 14 parts, denoted by $A_0^\boxplus, \dots, A_3^\boxplus, B_1^\boxplus, \dots, B_5^\boxplus, C^\boxplus, D^\boxplus, E_1^\boxplus, E_2^\boxplus, F^\boxplus$. Each part has measure $1/27$ except for $E_1^\boxplus$ and $E_2^\boxplus$ that have measure $11/27$ and $4/27$, respectively. Degrees of the vertices in different parts are listed in Table 3.1. We do not provide the exact values $e_1$ and $e_2$ of degrees of vertices in $E_1^\boxplus$ and $E_2^\boxplus$, respectively. Instead, we observe that $e_1 \leq 10/27$ and $e_2 \leq 1/27$ (since the neighborhood of every vertex of $E_1^\boxplus$ or $E_2^\boxplus$ has neighborhood of measure at most $10/27$ or $1/27$, respectively), therefore, $e_1, e_2$ are different from degrees of vertices in parts $A_0^\boxplus, \dots, A_3^\boxplus, B_1^\boxplus, \dots, B_5^\boxplus, C^\boxplus$ and $e_2$ is also different from $D^\boxplus$ and $F^\boxplus$. Moreover, from the construction of $W_\boxplus$, it follows that $e_1 > 5/27$ and therefore it is different from $e_2$ and from degrees of vertices in $D^\boxplus$ and $F^\boxplus$.

We describe the graphon $W_\boxplus$ as a collection of functions $W_\boxplus^{X \times Y}$ on products of its parts $X, Y \in \{A_0^\boxplus, \dots A_3^\boxplus, B_1^\boxplus, \dots B_5^\boxplus, C^\boxplus, D^\boxplus, E_1^\boxplus, E_2^\boxplus, F^\boxplus\}$. For

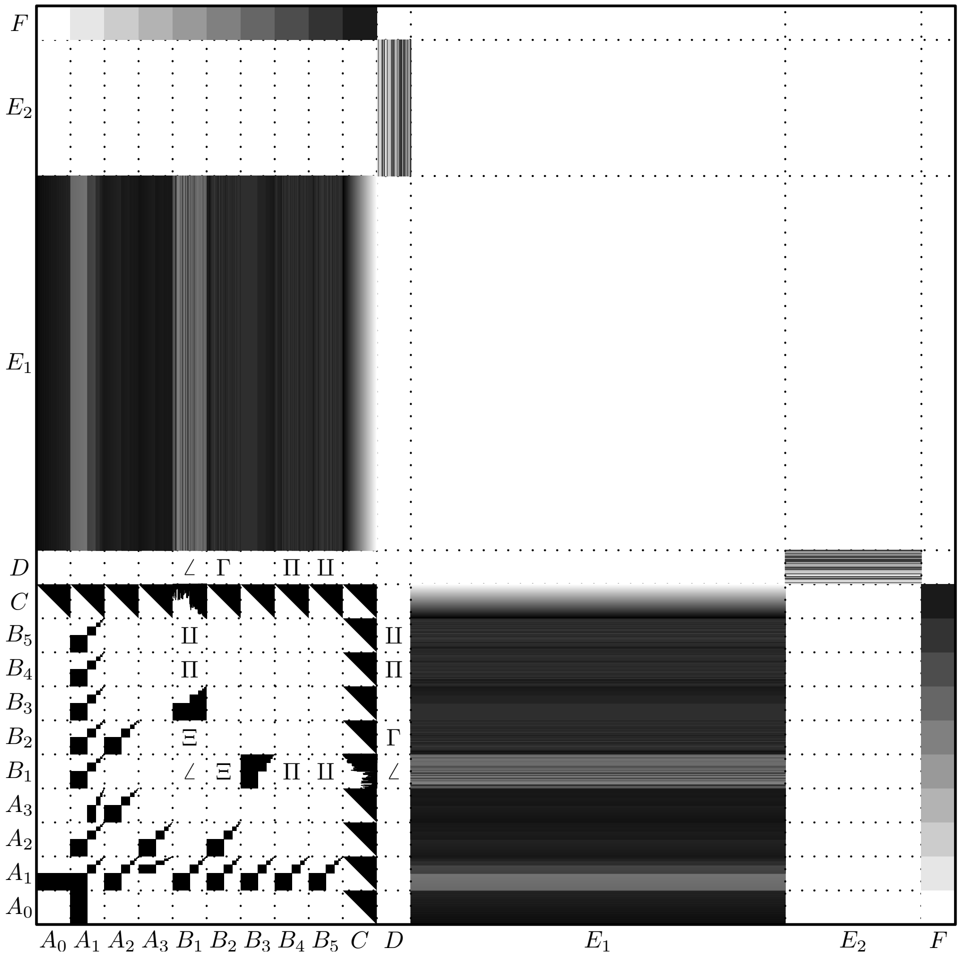


Figure 3.1: The hypercubical graphon.

better readability, we define these as functions from $[0,1]^2$ to $[0,1]$, assuming that we have a fixed bijective (scaling) map $\eta_X$ from each part $X$ to $[0,1]$ such that $\lambda(\eta_X^{-1}(S)) = \lambda(S)\lambda(X)$ for every measurable set $S \subseteq [0,1]$. So, $W_\boxplus(x,y) = W_\boxplus^{X\times Y}(\eta_X(x),\eta_Y(y))$ for $x \in X$ and $y \in Y$. Note that, unlike graphons, the functions $W_\boxplus^{X\times Y}$ need not be symmetric, instead they satisfy $W_\boxplus^{X\times Y}(x,y) = W_\boxplus^{Y\times X}(y,x)$.

For $x \in [1-2^{-k}, 1-2^{-(k+1)})$, let $\widehat{x} = (x-(1-2^{-k}))\cdot 2^{k+1}$ (the relative position of $x$ within the interval) and $\langle x\rangle = k+1$. Informally speaking, if we imagine $[0,1]$ as partitioned into consecutive intervals of measures $1/2$, $1/4$, etc., $\langle x\rangle$ indicates to which of the intervals $x$ belongs. Observe that $x = 1-2^{1-\langle x\rangle} + \widehat{x}/2^{\langle x\rangle}$ for every $x \in [0,1)$.

| part | $A_0^{\boxplus}$ | $A_1^{\boxplus}$ | $A_2^{\boxplus}$ | $A_3^{\boxplus}$ | $B_1^{\boxplus}$ | $B_2^{\boxplus}$ | $B_3^{\boxplus}$ | $B_4^{\boxplus}$ | $B_5^{\boxplus}$ |
|---|---|---|---|---|---|---|---|---|---|
| degree | $\frac{110}{270}$ | $\frac{111}{270}$ | $\frac{112}{270}$ | $\frac{113}{270}$ | $\frac{114}{270}$ | $\frac{115}{270}$ | $\frac{116}{270}$ | $\frac{117}{270}$ | $\frac{118}{270}$ |

| part | $C^{\boxplus}$ | $D^{\boxplus}$ | $E_1^{\boxplus}$ | $E_2^{\boxplus}$ | $F^{\boxplus}$ |
|---|---|---|---|---|---|
| degree | $\frac{119}{270}$ | $\frac{40}{270}$ | $e_1$ | $e_2$ | $\frac{45}{270}$ |

Table 3.1: Degrees of the parts of $W_{\boxplus}$.

The *diagonal checker* function $\varkappa : [0,1]^2 \to [0,1]$ is defined as follows (see Figure 3.2):

$$\varkappa(x,y) = \begin{cases} 1 & \text{if } \langle x \rangle = \langle y \rangle \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in [0,1]^n$, $n \in \mathbb{N}^*$, we denote its $i$-th coordinate of $x$ by $(x)_i$. A *recipe* $\mathfrak{R}$ is a set of measure preserving maps $r_n : [0,1] \to [0,1]^n$ for $n \in \mathbb{N}^*$. Observe that $\mathfrak{R} = \{r_n | n \in \mathbb{N}^*\}$ is a recipe if and only if

$$\lambda(\{x | \forall\ i \in [k]\ (r_n(x))_i \leq a_i\}) = \prod_{i=1}^{k} a_i \text{ for every } (a_1, \dots a_k) \in [0,1]^k \tag{3.1}$$

for every $k \in [n]$ (recall that we define $[\infty] := \mathbb{N}$). A recipe is *bijective* if all the functions are bijective.

For a fixed bijective recipe $\mathfrak{R}$, the graphon $W_{\boxplus}$ is defined as follows:

$$W_{\boxplus}^{A_0^{\boxplus} \times A_1^{\boxplus}}(x,y) = \begin{cases} 1 & \text{for } (x,y) \in [0,1] \times [0,1/2], \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

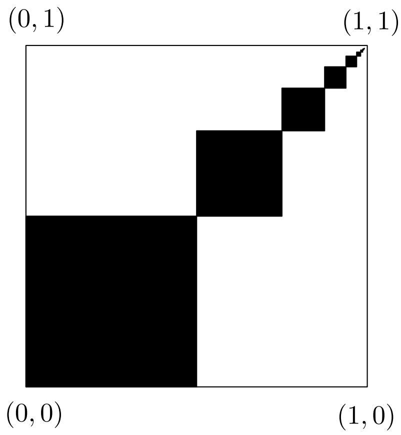


Figure 3.2: The diagonal checker function $\varkappa$.

$$W_{\boxplus}^{A_1^{\boxplus}\times A_1^{\boxplus}} = W_{\boxplus}^{A_1^{\boxplus}\times A_2^{\boxplus}} = W_{\boxplus}^{A_1^{\boxplus}\times B_1^{\boxplus}} = W_{\boxplus}^{A_1^{\boxplus}\times B_2^{\boxplus}} = W_{\boxplus}^{A_1^{\boxplus}\times B_3^{\boxplus}} = W_{\boxplus}^{A_1^{\boxplus}\times B_4^{\boxplus}} = W_{\boxplus}^{A_1^{\boxplus}\times B_5^{\boxplus}} = W_{\boxplus}^{A_2^{\boxplus}\times A_3^{\boxplus}} = W_{\boxplus}^{A_2^{\boxplus}\times B_2^{\boxplus}} = \varkappa.$$

For $X \in \{A_0^{\boxplus}, \dots, A_3^{\boxplus}, B_2^{\boxplus}, \dots, B_5^{\boxplus}, C^{\boxplus}\}$, let:

$$W_{\boxplus}^{C^{\boxplus}\times X}(x,y) = \begin{cases} 1 & \text{for } x+y \geq 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that for $x \in [1-2^{-k}, 1-2^{-(k+1)})$, $\widehat{x} = (x-(1-2^{-k}))\cdot 2^{k+1}$ and $\langle x\rangle = k+1$.

$$W_{\boxplus}^{A_1^{\boxplus}\times A_3^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle = \langle y\rangle + 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{C^{\boxplus}\times B_1^{\boxplus}}(x,y) = \begin{cases} 1 & \text{for } (1-2^{1-\langle y\rangle}) + (r_{\langle y\rangle}(\widehat{y}))_1 \cdot 2^{-\langle y\rangle} + x \geq 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{B_1^{\boxplus}\times B_1^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \leq \langle y\rangle \text{ and } (r_{\langle x\rangle}(\widehat{x}))_i \leq (r_{\langle y\rangle}(\widehat{y}))_i \text{ for every } i \leq \langle x\rangle, \\ 1 & \text{if } \langle x\rangle > \langle y\rangle \text{ and } (r_{\langle x\rangle}(\widehat{x}))_i \geq (r_{\langle y\rangle}(\widehat{y}))_i \text{ for every } i \leq \langle y\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{B_1^{\boxplus}\times B_2^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \geq \langle y\rangle \text{ and } \widehat{y} \leq (r_{\langle x\rangle}(\widehat{x}))_{\langle y\rangle}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{B_1^{\boxplus}\times B_3^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \geq \langle y\rangle, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{B_1^{\boxplus}\times B_4^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \geq \langle y\rangle \text{ and } \widehat{y} \leq \prod_{i=1}^{\langle y\rangle} (r_{\langle x\rangle}(\widehat{x}))_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{B_1^{\boxplus}\times B_5^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \geq \langle y\rangle \text{ and } \widehat{y} \leq \prod_{i=1}^{\langle y\rangle} (1-(r_{\langle x\rangle}(\widehat{x}))_i), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{D^{\boxplus}\times B_1^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \widehat{y} \leq (r_{\infty}(x))_i \text{ for every } \ i \leq \langle y\rangle, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{D^{\boxplus}\times B_2^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \widehat{y} \le (r_\infty(x))_{\langle y\rangle}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{D^{\boxplus}\times B_4^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \widehat{y} \le \prod_{i=1}^{\langle y\rangle}(r_\infty(x))_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{D^{\boxplus}\times B_5^{\boxplus}}(x,y) = \begin{cases} 1 & \text{if } \widehat{y} \le \prod_{i=1}^{\langle y\rangle}(1-(r_\infty(x))_i), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For every $X \in \{A_0^{\boxplus},\dots,A_3^{\boxplus},B_1^{\boxplus},\dots,B_5^{\boxplus},C^{\boxplus}\}$:

$$W_{\boxplus}^{E_1^{\boxplus}\times X}(x,y) = 1 - 1/11 \sum_{Y\in\{A_0^{\boxplus},\dots,A_3^{\boxplus},B_1^{\boxplus},\dots,B_5^{\boxplus},C^{\boxplus},D^{\boxplus}\}} \deg_Y y.$$

$$W_{\boxplus}^{E_2^{\boxplus}\times D^{\boxplus}}(x,y) = 1 - 1/4 \sum_{Y\in\{B_1^{\boxplus},B_2^{\boxplus},B_4^{\boxplus},B_5^{\boxplus}\}} \deg_Y y.$$

$W_{\boxplus}^{F^{\boxplus}\times A_1^{\boxplus}}(x,y) = 1/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F^{\boxplus}\times A_2^{\boxplus}}(x,y) = 2/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F^{\boxplus}\times A_3^{\boxplus}}(x,y) = 3/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F^{\boxplus}\times B_1^{\boxplus}}(x,y) = 4/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F^{\boxplus}\times B_2^{\boxplus}}(x,y) = 5/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F^{\boxplus}\times B_3^{\boxplus}}(x,y) = 6/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F^{\boxplus}\times B_4^{\boxplus}}(x,y) = 7/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F^{\boxplus}\times B_5^{\boxplus}}(x,y) = 8/10$ for all $(x,y)\in[0,1]^2$, and

$W_{\boxplus}^{F^{\boxplus}\times C^{\boxplus}}(x,y) = 9/10$ for all $(x,y)\in[0,1]^2$.

$W_{\boxplus}$ is identically equal to 0 on all the pairs of parts that are not listed above and that are not symmetric to the pairs listed.

Note that the definition of $W_{\boxplus}$ is dependent on the choice of a bijective recipe $\mathfrak{R}$. However, it can be shown that the graphons obtained for different bijective recipes are weakly isomorphic. (In fact, the statement stays true even if $\mathfrak{R}$ is a recipe that is not bijective.) Therefore we do not include the dependence on the recipe in the notation for the hypercubical graphon.

Before proceeding further, let us introduce additional notation that makes use of the structure of the diagonal checker subgraphons. We denote the set of vertices $x$ of $A_i^{\boxplus}$, $i \in \{1,2\}$, with $\deg_{A_1^{\boxplus}} x = 2^{-k}$ in $A_1^{\boxplus}$ by $A_{i,k}^{\boxplus}$ and we call $A_{i,k}^{\boxplus}$ the *k-th level* of $A_i^{\boxplus}$. We define levels $B_{j,k}^{\boxplus}$ of $B_j^{\boxplus}$ for $j \in \{1,\dots,5\}$ in the same way.

Similarly, we denote $A_{3,k}^{\boxplus}$ the set of vertices of $A_3^{\boxplus}$ of relative degree $2^{-k}$ in $A_2^{\boxplus}$ and we call the set of these vertices the *k-th level* of $A_3^{\boxplus}$. Note that measure of the $k$-th level is $2^{-k}/27$ for every $k$ and every $A_i^{\boxplus}$, $i \in \{1,2,3\}$ and $B_j^{\boxplus}$, $j \in \{1,\dots,5\}$.

The following proposition and Theorem 25 imply Theorem 5.

**Proposition 24.** *Both* $T(W_{\boxplus})$ *and* $\overline{T}(W_{\boxplus})$ *contain a subspace homeomorphic to* $[0,1]^{\mathbb{N}}$.

*Proof.* Every vertex of the part $D^{\boxplus}$ of the graphon $W_{\boxplus}$ is typical. In addition, the restrictions of the spaces $T(W_{\boxplus})$ and $\overline{T}(W_{\boxplus})$ to $\{f_x^{W_{\boxplus}}(y) := W_{\boxplus}(x,y),\ x \in D^{\boxplus}\}$ are the same. Recall that $\lambda(B_{j,i}^{\boxplus}) = 2^{-i}/27$ for $j \in \{1,2,4,5\}$,

$$\deg^{W_{\boxplus}}_{B_{1,i}^{\boxplus}} x = \deg^{W_{\boxplus}}_{B_{4,i}^{\boxplus}} x = \prod_{k\in[i]} \deg^{W_{\boxplus}}_{B_{2,i}^{\boxplus}} x \text{ and}$$

$$\deg^{W_{\boxplus}}_{B_{5,i}^{\boxplus}} x = \prod_{k\in[i]} (1 - \deg^{W_{\boxplus}}_{B_{2,i}^{\boxplus}} x)$$

for every $x \in D^{\boxplus}$ and $i \in \mathbb{N}$. Also note that

$$\begin{aligned}\lambda(E_2^{\boxplus}) \left|\deg^{W_{\boxplus}}_{E_2^{\boxplus}} x - \deg^{W_{\boxplus}}_{E_2^{\boxplus}} x'\right| &= \left|\sum_{j\in\{1,2,4,5\}} \sum_{i=1}^{\infty} \lambda(B_{j,i}^{\boxplus}) \deg^{W_{\boxplus}}_{B_{j,i}^{\boxplus}} x - \deg^{W_{\boxplus}}_{B_{j,i}^{\boxplus}} x'\right| \\ &\le \sum_{j\in\{1,2,4,5\}} \sum_{i=1}^{\infty} \lambda(B_{j,i}^{\boxplus}) \left|\deg^{W_{\boxplus}}_{B_{j,i}^{\boxplus}} x - \deg^{W_{\boxplus}}_{B_{j,i}^{\boxplus}} x'\right|\end{aligned}$$

for every $x, x' \in D^{\boxplus}$. This leads us to the following estimates for all $x, x' \in D^{\boxplus}$.

$$\begin{aligned}||f_x^{W_{\boxplus}} - f_{x'}^{W_{\boxplus}}||_1 &\ge \sum_{i=1}^{\infty} \lambda(B_{2,i}^{\boxplus}) \left|\deg^{W_{\boxplus}}_{B_{2,i}^{\boxplus}} x - \deg^{W_{\boxplus}}_{B_{2,i}^{\boxplus}} x'\right| \\ &= \sum_{i=1}^{\infty} \frac{2}{27} \left|\deg^{W_{\boxplus}}_{B_{2,i}^{\boxplus}} x - \deg^{W_{\boxplus}}_{B_{2,i}^{\boxplus}} x'\right|\end{aligned}$$

$$
\begin{aligned}
||f_x^{W_\boxplus} - f_{x'}^{W_\boxplus}||_1 &\leq 2\left(\sum_{j\in\{1,2,4,5\}}\sum_{i=1}^{\infty}\lambda(B_{j,i}^{\boxplus})\left|\deg_{B_{j,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{j,i}^{\boxplus}}^{W_\boxplus} x'\right|\right)\\
&= 2\left(\sum_{i=1}^{\infty}\lambda(B_{2,i}^{\boxplus})\left|\deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x'\right|\right.\\
&+ \sum_{i=1}^{\infty}\lambda(B_{1,i}^{\boxplus})\left|\prod_{k=1}^{i}\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \prod_{k=1}^{i}\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right|\\
&+ \sum_{i=1}^{\infty}\lambda(B_{4,i}^{\boxplus})\left|\prod_{k=1}^{i}\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \prod_{k=1}^{i}\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right|\\
&+ \left.\sum_{i=1}^{\infty}\lambda(B_{5,i}^{\boxplus})\left|\prod_{k=1}^{i}(1-\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x) - \prod_{k=1}^{i}(1-\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x')\right|\right)\\
&= \frac{2}{27}\left(\sum_{i=1}^{\infty}2^{-i}\left|\deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x'\right|\right.\\
&+ 2\sum_{i=1}^{\infty}2^{-i}\left|\prod_{k=1}^{i}\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \prod_{k=1}^{i}\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right|\\
&+ \left.\sum_{i=1}^{\infty}2^{-i}\left|\prod_{k=1}^{i}(1-\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x) - \prod_{k=1}^{i}(1-\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x')\right|\right)\\
&\leq \frac{2}{27}\left(\sum_{i=1}^{\infty}2^{-i}\left|\deg_{B_{j,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{j,i}^{\boxplus}}^{W_\boxplus} x'\right|\right.\\
&+ 2\sum_{i=1}^{\infty}2^{-i}\sum_{k=1}^{i}\left|\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right|\\
&+ \left.\sum_{i=1}^{\infty}2^{-i}\sum_{k=1}^{i}\left|(1-\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x) - (1-\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x')\right|\right)\\
&= \frac{14}{27}\sum_{i=1}^{\infty}2^{-i}\left|\deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x'\right|
\end{aligned}
$$

$$
\begin{aligned}
d_W(f_x^{W_\boxplus}, f_{x'}^{W_\boxplus}) &\leq ||f_x^{W_\boxplus} - f_{x'}^{W_\boxplus}||_1\\
&= \frac{14}{27}\sum_{i=1}^{\infty}2^{-i}\left|\deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x'\right|
\end{aligned}
$$

$$
\begin{aligned}
d_W(f_x^{W_\boxplus}, f_{x'}^{W_\boxplus}) &\geq \sum_{i=1}^{\infty} \int\limits_{A_{2,i}^\boxplus} \left| \int\limits_{[0,1]} W_\boxplus(z,y) \left( f_x^{W_\boxplus}(y) - f_{x'}^{W_\boxplus}(y) \right) \mathrm{d}y \right| \mathrm{d}z \\
&= \sum_{i=1}^{\infty} \int\limits_{A_{2,i}^\boxplus} \lambda(B_{2,i}^\boxplus) \left| \deg_{B_{2,i}^\boxplus}^{W_\boxplus} x - \deg_{B_{2,i}^\boxplus}^{W_\boxplus} x' \right| \mathrm{d}z \\
&= \sum_{i=1}^{\infty} \lambda(A_{2,i}^\boxplus)\lambda(B_{2,i}^\boxplus) \left| \deg_{B_{2,i}^\boxplus}^{W_\boxplus} x - \deg_{B_{2,i}^\boxplus}^{W_\boxplus} x' \right| \\
&= \frac{1}{27} \sum_{i=1}^{\infty} 4^{-i} \left| \deg_{B_{2,i}^\boxplus}^{W_\boxplus} x - \deg_{B_{2,i}^\boxplus}^{W_\boxplus} x' \right|
\end{aligned}
$$

Since the map $H$ from the restriction of $T(W)$ to $\{f_x^{W_\boxplus},\ x \in D^\boxplus\}$ to $[0,1]^{\mathbb{N}}$ defined as

$$
H(f_x^{W_\boxplus}) = (\deg_{B_{2,i}^\boxplus}^{W_\boxplus} x)_{i=1}^{\infty}
$$

is a homeomorphism by the definition of $W_\boxplus$ and $\mathfrak{R}$, the statement of the proposition follows. $\square$

**Theorem 25.** *The hypercubical graphon $W_\boxplus$ is finitely forcible.*

The rest of the chapter is devoted to the proof of Theorem 25.

## 3.3 Constraints

In this section, we present the constraints that finitely force the graphon $W_\boxplus$, as we prove in the next section.

The set of the constraints that finitely force the graphon $W_\boxplus$ is denoted by $\mathcal{C}_\boxplus$. The constraints in $\mathcal{C}_\boxplus$ are split into groups according to what features of a graphon they force:

*Partition constraints* force that every graphon satisfying $\mathcal{C}_\boxplus$ can be partitioned into parts of the sizes and degrees of vertices as in $W_\boxplus$. The existence of such constraints follows from Lemma 21.

In the rest of the section, we assume that $W$ is a partitioned graphon and we will refer to the parts of $W$ as $A_0, \ldots, A_3, B_1, \ldots, B_5, C, D, E_1, E_2, F$, respectively.

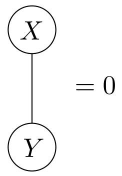


Figure 3.3: Constraint forcing zero edge density.

*Zero constraints* force that $W$ equals 0 almost everywhere on

- $A_0 \times (A_0 \cup A_2 \cup A_3 \cup B_1 \cup B_3 \cup B_4 \cup B_5 \cup D \cup E_2 \cup F)$,
- $A_1 \times (D \cup E_2)$,
- $A_2 \times (A_2 \cup B_1 \cup \cdots \cup B_5 \cup D \cup E_2)$,
- $A_3 \times (A_3 \cup B_1 \cup \cdots \cup B_5 \cup D \cup E_2)$,
- $B_2 \times (B_2 \cup \cdots \cup B_5 \cup E_2)$,
- $B_3 \times (B_3 \cup B_4 \cup B_5 \cup D \cup E_2)$,
- $B_4 \times (B_4 \cup B_5 \cup E_2)$,
- $B_5 \times (B_5 \cup E_2)$,
- $C \times (D \cup E_2)$,
- $D \times (D \cup E_1)$,
- $E_1 \times (E_1 \cup E_2 \cup F)$,
- $E_2 \times (E_2 \cup F)$, and
- $F \times F$.

The constraint forcing the zero edge density between parts $X$ and $Y$ is depicted in Figure 3.3.

*Degree unifying constraints* force that the relative degree $\deg_{[0,1]\setminus(E_2\cup F)} x$ of almost every vertex $x \in [0,1] \setminus (D \cup E_1 \cup E_2 \cup F)$ is $1/2$ and that the degree $\deg y$ of almost every vertex $y \in D$ is $4/27$. These constraints are depicted in Figures 3.4 and 3.5.

*Degree distinguishing constraints* force that the structure between $F$ and the remaining parts of a graphon consists of pseudorandom bipartite subgraphons of densities given in Table 3.2.

By Lemma 23, this can be forced by finitely many constraints. An example of the constraints for $F \times C$ is depicted in Figure 3.6.

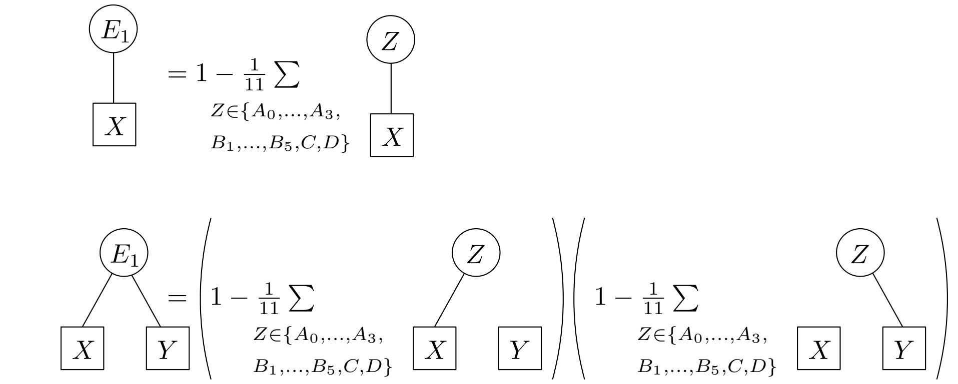


Figure 3.4: The degree unifying constraints contain the depicted constraints for all the choices of $X$ and $Y$ in $\{A_0, \ldots, A_3, B_1, \ldots, B_5, C\}$.

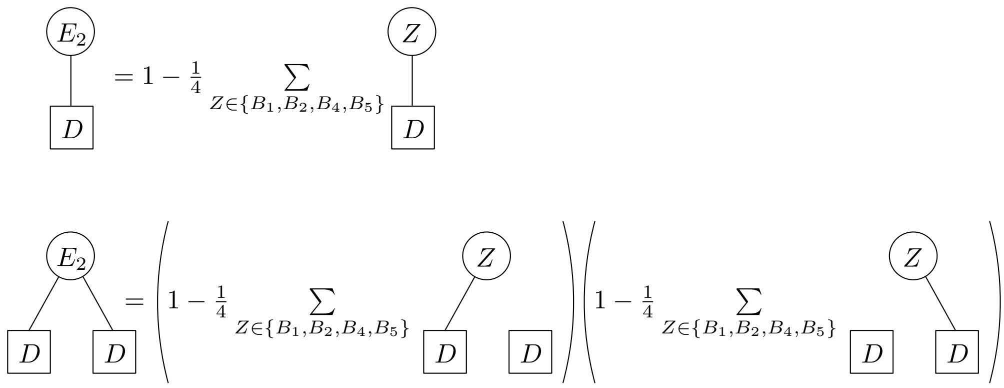


Figure 3.5: The degree unifying constraints for $D$.

| part | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $C$ |
|---|---|---|---|---|---|---|---|---|---|---|
| density | 0 | $\frac{1}{10}$ | $\frac{2}{10}$ | $\frac{3}{10}$ | $\frac{4}{10}$ | $\frac{5}{10}$ | $\frac{6}{10}$ | $\frac{7}{10}$ | $\frac{8}{10}$ | $\frac{9}{10}$ |

Table 3.2: Densities between the part $F$ and the other parts.

*Triangular constraints* force the structure on $C \times X$ is as in $W_{\boxplus}$ for every $X \in A_0, \ldots, A_3, B_1, \ldots, B_5, C$. From the proof of the Theorem 5.1 in [65] it follows that there exist finitely many constraints forcing the triangular structure on $C \times C$. The triangular constraints forcing the structure elsewhere are depicted in Figure 3.7. They ensure that the triangular structure of $C \times C$ is replicated to other parts of the graphon.

*Main diagonal checker constraints* force the diagonal checker structure of $A_1 \times A_1$. They are depicted in Figure 3.8.

*Complete bipartition constraints* force, in particular, that the subgraphons on $A_1 \times A_2$, $A_1 \times A_3$, $A_1 \times B_1$, ..., $A_1 \times B_5$, $A_2 \times A_3$ and $A_2 \times B_2$ are unions of complete bipartite subgraphons. The constraints are given in Figure 3.9.

*Auxiliary diagonal checker constraints* determine the sizes of the sides of complete bipartite subgraphons in $A_1 \times A_2$, $A_1 \times A_3$, $A_1 \times B_1, \ldots, A_1 \times B_5$, $A_2 \times A_3$ and $A_2 \times B_2$. They are depicted in Figure 3.10.

*First level constraints* force the structure of $A_0 \times A_1$ and they are depicted in Figure 3.11.

*Stair constraints* force the structure of $B_1 \times B_3$ and $B_2 \times B_3$. They are depicted in Figure 3.12.

*Coordinate constraints* force some features of structure of $B_1 \times (B_2 \cup B_4 \cup B_5)$ and $D \times (B_2 \cup B_4 \cup B_5)$. They can be found in Figure 3.13.

*Distribution constraints* determining the relative degrees of vertices of $B_2$ in $B_1$ and $D$ are depicted in Figure 3.14.

*An initial coordinate constraint* determines the relative degrees of vertices of $B_1$ in a subset of $B_2$. It is depicted in Figure 3.15.

*Product constraints* force the structure of $B_1 \times B_4$, $D \times B_4$, $B_1 \times B_5$ and $D \times B_5$. They are depicted in Figures 3.16, 3.17.

*Projection constraints* force the structure of $B_1 \times B_1$. They are depicted in Figures 3.18 and 3.19.

*The infinite constraints* force the structure between $D$ and the parts $B_1$ and $B_2$ of the graphon. They are depicted in Figure 3.20.

This completes the list of the constraints in $\mathcal{C}_{\boxplus}$.

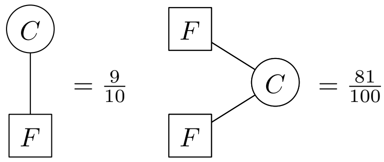


Figure 3.6: The degree distinguishing constraints for $F \times C$.

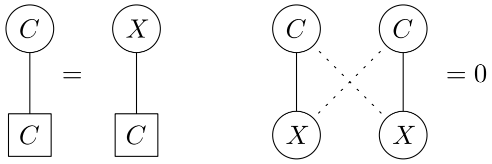


Figure 3.7: The triangular constraints include the depicted constraints for all the choices of $X$ in $\{A_0, \ldots, A_3, B_1, \ldots, B_5\}$.

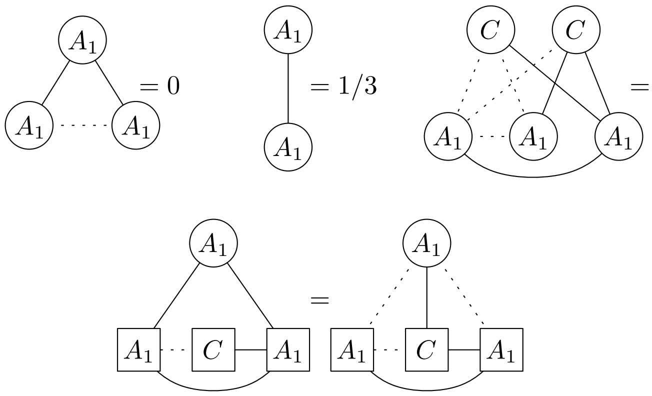


Figure 3.8: The main diagonal checker constraints.

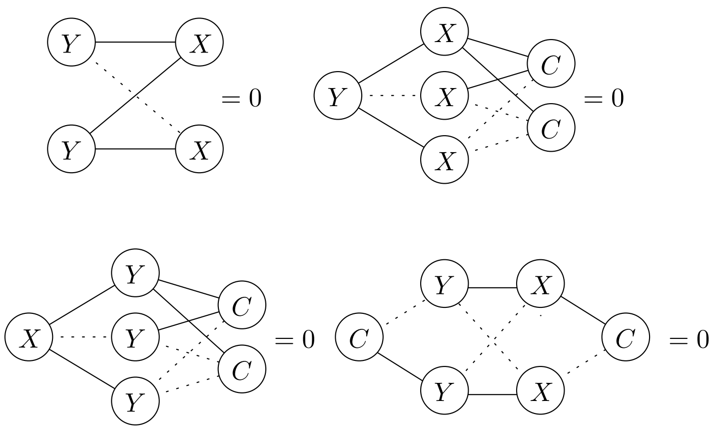


Figure 3.9: The complete bipartition constraints consist of the first two constraints for $(X, Y) \in \{(A_1, A_2), (A_1, A_3), (A_1, B_1), \ldots, (A_1, B_5), (A_2, A_3), (A_2, B_2)\}$ and the second two constraints for $(X, Y) \in \{(A_1, A_2), (A_1, A_3), (A_1, B_2), \ldots, (A_1, B_5), (A_2, A_3), (A_2, B_2)\}$.

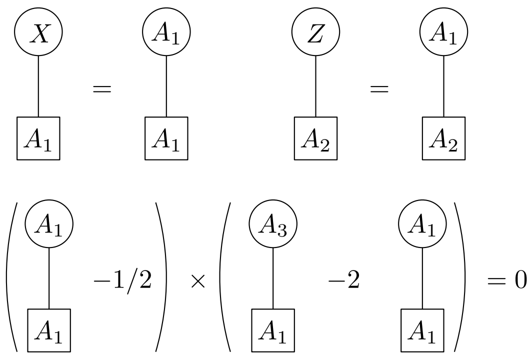


Figure 3.10: The auxiliary diagonal checker constraints consist of the depicted constraints, where $Y$ in the first constraint attains all values in $\{A_2, B_1, \ldots, B_5\}$ and $Z$ in the second constraint attains all values in $\{A_3, B_2\}$.

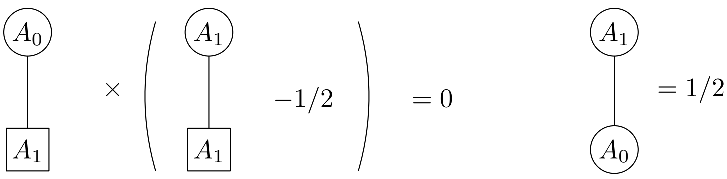


Figure 3.11: The first level constraints.

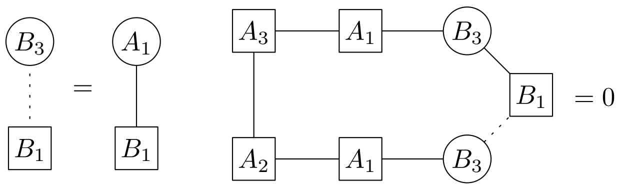


Figure 3.12: The stair constraints.

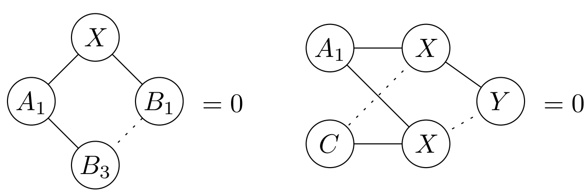


Figure 3.13: The coordinate constraints consist of the depicted constraints, where $X$ and $Y$ attain all values in $\{B_2, B_4, B_5\}$, $\{B_1, D\}$ respectively.

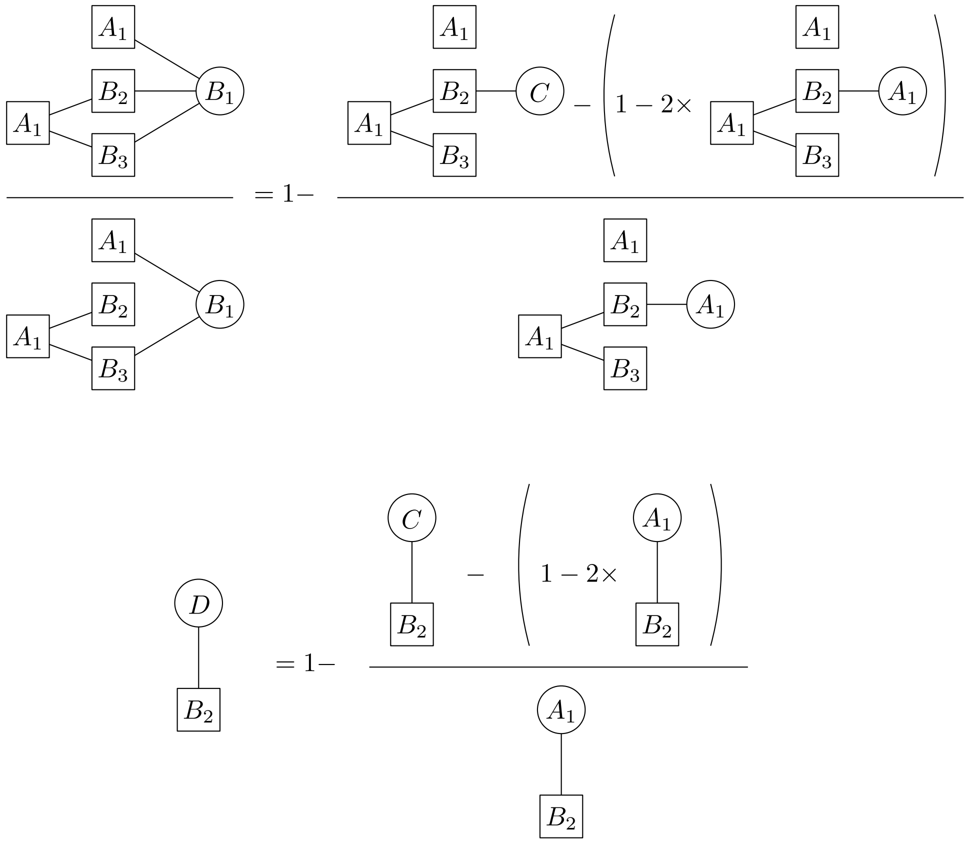

Figure 3.14: The distribution constraints.

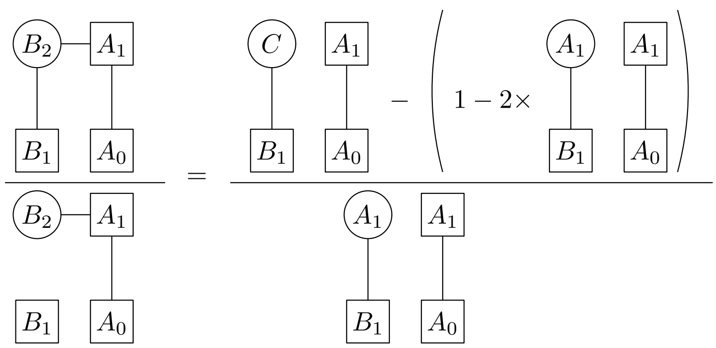

Figure 3.15: The initial coordinate constraint.

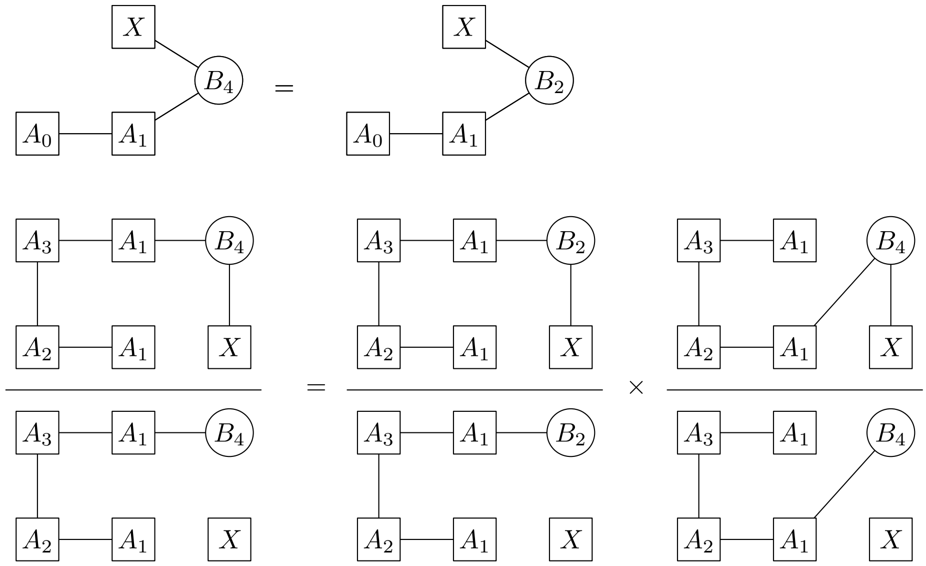


Figure 3.16: The product constraints forcing $B_1 \times B_4$ and $D \times B_4$ consist of the depicted constraints, where $X \in \{B_1, D\}$.

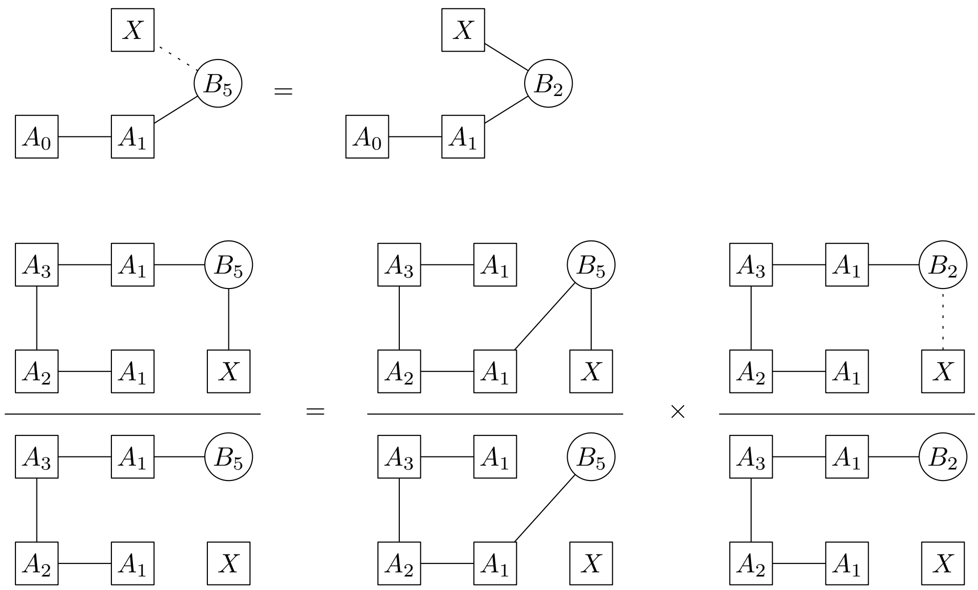


Figure 3.17: The product constraints forcing $B_1 \times B_5$ and $D \times B_5$ consist of the depicted constraints, where $X \in \{B_1, D\}$.

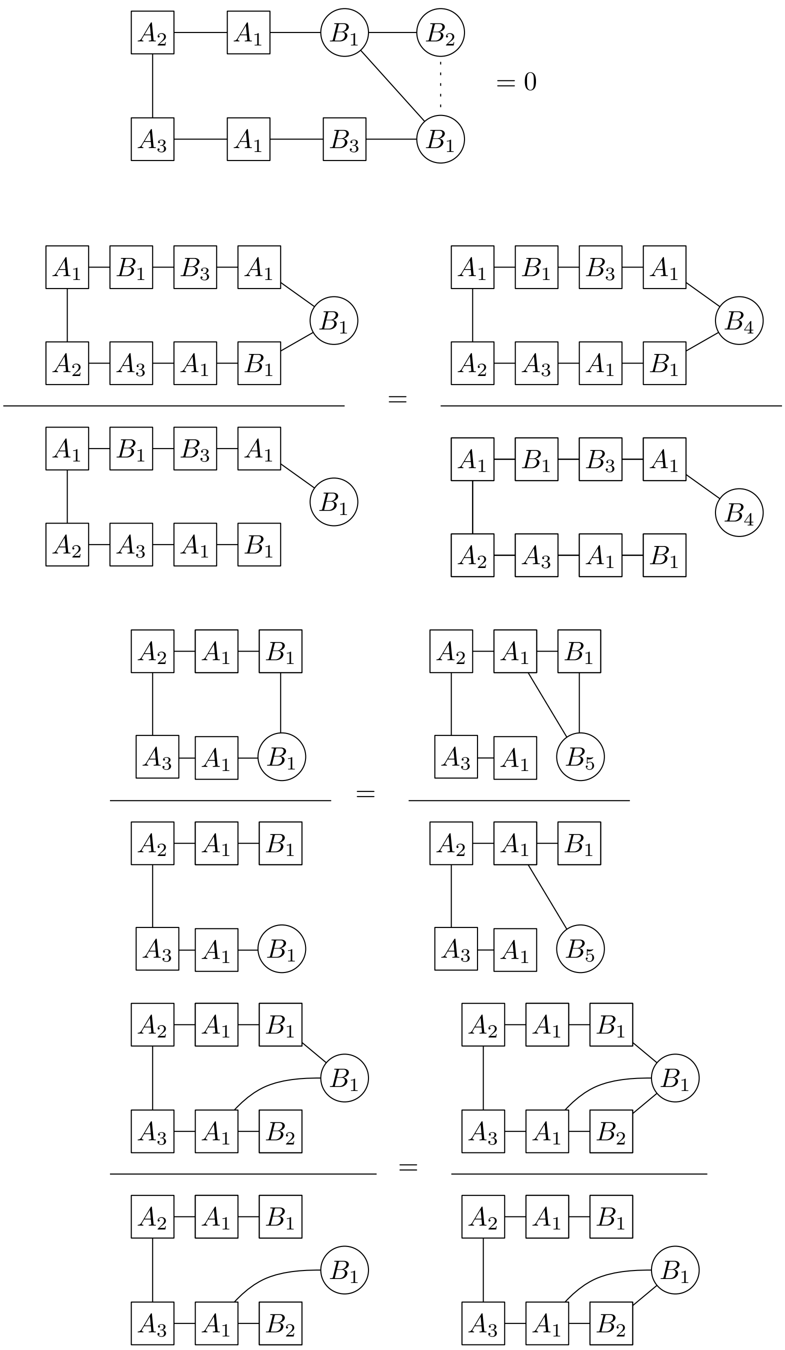


Figure 3.18: The first four projection constraints.

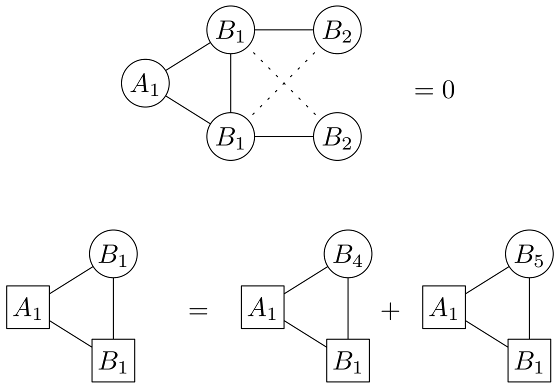


Figure 3.19: The last two projection constraints.

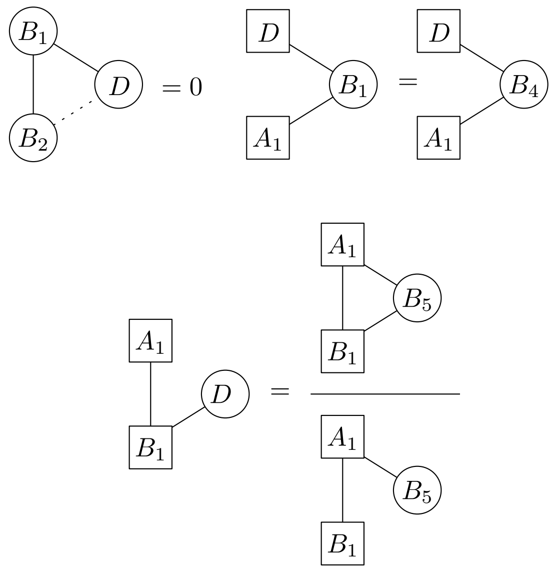


Figure 3.20: Infinite constraints.

## 3.4 Proof of Theorem 25

Let $W$ be a graphon satisfying all constraints of $\mathcal{C}_{\boxplus}$. Since $W$ satisfies the partition constraints, the interval $[0,1]$ can be partitioned into parts of the sizes as in $W_{\boxplus}$ such that the degrees of vertices in these parts are as in $W_{\boxplus}$. In particular, there exists a measure preserving map $\varphi : [0,1] \to [0,1]$ such that the subsets of $[0,1]$ corresponding to the parts of $W_{\boxplus}$ map to the corresponding parts of $W$. From now on, we denote the parts of $W^{\varphi}$ corresponding to the respective parts of $W_{\boxplus}$ by $A_0, \ldots, A_3, B_1, \ldots, B_5, C, D, E_1, E_2, F$. Note that by the definition of $\varphi$, the corresponding parts in $W_{\boxplus}$ and in $W^{\varphi}$ represent the same subsets of $[0,1]$, for instance $A_0 = A_0^{\boxplus}$. We write $A_0, \ldots, F$ in the context of the graphon $W^{\varphi}$ and $A_0^{\boxplus}, \ldots, F^{\boxplus}$ in the context of the graphon $W_{\boxplus}$.

By Monotone Reordering Theorem (see [63] for more details), there exist measure preserving maps $\psi_X : X \to X^{\boxplus}$ for $X = A_0, \ldots, A_3$, $B_2, \ldots, B_5$, $C$, $E_1$, $E_2$, $F$ and non-decreasing functions $f_X : X^{\boxplus} \to [0,1]$, such that $f_X(\psi_X(x)) = \deg_C^{W^{\varphi}} x$ for every $x \in X$.

By analogy to $B_{1,n}^{\boxplus}$ and $B_{2,n}^{\boxplus}$, we define $B_{1,n}$ and $B_{2,n}$ to be the vertices of $B_1$ and $B_2$, respectively, that have relative degree $1/2^n$ in $A_1$ in $W^{\varphi}$. Let $\mathfrak{R} = \{r_1, r_2, \ldots\}$ be the bijective recipe used to define $W_{\boxplus}$.

Let $\eta_{B_{1,n}}$ be a bijective maps from $B_{1,n}$ to $[0,1]$ such that

$$\lambda(\eta_{B_{1,n}}^{-1}(S)) = \lambda(S)\lambda(B_{1,n})$$

for every measurable set $S \subseteq [0,1]$ for every $n \in \mathbb{N}$. Let $g_n : [0,1] \to [0,1]^n$ be a function defined as

$$g_n(x) = (\deg_{B_{2,i}}^{W^{\varphi}} \eta_{B_{1,n}}^{-1}(x))_{i \in [n]}$$

for $n \in \mathbb{N}$ and the function $g_\infty : [0,1] \to [0,1]^{\mathbb{N}}$ defined as

$$g_\infty(x) = (\deg_{B_{2,i}}^{W^{\varphi}} \eta_D^{-1}(x))_{i \in \mathbb{N}}.$$

Later in this section (in Subsections 3.4.11 and 3.4.12), we show that these functions form a recipe $\mathfrak{G}$, i.e., every $g_n \in \mathfrak{G}$, $n \in \mathbb{N}^*$, satisfies (3.1) for every $k \in [n]$. Note that we will only prove that $\mathfrak{G}$ is a recipe, not a bijective recipe. The fact that $\mathfrak{G}$ is a recipe will imply that $\psi_{B_1}$ and $\psi_D$ defined as follows are measure preserving maps assuming that $\lambda(\psi_{B_1}(B_{1,n})) = \lambda(B_{1,n})$ for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ and $x \in B_{1,n}$, we define

$$\psi_{B_1}(x) := \eta_{B_1}^{-1}\left(1 - \frac{1}{2^{n-1}} + \frac{r_n^{-1}((\deg_{B_{2,i}}^{W^\varphi} x)_{i\in[n]})}{2^n}\right).$$

Observe that $\psi_{B_1}(B_{1,n}) \subseteq B_{1,n}^{\boxplus}$ for every $n$. For $x \in B_1$ that does not belong to any $B_{1,n}$, we define $\psi_{B_1}(x)$ to be equal to the same arbitrary vertex of $B_1^{\boxplus}$. We later prove that the set of such $x$ has measure zero (and therefore $\lambda(\psi_{B_1}^{-1}(B_{1,n}^{\boxplus})) = \lambda(B_{1,n}) = 2^{-n}/27$).

Similarly, for $D^{\boxplus}$, we define

$$\psi_D(x) = \eta_D^{-1}(r_\infty^{-1}((\deg_{B_{2,i}}^{W^\varphi} x)_{i\in\mathbb{N}})).$$

Let $\psi$ be a map from $[0,1]$ to $[0,1]$ consisting of maps $\psi_X$ for $X \in \{A_0, \dots, A_3, B_1, \dots, B_5, C, D, E_1, E_2, F\}$. Note that if $\psi_{B_1}$ and $\psi_D$ are measure preserving maps, $\psi$ is a measure preserving map, too.

In the rest of the section, we show that $\mathcal{C}_{\boxplus}$ force the graphon $W$ to be weakly isomorphic to $W_{\boxplus}$. Clearly, if the constraints $\mathcal{C}_{\boxplus}$ force that $W$ has a certain property, $W^\varphi$ has the same property. Therefore, we speak directly about properties $W^\varphi$ in our arguments.

We will be proving that $W_{\boxplus}^{\psi}$ and $W^\varphi$ are equal almost everywhere for different subgraphons. To do this, we do not need to assume that $\mathfrak{G}$ is a recipe. That is needed for showing that $\psi$ is a measure preserving map, i.e., that $W_{\boxplus} = W^\varphi$ almost everywhere implies that $W_{\boxplus}$ and $W^\varphi$ are weakly isomorphic.

### 3.4.1 Forcing $[0,1] \times C$—triangular constraints

The first constraint in Figure 3.7 forces that almost every vertex $c \in C$ has the same relative degree in $C$ and in the parts $A_0, \dots, A_3, B_1, \dots, B_5$ of the graphon. The second constraint yields that either $N_C(x \setminus y)$ or $N_C(y \setminus x)$ has measure zero for almost every pair $x, y \in X$. This implies that the graphon $W^\varphi$ has values 0 and 1 almost everywhere on $X \times C$. The choice of $\psi$ implies that $W^\varphi$ and $W_{\boxplus}^{\psi}$ are equal almost everywhere on $X \times C$ for $X \in \{A_0, \dots, A_3, B_2, \dots, B_5\}$.

We show that $W^\varphi$ and $W_{\boxplus}^{\psi}$ are equal almost everywhere also on $B_1 \times C$ later in the proof.

The subgraphon on $X \times C$ determines the order on the vertices of $X$ according to their relative degrees in $C$. We often use this fact when forcing other parts of the graphon. In this context, we will write $x \prec_X y$ instead of $\deg_C x < \deg_C y$ for $x, y \in X$. Abusing the notation, we will also write $Y \prec_X Z$ for $Y, Z \subseteq X$ such that $y \prec_X z$ for every $y \in Y$ and every $z \in Z$.

### 3.4.2 Forcing the structure on $A_1 \times A_1$

We now show that the main diagonal checker constraints, which are depicted in Figure 3.8 force that $W^\varphi$ and $W^\psi_\boxplus$ agree almost everywhere on $A_1 \times A_1$. Our line of arguments follows that in [42]. So, we only sketch the arguments.

The first condition in Figure 3.8 implies that $W^\varphi$ on $A_1 \times A_1$ is a union of disjoint cliques (and zero almost everywhere else). In particular, $W^\varphi$ is 0 or 1 almost everywhere on $A_1 \times A_1$. The second constraint determines the edge density on $A_1 \times A_1$. The third constraint implies that the cliques are disjoint intervals with respect to the ordering given by the part $C$ (up to sets of measure zero).

Let $\mathcal{J}$ be the set of nonempty intervals corresponding to the cliques forming $W$ on $A_1 \times A_1$. The intervals of $\mathcal{J}$ are linearly ordered by $\prec_{A_1}$. Let $m_J$ denote the measure of an interval $J \in \mathcal{J}$. The fourth constraint forces that for almost every two vertices $a_1 \prec_{A_1} a_2$ in one of the cliques in $A_1$,

$$\lambda(\{a \in A_1 | W^\varphi(a, a_1) = W^\varphi(a, a_2) = 1\})$$

$$= \lambda(\{a \in A_1 | W^\varphi(a, a_1) = 0, W^\varphi(a, a_2) = 0 \text{ and } a_1 \prec_{A_1} a\}).$$

Therefore, $m_I = \sum\limits_{I \prec_{A_1} J} m_J$. According to the second constraint, $\sum\limits_{J \in \mathcal{J}} m_J^2 = 1/3$. Thus, the $k$-th interval of $\mathcal{J}$ has measure $2^{-k}$ for every $k$. We conclude that $W^\varphi$ agrees with $W^\psi_\boxplus$ almost everywhere on $A_1 \times A_1$.

### 3.4.3 Forcing the remaining diagonal checker subgraphons

We now use the bipartition constraints, which are depicted in Figure 3.9 to force the structure of $A_1 \times A_2$, $A_1 \times B_1, A_1 \times B_2, A_1 \times B_3, A_1 \times B_4, A_1 \times B_5$, $A_2 \times A_3$ and $A_2 \times B_2$. The constraints are identical for all the pairs except for $A_1 \times B_1$. So we present the argument using $X \times Y$ for any of the above listed pairs except $A_1 \times B_1$. The case $A_1 \times B_1$ is discussed separately afterwards.

The first constraint in Figure 3.9 forces that $W^\varphi$ on $X \times Y$ is a union of disjoint complete bipartite subgraphons (and zero almost everywhere else). The second and the third constraints imply that the sides of these complete bipartite subgraphons form intervals in $X$ and $Y$ with respect to the ordering given by $C$ (up to sets of measure zero). The fourth constraint implies that the intervals are in the same order (with respect to $C$) in both $X$ and $Y$, i.e., if $I_1 \times J_1$ and $I_2 \times J_2$ are complete bipartite subgraphons, $I_1, I_2 \subseteq X$, $J_1, J_2 \subseteq Y$ and $I_1 \prec_X I_2$, then $J_1 \prec_Y J_2$.

It remains to determine the measures of the sides of the complete bipartite subgraphons.

Recall that we have shown that $W^\varphi$ agrees with $W^\psi_\boxplus$ almost everywhere on $A_1 \times A_1$. We show that the set of constraints depicted in Figure 3.10 forces that $W^\varphi$ and $W^\psi_\boxplus$ agree almost everywhere on $A_1 \times A_2$, $A_1 \times A_3$, $A_1 \times B_1, \dots, A_1 \times B_5$, $A_2 \times A_3$ and $A_2 \times B_2$.

The first constraint in Figure 3.10 implies that almost all the vertices of $A_1$ have the same relative degree in $A_1$ and in $Y$ for $Y \in \{A_2, B_1, \dots, B_5\}$. This determines the measures of the sides of complete bipartite subgraphons of $A_1 \times Y$, yielding that $W^\varphi$ and $W^\psi_\boxplus$ agree almost everywhere on $A_1 \times Y$ for every $Y \in \{A_2, B_2, \dots, B_5\}$.

The second constraint in Figure 3.10 determines the measures of the sides of complete bipartite subgraphons in $A_2 \times A_3$ and $A_2 \times B_2$ in $A_3$, $B_2$, respectively, yielding that $W^\varphi$ and $W^\psi_\boxplus$ agree almost everywhere on $A_2 \times A_3$ and $A_2 \times B_2$.

The third constraint forces that almost every vertex of $A_1$ has relative degree $1/2$ in $A_1$ or its relative degree in $A_3$ is twice as large as in $A_1$. Since the sum of measures of all the sides of bipartite subgraphons in $A_3$ is one, it follows that vertices of $A_1$ with relative degree $1/2$ in $A_1$ have relative degree 0 in $A_3$.

It remains to analyze the graphon on $A_1 \times B_1$. As before, the first two constraints in Figure 3.9 for $X = A_1$ and $Y = B_1$ force that $W^\varphi$ on $A_1 \times B_1$ is a union of disjoint complete bipartite subgraphons and that the sides of the complete bipartite subgraphons in $A_1$ form intervals with respect to the ordering given by $C$. The first constraint in Figure 3.10 for $Y = B_1$ implies that almost all the vertices of $A_1$ have the same relative degree in $B_1$ and in $A_1$. The choice of $\psi_{B_1}$ implies that $W^\varphi$ and $W^\psi_\boxplus$ agree almost everywhere on $A_1 \times B_1$.

### 3.4.4 Forcing the structure of $A_0 \times A_1$

We next show that the first level constraints, which are depicted in Figure 3.11 force $W^\varphi$ to be equal to $W^\psi_\boxplus$ almost everywhere on $A_0 \times A_1$. The first constraint implies that $\deg_{A_0} x = 0$ or $\deg_{A_1} x = 1/2$ for almost every vertex in $x \in A_1$. Since the set of vertices of relative degree $1/2$ in $A_1$ has measure $1/2$, the edge density on $A_0 \times A_1$ forced by the second constraint implies that $W^\varphi(x, y) = 1$ for almost every $x \in A_0$ and $y \in A_1$ with $\deg_{A_1} y = 1/2$. Therefore, $W^\varphi$ is equal to $W^\psi_\boxplus$ almost everywhere on $A_0 \times A_1$.

### 3.4.5 Partitioning $W$ into levels

The structure of $W^\varphi$ established so far allows us to split the parts $A_1$, $A_2$, $A_3$, $B_1, \dots, B_5$ of $W^\varphi$ into *levels*, in the same way as the parts of $W_\boxplus$ are split. We denote these levels by $A_{i,k}$ and $B_{j,k}$ for the parts $A_i$ and $B_j$ respectively.

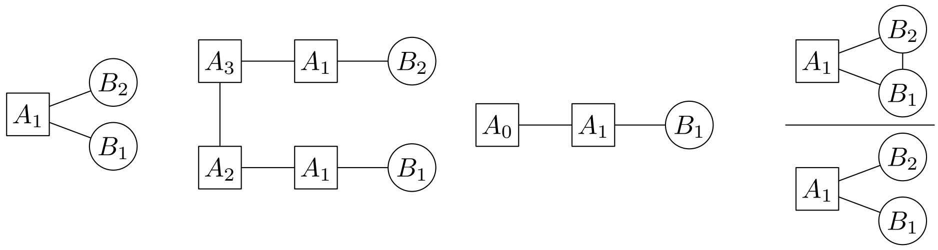


Figure 3.21: Density expressions specifying levels of vertices.

Formally, $A_{i,k}$, $i \in [2]$, is formed by $x \in A_i$ such that $\deg_{A_1} x = 2^{-k}$, $A_{3,k}$ is formed by $x \in A_3$ such that $\deg_{A_2} x = 2^{-k}$ and $B_{j,k}$, $j \in [5]$, is formed by $x \in B_j$ such that $\deg_{A_1} x = 2^{-k}$. Note that this coincides with our previous definition of $B_{1,k}$.

Note that the measure of $A_{i,k}$ and $B_{j,k}$ is $2^{-k}$ for every $k \in \mathbb{N}$, $i \in [3]$ and $j \in [5]$. Consequently, almost every vertex of the aforementioned parts belongs to some level. Let us give an example of use of this notation: $A_1 \times A_2$ consists of complete bipartite subgraphons with sides $A_{1,k}$ and $A_{2,k}$ for every $k \in \mathbb{N}$ and $A_1 \times A_3$ consists of complete bipartite subgraphons with sides $A_{1,k+1}$ and $A_{3,k}$ for every $k \in \mathbb{N}$.

The structure of the graphon $W^{\varphi}$ established so far allows to express relations between vertices from different parts of a graphon with respect to their containment in different levels. Some examples are given in Figure 3.21: the value of the first expression is equal to the probability that a random vertex of $B_1$ and a random vertex in $B_2$ belong to the same level. Similarly, the second expression is equal to the probability that a random vertex of $B_1$ and a random vertex of $B_2$ belong to the $i$-th and $(i+1)$-th level for some $i$. The third expression is equal to the probability that a random vertex of $B_1$ is in the first level. Finally, the last expression is equal to the probability that two random vertices of $B_1$ and $B_2$ of the same level are connected by an edge.

### 3.4.6 Stair constraints

Now, we focus on the stair constraints, which are depicted in Figure 3.12. They are intended to force the desired structure on $B_1 \times B_3$. The first constraint in Figure 3.12 determines the relative degrees of vertices of $B_1$ in $B_3$. The second constraint forces for almost every vertex $x \in B_{1,k_0}$ that if $x$ has nonzero relative degree in $B_{3,k}$, it has relative degree 1 in $B_{3,k-1}$, for every $k_0$ and $k \in [k_0]$. Consequently, such $x$ has relative degree 1 in every $B_{3,m}$, $m < k$.

Together, the constraints imply that for almost every $x \in B_{1,k_0}$ and almost every $y \in B_{3,k}$

$$W^\varphi(x,y) = \begin{cases} 1 & \text{if } k \leq k_0, \text{ and} \\ 0 & \text{if } k > k_0. \end{cases}$$

It follows that $W^\varphi$ agrees with $W^\psi_\boxplus$ almost everywhere on $B_1 \times B_3$.

### 3.4.7 Coordinate constraints

The coordinate constraints from Figure 3.13 force basic structure between the parts $B_1$ and $D$ on one side and the parts $B_2$, $B_4$ and $B_5$ on the other side. Here again, the constraints are identical for several pairs or parts, so we present the argument for $B_1 \times X$ in the case of the first constraint depicted in Figure 3.13 and $Y \times X$ in the case of the second constraint depicted in Figure 3.13, where $X \in \{B_2, B_4, B_5\}$ and $Y \in \{B_1, D\}$. The first constraint implies that almost every vertex $b$ of $B_1$ can have nonzero relative degree in $X_k$ only if it has nonzero relative degree in $B_{3,k}$, i.e., by 3.4.6, only if $b \in B_{1,k_0}$ for $k \leq k_0$. The second constraint implies that $N_Y(b' \setminus b)$ has measure zero for every $k$ and almost every two $b, b' \in X_k$ such that $b \prec_X b'$. This implies that $W^\varphi$ is equal to 0 or 1 almost everywhere on $Y \times X$.

The definition of $\psi$ on $B_1$ and $D$, and the just shown properties yield that $W^\varphi = W^\psi_\boxplus$ almost everywhere on $B_1 \times B_2$ and $D \times B_2$.

### 3.4.8 Initial coordinate constraint

The initial coordinate constraint can be found in Figure 3.15. It forces that

$$\deg_{B_{2,1}} b = \frac{\deg_C b - (1 - 2\deg_{A_1} b)}{\deg_{A_1} b}$$

for almost every $b \in B_1$. This implies that $W^\varphi$ agrees with $W^\psi_\boxplus$ almost everywhere on $B_1 \times C$. This and the triangular constraints for $B_1 \times C$ yield that every $g_n \in \mathfrak{G}$ satisfies (3.1) for $k = 1$.

### 3.4.9 Distribution constraints

The first constraint in Figure 3.14 implies that the relative degrees of vertices of $B_{2,k_0}$ in $B_{1,k}$, $k_0 \leq k$, are uniformly distributed. In particular, it holds for every $k \in \mathbb{N}$ and every $k_0 \in [k]$ that

$$\deg_{B_{1,k}} b = 1 - \frac{\deg_C b - (1 - 2\deg_{A_1} b)}{\deg_{A_1} b}$$

for almost every $b \in B_{2,k_0}$. The definition of $B_{2,k_0}$ yields that $\deg_{A_1} b = 2^{-k_0}$ for almost every $b \in B_{2,k_0}$. So, we get that

$$\deg_{B_{1,k}} b = 1 - 2^{k_0}\left(\deg_C b - \left(1 - 2^{-(k_0-1)}\right)\right)$$

for every $k \in \mathbb{N}$, every $k_0 \in [k]$ and almost every $b \in B_{2,k_0}$.

This means that relative degree of almost every $b \in B_{2,k_0}$ in $B_{1,k}$ decreases linearly from 1 to 0 with its position within $B_{2,k_0}$ given by $\prec_{B_2}$.

The second constraint in Figure 3.14 implies that the same is true for degrees of vertices of $B_{2,k}$ in $D$.

### 3.4.10 Product constraints

The product constraints, which are depicted in Figures 3.16 and 3.17, imply that $W^\varphi$ and $W_\boxplus^\psi$ are equal almost everywhere on $B_1 \times B_4$, $B_1 \times B_5$, $D \times B_4$ and $D \times B_5$.

Recall that the structure of the graphon $W^\varphi$ established so far implies that $W$ has only values 0 and 1 almost everywhere on $B_1 \times B_4$, $B_1 \times B_5$, $D \times B_4$ and $D \times B_5$ and the neighborhood of almost every vertex of $b \in B_1$ and $d \in D$ in $B_4$ and $B_5$ is determined by its relative degree up to a set of measure zero. Therefore, it is sufficient to show that these relative degrees are determined by the product constraints. We present the argument for $B_1$, the argument for $D$ is analogous.

The first constraint in Figure 3.16 implies that $\deg_{B_{4,1}} b = \deg_{B_{2,1}} b$ for almost every vertex $b \in B_1$. The second constraint forces $\deg_{B_{4,i}} b = \deg_{B_{2,i}} b \cdot \deg_{B_{4,i-1}} b$ for almost every $b \in B_1$ and $i > 1$.

Similarly, the constraints depicted in Figure 3.17 imply that

$$1 - \deg_{B_{5,1}} b = \deg_{B_{2,1}} b$$

for almost every vertex $b \in B_1$, and

$$\deg_{B_{5,i}} b = (1 - \deg_{B_{2,i}} b) \cdot \deg_{B_{5,i-1}} b$$

for almost every $b \in B_1$ and $i > 1$.

This implies that $W^\varphi$ and $W_\boxplus^\psi$ are equal almost everywhere on $B_1 \times B_4$, $B_1 \times B_5$, $D \times B_4$ and $D \times B_5$.

### 3.4.11 Projection constraints

We now establish that the projection constraints, which can be found in Figure 3.18, force that $W^\varphi$ equals to $W^\psi_\boxplus$ almost everywhere on $B_1 \times B_1$. We define $\tilde{g}_k(x)$ to be $g_k(\eta_{B_{1,k}}(x))$ for $x \in B_{1,k}$ to simplify our notation throughout the subsection (recall that $g_k(\eta_{B_{1,k}}(x)) = (\deg_{B_{2,i}} x)_{i\in[k]})$. In what follows we write $\leq_k$ and $\geq_k$ for lexicographic order of the first $k$ elements of sequences. That is, $(a_i)_{i\in[k_1]} \leq_k (b_i)_{i\in[k_2]}$, $k_1, k_2 \geq k$ if $a_i \leq b_i$ for every $i \in [k]$. In particular, we write $\tilde{g}_{k_1}(x) \leq_k \tilde{g}_{k_2}(y)$, $k_1, k_2 \geq k$ if $(\tilde{g}_{k_1}(x))_i \leq (\tilde{g}_{k_2}(y))_i$ for every $i \in [k]$.

We start by showing that $W^\varphi$ equals 0 or 1 almost everywhere on $B_{1,k} \times B_{1,k'}$ for every $k \in \mathbb{N}$ and $k' > k$ and that

$$\lambda\{b \in B_{1,k} |\ \tilde{g}_k(b) \leq_k (a_i)_{i\in[k]}\} = \lambda(B_{1,k}) \prod_{i\in[k]} a_i \quad \text{for every } (a_i)_{i\in[k]} \in [0,1]^k. \tag{3.2}$$

Note that (3.2) implies that

$$\tilde{g}_k(B_{1,k} \setminus Z) \text{ is dense in } [0,1]^k \text{ for every } k \in \mathbb{N} \text{ and } Z \text{ of measure zero.} \tag{3.3}$$

Our argument proceeds by induction on $k$. Recall that the initial coordinate constraints guarantee that (3.2) holds for $k = 1$.

We now focus on the induction step. The first constraint in Figure 3.18 forces that the set $N_{B_2}(b \setminus b')$ has measure zero for almost every pair of vertices $b \in B_{1,k}$ and $b' \in B_{1,k'}$, $k < k'$, such that $W^\varphi(b, b') > 0$. This implies for almost every pair $b \in B_{1,k}$ and $b' \in B_{1,k'}$, $k < k'$, with $W^\varphi(b, b') > 0$, that $\deg_{B_{2,i}} b \leq \deg_{B_{2,i}} b'$ for every $i \in [k]$. In other words,

$$N_{B_{1,k}}(b') \setminus \{b \in B_{1,k} |\ \tilde{g}_k(b) \leq_k \tilde{g}_{k'}(b')\}$$

has measure zero for almost every $b' \in B_{1,k'}$ and

$$N_{B_{1,k'}}(b) \setminus \{b' \in B_{1,k'} |\ \tilde{g}_{k'}(b') \geq_k \tilde{g}_k(b)\}$$

has measure zero for almost every $b \in B_{1,k}$, $k < k'$.

The second constraint forces that

$$\deg_{B_{1,k}} b' = \deg_{B_{4,k}} b'$$

for almost every $b' \in B_{1,k'}$, $k' > k$. We have shown in Subsection 3.4.10 that

$$\deg_{B_{4,k}} b' = \prod_{i\in[k]} (\tilde{g}_{k'}(b'))_i \tag{3.4}$$

for almost every $b' \in B_{1,k'}$, $k' > k$. It follows that if (3.2) holds for $k$, then the second constraint implies for every $k' > k$ that $W^\varphi$ equals to 0 or 1 almost everywhere on $B_{1,k} \times B_{1,k'}$, and that

$$N_{B_{1,k}}(b') \triangle \{b \in B_{1,k} | \ \tilde{g}_k(b) \leq_k \tilde{g}_{k'}(b')\}$$

has measure zero for almost every $b' \in B_{1,k'}$, and that

$$N_{B_{1,k'}}(b) \triangle \{b' \in B_{1,k'} | \ \tilde{g}_{k'}(b') \geq_k \tilde{g}_k(b)\} \tag{3.5}$$

has measure zero for almost every $b \in B_{1,k}$.

To complete the induction step, we should show that (3.2) holds for $k+1$ assuming it holds for $k$. The third constraint depicted in Figure 3.18 guarantees that

$$\deg_{B_{1,k+1}} b = \deg_{B_{5,k}} b = \prod_{i \in [k]} (1 - (\tilde{g}_k(b))_i)$$

for almost every $b \in B_{1,k}$. This combined with (3.2) yields that

$$\lambda(\{b' \in B_{1,k'} | \ \tilde{g}_{k'}(b') \geq_k \tilde{g}_k(b)\}) = \prod_{i \in [k]} (1 - (\tilde{g}_k(b)_i)) \tag{3.6}$$

for almost every $b \in B_{1,k}$.

The fourth constraint implies that

$$\deg_{B_{1,k+1}} b = \deg_{B^x_{1,k+1}} b \tag{3.7}$$

for almost every $b \in B_{1,k}$ and $x \in B_{2,k+1}$ where $B^x_{1,k+1}$ is the set of vertices $y \in B_{1,k+1}$ with $W^\varphi(x,y) = 1$ (recall that $W^\varphi$ is equal to 0 or 1 almost everywhere on $B_1 \times B_2$ as shown in Subsection 3.4.7). Note that the structure of the graphon established in Subsection 3.4.7 yields that $B^x_{1,k+1}$ is the set of vertices $y \in B_{1,k+1}$ of the relative degree at least $a_x$ in $B_{2,k+1}$ for some $a_x \in [0,1]$ (up to a set of measure zero and for almost every $x \in B_{2,k+1}$). Hence, the equality (3.7) guarantees that

$$\begin{aligned} &\frac{\lambda(\{b' \in B_{1,k+1} | \ \tilde{g}_{k+1}(b') \geq_k \tilde{g}_k(b)\})}{\lambda(B_{1,k+1})} \\ &= \frac{\lambda(\{b \in B_{1,k+1} | \ \tilde{g}_{k+1}(b') \geq_k \tilde{g}_k(b) \text{ and } (\tilde{g}_{k+1}(b'))_{k+1} \geq a_{k+1}\})}{\lambda(\{b' \in B_{1,k+1} | (\tilde{g}_{k+1}(b'))_{k+1} \geq a_{k+1}\})} \end{aligned} \tag{3.8}$$

for almost every $b \in B_{1,k}$ and every $a_{k+1} \in [0,1]$.

The structure of the graphon established in Subsection 3.4.9 implies that

$$\lambda(\{b' \in B_{1,k+1} | (\tilde{g}_{k+1}(b'))_{k+1} \geq a_{k+1}\}) = (1 - a_{k+1})\lambda(B_{1,k+1})$$

for every $a_{k+1} \in [0,1]$. This combined with (3.6) and (3.8) yields that

$$\begin{aligned}&\frac{\prod_{i\in[k]}(1-(\tilde{g}_k(b)_i))}{\lambda(B_{1,k+1})}\\ &= \frac{\lambda(\{b \in B_{1,k+1}| \ \tilde{g}_{k+1}(b') \geq_k \tilde{g}_k(b) \text{ and } (\tilde{g}_{k+1}(b'))_{k+1} \geq a_{k+1}\})}{(1-a_{k+1})\lambda(B_{1,k+1})}\end{aligned}$$

for almost every $b \in B_{1,k}$ and every $a_{k+1} \in [0,1]$. We conclude using (3.3) that

$$\lambda(\{b' \in B_{1,k+1}| \ \tilde{g}_{k+1}(b') \geq_{k+1} (a_i)_{i\in[k+1]}\}) = \prod_{i\in[k+1]} (1 - a_i)$$

for every $(a_i)_{i\in[k+1]} \in [0,1]^{k+1}$. By the principle of inclusion and exclusion, the equality (3.2) holds for $k+1$. The completion of the induction step yields that $g_n$ satisfies (3.1) for all $k, n \in \mathbb{N}$, in particular, $\psi_{B_1}$ is a measure preserving map.

We have shown that $B_{1,k} \times B_{1,k'}$ in $W^\varphi$ and $W^\psi_\boxplus$ agree almost everywhere for $k \neq k'$. It remains to analyze the structure of the graphon $W^\varphi$ on $B_{1,k} \times B_{1,k}$ for $k \in \mathbb{N}$. The first constraint in Figure 3.19 forces for every $k \in \mathbb{N}$ that $N_{B_2}(b' \setminus b)$ or $N_{B_2}(b \setminus b')$ has measure zero for almost all $b, b' \in B_{1,k}$ with $W^\varphi(b,b') > 0$. Since $\psi_{B_1}$ is a measure preserving map and the graphons $W^\varphi$ and $W^\psi_\boxplus$ are equal almost everywhere on $B_1 \times B_2$, $B_1 \times B_4$ and $B_1 \times B_5$, it follows that

$$\begin{aligned}\lambda(N_{B_{1,k}}(b)) &\leq \lambda(B_{1,k}) \left( \prod_{i=1}^{k} \deg_{B_{2,i}} b + \prod_{i=1}^{k} (1 - \deg_{B_{2,i}} b) \right)\\ &= \lambda(B_{1,k}) \left( \deg_{B_{4,k}} b + \deg_{B_{5,k}} b \right)\end{aligned}$$

for almost every $b \in B_{1,k}$.

The second constraint in Figure 3.19 implies that $\deg_{B_{1,k}} b = \deg_{B_{4,k}} b + \deg_{B_{5,k}} b$ for almost every $b \in B_{1,k}$. Hence, $W^\varphi$ has to be equal to 1 almost everywhere on $B_{1,k} \times B_{1,k}$, where it does not have to be zero by the fourth constraint. Therefore, the last two projection constraints imply that $W^\varphi$ equals to $W^\psi_\boxplus$ almost everywhere on $B_{1,k} \times B_{1,k}$ and thus on the whole $B_1 \times B_1$.

### 3.4.12 Infinite constraints

In this subsection, we prove that $W^\varphi$ equals to $W^\psi_\boxplus$ almost everywhere on $B_1 \times D$, by proving they are equal almost everywhere on $B_{1,k} \times D$ for every $k \in \mathbb{N}$. We also prove that $\psi_D$ is a measure preserving map by showing that $g_\infty$ satisfies (3.1) for every $k$. Let $k \in \mathbb{N}$ be fixed for the rest of the subsection.

Let $d \in D$ and $b \in B_{1,k}$. We define

$$M^k_{B_1}(d) = \{b \in B_{1,k} |\, \deg_{B_{2,i}} b \le \deg_{B_{2,i}} d \ \forall i \in [k]\} \text{ and}$$

$$M^k_D(b) = \{d \in D |\, \deg_{B_{2,i}} d \ge \deg_{B_{2,i}} b \ \forall i \in [k]\}.$$

Note that (3.2) is equivalent to

$$\lambda(\{b \in B_{1,k} |\, \deg_{B_{2,i}} b \le a_i \ \forall i \in [k]\} = \lambda(B_{1,k}) \cdot \prod_{i=1}^{k} a_i$$

for every $(a_i)_{i\in[k]} \in [0,1]^k$. Therefore, it holds that

$$\lambda(M^k_{B_1}(d)) = \lambda(B_{1,k}) \cdot \prod_{i=1}^{k} \deg_{B_{2,i}} d$$

for almost every $d \in D$.

The first constraint in Figure 3.20 forces that $N_{B_2}(b \setminus d)$ has measure zero for almost every $b \in B_1$, $d \in D$ with $W^\varphi(b,d) > 0$. It follows that $N_{B_{1,k}}(d) \setminus M^k_{B_1}(d)$ and $N_D(b) \setminus M^k_D(b)$ have measure zero for almost every $b \in B_{1,k}$ and $d \in D$.

The second constraint in Figure 3.20 implies that $\deg_{B_{1,k}} d = \deg_{B_{4,k}} d$ for almost every $d \in D$. We have shown in Subsection 3.4.10 that $\deg_{B_{4,k}} d = \prod_{i=1}^k \deg_{B_{2,i}} d$ for almost every $d \in D$. Therefore,

$$\begin{aligned}\lambda\left(N_{B_{1,k}}(d)\right) &\ge \lambda(B_{1,k}) \cdot \deg_{B_{1,k}} d = \lambda(B_{1,k}) \cdot \deg_{B_{4,k}} d \\ &= \lambda(B_{1,k}) \prod_{i=1}^{k} \deg_{B_{2,i}} d = \lambda\left(M^k_{B_1}(d)\right)\end{aligned}$$

for almost every $d \in D$. Since the measure of $N_{B_{1,k}}(d) \setminus M^k_{B_1}(d)$ is zero for almost every $d \in D$, it follows that $\lambda(N_{B_{1,k}}(d) \triangle M^k_{B_1}(d)) = 0$ for almost every $d \in D$ and $W^\varphi(b,d) = 1$ for almost every $d \in D$ and $b \in N_{B_{1,k}}(d)$.

We have shown that $W^\varphi(b,d) = 1$ for almost any $b \in B_{1,k}$ and $d \in D$ such that $\deg_{B_{2,i}} b \le \deg_{B_{2,i}} d$ for every $i \in [k]$, and it is zero almost everywhere else in $B_1 \times D$. This implies that $W^\varphi$ equals to $W^\psi_\boxplus$ almost everywhere on

$B_1^{\boxplus} \times D^{\boxplus}$. We now show that $\psi_D$ is a measure preserving map. In particular, we need to show that $g_\infty$ satisfies (3.1) for every $k \in \mathbb{N}$.

The third constraint in Figure 3.20 implies that

$$\deg_D b = \deg_{B_{5,k}} b = \prod_{i=1}^{k} (1 - \deg_{B_{2,i}} b)$$

for almost every $b \in B_{1,k}$. Thus,

$$\lambda\left(M_D^k(b)\right) = \lambda\left(N_D(b)\right) = \lambda(D) \cdot \deg_D b = \lambda(D) \prod_{i=1}^{k} (1 - \deg_{B_{2,i}} b)$$

for almost every $b \in B_{1,k}$. We deduce by the principle of inclusion and exclusion that

$$\frac{\lambda(\{d \in D | \deg_{B_{2,i}} d \leq \deg_{B_{2,i}} b \ \forall i \in [k]\})}{\lambda(D)} = \prod_{i=1}^{k} \deg_{B_{2,i}} b \tag{3.9}$$

for almost every $b \in B_{1,k}$.

Finally, observe that the definition of $g_\infty$ yields

$$g_\infty(\eta_D(d)) = (\deg_{B_{2,i}} d)_{i \in \mathbb{N}} \text{ for } d \in D.$$

It follows from (3.3) and (3.9) that $g_\infty$ is measure preserving.

### 3.4.13 Structure involving the parts $E_1, E_2$ and $F$

Let $I = [0, 1] \setminus (E_1 \cup E_2 \cup F)$. The degree unifying constraints, which are depicted in Figure 3.4, imply that for every $X, Y \in \{A_0, \ldots, A_3, B_1, \ldots, B_5, C\}$ and almost every $x \in X, y \in Y$:

$$\int_{E_1} W^\varphi(x, z) \, \mathrm{d}z = (1 - \deg_I x) \text{ and}$$

$$\int_{E_1} W^\varphi(x, z) W^\varphi(y, z) \, \mathrm{d}z = (1 - \deg_I x)(1 - \deg_I y).$$

Following the reasoning given in [65, proof of Lemma 3.3], this implies that

$$\int_{E_1} (W^\varphi(x, z))^2 \, \mathrm{d}z = (1 - \deg_I x)^2$$

for almost every $x \in X$. The Cauchy-Schwarz inequality yields that $W^{\varphi}(x,z) = 1 - \deg_I x$ for almost every $x \in X$ and $z \in E_1$. This implies that

$$\deg_{[0,1]\setminus(E_2 \cup F)} x = 1/2$$

for almost every $x \in I \setminus D$. Since $W^{\psi}_{\boxplus} = W^{\varphi}$ almost everywhere on $I^2$, almost every $x \in I$ has the same relative degree on $I$ in both $W^{\psi}_{\boxplus}$ and $W^{\varphi}$ which yields that $W^{\psi}_{\boxplus} = W^{\varphi}$ almost everywhere on $I \times E_1$.

Similarly, the constraints depicted in Figure 3.5 imply that

$$\deg_{B_1 \cup B_2 \cup B_4 \cup B_5 \cup E_2} x = 1/2$$

for almost every $x \in D$ and that $W^{\psi}_{\boxplus} = W^{\varphi}$ almost everywhere on $D \times E_2$.

Finally, the two degree distinguishing constraints in Figure 3.6 force that $W^{\varphi}$ on $[0,1] \times F$ is formed by pseudorandom bipartite subgraphons $X \times F$ for $X = A_1, \ldots, A_3, B_1, \ldots, B_5, C$, with densities given by Table 3.2. Thus, $W^{\varphi}$ is equal to $W^{\psi}_{\boxplus}$ almost everywhere on $[0,1] \times F$.

# Chapter 4

# Erdős-Lovász-Spencer theorem for permutations

In this chapter, we give a permutation analogue of the result of Erdős, Lovász and Spencer about subgraph densities in a graph [35]. In particular, we show that the body of possible densities of any $k$ indecomposable permutations in permutons has a non-empty interior and is full dimensional.

We start by introducing notion of densities of subpermutations in a permutation corresponding to the induced subgraph density, the homomorphism density and the subgraph density for graphs studied in [35].

## 4.1 Permutation densities

Let $\pi$ be a permutation of order $k$ and $\sigma$ be a permutation of order $n$. We introduce three ways in which $\pi$ can appear in $\sigma$: as a subpermutation, through a monomorphism and through a homomorphism. First, note that the notion of subpermutation an be equivalently defined as follows; a permutation $\pi$ is a *subpermutation of* $\sigma$ if there exists a strictly increasing function $f : [k] \to [n]$, such that $\pi(i) > \pi(j)$ if and only if $\sigma(f(i)) > \sigma(f(j))$ for every $i, j \in [k]$. Let $\mathrm{Occ}(\pi, \sigma)$ be the set of all such functions $f$ from $[k]$ into $[n]$ and let $\Lambda(\pi, \sigma) = |\,\mathrm{Occ}(\pi, \sigma)|$. Then, the *density* of $\pi$ in $\sigma$ can be computed as

$$t(\pi, \sigma) = \begin{cases} \Lambda(\pi, \sigma)\binom{n}{k}^{-1} & \text{if } k \leq n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

A non-decreasing function $f : [k] \to [n]$ is a *homomorphism* of $\pi$ to $\sigma$ if $\sigma(f(i)) > \sigma(f(j))$ for every $i, j \in [k]$ such that $i < j$ and $\pi(i) > \pi(j)$, that is, $f$ preserves inversions. A *monomorphism* is a homomorphism that is injective.

Let $\mathrm{Hom}(\pi, \sigma)$ and $\mathrm{Mon}(\pi, \sigma)$ denote the sets of homomorphisms and monomorphisms of $\pi$ to $\sigma$, respectively, and let $\Lambda_{\mathrm{hom}}(\pi, \sigma)$ and $\Lambda_{\mathrm{mon}}(\pi, \sigma)$ denote the sizes of the respective sets. Note that $\mathrm{Occ}(\pi, \sigma) \subseteq \mathrm{Mon}(\pi, \sigma) \subseteq \mathrm{Hom}(\pi, \sigma)$. The *homomorphism density* $t_{\mathrm{hom}}$ and *monomorphism density* $t_{\mathrm{mon}}$ are defined as follows:

$$t_{\mathrm{mon}}(\pi, \sigma) = \begin{cases} \Lambda_{\mathrm{mon}}(\pi, \sigma)\binom{n}{k}^{-1} & \text{if } k \leq n \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

$$t_{\mathrm{hom}}(\pi, \sigma) = \Lambda_{\mathrm{hom}}(\pi, \sigma)\binom{n+k-1}{k}^{-1}.$$

The three densities that we have just introduced are analogues of densities for graphs studied in [35].

Let $q$ be an integer and let $\{\tau_1, \ldots, \tau_r\}$ be the set of all non-trivial indecomposable permutations of order at most $q$. We consider the following three vectors

$$\begin{aligned} \mathbf{t}^q(\sigma) &= (t(\tau_1, \sigma), \ldots, t(\tau_r, \sigma)), \\ \mathbf{t}^q_{\mathrm{mon}}(\sigma) &= (t_{\mathrm{mon}}(\tau_1, \sigma), \ldots, t_{\mathrm{mon}}(\tau_r, \sigma)), \text{ and} \\ \mathbf{t}^q_{\mathrm{hom}}(\sigma) &= (t_{\mathrm{hom}}(\tau_1, \sigma), \ldots, t_{\mathrm{hom}}(\tau_r, \sigma)). \end{aligned}$$

Our aim is to understand possible densities of subpermutations in large permutations. This leads to the following definitions, which reflect the possible asymptotic densities of the indecomposable permutations of order at most $q$ in permutations:

$$\begin{aligned} T^q &= \{\mathbf{v} \in \mathbb{R}^r \mid \exists (\sigma_n)_{n\in\mathbb{N}} \text{ such that } \mathbf{t}^q(\sigma_n) \to \mathbf{v} \text{ and } |\sigma_n| \to \infty\}, \\ T^q_{\mathrm{mon}} &= \{\mathbf{v} \in \mathbb{R}^r \mid \exists (\sigma_n)_{n\in\mathbb{N}} \text{ such that } \mathbf{t}^q_{\mathrm{mon}}(\sigma_n) \to \mathbf{v} \text{ and } |\sigma_n| \to \infty\}, \text{ and} \\ T^q_{\mathrm{hom}} &= \{\mathbf{v} \in \mathbb{R}^r \mid \exists (\sigma_n)_{n\in\mathbb{N}} \text{ such that } \mathbf{t}^q_{\mathrm{hom}}(\sigma_n) \to \mathbf{v} \text{ and } |\sigma_n| \to \infty\}. \end{aligned}$$

The subpermutation density $t(\tau, \Phi)$ of a permutation $\tau$ of order $n$ in a permuton $\Phi$ is the probability that a $\Phi$-random permutation of order $n$ is $\tau$. Likewise, we can define the *monomorphism density* of $\tau$ as the probability that the identity mapping to a random $\Phi$-permutation is a monomorphism of $\tau$.

Since we view permutons as representing large permutations, if we define homomorphism densities in a natural way, they would coincide with monomorphism densities. So, we restrict our study to subpermutation densities and monomorphism densities in permutons. By analogy to the finite case, we define the vectors

$$\mathbf{t}^q(\Phi) = (t(\tau_1, \Phi), \ldots, t(\tau_r, \Phi)) \text{ and}$$
$$\mathbf{t}^q_{\text{mon}}(\Phi) = (t_{\text{mon}}(\tau_1, \Phi), \ldots, t_{\text{mon}}(\tau_r, \Phi)),$$

where $q \in \mathbb{N}$ and $\{\tau_1, \ldots, \tau_r\}$ is the set of all non-trivial indecomposable permutations of order at most $q$.

A sequence of $\Phi$-random permutations of increasing orders converges with probability one with its limit being $\Phi$ with probability one. Therefore, for every permuton $\Phi$ and every finite set of permutations $S$ and every $\varepsilon > 0$, there exists a permutation $\varphi$ such that $|t(\pi, \Phi) - t(\pi, \varphi)| < \varepsilon$ for every $\pi \in S$. This yields an alternative description of $T^q$ as the set $\{\mathbf{t}^q(\Phi) \mid \Phi \in \mathfrak{P}\}$. Similarly, $T^q_{\text{mon}} = \{\mathbf{t}^q_{\text{mon}}(\Phi) \mid \Phi \in \mathfrak{P}\}$.

Now we give three observations on how the sets $T^q$, $T^q_{\text{mon}}$ and $T^q_{\text{hom}}$ relate to each other.

**Observation 26.** *The sets $T^q_{\text{mon}}$ and $T^q_{\text{hom}}$ are equal for every $q \in \mathbb{N}$.*

*Proof.* Observe that for every fixed integer $k$,

$$\Lambda_{\text{hom}}(\tau, \sigma) - \Lambda_{\text{mon}}(\tau, \sigma) \leq \binom{k}{2} n^{k-1} = O(n^{k-1}),$$

for every $\sigma$ of order $n$ and $\tau$ of order $k$.

Hence, for every permutation $\tau$ and every real $\varepsilon > 0$ there exists $n_0$ such that $|t_{\text{mon}}(\tau, \sigma) - t_{\text{hom}}(\tau, \sigma)| < \varepsilon$ for every permutation $\sigma$ with $|\sigma| > n_0$. The statement now follows. □

In view of Observation 26, we will discuss only $T^q_{\text{mon}}$ in the rest of the chapter.

**Observation 27.** *For every $q \in \mathbb{N}$, the set $T^q_{\text{mon}}$ is closed.*

*Proof.* Let $(\mathbf{w}_n)_{n \in \mathbb{N}} \subseteq T^q_{\text{mon}}$ be a convergent sequence and let $\mathbf{w} = \lim_{n \to \infty} \mathbf{w}_n$. For each $n$, choose $\sigma_n$ such that $\|\mathbf{t}^q_{\text{mon}}(\sigma_n) - \mathbf{w}_n\| \leq 1/n$. Then, the sequence $(\mathbf{t}^q_{\text{mon}}(\sigma_n))_{n \in \mathbb{N}}$ converges to $\mathbf{w}$, i.e., $\mathbf{w} \in T^q_{\text{mon}}$. □

**Observation 28.** *The set $T^q$ is a non-singular linear transformation of $T^q_{\text{mon}}$ for every $q \in \mathbb{N}$.*

*Proof.* Note that $\Lambda_{\text{mon}}(\pi, \sigma) = \sum_{\pi' \in S} \Lambda(\pi', \sigma)$, where $S$ is a set of permutations $\pi'$ of the same order as $\pi$ such that the identity mapping is a monomorphism from $\pi$ to $\pi'$. Consequently, $t_{\text{mon}}(\pi, \sigma) = \sum_{\pi' \in S} t(\pi', \sigma)$. This gives that $T^q_{\text{mon}}$ is a linear transformation of $T^q$. Observe that if we order $\tau_1, \ldots, \tau_r$ by the

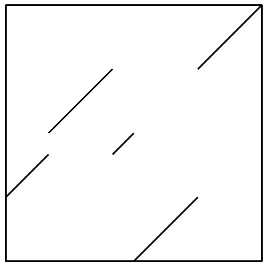

Figure 4.1: The permuton $\Phi_\sigma^{\mathbf{v}}$ for $\sigma = (2,4,3,1)$ and $\mathbf{v} = (1/6, 1/4, 1/12, 1/4)$.

number of inversions, the coefficient matrix of the induced linear mapping is upper triangular with diagonal entries equal to 1. We conclude that the linear transformation of $T^q$ is non-singular. $\square$

We conclude this section by deriving formulas for densities of indecomposable permutations in direct products of permutons and in *step-up permutons*, which are permutons with simple structure corresponding to a weighted permutation.

Recall that a permutation $\sigma$ of order $n$ is *indecomposable* if there is no $m < n$ such that $\sigma([m]) = [m]$.

**Observation 29.** *Let $\tau$ be a non-trivial indecomposable permutation of order $k$ and let $m$ be a positive integer. Let $\Phi_1, \dots, \Phi_m$ be permutons and let $(x_1, \dots, x_m) \in \mathbb{R}_+^m$ be such that $\sum_{i \in [m]} x_i = 1$. The permuton $\Phi = \bigoplus_{i\in[m]} x_i \Phi_i$ satisfies*

$$t(\tau, \Phi) = \sum_{i=1}^{m} x_i^k t(\tau, \Phi_i).$$

Observation 29 is based on the fact that if $k$ random points with distribution $\Phi_\sigma^{\mathbf{p}}$ induce an indecomposable permutation $\tau$, then all the points lie in the same square corresponding to one of the permutons $\Phi_i$.

Let $\sigma$ be a permutation of order $n$ and let $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}_+^n$ be such that $\sum_{i\in[n]} v_i \le 1$, where $\mathbb{R}_+$ is the set of positive reals. The *step-up permuton* of $\sigma$ and $\mathbf{v}$ is the permuton $\Phi_\sigma^{\mathbf{v}} = \sum_{i\in[n+1]} v_i \Upsilon_{A_i}$, where $A_i$ is a segment between the points $(\sum_{j<i} v_j, \sum_{\sigma(j)<\sigma(i)} v_j)$ and $(\sum_{j\le i} v_j, \sum_{\sigma(j)\le\sigma(i)} v_j)$ for $i \in [n]$, $v_{n+1} = 1 - \sum_{j\in[n]} v_j$ and $A_{n+1}$ is the segment between the points $(\sum_{j\in[n]} v_j, \sum_{j\in[n]} v_j)$ and $(1,1)$. Note that this is indeed a permuton. See Figure 4.1 for an example.

For a permutation $\tau$ of order $k$, we call a partition $\mathcal{P} = \{P_1, \dots, P_\ell\}$ of $[k]$ *$\tau$-compressive* if

- $P_i$ is an interval for every $i \in [\ell]$,

- $a < b$ for every $a \in P_i$ and $b \in P_j$ with $i < j$, and
- for every $i \in [\ell]$, there exists an integer $c_i$, such that $\tau(a) = a + c_i$ for every $a \in P_i$. (In particular, $\tau(P_i)$ is interval for every $i \in [\ell]$.)

We denote the set of all $\tau$-compressive partitions by $\mathcal{R}(\tau)$. Note that for every permutation $\tau$, there exist at least one $\tau$-compressive partition: the partition into singletons.

For a permutation $\tau$ of order $k$ and a $\tau$-compressive partition $\mathcal{P} = \{P_1, \ldots, P_\ell\}$, let $\tau/\mathcal{P}$ be a subpermutation of $\tau$ of order $\ell$ induced by $\{a_1, \ldots, a_\ell\}$ where $a_i \in P_i$ for every $i \in [\ell]$. Note that $\tau/\mathcal{P}$ is unique, in particular, it is independent of the choice of the elements $a_i$.

In other words, the permutation $\tau/\mathcal{P}$ is a permutation that can be obtained from $\tau$ by shrinking each interval $P_i$ and its image into single points, without changing the relative order of the elements of the permutation.

**Observation 30.** *Let $\tau$ be a non-trivial indecomposable permutation of order $k$, $\sigma$ a permutation of order $n \geq k$ and let $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n_+$ be such that $\sum_{i\in[n]} p_i \leq 1$. It follows that*

$$t(\tau, \Phi^{\mathbf{p}}_\sigma) = k! \sum_{\mathcal{P}\in\mathcal{R}(\tau)} \sum_{\psi\in \mathrm{Occ}(\tau/\mathcal{P},\sigma)} \prod_{i=1}^{|\mathcal{P}|} p_{\psi(i)}^{|P_i|} .$$

Informally speaking, Observation 30 holds because for a fixed indecomposable permutation $\tau$ of order $k$, $k$ random points chosen based on the distribution $\Phi^{\mathbf{p}}_\sigma$ induce $\tau$ if and only if none of the $k$ points lies on the last segment of the support of $\Phi^{\mathbf{p}}_\sigma$ and there is a $\tau$-compressive partition $\mathcal{P}$ such that points corresponding to the elements of $\tau$ in the same part of $\mathcal{P}$ lie on the same segment of the support of $\Phi^{\mathbf{p}}_\sigma$, and the elements of $\sigma$ corresponding to these segments induce $\tau/\mathcal{P}$ in $\sigma$.

Analogues of Observations 29 and 30 for densities of monomorphisms also hold.

## 4.2 Properties of the sets $T^q$ and $T^q_{\mathrm{mon}}$

In this section, we show that densities of non-trivial indecomposable permutations are mutually independent and, more generally, that $T^q$ contains a ball. We start by considering the linear span of $T^q$.

**Lemma 31.** *For every $q \in \mathbb{N}$, $\mathrm{span}(T^q) = \mathbb{R}^r$, where $r$ is the number of non-trivial indecomposable permutations of order at most $q$.*

*Proof.* Let $\{\tau_1, \ldots, \tau_r\}$ be the set of all non-trivial indecomposable permutations of order at most $q$. For a contradiction, suppose that $\operatorname{span}(T^q)$ has dimension less than $r$, i.e., there exist reals $c_1, \ldots, c_r$, not all of which are zero, such that

$$\sum_{i=1}^{r} c_i v_i = 0$$

for every $(v_1, \ldots, v_r) \in \operatorname{span}(T^q)$. Therefore,

$$\sum_{i=1}^{r} c_i t(\tau_i, \Phi) = 0$$

for every permuton $\Phi \in \mathfrak{P}$.

Consider the permutations $\tau_i$ such that $c_i \neq 0$. Among these pick a $\tau_k$ of maximum order. Observation 30 yields that the following holds for $s = |\tau_k|$ and every $\mathbf{x} = (x_1, \ldots x_s) \in \mathbb{R}_+^s$ such that $\sum_{i=1}^{s} x_i \leq 1$:

$$\sum_{i=1}^{r} c_i t(\tau_i, \Phi_{\tau_k}^{\mathbf{x}}) = \sum_{i=1}^{r} c_i |\tau_i|! \sum_{\mathcal{P} \in \mathcal{R}(\tau_i)} \sum_{\psi \in \operatorname{Occ}(\tau_i/\mathcal{P}, \tau_k)} \prod_{j=1}^{|\mathcal{P}|} x_{\psi(j)} = p(x_1, \ldots, x_s),$$

where $p$ is a polynomial. We now argue that $p$ is a polynomial of degree $s$ (and therefore it is a non-zero polynomial). Clearly, the polynomial $p$ has degree at most $s$. Since $\operatorname{Occ}(\tau', \tau_k) = \emptyset$ for every $\tau'$ of order $s$ such that $\tau' \neq \tau_k$, $c_k s! x_1 x_2 \cdots x_s$ is the only term of $p$ containing the monomial $x_1 x_2 \cdots x_s$ with nonzero coefficient. Therefore, there exists $\mathbf{x}$ such that $\sum_{i=1}^{r} c_i t(\tau_i, \Phi_{\tau_k}^{\mathbf{x}}) \neq 0$, which is a contradiction. □

The following theorem is the main result of this chapter. It shows that the interior of $T^q$ is non-empty. Observation 28 yields the same conclusion for $T^q_{\mathrm{mon}}$. In the statement of the following theorem and its proof, we write $B(\mathbf{w}, \varepsilon)$ for the ball of radius $\varepsilon$ around $\mathbf{w}$ in $\mathbb{R}^r$.

**Theorem 32.** *For every integer $q \geq 2$, there exist a vector $\mathbf{w} \in T^q$ and $\varepsilon > 0$ such that $B(\mathbf{w}, \varepsilon) \subseteq T^q$.*

*Proof.* Let $\{\tau_1, \ldots, \tau_r\}$ be the set of all non-trivial indecomposable permutations of order at most $q$ and let $\Phi_1, \ldots, \Phi_r$ be permutons such that $\{\mathbf{t}^q(\Phi_i) \mid i \in [r]\}$ spans $\mathbb{R}^r$. (Existence of such permutons follows from Lemma 31.) Consider the matrix $V = (v_{i,j})_{i,j=1}^{r}$, where $v_{i,j} = t(\tau_j, \Phi_i)$. Observe that the matrix $V$ is non-singular.

Consider a vector $\mathbf{x} = (x_1, \ldots, x_r) \in (0, r^{-1})^r$ and let

$$\Phi^{\mathbf{x}} = \left(1 - \sum_{i \in [r]} x_i\right) I \oplus \left(\bigoplus_{i \in [r]} x_i \Phi_i\right).$$

Recall that $I$ denotes the unique permuton with support consisting of the segment between $(0,0)$ and $(1,1)$ and observe that $t(\tau, I) = 0$ for every non-trivial indecomposable permutation $\tau$. Thus, Observation 29 yields that

$$t(\tau_j, \Phi^{\mathbf{x}}) = \sum_{i=1}^{r} x_i^{|\tau_j|} t(\tau_j, \Phi_i) = \sum_{i=1}^{r} x_i^{|\tau_j|} v_{i,j}.$$

Let $\Psi$ be a map from $\mathbb{R}^r$ to $\mathbb{R}^r$ such that

$$\Psi_j(\mathbf{x}) = \sum_{i=1}^{r} x_i^{|\tau_j|} v_{i,j} \text{ for all } j \in [r].$$

Since we have $\Psi(\mathbf{x}) = \mathbf{t}^q(\Phi^{\mathbf{x}})$, we get that

$$\Psi((0, r^{-1})^r) = \{\Psi(\mathbf{x}) \mid \mathbf{x} \in (0, r^{-1})^r\} \subseteq T^q.$$

The Jacobian $\mathrm{Jac}(\Psi)(\mathbf{x})$ is a polynomial in $x_1, \ldots, x_r$. Since for $x_1 = \cdots = x_r = 1$ we have

$$\mathrm{Jac}(\Psi) = \det(v_{i,j} \cdot |\tau_j|)_{i,j=1}^{r} = \left(\prod_{j=1}^{r} |\tau_j|\right) \det V \neq 0,$$

$\mathrm{Jac}(\Psi)$ is a non-zero polynomial.

Hence, there exists $\mathbf{x} \in (0, r^{-1})^r$ for which $\mathrm{Jac}(\Psi)(\mathbf{x}) \neq 0$. Consequently, $T^q$ contains a ball around $\mathbf{w}$ for $\mathbf{w} = \Psi(\mathbf{x})$. □

Theorem 32 implies that for every finite family $\mathcal{A}$ of indecomposable permutations, there exist permutons $\Phi$ and $\Phi'$ and an indecomposable permutation $\tau$ such that $t(\pi, \Phi) = t(\pi, \Phi')$ for every $\pi \in \mathcal{A}$ and $t(\tau, \Phi) \neq t(\tau, \Phi')$. The following lemma shows that an analogous statement holds for any finite family of permutations, not only for indecomposable permutations.

**Lemma 33.** *For every finite set of permutations $\mathcal{A} = \{\tau_1, \ldots, \tau_k\}$, there exists a permutation $\tau$ and permutons $\Phi$ and $\Phi'$ such that $t(\tau_i, \Phi) = t(\tau_i, \Phi')$ for every $i \in [k]$ and $t(\tau, \Phi) \neq t(\tau, \Phi')$.*

*Proof.* Let $\mathcal{B} = \{\pi_1, \ldots, \pi_{k+1}\}$ be a family of permutations each of order $n$ with $n > |\tau_i|$ for every $i \in [k]$, such that no $\pi \in \mathcal{B}$ maps a proper interval onto an interval. Permutations with this property are called *simple*. By the result of Albert, Atkinson and Klazar [8, Theorem 5], a random permutation of order $n$ is simple with probability bounded away from zero as $n$ tends to infinity (in particular, the probability tends to $e^{-2}$). Therefore such a family $\mathcal{B}$ of simple permutations exists for $n$ sufficiently large.

Let $\mathbf{n} = \underbrace{(1/n, \ldots, 1/n)}_{n\times}$ and

$$\Phi^{\mathbf{u}} = \left(1 - \sum_{i\in[k+1]} u_i\right) I \oplus \left(\bigoplus_{i\in[k+1]} u_i \Phi^{\mathbf{n}}_{\pi_i}\right)$$

for $\mathbf{u} = (u_1, \ldots, u_{k+1}) \in (0, \frac{1}{k+1}]^{k+1}$. Observe that if $\pi$ is a simple permutation, it is also indecomposable and the only $\pi$-compressive partition is the partition into singletons (specifically, the partition consisting of $P_i = \{i\}$, $i \in [|\pi|]$). Hence, by Observations 29 and 30, $t(\pi_i, \Phi^{\mathbf{u}}) = n!(u_i/n)^n$ for every $i \in [k+1]$. The function $\mathbf{u} \mapsto t(\tau_j, \Phi^{\mathbf{u}})$ is continuous for every $j \in [k]$. We consider the continuous map $\Gamma$ from $(0, 1/(k+1)]^{k+1}$ to $\mathbb{R}^k$ such that

$$\Gamma(\mathbf{u}) = (t(\tau_1, \Phi^{\mathbf{u}}), \ldots, t(\tau_k, \Phi^{\mathbf{u}})).$$

Now, consider any $k$-dimensional sphere in $(0, 1/(k+1)]^{k+1}$. The Borsuk-Ulam Theorem [23] yields the existence of two distinct points on its surface that are mapped by $\Gamma$ to the same point in $[0, 1]^k$. Hence, there exist distinct $\mathbf{v} = (v_1, \ldots, v_{k+1})$ and $\mathbf{v}' = (v'_1, \ldots, v'_{k+1})$ such that $t(\tau_j, \Phi^{\mathbf{v}}) = t(\tau_j, \Phi^{\mathbf{v}'})$ for every $j \in [k]$. However, if, say $v_i \neq v'_i$, then $t(\pi_i, \Phi^{\mathbf{v}}) = n!(v_i/n)^n \neq n!(v'_i/n)^n = t(\pi_i, \Phi^{\mathbf{v}'})$. Therefore, $\tau = \pi_i$, $\Phi = \Phi^{\mathbf{v}}$, and $\Phi' = \Phi^{\mathbf{v}'}$ satisfy the assertion of the theorem. $\square$

# Chapter 5

# Non-forcible approximable parameter

In this chapter, we construct a bounded permutation parameter which is finitely approximable but not finitely forcible, answering a question of Hoppen et al. [54, Question 5.5] whether such parameter exists.

For this chapter, we fix a sequence $(\tau_k)_{k\in\mathbb{N}}$ of permutations of strictly increasing orders and sequences of permutons $(\Phi_k)_{k\in\mathbb{N}}$ and $(\Phi'_k)_{k\in\mathbb{N}}$ that satisfy the following: For every $k > 1$, $t(\sigma, \Phi_k) = t(\sigma, \Phi'_k)$ for every permutation $\sigma$ of order at most $|\tau_{k-1}|$, and $t(\tau_k, \Phi_k) > t(\tau_k, \Phi'_k)$. Such a sequences $(\tau_k)_{k\in\mathbb{N}}$, $(\Phi_k)_{k\in\mathbb{N}}$ and $(\Phi'_k)_{k\in\mathbb{N}}$ exist by Lemma 33. Let $\gamma_k = t(\tau_k, \Phi_k) - t(\tau_k, \Phi'_k)$ for every $k \in \mathbb{N}$.

Let $(\alpha_i)_{i\in\mathbb{N}}$ be a sequence of positive reals satisfying $\sum_{i\in\mathbb{N}} \alpha_i < 1/2$ and $\sum_{i>k} \alpha_i < \alpha_k\gamma_k/4$ for every $k$. The main result of this section is that the permutation parameter

$$f_\bullet(\sigma) = \sum_{i\in\mathbb{N}} \alpha_i t(\tau_i, \sigma)$$

is finitely approximable but not finitely forcible.

**Lemma 34.** *The permutation parameter $f_\bullet$ is finitely approximable.*

*Proof.* Consider fixed $\varepsilon > 0$. Since the sum $\sum_{i\in\mathbb{N}} \alpha_i$ converges, there exists $k$ such that $\sum_{i>k} \alpha_i < \varepsilon/2$. Set $\mathcal{A} = \{\tau_1, \ldots, \tau_k\}$ and $\delta = \varepsilon$. Consider two permutations $\sigma$ and $\pi$ that satisfy $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}$.

We obtain that

$$\begin{aligned}
|f_\bullet(\sigma) - f_\bullet(\pi)| &= \left|\sum_{i\in\mathbb{N}} \alpha_i(t(\tau_i,\sigma) - t(\tau_i,\pi))\right| \\
&\le \sum_{i\in\mathbb{N}} \alpha_i \left|t(\tau_i,\sigma) - t(\tau_i,\pi)\right| \\
&< \sum_{i\le k} \alpha_i\delta + \sum_{i>k} \alpha_i|t(\tau_i,\sigma) - t(\tau_i,\pi)| \\
&< \delta/2 + \sum_{i>k} \alpha_i \cdot 1 < \varepsilon.
\end{aligned}$$

It follows that the parameter $f_\bullet$ is finitely approximable. □

In the following lemma, we show that $f_\bullet$ is not finitely forcible.

**Lemma 35.** *The permutation parameter $f_\bullet$ is not finitely forcible.*

*Proof.* Suppose that $f_\bullet$ is finitely forcible and that $\mathcal{A}$ is a forcing family for $f_\bullet$. Let $k$ be such that the maximum order of a permutation in $\mathcal{A}$ is at most $|\tau_{k-1}|$. Then we have $t(\rho,\Phi_k) = t(\rho,\Phi'_k)$ for every $\rho \in \mathcal{A}$, $t(\tau_i,\Phi_k) = t(\tau_i,\Phi'_k)$ for every $i < k$, and $t(\tau_k,\Phi_k) - t(\tau_k,\Phi'_k) = \gamma_k$.

Let $\varepsilon = \alpha_k\gamma_k/4$. Let $\delta > 0$ be as in the definition of finite forcibility of $f_\bullet$. Without loss of generality we may assume that $\delta < \varepsilon$.

There exist a $\Phi_k$-random permutation $\sigma$ and a $\Phi'_k$-random permutation $\sigma'$ such that $|t(\rho,\sigma) - t(\rho,\sigma')| < \delta$ for every $\rho \in \mathcal{A}$, $|t(\tau_i,\sigma) - t(\tau_i,\sigma')| < \delta$ for every $i < k$ and $t(\tau_k,\sigma) - t(\tau_k,\sigma') > \gamma_k - \delta > 3\gamma_k/4$. Let us estimate the sum in the definition of $f_\bullet$ with the $k$-th term missing.

$$\begin{aligned}
&\left|\sum_{i\in\mathbb{N}, i\neq k} \alpha_i(t(\tau_i,\sigma) - t(\tau_i,\sigma'))\right| \\
&\qquad = \left|\sum_{i<k} \alpha_i \left(t(\tau_i,\sigma) - t(\tau_i,\sigma')\right) + \sum_{i>k} \alpha_i \left(t(\tau_i,\sigma) - t(\tau_i,\sigma')\right)\right| \\
&\qquad < \sum_{i<k} \alpha_i\delta + \sum_{i>k} \alpha_i < \frac{\alpha_k\gamma_k}{8} + \frac{\alpha_k\gamma_k}{4} < \frac{\alpha_k\gamma_k}{2}
\end{aligned}$$

This leads to the following

$$\begin{aligned}
|f_\bullet(\sigma) - f_\bullet(\sigma')| &= \left| \sum_{i \in \mathbb{N}} \alpha_i \left( t(\tau_i, \sigma) - t(\tau_i, \sigma') \right) \right| \\
&\geq \alpha_k \left( t(\tau_k, \sigma) - t(\tau_k, \sigma') \right) - \left| \sum_{i \in \mathbb{N}, i \neq k} \alpha_i \left( t(\tau_i, \sigma) - t(\tau_i, \sigma') \right) \right| \\
&> \frac{3}{4} \alpha_k \gamma_k - \frac{\alpha_k \gamma_k}{2} = \frac{\alpha_k \gamma_k}{4} = \varepsilon.
\end{aligned}$$

This contradicts our assumption that $f_\bullet$ is finitely forcible. $\square$

Lemmas 34 and 35 yield the Theorem 8, stating that there exists a bounded permutation parameter $f$ that is finitely approximable but not finitely forcible.

Recall that, by [54, Proposition 5.4] the testable bounded permutation parameters are precisely the finitely approximable ones. Thus, Theorem 8 implies that a finitely forcible bounded permutation parameter does not have to be testable.

# Chapter 6

# Property testing algorithms for permutations

In this chapter, we present a proof that all hereditary permutation properties are strongly testable, i.e., testable with respect to the Kendall's tau distance (Theorem 7). Hoppen et al. [54] observed that this result would be implied by the following statement, asserting that every permutation close to a hereditary property in the rectangular distance is also close in the Kendall's tau distance.

**Conjecture 2** (Hoppen, Kohayakawa, Moreira and Sampaio [54], Conjecture 5.3)**.** *Let $\mathcal{P}$ be a hereditary property. For every positive real $\varepsilon_0$, there exists $\delta_0$ such that any permutation $\pi$ satisfying* $\mathrm{dist}_\square(\pi, \mathcal{P}) < \delta_0$ *also satisfies* $\mathrm{dist}_K(\pi, \mathcal{P}) < \varepsilon_0$.

The conjecture is an analogue of the known relation between the cut distance and the edit distance to hereditary graph properties from [69]. Our method actually gives the proof of this conjecture which we state as Theorem 40. However, we include the proof of Theorem 7 instead of just stating that it can be derived from Theorem 40 for completeness.

## 6.1 Branchings

In this section, we present the notion of branchings which are rooted trees approximately describing hereditary properties. This notion is key in our analysis of hereditary properties. Let us start with a formal definition of $k$-sequences. For an integer $k$, a *$k$-sequence* $A$ is a sequence $A_1, \ldots, A_\ell$ of non-empty subsets of $[k]$. We refer to $\ell$ as the *length* of $A$ and we write $|A|$ for the length of $A$. The *basic* $k$-sequence is the $k$-sequence of length one comprised of the set $[k]$. A $k$-sequence $A$ is *simple* if each $A_i$ has size one. Finally, a $k$-sequence $A$ is *monotone* if every pair $x \in A_i$ and $x' \in A_{i'}$ with $1 \le i < i' \le |A|$ satisfies that $x < x'$.

Before we proceed further, we have to introduce some auxiliary notation. If $A$ is a $k$-sequence, then we write $|A|_i$ for the sum $|A_1| + \cdots + |A_i|$. For completeness, we define $|A|_0 = 0$.

Fix a $k$-sequence $A$. Let $A_i = \{x^i_1, \ldots, x^i_{|A_i|}\}$ where $x^i_1 < \cdots < x^i_{|A_i|}$. For an integer $m$, we define a function $g^{A,m} : [m \cdot |A|_{|A|}] \to [k]$ as

$$g^{A,m}(j) = x^i_{(j - m \cdot |A|_{i-1})) \bmod |A_i|}$$

where $i$ is the largest integer such such that $m \cdot |A|_{i-1} < j$. For example, if $A = \{1, 2, 3\}, \{1, 4\}, \{3\}$, then

$$g^{A,4}(1), \ldots, g^{A,4}(24) = \\ 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 4, 1, 4, 1, 4, 1, 4, 3, 3, 3, 3\ .$$

Note that the sequence $g^{A,m}(1)g^{A,m}(2) \ldots g^{A,m}(m \cdot |A|_{|A|})$ has $|A|$ blocks such that the $i$-th block consists of $m$ parts each containing the elements of $A_i$ in the increasing order.

A permutation $\pi$ is an *$m$-expansion* of a $k$-sequence $A$ if the following holds:

- the order of $\pi$ is $m \cdot |A|_{|A|}$, and
- if $g^{A,m}(j) < g^{A,m}(j')$ for $j, j' \in \{1, \ldots, m|A|_{|A|}\}$, then $\pi(j) < \pi(j')$.

For example, one of the 3-expansions of the 2-sequence $\{1, 2\}, \{1\}$ is the permutation $4, 8, 2, 7, 3, 9, 1, 6, 5$. In other words, if a permutation $\pi$ is an $m$-expansion of $A$, then the range of $\pi$ can be viewed as partitioned into $k$ parts such the following holds: the permutation $\pi$ consists of $|A|$ groups (in the example, these are $4, 8, 2, 7, 3, 9$ and $1, 6, 5$) where the $i$-th group has $m$ blocks of length $|A_i|$ each and the values of $\pi$ in each block belong to the parts of the range of $\pi$ with indices in $A_i$ in the increasing order. The number of $m$-expansions of a $k$-sequence $A$ is equal to

$$\prod_{j=1}^{k} (m \cdot |\{i \text{ such that } i \in [|A|] \text{ and } j \in A_i\}|)!\ .$$

Let $\mathcal{P}$ be a hereditary property. A $k$-sequence $A$ is *$\mathcal{P}$-good* if there exists an $m$-expansion of $A$ in $\mathcal{P}$ for every integer $m$. Otherwise, the $k$-sequence $A$ is *$\mathcal{P}$-bad*. So, if $A$ is $\mathcal{P}$-bad, there exists an integer $m$ such that no $m$-expansion of $A$ is in $\mathcal{P}$. The smallest such integer $m$ is called the *$\mathcal{P}$-order* of $A$ and it is denoted by $\langle A\rangle_{\mathcal{P}}$; if $\mathcal{P}$ is clear from the context, we just write $\langle A\rangle$. Observe that if $A$ is $\mathcal{P}$-bad, then no $m$-expansion of $A$ is in $\mathcal{P}$ for every $m \geq \langle A\rangle$ (here, we use that $\mathcal{P}$ is hereditary).

If $A$ is a $\mathcal{P}$-bad $k$-sequence, then any $k$-sequence $A'$ obtained from $A$ by replacing one element, say $A_i$, by a sequence of at least one and at most $|A_i|\langle A\rangle$ proper subsets of $A_i$ is called a *$\mathcal{P}$-reduction* of $A$. For example, if the 3-sequence $A = \{1\}, \{2,3\}, \{1,3\}$ is $\mathcal{P}$-bad and its $\mathcal{P}$-order is two, then one of its $\mathcal{P}$-reductions is $\{1\}, \{2\}, \{2\}, \{3\}, \{1,3\}$.

The *$k$-branching* of a hereditary property $\mathcal{P}$ is a rooted tree $\mathcal{T}$ such that

- each node $u$ of $\mathcal{T}$ is associated with a $k$-sequence $A^u$,
- the root of $\mathcal{T}$ is associated with the basic $k$-sequence,
- if the $k$-sequence $A^u$ of a node $u$ is $\mathcal{P}$-good or simple, then $u$ is a leaf, and
- if the $k$-sequence $A^u$ of a node $u$ is $\mathcal{P}$-bad and it is not simple, then the number of children of $u$ is equal to the number of $\mathcal{P}$-reductions of $A^u$ and the children of $u$ are associated with the $\mathcal{P}$-reductions.

Note that the $k$-branching, i.e., the tree and the association of its nodes with $k$-sequences, is uniquely determined by the property $\mathcal{P}$ and the integer $k$.

Let us argue that the $k$-branching of every hereditary property $\mathcal{P}$ is *finite*. We define the *score* of a $k$-sequence $A$ to be the sequence $m_1, \dots, m_k$ where $m_j$ is the number of $A_i$'s of cardinality $k+1-j$. Observe that the score of a $\mathcal{P}$-reduction of a $\mathcal{P}$-bad $k$-sequence $A$ is always lexicographically smaller than that of $A$. Since the lexicographic ordering on the scores is a well-ordering, the $k$-branching is finite for every hereditary property $\mathcal{P}$.

Let $\mathcal{T}$ be the $k$-branching of a hereditary property $\mathcal{P}$. We now assign to every node $u$ of the $k$-branching of $\mathcal{P}$ an integer weight $w_u$. The weight of a leaf node $u$ is one if $A^u$ is $\mathcal{P}$-good. Otherwise, the weight of a leaf node $u$ is $k\langle A^u\rangle$. If $u$ is an internal node, then $w_u$ is equal to $\langle A^u\rangle km$ where $m$ is the maximum weight of a child of $u$. In particular, the weight of $u$ is at least the weight of any of its children.

## 6.2 Decompositions

In this section, we introduce a grid-like way of decomposing permutations which we use in our proof. The domain of a permutation will be split into $K$ equal size parts and the range into $k$ such parts with $k \leq K$.

We start with some auxiliary notation. Recall that $[a]$ denotes all integers from 1 to $a$. We extend this notation by writing $[a]_{i/b}$ for the set of all integers $k \in [a]$ such that $i - 1 < k/\lfloor a/b \rfloor \leq i$, i.e., $[a]_{i/b}$ is the $i$-th part after dividing $[a]$ into $b$ equal-sized parts (with $b+1$-st part containing the remaining elements). For example, $[25]_{2/6} = \{5, 6, 7, 8\}$. Observe that $|[a]_{1/b}| = \cdots = |[a]_{b/b}| = \lfloor a/b \rfloor$ and $|[a]_{b+1/b}| \leq b - 1$.

Fix now a permutation $\pi$ of order $n$ and integers $K \in [n]$, $i \in [K]$, $k \in [K]$ and $j \in [k]$. We define $R_{i,j}(\pi)$ as

$$R_{i,j}(\pi) = \{x \in [n]_{i/K} \text{ such that } \pi(x) \in [n]_{j/k}\}$$

and we set

$$\rho_{i,j}(\pi) = \frac{|R_{i,j}(\pi)|}{\lfloor n/K \rfloor} .$$

Vaguely speaking, $\rho_{i,j}(\pi) \in [0, 1]$ is the density of $\pi$ in the part of the $K \times k$ grid at the coordinates $(i, j)$. The values of $K$ and $k$ will always be clear from the context.

To get used to the definition of the sets $R_{i,j}$ and the quantities $\rho_{i,j}$, we now prove a simple auxiliary lemma.

**Lemma 36.** *Let $k$ and $K$ be positive integers and let $\varepsilon' \leq 1/(k+1)$ be a positive real. For every permutation $\pi$ of order at least $k(k+1)K$ and every $x \in [K]$, there exists $y \in [k]$ such that $\rho_{x,y}(\pi) \geq \varepsilon'$.*

*Proof.* Observe that

$$\begin{aligned} |R_{x,1}(\pi)| + \cdots + |R_{x,k}(\pi)| &\geq \lfloor |\pi|/K \rfloor - k \\ &\geq \left(1 - \frac{1}{k+1}\right) \lfloor |\pi|/K \rfloor . \end{aligned}$$

Since $\varepsilon' \leq 1/(k+1)$, there must exist $y$ such that $\rho_{x,y}(\pi) \geq \varepsilon'$ by the pigeonhole principle. $\square$

Fix a permutation $\pi$, integers $k$, $K$ and $M$ such that $1 \leq k \leq K \leq |\pi|$, and a real $0 \leq \varepsilon' < 1$. If $A$ is a $k$-sequence, then we say that a $K$-sequence $B$ is $(A, M, \varepsilon')$-*approximate* for $\pi$ if the following holds:

- the length of $B$ is $|A|$,
- $B$ is monotone,
- $|B|_{|B|} = \sum_{i=1}^{|B|} |B_i| \geq K - M$, and
- for every $i \in [|A|]$, if $x \in B_i$ and $y \in [k] \setminus A_i$, then $\rho_{x,y}(\pi) < \varepsilon'$.

In other words, an $(A, M, \varepsilon')$-approximate $K$-sequence $B$ decomposes the whole index set $[K]$ except for at most $M$ indices into $|A|$ parts such that the indices contained in the parts determined by $B$ are in the increasing order and for $x \in B_i$, the only dense sets $R_{x,y}(\pi)$ are those with $y \in A_i$.

Suppose that a $k$-sequence $A$ is $\mathcal{P}$-bad for a hereditary property $\mathcal{P}$. We say that a $K$-sequence $B$ is $(A, \varepsilon')$-*witnessing* for $\pi$ if the following holds:

- the length of $B$ is $|A|$,
- there exist integers $1 \le x_1 < \ldots < x_{|A|_{|A|} \cdot \langle A \rangle} \le K$ such that $x_j \in B_i$ if $|A|_{i-1}\langle A \rangle < j \le |A|_i \langle A \rangle$, and
- $\rho_{x_j, g^{A,\langle A \rangle}(j)}(\pi) \ge \varepsilon'$ for every $j \in [|A|_{|A|} \cdot \langle A \rangle]$ (the definition of the function $g$ can be found in Section 6.1).

In other words, a $K$-sequence $B$ which decomposes the index set $[K]$ is $(A, \varepsilon')$-witnessing, if it is possible to find indices such that there are $|A_i|\langle A \rangle$ indices $x_j$ in each $B_i$ and all the sets $R_{x_j, g^{A,\langle A \rangle}(j)}(\pi)$ are dense. The motivation for this definition is the following: if $B$ is $(A, \varepsilon')$-*witnessing*, then each set $R_{x_j, g^{A,\langle A \rangle}(j)}(\pi)$ has at least $\varepsilon' \lfloor |\pi|/K \rfloor$ elements and consequently at least $(\varepsilon' \lfloor |\pi|/K \rfloor)^{|A|\langle A \rangle}$ subsets of $[|\pi|]$ induce subpermutations that are $\langle A \rangle$-expansions of $A$. This will allow us to deduce that a random subpermutation of sufficiently large order does not have the property $\mathcal{P}$ with high probability.

We now state a lemma saying that if a $K$-sequence $B$ is approximate but not witnessing with respect to a $k$-sequence $A$ for a permutation $\pi$, then there exists a reduction $A'$ of $A$ and a $K$-sequence $B'$ such that $B'$ is approximate with respect to $A'$.

**Lemma 37.** *Let $\mathcal{P}$ be a hereditary property, let $k$, $K$, $m$ and $M$ be positive integers and let $\varepsilon' \le 1/(k+1)$ be a positive real. Let $A$ be a $\mathcal{P}$-bad $k$-sequence and $B$ a monotone $K$-sequence with $|A| = |B|$. If the $K$-sequence $B$ is $(A, M, \varepsilon')$-approximate for a permutation $\pi$, $|\pi| \ge k(k+1)K$, $B$ is not $(A, \varepsilon')$-witnessing for $\pi$ and $|B_i| \ge mk\langle A \rangle$ for every $i \in [|B|]$, then there exist a $\mathcal{P}$-reduction $A'$ of $A$ and a monotone $K$-sequence $B'$ such that*

- *the lengths of $A'$ and $B'$ are the same,*
- *$B'$ is $(A', M + mk\langle A \rangle, \varepsilon')$-approximate for $\pi$, and*
- *$|B'_i| \ge m$ for every $i \in [|B'|]$.*

*Proof.* If $B$ is not $(A, \varepsilon')$-witnessing for $\pi$, then there exists an index $j \in [|B|]$ such that there is no $|A_j|\langle A \rangle$-tuple $x_1 < \cdots < x_{|A_j|\langle A \rangle}$ in $B_j$ satisfying $\rho_{x_i, y_i}(\pi) \ge \varepsilon'$ where $y_i = g^{A,\langle A \rangle}(|A|_{j-1}\langle A \rangle + i)$. Fix such an index $j$ for the rest of the proof.

If $|A_j| = 1$, then an $\langle A\rangle$-tuple with the properties given in the previous paragraph is formed by any $\langle A\rangle$ elements of $B_j$ by Lemma 36. So, we assume that $|A_j| \geq 2$ in the rest of the proof. Define $x_1$ to be the smallest index in $B_j$ such that $\rho_{x_1,y_1}(\pi) \geq \varepsilon'$. Suppose that we have defined the indices $x_1, \dots, x_i$ and define $x_{i+1}$ to be the smallest index in $B_j$ that is larger than $x_i$ such that $\rho_{x_{i+1},y_{i+1}}(\pi) \geq \varepsilon'$. If no such index exists, we stop constructing the sequence. Let $\ell$ be the number of the indices defined. By the choice of $j$, $\ell < |A_j|\langle A\rangle$. For completeness, set $x_0 = 0$ and $x_{\ell+1} = K + 1$.

Define $C_i$, $i \in [\ell+1]$, to be the set of the elements of $B_j$ strictly between $x_{i-1}$ and $x_i$. If the subset $C_i$ has size less than $m$, remove it from the sequence and let $C'_1, \dots, C'_{\ell'}$ be the resulting sequence. Observe that

$$\begin{aligned} |B_j| - \textstyle\sum_{i=1}^{\ell'} |C'_i| &\leq \ell + (\ell+1)(m-1) \\ &\leq (\ell+1)m - 1 \\ &\leq m|A_j|\langle A\rangle - 1 \\ &\leq mk\langle A\rangle - 1 \end{aligned} \tag{6.1}$$

since the sets $C'_1, \dots, C'_{\ell'}$ contain all the elements of $B_j$ except for the elements $x_1, \dots, x_\ell$ and the elements contained in the sets $C_1, \dots, C_{\ell+1}$ with cardinalities at most $m-1$. In particular, we can infer from $|B_j| \geq mk\langle A\rangle$ that $\ell' \geq 1$.

Next, define $C''_i$, $i \in [\ell']$, to be the set of $y \in [k]$ such that there exists $x \in C'_i$ with $\rho_{x,y}(\pi) \geq \varepsilon'$. Lemma 36 implies that the sets $C''_1 \dots, C''_{\ell'}$ are non-empty. We infer from the way we have chosen the indices $x_1, \dots, x_\ell$ that each set $C''_i$ is a proper subset of $A_j$. Finally, define the $k$-sequence $A'$ to be the $K$-sequence $A$ with $A_j$ replaced with $C''_1, \dots, C''_{\ell'}$ and the $K$-sequence $B'$ to be the $K$-sequence $B$ with $B_j$ replaced with $C'_1, \dots, C'_{\ell'}$. By the definition of $C''_1, \dots, C''_{\ell'}$ and by (6.1), the $K$-sequence $B'$ is $(A', M + mk\langle A\rangle, \varepsilon')$-approximate for $\pi$. By the choice of $C'_1, \dots, C'_{\ell'}$, we have that $|B'_i| \geq m$ for every $i \in [|B'|]$. Finally, since $\ell' \leq \ell \leq |A_j|\langle A\rangle$ and every $C''_i$ , $i \in [\ell']$, is a proper subset of $A_j$, $A'$ is $\mathcal{P}$-reduction of $A$. □

We finish this section with the following lemma on approximating the structure of a sufficiently large permutation $\pi$ with respect to a hereditary property.

**Lemma 38.** *Suppose $\mathcal{P}$ is a hereditary property. For all integers $k$ and reals $\varepsilon$ and $\varepsilon'$ such that $0 < \varepsilon \leq 1$ and $0 < \varepsilon' \leq 1/(k+1)$, there exists an integer $K$ such that for every permutation $\pi$ of order at least $k(k+1)K$, there exist a $k$-sequence $A$ and a $K$-sequence $B$ with the same lengths such that*

- *$A$ is $\mathcal{P}$-bad and $B$ is $(A, \varepsilon')$-witnessing for $\pi$, or*
- *$A$ is $\mathcal{P}$-good and $B$ is $(A, \lfloor \varepsilon K \rfloor, \varepsilon')$-approximate for $\pi$.*

*Proof.* Let $\mathcal{T}$ be the $k$-branching with respect to $\mathcal{P}$. Let $d$ be the depth of $\mathcal{T}$, i.e., the maximum number of vertices on a path from the root to a leaf, and let $w_0$ be the weight of the root of $\mathcal{T}$. We show that $K := \lceil dw_0/\varepsilon \rceil$ has the properties claimed in the statement of the lemma.

Let $\pi$ be a permutation of order at least $k(k+1)K$. Based on $\pi$, we define a path from the root to one of the nodes in $\mathcal{T}$ in a recursive way. In addition to choosing the nodes $u^i$ on the path, we also define monotone $K$-sequences $B^i$ such that $B^i$ is $(A^{u^i}, i \cdot w_0, \varepsilon')$-approximate for $\pi$ and $|B^i_j| \geq w_{u^i}$ for every $j \in [|B^i|]$.

Let $u^0$ be the root of $\mathcal{T}$ and set $B^0$ to be the basic $K$-sequence. Clearly, $B^0$ is $(A^{u^0}, 0, \varepsilon')$-approximate for $\pi$. Suppose that the node $u^i$ on the path has already been chosen and we now want to choose the next node. If $u^i$ is a leaf node, we stop. If $u^i$ is not a leaf node, then the $k$-sequence $A^{u^i}$ must be $\mathcal{P}$-bad. If $B^i$ is $(A^{u^i}, \varepsilon')$-witnessing for $\pi$, we also stop. Otherwise, Lemma 37 applied with $m$ equal to the maximum weight of a child of $u^i$ (note that $|B^i_j| \geq mk\langle A^{u^i}\rangle$ for every $j \in [|B^i|]$) implies that there exist a $\mathcal{P}$-reduction $A'$ of $A^{u^i}$ and a $K$-sequence $B^{i+1}$ such that $B^{i+1}$ is $(A', i \cdot w_0 + mk\langle A^{u^i}\rangle, \varepsilon')$-approximate for $\pi$ and $|B^{i+1}_j| \geq m$ for every $j \in [|B^{i+1}|]$. Choose $u^{i+1}$ to be the child of $u^i$ such that $A^{u^{i+1}} = A'$. Since $mk\langle A^{u^i}\rangle \leq w_0$, we obtain that $B^{i+1}$ is $(A^{u^{i+1}}, (i+1)w_0, \varepsilon')$-approximate for $\pi$.

Let $\ell$ be the length of the constructed path. We claim that the $k$-sequence $A^{u^\ell}$ and the $K$-sequence $B^\ell$ have the properties described in the statement of the lemma.

If $u^\ell$ is not a leaf node, then $A^{u^\ell}$ is $\mathcal{P}$-bad and $B^\ell$ is $(A^{u^\ell}, \varepsilon')$-witnessing for $\pi$ (since we have stopped at $u^\ell$). If $u^\ell$ is a leaf node and $A^{u^\ell}$ is $\mathcal{P}$-bad, then $B^\ell$ is $(A^{u^\ell}, \varepsilon')$-witnessing for $\pi$ by Lemma 37 applied for $m = 1$ ($A^{u^\ell}$ cannot have a $\mathcal{P}$-reduction because it is simple). Finally, if $u^\ell$ is a leaf node and $A^{u^\ell}$ is $\mathcal{P}$-good, $B^\ell$ is $(A^{u^\ell}, \lfloor \varepsilon K \rfloor, \varepsilon')$-approximate for $\pi$ since $dw_0 \leq \lfloor \varepsilon K \rfloor$. □

## 6.3 Testing

In this section, we establish our main result. The next lemma, which says that every permutation that is far from a hereditary property $\mathcal{P}$ in the Kendall's tau distance has a witnessing $K$-sequence for a suitable choice of $k$ and $K$, is the core of our proof.

**Lemma 39.** *Let $\mathcal{P}$ be a hereditary property of permutations. For every real $\varepsilon_0 > 0$, there exist integers $k$, $K$ and $M$, and a real $\varepsilon' > 0$ such if $\pi$ is a permutation of order at least $M$ with $\mathrm{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$, then there exist a $\mathcal{P}$-bad $k$-sequence $A$ and a $K$-sequence $B$ with the same length such that $B$ is $(A, \varepsilon')$-witnessing for $\pi$.*

*Proof.* Without loss of generality, we can assume that $\varepsilon_0 < 1$. Set $k = \lceil 10/\varepsilon_0 \rceil$, $\varepsilon = \varepsilon_0/10$ and $\varepsilon' = \varepsilon_0/(10k + 10) \leq 1/(k + 1)$. Let $K$ be the integer from the statement of Lemma 38 applied for $\mathcal{P}$, $k$, $\varepsilon$ and $\varepsilon'$. Using this value, set

$$M = \max\left\{ k(k+1)K,\ \left\lceil \frac{10k}{\varepsilon_0} \right\rceil,\ \left\lceil \frac{10K}{\varepsilon_0} \right\rceil \right\}.$$

We show that this choice of $k$, $K$, $M$ and $\varepsilon'$ satisfies the assertion of the lemma.

Let $\pi$ be a permutation of order $n \geq M$. Apply Lemma 38 to $\pi$. Let $A$ be the $k$-sequence and $B$ the $K$-sequence as in the statement of the lemma. Either $A$ is $\mathcal{P}$-bad and $B$ is $(A, \varepsilon')$-witnessing for $\pi$, which is the conclusion of the lemma, or $A$ is $\mathcal{P}$-good and $B$ is $(A, \varepsilon K, \varepsilon')$-approximate for $\pi$. Hence, we assume the latter and deduce that $\mathrm{dist}_K(\pi, \mathcal{P}) < \varepsilon_0$.

To reach our goal, we define two auxiliary functions $f_B : [n] \to [|B|]$ and $f_A : [n] \to [k]$. Informally speaking, when searching for a permutation in $\mathcal{P}$ close to $\pi$, we consider an $m$-expansion of $A$ for a very large integer $m$ and we show that one of its subpermutations is close to $\pi$. As explained after the definition of an $m$-expansion, every $m$-expansion can be viewed as consisting of $|A| = |B|$ blocks where the $i$-th block has $m \cdot |A_i|$ elements. In the subpermutation we construct, we choose the element corresponding to $x \in [n]$ in the $f_B(x)$-th block of an $m$-expansion of $A$ and the value of $g^{A,m}$ for this element will be the $f_A(x)$-th smallest element of $A_{f_B(x)}$.

Let us now proceed in a formal way. First, we define the function $f_B$. Let $x \in [n]$ and let $i$ be the integer such that $x \in [n]_{i/K}$. Let $j$ be the largest integer such that $i$ is smaller than all the elements of $B_j$; if no such set exists, let $j = |B| + 1$. Set $f_B(x) = \max\{1, j - 1\}$. Clearly, $f_B$ is non-decreasing and if $i \in B_j$, then $f_B(x) = j$ for every $x \in [n]_{i/K}$. We now proceed with defining the function $f_A$. If $i \in B_{f_B(x)}$, $\pi(x) \in [n]_{i'/k}$ such that $i' \in [k]$ and $\rho_{i,i'}(\pi) \geq \varepsilon'$, set $f_A(x) = i''$ where $i''$ is the number of elements of $A_{f_B(x)}$ smaller or equal to $i'$. Otherwise, set $f_A(x) = 1$.

Since $A$ is $\mathcal{P}$-good, there exists an $n$-expansion $\sigma$ of $A$ that is in $\mathcal{P}$. Set

$$z_x = |A|_{f_B(x)-1}n + x|A_{f_B(x)}| + f_A(x) \quad \text{for } x \in [n].$$

Observe that $1 \leq z_1 < \cdots < z_n \leq n \cdot |A|_{|A|}$. In the rest of the proof, we establish that the subpermutation $\pi'$ of $\sigma$ induced by $\{z_1, \ldots, z_n\}$ satisfies $\mathrm{dist}_K(\pi, \pi') \leq \varepsilon_0$. Since $\mathcal{P}$ is hereditary and $\sigma \in \mathcal{P}$, this implies $\mathrm{dist}_K(\pi, \mathcal{P}) \leq \varepsilon_0$.

We now define five types of pairs $(x, x')$, $1 \leq x < x' \leq n$. Suppose that $x \in [n]_{i/K}$, $\pi(x) \in [n]_{j/k}$, $x' \in [n]_{i'/K}$ and $\pi(x') \in [n]_{j'/k}$.

- The pair $(x, x')$ is of *Type I* if $i = K + 1$ or $i' = K + 1$.
- The pair $(x, x')$ is of *Type II* if $j = k + 1$ or $j' = k + 1$.
- The pair $(x, x')$ is of *Type III* if it is not of Type I and $i \notin B_{f_B(x)}$ or $i' \notin B_{f_B(x')}$.
- The pair $(x, x')$ is of *Type IV* if it is neither of Type I nor of Type II, and $\rho_{i,j} < \varepsilon'$ or $\rho_{i',j'} < \varepsilon'$.
- The pair $(x, x')$ is of *Type V* if it is not of Type II and $j = j'$.

We now estimate the number of pairs $(x, x')$, $1 \leq x < x' \leq n$, of each of the five types. The number of pairs of Type I is at most $K(n-1) \leq \varepsilon_0 n(n-1)/10$ since $|[n]_{K+1/K}| \leq K$. Similarly, the number of pairs of Type II is at most $k(n-1) \leq \varepsilon_0 n(n-1)/10$ since $|[n]_{k+1/k}| \leq k$. The number of pairs of Type III is at most $\varepsilon n(n-1) = \varepsilon_0 n(n-1)/10$ since $K - (|B_1| + \cdots + |B_{|B|}|) \leq \varepsilon K$.

For $i \in [K]$ and $j \in [k]$ with $\rho_{i,j}(\pi) < \varepsilon'$, the number of the choices of $x \in [n]_{i/K}$ with $\pi(x) \in [n]_{j/k}$ is at most $\varepsilon' n/K$. Hence, the number of $x$ with this property for some $i$ and $j$ is at most $\varepsilon' kn < \varepsilon_0 n/10$. Consequently, the number of pairs of Type IV is strictly less than $\varepsilon_0 n(n-1)/10$. Finally, for $x$ with $\pi(x) \in [n]_{j/k}$, the number of choices of $x' \neq x$ with $\pi(x') \in [n]_{j/k}$ is at most $n/k - 1$. Hence, the number of pairs of Type V is strictly less than $n(n/k - 1) \leq n(n-1)/k \leq \varepsilon_0 n(n-1)/10$.

We conclude that the number of pairs $(x, x')$, $1 \leq x < x' \leq n$, that are of at least of one of Types I–V is at most $\varepsilon_0 n(n-1)/2$.

We claim that if the pair $(x, x')$, $1 \leq x < x' \leq n$, is not of any of the Types I–V, then $\pi(x) < \pi(x')$ if and only if $\pi'(x) < \pi'(x')$. Let $i$, $i'$, $j$ and $j'$ be chosen as in the previous paragraph. Suppose $\pi(x) < \pi(x')$. If $(x, x')$ is not of any of the Types I–V, then it holds that $i \in B_{f_B(x)}$, $i' \in B_{f_B(x')}$, $j \neq j'$, $\rho_{i,j}(\pi) \geq \varepsilon'$ and $\rho_{i',j'}(\pi) \geq \varepsilon'$. This implies that the $f_A(x)$-th smallest element of $A_{f_B(x)}$ is smaller than the $f_A(x')$-th smallest element of $A_{f_B(x')}$. Consequently, $\pi'(x) < \pi'(x')$ by the choice of $z_x$ and $z_{x'}$. Analogously, one can show that if $\pi(x) > \pi(x')$, then $\pi'(x) > \pi'(x')$.

Since the number of pairs $(x, x')$, $1 \leq x < x' \leq n$, of at least one of the five types is at most $\varepsilon_0 n(n-1)/2$, we get that $\mathrm{dist}_K(\pi, \pi') < \varepsilon_0$ as desired. $\square$

We are now ready to prove the main result of this chapter, Theorem 7, which we restate below. Note that the theorem implies that hereditary properties of permutations are strongly testable through subpermutations: for $\varepsilon > 0$, the tester takes a random subpermutation of order $M_0$ from the statement of Theorem 7 and it accepts if the random subpermutation has the tested property and rejects otherwise.

**Theorem 7.** *Let $\mathcal{P}$ be a hereditary property. For every positive real $\varepsilon_0$, there exists $M_0$ such that if $\pi$ is a permutation of order at least $M_0$ with $\mathrm{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$, then a random subpermutation $\pi$ of order $M_0$ has the property $\mathcal{P}$ with probability at most $\varepsilon_0$.*

*Proof.* Without loss of generality, we assume that $\varepsilon_0 < 1$. Apply Lemma 39 to $\mathcal{P}$ and $\varepsilon_0$. Let $k$, $K$ and $M$ be the integers and let $\varepsilon'$ be the real as in the statement of the lemma. Note that we can also assume that $\varepsilon' < 1$. Set $M_0$ as

$$M_0 = \max\left\{M, K(K+1), \frac{\log \frac{kK}{\varepsilon_0}}{\log \frac{K+1}{K+1-\varepsilon'}}\right\} .$$

Let $\pi$ be a permutation of order $n \geq M_0$. Note that the probability that a random $M_0$-element subset $X$ of $[n]$ contains no element of a set $R_{i,j}(\pi)$ with $\rho_{i,j}(\pi) \geq \varepsilon'$ is at most

$$\begin{aligned}\left(1 - \frac{|R_{i,j}(\pi)|}{n}\right)^{M_0} &= \left(1 - \rho_{i,j}(\pi)\left\lfloor\frac{n}{K}\right\rfloor\frac{1}{n}\right)^{M_0} \\ &\leq \left(1 - \frac{\varepsilon'}{K+1}\right)^{M_0} \leq \frac{\varepsilon_0}{kK} .\end{aligned}$$

By the union bound, the probability that there exists $i \in [K]$ and $j \in [k]$ with $\rho_{i,j}(\pi) \geq \varepsilon'$ such that $X$ contains no element from the set $R_{i,j}(\pi)$ is at most $\varepsilon_0$. This implies that with probability at least $1 - \varepsilon_0$ a random $M_0$-element subset $X$ of $[n]$ contains at least one element from each set $R_{i,j}(\pi)$ with $\rho_{i,j}(\pi) \geq \varepsilon'$.

By Lemma 39, if $\mathrm{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$, there exists a $k$-sequence $A$ and a $K$-sequence $B$ such that $A$ is $\mathcal{P}$-bad and $B$ is $(A, \varepsilon')$-witnessing for $\pi$. Since a random $M_0$-element subset of $[n]$ contains an element from each $R_{i,j}(\pi)$ with $\rho_{i,j}(\pi) \geq \varepsilon'$ with probability at least $1 - \varepsilon_0$, a random $M_0$-element subpermutation of $\pi$ contains an $\langle A\rangle$-expansion of $A$ as a subpermutation with probability at least $1 - \varepsilon_0$. Consequently, a random $M_0$-element subpermutation of $\pi$ is not in $\mathcal{P}$ with probability at least $1 - \varepsilon_0$. $\square$

We are also in a position to prove that, for hereditary properties $\mathcal{P}$, the function $\mathrm{dist}_K(\pi, \mathcal{P})$ is continuous with respect to the metric given by $\mathrm{dist}_\square$ in the sense considered in [54].

**Theorem 40.** *Let $\mathcal{P}$ be a hereditary property. For every $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that any permutation $\pi$ satisfying* $\mathrm{dist}_\square(\pi, \mathcal{P}) < \delta_0$ *also satisfies* $\mathrm{dist}_K(\pi, \mathcal{P}) < \varepsilon_0$.

*Proof.* Apply Lemma 39 to $\mathcal{P}$ and $\varepsilon_0$. Let $k$, $K$ and $M$ be the integers and let $\varepsilon'$ be the real as in the statement of the lemma. Set $M_0$ to be the maximum of $M$ and $K+1$, and set $\delta_0$ to be the minimum of $1/M_0$ and $\frac{\varepsilon'}{4K}$.

Let $\pi$ be a subpermutation such that $\mathrm{dist}_\square(\pi, \mathcal{P}) < \delta_0$, i.e., there exists a permutation $\sigma \in \mathcal{P}$ with $|\pi| = |\sigma|$ and $\mathrm{dist}_\square(\pi, \sigma) < \delta_0$. If the order of $\pi$ is smaller than $M_0$, then $\pi$ and $\sigma$ must be the same which yields $\mathrm{dist}_\square(\pi, \mathcal{P}) = \mathrm{dist}_K(\pi, \mathcal{P}) = 0$. So, we can assume that the order of $\pi$ is at least $M_0$.

Assume to contrary that $\mathrm{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$. By Lemma 39, there exists a $\mathcal{P}$-bad $k$-sequence $A$ and a $K$-sequence $B$ such that $B$ is $(A, \varepsilon')$-witnessing for $\pi$. By the choice of $\delta_0$, $B$ is $(A, \varepsilon'/2)$-witnessing for $\sigma$ (recall that the order of $\pi$ is at least $K+1$). This yields that $R_{x_j, g^{A,\langle A\rangle}(j)}(\sigma) \neq \emptyset$ for every $j \in [|A|_\ell \cdot \langle A\rangle]$ where $x_j$ are chosen as in the definition of $(A, \varepsilon'/2)$-witnessing. In particular, $\sigma$ contains a subpermutation not in $\mathcal{P}$ (choose one element from each of the sets $R_{x_j, g^{A,\langle A\rangle}(j)}$ and consider the subpermutation induced by the chosen elements) which is impossible since $\sigma \in \mathcal{P}$ and $\mathcal{P}$ is hereditary. $\square$

# Bibliography


[1] N. Ailon and B. Chazelle: Information theory in property testing and monotonicity testing in higher dimension, *Inform. and Comput.* **204** (2006), 1704–1717.

[2] N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy: Efficient testing of large graphs, *Combinatorica* **20** (2000), 451–476.

[3] N. Alon, E. Fischer, I. Newman, and A. Shapira: A combinatorial characterization of the testable graph properties: it's all about regularity, *SIAM J. Comput.* **39** (2009), 143–167.

[4] N. Alon, M. Krivelevich, I. Newman, and M. Szegedy: Regular languages are testable with a constant number of queries, *SIAM J. Comput.* **30** (2001), 1842–1862.

[5] N. Alon and A. Shapira: Testing satisfiability, *J. Algorithms* **47** (2003), 87–103.

[6] N. Alon and A. Shapira: A characterization of the (natural) graph properties testable with one-sided error, *SIAM J. Comput.* **37** (2008), 1703–1727.

[7] N. Alon and A. Shapira: Every monotone graph property is testable, *SIAM J. Comput.* **38** (2008), 505–522.

[8] M.H. Albert, M.D. Atkinson, and M. Klazar: The enumeration of simple permutations, *J. Integer Seq.* **6** (2003), 03.4.4.

[9] R. Baber: Turán densities of hypercubes, preprint available as arXiv:1201.3587.

[10] R. Baber and J. Talbot: A solution to the 2/3 conjecture, *SIAM J. Discrete Math.* **28** (2014), 756–766.

[11] R. Baber and J. Talbot: Hypergraphs do jump, *Combin. Probab. Comput.* **20** (2011), 161–171.

[12] J. Balogh, P. Hu, B. Lidický, and H. Liu: Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube, *European J. Combin.* **35** (2014), 75–85.

[13] T. Batu, R. Kumar, and R. Rubinfeld: Sublinear algorithms for testing monotone and unimodal distributions, in: Proc. STOC'04, ACM, New York, 2004, 381–390.

[14] I. Benjamini, O. Schramm, A. Shapira: Every minor-closed property of sparse graphs is testable, *Adv. Math.* **223(6)** (2010), 2200–2218.

[15] A. Bhattacharyya, E. Grigorescu, P. Raghavendra, and A. Shapira: Testing odd-cycle freeness in Boolean functions, in: Proc. SODA'12, 1140–1149.

[16] A. Bhattacharyya, E. Grigorescu, and A. Shapira: A unified framework for testing linear-invariant properties, in: Proc. FOCS'10, 478–487.

[17] A. Bogdanov, K. Obata, and L. Trevisan: A lower bound for testing 3-colorability in bounded-degree graphs, in: Proc. FOCS'02, 93–102.

[18] B. Bollobás and O. Riordan: Sparse graphs: Metrics and random models, *Random Structures Algorithms* **39** (2011), 1–38.

[19] C. Borgs, J.T. Chayes, and L. Lovász: Moments of two-variable functions and the uniqueness of graph limits, *Geom. Funct. Anal.* **19** (2010), 1597–1619.

[20] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, B. Szegedy, and K. Vesztergombi: Graph limits and parameter testing, in: Proc. STOC'06, 261–270.

[21] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, *Adv. Math.* **219** (2008), 1801–1851.

[22] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: Convergent sequences of dense graphs II. Multiway cuts and statistical physics, *Ann. of Math.* **176** (2012), 151–219.

[23] K. Borsuk: Drei Sätze über die $n$-dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933), 177–190.

[24] F.R.K. Chung, R.L. Graham, and R.M. Wilson: Quasi-random graphs, *Combinatorica* **9** (1989), 345–362.

[25] J.N. Cooper: Quasirandom permutations, *J. Combin. Theory Ser. A* **106** (2004), 123–143.

[26] J.N. Cooper: A permutation regularity lemma, *Electron. J. Combin.* **13(1)** (2006), R#22, 20 pp.

[27] A. Czumaj and C. Sohler: On testable properties in bounded degree graphs, in Proc. SODA'07, 494–501.

[28] A. Czumaj, A. Shapira, and C. Sohler: Testing Hereditary Properties of Nonexpanding Bounded-Degree Graphs, *SIAM J. Comput.* **38(6)** (2009), 2499–2510.

[29] A. Czumaj and C. Sohler: Sublinear-time algorithms, Bulletin of the EATCS, 89 (2006), 23–47.

[30] A. Czumaj and C. Sohler: Testing expansion in bounded-degree graphs, in: Proc. FOCS'07, 570–578.

[31] P. Diaconis: Group representations in probability and statistics, in: Institute of Mathematical Statistics Lecture Notes–Monograph Series, vol. 11, Institute of Mathematical Statistics, Hayward, CA, 1988.

[32] P. Diaconis, S. Holmes, and S. Janson: Threshold graph limits and random threshold graphs, *Internet Math.* **5** (2009), 267–318.

[33] Y. Dodis, O. Goldreich, E. Lehman, S. Raskhodnikova, D. Ron, and A. Samorodnitsky: Improved testing algorithms for monotonicity, in: Proc. RANDOM'99, 97–108.

[34] G. Elek: On limits of finite graphs, *Combinatorica* **27** (2007), 503–507.

[35] P. Erdős, L. Lovász, and J. Spencer: Strong independence of graph copy functions, *Graph Theory and Related Topics, Academic Press, Amsterdam (1979)*, 165–172.

[36] F. Ergün, S. Kannan, R. Kumar, R. Rubinfeld, and M. Viswanathan: Spot-checkers, *J. Comput. System Sci.* **60** (2000), 717–751.

[37] E. Fischer: Testing graphs for colorability properties, in: Proc. SODA'01, 873–882.

[38] E. Fischer, E. Lehman, I. Newman, S. Raskhodnikova, R. Rubinfeld, and A. Samorodnitsky: Monotonicity testing over general poset domains, in: Proc. STOC'02, 474–483.

[39] E. Fischer and I. Newman: Testing versus estimation of graph properties, in: Proc. STOC'05, 138–146.

[40] R. Glebov, A. Grzesik, T. Klimošová, and D. Kráľ: Finitely forcible graphons and permutons, *J. Combin. Theory Ser. B* **110** (2015), 112–135.

[41] R. Glebov, T. Klimošová, D. Kráľ: Infinite dimensional finitely forcible graphon, preprint available as arXiv:1404.2743.

[42] R. Glebov, D. Kráľ, and J. Volec: Compactness and finite forcibility of graphons, preprint available as arXiv:1309.6695.

[43] O. Goldreich: Combinatorial property testing—a survey, in: Randomization Methods in Algorithm Design, 1998, 45–60.

[44] O. Goldreich, S. Goldwasser, E. Lehman, D. Ron, and A. Samordinsky: Testing monotonicity, *Combinatorica* **20** (2000), 301–337.

[45] O. Goldreich, S. Goldwasser, and D. Ron: Property testing and its connection to learning and approximation, JACM 45 (1998), 653–750.

[46] O. Goldreich and D. Ron: A sublinear bipartiteness tester for bounded degree graphs, *Combinatorica* **19** (1999), 335–373.

[47] O. Goldreich and D. Ron: Property Testing in Bounded-Degree Graphs, *Algorithmica* **32** (2002), 302–343.

[48] O. Goldreich and L. Trevisan: Three theorems regarding testing graph properties, *Random Structures Algorithms* **23** (2003), 23–57.

[49] A. Grzesik: On the maximum number of five-cycles in a triangle-free graph, *J. Combin. Theory Ser. B* **102** (2012), 1061–1066.

[50] H. Hatami, J. Hladký, D. Kráľ, S. Norine, and A. Razborov: Non-three-colorable common graphs exist, *Combin. Probab. Comput.* **21** (2012), 734–742.

[51] H. Hatami, J. Hladký, D. Kráľ, S. Norine, and A. Razborov: On the number of pentagons in triangle-free graphs, *J. Combin. Theory Ser. A* **120** (2013), 722–732.

[52] C. Hoppen, Y. Kohayakawa, C.G. Moreira, B. Ráth, and R.M. Sampaio: Limits of permutation sequences, *J. Combin. Theory Ser. B* **103** (2013), 93–113.

[53] C. Hoppen, Y. Kohayakawa, C.G. Moreira, and R.M. Sampaio: Limits of permutation sequences through permutation regularity, available as arXiv:1106.1663.

[54] C. Hoppen, Y. Kohayakawa, C.G. Moreira, and R.M. Sampaio: Testing permutation properties through subpermutations, *Theoret. Comput. Sci.* **412** (2011), 3555–3567.

[55] C. Hoppen, Y. Kohayakawa, C.G. Moreira, and R. M. Sampaio: Property testing and parameter testing for permutations, in: Proc. SODA'10, 66–75.

[56] S. Janson: Poset limits and exchangeable random posets, *Combinatorica* **31** (2011), 529–563.

[57] T. Klimošová and D. Kráľ: Hereditary properties of permutations are strongly testable, in: Proc. SODA'14, 1164–1173.

[58] Y. Kohayakawa: Limits of permutation sequences, presentation at the Conference on Graphs and Analysis, IAS, Princeton, US, June 2012.

[59] D. Kráľ, C.-H. Liu, J.-S. Sereni, P. Whalen, and Z. Yilma: A new bound for the 2/3 conjecture, *Combin. Probab. Comput.* **22** (2013), 384–393.

[60] D. Kráľ, L. Mach, and J.-S. Sereni: A new lower bound based on Gromov's method of selecting heavily covered points, *Discrete Comput. Geom.* **48** (2012), 487–498.

[61] D. Kráľ and O. Pikhurko: Quasirandom permutations are characterized by 4-point densities, *Geom. Func. Anal.* **23** (2013) 570–579.

[62] O. Lachish and I. Newman: Testing periodicity, in: Approximation, Randomization and Combinatorial Optimization, LNCS vol. 3624, Springer, Berlin, 2005, 366–377.

[63] L. Lovász: **Large networks and graph limits**, AMS, Providence, RI, 2012.

[64] L. Lovász and V.T. Sós: Generalized quasirandom graphs, *J. Combin. Theory Ser. B* **98** (2008), 146–163.

[65] L. Lovász and B. Szegedy: Finitely forcible graphons, *J. Combin. Theory Ser. B* **101** (2011), 269–301.

[66] L. Lovász and B. Szegedy: Limits of dense graph sequences, *J. Combin. Theory Ser. B* **96** (2006), 933–957.

[67] L. Lovász and B. Szegedy: Regularity partitions and the topology of graphons, in: I. Bárány and J. Solymosi (eds.): An irregular mind. Szemerédi is 70, Bolyai Society Mathematical Studies **21**, Springer, 2010, 415–446.

[68] L. Lovász and B. Szegedy: Szemerédi's regularity lemma for the analyst, *Geom. Funct. Anal.* **17** (2007), 252–270.

[69] L. Lovász and B. Szegedy: Testing properties of graphs and functions, *Israel J. Math.* **178** (2010), 113–156.

[70] S. Norine: private communication.

[71] O. Pikhurko and A. Razborov: Asymptotic structure of graphs with the minimum number of triangles, preprint available as arXiv:1204.2846.

[72] O. Pikhurko and E.R. Vaughan: Minimum number of $k$-cliques in graphs with bounded independence number, *Combinat. Prob. Comput.* **22** (2013), 910–934.

[73] C.B. Presutti and W.R. Stromquist: Packing rates of measures and a conjecture for the packing density of 2413, in: *Permutation patterns, London Mathematical Society Lecture Note Series* **376** (2010), 3–40.

[74] A. Razborov: Flag algebras, *J. Symbolic Logic* **72** (2007), 1239–1282.

[75] A. Razborov: On 3-hypergraphs with forbidden 4-vertex configurations, *SIAM J. Discrete Math.* **24** (2010), 946–963.

[76] A. Razborov: On the minimal density of triangles in graphs, *Combin. Probab. Comput.* **17** (2008), 603–618.

[77] V. Rödl and R.A. Duke: On graphs with small subgraphs of large chromatic number, *Graphs Combin.* **1** (1985), 91–96.

[78] V. Rödl: On universality of graphs with uniformly distributed edges, *Discrete Math.* **59** (1986), 125–134.

[79] D. Ron: Property testing, in: Handbook on Randomization, vol. II, 2001, 597–649.

[80] R. Rubinfeld: Sublinear time algorithms, in M. Sanz-Sole, J. Soria, J. L. Varona, and J. Verdera (eds.): Proc. International Congress of Mathematicians, 2006, volume III, 1095–1110.

[81] R. Rubinfeld and M. Sudan: Robust characterizations of polynomials with applications to program testing, *SIAM J. Comput.* **25** (1996), 252–271.

[82] A. Shapira: Green's conjecture and testing linear-invariant properties, in: Proc. STOC'09, 159–166.

[83] A. Shapira: A proof of Green's conjecture regarding the removal properties of sets of linear equations, *J. London Math. Soc.* **81** (2010), 355–373.

[84] A. Thomason: Pseudo-random graphs, in: A. Barlotti, M. Biliotti, A. Cossu, G. Korchmaros, and G. Tallini, (eds.), North-Holland Mathematics Studies **144**, North-Holland, 1987, 307–331.

[85] A. Thomason: Random graphs, strongly regular graphs and pseudorandom graphs, in: C. Whitehead (ed.), Surveys in Combinatorics 1987, London Mathematical Society Lecture Note Series, **123**, Cambridge Univ. Press, Cambridge, 1987, 173–195.