Vector and Tensor Fields

by

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Abstract.

This thesis consists of two unconnected parts.
In the first part we study the C^r-conjugacy classes of flows on two dimensional manifolds whose flow lines near a fixed point are diffeomorphic to the level surfaces of a Morse function near a critical point and which have no holonomy. We show how these can be decomposed into those in which every flow line is closed and those in which no flow line is closed. In the remainder of the thesis we consider the latter case and show that then the number of limit sets is finite. We describe their geometry and use the techniques of ergodic theory to show that the number of asymptotic cycles is finite in certain cases. We show that the asymptotic cycles are classifying for flows of this type on a manifold of genus 2 with exactly two non--trivial limit sets. Finally we give some new examples on manifolds of higher genus both of flows in which every flow line is dense and of flows in which each limit set is a closed, nowhere dense set which meets any transverse interval in a perfect set.
In the second part we consider differential operators which are functorially associated to Riemannian manifolds and which satisfy a regularity condition that arises in the proof of the index theorem via the heat equation. These are classified in terms of the O_n-equivariant representations of the general linear group.

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I should also like to thank Professor M.S.Narasimhan for his help in the second part of the thesis, the Science Research Council for their financial support during most of the preparation of this thesis, the Universities of Warwick & Paris XI and the Institut des Hautes Etudes Scientifiques for their stimulating mathematical environments and finally all my colleagues who have so willingly helped me.

Declaration.

The second part of this thesis was originally published in volume 10 of the Journal of Differential Geometry in December 1975. None of the material in the thesis has ever been used by me in any other context.

Contents

Contents

Contents

Contents

PART I

MORSE FOLIATIONS

Chapter 1. Introduction.

Consider a smooth codimension 1 foliation of a differentiable manifold M in which is embedded a closed two dimensional submanifold. By Sard's theorem, the embedding can be approximated by one in which the intersections of the leaves of the foliation trace out on the submanifold a flow which near a fixed point is like the level surfaces of a Morse function near a zero. By adjusting the embedding near a saddle point it can be assumed that no saddle point of the flow is joined to any other by a flow line.

In this thesis we study C^2 flows on two dimensional manifolds whose fixed points have these two properties. As they are studied from a foliations theoretical viewpoint they will be called Morse foliations.

Embeddings of two dimensional manifolds in foliated manifolds arise naturally in a number of ways. For example, if M is a fibre bundle with fibre a two dimensional manifold, then there are many embeddings of the fibre in M. Again, of $H^1(M, \mathbb{Z}) = 0$ and M is compact or has a non-closed leaf there is a transverse circle embedded in M which bounds an embedded two dimensional manifold. The study of the induced flow in this case is exploited in the proof of Novikov's theorem (see [10]).

Conversely any Morse foliation of a two dimensional manifold W is induced from the natural embedding of W in the normal bundle of the corresponding Haefliger structure (see [14] & 2.9)

We shall mainly be concerned with the holonomy group (see [10]),the limit sets and the invariants of Morse foliations.

In chapter 5 we prove that a Morse foliation with trivial holonomy groups can be decomposed into Morse foliations with every leaf closed and Morse foliations with no leaf closed.

The interesting Morse foliations to study are those with no holonomy and no closed leaf and the remainder of our results concern these.In chapter 7 we prove our second main result. Using the theorem of A.J.Schwartz (see [31]) and an elementary analysis of the point of first return function on a small transverse interval we show that in this situation there are only finitely many limit sets. In general a limit set is a nowhere dense set which meets any transverse submanifold in a perfect set.

This behaviour contrasts sharply with the situation on the sphere or torus (see [1] and [4]).On the other hand Hector ([12]) and Sacksteder ([20]) have given examples of codimension 1 foliations (in one case analytic) of three dimensional manifolds in which there are exceptional minimal sets.

In chapter eight we apply the techniques of ergodic theory to prove that in a certain restricted situation the number of asymptotic cycles of a Morse foliation is finite. The essential feature of these Morse foliations is that given any transverse circle meeting a single ω-limit set of a leaf in a set Ω, any holonomy invariant transverse measure and any point p of Ω then that circle can be approximated in measure by the disjoint union of iterates under the point of first return function of any arbitrarily small interval about p.

In chapter 10 we give the first known examples of Morse foliations with trivial holonomy groups and no closed leaf on 2-manifolds of genus greater than one. We also show that Morse foliations of the two manifold of genus 2 with no holonomy and no closed leaf which have exactly two limit sets (the other possibility is one limit set) are classified by their asymptotic cycles. A typical example of such a Morse foliation is shown in figure 9.5 in which the pairs of circles A_1, A_2 and E have to be identified by suitable diffeomorphisms.

Three questions are raised and left unanswered by the thesis. The first is whether Morse foliations without

holonomy and without closed leaves on manifolds of genus greater than one can be analytic. The second is whether it is possible for Morse foliations with no holonomy and no closed leaf to have a single limit set which is not the whole manifold.
Thirdly it is not known, in general, whether the asymptotic cycle of a leaf depends only on its ω-limit set. Indeed, except in a weak measure theoretic sense, it is not known if the number of asymptotic cycles (up to multiplication by positive scalars) is finite.

Chapter two sketches the theoretical foundations of the study and states the material assumed. Chapters three and four consider the behaviour near a centre and it is shown that if the holonomy groups of the Morse foliation are all trivial then the centres are of just two types. The rest of the thesis contains the results already mentioned.

Standard notation is used throughout the thesis. In particular, $\mathbb{R}$ denotes the real numbers and round brackets are used to denote either an interval or a point of $\mathbb{R}^2$ - depending on the context. Lemmas and propositions are numbered in the same sequence within each section of each chapter and diagrams are numbered within each chapter in a separate sequence. Numbers in square brackets refer to the bibliography.

Chapter 2. Morse foliations.

2.1 The manifolds M_g.

To fix our ideas we define for each integer $g \geqslant 0$ an orientable two dimensional manifold M_g of differentiability class C^r $(o \leqslant r \leqslant \omega)$ and genus g.

M_o is the 2-sphere $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with the usual differential structure and orientation induced from that on $\mathbb{R}^3$. It is simply connected and is therefore its own universal cover.

M_1 is the 2-torus which is defined as follows.
Let $\mathbb{Z} \oplus \mathbb{Z}$ act on $\mathbb{R}^2$ via:

$$((m,n),(x,y)) \in \mathbb{Z} \oplus \mathbb{Z} \times \mathbb{R}^2 \longmapsto (x+m, y+n) \in \mathbb{R}^2.$$

M_1 is the quotient space of $\mathbb{R}^2$ under this action. The projection $\rho_1 : \mathbb{R}^2 \longrightarrow M_1$ is a local homeomorphism. The C^r structure and orientation on M_1 are the unique ones making ρ_1 a local orientation preserving C^r diffeomorphism, where $\mathbb{R}^2$ has the usual C^r-structure and orientation.

M_g for $g \geqslant 2$ is defined as follows.
Let $H = \{z \in \mathbb{C} : |z| < 1\}$ be the hyperbolic plane with geodesics circles perpendicular to the boundary circle of H.

Let F be the unique geodesic sided regular polygon in H with centre at 0, $4g$ sides, angle sum 2π and a vertex on the positive real axis. Label and orient the sides $A_1, B_1, A_1', B_1', A_2, \ldots\ldots, A_g, B_g, A_g', B_g'$, in an anticlockwise direction. Let α_1 denote the unique orientation preserving isometry of H mapping A_1' onto A_1 in the opposite direction and β_1 the unique orientation preserving isometry of H mapping B_1 onto B_1' in the opposite direction.

Let $\quad K_g = Gp\{a_1, b_1 \; 1 \leqslant 1 \leqslant g \; : \; a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}\}$.

K_g acts on H via $a_1 \longmapsto \alpha_1$, $b_1 \longmapsto \beta_1$ and M_g is the quotient space of H under this action.

If $\rho_g : H \longrightarrow M_g$ is the projection, ρ_g is a local homeomorphism.

The C^r structure and orientation on M_g is the unique one which makes ρ_g a local C^r orientation preserving diffeomorphism, where H has the usual C^r structure and orientation as a submanifold of $\mathbb{C}$. Further details may be found in [34] chapter 4 section 9.

Now it is very well known that any orientatable two dimensional manifold is homeomorphic to M_g where g is the genus of the manifold (see for example [6]). It then follows from [18] that any C^r two dimensional manifold is C^r diffeomorphic to M_g $(0 \leqslant r \leqslant \infty)$.

This observation shows that we can regard the manifold

M_g as the join of g-1 tori :

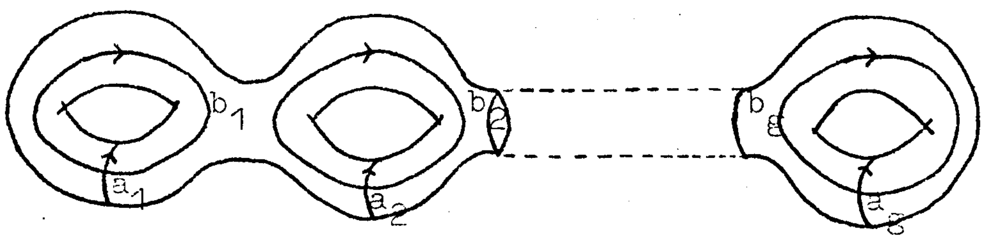


Fig. 2.1

2.2 Homology of M_g

It is well known that

$H_1(M_g, \mathbb{Z}) \cong 2g\mathbb{Z}$ (i.e. direct sum of 2g copies of $\mathbb{Z}$).
Give S^1 the usual orientation (i.e. that induced from the usual orientation on $\mathbb{R}^1$ under the covering map $\rho: \mathbb{R} \longrightarrow S^1$ $: t \longmapsto e^{2\pi i t}$), and let $\alpha \in H_1(S^1, \mathbb{Z})$ be the associated generator.
Any embedding $\iota: S^1 \longrightarrow M_g$ as a submanifold induces a class $\iota_*(\alpha) \in H_1(M_g, \mathbb{Z})$ which we shall refer to as the <u>homology class associated to $\iota(S^1)$.</u> Any integer homology class is an integer multiple of the homology class associated to an embedded circle in this way (see e.g. [36]).This homology class is zero if and only if there is a commutative diagram:

$$\begin{tikzcd} S^1 \arrow[rr, "\iota"] \arrow[dr, "j"'] & & M_g \\ & W \arrow[ur, "k"'] & \end{tikzcd}$$

where j is a diffeomorphism onto ∂W and k is an embedding of W, a two dimensional manifold with boundary, into M_g. Geometrically this means that when we "cut" along $\iota(S^1)$ in M_g we obtain manifolds diffeomorphic to $M_{g_1}\setminus$disc and $M_{g_2}\setminus$disc where $g_1+g_2 = g$, $\iota(S^1) = \partial$disc and W is diffeomorphic to $M_{g_1}\setminus$disc or $M_{g_2}\setminus$disc.
The homology classes associated to the circles a_i, b_i shown in figure 2.1 serve as a set of generators of $H_1(M_g,\mathbb{Z})$.

2.3 Intersection numbers.

The intersection number is a bilinear pairing

$$\bullet : H_1(M_g,\mathbb{Z})\times H_1(M_g,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$: \qquad ([\xi],[\eta]) \longmapsto [\xi]\bullet[\eta]$$

which is associative and antisymmetric (see [5]).
Geometrically, if $\iota_1, \iota_2 : S^1 \longrightarrow M_g$ are two embedded circles intersecting transversely, and if we traverse ι_1 once in the positive direction and count +1 for each time ι_2 crosses from the right and -1 for each time ι_2 crosses from the left, then adding these numbers gives the intersection number of the class associated to $\iota_1(S^1)$ with that associated to $\iota_2(S^1)$ up to sign.
Taking the classes $[a_i]$, $[b_i]$ associated to the circles a_i, b_i shown in figure 2.1 we see that:

$$[a_i]\bullet[a_j] = [b_i]\bullet[b_j] = 0$$
$$[a_i]\bullet[b_j] = -[b_j]\bullet[a_i] = \delta_{ij}.$$

2.4 Morse foliations.

In this section we define the fundamental objects of our study. We give a definition which belongs unequivocally to foliations theory and those who have other tastes may prefer the definitions given in sections 2.5 or 2.6.

A <u>Morse foliation</u> $\mathfrak{F}$ of class C^r $(2 \leq r \leq \omega)$ on the orientable 2-manifold M_g of genus g and class C^s $(s \geq r)$ is a set $\{f_i : V_i \longrightarrow \mathbb{R} : i \in I\}$ of Morse functions satisfying:

(i) $\{V_i\}_{i \in I}$ is an open cover of M_g.

(ii) $f_i : V_i \longrightarrow \mathbb{R}$ is a C^r Morse function.

(iii) If $x \in V_i \cap V_j$ there is a neighbourhood U of x in $V_i \cap V_j$ and an <u>orientation preserving</u> C^r diffeomorphism h_{ij} defined on a neighbourhood of $f_j(x)$ such that:

$$h_{ij}(f_j(y)) = f_i(y) \quad y \in U.$$

(iv) $\mathfrak{F}$ is maximal with respect to property (iii).

The Morse functions $f_i \in \mathfrak{F}$ are called <u>distinguished maps.</u> Condition (iv) means that any Morse function which is locally the composition of a C^r- orientation preserving diffeomorphism with some $f_i \in \mathfrak{F}$ also lies in $\mathfrak{F}$.

Condition (iii) needs further elaboration. We first remark that (iii) includes the orientation preserving property of the diffeomorphisms h_{ij} so that we always assume our Morse foliations to be "transversely oriented". This condition also implies that if p is a critical point of f_i and $p \in V_j$ then p is a critical point of f_j. Such a point will be called a <u>singular point</u> of $\mathfrak{F}$. Since the critical points of Morse functions are isolated and M_g is compact, there are only finitely many singular points. Finally note that the germ of h_{ij} at $f_j(x)$ is uniquely specified except in the case that x is a centre. This remark is elaborated in section 2.5 which follows.

2.5 Distinguished charts.

We shall suppose that $\mathbb{R}^2$ has co-ordinates (x,y). The following remarks are explained in [16].

Let $\mathfrak{F} = \{f_i : V_i \longrightarrow \mathbb{R} : i \in I\}$ be a C^r Morse foliation on M_g. If $p \in M_g$ is not a singular point and $p \in V_i$ then there is an orientation preserving chart centred at p whose image is a neighbourhood U of O in $\mathbb{R}^2$, such that

$$f_i \varphi^{-1}(x,y) = x \qquad (x,y) \in U.$$

Such a chart is called a <u>distinguished chart at p.</u>

If $p \in M_g$ is a singular point and $p \in V_i$ the <u>Morse index</u> of f_i at p is defined to be the maximum dimension of a subspace on which the Hessian of f_i is negative definite.

If $p \in V_j$ then since the h_{ij} are orientation preserving, the Morse index of f_j at p is equal to that of f_i at p and hence p has a well defined <u>Morse index.</u>
If this index is 0 or 2, p is called a <u>centre</u> and if it is 1, p is called a <u>saddle point</u>.

It follows from the Morse lemma, that if $p \in V_i$ is a singular point then there is an orientation preserving chart φ centred at p defined from a neighbourhood of p in M_g to a neighbourhood U of 0 in $\mathbb{R}^2$ such that throughout U

$f_i\varphi^{-1}(x,y) = f_i(p)+x^2+y^2$ if p is a centre of index 0,
$f_i\varphi^{-1}(x,y) = f_i(p)+x^2-y^2$ if p is a saddle point,
$f_i\varphi^{-1}(x,y) = f_i(p)-x^2-y^2$ if p is a centre of index 2.

Such a chart is called a <u>distinguished chart at p</u>.

Consideration of a distinguished chart at $p \in V_i$ shows why the germ of h_{ji} at $f_i(p)$ is well-defined except at a centre.

A Morse foliation can equally well be defined as a maximal atlas of distinguished charts but as the exact properties of such an atlas are somewhat inelegant (the overlap properties vary according to whether one is at a singular point or not) we omit them.

2.6 Vector fields tangent to a Morse foliation.

Consider the standard flows on $\mathbb{R}^2$:

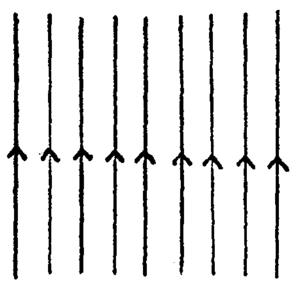


Fig. 2.2

Flow: $\varphi_t(x,y) = (x,y+t)$.

Field: $\partial/\partial y$.

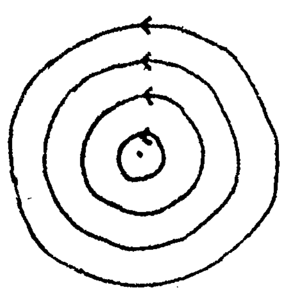


Fig. 2.3

Flow: $\varphi_t(x,y) =$

$(x\cos t - y\sin t, y\cos t + x\sin t)$.

Field: $-y\,\partial/\partial x + x\,\partial/\partial y$.

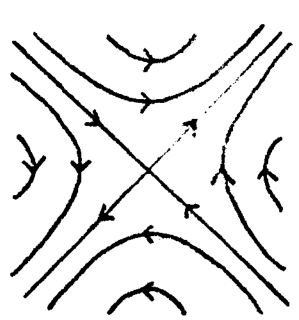
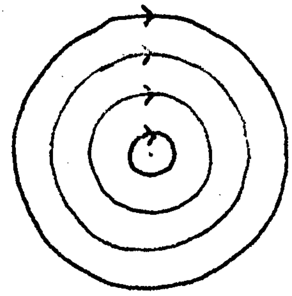


Fig. 2.4

Flow: $\varphi_t(x,y) =$

$(x\,\mathrm{ch}\,t + y\,\mathrm{sh}\,t, x\,\mathrm{sh}\,t + y\,\mathrm{ch}\,t)$.

Field: $y\,\partial/\partial x + x\,\partial/\partial y$.



Flow: $\varphi_t(x,y) = (x\cos t + y\sin t, y\cos t - x\sin t)$

Field: $y\,\partial/\partial x - x\,\partial/\partial y$.

Fig. 2.5

Let M_g be the two dimensional orientable manifold of genus g and differentiability class C^s $(s \geqslant 2)$ and let $\mathcal{F}$ be a C^r $(2 \leqslant r \leqslant s)$ Morse foliation on M_g.
For each point $p \in M_g$ let φ_p be a distinguished chart at p defined on a neighbourhood U_p of p.
According as p is non-singular or has Morse index 0,1 or 2, pull back the vector field given in figure 2.2, 2.3, 2.4, 2.5 via φ_p to obtain a vector field s_p on U_p.
Using a partition of unity subordinate to $\{U_p\}_{p \in M_g}$ we can piece together the local vector fields s_p to obtain a vector field X on M_g with the following properties:

(a) each distinguished map f is constant along the flow lines of X.

(b) $X(p) = 0$ if and only if p is a singular point of $\mathcal{F}$ (by the compatibility condition 2.4 (iii)).

This observation leads to the third possible definition of a Morse foliation. That is, it can be regarded as a flow of a vector field with the property that in a neighbourhood of a zero of the vector field the flow lines are diffeomorphic to the level surfaces of a Morse function near a critical point. However different Morse foliations may arise from the same flow-a point which is discussed in more detail in section 2.14.

2.7 The index theorem.

Having constructed a vector field as in 2.6, the index theorem for vector fields (see e.g. [7]) then shows that:

2-2g = number of centres - number of saddle points,

where g is the genus of the manifold.

In terms of distinguished maps this becomes

$$2-2g = C_0 - C_1 + C_2$$

where C_i is the number of singular points of the Morse foliation $\mathfrak{F}$ of Morse index i (i = 0,1,2).

2.8 Leaves and the leaf space.

Let M_g be the 2-manifold of genus g which is oriented and of differentiability class C^s ($s \geqslant 2$) as defined in section 2.1 and let $\mathfrak{F} = \{f_i : V_i \longrightarrow \mathbb{R}\}$ be a Morse foliation on M_g of class C^r ($2 \leqslant r \leqslant s$).

We define the <u>leaf manifold M_g^{α}</u> of $\mathfrak{F}$ as follows.

The points of M_g^{α} are the points of M_g.

A base of open sets for M_g^{α} consists of all sets of the form $U \cap f_i^{-1}(c)$ where $c \in \mathbb{R}$ and U is an open subset of M_g.

Although we have used terminology analogous to that in use in foliations theory, M_g^{α} is not usually a manifold (except possibly in the case $g=1$), since there can be no chart about a singular point. However if the singular points are removed from M_g^{α} we obtain a 1-dimensional manifold $\dot{M}_g^{\alpha}$, which is not second countable.

The centres are isolated points in M_g^{α}.

There is a natural continuous bijection

$$i : M_g^{\alpha} \longrightarrow M_g$$

which restricts to an immersion on $\dot{M}_g^{\alpha}$.

A <u>leaf</u> is a connected component of M_g^{α}. Any component of $\dot{M}_g^{\alpha}$ is a submanifold of M_g.

The vector fields constructed in 2.6 are all tangent to the leaves of $\mathfrak{F}$. Fix such a vector field $X_{\mathfrak{F}}$.

Then the associated flow

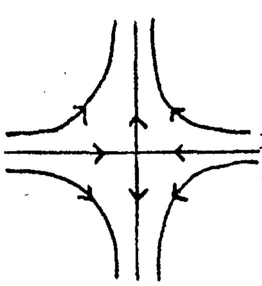
$$\Phi_{\mathfrak{F}} : M_g \times \mathbb{R} \longrightarrow M_g \ : \ (x,t) \longmapsto \Phi_{\mathfrak{F},t}(x)$$

parametrises each <u>non-singular leaf</u> (that is each leaf not containing a singular point).

Suppose now that each leaf contains at most one saddle point. Then each leaf which contains a saddle point is the disjoint union of a fixed point p of $\Phi_{\mathfrak{F}}$ and at most four and at least two trajectories of $\Phi_{\mathfrak{F}}$. These trajectories are called <u>separatrices</u> of $\mathfrak{F}$.

A separatrix is called <u>inward</u> if for any point x_0 lying on the separatrix $\lim_{t\to\infty} \Phi_{\mathfrak{F},t}(x_0) = p$ and <u>outward</u> if $\lim_{t\to-\infty} \Phi_{\mathfrak{F},t}(x_0) = p$.

Any separatrix is either inward or outward and one that is both is called a <u>loop separatrix</u> (see figure 2.6).



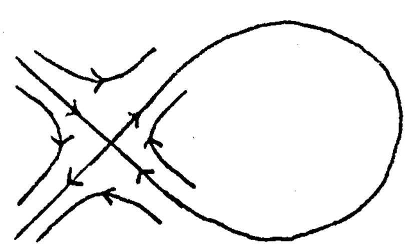


Fig. 2.6

We shall see that the loop separatrices play quite an important role in the theory of Morse foliations.

The flow $\Phi_{\mathfrak{F}}$ also parametrises the separatrices of $\mathfrak{F}$ so that if $\dot{M}_g$ is the complement in M_g of the singular points there is a consistent orientation on the leaves of the induced foliation there (see [14] for a survey of foliations).

<u>2.9 We shall now make the assumption that no leaf of any Morse foliation contains more than one singular point.</u>

Note that any Morse foliation $\mathfrak{F} = \{ f_i : U_i \longrightarrow \mathbb{R} \}_{i \in I}$ on M_g can be approximated by one satisfying this special condition (compare the results of [20] and [37]).
To see this, it is first necessary to embed M_g in a (genuinely) foliated three dimensional manifold $(N, \mathcal{G})$ constructed as follows.
Choose a finite subcover $U_{i_1}, \ldots, U_{i_n}$ of $\{U_i\}_{i \in I}$ such that each singular point of $\mathfrak{F}$ is contained in a single set U_{i_k}. For each k $(1 \leq k \leq n)$ let W_{i_k} be a neighbourhood of $\{(x, f_{i_k}(x)) : x \in U_{i_k}\}$ in $U_{i_k} \times \mathbb{R}$ with the property that for each point $x \in U_{i_k} \cap U_{i_l}$ the unique local diffeomorphism $g_{i_k i_l}$ of $\mathbb{R}$ with

$$f_{i_k} = g_{i_k i_l} f_{i_l} \quad \text{on a neighbourhood of } x$$

is defined at each point y with $(x,y) \in W_{i_k}$.
Then we let

$$N = \frac{\coprod_{k=1}^{n} W_{i_k}}{(x,y) \in W_{i_k} \sim (x, g_{i_l i_k}(y)) \in W_{i_l}}$$

and take the distinguished map of $\mathcal{G}$ on W_{i_k} to be the projection on $\mathbb{R}$.
Now there is an embedding

$$i : M \longrightarrow N : x \in U_{i_k} \longmapsto (x, f_{i_k}(x))$$

with $i^* \mathcal{G} = \mathfrak{F}$.
Then by making arbitrarily small adjustments to i near

each saddle point we can ensure that the induced Morse foliations satisfy our condition and approximate $\mathfrak{F}$.

2.10 The regular covering space $\widetilde{M}_g^{\alpha}$ of M_g^{α}.

In this section we define a covering space which will enable us to give a rigorous definition of holonomy in section 2.11.

We define the space $\widetilde{M}_g^{\alpha}$ as follows.
Let $\mathfrak{F}$ be a Morse foliation on M_g.
The points of $\widetilde{M}_g^{\alpha}$ consist of pairs $[x,f]$ where $x \in M_g$ and f is a germ at x in M_g of a distinguished map.
The open sets of $\widetilde{M}_g^{\alpha}$ have a subbase consisting of sets

$$\widetilde{U}(f) = \{[x,f] : [x,f] \in \widetilde{M}_g^{\alpha}, \ x \in \check{U}\}$$

where $\check{U}$ is an open set of M_g^{α} contained in the domain of f.

<u>Lemma</u>: $\widetilde{M}_g^{\alpha}$ is Hausdorff and the map

$$\alpha : \widetilde{M}_g^{\alpha} \longrightarrow M_g^{\alpha} \ : \ [x,f] \longrightarrow x$$

is a regular covering map.

<u>Proof</u>: Since M_g is Hausdorff and centres are isolated points in M_g it is sufficient to show that distinct points $[x,f]$, $[x,g]$ of M_g ,where x is not a centre, can be separated by open sets. Suppose that f and g are both defined on an open neighbourhood U of x in M_g, containing no centre.

Let $\tilde{U} = U \cap f^{-1}(c) = U \cap g^{-1}(c')$ be an open neighbourhood of x in M_g^α.

We show that $\tilde{U}(f) \cap \tilde{U}(g) \neq \phi$ for all open neighbourhoods U of x in M_g leads to a contradiction.

Since $[x,f]$ and $[x,g]$ are distinct we can choose a diffeomorphism k of a neighbourhood $W_{c'}$ of c' in $\mathbb{R}$ onto a neighbourhood W_c of c in $\mathbb{R}$ whose germ at c' is not that of the identity and choose U so small that $gU \subseteq W_{c'}$, and

$$f(y) = k(g(y)) \quad y \in U \quad \text{(i)}.$$

If $[z,h] \in \tilde{U}(f) \cap \tilde{U}(g)$ then $h(z)=f(z)=g(z)=c=c'$ and there is a neighbourhood V of z in U such that

$$f(y) = h(y) = g(y) \quad y \in V \quad \text{(ii)}.$$

Now since z is not a centre g(V) is a neighbourhood of c' in $W_{c'}$. Statement (ii) then implies that $k(e) = e \quad e \in g(V)$ contradicting the assumption that the germ of k at c' is not that of the identity.

This completes the proof that $\tilde{M}_g^\alpha$ is Hausdorff.

We now show that α is a covering map.

Let $[p,f] \in \tilde{M}_g^\alpha$ and let f be defined on the domain U of a distinguished chart φ at p.

For each germ of a diffeomorphism h defined at f(p) let:

$U_h = \{[x,g] : x \in U \cap f^{-1}(f(p)), g=hf \text{ as germs at } x \in U\}$, and

$V = U \cap f^{-1}(f(p))$ an open subset of M_g^α.

Then $\alpha | U_h : U_h \longrightarrow V$ is a homeomorphism.

Also $U_{h_1} \cap U_{h_2} \neq \emptyset \Leftrightarrow U_{h_1} = U_{h_2}$.
Finally note that if $[x,k] \in \alpha^{-1}(\tilde{V})$ k is the germ of a distinguished map at x and $f(x) = f(p)$. It follows from the definition of a Morse foliation that there is a germ of a diffeomorphism h at $f(p)$ such that $k = hf$ near x.
Thus $\alpha^{-1}(\tilde{V}) = \bigcup_h U_h$.

It remains to show that the covering is regular i.e. that if $[x_0,f_1],[x_0,f_2] \in \tilde{M}_g^\alpha$ there is a homeomorphism φ of $\tilde{M}_g$ such that:

(i) $\varphi[x_0,f_1] = \varphi[x_0,f_2]$

(ii) $\alpha \circ \varphi = \alpha$.

Let $\beta : \tilde{M}_g^\alpha \longrightarrow \mathbb{R} : [x,f] \longmapsto f(x)$.

Then $\beta^{-1}(f_1(x_0))$ and $\beta^{-1}(f_2(x_0))$ are open and closed subsets of $\tilde{M}_g$.

Choose h such that $f_2 = hf_1$ near x_0.

If $f_2(x_0) \neq f_1(x_0)$ let $\varphi[x,f] = \begin{cases} [x,hf] & \text{on } \beta^{-1}f_1(x_0) \\ [x,h^{-1}f] & \text{on } \beta^{-1}f_2(x_0) \\ [x,f] & \text{otherwise.} \end{cases}$

If $f_1(x_0) = f_2(x_0)$ let $\varphi[x,f] = \begin{cases} [x,hf] & \text{on } \beta^{-1}f_1(x_0) \\ [x,f] & \text{otherwise} \end{cases}$

φ is the required covering map.

2.11 Holonomy

Our definition of holonomy is due to Haefliger ([3] or [9]).

In this section holonomy is defined using the lifting property of the covering map

$$\alpha : \tilde{M}_g^{\mathscr{L}} \longrightarrow M_g^{\mathscr{L}} .$$

A geometric interpretation of the notion is given below in section 2.13.

Let $x_o \in M_g$ and let γ be a loop at x_o. γ lifts uniquely to a path $\tilde{\gamma}$ in $\tilde{M}_g$: $\tilde{\gamma}(t) = [\gamma(t), f_t]$ with f_o previously chosen.

Further, it follows from the properties of regular covering spaces (see [13]) that if $\gamma \simeq \gamma'$ rel x_o then $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$ and $\tilde{\gamma} \simeq \tilde{\gamma}'$ rel. 0,1.

If L_{x_o} is the leaf containing x_o and f_o is fixed, this process defines a map

$$h_{[x_o,f_o]} : \Pi_1(L_{x_o}, x_o) \longrightarrow G_{f_o(x_o)} ,$$

where $G_{f_o(x_o)}$ is the group of germs of C^r diffeomorphisms mapping $f_o(x_o)$ onto itself and $h_{[x_o,f_o]}([\gamma]) = h$ where h is the unique germ with $f_1 = hf_o$ near x_o, if x_o is not a centre and $h_{[x_o,f_o]}([\gamma])$ is the germ of the identity if x_o is a centre.

The definition is reasonable for a centre since $\tilde{\gamma}(t) = [x_0, f_0]$ $0 \leq t \leq 1$ in this case.

If f_0' is another germ at x_0, let h' be the unique germ at $f_0(x_0)$ with $f_0' = h'f_0$, assuming that x_0 is not a centre. If $\tilde{\gamma}$ is the unique lift of γ starting at $[x_0, f_0]$, so that $\tilde{\gamma}(t) = [\gamma(t), f_t]$, then the unique lift $\tilde{\gamma}'$ of γ starting at $[x_0, f_0']$ is given by $\tilde{\gamma}'(t) = [\gamma(t), h'f_t]$.
Thus $h_{[x_0, f_0']}([\gamma]) = h' h_{[x_0, f_0]}([\gamma]) \circ (h')^{-1}$.
Hence in particular, $h_{[x_0, f_0]}$ is an antihomomorphism of $\pi_1(L_{x_0}, x_0)$ into $G_{f_0(x_0)}$ and the images of $\pi_1(L_{x_0}, x_0)$ are isomorphic.

The <u>holonomy group</u> of L_{x_0} is the isomorphism class of the image in $G_{f_0(x_0)}$ of $\pi_1(L_{x_0}, x_0)$.

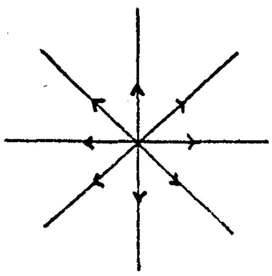
Now (see [35]) the equivalence classes under inner automorphisms of $G_{f_0(x_0)}$ of homomorphisms of $\pi_1(L_{x_0}, x_0)$ into $G_{f_0(x_0)}$ are in bijective correspondence with elements of $H^1(L_{x_0}, G_{f_0(x_0)})$ - the set of isomorphism classes of principal $G_{f_0(x_0)}$ bundles on L_{x_0}, where $G_{f_0(x_0)}$ has the discrete topology.
If Γ_1^r is the topological groupoid of C^r diffeomorphisms of $\mathbb{R}^1$ (see [8]), $\mathcal{F}$ restricts to a Γ_1^r - structure on L_{x_0} which has a representative in the groupoid taken with the

discrete topology. It should be no surprise that this is the element just obtained.

2.12 Transverse vector fields

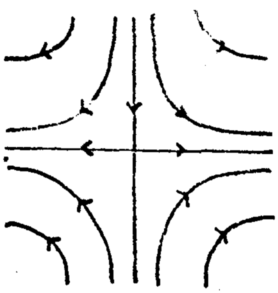
Let $\mathfrak{F}$ be a C^r Morse foliation on M_g, the orientable 2-manifold of genus g.
If $p \in M_g$ is a non-singular point of $\mathfrak{F}$, a <u>transverse interval</u> at p is a C^r embedded interval whose tangent vector at every point, together with the tangents to the leaf through that point span the tangent space to M_g.

Consider the vector fields on $\mathbb{R}^2$:



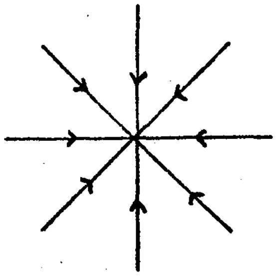
Field: $x\,\partial/\partial x + y\,\partial/\partial y$
Flow : (xe^t, ye^t)

Fig. 2.7



Field: $x\,\partial/\partial x - y\,\partial/\partial y$
Flow : (xe^t, ye^{-t})

Fig. 2.8



Field: $-x\,\partial/\partial x - y\,\partial/\partial y$

Flow : (xe^{t}, ye^{-t})

Fig. 2.9

A <u>transverse vector field</u> $X_{\mathfrak{F}}^{\perp}$ on M_g is one constructed as follows.

Choose a finite cover of M_g by distinguished charts such that a neighbourhood of each singular point is contained in a unique chart.

Define local vector fields on each chart by pulling back the field of figure 2.7, 2.8, 2.9, if the chart contains a singular point of Morse index 0,1,2 respectively and on any other chart choosing a flow whose trajectories are locally transverse intervals and such the pairs (tangent to transverse flow, tangent to $\mathfrak{F}$) lie in the orientation of M_g.

$X_{\mathfrak{F}}^{\perp}$ is formed from these local fields using a C^r partition of unity.

We make several observations.

$X_{\mathfrak{F}}^{\perp}$ is C^r if $\mathfrak{F}$ is C^r.

In a neighbourhood of a singular point of $\mathfrak{F}$, $X_{\mathfrak{F}}^{\perp}$ is locally diffeomorphic to one of the flows in 2.7, 2.8 or 2.9.

Finally note that if we are given a transverse circle or a compact union of transverse circles, together with a finite number of transverse intervals we can extend these to the flow of a transverse vector field.

To conclude this section, the following definition is useful.

A <u>transverse interval</u> at a singular point p is the homeomorphic image $c([0,1))$ of a homeomorphism

$$c : [0,1) \longrightarrow M_g$$

onto its image such that:

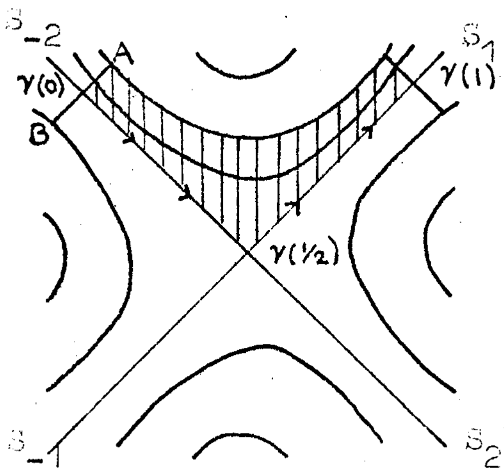
(i) $c|(0,1)$ is an embedding of $(0,1)$ into a trajectory of a transverse vector field.

(ii) $c(0) = p$.

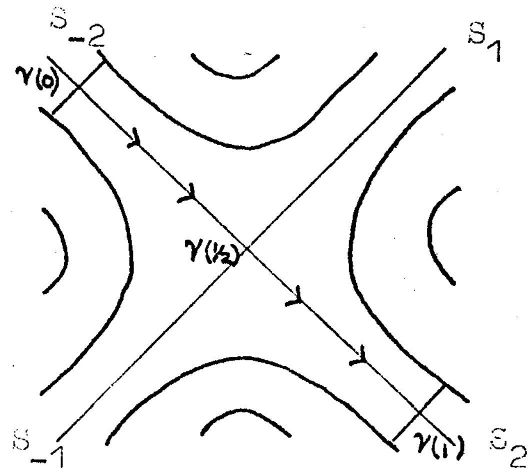
2.13 The holonomy lemma.

Suppose that $\mathfrak{F}$ is a (C^2) Morse foliation on M_g. Given a leaf L containing a saddle point and a path γ in L from p_1 to p_2 (non-singular points) which passes through the saddle point, it is false that a transverse interval at p_1 sweeps out a "strip" when translated along γ. However if γ has the property that each passage through the saddle point is either contained in a fixed pair of adjacent separatrices or the opposite pair then half a transverse interval will sweep out such a strip.

This condition on paths is made precise in part (i) of the definition of an admissible curve below.



(a) half strip swept out.



(b) no strip swept out.

Fig. 2.10

So let s be a saddle point on a leaf L and φ a distinguished chart at s. Consider the four subsets of the range of φ :

$S_1 = \{(x,x) : x > 0\}$ $\qquad$ $S_{-1} = \{(x,x) : x < 0\}$

$S_2 = \{(x,-x) : x > 0\}$ $\qquad$ $S_{-2} = \{(x,-x) : x < 0\}$.

Define $I(S_i, S_j) = \begin{cases} 0 & \text{if } |i| = |j| \\ 1 & \text{if } ij > 0 \\ -1 & \text{if } ij < 0 . \end{cases}$

Let $K = [0,1]$ or $K = S^1 = [0,1]/_{0=1}$.

Let $\gamma : K \longrightarrow L$ be continuous and let f_0 be a distinguished map at $\gamma(0)$.

Then γ is <u>admissible</u> if and only if:

(i) Either $s \notin \gamma(K)$ or there is a number ε_γ which is +1 or -1 and such that whenever $t_1 < t_2$ and $\gamma([t_1, t_2])$ lies in the domain of φ with $\gamma(t_1) \in S_{i_1}$, $\gamma(t_2) \in S_{i_2}$ then:

either $i_1 = i_2$

or $I(S_{i_1}, S_{i_2}) = \varepsilon_\gamma$.

(ii) $h_{[\gamma(0), f_0]}(\gamma_* \pi_1(K, 0))$ is the germ of the identity map.

If γ is admissible we can define the <u>index</u> ε_γ of γ to be 0 if $s \notin \gamma(K)$ and as in (i) if $s \in \gamma(K)$.

Finally we need a notion of which half of a transverse interval at $\gamma(t)$ we can define a strip through, and such an interval will be called an <u>admissible</u> transverse interval.

If γ is an admissible path and $\varepsilon_\gamma = 0$, any transverse interval at $\gamma(t)$ is admissible. If $\varepsilon_\gamma = \pm 1$, orient the transverse intervals at $\gamma(t)$ by letting the pairs (tangent to interval, tangent to $\mathcal{F}$) lie in the orientation of M_g. Then if $\varepsilon_\gamma = +1$ take the right hand (positive) half (including $\gamma(t)$) and if $\varepsilon_\gamma = -1$ take the left hand (negative) half. For example, the half interval $(\gamma(0), A)$ in figure 2.10(a) is admissible, whilst $(\gamma(0), B)$ is not.

<u>Holonomy Lemma:</u> Let M_g be the orientable 2-dimensional C^s manifold of genus g ($s \geq 2$) and let $\mathcal{F}$ be a C^r Morse foliation on M_g ($2 \leq r \leq s$) in which no leaf contains more than one singular point.

Let L be a leaf of $\mathcal{F}$ and let $X^{\perp}_{\mathcal{F}}$ be a transverse vector field.

Let $\gamma : K \longrightarrow L$ be a continuous admissible map (where $K = S^1$ or $[0,1]$) of index ε_γ.

Then there is an admissible transverse interval V at $\gamma(0)$ contained in a trajectory of $X^{\perp}_{\mathcal{F}}$ and a continuous map

$$H : K \times V \longrightarrow M_g$$

such that:

(i) $H(0,v) = v$ all $v \in V$,

(ii) $H(t,v)$ lies in a non-singular leaf L_v of $\mathcal{F}$ for $v \neq \gamma(0)$ which depends only on $v \in V$,

(iii) $H(t,\gamma(0)) = \gamma(t)$,

(iv) For each $t \in K$ $H(\{t\} \times V)$ is an admissible transverse interval at $\gamma(t)$ contained in a trajectory of $X^{\perp}_{\mathcal{F}}$ (possibly with a singular point added) and

$$H_t : V \longrightarrow H(\{t\} \times V) : v \longmapsto H(t,v)$$

is a C^r diffeomorphism, whose germ at $\gamma(0)$ depends basically only on γ.

Moreover:

(v) If K' is the set of points of K not mapped by γ to a singular point and $\gamma|K'$ is C^r then so is $H|K' \times V$.

(vi) If γ is a homeomorphism onto its image and $\gamma|K'$ is a C^t embedding ($0 \leqslant t \leqslant r$) then H is a homeomorphism & $H|K' \times V$ is a C^t diffeomorphism.

Proof: The idea is to define H locally and use the covering of section 2.10 to piece these maps together.

So let M_g^α be the leaf space of $\mathcal{F}$, $\tilde{M}_g^\alpha$ the covering space defined in section 2.10.

γ can be regarded as a map into M_g^α.

Let f_o be a distinguished map at $\gamma(0)$ and lift γ to a path $\tilde{\gamma}(t) = [\gamma(t), f_t]$.

Let W_t be an open neighbourhood of $\gamma(t)$ sufficiently small and of the right shape that:

(1) There is a distinguished chart φ_t at $\gamma(t)$ defined on W_t such that

$$f_t\varphi_t^{-1}(x,y) = \begin{cases} f_t(\gamma(t)) + x & \text{if } \gamma(t) \text{ is non-singular} \\ f_t(\gamma(t)) + x^2 - y^2 & \text{if } \gamma(t) \text{ is a saddle point} \end{cases}$$

whenever $(x,y) \in \varphi_t W_t$.

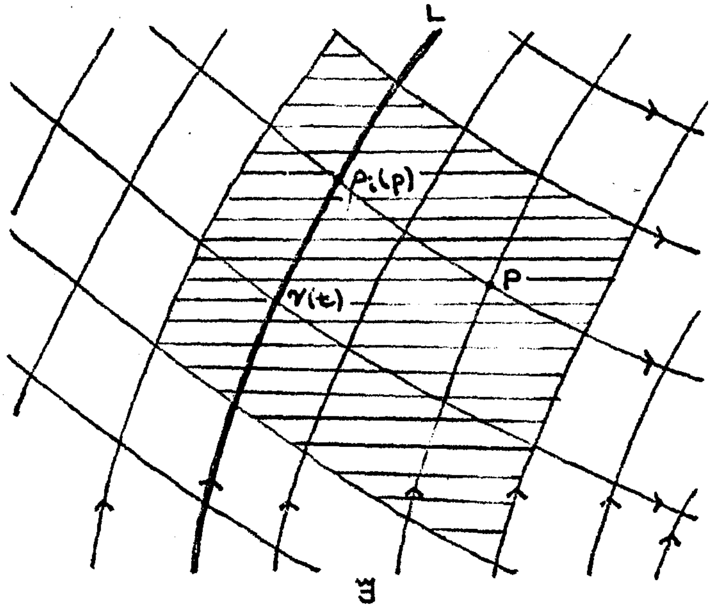
(2) There is an orientation preserving chart Υ_t defined on W_t in which

$\Upsilon_t(\gamma(t)) = 0$

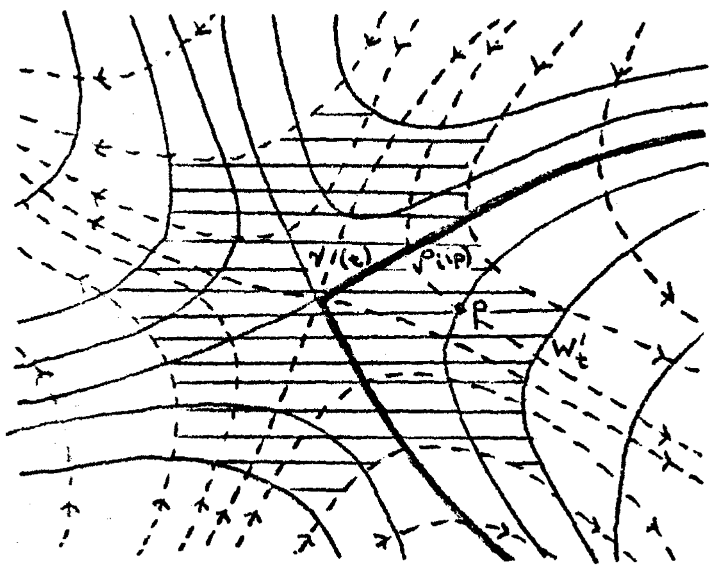
and $X_{\mathcal{F}}^{\perp}$ is given by $\partial/_{\partial x}$ if $\gamma(t)$ is non-singular and $x\,\partial/_{\partial x} - y\,\partial/_{\partial y}$ if $\gamma(t)$ is a saddle point.

(3) W_t is a union of segments of leaves of $\mathcal{F}$ and trajectories of $X_{\mathcal{F}}^{\perp}$ as shown in figure 2.11.

(4) $\gamma(K) \cap W_t$ is connected.



$\gamma(t)$ non-singular



$\gamma(t)$ a saddle point

Fig. 2.11

Now choose a partition $0 = t_0 < t_1 < \ldots . < t_n = 1$ such that $\gamma([t_i, t_{i+1}]) \subseteq W_i$ where $W_i = W_t$ some $t \in [t_i, t_{i+1}]$.

We are now able to give the local definition of H.

Let $f_i = f_t$ where t is such that $W_i = W_t$.

If $\gamma[t_i, t_{i+1}]$ contains no singular point then the map

$$\rho_i : W_i \longrightarrow \gamma[t_i, t_{i+1}]$$

mapping $p \in W_i$ to the unique point of $\gamma[t_i, t_{i+1}]$ lying on the same segment of trajectory of $X_{\mathfrak{z}}^{\perp}$ in W_i as does p (see figure 2.11) is a well defined submersion.

Thus $\rho_i \times f_i : W_i \longrightarrow \gamma([t_i, t_{i+1}]) \times f_i W_i$ is a C^r diffeomorphism.

If $s \in \gamma([t_i, t_{i+1}])$ is a saddle point, let R^2 have co-ordinates (x,y) with $\pi_1(x,y) = x$, $\pi_2(x,y) = y$.

Then the map

$$\rho_i : W_i' \longrightarrow \gamma[t_i, t_{i+1}]$$

which maps $p \in W_i'$ to the unique point of $\gamma([t_i, t_{i+1}])$ with $(\pi_1 \Upsilon_i(\rho_i(p)))(\pi_2 \Upsilon_i(\rho_i(p))) = (\pi_1 \Upsilon_i(p))(\pi_2 \Upsilon_i(p))$, where W_i' is the quadrant of W_i bounded by $\gamma([t_i, t_{i+1}])$ (see figure 2.[illegible]), is a surjective map which is C^r on $W_i'' = W_i' \setminus \{p : \pi_1 \Upsilon_i(p) \pi_2 \Upsilon_i(p) = 0\}$.

Thus $\rho_i \times f_i : W_i' \longrightarrow \gamma([t_i, t_{i+1}]) \times f_i W_i'$ is a homeomorphism which restricts to a C^r diffeomorphism on W_i''.

Shrinking down W_i if necessary, we can assume that

$$f_i \mid W_i \cap W_j = f_j \mid W_i \cap W_j$$
$$\rho_i \mid W_i \cap W_j = \rho_j \mid W_i \cap W_j$$

since the f_i were obtained from a lifting of a continuous path.

Then if V is sufficiently small and admissible we may define

$$H : K \times V \longrightarrow M_g \quad \text{by} \quad H(t,v) = (\rho_i \times f_i)^{-1}(\gamma(t), f_o(v))$$

for $t \in [t_i, t_{i+1}]$.

H then has the required properties.

In particular the uniqueness in (iv) follows from the fact that since the germ of $f_t H_t$ is a locally constant function of t, it is constant and equal to f_o.

2.14 Diffeomorphisms and conjugacy.

Let $\mathfrak{F} = \{f_i : V_i \longrightarrow \mathbb{R}\}$ be a Morse foliation on M_g, and suppose $\rho : M_g \longrightarrow M_g$ is an orientation preserving diffeomorphism of class C^t (for suitable t).

We define a new Morse foliation on M_g, the Morse foliation <u>induced by ρ</u> to be

$$\rho^* \mathfrak{F} = \{ f_i \circ \rho : \rho^{-1} V_i \longrightarrow \mathbb{R} \}.$$

If ρ' is another orientation preserving diffeomorphism of M_g we have

$$(\rho' \circ \rho)^* \mathfrak{F} = \rho^* \rho'^* \mathfrak{F}.$$

If $\mathfrak{F}, \mathfrak{F}'$ are two Morse foliations, we say that $\mathfrak{F}$ and $\mathfrak{F}'$ are <u>C^t-conjugate</u> if there is a C^t diffeomorphism ρ of M_g

preserving orientation such that

$$\rho^* \mathcal{F}' = \mathcal{F} .$$

If ρ is isotopic to the identity $\mathcal{F}$ and $\mathcal{F}'$ are said to be <u>completely equivalent</u>.

It is important to note at this point that two Morse foliations $\mathcal{F}$, $\mathcal{F}'$ are not necessarily conjugate just because there is a diffeomorphism of M_g mapping the leaves of $\mathcal{F}$ onto the leaves of $\mathcal{F}'$.

Indeed it is always true that given a Morse foliation of class C^r with at least one saddle point there is a non-conjugate foliation $\mathcal{F}'$ having the same leaves as $\mathcal{F}$.

To see this consider the Morse foliation of $\mathbb{R}^2$ given by lines xy = constant.

Let $h : \mathbb{R} \longrightarrow [0,1]$ be a C^∞ function such that

a) $h(x) = 0 \quad x \geqslant 0$ & $x \leqslant -1$

b) the germ of h at 0 is <u>not</u> equal to 0.

Now define $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$H(x,y) = \begin{cases} (x,y) & \text{if } x \geqslant 0 \text{ or } xy \geqslant 0 \\ (x+xh(xy),y) & \text{if } x \leqslant 0 \text{ \& } y \geqslant 0 \end{cases} .$$

Then there is an open neighbourhood N of 0 in $\mathbb{R}^2$ such that $H|N$ is a C^∞ diffeomorphism of N onto $H(N)$.

Now let θ be the rotation of $\mathbb{R}^2$ through $-\frac{1}{4}\pi$.

Replacing a distinguished chart φ at a saddle point p

with suitable range by $\theta(H|N)\theta^{-1}\varphi$ gives the inequivalent Morse foliation $\mathcal{F}'$.

This departure of the model from the intuitive conception only appears to matter in considerations of the holonomy around a loop separatrix. However, as is shown in the next section, we can always choose a Morse foliation which reflects the intuitive situation, with the possible loss of one degree of smoothness.

2.15 Practical interpretation of holonomy.

We wish to link the holonomy lemma with the holonomy group.

Suppose $K = [0,1]$ and γ is a path in a leaf of $\mathcal{F}$ with $\gamma(0) = \gamma(1)$. Let H be the map

$$H : [0,1]\times V \longrightarrow M_g$$

obtained in the holonomy lemma for some transverse vector field and write $h = H_1$.

Then writing $k = h_{[\gamma(0),f_0]}([\gamma])$ we see that

$$k\circ f_0\circ h|V = f_0|V$$

as germs at $\gamma(0)$ in V so that if $\gamma([0,1])$ contains no saddle point, k and h^{-1} agree as germs in a suitable co-ordinate system. If $\gamma([0,1])$ contains a saddle point, k and h^{-1} agree as one-sided germs.

The holonomy also has significance when there are one or more loop separatrices at a saddle point s.
To simplify matters let φ be a chart at s in which the map f_o defined by $f_o(x,y) = xy$ is distinguished.
Suppose that there is a loop separatrix at s.
Choose points A_o, B_o in that separatrix with $\varphi(A_o) = (0,1)$ $\varphi(B_o) = (1,0)$ and transverse intervals A_1A_2 at A_o & B_1B_2 at B_o (see figure 2.12(a)) with image under φ similarly named (see figure 2.13).

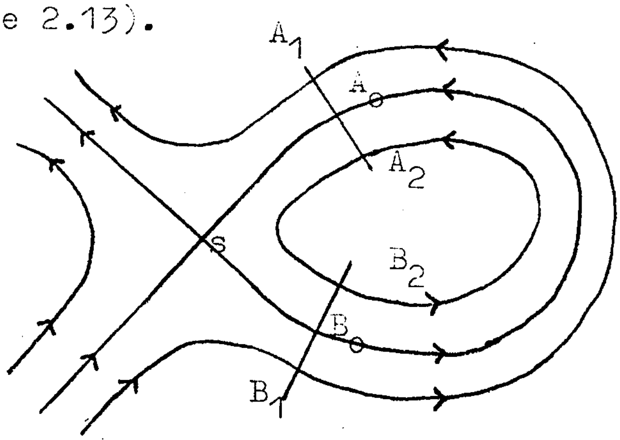


Fig 2.12(a)

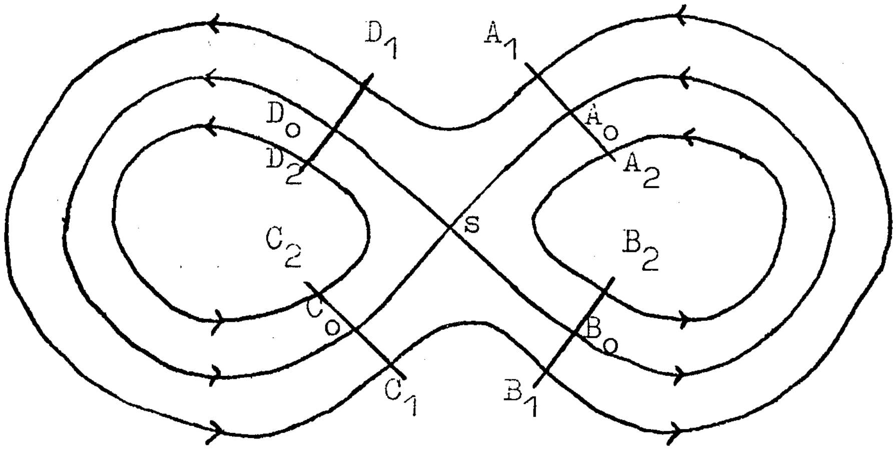


Fig 2.12(b)

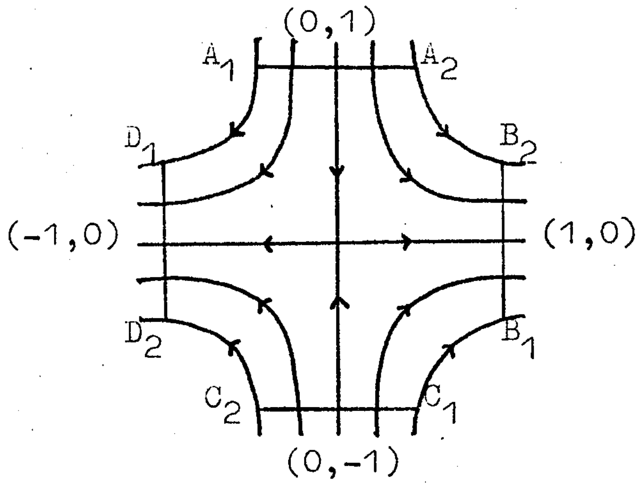


Fig. 2.13

Let $\gamma : [0,1] \longrightarrow M_g$ be an embedded path in the loop separatrix starting at A_0 and ending at B_0, and passing outside the domain of φ.

The holonomy construction determines a diffeomorphism of A_1A_2 into B_1B_2 given in the chart φ by

$$(x,1) \longmapsto (1,\rho(x))$$

where ρ is a germ at 0 of an orientation preserving diffeomorphism of $\mathbb{R}$.

If γ is now completed to a parametrisation of the loop separatrix then the image of $[\gamma]$ under the holonomy map is essentially the germ of ρ .

If there is no other loop separatrix at s and every leaf on one side of the loop is closed, as in figure 2.12(a) then intuitively there is no holonomy. However the germ of ρ may not be that of the identity, although it will of course be the identity on one side of zero.

We can then replace the Morse foliation by one in which the holonomy is formally trivial as follows.

The map ρ satisfies :

$$\rho(x) = x \quad \text{if } x \geqslant 0.$$

Thus
$$\rho^{-1}(x) = x \quad \text{if } x \geqslant 0.$$

Write $\rho^{-1}(x) = x + x\eta(x)$.

Then if ρ is C^r η is C^{r-1} and has all jets up to and including the (r-1)th zero at 0.

Define a local diffeomorphism Υ of $\mathbb{R}^2$ at 0 by:

$$\Upsilon(x,y) = \begin{cases} (x,y) & x \leqslant 0 \text{ or } (y \geqslant 0 \ \& \ x \geqslant 0) \\ (x+x\eta(xy),y) & x \geqslant 0 \text{ and } y \leqslant 0. \end{cases}$$

writing $\Upsilon(x,y) = (\Upsilon_1(x,y),\Upsilon_2(x,y))$ and replacing f_o by

$$(x,y) \longmapsto \Upsilon_1(x,y)\Upsilon_2(x,y)$$

then gives the required C^{r-1} Morse foliation.

If there are two loop separatrices at s and the situation is exactly as in figure 2.12(b) with all nearby leaves closed we can again choose a Morse foliation with the same leaves and no holonomy around the loops by modifying the original foliation just in the quadrants D_1A_1 and C_1B_1.

2.16 α and ω limit sets.

The α and ω limit sets of a Morse foliation $\mathfrak{F}$ play a crucial role in describing the conjugacy classes of Morse foliations. They encapsulate the asymptotic behaviour of the leaves and are defined as follows.

Let $\Phi_{\mathfrak{F}}$ be a flow tangent to $\mathfrak{F}$ obtained as in section 2.6. Let l be a non-singular leaf, separatrix or singular point and suppose $\qquad l = \{\Phi_{\mathfrak{F}}(x_o, t) : t \in \mathbb{R}\}$.

The ω-limit set $\omega(l)$ of l and α-limit set $\alpha(l)$ of l are defined by:

$$\omega(l) = \bigcap_{t \in \mathbb{R}} \text{closure}(\Phi_{\mathfrak{F}}(x_o \times [t,\infty))),$$
$$\alpha(l) = \bigcap_{t \in \mathbb{R}} \text{closure}(\Phi_{\mathfrak{F}}(x_o \times (-\infty, t])).$$

Since $\Phi_{\mathfrak{F}}$ was chosen according to the orientation on M_g and transverse orientation on $\mathfrak{F}$, the distinction between α and ω-limit sets is well-defined.

These sets have the following properties:

(i) $\omega(l)$ is a union of singular points, separatrices and non-singular leaves.

(ii) $\bar{l} = l \cup \omega(l) \cup \alpha(l)$.

(iii) If l is a circle leaf $\bar{l} = \omega(l) = \alpha(l)$.

(iv) If l is an inward separatrix at s then $\omega(l) = s$, if l is an outward separatrix at s then $\alpha(l) = s$, if l is a loop separatrix at s $\omega(l) = \alpha(l) = s$.

(v) If s is a saddle point in $\omega(l)$ and l is not an inward separatrix then $\omega(l)$ contains in addition a pair of adjacent separatrices.

2.17 The Poincaré-Bendixson Theorem.

The Poincaré-Bendixson theorem describes the global behaviour of vector fields on the plane or sphere and has been generalised to higher dimensional foliations in many ways (see [21], [27], [29], [31]). The original papers of Poincaré and Bendixson can be found in [1] and [24] and a modern treatment in [3]. We state below the result for Morse foliations on the sphere M_0 (in which each leaf contains at most one singular point).

A leaf l is said to be <u>proper</u> if its topology as a manifold agrees with its topology as a subset of M_g.

<u>Theorem:</u> Let $\mathfrak{F}$ be a Morse foliation of the sphere M_0 such that no leaf contains more than one singular point. Then:

1. Every leaf is proper.
2. For every singular point, non-singular leaf or separatrix l one and only one of the following occurs:

(i) $l = \omega(l)$ and l is a singular point or circle.

(ii) $\alpha(l) = \omega(l)$ and $l \cap \omega(l) = \phi$. Then l is a loop separatrix, (see 2.16(iv)).

(iii) $l, \omega(l)$ and $\alpha(l)$ are mutually disjoint and either a) $\omega(l)$ is a saddle point,

or b) $\omega(l)$ is a circle,

or c) $\omega(l)$ is the union of a saddle point and one or two loop separatrices.

in cases b) and c) the leaf containing $\omega(l)$ has non-trivial holonomy group and l spirals towards $\omega(l)$ as $t \to \infty$. In other words, if $p \in \omega(l)$ and T is a small transverse interval about p the successive intersections of l and T tend monotonically to p from one side as $t \longrightarrow \infty$.

2.18 The theorem of A.J.Schwartz.

This theorem applies to any C^2 flow on a 2-dimensional manifold and a proof can be found in [31]. Since every Morse foliation $\mathfrak{F}$ is C^2 we can apply this theorem, obtaining the result below.

<u>Theorem</u>: Let $\mathfrak{F}$ be a ($C^r, r \geqslant 2$) Morse foliation on M_g, the oriented two dimensional manifold of genus g (satisfying 2.9). If l is a singular point, separatrix or non-singular leaf of $\mathfrak{F}$, one and only one of the following occurs:

(i) $\omega(l) = M_g$ and $g=1$ i.e. M_g is the torus.

(ii) $\omega(l)$ is a circle and if l is not a circle, l spirals towards $\omega(l)$ which has non trivial holonomy group.

(iii) $\omega(l)$ contains a singular point.

Thus if M_g is the join of more than one torus and every leaf has trivial holonomy group, (iii) is the only possibility. Even if $\omega(l)$ is not just a saddle point we shall see that the saddle points in $\omega(l)$ determine it.

Chapter 3. Centres.

In Morse foliations of manifolds with positive genus, each centre is associated with a saddle point. In this chapter we obtain a detailed description of the behaviour near a centre and thus exhibit this association.

In our treatment of Morse foliations with trivial holonomy it is crucial that the behaviour can be deduced from that of Morse foliations without centres on a manifold of the same or smaller genus. On the other hand the construction of appendix 4 can be used to build a Morse foliation of the torus in which the ω-limit set of every leaf is a perfect, closed, nowhere dense set as in figure 3.1. Any Morse foliation without centres would either have all leaves

identify ▸▸ & ▸
to obtain a
torus

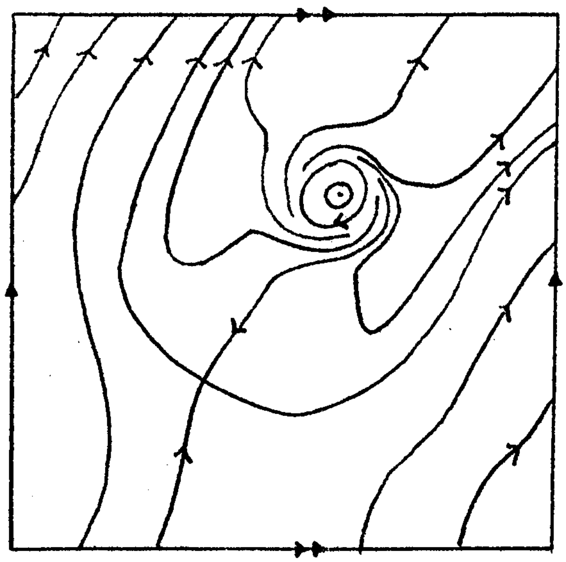


Fig. 3.1

dense or the limit set of every leaf is a circle, as follows from A.J.Schwartz's theorem (2.18) and was first proved by Denjoy ([4]). This is quite different behaviour to that of the original foliation.

Our approach is to use the fact that a centre lies in a disc foliated by circles and to extend this disc to a maximal one using the holonomy lemma.
This information is contained in the following:

<u>Proposition 3.1:</u> Let $\mathcal{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g (satisfying 2.9).
Let c be a centre of $\mathcal{F}$ and let

$\mathcal{C} = \{D \subseteq M_g$: D is an open embedded C^r disc which is a union of c and non-singular leaves and ∂D is a circle leaf $\}$.

Then if $Q_c = \bigcup \mathcal{C}$, one and only one of the following possibilities occurs:

(1) $g = 0, M_g$ is the sphere and ∂Q_c is a centre.

(2) $Q_c \in \mathcal{C}$ so that ∂Q_c is a circle leaf.

In this case, either there is a separatrix in $\overline{Q}_c^{\,c}$ which has ∂Q_c as its α or ω limit set or every leaf near ∂Q_c but not in Q_c is non-singular, spirals towards ∂Q_c at one end and to the union of one or two loop separatrices at the other end (see figure 3.2).

Fig. 3.2

(3) Q_c contains a saddle point s and one or two loop separatrices (see figure 3.3).

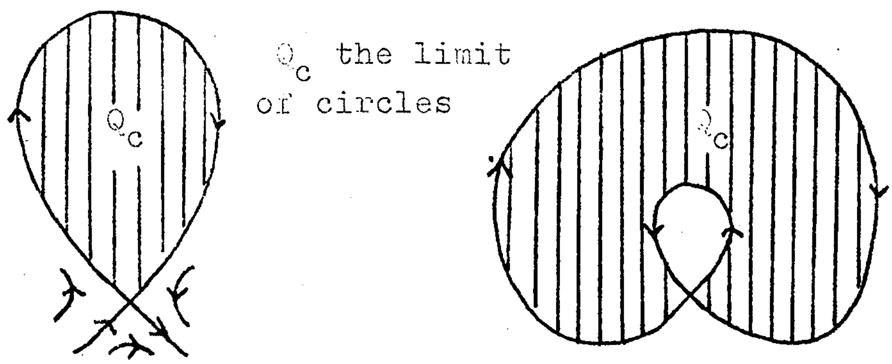


Fig. 3.3

<u>Proof</u>: Since each embedded C^r disc $D \in \mathcal{C}$ is a union of non-singular leaves, separatrices and singular points, the holonomy lemma shows that this is also true of ∂Q_c. Since ∂Q_c is closed, the ω and α limit set of any separatrix or non-singular leaf in ∂Q_c is also in ∂Q_c. Having noted these facts we show that if (1) does not hold then (2) or (3) must.

The proof proceeds in four steps.

In step 1 we show that if ∂Q_c contains a circle leaf then $Q_c \in \mathcal{C}$ and that every circle leaf near ∂Q_c lies in $\overline{Q}_c$. The proof is topologically straightforward, but requires slightly more care in ensuring that Q_c is embedded as a C^r disc.

In step 2 we show that if $Q_c \in \mathcal{C}$ then either there is a separatrix in $\overline{Q}_c^{\,c}$ near ∂Q_c or every leaf in $\overline{Q}_c^{\,c}$ near ∂Q_c has ∂Q_c as limit set at one end and limit set at the other end either a circle leaf or one of the sets in figure 3.2.

In step 3 we show that if $Q_c \in \mathcal{C}$ and every leaf in $\overline{Q}_c^{\,c}$ near ∂Q_c has a circle in both limit sets, then Q_c lies in the interior of a disc in $\mathcal{C}$. This is impossible and thus (2) is proved.

In step 4 we show that if ∂Q_c contains a saddle point, then the situation is as shown in figure 3.3.

<u>Step 1</u>. Suppose ∂Q_c contains a circle leaf. Suppose that Q_c does not belong to $\mathcal{C}$.

Then ∂Q_c is a limit of circle leaves bounding discs in $\mathcal{C}$.

By lemma 1 of appendix 1 there is a C^r embedding

$$\Psi : S^1 \times (-1,1) \longrightarrow M_g$$

such that $\Psi(S^1 \times \{0\}) = \partial Q_c$

$\Psi(S^1 \times \{-\tfrac{1}{2}\})$ is a circle leaf bounding a disc in $\mathcal{C}$.

By lemma 2 in appendix 1 there is a C^r embedding φ' of the unit disc in $\mathbb{R}^2$ onto Q_c.
Hence $Q_c \in \mathcal{C}$ contradicting our assumption.
Thus $Q_c \in \mathcal{C}$.
Similar arguments show that if there are circle leaves arbitrarily close to ∂Q_c in $\overline{Q}_c^{\,c}$ then they bound discs in $\mathcal{C}$ - an impossibility since such a disc would contain $\overline{Q}_c$ in its interior.

Step 2. Suppose that no separatrix in $\overline{Q}_c^{\,c}$ has ∂Q_c as limit set at one end.
The holonomy lemma and the fact that there are no circle leaves in $\overline{Q}_c^{\,c}$ arbitrarily close to ∂Q_c show that there is a C^r-embedded transverse circle in $\overline{Q}_c^{\,c}$ which approximates ∂Q_c and is such that any leaf cutting it has ∂Q_c as limit set at one end, (see figure 3.4).

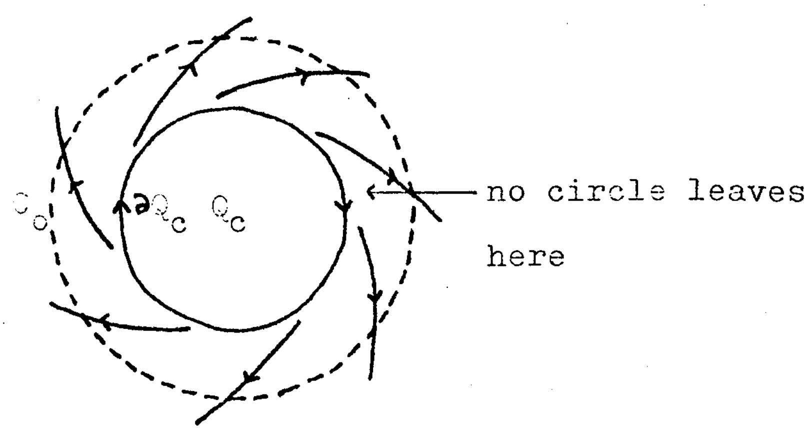


Fig. 3.4

Let p be a point in the limit set at the other end from ∂Q_c of a leaf l_o cutting the transverse circle C_o. Suppose that p is non-singular and let T be a transverse interval at p, lying outside the disc bounded by C_o. Suppose l_o cuts T for the first time at p_1 and next at p_2 (assuming $p \in \omega(l_o)$).
Let T_o be the subinterval of T with endpoints p_1 and p_2 (see figure 3.5).

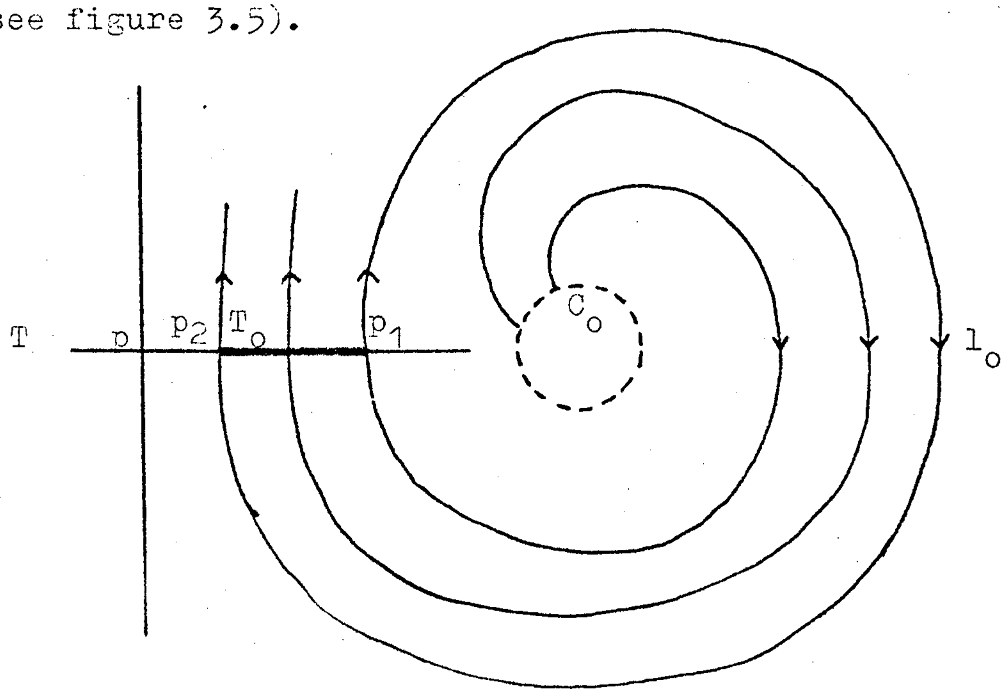


Fig. 3.5

Then the holonomy lemma shows that every leaf leaving C_o cuts T_o in exactly one point (except for l_o) and every trajectory through a point of T_o meets C_o in exactly one point (except for endpoints).
Thus every leaf on one side of p and cutting T has limit set ∂Q_c at one end and has the non-singular leaf or

separatrix through p in its limit set at the other end. Further no point in the limit set of the non-singular leaf or separatrix through p can cut T in any point other then p.
Hence either p lies on a circle leaf or a loop separatrix. If p lies on a circle leaf the holonomy lemma shows that every leaf leaving C_o is eventually "trapped" in a small neighbourhood of the circle leaf containing p given by the holonomy lemma. Thus this circle leaf is the entire limit set of every leaf cutting C_o.
Otherwise $\omega(l_o)$ contains one or two loop separatrices and another application of the holonomy lemma shows that the situation is as described in figure 3.2. This completes the proof of step 2.

<u>Step 3</u>. Suppose that ∂Q_c is a circle leaf and every leaf near ∂Q_c has ∂Q_c as limit set at one end and a circle C as limit set at the other end.
We have to show that C bounds a disc in $\mathcal{C}$ for this implies that $\overline{Q}_c \subseteq \operatorname{int} \bigcup \mathcal{C} = Q_c$ - a contradiction.

By lemma 3 of appendix 1, there are C^r embeddings

$$\Psi_1, \Psi_2 \; : \; S^1 \times (-1,1) \longrightarrow M_g \quad ,$$

whose images do not meet, with the following properties:

(i) $\Psi_1(S^1 \times \{-\frac{1}{2}\}) = \partial Q_c$

(ii) $\Psi_2(S^1 \times \{\frac{1}{2}\}) = C$

(iii) All the circles $\Psi_1(S^1 \times \{t\}), \Psi_2(S^1 \times \{s\})$ for $0 \leqslant t \leqslant \frac{1}{2}$, $-\frac{1}{2} \leqslant s \leqslant 0$ are transverse.

The situation is shown in figure 3.6 .

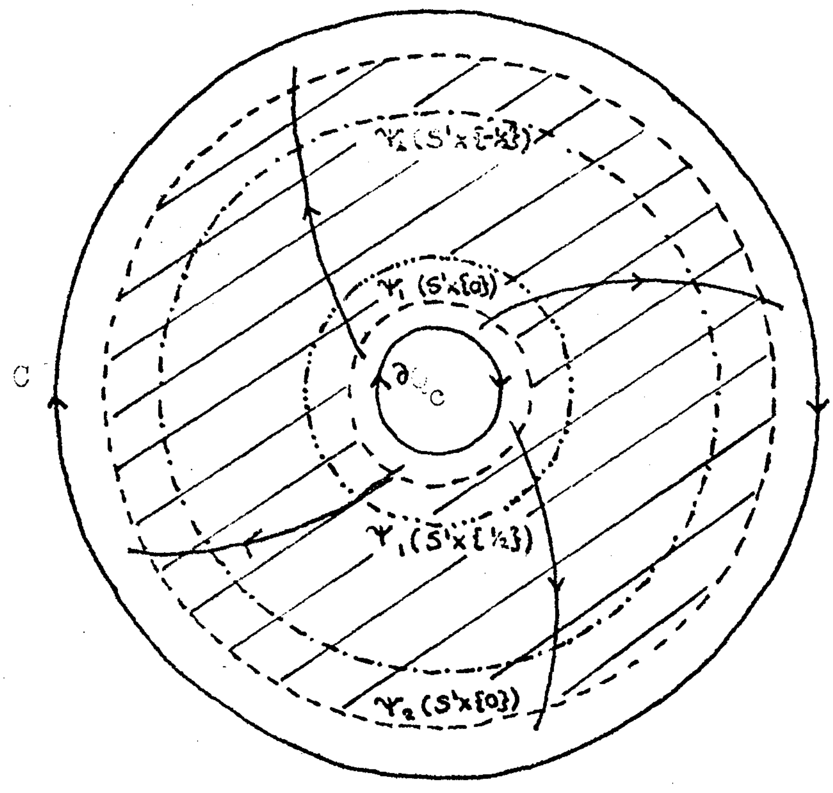


Image of Ψ_3 shaded

Fig. 3.6

By the holonomy lemma, using a suitable transverse vector field in which all the circles of (iii) above are trajectories and adjusting the resultant map we obtain a C^r embedding

$$\Psi_3 : S^1 \times [-1,1] \longrightarrow M_g$$

with the following properties:

(iv) each $\theta \in S^1, \Psi_3(\{\theta\} \times [-1,1])$ is contained in a single leaf of $\mathcal{F}$.

(v) All the circles $\Psi_3(S^1 \times \{t\})$ are transverse and $\Psi_3(S^1 \times \{-1\}) = \Psi_1(S^1 \times \{0\})$, $\Psi_3(S^1 \times \{-\frac{1}{2}\}) = \Psi_1(S^1 \times \{\frac{1}{2}\})$, $\Psi_3(S^1 \times \{\frac{1}{2}\}) = \Psi_2(S^1 \times \{-\frac{1}{2}\})$, $\Psi_3(S^1 \times \{1\}) = \Psi_2(S^1 \times \{0\})$.

Let φ be the embedding of Q_c.
Application of lemma 2 of appendix 1 to φ and Ψ_1 yields an embedding φ' of a disc bounded by $\Psi_1(S^1 \times \{\frac{1}{2}\})$.
Repitition with φ' and Ψ_3 yields φ'' an embedding of a disc bounded by $\Psi_2(S^1 \times \{-\frac{1}{2}\})$.
Finally repetition with φ'' and Ψ_2 yields an embedding of a disc bounded by C so that $\overline{Q_c} \subseteq \operatorname{int} \mathcal{U}_{\mathcal{C}}$ as required.

Step 4. We have to show that if ∂Q_c contains no circle leaf then it contains only singular leaves and every separatrix in ∂Q_c is a loop separatrix. "Trapping" arguments used in step 2 then give the required result.
It is at this point that we use the fact that $\mathcal{F}$ is C^2 and hence subject to A.J.Schwartz's theorem.

Let l be a non-singular leaf or outward separatrix in ∂Q_c.
By the theorem of A.J.Schwartz there is a saddle point

s in $\omega(1)$.

If 1 is not an inward separatrix at s, 1 makes two successive passages AB, CD past s in a single quadrant cutting a transverse interval T at s in points p_1, p_2 as shown in figure 3.7.

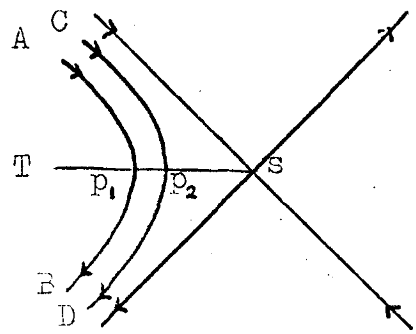


Fig. 3.7

Let $X_{\mathfrak{F}}^{\perp}$ be a transverse vector field having T as a trajectory. Now the interval (p_1, p_2) meets some disc in $\mathcal{C}$ since 1 is the limit of discs in $\mathcal{C}$.

This implies that there is a trajectory of $X_{\mathfrak{F}}^{\perp}$ which cuts a circle leaf twice - which is impossible by the transverse orientation of $\mathfrak{F}$.

Hence 1 is an inward separatrix at s.

This completes the proof of proposition 3.1.

Chapter 4 Centres in Morse foliations with no holonomy.

Assumption: From now on we shall consider C^r transversely oriented Morse foliations $\mathfrak{F}$ with no leaf containing more than one saddle point and in which the holonomy group of each leaf is trivial. The latter assumption will be stated as "$\mathfrak{F}$ has no holonomy".

Definition 4.1.1: Let c be a centre of a Morse foliation on M_g, the oriented 2-manifold of genus g, and suppose that no leaf of $\mathfrak{F}$ contains more than one saddle point.
Let $\mathcal{C}$ be the collection of all distinguished charts (φ, U) at c.
D_c is defined to be the set

$$D_c = \bigcup_{(\varphi, U)\in\mathcal{C}} U$$

Lemma 4.1.2: Let $\mathfrak{F}$ and D_c be as defined in definition 4.1.1 and let Q_c be as defined in the statement of proposition 3.1. Then $D_c = Q_c$ and hence the situation is as in proposition 3.1 (1) or figure 3.3.
Proof: Clearly $D_c \subseteq Q_c$, and ∂D_c is a union of non-singular leaves, separatrices and singular points.
If ∂D_c is not a centre, we have to show that it does not contain any circle leaf.
If C is any circle leaf in the boundary of D_c, the holonomy lemma gives a C^r embedding

$$\Psi: S^1 \times (-1,1) \longrightarrow M_g$$

onto a neighbourhood of C with each set $\Psi(S^1 \times \{t\})$

a leaf of $\mathfrak{F}$.

Lemma 2 of appendix 1 then gives an embedding

$$\varphi' : \{x \in \mathbb{R}^2 : \|x\| < 1\} \longrightarrow M_g$$

which agrees with some distinguished chart at c near 0, and contains c in its image.

The proof of lemma 2 then shows that φ' is a distinguished chart at c - a contradiction.

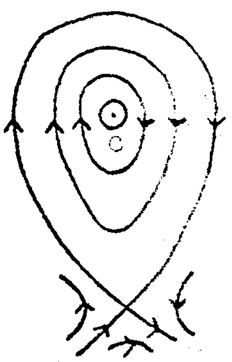
Thus ∂D_c contains a saddle point by the theorem of A.J. Schwartz. This proves the lemma.

<u>Definition 4.2</u>: Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation with no holonomy and no leaf containing more than one saddle point.

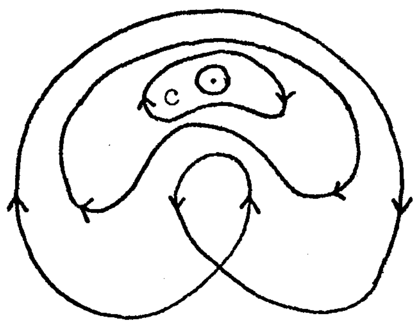
Let c be a centre of $\mathfrak{F}$.

Then c is of <u>type 1</u> if ∂D_c contains a single loop separatrix and is of <u>type 2</u> if ∂D_c contains two loop separatrices.

The situation is illustrated in figure 4.1 below:



Centre of type 1.



Centre of type 2.

Fig. 4.1

4.3 Standard models for behaviour near a centre.

In this section we fix the properties of three standard models of partial Morse foliations near a centre. The precise constructions are given in appendix 2.

1. The first example is the Morse foliation $\mathcal{S}$ of the sphere

$$M_0 = S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

given by the circles z = constant.
The centre $(0,0,1)$ has Morse index 2 and the centre $(0,0,-1)$ has Morse index 0.
See figure 4.2.

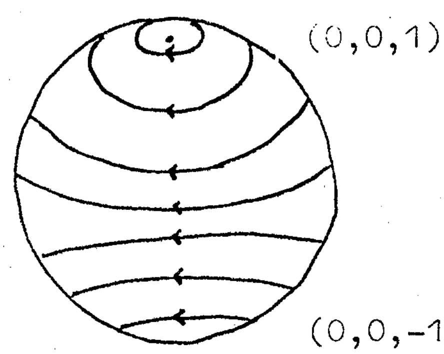


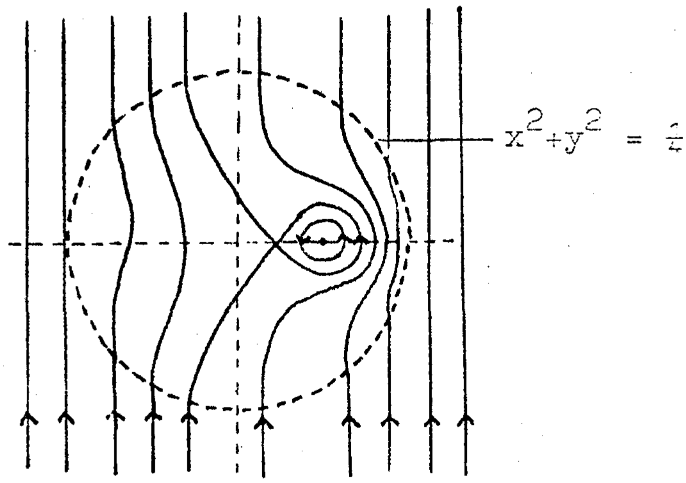
Fig. 4.2. The Morse foliation $\mathcal{S}$.

2. The second examples are of Morse foliations $\mathcal{B}^+$ & $\mathcal{B}^-$ on the square $(-1,1) \times (-1,1) \subseteq \mathbb{R}^2$.
The foliation $\mathcal{B}^+$ has a single centre of type 1, is symmetric about the line $y = 0$, is equal to the foliation by lines x = constant outside the circle $x^2+y^2 = \frac{1}{4}$, and

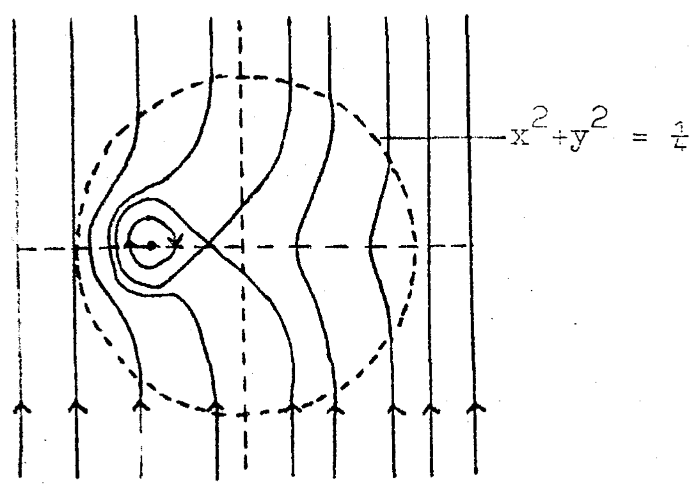
has a centre at the point $(0,\lambda)$ and a saddle point at $(0,\lambda')$ where $0 < \lambda' < \lambda$.

$\mathcal{B}^-$ is $\mathcal{B}^+$ rotated through an angle π with the opposite orientation on the leaves.

These are illustrated in figure 4.3.



Morse foliation $\mathcal{B}^+$



Morse foliation $\mathcal{B}^-$

Fig. 4.3

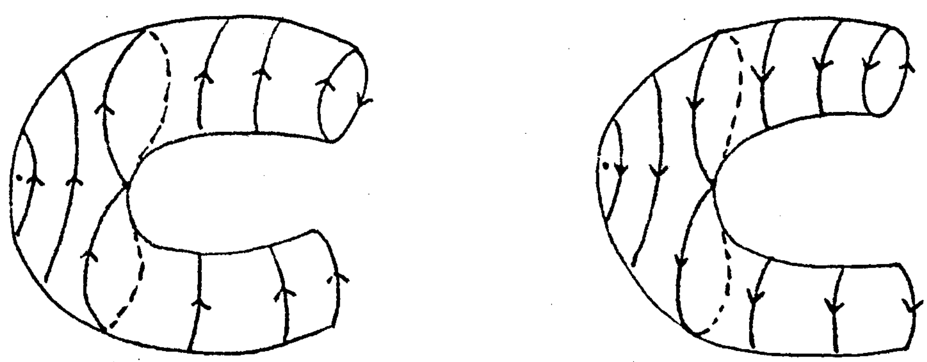
3. The third examples are of Morse foliations $\mathcal{E}^+, \mathcal{E}^-$ on the half torus, which is the image under the covering map

$$\rho_1 : \mathbb{R}^2 \longrightarrow M_1 = \text{torus}$$

of the strip $0 < y < \frac{1}{2}$.

These have one centre of type 2 and one saddle point and agree with the foliation by circles $\rho_1((-\infty,\infty) \times \{y\})$ for y near 0 or $\frac{1}{2}$.

The situation is illustrated in figure 4.4.



The Morse foliation $\mathcal{E}^+$. The Morse foliation $\mathcal{E}^-$.

Fig. 4.4

It is easy to see that there is a natural way to replace the Morse foliations $\mathcal{B}$ and $\mathcal{E}$ by foliations without singularity.

In the rest of the chapter we show that any centre of type 1 is locally C^r-conjugate to one of the examples $\mathcal{B}^+$ or $\mathcal{B}^-$ and any centre of type 2 to one of the examples $\mathcal{E}^+$ or $\mathcal{E}^-$.

In the case of a centre of type 1 we can replace the centre by a foliation without singularities, and this can be done uniquely up to C^r-complete equivalence.
The trouble about doing this with centres of type two is that the resultant Morse foliation will not be transversely oriented. However, we shall see that in decomposing a Morse foliation we can always deal with a centre of type 1. This is because a sphere always has a centre of type 1.

<u>Proposition 4.4</u>: (i) There is a unique C^r-complete equivalence class of Morse foliations on the sphere M_0 with no holonomy and no saddle point ($r \geqslant 2$).
(ii) There is a unique C^r conjugacy class of Morse foliations on the torus M_1 with no singular point, no holonomy and at least one closed leaf ($r \geqslant 2$).
To see that there are many complete equivalence classes, look in [17].
<u>Proof</u>: (i) Let $\mathfrak{F}$ be a C^r Morse foliation of the sphere M_0 with no saddle point.
Then $\mathfrak{F}$ has a centre c_0 of Morse index 0 and one of Morse index 2.
Let $\mathcal{S}$ be the standard Morse foliation of M_0 defined in section 4.3 and let z_0 denote the point $(0,0,-1)$ and z_2 the point $(0,0,1)$.

We construct a C^r-diffeomorphism φ of M_0 onto itself with $\varphi^*\mathfrak{F} = \mathfrak{G}$.

Since φ is isotopic to the identity by lemma 2 of appendix 3, the result follows.

By lemma 4.2 we can choose distinguished charts Ψ_0, Ψ_2 at c_0, c_2 and ρ_0, ρ_2 at z_0, z_2, whose images overlap and whose range is the unit disc in $\mathbb{R}^2$.

Without loss of generality we may also assume that if

$$B_1 = \{x \in \mathbb{R}^2 : \|x\| < 1\} \quad B_{\frac{1}{2},1} = \{x \in \mathbb{R}^2 : \tfrac{1}{2} < \|x\| < 1\}$$

then

$$\Psi_0^{-1}B_1 \cap \Psi_2^{-1}B_1 = \Psi_0^{-1}B_{\frac{1}{2},1} = \Psi_2^{-1}B_{\frac{1}{2},1}$$

$$\rho_0^{-1}B_1 \cap \rho_2^{-1}B_1 = \rho_0^{-1}B_{\frac{1}{2},1} = \rho_2^{-1}B_{\frac{1}{2},1}$$

(see figure 4.5)

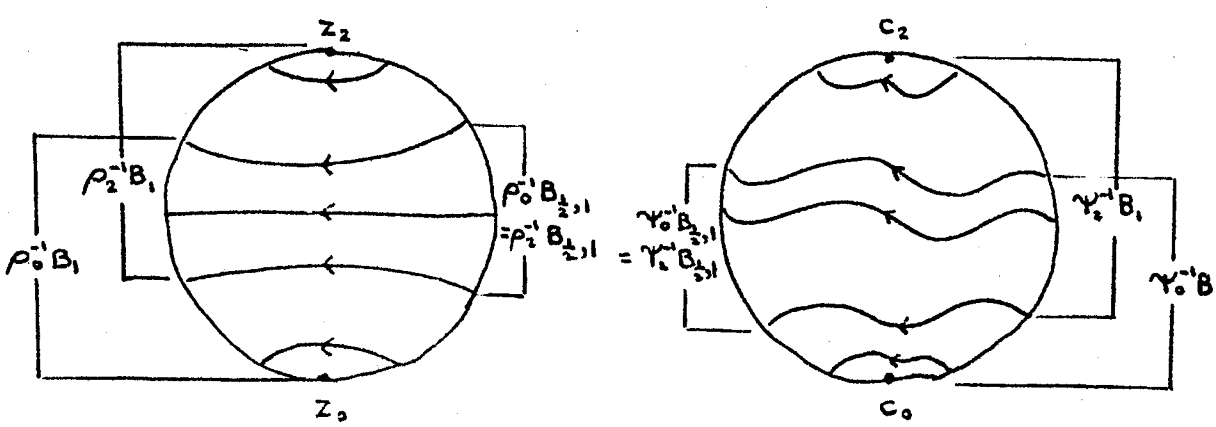


Fig. 4.5

Since $B_{\frac{1}{2},1}$ can be identified with $S^1 \times (-1,1)$ in such a way that the circles x^2+y^2 = constant become circles $S^1 \times \{t\}$, it follows from lemma 4 of appendix 2 that there is a C^r diffeomorphism

$$\lambda : B_{\frac{1}{2},1} \longrightarrow B_{\frac{1}{2},1}$$

with $\quad \lambda = \text{identity map near } x = \frac{1}{2}$

$$\lambda = \rho_2\rho_0^{-1}\Psi_0\Psi_2^{-1} \quad \text{near } x = 1.$$

Then the required diffeomorphism may be defined by

$$\varphi(x) = \begin{cases} \rho_0^{-1}\Psi_0(x) & x \in \Psi_0^{-1}B_{\frac{1}{2}} \\ \rho_2^{-1}\lambda\Psi_2(x) & x \in \Psi_2^{-1}B_{\frac{1}{2},1} \\ \rho_2^{-1}\Psi_2(x) & x \in \Psi_2^{-1}B_{\frac{1}{2}} \end{cases}$$

where $B_{\frac{1}{2}} = \{x \in \mathbb{R}^2 : \|x\| \leqslant \frac{1}{2}\}$.

(ii)The proof of this part relies on the results of chapter 5 but we sketch a proof here.

So let $\mathcal{F}$ be a Morse foliation of the torus with at least one circle leaf and with no holonomy. By the theorem of A.J.Schwartz, if $\mathcal{F}$ has no singular point, every leaf of $\mathcal{F}$ is a circle leaf.

Cutting along such a leaf and gluing in centres (see 5.1.1) produces a Morse foliation without holonomy or any saddle point on the sphere, which is well-defined up to C^r-conjugacy (see 5.1.2).

By part (i) there is a unique C^r-conjugacy class of such Morse foliations of the sphere.

But $\mathcal{F}$ is obtained, up to C^r-conjugacy, by gluing together the centres of this foliation on the sphere (see 5.2.1 and 5.2.2).

Hence $\mathcal{F}$ is unique up to C^r-conjugacy.

Definition 4.5.1: Let $\mathcal{F}$ be a Morse foliation on the oriented 2-manifold M_g of genus g.
We define an equivalence relation $\sim$ on the circle leaves of $\mathcal{F}$ by:

$l \sim l'$ if & only if there is a C^r embedding

$H : S^1 \times (-1,1) \longrightarrow M_g$

with $H(S^1 \times \{t\})$ a leaf for each $t \in (-1,1)$

and $\left.\begin{array}{l} H(S^1 \times \{\eta\}) = l \\ H(S^1 \times \{-\eta\}) = l' \end{array}\right\} \quad \eta \in [0,1)$.

The equivalence class of l is denoted by U_l.

Lemma 4.5.2: Let $\mathcal{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy and no leaf containing more than one saddle point.

Then: (i) $\sim$ is an equivalence relation.

(ii) U_l is open.

(iii) Either $g = 1$ and $U_l = M_g$

or $U_l \setminus l$ has precisely two components.

In the latter case, the boundary of each component consists either of a centre or of the disjoint union of l, a saddle point and one or two loop separatrices.

Proof: (i) This is immediate from elementary considerations and lemma 4 of appendix 1.

(ii) It is clear from the definition that U_l is open since all leaves in $H(S^1 \times (-1,1))$ are in U_l.

(iii) The proof of this part relies on the results of chapter 5.

According to lemma 5.1.1, by cutting along a closed leaf l and "gluing in" two centres c_1, c_2 we obtain either a Morse foliation on a manifold of genus g-1 or Morse foliations on manifolds of genus g-u, u $(0 \leqslant u \leqslant g)$ each with exactly one of the centres c_1, c_2.

The result now follows from lemma 4.1.2 by inspection of the boundaries of the discs D_{c_1}, D_{c_2}.

<u>Lemma 4.6.1</u>: Let $\mathfrak{F}$ be a $C^r (r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy and no leaf containing more than one saddle point.

Let c be a centre of $\mathfrak{F}$ of type 1.

Then there is an open neighbourhood U of $\overline{D}_c$ and a C^r orientation preserving diffeomorphism:

$$\Theta : U \longrightarrow (-1,1) \times (-1,1)$$

such that $\Theta^* \mathcal{B}^+ = \mathfrak{F}|U$ or $\Theta^* \mathcal{B}^- = \mathfrak{F}|U$ according as c has Morse index 0 or 2.

The neighbourhood U may be chosen arbitrarily small.

<u>Proof</u>: Without loss of generality assume that c has Morse index 0.

The idea of the proof is to construct a diffeomorphism on parts of a neighbourhood U and either to modify them on overlaps or to ensure , by using a transverse vector field, that they already agree.

We first chop up $(-1,1)\times(-1,1)$ foliated by $\mathcal{E}^+$ into regions $A_1,\ldots,A_7$ as shown in figure 4.6.

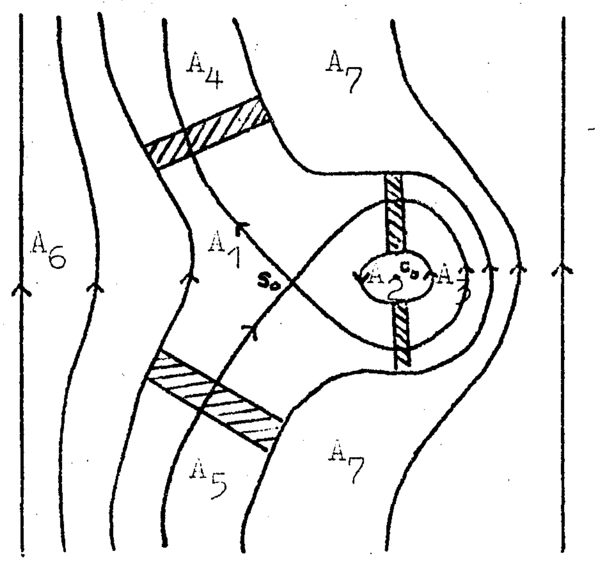


Fig. 4.6

Specifically, there are C^r orientation preserving diffeomorphisms:

$\varphi_1 : A_1 \longrightarrow \{(x,y)\in\mathbb{R}^2 : |x^2-y^2|<\frac{1}{4}, |x+y|<1, |x-y|<1\}$

a distinguished chart at the saddle point s_o,

$\varphi_2 : A_2 \longrightarrow \{(x,y)\in\mathbb{R}^2 : \; x^2+y^2<1\}$ a distinguished chart at c_o, the centre of $\mathcal{E}^+$.

$\varphi_i : A_i \longrightarrow [0,1]\times(-1,1)$ $i = 3,4,5,6,7$ mapping a single segment of leaf onto $[0,1]\times\{x\}$.

We also assume the following overlap conditions:

$A_1\cap A_3 = \varphi_3^{-1}([0,1/8]\times(-1,1)\cup[7/8,1]\times(-1,1)).$

$A_1\cap A_4 = \varphi_4^{-1}([0,1/8]\times(-1,1))$

$A_1\cap A_5 = \varphi_5^{-1}([7/8,1]\times(-1,1))$

as shaded in figure 4.6, and

$$(A_1 \cup A_3) \cap A_2 = \varphi_2^{-1}(\{(x,y) \in \mathbb{R}^2 : \tfrac{1}{2} < x^2+y^2 < 1\}),$$
$$(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) \cap A_6 = \varphi_6^{-1}([\tfrac{1}{2},1] \times (-1,1)),$$
$$(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) \cap A_7 = \varphi_7^{-1}([0,\tfrac{1}{2}] \times (-1,1)).$$

We now construct analogous regions A_i' in M_g and use lemmas 4 and 5 of appendix 1 to modify the overlaps - except for A_6 and A_7 where we use a transverse vector field to ensure that the overlaps agree.

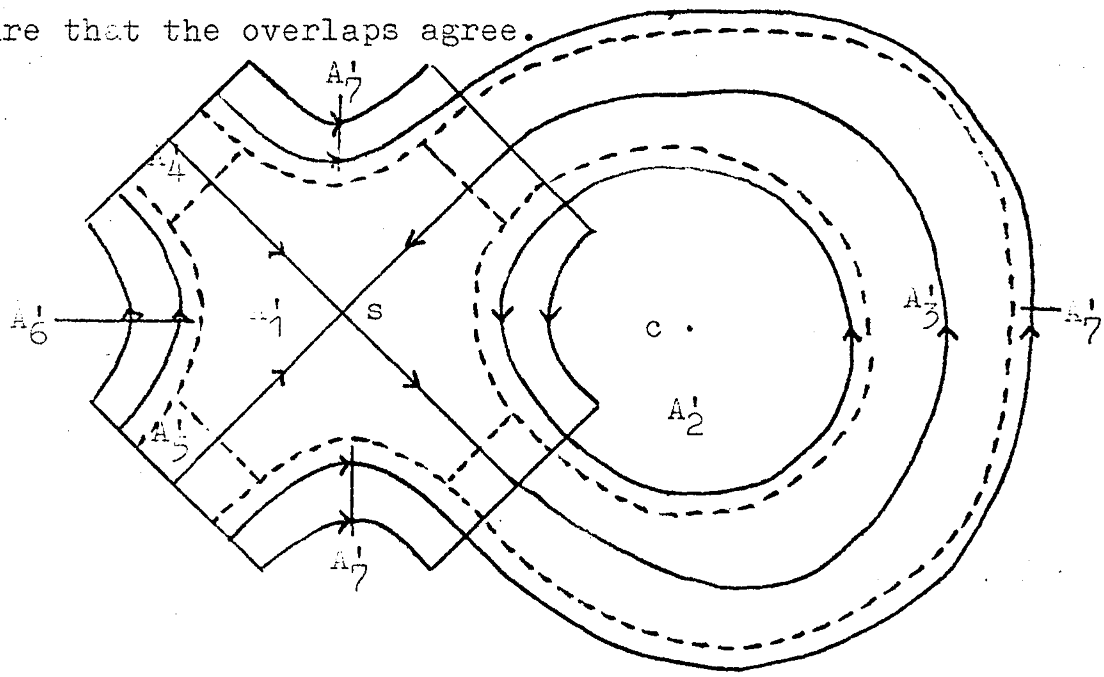


Fig. 4.7

Let c be the centre of $\mathfrak{Y}$ and s the unique saddle point in ∂D_c.

Let Υ be a distinguished chart at s and suppose Υ maps the segments of loop separatrix to the same pair of half-lines as φ_1 maps the loop separatrix of $\mathfrak{B}^+$.

In fact we may assume that the image of Υ is

$$\{(x,y) \in \mathbb{R}^2 : |x^2-y^2| < 4 \ , \ |x \pm y| < 4\}.$$

We may assume that the holonomy is defined from the transverse interval T_1 to the transverse interval T_2 (see figure 4.7) where

$$T_1 = \{(x,y) \in \mathbb{R}^2 : y=-x+2 \ , \ |x-1| < \tfrac{1}{4}\} \ ,$$
$$T_2 = \{(x,y) \in \mathbb{R}^2 : y= x-2 \ , \ |x-1| < \tfrac{1}{4}\} \ .$$

Let $A_1' = \Psi^{-1} \{(x,y) \in \mathbb{R}^2 : |x^2-y^2| < \frac{1}{4}, |x \pm y| < 1\}$,

$\varphi_1' = \Psi | A_1'$ and $\Theta | A_1' = \varphi_1' \varphi^{-1}$.

Suppose that for all i the maps φ_i are chosen so that the partial transverse vector field on M_g given by

$$\Theta_*^{-1} \varphi_{3*}^{-1} \ (\partial/\partial x) \ \text{on} \ \Theta^{-1}(A_1 \cup A_3)$$
$$\Theta_*^{-1} \varphi_{4*}^{-1} \ (\partial/\partial x) \ \text{on} \ \Theta^{-1}(A_1 \cup A_4)$$
$$\Theta_*^{-1} \varphi_{5*}^{-1} \ (\partial/\partial x) \ \text{on} \ \Theta^{-1}(A_1 \cup A_5)$$

extends to a transverse vector field $X_{\mathcal{F}}^{\perp}$ on all of M_g.

Then using the holonomy lemma with respect to $X_{\mathcal{F}}^{\perp}$ and the fact that $\mathcal{F}$ has no holonomy (see 2.15) we can construct a C^r diffeomorphism

$$\varphi_3' : A_3' \longrightarrow [0,1] \times (-1,1) \quad \text{such that}$$
$$\varphi_3' \, | A_1' \cap A_3' = \varphi_3 \Theta | A_1' \cap A_3'$$

where A_3' is as shown in figure 4.7.

Hence we can extend Θ to $A_1' \cup A_3'$ by taking

$$\Theta = \varphi_3^{-1} \varphi_3' \quad \text{on } A_3.$$

Similarly using $X_{\mathcal{F}}^{\perp}$ and the holonomy lemma we can extend Θ to regions A_4' and A_5' as shown in figure 4.7.

Lemma 4 of appendix 1 and lemma4.1.2 then allow us to extend ® to $A_2^!$ by choosing a distinguished chart $\varphi_2^!$ with domain $A_2^!$ at c and $A_2^! \cap (A_1^! \cup A_3^!) = \varphi_2^{!\,-1} B_{\frac{1}{2},1}$.

Using the holonomy lemma we can construct diffeomorphisms

$$\varphi_i^! : A_i^! \longrightarrow [0,1] \times (-1,1) \qquad i = 6,7$$

mapping segments of leaf to segments $[0,1] \times \{t\}$ where the regions $A_6^!, A_7^!$ are as shown in figure 4.7 and satisfy analogous overlap conditions to A_6 and A_7. Using these maps and lemma 5 of appendix 1 finally allows us to extend ® to all of $A_1^! \cup \ldots\ldots \cup A_7^!$.

Now ® maps leaves to leaves but at singular points it preserves distinguished charts and hence distinguished maps. Hence $®^*\mathcal{D}^+ = A_1^! \cup \ldots\ldots \cup A_7^!$.

Finally note that taking the domain of Υ sufficiently small we can make $A_1^! \cup \ldots\ldots \cup A_7^! \subseteq \text{domain}\,\Upsilon \cup \overline{D}_c \cup A_3^!$ as small as we like.

Our next task is to show that centres of type 1 can be removed or added in a unique way up to C^r complete equivalence. Before we state and prove this we need the following lemma.

Lemma 4.6.2: Let $\mathcal{Q}$ be the foliation of $(-1,1) \times (-1,1)$ by lines x = constant.

Let $\mathcal{X}$ be a C^r flow on $(-1,1)\times(-1,1)$ equalling $\mathfrak{a}$ outside $(-\frac{3}{4},\frac{3}{4})\times(-\frac{3}{4},\frac{3}{4})$.

Then there is a C^r diffeomorphism

$$\varphi_{\mathcal{X}} : (-1,1)\times(-1,1) \longrightarrow (-1,1)\times(-1,1)$$

such that (i) $\varphi_{\mathcal{X}}{}^*\mathfrak{a} = \mathcal{X}$,

(ii) $\varphi_{\mathcal{X}}$ is the identity map near the boundary (in $\mathbb{R}^2$) of $(-1,1)\times(-1,1)$.

Proof: $\mathcal{X}$ is a C^r map

$$: (-1,1)\times(-1,1)\times\mathbb{R} \longrightarrow (-1,1)\times(-1,1)$$

$$: ((x,y),t) \longmapsto \mathcal{X}_t((x,y)) \quad .$$

Then define

$$\varphi_{\mathcal{X}} : (-1,1)\times(-1,1) \longrightarrow (-1,1)\times(-1,1)$$

by $\quad \varphi_{\mathcal{X}}(x,y) = \mathcal{X}_y(x,0)$.

Since $\mathcal{X}$ has no singular points $\varphi_{\mathcal{X}}$ is the required map.

Proposition 4.6.3: A. Let $\mathfrak{F}$ be a C^r $(r\geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy and no leaf containing more than one saddle point.

Let c be a centre of $\mathfrak{F}$ of type 1. Then up to C^r-complete equivalence there is a unique way of removing c from $\mathfrak{F}$.

That is, up to C^r-complete equivalence there is a unique Morse foliation $\mathfrak{F}'$ on M_g satisfying:

1. $\mathfrak{F}'$ has one less centre and one less saddle point

than $\mathcal{F}$.

2. There is an open neighbourhood U of $\overline{D}_c$ and a C^r orientation preserving diffeomorphim

$$\varphi : U \longrightarrow (-1,1) \times (-1,1)$$

such that a) $\mathcal{F} | M_g \setminus U_{\frac{3}{4}} = \mathcal{F}' | M_g \setminus U_{\frac{3}{4}}$

b) $\mathcal{F} | U = \varphi^* \mathcal{S}^+$ or $\varphi^* \mathcal{S}^-$ as appropriate, where $U_{\frac{3}{4}} = \varphi^{-1}((-\frac{3}{4},\frac{3}{4}) \times (-\frac{3}{4},\frac{3}{4}))$.

3. The leaf of $\mathcal{F}|U$ cutting $\varphi^{-1}((-\frac{3}{4},\frac{3}{4}) \times \{-\frac{3}{4}\})$ at $\varphi^{-1}(x,\frac{3}{4})$ cuts $\varphi^{-1}((-\frac{3}{4},\frac{3}{4}) \times \{\frac{3}{4}\})$ at $\varphi^{-1}(x,\frac{3}{4})$.

B. Conversely, let $\mathcal{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g with the same properties as in A. and let l be a circle leaf. Then up to C^r-complete equivalence there is a unique way of adding a centre of type 1 and Morse index 0 to l.

That is, up to C^r-complete equivalence there is a unique C^r Morse foliation $\mathcal{F}'$ on M_g such that:

1. $\mathcal{F}'$ has one more centre of type 1 and Morse index 0, and one more saddle point than $\mathcal{F}$.

2. There is a C^r diffeomorphism

$$\varphi : U \longrightarrow (-1,1) \times (-1,1)$$

where U is a neighbourhood of some point $x_0 \in l$

such that a) $\mathcal{F} | M_g \setminus U_{\frac{3}{4}} = \mathcal{F}' | M_g \setminus U_{\frac{3}{4}}$

b) φ is a distinguished chart for $\mathcal{F}$ at x_0

where $U_{\frac{3}{4}} = \varphi^{-1}((-\frac{3}{4},\frac{3}{4}) \times (-\frac{3}{4},\frac{3}{4}))$.

3. The leaf of $\mathfrak{F}'|U$ cutting $\varphi^{-1}((-\frac{3}{4},\frac{3}{4})\times\{-\frac{3}{4}\})$ at $\varphi^{-1}(x,-\frac{3}{4})$ cuts $\varphi^{-1}((-\frac{3}{4},\frac{3}{4})\times\{\frac{3}{4}\})$ at $\varphi^{-1}(x,\frac{3}{4})$.

4. The additional saddle point lies on a leaf agreeing with 1 outside U.

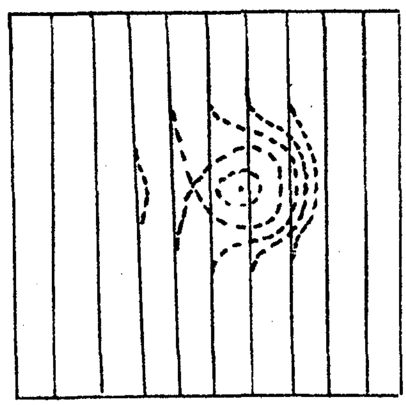


Fig. 4.9

Proof: A. The existence of the foliation $\mathfrak{F}'$ is immediate from lemma 4.8 and the fact that the Morse foliations $\mathfrak{F}^+$, $\mathfrak{F}^-$ on $(-1,1)\times(-1,1)$ agree with that given by lines x = constant outside the circle $x^2+y^2=\frac{1}{2}$.

Now we prove uniqueness, supposing that c has Morse index 0. Suppose that $\mathfrak{F}_1'$, $\mathfrak{F}_2'$ are two Morse foliations which satisfy conditions 1., 2., 3. of the statement and let

$$\varphi_1,\varphi_2 : U_1,\ U_2 \longrightarrow (-1,1)\times(-1,1)$$

be the corresponding diffeomorphisms as defined in part 2.

By the hypotheses of this proposition and the proof of lemma 4.8 we can find a neighbourhood U_0 of $\overline{D}_c$ with $U_0 \subseteq \varphi_1^{-1}((-\frac{3}{4},\frac{3}{4})\times(-\frac{3}{4},\frac{3}{4}))\cap\varphi_2^{-1}((-\frac{3}{4},\frac{3}{4})\times(-\frac{3}{4},\frac{3}{4}))$

and $\varphi_o : U_o \longrightarrow (-1,1) \times (-1,1)$

a C^r diffeomorphism such that

$$\varphi_o^* \partial^+ = \mathfrak{F} | U_o .$$

There is a well-defined C^r Morse foliation $\mathfrak{F}_o'$ on M_g given by $\mathfrak{F}$ outside U_o and by $\varphi_o^* \mathfrak{A}$ on U_o.

We show that $\mathfrak{F}_1'$ and $\mathfrak{F}_2'$ are C^r-completely equivalent to $\mathfrak{F}_o'$.

Now by lemma 4.6.2 and assumption 3 of the statement we may assume that

$$\varphi_1^* \mathfrak{A} = \mathfrak{F}_1' \mid U_1 \quad \text{and} \quad \varphi_2^* \mathfrak{A} = \mathfrak{F}_2' | U_2$$

where $\mathfrak{A}$ is the flow on $(-1,1) \times (-1,1)$ given by lines x = constant.

By lemma 4.6.2 we can find C^r diffeomorphisms

$$\rho_i : (-1,1) \times (-1,1) \longrightarrow (-1,1) \times (-1,1) \quad i=1,2$$

such that ρ_i agrees with the identity map outside $(-\frac{3}{4},\frac{3}{4}) \times (-\frac{3}{4},\frac{3}{4})$ and

$$\rho_i^* \mathfrak{A} = \varphi_i^{-1\,*} \mathfrak{F}_o' \quad i=1,2.$$

Then by lemma 1 of appendix 3, $\varphi_i^{-1} \rho_i^{-1} \varphi_i$ extends by the identity map to a C^r diffeomorphism

$$\gamma_i : M_g \longrightarrow M_g$$

isotopic to the identity.

Then $\gamma_i^* \mathfrak{F}_i' = \mathfrak{F}_o' \quad i=1,2$ as required.

B. The existence of $\mathfrak{F}'$ is straightforward - simply replace the foliation on a distinguished chart at a point of 1 by standard example $\mathcal{D}^+$.

Now we prove uniqueness.

Suppose that $\mathcal{F}_1'$ and $\mathcal{F}_2'$ are Morse foliations with one more centre of type 1 and Morse index 0 and one more saddle point than $\mathcal{F}$ which satisfy conditions 1,2 and 3 of the statement.

Let $\varphi_1, \varphi_2 : U_1, U_2 \longrightarrow (-1,1) \times (-1,1)$ be the corresponding diffeomorphisms as in part 2. of the statement.

Suppose first that $U_1 = U_2 = U_o$ (say).

By similar methods to the proof of lemma 4.8 we can find C^r diffeomorphisms

$$\rho_i : (-1,1) \times (-1,1) \longrightarrow (-1,1) \times (-1,1) \quad i=1,2$$

which agree with the identity outside $(-\frac{3}{4},\frac{3}{4}) \times (-\frac{3}{4},\frac{3}{4})$ and satisfy

$$\rho_i^* \varphi_i^{-1*} \mathcal{F}_i' = \mathcal{D}^+ \quad i=1,2$$

Similarly we can find a C^r diffeomorphism

$$\rho_o : (-1,1) \times (-1,1) \longrightarrow (-1,1) \times (-1,1)$$

agreeing with the identity outside $(-\frac{3}{4},\frac{3}{4}) \times (-\frac{3}{4},\frac{3}{4})$ such that $\rho_o^* \mathcal{D}^+ = (\varphi_2 \varphi_1^{-1})^* \mathcal{D}^+$.

then by lemma 1 of appendix 3 the diffeomorphism

$$\varphi_1^{-1} \rho_1 \rho_o \varphi_1 \varphi_2^{-1} \rho_2^{-1} \varphi_2 : U_o \longrightarrow U_o$$

extends to a C^r diffeomorphism

$$R : M_g \longrightarrow M_g$$

which is isotopic to the identity and satisfies

$$R^*\mathfrak{F}'_1 = R^*\mathfrak{F}'_2 \quad .$$

Hence it remains to show that we can assume that $U_1 = U_2$. To do this we first construct a neighbourhood W of a segment of l as shown in figure 4.10.

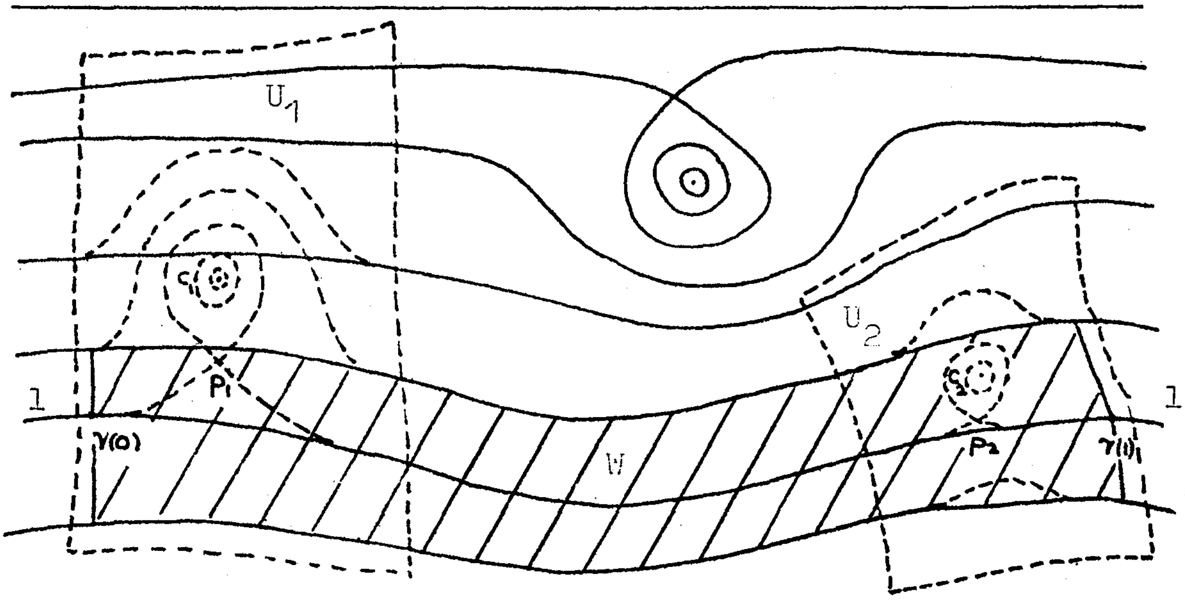


Fig. 4.10

Let p_1 denote the additional saddle point of $\mathfrak{F}'_1$ and p_2 that of $\mathfrak{F}'_2$.

Let l_1 denote the leaf of $\mathfrak{F}'_1$ containing p_1 (which agrees with l outside U_1) and l_2 the leaf of $\mathfrak{F}'_2$ containing p_2. Let $\varphi_1^{-1}(\{k_1\} \times (-1,1))$ denote the component of $l \cap U_1$ which agrees with the segment of $l_1 \cap U_1$ containing p_1 on $\varphi_1^{-1}(\{k_1\} \times (-1,-\frac{3}{4}) \cup \{k_1\} \times (\frac{3}{4},1))$.

Let $\varphi_2^{-1}(\{k_2\} \times (-1,1))$ denote the corresponding component of $l \cap U_2$.

Interchanging the subscripts 1 and 2 if necessary, we can find a positively oriented embedded curve

$$\gamma : [0,1] \longrightarrow l$$

such that $\gamma(0) = \varphi_1^{-1}(k_1, -7/8)$

$\gamma(1) = \varphi_2^{-1}(k_2, 7/8)$ or $\varphi_1^{-1}(k_1, 7/8)$ as appropriate so that

$$\varphi_1^{-1}(\{k_1\} \times (-7/8,1)) \cup \varphi_2^{-1}(\{k_2\} \times (-1,7/8)) \subseteq \gamma([0,1]).$$

We choose W to be given by the holonomy map of $\mathfrak{F}$ along γ of a sufficiently small transverse interval at $\gamma(0)$.

In fact we require

$W \cap \varphi_1^{-1}((-1,1) \times \{-7/8\}) \subseteq \varphi_1^{-1}((-1,1) \times \{-7/8\})$ and

$W \cap \varphi_2^{-1}((-1,1) \times \{7/8\}) \subseteq \varphi_2^{-1}((-1,1) \times \{7/8\})$ or

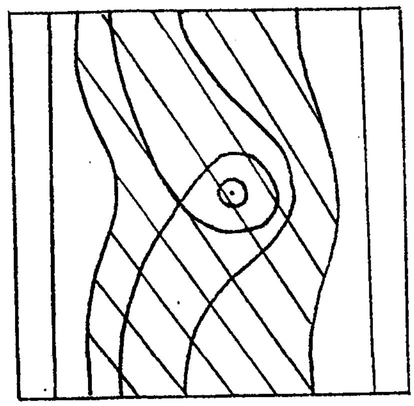
$W \cap \varphi_1^{-1}((-1,1) \times \{7/8\}) \subseteq \varphi_1^{-1}((-1,1) \times \{7/8\})$ as appropriate.

Let c_1, c_2 denote the additional centres of $\mathfrak{F}_1'$, $\mathfrak{F}_2'$.

Let K_1 be the union of $\overline{D}_{c_1}$ and all segments of $\mathfrak{F}_1'$ meeting $W \cap \varphi_1^{-1}((-1,1) \times \{-7/8\})$.

Let K_2 denote the union of $\overline{D}_{c_2}$ and all segments of $\mathfrak{F}_2'$ meeting $W \cap \varphi_2^{-1}((-1,1) \times \{7/8\})$ as in figure 4.11.

Let $\mathfrak{X}_i$ $(i=1,2)$ be a C^r Morse foliation agreeing with $\mathfrak{F}_i'$ outside K_i and without singularities on K_i.



K_i shaded

Fig. 4.11

By lemma 4.9 and lemma 1 of appendix 3 we can find C^r diffeomorphisms

$$\rho_i : M_g \longrightarrow M_g \quad i = 1,2$$

isotopic to the identity and equal to the identity outside $\varphi_i^{-1}((-\frac{3}{4},\frac{3}{4}) \times (-\frac{3}{4},\frac{3}{4}))$

such that $\qquad \mathcal{F} = \rho_i^* \mathcal{X}_i \qquad i = 1,2.$

Now $\rho_i^* \mathcal{F}_i'$ agrees with $\mathcal{F}$ outside $\rho_i^{-1} K_i$ $(i=1,2)$.

However, the choice of K_i shows that $\rho_i^{-1} K_i \subsetneq W$.

Hence we can assume that $U_1 = U_2$.

<u>Corollory 4.6.4</u>: Let $\widetilde{\mathcal{F}}^{1,j}_{g,\sigma}$ denote the set of equivalence classes of pairs $(\mathcal{F},c)$ where $\mathcal{F}$ is a C^r Morse foliation on M_g with no holonomy, with no leaf containing more than one saddle point, with σ centres and in which c is a centre of $\mathcal{F}$ of type 1 and Morse index j (j=0 or 2).

$(\mathcal{F},c)$ and $(\mathcal{F},c')$ are equivalent if there is a C^r diffeomorphism f of M_g which is isotopic to the identity and which satisfies $\quad f(c) = c'$ and $f^* \mathcal{F}' = \mathcal{F}$.

Let $\mathfrak{F}_{g,\sigma}$ denote the set of C^r-complete equivalence classes of Morse foliations on M_g which have no holonomy, no leaf containing more than one saddle point and σ centres. Then there are bijections:

$$p^j_{g,\sigma} \;:\; \tilde{\mathfrak{F}}^{1,j}_{g,\sigma} \longrightarrow \mathfrak{F}_{g,\sigma-1} \qquad j = 0,2$$

$$q^j_{g,\sigma} \;:\; \mathfrak{F}_{g,\sigma-1} \longrightarrow \tilde{\mathfrak{F}}^{1,j}_{g,\sigma} \qquad j = 0,2 \;,$$

such that $\quad q^j_{g,\sigma}\, p^j_{g,\sigma} = \text{identity} \qquad j = 0,2$

$$p^k_{g,\sigma}\, q^j_{g,\sigma} = \text{identity} \qquad j,k = 0,2.$$

Proof: It is left to the reader to check that the proposition defines such maps and to prove the equalities.

We now consider centres of type 2. It turns out that in this case we cannot remove the centre without destroying the transverse orientability of the Morse foliation. Thus we have to express the uniqueness up to C^r-complete equivalence of the behaviour near a centre of type 2 in a different way to that for centres of type 1.

Definition 4.7: Let $\mathfrak{Z}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy and with no leaf containing more than one saddle point. Let c be a centre of $\mathfrak{Z}$ of type 2 and let

$$U_c = \overline{D}_c \cup U_{l_1} \cup U_{l_2}$$

where l_1 and l_2 are circle leaves which lie in the complement of $\overline{D}_c$ and approximate each of the loop separatrices in ∂D_c (see figure 4.12).

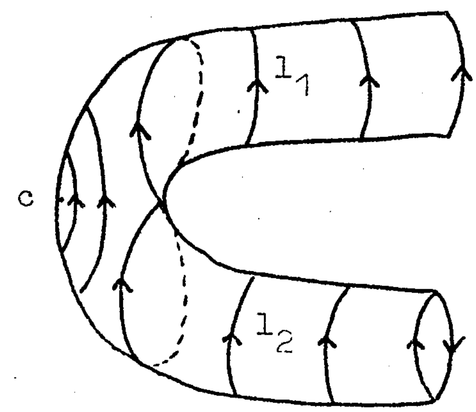


Fig. 4.12

If $g=1$, the fact that $\mathfrak{F}$ is transversely oriented precludes U_c from being all of M_g.

Thus it is always true that ∂U_c is a non-empty union of singular points and loop separatrices.

<u>Lemma 4.8.1</u>: Let $\mathfrak{F}$ be a C^r Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no leaf containing more than one saddle point and with a centre c of type 2 and Morse index 0.

Let U be a saturated neighbourhood of $\overline{D}_c$ whose closure is contained in U_c.

Let $T_{\frac{1}{2}}$ be the half torus.

Then there is a C^r diffeomorphism

$$\varphi : U \longrightarrow T_{\frac{1}{2}}$$

such that $\mathfrak{F}|U = \varphi^*\mathcal{E}+$, where $\mathcal{E}+$ is the standard Morse foliation on $T_{\frac{1}{2}}$ defined in section 4.3.

Proof: This is similar to the proof of lemma 4.6.2 and is left to the reader.

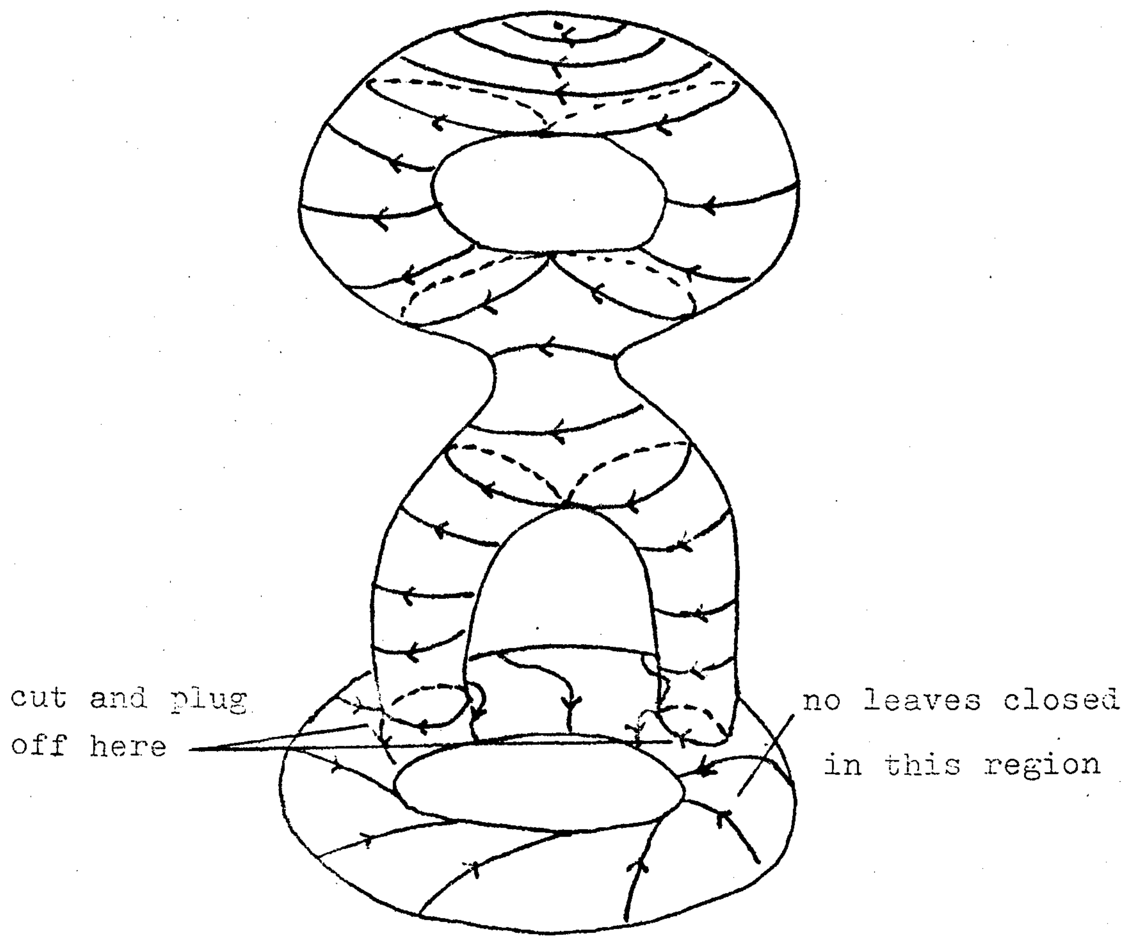
Proposition 4.8.2: Let M_g be the oriented 2-manifold of genus g and let $\mathcal{F}_1, \mathcal{F}_2$ be C^r $(r \geqslant 2)$ Morse foliations on M_g with no holonomy and no leaf containing more than one saddle point.
Let c_1, c_2 be centres of $\mathcal{F}_1, \mathcal{F}_2$ respectively each of type 2 and Morse index 0.
Suppose that $U_{c_1} = U_{c_2}$ and $\mathcal{F}_1$ and $\mathcal{F}_2$ agree on a saturated neighbourhood of the complement of U_c.
Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are C^r-completely equivalent.
Proof: This uses lemma 4.8.2 and is left to the reader.

Chapter5. The decomposition theorem.

We have already made a start in classifying Morse foliations without holonomy up to C^r conjugacy, indeed we have seen that up to C^r complete equivalence the behaviour at a centre is of two types, once the Morse index of the centre is fixed.

We have also seen that any circle leaf l has a maximal open neighbourhood U_l consisting entirely of circle leaves. In general U_l is a cylinder whose boundaries consist either of a centre or of a saddle point and one or two loop separatrices. If a boundary component of U_l has two loop separatrices there are in general circle leaves near ∂U_l not in U_l. Thus there is some leaf l' with U_l and $U_{l'}$ abutting. Inductively adding on sets $U_{l'}$ we obtain a "tree" made up of cylinders foliated by circles joined to each other by loop separatrices and such that each boundary component is either a centre or has no nearby circle leaves not in the "tree". Adding in centres in the boundary and plugging off the remaining boundary components of the "tree" with centres and the holes left also with centres produces a new Morse foliation. Repeating this procedure and shrinking away centres of type 1 we end up with a number of 2-manifolds and Morse foliations without holonomy either having all leaves closed or having no leaf closed.

In this chapter we shall construct such a decomposition and prove it to be unique up to C^r conjugacy.
In figure 5.1 below we give an example of this procedure.



Cutting & plugging off produces a torus with three centres & three saddle points. Remainder of manifold is a torus with two centres of type one and no other closed leaves.

Fig. 5.1

Definition 5.1.1: Suppose $\mathcal{F}$ is a C^r $(r \geq 2)$ Morse foliation on the oriented 2-manifold M_g of genus g, without holonomy. Suppose l is a closed leaf of M_g, then cutting along l and gluing in two discs produces a manifold $M_{g,l}$. $M_{g,l}$ is foliated by foliating the discs with circles and a centre.

Rigorously we proceed as follows.

Let $B_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\} \subseteq B_1$.

Let $B_1 \times \{1\}$ and $B_1 \times \{2\}$ be two copies of B_1 with foliations $\mathcal{F}_1, \mathcal{F}_2$ given by functions $(x,y) \longrightarrow -(x^2 + y^2)$ and $(x,y) \longrightarrow (x^2 + y^2)$ respectively.

We define a new oriented C^r manifold $M_{g,l}$ with a Morse foliation as follows:

$M_{g,l} = M_g \setminus l \ \cup B_1 \times \{1\} \cup B_1 \times \{2\}$ as a set.

Let $\gamma : S^1 \longrightarrow l$ be a C^r orientation preserving embedding and $H : S^1 \times (-\frac{1}{2},\frac{1}{2}) \longrightarrow M_g$ a C^r orientation preserving embedding such that:

(i) $H((x,y),t) \in l_t$ a leaf depending only on t.

(ii) $H((x,y),0) = \gamma(x,y)$.

Define $H_1 : S^1 \times (-\frac{1}{2},\frac{1}{2}) \longrightarrow M_{g,l}$ by

$$H_1((x,y),t) = \begin{cases} H((x,y),t) & t < 0 \\ (((1-t)x,(1-t)y),1) & t \geq 0 \end{cases}$$

and $H_2 : S^1 \times (-\frac{1}{2},\frac{1}{2}) \longrightarrow M_{g,l}$ by

$$H_2((x,y),t) = \begin{cases} H((x,y),t) & t > 0 \\ (((1+t)x,-(1+t)y),2) & t \leq 0. \end{cases}$$

Then the differentiable structure on $M_{g,1}$ is given by taking a chart at $x \in M_g \setminus 1$ to be any chart for M_g contained in $M_g \setminus 1$, obvious charts for points in $B_1 \setminus S^1 \times \{i\}$ $(i=1,2)$ and for $x \in S^1 \times \{i\}$ take H_i^{-1} as a chart $(i=1,2)$. Similarly, the distinguished maps for $\mathfrak{F}_1$ are obtained by adjoining those for $\mathfrak{F}|M_g \setminus 1$, $\mathfrak{F}_i | B_1 \setminus S^1 \times \{i\}$ $(i=1,2)$ to the functions $\pi \cdot H_i^{-1}$ $(i=1,2)$ where π is the projection onto $(-\frac{1}{2},\frac{1}{2})$.

Note that $M_{g,1}$ may have one or two components.
It also satisfies:
(1) $M_g \setminus 1, B_1 \setminus S^1 \times \{1\}, B_1 \setminus S^1 \times \{2\}$ are open submanifolds and the inclusion maps are maps of Morse foliations.
(2) $S^1 \times \{1\}$ and $S^1 \times \{2\}$ are leaves of $\mathfrak{F}_1$.

Lemma 5.1.2: The C^r structure on $M_{g,1}$ and Morse foliation $\mathfrak{F}_1$ defined in 5.1.1 are the unique ones on $M_{g,1}$ up to C^r diffeomorphism satisfying properties (1) and (2) immediately above.

Proof: Let $\mathfrak{D}$ be the C^r structure on $M_{g,1}$ and $\mathfrak{F}_1$ the Morse foliation defined in 5.1.1.
Let $\mathfrak{D}'$ and $\mathfrak{F}_1'$ be any others satisfying (1) and (2).
We construct a diffeomorphism

$$\Upsilon : (M_{g,1},\mathfrak{D}) \longrightarrow (M_{g,1},\mathfrak{D}')$$

such that $\Upsilon^* \mathfrak{F}_1' = \mathfrak{F}_1$.

Let $\rho_1 : S^1 \longrightarrow (M_{g,1}, \mathcal{D}')$ be a smooth embedding onto the leaf $S^1 \times \{1\}$ of $\mathcal{F}_1'$ and

$K_1 : S^1 \times (-\frac{1}{2}, \frac{1}{2}) \longrightarrow (M_{g,1}, \mathcal{D}')$

an embedding such that

$K_1(x,0) = \rho_1(x)$

$K_1(x,t)$ lies in a leaf of $\mathcal{F}_1'$ independent of x.

Without loss of generality we may choose a diffeomorphism h_1 of $(-\frac{1}{2}, \frac{1}{2})$ into itself such that K_1 and $H_1 \circ (\mathrm{id} \times h_1)$ have the same image and h_1 = identity near 0.

Then $K_1^{-1} H_1 (\mathrm{id} \times h_1) : S^1 \times (-\frac{1}{2}, \frac{1}{2}) \setminus S^1 \times \{0\}$

$\longrightarrow S^1 \times (-\frac{1}{2}, \frac{1}{2}) \setminus S^1 \times \{0\}$

is an orientation preserving diffeomorphism preserving the foliation by leaves $S^1 \times \{t\}$.

By a double application of lemma 4, appendix 1 we can find a C^r orientation preserving diffeomorphism

$P_1 : S^1 \times (-\frac{1}{2}, \frac{1}{2}) \longrightarrow S^1 \times (-\frac{1}{2}, \frac{1}{2})$ such that

$$P_1 = \begin{cases} \text{identity map near } S^1 \times \{0\} \\ K_1^{-1} H_1 (\mathrm{id} \times h_1) \text{ near } S^1 \times \{\pm\frac{1}{2}\}. \end{cases}$$

Similarly choose $K_2, h_2 P_2$ for $S^1 \times \{2\}$.

Now define Ψ by

$$\Psi(x) = \begin{cases} K_1 P_1 (\mathrm{id} \times h_1^{-1}) H_1^{-1}(x) & x \in \text{image } H_1(\mathrm{id} \times h_1) \\ K_2 P_2 (\mathrm{id} \times h_2^{-1}) H_2^{-1}(x) & x \in \text{image } H_2(\mathrm{id} \times h_2) \\ x \text{ otherwise.} \end{cases}$$

Ψ equals the identity outside a small neighbourhood of $S^1 \times \{1\} \cup S^1 \times \{2\}$.

<u>Lemma 5.1.3:</u> Up to C^r diffeomorphism, $M_{g,l}$ and $\mathfrak{F}_l$ depend only on U_l.

<u>Proof:</u> Let $l' \subseteq U_l$.

Without loss of generality, we can assume that there is an orientation preserving diffeomorphism

$$H : S^1 \times (-\tfrac{1}{2},\tfrac{1}{2}) \longrightarrow M_g$$

such that $H(S^1 \times \{-\tfrac{1}{4}\}) = l$

$H(S^1 \times \{\tfrac{1}{4}\}) = l'$

$H(S^1 \times \{t\})$ is a leaf each t.

Let h_1 be an orientation preserving diffeomorphism of $(-\tfrac{1}{2},\tfrac{1}{2})$ equal to the identity map near $\pm\tfrac{1}{2}$ which maps 0 to $-\tfrac{1}{4}$.

Let h_2 be a similar map, mapping 0 to $\tfrac{1}{4}$.

Then $H(\mathrm{id}\times h_1)$ can be used to define the structure of $M_{g,l}$ and $H(\mathrm{id}\times h_2)$ to define that of $M_{g,l'}$.

Define Ψ: $M_{g,l} \longrightarrow M_{g,l'}$ by

$$\Psi(x) = \begin{cases} H(\mathrm{id}\times h_2h_1^{-1})H^{-1}(x) & x \in H(\mathrm{id}\times h_1)(Y) \\ x & \text{otherwise} \end{cases}$$

where $Y = S^1 \times (-\tfrac{1}{2},0) \cup S^1 \times (0,\tfrac{1}{2})$.

Then Ψ is a diffeomorphism with respect to the structures referred to above and $\Psi^*\mathfrak{F}_l' = \mathfrak{F}_l$.

5.1.4 Having defined the 2-manifold $M_{g,l}$ and foliation $\mathfrak{F}_l$ we must relate these to the standard oriented 2-manifolds M_g.

Now $M_{g,l}$ is an oriented 2-manifold with one or two components.

If $M_{g,l}$ has a single component ,then by index number arguments applied to $\mathfrak{F}_l$,it has genus g-1.

If $M_{g,l}$ has two components M_1, M_2 then without loss of generality $B_1 \times \{1\} \subseteq M_1$ and $B_1 \times \{2\} \subseteq M_2$.

Further if M_1 has genus g_1 and M_2 has genus g_2 then index sum arguments show that $g_1 + g_2 = g$.

By the remarks of chapter 2,this second case occurs if & only if the leaf l represents the zero homology class in $H_1(M_g, \mathbb{Z})$.

It follows from the above lemmas that the above process defines a unique C^r conjugacy class of Morse foliations $\mathfrak{F}_l$ on M_{g-1} if l is not homologous to zero and on $M_{g_1} \amalg M_{g_2}$ if l is homologous to zero.

Conversely we shall see in the next few sections that this procedure can be reversed. By removing two centres and identifying the two circle boundaries we retrieve a manifold which is C^r diffeomorphic to M_g and Morse foliation C^r conjugate to $\mathfrak{F}$.

<u>Definition 5.2.1</u>: We now define the notion of gluing centres together.

Let $M = M_{g_1} \amalg M_{g_2}$ or M_g be given a Morse foliation $\mathfrak{F}$.

Let B_r denote the closed r-ball in $\mathbb{R}^2$ and $\overset{\circ}{B}_r$ the open r-ball.

Let c_1 be a centre of Morse index 2 and c_2 one of Morse index 0 in $\mathfrak{F}$.

Let $\varphi_1 : U_1 \longrightarrow \overset{\circ}{B}_{3/2}$ be a distinguished chart at c_1

and $\varphi_2 : U_2 \longrightarrow \overset{\circ}{B}_{3/2}$ a distinguished chart at c_2.

Suppose without loss of generality that $U_1 \cap U_2 = \emptyset$.

We define a new manifold $M^{c_1 \# c_2}$ obtained by gluing the centres c_1 and c_2 together plus a Morse foliation $\mathfrak{F}^{c_1 \# c_2}$, as follows:

$$M^{c_1 \# c_2} = \frac{M \setminus (\varphi_1^{-1}(\overset{\circ}{B}_{3/2} \setminus \overset{\circ}{B}_1) \cup \varphi_2^{-1}(\overset{\circ}{B}_{3/2} \setminus \overset{\circ}{B}_1))}{x \sim y \text{ iff } \varphi_1(x) = 1,\ \varphi_2(y) = 1,\ \varphi_1(x) = \varphi_2(y) \text{ or vice versa}}$$

as a set.

For points $x \in M^{c_1 \# c_2} \setminus \varphi_1^{-1}(\partial B_1)$ charts and distinguished maps are defined as in M.

For points in $\varphi_1^{-1}(\partial B_1)$ we can take H^{-1} as a chart where $H : S^1 \times (-\frac{1}{2}, \frac{1}{2}) \longrightarrow M^{c_1 \# c_2}$ is defined by

$$H((x,y),t) = \begin{cases} \varphi_1((1-t)x, (1-t)y) & t < 0 \\ \varphi_2((1+t)x, -(1+t)y) & t > 0. \end{cases}$$

Further if π is the projection onto $(-\frac{1}{2},\frac{1}{2})$ we can take πH^{-1} as a distinguished map for points in $\varphi_1^{-1}(\partial B_1)$. This defines the Morse foliation $\mathcal{Y}^{c_1 \# c_2}$ with two less centres than $\mathcal{Y}$ and the same number of saddle points.

$M^{c_1 \# c_2}$ and $\mathcal{Y}^{c_1 \# c_2}$ satisfy the following properties:

(1) The inclusion of $M \setminus (\varphi_1^{-1}(\overset{o}{B}_{3/2} \setminus B_1) \cup \varphi_2^{-1}(\overset{o}{B}_{3/2} \setminus B_1))$ is an embedding of a submanifold preserving a Morse foliation.

(2) The image of $\varphi_1^{-1}(\partial B_1)$ is a leaf of $\mathcal{Y}^{c_1 \# c_2}$.

<u>Lemma 5.2.2:</u> The C^r differential structure on $M^{c_1 \# c_2}$ and Morse foliation on it $\mathcal{Y}^{c_1 \# c_2}$ are the unique ones on the set $M^{c_1 \# c_2}$ up to C^r diffeomorphism satisfying the conditions 5.2.1 (1) & (2) immediately above.

<u>Proof:</u> This is left to the reader and is similar to that of lemma 5.1.2.

<u>Lemma 5.2.3:</u> Up to C^r diffeomorphism $M^{c_1 \# c_2}$ and $\mathcal{Y}^{c_1 \# c_2}$ depend only on c_1 and c_2 i.e. they are independent of φ_1 and φ_2.

<u>Proof:</u> Let ψ_1, ψ_2 be another pair of distinguished charts at c_1, c_2 respectively with domains V_1, V_2.

Let $M_\varphi^{c_1 \# c_2}$ denote the manifold $M^{c_1 \# c_2}$ with C^r differential structure defined by φ_1, φ_2 as in definition 5.2.1 and $M_\psi^{c_1 \# c_2}$ that defined by ψ_1, ψ_2.

Without loss of generality we can assume that

$$\varphi_2 = \psi_2\ ,\quad U_2 = V_2 \quad \text{and} \quad V_1 \subseteq U_1.$$

Now $\varphi_1\psi_1^{-1} : \overset{\circ}{B}_{3/2} \longrightarrow \overset{\circ}{B}_{3/2}$ preserves the Morse foliation by circles.

Let $\rho : \overset{\circ}{B}_{3/2} \longrightarrow \overset{\circ}{B}_{3/2}$ be a C^r diffeomorphism preserving the Morse foliation by circles with:

$$\rho(x) = \begin{cases} x & \text{near } \|x\| = 3/2 \\ \varphi_1\psi_1^{-1}(x) & \text{near } \|x\| = 1. \end{cases}$$

Now define a C^r diffeomorphism

$$R : M_\varphi^{c_1\#c_2} \longrightarrow M_\psi^{c_1\#c_2} \quad \text{by}$$

$$R(x) = \begin{cases} \varphi_1^{-1}\rho\varphi_1(x) & x \in \varphi_1^{-1}(\overset{\circ}{B}_{3/2} \setminus \overset{\circ}{B}_1) \\ x & \text{otherwise.} \end{cases}$$

then R is the required diffeomorphism and

$$R^* \mathcal{F}_\psi^{c_1\#c_2} = \mathcal{F}_\varphi^{c_1\#c_2} .$$

<u>Proposition 5.3:</u> Let $\mathcal{F}$ be a Morse foliation without holonomy of class C^r $(r \geqslant 2)$ on M_g, the oriented 2-manifold of genus g.

Let l be a closed leaf of $\mathcal{F}$ without singular points and let $c_1(l), c_2(l)$ be the additional centres in $\mathcal{F}_l$.

Then $(M_l, \mathcal{F}_l)$ depends up to C^r conjugacy only on the C^r conjugacy class of $(M_g, \mathcal{F})$.

Conversely if M, c_1, c_2 are as in definition 5.5 let $l(c_1, c_2)$ be a closed leaf in $H(S^1 \times (-\frac{1}{2}, \frac{1}{2}))$.

Then $(M^{c_1\#c_2}, \mathcal{F}^{c_1\#c_2})$ depends only on the C^r-conjugacy class of $(M, \mathcal{F})$.

Further up to C^r-conjugacy:

$$((M_1)^{c_1\# c_2},(\mathfrak{F}_1)^{c_1\# c_2}) \qquad = (M_g,\mathfrak{F})$$

$$((M^{c_1\# c_2})_{l(c_1,c_2)},(\mathfrak{F}^{c_1\# c_2})_{l(c_1,c_2)}) \qquad = (M,\mathfrak{F}).$$

Proof: The proof is entirely routine.

In the course of the proof of the decomposition theorem we shall need the following lemma:

Lemma 5.4.1: Let $\mathfrak{F}$ be a C^r $(r \geq 2)$ Morse foliation on the sphere with no leaf containing more than one saddle point. If $\mathfrak{F}$ has more than two centres then it has at least two centres of type 1.

Proof: Suppose that $\mathfrak{F}$ has σ saddle points, where $\sigma \geq 1$. Then $\mathfrak{F}$ has $\sigma+2$ centres.

If $\sigma = 1$ $\mathfrak{F}$ has two centres of type 1 and one of type 2.

If $\sigma > 1$, suppose inductively that the result is true for Morse foliations with $\sigma-1$ saddle points.

Let c be a centre of $\mathfrak{F}$ of type 2.

Let U_c be the cylinder associated to c as in definition 4.7.

If one or both of the boundary components of U_c consists of a single centre then clearly $\mathfrak{F}$ has a centre of type 1. Otherwise U_c has two boundary components and we can choose circle leaves l_1, l_2 in U_c approximating ∂U_c.

Gluing in centres along l_1 and l_2 produces three Morse foliated spheres.

One of these spheres contains c, two "glued in" centres and one saddle point.

Each of the other two spheres contains exactly one "glued in" centre and at least one & at most $\sigma-1$ saddle points.

The result now follows by induction.

<u>Corollory</u>: Let $\mathcal{F}$ be a C^r $(r \geqslant 2)$ Morse foliation of M_g, the oriented 2-manifold of genus g, with no holonomy and no leaf containing more than one saddle point.

Suppose that $\mathcal{F}$ has at least one closed leaf and that every closed leaf of $\mathcal{F}$ is homotopic to zero.

Then $\mathcal{F}$ has at least one centre of type 1.

<u>Proof</u>: Let l be any closed leaf of $\mathcal{F}$.

Glue in centres along l.

Since l is homotopic to zero, at least one of the resulting manifolds is a sphere.

If this sphere contains two centres of type 1, the result follows.

Otherwise l lies in the disc D_c associated to some centre c.

If c is of type 1 the result follows.

If c is of type 2, ∂D_c is a figure of eight and there is a closed leaf l' approximating one loop of this figure and lying outside D_c.

Now l' cannot lie in a disc $D_{c'}$ associated to a centre c' of type 2.

Repitition with l' of the above argument for l then gives the desired result.

The Decomposition Theorem.

Proposition 5.4.2: Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy and no leaf containing more than one saddle point.
Then, up to C^r-conjugacy, $(M_g, \mathfrak{F})$ is uniquely constructed as follows.
Take two (not necessarily connected) closed 2-manifolds each Morse foliated with no holonomy and no leaf containing more than one saddle point.
Suppose that the first manifold has every leaf closed and that the second has only those leaves lying near a centre of type 1 closed.
Then $(M_g, \mathfrak{F})$ is constructed by gluing centres of the first Morse foliated manifold to centres of the second.

Explicitly, $(M_g, \mathfrak{F})$ is uniquely constructed as follows:
Choose Morse foliations $(M_{h_i}, \mathcal{G}_i)$, $(M_{g_j}, \mathfrak{F}_j)$ with $0 \leqslant i \leqslant s$, $0 \leqslant j \leqslant t$ integers and $h_i > 0$, $g_j \geqslant 0$ without holonomy such that $\mathcal{G}_i$ has no leaf closed and $\mathfrak{F}_j$ has every leaf closed.
Use proposition [illegible].6.3 to add k_i centres of type 1 to k_i distinct non-singular leaves of $\mathcal{G}_i$.
Then $(M_g, \mathfrak{F})$ is obtained by gluing a_{ij} centres of $\mathcal{G}_i$ to centres of $\mathfrak{F}_j$ $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$.

Conversely such a process will produce a Morse foliation of M_g with σ centres provided the following constraints are satisfied:

a) $\sum_{i=1}^{s} a_{ij} \leqslant \sigma_j$, $\sum_{j=1}^{t} a_{ij} \leqslant k_i$ where $\mathcal{F}_j$ has σ_j centres.

b) $\sigma_j = 0$ if & only if $s=0$, $t=1$.

c) $g_j = 0$, $\sigma_j = 2 \Longrightarrow a_{ij} \neq 0$ for at least two distinct values of i.

d) $\sigma = \sum_{i=1}^{s} k_i + \sum_{j=1}^{t} \sigma_j - 2\sum_{i=1}^{s}\sum_{j=1}^{t} a_{ij}$

$g = 1 + \sum_{i=1}^{s}(h_i - 1) + \sum_{j=1}^{t}(g_j - 1) + \sum_{i=1}^{s}\sum_{j=1}^{t} a_{ij}$.

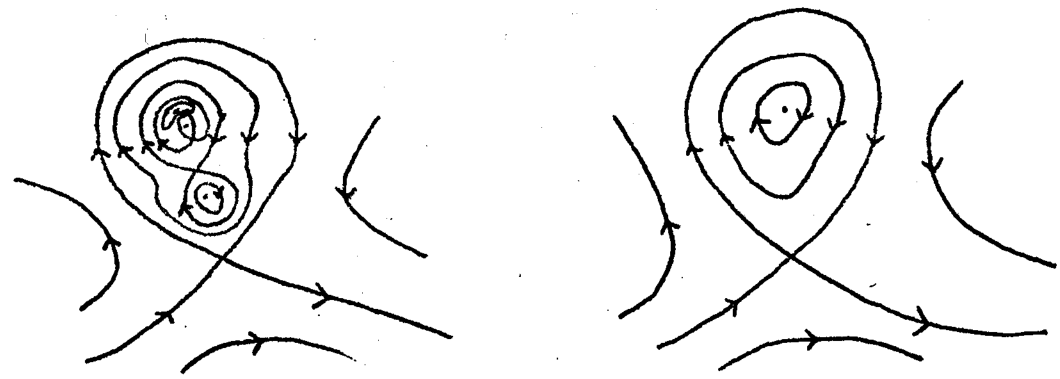
e) To ensure connectedness we require:

given i_1, i_2 $1 \leqslant i_1, i_2 \leqslant s$ $\exists$ $j'_1, \ldots\ldots, j'_{p-1}$;
$i'_1, \ldots\ldots, i'_p$ with
$i_1 = i'_1$, $i_2 = i'_p$ and $a_{i'_v j'_v} \neq 0$ $a_{i'_{v+1} j'_v} \neq 0$ $1 \leqslant v \leqslant p-1$.

Proof: If $g=0$ the result is proved so we assume $g > 0$.
We first locate the manifolds $(M_{h_i}, \mathcal{G}_i)$. This is done by cutting along closed leaves and gluing in centres. Following such a proceedure we end up with the $(M_{h_i}, \mathcal{G}_i)$ with centres of type one added, except that the discs foliated with single centres are replaced by some more general Morse foliation of the disc (see figure 5.2).

Choose in M_g a maximal collection of closed leaves $l_1, \ldots, l_r$ representing linearly independent homology classes.

Glue in centres along $l_1, \ldots, l_r$ thus obtaining a Morse



Initial decomposition Final decomposition

Fig. 5.2

foliation of M_{g-r} with $\sigma+2r$ centres, in which every circle leaf is homologous to zero.

Now cut along circle leaves homologous to zero but not homotopic to zero and glue in centres until this can no longer be done.

This gives $\mathfrak{F}_1,\ldots\ldots,\mathfrak{F}_n$ Morse foliations on oriented 2-manifolds $M_{\gamma_1},\ldots\ldots,M_{\gamma_n}$ with a total of $\sigma+2r+2n-2$ centres and $\gamma_1+\ldots\ldots+\gamma_n = g-r$.

Note that M_{γ_i} has every leaf closed if & only if $\gamma_i = 0$ and $i=n=1$. For we can remove all centres of type 1 and then by lemma 5.4.1 the resultant foliation either has no saddle points, in which case $\gamma_i = 0$ or has no closed leaf, which is impossible. But $\gamma_i = 0$ and $i > 1$ implies that M_{γ_i} was obtained by cutting along a circle leaf homotopic to zero.

Thus either M_g has every leaf closed or for each i, M_{γ_i} has a non-closed leaf, and thus we may assume the latter.

We now show that each closed leaf l is contained in an open disc D_l Morse foliated by closed leaves whose boundary is a saddle point together with a single loop separatrix contained in a non-closed leaf (see figure 5.2). In fact since any such leaf l is contained in an open disc Morse foliated by closed leaves we may choose D_l maximal.

Since $\gamma_i \neq 0$ the boundary of D_l is a union of saddle points and loop separatrices. Since D_l is maximal the boundary of D_l is as required.

We are now ready to undertake the decomposition.

Let c_{ij} $0 \leqslant j \leqslant t_i$ denote the distinct sets ∂D_l in M_{γ_i} satisfying one of the two additional conditions:

(i) D_l contains more than one centre.

(ii) D_l contains a centre which was glued in at some stage in the decomposition.

Then c_{ij} $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant t_i$ correspond to well defined sets d_{ij} in M_g which consist of a saddle point and a loop separatrix contained in a non-closed leaf.

Now choosing closed leaves d'_{ij} approximating d_{ij} and gluing in centres in $\mathfrak{J}$ decomposes $(M_g,\mathfrak{J})$ into Morse foliated manifolds $(M_{g_i},\mathfrak{J}_i)$ $1 \leqslant i \leqslant t$ and $(M_{\gamma_i},\mathcal{G}'_i)$ $1 \leqslant i \leqslant n$ without holonomy in which $\mathfrak{J}_i$ has every leaf closed, and $\mathcal{G}'_i$ is the Morse foliation on M_{γ_i} obtained above but with each disc D_1 replaced with a disc foliated by circles and a single centre.

Now by proposition 4.6.3 since every centre of $\mathcal{G}_i^!$ is of type 1 we may remove it and removing every centre from $\mathcal{G}_i^!$ in this way yields a Morse foliation $\mathcal{G}_i$ without closed leaf and without holonomy.

Setting $h_i = \gamma_i$, $n = s$,we have the required decomposition.

We now prove uniqueness.

First note that from the proof that the given decomposition of $(M_g,\mathfrak{F})$ is clearly the unique one up to C^r diffeomorphism for it was obtained by gluing in centres along closed leaves 1 which are well defined up to U_1 - the maximal annulus containing 1 which is foliated by circles.

Now if f is a C^r diffeomorphism of M_g,the decomposition is obtained by gluing in centres along closed leaves 1 of $f^*\mathfrak{F}$.

But then the decomposition of $\mathfrak{F}$ is obtained by gluing in centres along the circles fl - leaves of $\mathfrak{F}$.

Hence f defines a diffeomorphism of the factors obtained in the decomposition of $f^*\mathfrak{F}$ onto those obtained in the decomposition of $\mathfrak{F}$.

This completes the proof of uniqueness.

Chapter 6 Morse foliations with all leaves closed.

We saw in the last chapter that any Morse foliation without holonomy can be decomposed into Morse foliations without holonomy and either with every leaf closed or with no leaf closed. This decomposition respects C^r conjugacy. Thus in studying the C^r conjugacy classes of Morse foliations without holonomy we need only consider these two restricted cases.

The case of no closed leaf is complicated and not yet fully understood. It will be considered in subsequent chapters. In the present chapter we consider Morse foliations with no holonomy and all leaves closed (see 2.15 for an explanation of why these conditions are both included), up to C^r conjugacy. These are relatively managable. In the first proposition we consider the number of C^r conjugacy classes and in the second we consider some invariants for these.

Proposition 6.1: Let $\mathfrak{F}$ be a Morse foliation on the oriented 2-manifold M_g of genus g, without holonomy, with no closed leaf and with no leaf containing more than one saddle point. Then there is a Morse foliation $\mathfrak{F}_0$ on the sphere with no holonomy and every leaf closed such that $\mathfrak{F}$ is obtained by gluing together g pairs of centres of $\mathfrak{F}_o$.

Proof: By lemma 5.2.2 there must be g closed leaves $l_1, \ldots, l_g$ of $\mathfrak{F}$ which are linearly independent in $H_1(M_g, \mathbb{Z})$. Cutting along them and gluing in centres proves the desired result.

Corollary: Let $(M_g, \mathfrak{F})$ be as in the statement of the proposition. Then for a fixed number of saddle points there are only finitely many C^r-conjugacy classes of such Morse foliations (see proposition 4.4).

Proof: From the proposition we see that it is sufficient to prove the result for the sphere.

Now given a Morse foliation of the sphere without holonomy either there is a centre of type 1, by lemma 5.4.1 or there are no saddle points.

In any case, we can use proposition 4.6.3 to successively remove the centres of type 1 and we eventually arrive at the unique Morse foliation (up to C^r conjugacy) with no saddle points.

In the reverse procedure we successively add centres of type 1 to circle leaves l. At each stage, the C^r-conjugacy class depends only on U_l (see definition 4.5.1).

Since there are only finitely many such sets, the result is proved.

Proposition 6.2: Let $\mathcal{F}$ be a C^r ($r \geqslant 2$) Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, every leaf closed and no leaf containing more than one saddle point.
Then the number n of non-zero homology classes represented by the closed non-singular leaves of $\mathcal{F}$ is a C^r-conjugacy invariant of $\mathcal{F}$ and satisfies

$$g \leqslant n \leqslant 3g-3 \qquad g \geqslant 2$$
$$n = g \qquad g = 0,1.$$

Further each such value of n is attained for any predetermined number of centres.

If $g \leqslant 2$ and the number of centres is minimal, n is a complete invariant.
If $g \geqslant 3$ this is not the case.
Proof: First note that by lemma 4.5.2 the homology class of a circle leaf depends only on U_1. Indeed if c is a centre of type two, the homology class of a leaf l in U_c which is not homologous to zero (see 4.7 for definition of U_c) depends only on U_c.

To show that $n \leqslant 3g-3$ we first remove all centres of type 1 from $\mathcal{F}$ using proposition 4.6.3 and this does not alter n. In the resulting Morse foliation we take a maximal collection $U_1, \ldots, U_m$ of pairwise disjoint open cylinders in the manifold such that

either $U_i = U_l$ for some closed leaf l which is not contained in any set U_c

or U_i is a maximal connected union of intersecting sets U_c (see figure 6.1) .

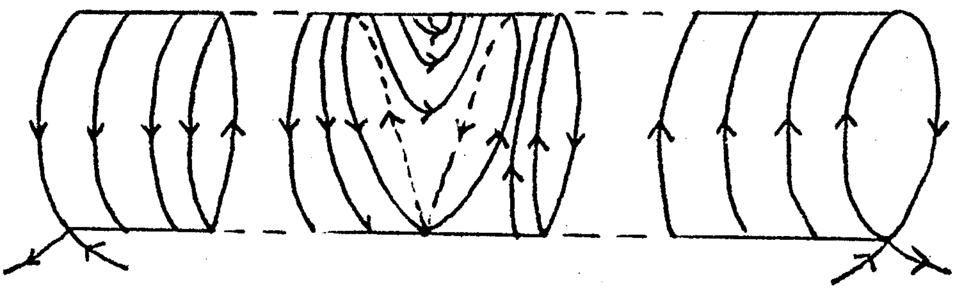


Fig. 6.1 U_i

Then every circle leaf l lies in some U_i and if c is a centre $\overline{D}_c \subseteq U_i$ for some i.

Further the homology class of each circle leaf l in U_i which is not homotopic to zero depends only on U_i.

Now each component of the boundary of U_i consists of one or two loop separatrices and a saddle point. In this way each U_i is associated to one third of two saddle points (possibly the same) in the complement of $U_1 \cup \ldots \cup U_m$.

Hence $n \leqslant m = (3/2)(2g-2) = 3g-3$.

That $n \geqslant g$ is clear since if $l_1, \ldots, l_v$ are closed leaves representing a maximal linearly independent set in $H_1(M_g)$ then cutting along $l_1, \ldots, l_v$ in succession and gluing in centres, produces by lemma 5.4.1 a Morse foliation on the sphere.

Hence $r = g$.

We now show that any value of n in the given range can be attained.

This is done by induction.

We define operations which add 1 to the genus of the manifold and 1,2 or 3 to n.

To add 1 to n, glue a torus foliated with a single centre, which is of type 1 to a centre of type 1 added to a circle leaf of $\mathfrak{F}$ as in figure 6.2.

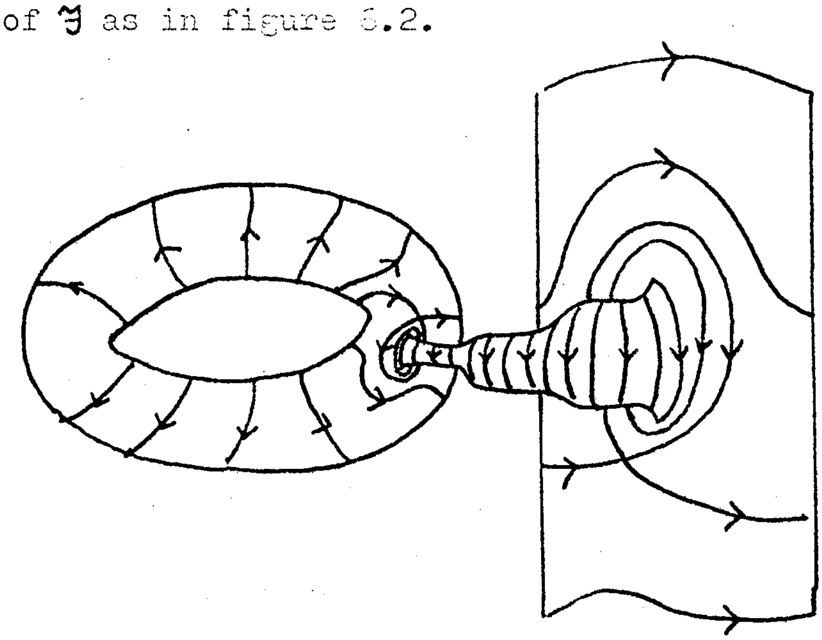


Fig. 6.2

To add 2 to n add centres of type 1 and opposite Morse indices to circle leaves $1, 1'$ with $U_1 = U_{1'}$, and glue them together.

This works provided $g \neq 0$, see figure 6.3.

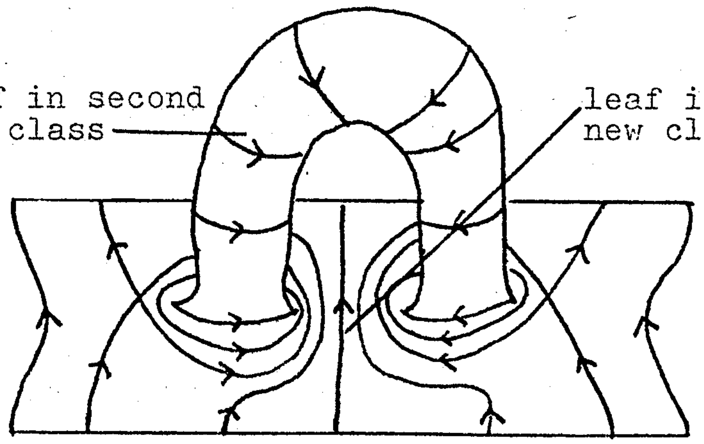
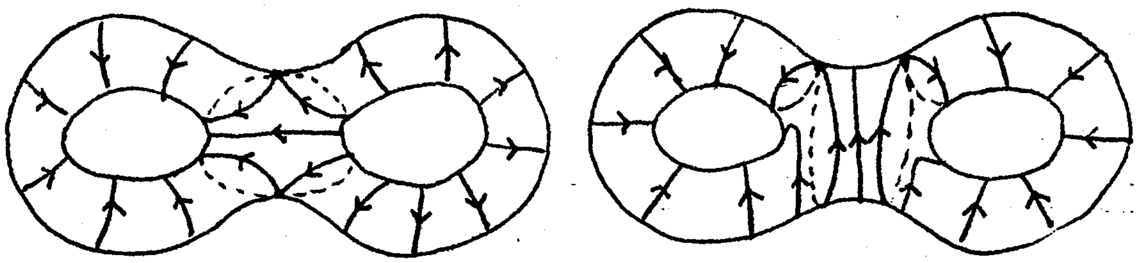


Fig. 6.3

To add 3 to n we do the same as in the case for adding two except thet l and l' are chosen to represent linearly independent homology classes. This works provided $g \geqslant 2$.

Now proposition 4.4 gives the result if g=0 or 1 and it then follows for $g > 1$ by the preceding remarks and induction.

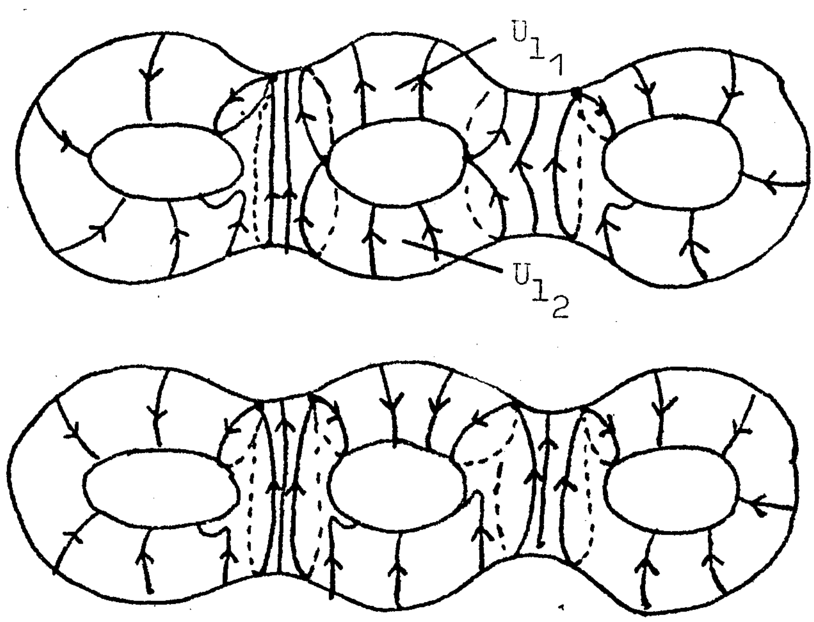
Finally we wish to show that n is classifying for $g \leqslant 2$ but not for $g > 2$ if the number of centres is minimal. This follows from proposition 4.4 for g = 0 or 1. For g=2 the two classes are shown in figure 6.4.



n=3 n=2

Fig. 6.4

For $g > 2$ we have to find two non-conjugate Morse foliations with no holonomy, every leaf closed and the closed leaves representing the same number of distinct homology classes. This is left to the reader, but an example in genus 3 with $n = 3$ is indicated in figure 6.5.



In first picture there are two U_1 representing non-zero homology classes with ∂U_1 two halves of different figures of eight (U_{1_1} & U_{1_2}).

Fig. 6.5

Chapter 7. Geometric structure of Morse foliations with no closed leaf.

From now on we consider only Morse foliations with no holonomy, no closed leaf and no leaf containing more than one saddle point. The first two of these conditions imply that there are no loop separatrices since any loop separatrix has nearby closed leaves. The no holonomy assumption is necessary since the latter two assumptions do not preclude a loop separatrix on a non-closed leaf from having holonomy. The three conditions together are equivalent to the single condition that the induced foliation of the non-singular manifold has no closed leaf.

Lemma 7.1.1: Let $\mathfrak{F}$ be a C^r $(r \geq 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, and suppose that $\mathfrak{F}$ satisfies the conditions immediately above.
Then there are outward separatrices $s_1, \ldots, s_{k_{\mathfrak{F}}}$ such that:
(i) $\{\omega(s_1), \ldots, \omega(s_{k_{\mathfrak{F}}})\}$ is the set of distinct minimal (under the ordering by inclusion) elements of:

$$\delta = \{\omega(s) : s \text{ is an outward separatrix}\} .$$

(ii) $s_i \subseteq \omega(s_i)$.
(iii) $\omega(s_i) \cap \omega(s_j)$ consists only of saddle points if $i \neq j$.
(iv) $\omega(s_i) \cap \omega(s_j) \cap \omega(s_w) = \emptyset$ if $i \neq j \neq w \neq i$.
(v) Either $k_{\mathfrak{F}} = 1$ and $\omega(s_1) = M_g$ or $\omega(s_i)$ is a closed non-empty nowhere dense set which meets every transverse interval in a perfect set.

(vi) If l is any non-singular leaf or outward separatrix then for some i $\omega(s_i) \subseteq \omega(l)$ and if $l \subseteq \omega(s_i)$ $\omega(l) = \omega(s_i)$.

Proof: Note first that if l is any non-singular leaf or outward separatrix, there is a saddle point p in $\omega(l)$ by the theorem of A.J.Schwartz. Since l is not a loop separatrix l passes through some quadrant at p infinitely many times as $t \longrightarrow \infty$.

It follows that $\omega(l)$ contains at least one inward and at least one outward separatrix.

Now let $\tilde{s}_1, \ldots, \tilde{s}_{k_\vartheta}$ be outward separatrices such that

$$\{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_{k_\vartheta})\}$$

is the complete set of distinct minimal elements of ϑ.

Let $s_i \subseteq \omega(\tilde{s}_i)$ be an outward separatrix.

Then the minimality of $\omega(\tilde{s}_i)$ implies that $\omega(s_i) = \omega(\tilde{s}_i)$.

We show that $\omega(s_1), \ldots, \omega(s_{k_\vartheta})$ have the required properties.

(i) and (ii) are satisfied by definition.

(iii) follows from the minimality.

(iv) follows from the fact that at least two of the separatrices at a saddle point in $\omega(s_i)$ also lie in $\omega(s_i)$.

(vi) follows from the choice of the sets $\omega(s_i)$ and the fact that for any non-singular leaf or outward separatrix l there is an outward separatrix s with $s \subseteq \omega(s) \subseteq \omega(l)$ (and if $l \subseteq \omega(s_i)$ we may take $s = s_i$).

It remains to prove (v).

First note that if $\omega(s_1) = M_g$ then $k_{\mathcal{F}} = 1$ by (iii).
Suppose that $\omega(s_i)$ is a proper subset of M_g.
Clearly $\omega(s_i)$ is non-empty and closed.
Further $s_i \subseteq \omega(s_i)$ implies that any transverse interval meets $\omega(s_i)$ in a perfect set.
Now $\omega(s_i)$ is a union of non-singular leaves, saddle points and separatrices. If $\partial\omega(s_i)$ contains a non-singular leaf or separatrix $\partial\omega(s_i) = \omega(s_i)$ by minimality and hence $\omega(s_i)$ is nowhere dense.
Otherwise $\partial\omega(s_i)$ consists of finitely many saddle points.
However this is impossible since a finite number of points cannot separate a 2-manifold.
This completes the proof.

The technical lemma which follows is in fact true for any Morse foliation $\mathcal{F}$ in which no leaf contains more than one saddle point. The proof in the general case is essentially the same as the case we give.

Lemma 7.1.1½: Let $\mathcal{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point.
Let T_1, T_2, T_3 be open transverse intervals with $\overline{T}_3 \subseteq T_2$.
Let $p < q$ be points of T_1 and suppose that every leaf cutting (p,q) subsequently cuts T_3.
Then:
either (i) The non-singular leaf or separatrix through p subsequently cuts T_2 (see figure 7.0(i)).

or (ii) p lies on an inward separatrix that never subsequently cuts T_2. There is an outward separatrix in the same leaf which cuts T_2, as in figure 7.0(ii).

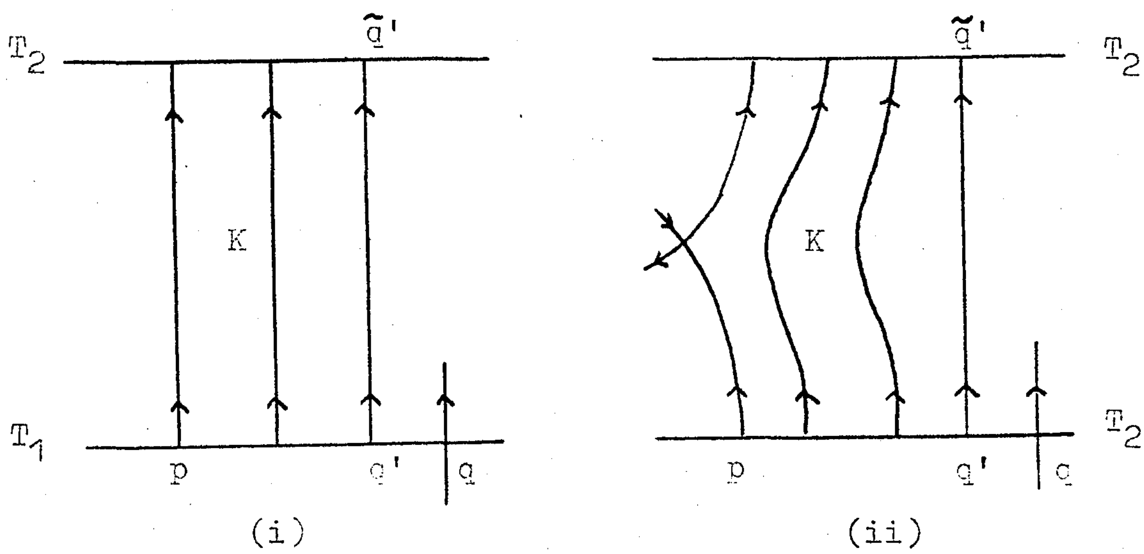


Fig. 7.0

Proof: Choose $q' \in T_1$ with $p < q' < q$ and a transverse vector field $X_g^{\perp}$ containing T_1 & T_2 in trajectories.

Suppose that the non-singular leaf or outward separatrix through q' cuts T_2, for the first time after passing through q', at a point $\tilde{q}'$.

Choose a parametrisation $\gamma(t)$ of the portion of this leaf between q' and $\tilde{q}'$ by the unit interval $[0,1]$.

The holonomy lemma then determines a C^r-map

$$H : [0,1] \times (p,q'] \longrightarrow M_g$$

such that (i) $H(t,q') = \gamma(t)$

(ii) $H(0,x) = x \in T_1$

(iii) $H(1,x) \in T_3 \subseteq T_2$

(iv) For each point x of $(p,q']$ the set $H([0,1] \times \{x\})$ is a single segment of leaf.

Let $K = H([0,1] \times (p,q'])$ be the image of H, as in figure 7.0.

Suppose that the lemma is false.

If p lies on a non-singular leaf or outward separatrix l then l never subsequently cuts T_2.

If p lies on an inward separatrix, there is an outward separatrix l lying in the same leaf as p_2 which never cuts T_2 as in figure $7.\frac{1}{2}$.

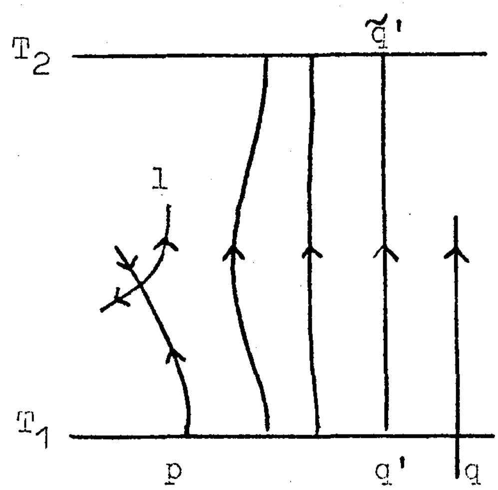


Fig. $7.\frac{1}{2}$

In either case it is clear that $\omega(l) \subseteq \overline{K}$.

Let m be a non-singular leaf or outward separatrix in $\omega(l)$.

By lemma 7.1.1 we can assume $m \neq l$.

If m meets K it cuts $(p,q']$.

Now m cannot cut (p,q') since then $m \subseteq \omega(l)$ would imply that l cuts (p,q') infinitely many times.

By lemma 7.1.1 $\omega(l)$ is large enough that we may in fact assume that m does not meet $\overline{K}$ and hence, in particular, $[p,q']$.

Let $x \in m$ and let W be a transverse interval at x.

Without loss of generality, shrinking T_1 & T_2 if necessary,

W is contained in a distinguished chart at x which does not meet T_1 or T_2.
We may also assume that l limits on x from the right.
Since points of l subsequent to p bound K, K meets W between each pair of points of intersection of l with W.
Since we may then assume that the right hand end of W is in the complement of K and since also W does not meet T_1 or T_2, W meets every segment in K between any pair of points of intersection of l with W.
Hence each segment of K contains a sequence of points with limit x.
But clearly any such limit point must lie on a segment of K.
This contradicts the fact that m does not meet K and hence proves the lemma.

<u>Lemma 7.1.2</u>: Let $\mathfrak{Z}$ and $s_1,\dots,s_{k_{\mathfrak{Z}}}$ be as in lemma 7.1.1 and let $\Omega = \omega(s_1) \cup \dots \cup \omega(s_{k_{\mathfrak{Z}}})$.
Let T' be a transverse interval and let T be an open transverse interval with $\overline{T} \subseteq T'$.
Suppose that the endpoints of T lie in Ω but $\Omega \cap T = \emptyset$.
Then any leaf meets T only finitely many times.
<u>Proof</u>: Suppose that $T' = (-1,1)$ and $T = (a,b)$ where $-1 < a < b < 1$, as in figure 7.1.
Let $p_1 < \dots < p_d$ denote the points at which those inward separatrices which cross (a,b) cross it for the last time.
Suppose that there is a leaf m which cuts (p_i, p_{i+1}) at a

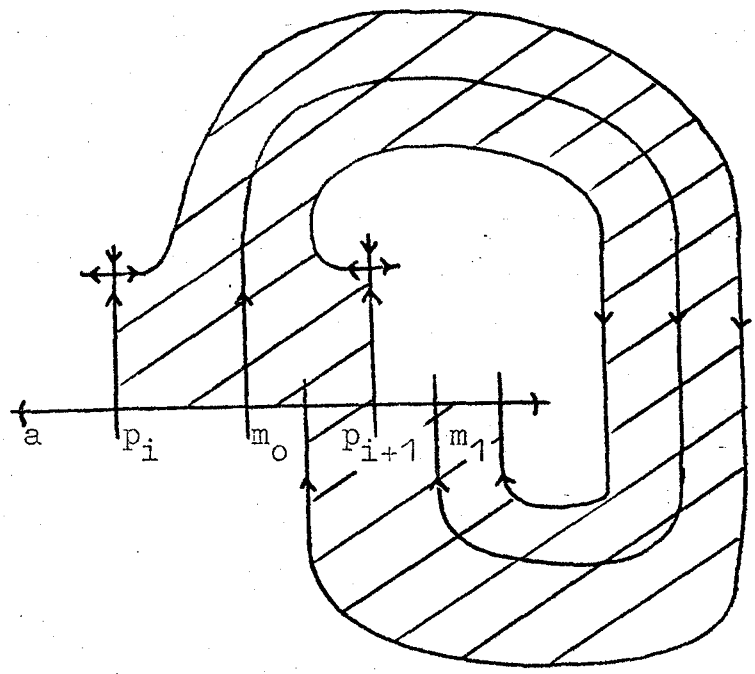


Fig. 7.1

point m_o and which then returns to (a,b) at a point m_1. Then every leaf near m_o returns to (a,b). In fact since no leaf cutting (p_i, p_{i+1}) runs into a separatrix before cutting (p_i, p_{i+1}) again lemma 7.1.$\frac{1}{2}$ shows that every leaf which cuts (p_i, p_{i+1}) returns to (a,b) as in figure 7.1.

Now parametrise the segment of m from m_o to m_1 by m_t ($t \in [0,1]$) and choose a transverse vector field $X_g^{\perp}$ which has T as part of a trajectory.

Then the holonomy lemma provides a map

$$H : [0,1] \times (p_i, p_{i+1}) \longrightarrow M_g$$

such that:

(i) $H(0,x) = x$,

(ii) $H(t,m_o) = m_t$,

(iii) $H(t,x)$ lies in a leaf which depends only on x ,

(iv) each segment $H([0,1] \times \{x\})$ meets (a,b) only at $H(0,x) = x$ and $H(1,x)$.

Let H_i be the closure of the image of H (shaded in figure 7.1).

If no leaf cutting (p_i, p_{i+1}) cuts (a,b) again let $H_i = \emptyset$.
Note also that since no leaf is closed either $p_1 = a$ or no leaf cutting (a,p_1) cuts (a,b) again. The same is true of b.

We have now shown that any leaf m which cuts (a,b) infinitely many times remains permanently in $H_1 \cup \ldots \cup H_d$ after its first crossing of (a,b).
Since $(a,b) \cap \Omega = \emptyset$ figure 7.1 shows that $H_1 \cup \ldots \cup H_d$ meets Ω only in that part of the boundary of $H_1 \cup \ldots \cup H_d$ which is made up from segments of leaf.
Hence $\omega(m) \cap \Omega$ is the union of a finite number of saddle points and separatrices and therefore meets T in a countable set.
On the other hand by lemma 7.1.1 (vi) there is an i such that $\omega(s_i) \subseteq \omega(m) = \omega(m) \cap \Omega$. Thus by lemma 7.1.1 (v) $\omega(m) \cap \Omega \cap T$ contains a perfect set which by a well-known theorem of Cantor must be uncountable.
This is a contradiction.

<u>Corollory</u>: If l is any non-singular leaf or outward separatrix of $\mathfrak{F}$ then

$$\omega(l) = \omega(s_{i_1}) \cap \ldots \cap \omega(s_{i_e})$$

for some integers i_j $\quad 1 \leqslant j \leqslant e \quad 1 \leqslant i_j \leqslant k_{\mathfrak{J}}$.

Proof: First note that $\omega(l) \subseteq \omega(s_1) \cup \ldots \cup \omega(s_{k_{\mathfrak{J}}})$ since if m is a leaf in $\omega(l) \setminus (\omega(s_1) \cup \ldots \cup \omega(s_{k_{\mathfrak{J}}}))$ by considering $\omega(m)$ we can find a transverse interval T about m satisfying the hypotheses of the lemma. Then since $m \in \omega(l)$ l cuts T infinitely many times contradicting the lemma.

Further, the choice of the sets $\omega(s_i)$ shows that if $\omega(l)$ meets $\omega(s_i)$ for some i then either $\omega(s_i)$ is contained in $\omega(l)$ or $\omega(l) \cap \omega(s_i)$ consists just of saddle points and each of these lies in some set $\omega(s_j)$ which is contained in $\omega(l)$.

This completes the proof of the corollory.

Lemma 7.1.3: Let $\mathfrak{J}$; $s_1, \ldots, s_{k_{\mathfrak{J}}}$ be as in lemma 7.1.1 and let $\Omega = \omega(s_1) \cup \ldots \cup \omega(s_{k_{\mathfrak{J}}})$.

Let T' be a transverse interval and T an open transverse interval with $\overline{T} \subseteq T'$ whose endpoints are in Ω but which does not meet Ω .

Suppose that there is a non-singular leaf or outward separatrix which cuts T at distinct points m_0, m_1.

Then there is a point of T between m_0 and m_1 lying on an inward separatrix.

Proof: We prove the result by supposing that some non-singular leaf or outward separatrix m cuts T at points m_0, m_1 between which no point lies on an inward separatrix.

We show that this implies that m cuts T infinitely many times, contradicting the conclusion of lemma 7.1.2.

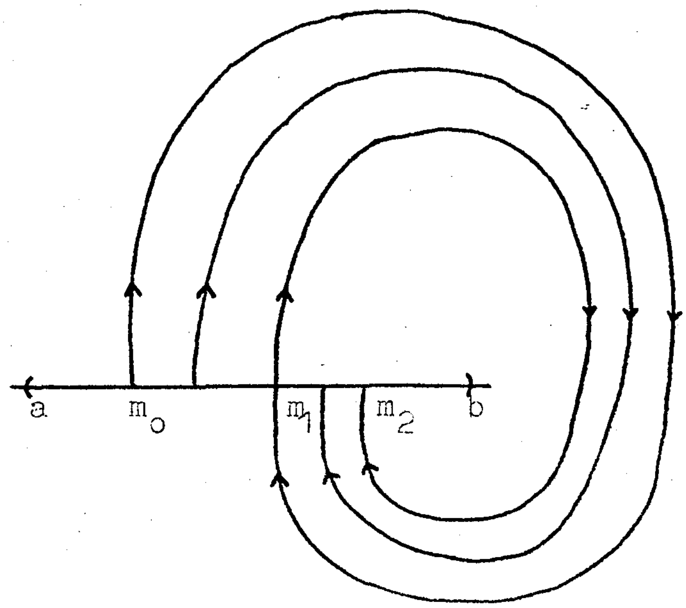


Fig. 7.2

Let T = (a,b) and suppose that $m_o < m_1$.
Suppose that m cuts T first at m_o and then at m_1.
As in figure 7.2 every leaf near m_o cutting (m_o,m_1) returns to (a,b) in the interval (m_1,b).
Since $m \cap \Omega = \phi$, a&b$\in\Omega$ and there are no inward separatrices cutting (m_o,m_1), every leaf cutting (m_o,m_1) returns to (a,b) at a point of (m_1,b) by lemma 7.1.$\frac{1}{2}$.
By continuity and the fact that $m \cap \Omega = \phi$, m cuts T for a third time at a point m_2 as in figure 7.2.
The hypotheses of the lemma show that we can repeat this argument for (m_1,m_2). Thus we obtain a sequence of distinct points $\{m_i\}_{i \geqslant 0}$ of $m \cap T$ contradicting lemma 7.1.2.

Lemma 7.1.4: Let $\mathcal{F}$; $s_1,\dots,s_{k_{\mathcal{F}}}$ be as in lemma 7.1.1 and let $\Omega = \omega(s_1)\cup\dots\cup\omega(s_{k_{\mathcal{F}}})$.

Let l be any non-singular leaf or outward separatrix which does not lie in Ω.

Parametrise l by l_t.

Then there is a real number t_o, a closed transverse interval I at l_{t_o} whose endpoints are in distinct leaves m_1, m_2 of $\omega(s_i)$ for some i & whose interior is in the complement of $\omega(s_1)\cup\dots\cup\omega(s_{k_{\mathcal{F}}})$ and a diffeomorphism

$$H : [t_o,\infty)\times I \longrightarrow M_g$$

with the properties:

(i) $H(t_o,x) = x$,

(ii) $H(t,l_{t_o}) = l_t$,

(iii) $H(t,x)$ lies in a single leaf for x fixed,

(iv) $H_t : I \longrightarrow H_t I$ is a diffeomorphism of I onto a transverse interval about l_t with the same properties as I.

Proof: Let $p\in\omega(l)$ be a non-singular point and let $T = (-1,1)$ be a transverse interval at p (with p corresponding to 0).

Without loss of generality we can assume that there is a sequence of points of $(0,1)\subseteq T$ on l tending to p as $t\to\infty$.

We shall construct inductively a sequence of distinct intervals $I_n = (a_n,b_n)\subseteq T$ tending to p from the right with $a_n, b_n \in\Omega$, $(a_n,b_n)\cap\Omega=\emptyset$ and (a_n,b_n) meeting l, as follows (see figure 7.3).

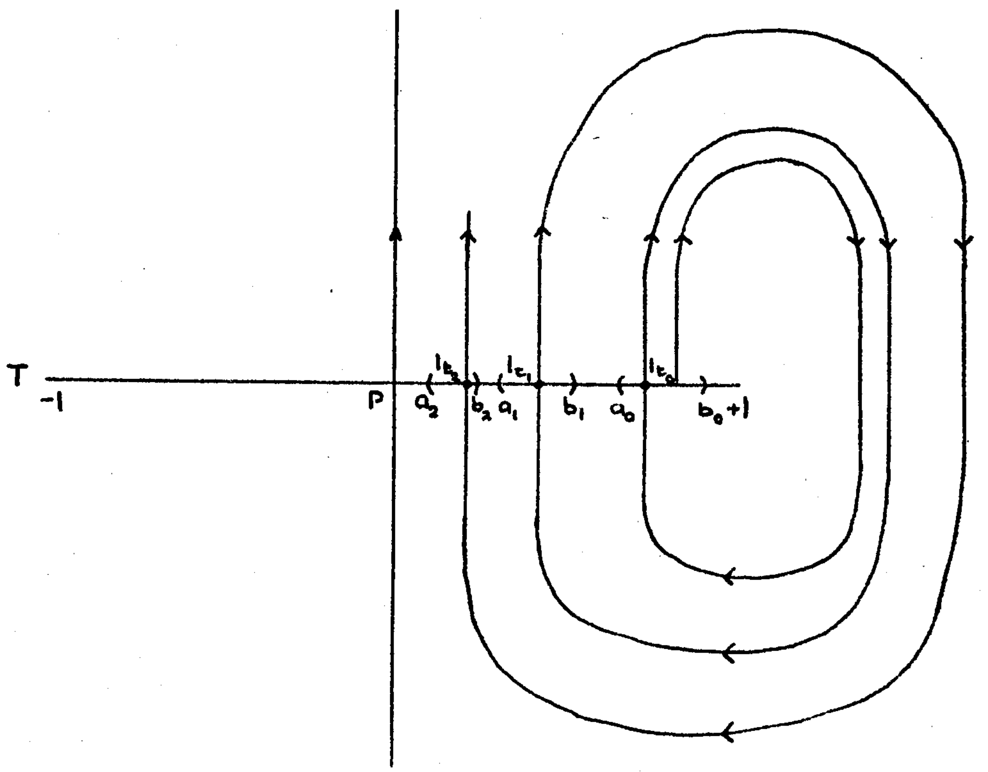


Fig. 7.3

We take I_0 to be any interval with the properties just mentioned which contains a point of l.

Suppose that I_n has been chosen.

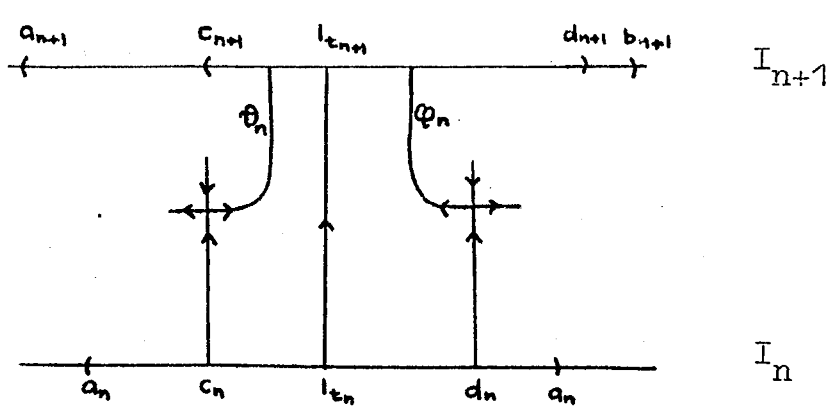


Fig. 7.4

By lemma 7.1.2 only finitely many points of (a_n, b_n) lie on inward separatrices. Thus we may choose points c_n, d_n of (a_n, b_n) with the following properties:

(i) l cuts (a_n, b_n) for the last time in (c_n, d_n) at a point l_n ,

(ii) no point of (c_n, d_n) lies on an inward separatrix ,

(iii) either $c_n = a_n$ or c_n lies on an inward separatrix ,

(iv) either $d_n = b_n$ or d_n lies on an inward separatrix.

Note that by lemma 7.1.3 c_n and d_n lie on distinct separatrices.

If c_n lies on an inward separatrix we let p_n be the saddle point that this separatrix runs into. Similarly q_n is the saddle point corresponding to b_n.

Note that if p_n and q_n both exist then by the choice of c_n & d_n and lemma 7.1.3, $p_n \neq q_n$.

If c_n lies on an inward separatrix let θ_n be that outward separatrix at p_n near which there are leaves that emanate from (c_n, d_n) as shown in figure 7.4. If c_n does not lie on an inward separatrix, so that $c_n = a_n$,we let θ_n be the leaf through a_n. Similarly choose φ_n corresponding to d_n as shown in figure 7.4.

Then we can choose I_{n+1} to have the following properties:

(i) $0 < a_{n+1} < b_{n+1} < a_n < b_n < 1$ since l tends to p from the right and (c_n,d_n) contains no inward separatrices ,

(ii) $I_{n+1} \cap \Omega = \emptyset$,

(iii) a_{n+1} , $b_{n+1} \in \Omega$,

(iv) Θ_n and φ_n cut I_{n+1} ,

(v) If Θ_n or φ_n cuts I_n it cuts I_{n+1} after it has cut I_n.

By the holonomy lemma and the fact that no inward separatrix cuts (c_n,d_n) every leaf cutting (c_n,d_n) subsequently cuts I_{n+1} and cuts it for the last time in an interval whose endpoints lie on Θ_n & φ_n and which is contained in (c_{n+1},d_{n+1}) ,as in figure 7.4.

We show that for sufficiently large n $a_n = c_n$, $b_n = d_n$ and a_n & b_n lie on a non-singular leaf or outward separatrix.

Let K_n be the number of points in (c_n,d_n) lying on an outward separatrix.

From figure 7.4 we see that $K_{n+1} \geqslant K_n$ and $K_{n+1} = K_n$ if & only if $a_n=c_n$, $b_n=d_n$ and a_n&b_n both lie on an outward separatrix or non-singular leaf.

If K_n increases without limit it follows from the fact that there are only finitely many separatrices and lemma 7.1.2 that for sufficiently large n there are two points of (c_n,d_n) lying on the same outward separatrix.

Since no point of (c_n,d_n) lies on an inward separatrix this contradicts lemma 7.1.3 .

Hence there is an integer N such that $n \geq N$ implies that $a_n = c_n$, $b_n = d_n$ and a_n & b_n lie on a non-singular leaf or outward separatrix.

Further for $n \geq N$ no non-singular leaf or separatrix cuts $[a_n, b_n]$ more than once by lemma 7.1.3, the fact that no point of $[a_n, b_n]$ lies on an inward separatrix and the fact that no leaf in Ω is isolated.

Now let $I = I_N$.

Let l cut I_N at l_{t_o} and I_{N+n} at l_{t_n} .

Then the holonomy construction with respect to a fixed transverse vector field in which I is part of a trajectory gives a diffeomorphism

$$H_n : [t_n, t_{n+1}] \times [a_{N+n}, b_{N+n}] \longrightarrow M_g$$

for each $n \geq 0$ such that:

(i) $H_n(t_n, x) = x$,

(ii) $H_n(t, l_{t_n}) = l_t$,

(iii) $H_n(t,x)$ lies on a leaf which depends only on x.

The image of H_n is shown in figure 7.5.

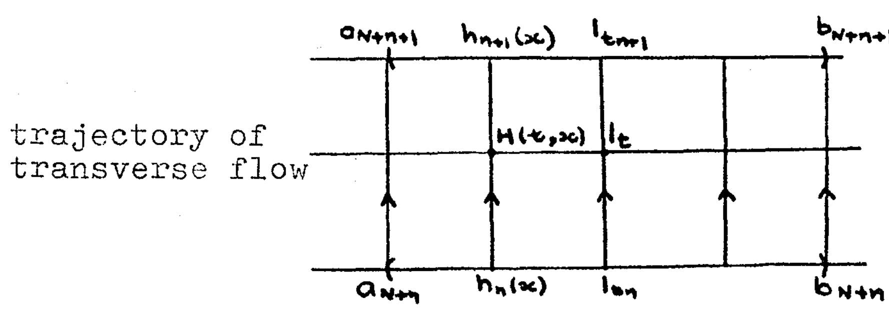


Fig. 7.5

Let $h_i : [a_N, b_N] \longrightarrow [a_{N+i}, b_{N+i}]$ be defined inductively by: $h_o(x) = x$

$h_{i+1}(x) = H_i(t_{i+1}, h_i(x))$.

Note that $h_i(l_{t_o}) = l_{t_i}$.

Then the required diffeomorphism

$$H : [t_o, \infty) \times I \longrightarrow M_g$$

is given by $\quad H(t,x) = H_n(t, h_n(x)) \qquad t \in [t_n, t_{n+1}]$.

We have already remarked that the endpoints of I lie on distinct leaves and by construction these lie in Ω.

The only non-obvious point remaining to be checked is that m_1 and m_2 (which lie in Ω) both lie in the same set $\omega(s_i)$.

Now $\omega(m_1) = \omega(s_{i_1})$ and $\omega(m_2) = \omega(s_{i_2})$ for some integers i_1 & i_2, however it is easy to see that any non-singular leaf or outward separatrix meeting the image of H has the same ω-limit set as m_1 or m_2 . Hence $i_1 = i_2$.

Since $m_j \subseteq \omega(s_{i_j})$ this completes the proof.

<u>Proposition 7.1.5:</u> Let $\mathfrak{F}$ be a C^r ($r \geqslant 2$) Morse foliation with no closed leaf, no leaf containing more than one saddle point and no holonomy on M_g, the oriented 2-manifold of genus g.

Let $s_1, \ldots, s_{k_{\mathfrak{F}}}$ be outward separatrices as in lemma 7.1.1. Then the ω-limit set of any non-singular leaf or outward separatrix is one of the sets $\omega(s_i)$.

Similarly inward separatrices $t_1,\dots,t_{k_{\mathfrak{F}}}$ can be chosen satisfying analogous properties to those of $s_1,\dots,s_{k_{\mathfrak{F}}}$ for negative time and with $\alpha(t_i) = \omega(s_i)$.

Hence in particular the α-limit set of any non-singular leaf or inward separatrix is one of the sets $\omega(s_i)$.

Proof: Let l be any non-singular leaf or outward separatrix. If $l \subseteq \omega(s_i)$ for some i then $\omega(l) = \omega(s_i)$ by 7.1.1 (vi). Otherwise $l \nsubseteq \omega(s_1) \cup \dots \cup \omega(s_{k_{\mathfrak{F}}})$ and lemma 7.1.4 gives a whole strip of leaves about l bounded by leaves m_1 and m_2 lying in $\omega(s_i)$ for some i. Hence in this case too $\omega(l) = \omega(s_i)$.

The existence of inward separatrices $t_1,\dots,t_{e_{\mathfrak{F}}}$ satisfying analogous properties to those of $s_1,\dots,s_{k_{\mathfrak{F}}}$ is obvious by reversing time.

It remains to show that $e_{\mathfrak{F}} = k_{\mathfrak{F}}$ and that $\omega(s_i) = \alpha(t_i)$ after reordering.

Now for each i $\alpha(t_i)$ meets each transverse interval in a perfect set. Hence $\alpha(t_i)$ contains a non-singular leaf l. By minimality $\omega(l) \subseteq \alpha(t_i) = \alpha(l)$.

Now there is an integer j_i with $\omega(l) = \omega(s_{j_i})$ hence $\omega(s_{j_i}) \subseteq \alpha(t_i)$.

Reversing time the same argument also shows that every set $\omega(s_i)$ contains some set $\alpha(t_{f_i})$.

The result then follows from the minimality of these sets.

Corollary: If $\mathfrak{F}$ satisfies the hypotheses of the proposition and some leaf of $\mathfrak{F}$ is dense then every leaf is dense.

Proof: Immediate from the proposition.

A similar result for a general flow on a 2-manifold can be found in [30] under the additional hypothesis that the ω-limit sets meet the non-singular manifold in a compact set, a condition which is never met in our case.

7.2 Transverse circles and the bound $k_{\mathfrak{F}} \leq g$.

Consider a Morse foliation $\mathfrak{F}$ of class C^r ($r \geq 2$) with no holonomy, no closed leaf and no leaf containing more than one saddle point on the oriented 2-manifold M_g of genus g. Let $s_1, \ldots, s_{k_{\mathfrak{F}}}$ be the outward separatrices defined in lemma 7.1.1. $k_{\mathfrak{F}}$ is a C^r-conjugacy invariant of $\mathfrak{F}$.
It follows from the results in [15] (where 2g-1 is mis-printed for g) or [23] that $k_{\mathfrak{F}} \leq g$. We can see this as follows.

Choose $p \in s_1$ a non-singular point and T a transverse interval at p which does not meet the closed set
$\omega(s_2) \cup \ldots \cup \omega(s_{k_{\mathfrak{F}}})$.
Now s_1 meets T again at subsequent time at a point q.
As usual in foliations theory by taking a small strip of segments about the segment of s_1 from p to q we can construct a transverse circle A_1 meeting only leaves which cross the interval (p,q) of T as in figure 7.6.

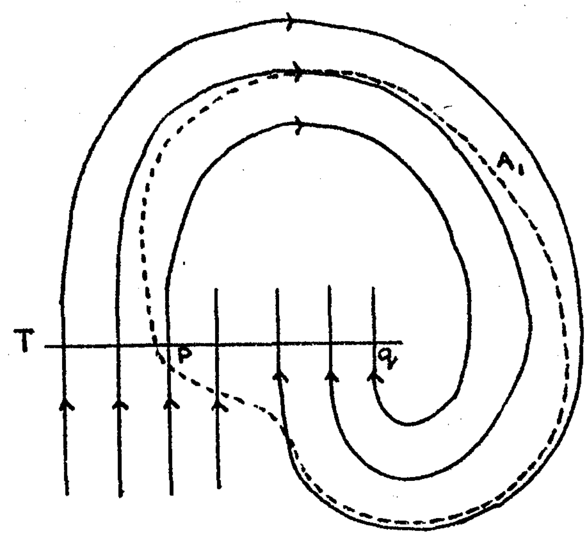


Fig. 7.6

By a similar method, taking the strip sufficiently small, we can then construct inductively transverse circles A_i $1 \leqslant i \leqslant k_g$ such that $A_i \cap A_j = \phi$ $i \neq j$ and such that every non-singular leaf or separatrix in $\omega(s_i)$ cuts A_i infinitely many times.

Now let $l_i \subseteq \omega(s_i)$ be a non-singular leaf and suppose that l_i cuts A_i at successive times t_0, t_1 at points p_0, p_1 and at no time t with $t_0 < t < t_1$.
By the holonomy lemma there is a strip of segments containing the segment of l_i from p_0 to p_1 in its interior and in which each segment cuts A_i exactly twice - once at each end. By the choice of the A_i we may choose this strip so small that no segment in it meets any other transverse circle A_j.

Using this strip we may then construct a second transverse circle B_i which does not cut any other circle A_j $j \neq i$ and cuts A_i transversely exactly once, as in figure 7.7. Choosing B_i inductively we can assume that $B_i \cap B_j = \emptyset$ $i \neq j$.

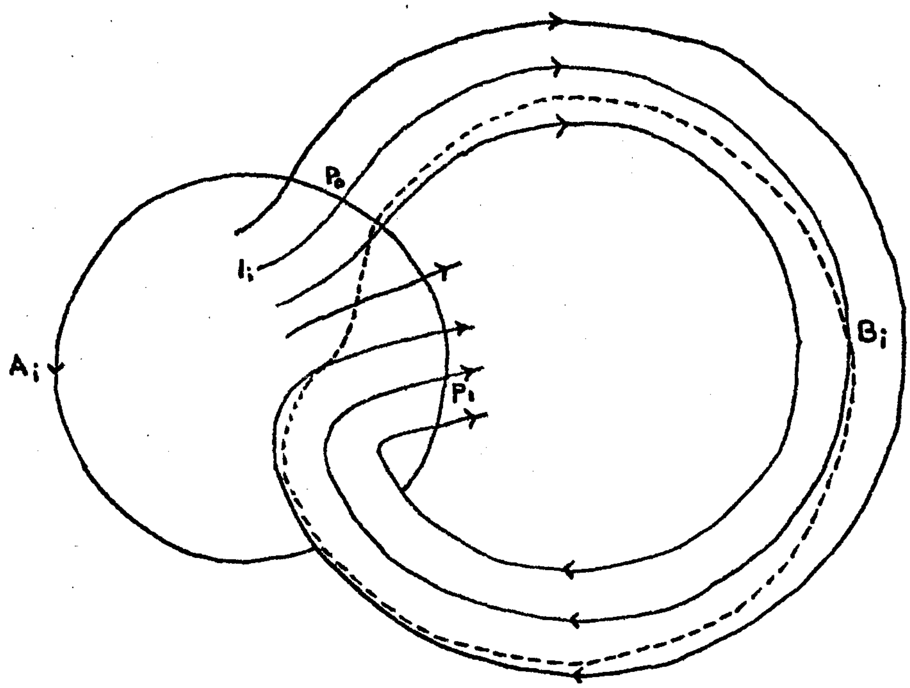


Fig. 7.7

Then orienting the transverse circles so that the pairs (tangent to circle, tangent to $\mathfrak{F}$) lie in the orientation of M_g, the homology classes $[A_i]$, $[B_i]$ (see 2.2) satisfy:

$$[A_i] \cdot [B_j] = -\delta_{ij}$$
$$[A_i] \cdot [A_j] = [B_i] \cdot [B_j] = 0 .$$

It follows that the classes $[A_1], \ldots, [A_{k_{\mathfrak{F}}}]$ and $[B_1], \ldots, [B_{k_{\mathfrak{F}}}]$ are linearly independent in $H_1(M_g, \mathbb{Z})$.

Hence $2k_{\mathfrak{F}} \leqslant \dim H_1(M_g, \mathbb{Z}) = 2g$.

The definition of these classes was somewhat arbitrary. However in certain cases at least, we shall see in chapter 8 that it is possible to define $k_{\mathfrak{F}}$ classes in $H_1(M_g, \mathbb{R})$ which in some sense carry all the information of the possible classes $[A_i]$ and $[B_i]$. In order to do this we need to study the point of first return function and this is defined below.

7.3.1 The point of first return function.

Let $\mathfrak{F}$ be a Morse foliation on M_g with no holonomy, no closed leaf and no leaf containing more than one saddle point.

Let A be a transverse circle containing no saddle point. Orient A so that the pairs (tangent to A, tangent to $\mathfrak{F}$) lie in the orientation of M_g. Let $p_1, \ldots, p_n$ be the last points at which the inward separatrices of $\mathfrak{F}$ cut A, in order around A.

It follows from the holonomy lemma that if some leaf cutting (p_i, p_{i+1}) returns to A then so does every leaf which meets (p_i, p_{i+1}) (we identify p_1 & p_{n+1}).

Let $i_1, \ldots, i_u$ be those i for which every leaf in (p_i, p_{i+1}) returns to A.

Then there is a function

$$f : \bigcup_{j=1}^{u} (p_{i_j}, p_{i_j+1}) \longrightarrow A$$

defined by taking for $x \in (p_{i_j}, p_{i_j+1})$ $f(x)$ to be the point of A at which the leaf through x next cuts A.

f is called the <u>point of first return function</u>.

Similarly we have the point of previous return function.

<u>Lemma 7.3.2</u>: Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on the oriented 2-manifold M_g of genus g as in 7.3.1.

Let A be a transverse circle to $\mathfrak{F}$, and f the point of first return function on A. Then:

(i) f is C^r,

(ii) $\lim_{x \to p_i^+} D^s f(x)$ and $\lim_{x \to p_{i+1}^-} D^s f(x)$ $s \leqslant r$ exist if f is defined on (p_i, p_{i+1}).

(iii) There is a real number $L > 0$ such that $|Df(x)| \geqslant L$ for all $x \in \operatorname{dom} f$.

<u>Proof</u>: Away from a saddle point the overlap maps are of the form $\quad (x,y) \longmapsto (h_1(x,y), h_2(y))$

and so f is locally the composition of a finite number of h_2's. This proves part (i).

Now the holonomy past a saddle point is essentially the identity map. Hence f extends to a diffeomorphism in a neighbourhood of an endpoint of any interval in the domain of f.

This observation proves parts (ii) and (iii).

Chapter 8. Measure and Holonomy.

In chapter 7 we considered the elementary properties of Morse foliations with no holonomy and no closed leaf. However that chapter left unanswered a number of questions of a general nature:

1) Is it true (in the notation of 7.2) that $k_{\mathfrak{F}} = 1$ if & only if every leaf is dense ?

More generally:

2) Is it true that $\omega(l) = \alpha(l)$ for a non-singular leaf l if & only if $l \subseteq \omega(l)$?

3) Is it true that the asymptotic cycle of a leaf l, as defined for example in [32] depends only on $\omega(l)$?

In the remainder of this thesis we propose first to give solutions to all these questions although in a restricted sense. If we place an additional restriction on our Morse foliations it turns out that the first question is inappropriate but that the second and third questions can be answered in the affirmative. We then apply these results to 2-manifolds of genus 2. The resulting analysis gives a method for constructing a large number of examples of Morse foliations without holonomy or a closed leaf on 2-manifolds of genus two or higher. Other research workers looked for such examples without success so that our examples are the first of their kind.

Definition 8.1.1: Let $\mathfrak{F}$ be a Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point. Then $\mathfrak{F}$ has order preserving holonomy if the point of first return function f on any transverse circle A is order preserving. In other words if a,b are distinct points in the domain of f and A has been given an orientation then:

$$f([a,b] \cap \mathrm{dom} f) = [f(a),f(b)] \cap \mathrm{im} f \ .$$

This condition is automatically satisfied if $\mathfrak{F}$ has no singular points since then the domain of $\mathfrak{F}$ is the whole of A and as $\mathfrak{F}$ is transversely oriented f is orientation preserving. On the other hand the condition is also highly restrictive:

Lemma 8.1.2: Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point.
Then $k_{\mathfrak{F}} = 1$ and $\mathfrak{F}$ has order preserving holonomy if & only if g = 1.
Proof: Consider a transverse circle A which meets the set $\omega(s_1)$ (as defined in 7.1.5) and let f be the point of first return function on A.
Suppose that $k_{\mathfrak{F}} = 1$ so that $\omega(s_1)$ is the unique ω-limit set for $\mathfrak{F}$.

Then every leaf cuts A infinitely many times.

Since every point of A which is not the last point of intersection of an inward separatrix with A lies on a leaf which returns to A, the domain of f is of form:

$$(a_1,a_2)\cup(a_2,a_3)\cup\cdots\cdots\cup(a_{4g-4},a_1)\ .$$

If f has order preserving holonomy f has range

$$(b_1,b_2)\cup(b_2,b_3)\cup\cdots\cdots\cup(b_{4g-4},b_1)$$

where $\qquad f(a_i,a_{i+1}) = (b_i,b_{i+1})$.

This implies that f extends to a continuous function defined on all of A.

If $g>1$ each point a_i lies on an inward separatrix and if f is order preserving f must be discontinuous at a_i (see figure 8.1).

Hence $g = 1$.

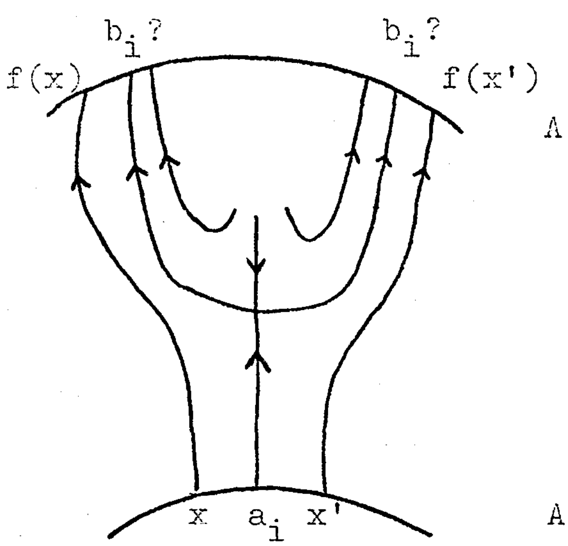


Fig. 8.1

Conversely if $g = 1$ calculation of the Euler characteristic shows that $\mathfrak{F}$ has no singular points and it is then clear that $\mathfrak{F}$ has order preserving holonomy.

Proposition 8.1.3: Let $\mathcal{F}$ be a C^r $(r \geq 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point.

Suppose that $\mathcal{F}$ has order preserving holonomy.

Then if l is a non-singular leaf of $\mathcal{F}$.

$$\omega(l) = \alpha(l) \Longleftrightarrow l \subseteq \omega(l).$$

Proof: Clearly $l \subseteq \omega(l) \Rightarrow \omega(l) = \alpha(l)$ by 7.1.1 (vi).

We prove the other implication by contradiction.

In the notation of 7.1.1 and 7.1.5, suppose that there is a non-singular leaf l_o with $\omega(l_o) = \alpha(l_o)$, but $l_o \not\subseteq \omega(l_o)$.

Then $\omega(l_o) = \omega(s_i)$ for some i and there is a transverse circle A_i as in section 7.2 which intersects leaves in $\omega(s_j)$ if & only if $i = j$.

We consider the point of first return function f on A_i and obtain a contradiction using a method analogous to that of Siegel [33].

Give A_i an orientation.

It follows from lemma 7.1.4 applied for both positive and negative time that if $p_o \in A_i \cap l_o$ there is an interval (x_o, y_o) (x_o & y_o are not generally in $\omega(s_i)$) about p_o in A_i meeting only non-singular leaves whose ω and α-limit set is $\omega(s_i)$ but which do not lie in $\omega(s_i)$.

Further the interval (x_o, y_o) can be chosen so small that it satisfies:

a) $(x_o,y_o) \subseteq \bigcap_{m \in \mathbb{Z}} \mathrm{domf}^m$.

b) The transverse intervals $f^m(x_o,y_o)$ are mutually disjoint. This can be achieved if (x_o,y_o) is sufficiently small since l_o is not closed and if $|m|$ is large the intervals $f^m(x_o,y_o)$ are automatically disjoint by lemma 7.1.4.

c) Lemma 7.1.4 implies that for sufficiently large n there are intervals (u_n,v_n) and (u_{-n},v_{-n}) in A_i with endpoints in $\omega(s_i)$ such that:

(i) $f^n(x_o,y_o) \subseteq (u_n,v_n) \subseteq \bigcap_{m \geqslant 0} \mathrm{domf}^m$ and

$f^m(u_n,v_n) = (u_{m+n},v_{n+m})$,

(ii) $f^{-n}(x_o,y_o) \subseteq (u_{-n},v_{-n}) \subseteq \bigcap_{m \leqslant 0} \mathrm{domf}^{-m}$ and

$f^{-m}(u_{-n},v_{-n}) = (u_{-m-n},v_{-m-n})$,

(iii) the intervals $f^m(u_n,v_n)$ $m \geqslant 0$ are mutually disjoint ,

(iv) the intervals $f^{-m}(u_{-n},v_{-n})$ $m \leqslant 0$ are mutually disjoint ,

(v) every point in (u_n,v_n) lies on a non-singular leaf or outward separatrix l with $\omega(l) = \omega(s_i)$ and every point in (u_{-n},v_{-n}) lies on a non-singular leaf or inward separatrix l with $\alpha(l) = \omega(s_i)$ by the assumption on $\omega(l_o)$ and lemma 7.1.4.

Fix q large enough for the intervals (u_q,v_q) and (u_{-q},v_{-q}) to be defined.

Let $g = f^q$ and $g^m(x_o,y_o) = (x_m,y_m)$ $m \in \mathbb{Z}$.

Then: (i) The intervals (x_m,y_m) $m \in \mathbb{Z}$ are mutually disjoint and contained in $\bigcap_{n\in\mathbb{Z}} \mathrm{dom}g^n$,

(ii) $(x_m,y_m) \subsetneq (u_{qm},v_{qm})$ if $m\neq 0$.

(iii) We leave the reader to check that $\omega(s_i)\cap A_i$ is the accumulation set of $\{g^m(x) : m \geqslant 0\}$ if $x \in \bigcap_{m\geqslant 0} \mathrm{dom}g^m$ and of $\{g^m(y) : m \leqslant 0\}$ if $y \in \bigcap_{m\leqslant 0} \mathrm{dom}g^m$.

We show that for N sufficiently large, there is an integer $n > N$ such that either all the intervals (x_{-k},y_{n-k}) or all the intervals (x_{n-k},y_{-k}) $k = 1,\dots,n$ are disjoint.

This condition means that if we choose a point p_m in (x_m,y_m) for each m with $m \leqslant n$ then in the ordering of the points p_m on A_i p_{n-k} either appears immediately after p_{-k} for all k $(1\leqslant k\leqslant n)$ or immediately before p_{-k} for all k $(1\leqslant k\leqslant n)$. (*)

There is a unique integer m_o such that $1 \leqslant |m_o| \leqslant N$ and the interval (x_o,x_{m_o}) contains no point x_j with $0\leqslant |j|\leqslant N$.

Condition (ii) immediately above shows that u_{qm_o} lies in the interval (x_o,x_{m_o}). The fact that u_{qm_o} is in $\omega(s_i)$ and condition (iii) immediately above shows that there is an integer h' with $|h'| > N$ and $x_{h'} \in (x_o,x_{m_o})$.

Let h be an integer with $|h| > N$, $x_h \in (x_o,x_{m_o})$ and $|h|$

minimal (if h and $-h$ satisfy this criterion choose h if x_h is nearest x_o and $-h$ otherwise).

I claim that if $h > 0$ all the intervals $(x_{-k}, y_{|h|-k})$ $k = 1,\ldots,|h|$ are disjoint and if $h < 0$ all the intervals $(x_{|h|-k}, y_{-k})$ $k = 1,\ldots,|h|$ are disjoint.

If $h > 0$ and the claim is false there are integers k_1, k_2 with $1 \leqslant k_1 \neq k_2 \leqslant |h|$ and $x_{-k_2} \in (x_{-k_1}, y_{|h|-k_1})$. Then $x_{k_1-k_2} \in (x_o, y_h)$ (this the only place that we use the fact that f and hence g is order preserving) and so $k_1 = k_2 + h$ contradicting $1 \leqslant k_1 \leqslant |h|$ and $k_2 \geqslant 1$.

If $h < 0$ and the claim is false there are integers k_1, k_2 with $1 \leqslant k_1 \neq k_2 \leqslant |h|$ and $x_{k_1-k_2+h} \in (x_o, y_h)$ which is again a contradiction.

This proves (*).

By lemma 7.3.2 and the fact that $g = f^q$, $\log Dg$ has bounded variation V.

Let N be any integer and n as in (*) above.

Then if δ_j = length of (x_j, y_j)

$$\delta_j = \delta_o Dg^j(\eta_j) \quad \text{for some } \eta_j \in (x_o, y_o).$$

Hence

$$\log \frac{\delta_o^2}{\delta_n \delta_{-n}} = \log \frac{1}{Dg^n(\eta_n) Dg^{-n}(\eta_{-n})}$$

$$= \log \frac{Dg^n(g^{-n}(\eta_{-n}))}{Dg^n(\eta_n)}$$

$$= \sum_{k=1}^{n} (\log Dg(g^{-k}(\eta_{-n})) - \log Dg(g^{n-k}(\eta_n)))$$

$$\leqslant \sum_{k=1}^{n} |\log Dg(g^{-k}(\eta_{-n})) - \log Dg(g^{n-k}(\eta_n))|$$

$$\leqslant V$$

by (*) and the bounded variation of logDg.

But $\lim\limits_{n\to\infty} \dfrac{\delta_o{}^2}{\delta_n\delta_{-n}} = +\infty$ since the intervals (x_j, y_j) are disjoint, which is a contradiction and hence proves the result.

<u>Definition 8.2</u>: Let $\mathfrak{F}$ be a Morse foliation on M_g.

A <u>transverse measure</u> μ on $\mathfrak{F}$ assigns to each transverse submanifold (open interval or circle) K, a Borel measure μ_K on K which is finite on compact sets.

A transverse measure μ is <u>holonomy invariant</u> if whenever K_1, K_2 are transverse submanifolds and the holonomy map

$$h : K_1 \longrightarrow K_2$$

is defined then

$$\mu_{K_1}(A) = \mu_{K_2}(h(A))$$

where $A \subseteq K_1$ is any μ_{K_1} measurable set.

We shall denote each measure μ_K by μ.

A point p lies in the <u>support</u> of a holonomy invariant transverse measure μ if for each transverse submanifold K containing p, $\mu(K) > 0$.

The support of a holonomy invariant transverse measure is a union of non-singular leaves and separatrices. Any point in the point set boundary of the support of such a measure μ is a saddle point in the ω or α limit set of a separatrix in the support of μ.

Lemma 8.3: Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point. Then for every non-singular leaf or outward separatrix l, there exists a holonomy invariant transverse measure μ on $\mathfrak{F}$ whose support has closure equal to $\omega(l)$.

Proof: Let $s_1, \ldots, s_{k_{\mathfrak{F}}}$ be outward separatrices whose ω-limit sets form the entire collection of ω-limit sets of leaves of $\mathfrak{F}$ as described in 7.1.1 and 7.1.5.

Let $A_1, \ldots, A_{k_{\mathfrak{F}}}$ be transverse circles such that

$$A_i \cap A_j = \emptyset \qquad \text{if } i \neq j$$

$$A_i \cap \omega(s_j) \neq \emptyset \qquad \text{if \& only if } i = j$$

as described in 7.2.

Without loss of generality $\omega(l) = \omega(s_1)$ and in order to define the required transverse measure μ, it is clearly sufficient to define measures on $A_1, \ldots, A_{k_{\mathfrak{F}}}$ which are invariant under the point of first return functions on each of these circles.

We take $\mu = 0$ on $A_2, \ldots, A_{k_{\mathfrak{F}}}$.

For each point $p \in \bigcap_{m \geqslant 0} \operatorname{dom} f^m$ we shall define a measure μ_p on A_1 which is invariant under the point of first return function. To do this we define a linear functional Λ_p on $C(A_1)$ (the continuous real valued functions on A_1) as follows.

A_1 is a compact metric space.

A standard application of the Stone-Weirstrass theorem then implies that $C(A_1)$ is separable.

Let $\varphi_1, \ldots, \varphi_n, \ldots$ be a countable dense subset of $C(A_1)$.

For each positive integer n choose a sequence $\{r_{n,m}\}_{m \geq 1}$ of positive integers such that

$$\lim_{m\to\infty} \frac{1}{r_{n,m}} \sum_{i=0}^{r_{n,m}-1} \varphi(f^i(p))$$

exists.

Then given $\varphi \in C(A_1)$ we let

$$\Lambda_p(\varphi) = \lim_{n\to\infty} \frac{1}{r_{n,n}} \sum_{i=0}^{r_{n,n}-1} \varphi(f^i(p))$$

The Riesz representation theorem then gives a measure μ_p on A_1. This is the unique positive measure satisfying

$$\Lambda_p(\varphi) = \int_{A_1} \varphi \, d\mu_p \,.$$

Clearly μ_p has support equal to $\omega(1) \cap A_1$ and one can check that μ_p is holonomy invariant.

We shall see below in proposition 8.4 that if the holonomy map is order preserving then the measure given by lemma 8.3 is unique up to multiplication by a positive real number.

It then follows that in this case

$$\Lambda_p(\varphi) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(p))$$

is well defined for all points p and independent of p.

Hence the measure μ_p is well defined for all points p.

I do not know whether proposition 8.4 is true in general. It may be that the recent example of a "non-uniquely ergodic interval exchange transformation" given in [40] will suggest a way of constructing a Morse foliation with at least two ergodic invariant measures which have the same support.

However the recent paper [39] on interval exchange transformations does suggest a way of proving that the number of ergodic measures is finite.

The proof of this given in lemma 8.4.0 below is closely modelled on that of [39].

<u>Definition 8.4.00</u>: A holonomy invariant measure μ is <u>ergodic</u> if for any set X which is a union of non-singular leaves and separatrices and any transverse submanifold T

$$\begin{array}{ll} \text{either} & \mu(X \cap T) = \mu(T) \\ \text{or} & \mu(X \cap T) = \emptyset \end{array}$$

Lemma 8.4.0: Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point.

Let ω be the ω-limit set of some leaf of $\mathfrak{F}$ and let A be a transverse circle meeting ω but no other distinct limit set.

Let e be the minimum of the number of inward and the number of outward separatrices whose limit set is ω.

Then there are at most $e + 1$ ergodic Borel measures μ on A which are invariant under the point of first return function on A and satisfy $\mu(A) = 1$.

Proof: Let $\Omega = \omega \cap A$.

Since every separatrix whose limit set is not ω meets A only finitely many times we can assume that a separatrix meets A if and only if it has limit set ω.

Let μ be an invariant measure on A.

Let $K_1, \ldots, K_m$ be disjoint invariant sets in A with

$$\mu(K_i) > 0 \qquad 1 \leqslant i \leqslant m.$$

We show that $m \leqslant e + 1$ and this then implies the result. For if $\mu_1, \ldots \mu_m$ are distinct ergodic measures, it follows from the ergodic theorem ([2]) that there are invariant sets $K_1, \ldots, K_m$ satisfying

$$\mu_i(K_j) = 1 \quad i=j$$
$$= 0 \quad i \neq j \ .$$

Then setting $\mu = m^{-1}(\mu_1 + \dots + \mu_m)$ gives the required result.

We first choose for each j $(1 \leqslant j \leqslant m)$ an interval L_j in A with

$$\mu(L_j \cap K_j) > \tfrac{3}{4}\mu(L_j) \qquad (*) \ .$$

The existence of L_j is implied by the fact that μ is a Borel measure and hence for some open set U_j containing K_j

$$\mu(U_j) < (4/3)\mu(K_j).$$

U_j is a countable disjoint union of open intervals and one of them must satisfy $(*)$.

Now choose $\delta > 0$ such that for any j $(1 \leqslant j \leqslant m)$ $(*)$ holds with L_j' replacing L_j for any subinterval L_j' of L_j with

$$\mu(L_j \setminus L_j') < 2\delta .$$

Let f be the point of first return function on A.

If $e = 1$ it is clear that f is order preserving and the lemma follows from proposition 8.4.

If $e > 1$ we can choose an open interval $I = (a,b)$ satisfying

(i) $I \cap \Omega \neq \emptyset$.

(ii) a and b lie on distinct inward separatrices.

(iii) a and b are the last points of intersection of the inward separatrices on which they lie with the closed interval $[a,b]$.

(iv) $\mu(I) < \delta$ (since μ is a regular Borel measure and a point has μ-measure 0).

We can achieve these properties by initially choosing any interval J with $J \cap \Omega \neq \emptyset$. Since Ω contains an inward separatrix we can choose the last point of intersection of this separatrix with J to be one of the endpoints of I. Since $e > 1$ we can arrange for the other endpoint of I to be as stated.

Now let e^+ be the number of inward separatrices meeting A. Let
$$p_1 < p_2 < \dots < p_{e^+}$$
be the last points of intersection of each inward separatrix with the open interval (a,b).

Let
$$I_o = (a, p_1)$$
$$I_j = (p_j, p_{j+1}) \quad 1 \leqslant j < e^+$$
$$I_{e^+} = (p_{e^+}, b) \qquad .$$

For each j :

either $\quad$ f is defined nowhere on I_j, in which case $I_j \cap \Omega = \emptyset$ and we set $t_j = 0$

or $\quad$ f is defined throughout I_j and we let $t_j > 0$ be minimal such that $f^{t_j}(I_j \cap \mathrm{dom} f^{t_j}) \cap I \neq \emptyset$.

Note the following properties:

(1) f^t is defined throughout I_j for all $t \leqslant t_j$ since otherwise for some $t \leqslant t_j$ there is a point $p \in f^t I_j$ which is the last point of intersection of an inward separatrix with A. Then $f^{-t}(p) \in I_j$ is the last point of intersection of this inward separatrix with I contradicting the definition of the intervals I_j.

(2) $f^{t_j} I_j \subseteq I$.

For otherwise a or b lies in $f^{t_j} I_j$.

Then, since a and b were chosen to be the last points of intersection of inward separatrices with the closed interval $[a,b]$, this implies that $f^{-t_j}a$ or $f^{-t_j}b \in I_j$ is the last point of intersection of an inward separatrix with I. This contradicts the definition of the intervals I_j.

(3) Given a point p in Ω which does not lie on a separatrix there is an integer j $(0 \leqslant j \leqslant e^+)$ and an integer t $(0 \leqslant t < t_j)$ for which $p \in f^t I_j$.

This follows from property (i) of I since this implies that we can choose an integer $s \geqslant 0$ to be minimal with $f^{-s}(p) \in I$. Then $f^{-s}(p) \in I_j$ for some j implies $p \in f^s I_j$ and the minimality of s implies $s < t_j$.

(4) Let S be the set of points in Ω which lie on a separatrix.

Let $\mathcal{U} = \{ I_0, \ldots, f^{t_0 - 1} I_0; \; \ldots\ldots \; ; I_{e+}, \ldots, f^{t_{e+}-1} I_{e+} \}$.

Then $\mathcal{U}$ is a cover of $\Omega \setminus S$ by pairwise disjoint intervals of measure less than δ. (In fact we can replace S by a finite set but we do not need this accuracy).

This is immediate from (3), the disjointness of the sets $I_0, \ldots, I_{e+}$ and the choice of the integers t_j.

Note that $\mu(S) = 0$ since S is countable and that the measure of the complement of Ω is 0 by lemma 7.1.4 and the fact that if the iterates of any set under f are disjoint then their union must have measure 0 or infinity.

It follows from (4) that since L_j is an interval there is a finite union X_j of intervals in $\mathcal{U}$ such that

$$X_j \subseteq L_j \quad \text{and} \quad \mu(L_j \setminus X_j) < 2\delta.$$

Then by the choice of δ

$$\mu(X_j \cap L_j) > \tfrac{3}{4}\mu(X_j).$$

Hence for some interval $f^t I_{k_j}$ in X_j

$$\mu(f^t I_{k_j} \cap K_j) > \tfrac{3}{4}\mu(f^t I_{k_j}) \qquad (**) \quad .$$

Hence since μ and K_j are invariant

$$\mu(f^s I_{k_j} \cap K_j) > \tfrac{3}{4}\mu(f^s I_{k_j}) \quad \text{for all } s \ (0 \leqslant s < t_j).$$

Since the sets K_j are disjoint (**) can hold for given I_{k_j} for at most one set K_j.

For each j choose k_j such (**) is satisfied.
Then the map $j \longmapsto k_j$ is injective.
Hence $m \leqslant e^+ + 1$.
Applying the same argument for the point of previous intersection function shows that

$$m \leqslant e + 1 \quad \text{as required.}$$

<u>Corollory</u>: Let $\mathfrak{Z}$ be as in the statement of the lemma. Let $\omega(s_1), \ldots, \omega(s_{k_{\mathfrak{Z}}})$ be the distinct non-trivial ω-limit

sets of non-singular leaves or outward separatrices of $\mathcal{F}$ as in section 7.1.

Let c_i be the minimum of the number of inward separatrices with α-limit set $\omega(s_i)$ and the number of outward separatrices with ω-limit set $\omega(s_i)$.

Then, up to multiplication by positive scalars:

1. The number of ergodic holonomy invariant transverse measures with support $\omega(s_i)$ is at most $e_i + 1$.
2. The number of ergodic holonomy invariant transverse measures is at most $4g - 4 + k_{\mathcal{F}}$.

<u>Proof</u>: The results are immediate from the proposition, the fact that the support of any ergodic measure must be some set $\omega(s_i)$ and the existence of transverse circles as in section 7.2.

In the following proposition we prove that if $\mathfrak{F}$ has order preserving holonomy then any holonomy invariant transverse measure with support the ω-limit set of a single leaf is ergodic.

The important property which is implied by the existence of order preserving holonomy is the following.

Given any transverse circle A which meets a single ω-limit set in a set Ω, any point p of Ω and any holonomy invariant transverse measure; then A can be approximated as closely as desired in measure by the disjoint union of iterates under the point of first return function of any small interval containing p.

<u>Proposition 8.4</u>: Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point.

Suppose that $\mathfrak{F}$ has order preserving holonomy.

Then given any non-singular leaf or outward separatrix there is, up to multiplication by positive real numbers, a unique holonomy invariant transverse measure μ on $\mathfrak{F}$ whose support has closure equal to $\omega(l)$.

<u>Proof</u>: By proposition 7.1.5 we can assume $\omega(l) = \omega(s_1)$.

Let A_1 be a transverse circle meeting $\omega(s_i)$ if & only if $i = 1$ as in 7.2 (or 8.3 above) and let f be the point of first return function on A_1.

The proposition is proved by showing that any transverse measure μ restricts to an ergodic measure on A_1 (i.e. to a measure in which every invariant set has measure 0 or $\mu(A_1)$) and then applying the ergodic theorem (see [2]).

Note first that by lemma 7.1.4 any interval of A_1 in the complement of $\omega(s_1)\cap A_1$ has measure 0 and also that any point of A_1 has measure 0. This is because if all the iterates of a set under f are disjoint then their union must have measure zero or infinity.

Suppose that $\omega(A_1) = 1$.

Let $R \subseteq A_1 \cap \mathrm{dom} f$ be an invariant set i.e. $R = f(R)$ then without loss of generality we may assume that $R \subseteq \omega(s_1) \cap A_1$.

We must show that $\mu(R) = 0$ or $\mu(R) = 1$.

Suppose $\mu(R) > 0$ and let $\varepsilon > 0$.

Since μ is a Borel measure on a compact set μ is regular (see [26] p.47) and since also $\omega(s_1) \cap A_1$ is totally disconnected we can find a sequence $\{I_j\}_{j \geqslant 0}$ of disjoint intervals in A_1 with

$$R \subseteq \bigcup_{j=0}^{\infty} I_j \ , \quad \sum_{j=0}^{\infty} \mu(I_j) < \frac{\mu(R)}{1-\varepsilon} \quad \text{and } \mu(I_j) < \varepsilon \quad \text{all } j \geqslant 0.$$

Then $\mu(R) = \sum_{j=0}^{\infty} \mu(R \cap I_j) > (1-\varepsilon) \sum_{j=0}^{\infty} \mu(I_j)$.

Thus one of the intervals I_j, say $I_j = I$ satisfies

$$\mu(R \cap I) > (1-\varepsilon)\mu(I) \quad \text{(hence in particular } \mu(I) > 0).$$

Since (i) there are only countably many points of $\omega(s_1) \cap A_1$

not in $\bigcap_{m \in \mathbb{Z}} \operatorname{domf}^m$,

(ii) $\omega(s_1) \cap A_1$ is perfect and

(iii) any interval in the complement of $\omega(s_1) \cap A_1$ has measure zero ,

we can assume that the endpoints of I lie in $\omega(s_1) \cap \bigcap_{m \in \mathbb{Z}} \operatorname{domf}^m$.

Orient A_1 and let $I = (a,b)$, with $a,b \in \omega(s_1) \cap \bigcap_{m \in \mathbb{Z}} \operatorname{domf}^m$.
Either every non-singular leaf or outward separatrix in $\omega(s_1)$ limits on b from the right or b is the left-hand endpoint of an open interval in the complement of $\omega(s_1) \cap A_1$. In either case we can find an integer n_1 such that $\mu(b, f^{n_1}(a))$ is as small as we please.
Then either $\mu(I) > \frac{1}{3}$ or we can find an integer n_1 such that (a,b) and $(f^{n_1}(a), f^{n_1}(b))$ are disjoint and

$$\mu(a, f^{n_1}(b)) < 3\mu(I).$$

We leave it to the reader to show similarly that if $(m+2)\mu(I) \leqslant 1$ we can choose inductively an integer $n_m > n_{m-1}$ such that the intervals

$$(a,b)\ ,\ (f^{n_1}(a), f^{n_1}(b))\ , \ldots\ldots,\ (f^{n_m}(a), f^{n_m}(b))$$

are disjoint and $\mu(a, f^{n_m}(b)) < (m+2)\mu(I)$.
This process stops when $m = N = [(\mu(I))^{-1}] - 2$.
Hence the intervals $f^{n_i}(I \cap \operatorname{domf}^{n_i}) = (f^{n_i}(a), f^{n_i}(b)) \cap \operatorname{imf}^{n_i}$ (where $0 \leqslant i \leqslant N$ and we set $n_o = 0$) are mutually disjoint and $\mu(I \cup f^{n_1} I \cup \ldots \cup f^{n_N} I) \geqslant 1 - 2\mu(I) \geqslant 1 - 2\varepsilon$.

Then $\quad \mu(R) \geqslant \sum_{i=0}^{N} \mu(R \cap f^{n_i}(I))$

$$= \sum_{i=0}^{N} \mu(R \cap I) \quad \text{since } R \text{ is invariant}$$

$$\geqslant N(1-\varepsilon)\mu(I)$$

$$\geqslant (1-\varepsilon)(1-2\varepsilon) \;.$$

But ε was arbitrary hence $\mu(R) = 1$.

Now let μ, μ' be distinct invariant measures on A_1 satisfying $\mu(A_1) = \mu'(A_1) = 1$.

Let χ denote the characteristic function of a set.

Then by the ergodic theorem (see [2]) if λ is either μ or μ', and T is any λ-measurable set,

$$\lambda(T) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi(T \cap \{f^i(p)\}) \text{ for } \lambda\text{-almost all } p.$$

Then if $\mu \neq \mu'$ there is a set T such that $\mu(T) \neq \mu'(T)$.

Hence there are invariant sets S, S' with $\mu(S) = \mu'(S') = 1$

and $\quad \mu(T) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi(T \cap \{f^i(p)\}) \qquad p \in S$

$$\mu'(T) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi(T \cap \{f^i(p)\}) \qquad p \in S' \;.$$

Then $\mu(T) \neq \mu'(T) \Rightarrow S \cap S' = \emptyset$.

But then the measure $\frac{1}{2}\mu + \frac{1}{2}\mu'$ is not ergodic - contradiction.

Hence there is a unique invariant measure on A_1 with $\mu(A_1) = 1$.

If μ is any transverse measure which has support with closure equal to $\omega(1)$ and which is invariant under the holonomy map $\mu(A_i) = 0$ for $i > 1$.
Hence any transverse measure invariant under the holonomy map, whose support has closure equal to $\omega(1)$, is a multiple of the measure μ with $\mu(A_1) = 1$.

8.5 Rotation numbers.

Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point.
Suppose that $\mathfrak{F}$ has orientation preserving holonomy and let $\omega_1, \ldots, \omega_{k_{\mathfrak{F}}}$ be the complete set of distinct ω-limit sets of non-singular leaves or outward separatrices in $\mathfrak{F}$.
Let A be a transverse circle meeting leaves in precisely one set ω_i.
Let f be the point of first return function on A and let μ be a transverse holonomy invariant measure whose support has closure equal to ω_i.
Let $p \in A \cap \omega_i$ lie on a non-singular leaf or separatrix which returns to A.
Let
$$\alpha(A) = \frac{\mu((p, f(p)))}{\mu(A)} .$$
Then $\alpha(A)$ depends only on A and the orientation on A.

For let $q \in A \cap \omega_i$ lie on a non-singular leaf or outward separatrix, then for sufficiently large n, $f^n(q)$ and p lie in the same interval in the domain of f (see 7.3).
Then since f is defined throughout the interval of A between p and $f^n(q)$ we can assume that the points $p, f^n(\bar{q})$, $f(p), f^{n+1}(q)$ appear on A in precisely this order.
Then $\mu(p,f(p))$

$= \mu(p,f^n(q)) + \mu(f^n(q),f(p))$

$= \mu(f(p),f^{n+1}(q)) + \mu(f^n(q),f(p))$ (since f is holonomy invariant)

$= \mu(f^n(q),f^{n+1}(q))$.

Since f is order preserving and μ is invariant under f, $\mu(q,f(q)) = \mu(f^n(q),f^{n+1}(q))$ as required.
Assuming that A has the orientation in which the pairs (tangent to A, tangent to $\mathfrak{F}$) lie in the orientation of M_g, $\alpha(A)$ is uniquely defined and is called the <u>rotation number of f</u>.

Since $\mathfrak{F}$ has no closed leaves and no holonomy $\alpha(A)$ is irrational. For without loss of generality $\mu(A) = 1$ then if $\alpha(A)$ is a rational number m/n the fact that f is order preserving shows that for any point $x \in \omega_i \cap \bigcap_{n' \in \mathbb{Z}} \operatorname{dom} f^{n'}$ $\mu(x,f^n(x)) = 0$.
Since any open interval meeting ω_i has positive measure (ω_i is the support of μ) the intervals $(x,f^m(x))$ are

maximal open intervals in the complement of ω_i which are disjoint for distinct x. Hence there are uncountably many disjoint intervals in the complement of ω_i - which is impossible.

We remark that the rotation number as defined here is the same as that defined classically (see e.g. [19]) as we shall see in section 9.2 following.

Definition 8.6.1: Let $\mathfrak{F}$ be a C^r $(r \geq 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf, no leaf containing more than one saddle point and order preserving holonomy.
Let μ be a holonomy invariant transverse measure whose support is the support of some non-singular leaf or outward separatrix l.
We associate with μ a cohomology class $\Theta(\mu) \in H^1(M_g, \mathbb{R})$ which is an invariant of $\mathfrak{F}$. This invariant is called the asymptotic cycle associated to μ (or l) and is defined as follows. Various equivalent definitions of $\Theta(\mu)$ can be found in [22] or [23].

We realise $\Theta(\mu)$ as a homomorphism

$$\Theta(\mu) : \pi_1(M_g) \longrightarrow \mathbb{R} .$$

If $[\gamma] \in \pi_1(M_g)$ we can write

$$\gamma \simeq \gamma_1 * \delta_1 * \cdots\cdots * \gamma_n * \delta_n$$

where $\delta_i : [0,1] \longrightarrow M_g$ is a path lying in a leaf of $\mathfrak{F}$ and $\gamma_i : [0,1] \longrightarrow M_g$ is a path transverse to $\mathfrak{F}$.
Then setting $\varepsilon(\gamma_i) = +1$ if γ_i is traversed in the positive direction & $\varepsilon(\gamma_i) = -1$ if γ_i is traversed in the negative direction we make the definition:

$$\circledR(\mu)([\gamma]) = \sum_{i=1}^{n} \varepsilon(\gamma_i)\mu(\gamma_i([0,1])) .$$

If C is a transverse circle it is clear that

$$\circledR(\mu)([C]) = \mu(C) .$$

Now let A_i, B_i be transverse circles meeting leaves in ω_j if & only if $i=j$ as in 7.2.
Then there is a unique holonomy invariant transverse measure μ_i on $\mathfrak{F}$ such that $\mu_i(A_i) = 1$ and the support of μ_i has closure equal to ω_i (since $\mathfrak{F}$ has order preserving holonomy). By the choice of the circles $B_i, \mu_i(B_i) = \alpha(A_i)$, an irrational number.
Hence $\circledR(\mu_i) = \alpha_i + \alpha(A_i)\beta_i + K_i$ where α_i, β_i are the Poincare duals of A_i, B_i respectively and K_i lies in a subspace of $H^1(M_g, \mathbb{R})$ complementary to that generated by $\alpha_1, \ldots, \alpha_{k_{\mathfrak{F}}}$; $\beta_1, \ldots, \beta_{k_{\mathfrak{F}}}$.
Since any holonomy invariant transverse measure whose support has closure equal to the ω-limit set of a single non-singular leaf or outward separatrix is a positive

multiple of μ_i (proposition 8.4) we have:

<u>Proposition 8.6.2</u>: Let $\mathcal{F}$ be a C^r $(r \geq 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf, no leaf containing more than one saddle point and order preserving holonomy.

Then up to multiplication by positive scalars the asymptotic cycle of a non-singular leaf or outward separatrix l is an irrational real cohomology class which depends only on $\omega(l)$.

To end this section we remark that if the number of distinct ω-limit sets is equal to g (its maximum possible value) then from the remarks preceeding the proposition we see that the asymptotic cycles are given by:

$$\Theta(\mu_i) = \alpha_i + \alpha(A_i)\beta_i \qquad 1 \leq i \leq g \ .$$

8.7 Asymptotic cycles in general.

Even if we do not have order preserving holonomy we can define the asymptotic cycle of a non-singular leaf or outward separatrix l for "almost all leaves l".

In other words there is a set X consisting of non-singular leaves and separatrices such that:

(i) $\mu(X \cap T) = \mu(T)$ for any transverse interval T and any holonomy invariant transverse measure μ.

(ii) If $l \in X$ the asymptotic cycle of l is defined.

We define the asymptotic cycle as follows.

Let $\omega_1,\dots,\omega_{k_{\mathcal{F}}}$ be the set of distinct ω-limit sets of leaves of $\mathcal{F}$.

As in 7.2 let $A_1,\dots,A_{k_{\mathcal{F}}}$ be disjoint transverse circles with A_i meeting only leaves in ω_i.

Then a careful application of the results of Oxtoby [43] shows that for almost all $p \in A_i$ the measure μ_p of lemma 8.3 depends only on p and not on the sequence $r_{n,m}$.

Let l be the non-singular leaf or outward separatrix through p.

μ_p determines a transverse measure μ_l depending only on l.

Note that the closure of the support of μ_l is $\omega(l)$.

$\mathcal{A}(\mu_l)$, the <u>asymptotic cycle associated to l</u>, can then be defined exactly as in 8.6.1 with μ_l replacing μ.

A further careful application of [43] shows that μ_l is ergodic for almost all leaves l.

That is: for any holonomy invariant transverse measure λ the set of leaves for which the measure μ_l is ergodic meets any transverse submanifold T in a set of measure $\lambda(T)$.

This observation, together with lemma 8.4.0, gives the following:

<u>Proposition</u>: Let $\mathcal{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point.

Let $k_{\mathcal{Y}}$ be the number of distinct non-trivial limit sets of $\mathcal{Y}$ as in section 7.1.

Then there is a subset X of M_g satisfying:

(1) X is a union of non-singular leaves and separatrices.

(2) Given any holonomy invariant transverse measure μ and any transverse submanifold T

$$\mu(X \cap T) = \mu(T) .$$

(3) Up to multiplication by positive scalars there are at most $4g - 4 + k_{\mathcal{Y}}$ asymptotic cycles associated to the non-singular leaves or separatrices of $\mathcal{Y}$.

We warn the reader that unless every leaf of $\mathcal{Y}$ is dense there is no theoretical reason why X should not be a nowhere dense set.

Chapter 9. Morse foliations on manifolds of genus 2.

In this chapter we examine Morse foliations on M_2, the join of two tori, in which there are exactly two non-trivial limit sets. These results are applied in chapter 10 in the construction of Morse foliations with no holonomy, no closed leaf and no leaf containing more than one saddle point on a two manifold of any positive genus. In chapter 10 we also give a C^0-conjugacy classification of the Morse foliations on M_2 with exactly two non-trivial limit sets.

Lemma 9.1.1: Let $\mathcal{F}$ be a C^r ($r \geqslant 2$) Morse foliation on M_g, the oriented 2-manifold of genus g, with no holonomy, no closed leaf and no leaf containing more than one saddle point. Let E, F be transverse circles and suppose that some non-singular leaf or separatrix cuts E and then cuts F. Then either some inward separatrix cuts E and never subsequently cuts F or every non-singular leaf or separatrix which cuts E subsequently cuts F.
In the latter case E and F are homotopic.

Proof: Suppose that there is a leaf which cuts E at a point e and then cuts F.

The holonomy construction shows that every leaf cutting E in a neighbourhood of e subsequently cuts F.

It is immediate from lemma 7.1.$\frac{1}{2}$ that the only

obstruction to extending this neighbourhood is the existence of an inward separatrix which cuts E and never subsequently cuts F.
The required homotopy is given in the case stated by flowing along the leaves.

Lemma 9.1.2: Let $\mathfrak{F}$ be a C^r $(r \geqslant 2)$ Morse foliation on M_2, the join of two tori, with no holonomy, no leaf containing more than one saddle point and precisely two distinct non-trivial limit sets.
Then there is a transverse circle which represents the zero homology class in $H_1(M_2, \mathbb{Z})$.
Proof: We remark first that the non-existence of closed leaves is implied by the conditions of the lemma since the existence of a closed leaf together with the no holonomy assumption would imply the existence of infinitely many distinct non-trivial limit sets.

So let ω_1, ω_2 be the distinct non-trivial ω-limit sets of $\mathfrak{F}$. The properties of these sets were described in section 7.1. Let A_1, A_2 be transverse circles such that $A_1 \cap A_2 = \emptyset$, every leaf in ω_1 meets A_1 but not A_2 and conversely as in section 7.2.

Let p_1, p_2 be the saddle points of $\mathfrak{F}$ and let the inward and outward separatrices be denoted as shown in figure 9.1.

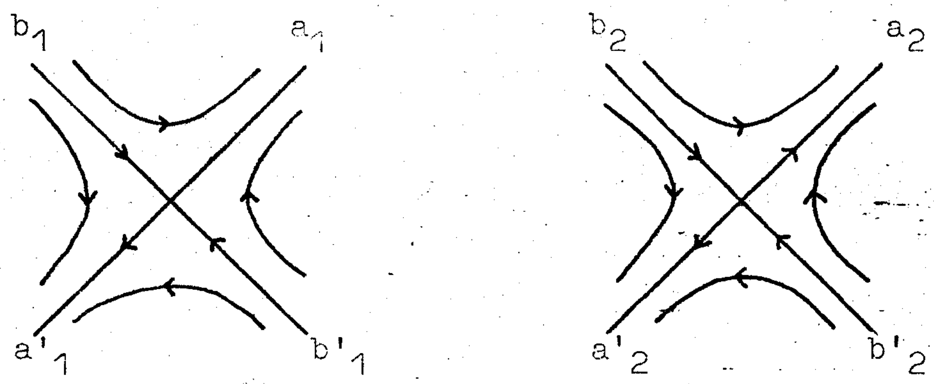


Fig. 9.1

Now ω_1 and ω_2 both contain at least one inward and at least one outward separatrix. Hence at least one inward separatrix cuts A_1 and not A_2 and conversely.

Hence either one, two or three inward separatrices cut A_1. If a single inward separatrix cuts A_1, every point of A_1 lies on a non-singular leaf or separatrix which returns to A_1. Hence, as in figure 9.2, there are two outward separatrices cutting A_1.

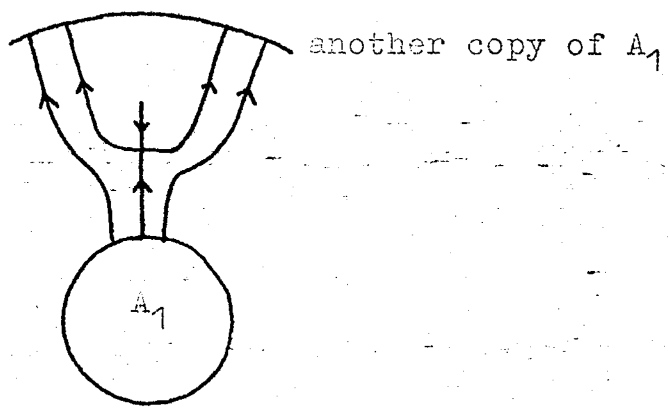


Fig. 9.2

We can therefore assume, reversing the orientation of the leaves if necessary, that the number of inward separatrices cutting A_1 is two or three.

Without loss of generality we now have two cases:

either only b_1 and b_2 cut A_1

or b_1 and b_1' cut A_1 and b_2' does not cut A_1.

We show first that the latter hypothesis implies the result and then that the former hypothesis is impossible.

Suppose that b_1 and b_1' cut A_1 for the last time at the points β_1, β_1' respectively.

Then, as shown in figure 9.3, we can find a transverse circle E such that each leaf cutting the interval (β_1, β_1') in A_1 subsequently cuts E and every point of E except one is a point of intersection of such a leaf.

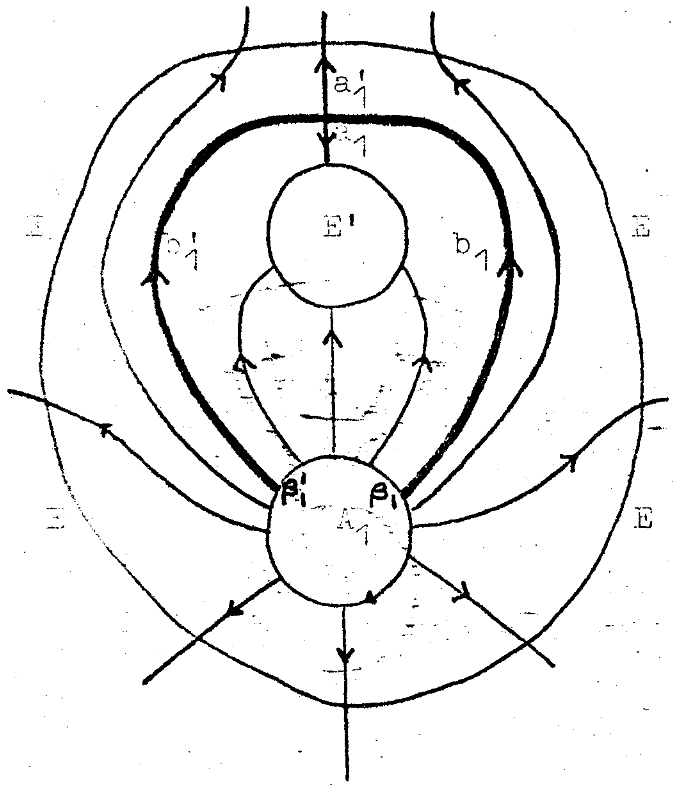


Fig. 9.3

Similarly let E' be a transverse circle such that every leaf leaving the interval (β_1', β_1) in A_1 subsequently cuts E' and every point of E' except the last point of intersection of a_1 with E' lies on such a leaf, as in figure 9.3.

We assert first that if C is either of the circles E, E', then C has the property that either every leaf cutting C subsequently cuts A_1 or there is some inward separatrix which cuts C and never subsequently cuts A_1.
For by lemma 9.1.1 if some leaf cuts C and subsequently cuts A_1 then our assertion must be true. Hence the only other possibility is that no leaf cutting C subsequently cuts A_1. If this is so the ω-limit set of every non--singular leaf or outward separatrix cutting C is ω_2 and hence every non-singular leaf or outward separatrix which cuts C subsequently cuts A_2. If our assertion is false it is also true that no inward separatrix cuts C, for such an inward separatrix would never subsequently cut A_1. Hence by lemma 9.1.1 if the assertion is false every non-singular leaf or separatrix cutting C subsequently cuts A_2. But this means that every non-singular leaf cutting A_2 has previously cut A_1 which is impossible since every non-singular leaf in ω_2 cuts A_2 and not A_1.
Hence the assertion is true.

I claim that the only inward separatrix which can cut C (= E or E') and which never subsequently cuts A_1 is b_2. For from figure 9.3 we see that b_1, b_1' cut A_1 for the last time at β_1, β_1' and never subsequently cut either E or E'. Also by the assumption at the beginning of the proof b_2' never cuts A_1 and hence never cuts either E or E'. But now b_2 cannot have the property that it cuts both E and E' and never subsequently cuts A_1 since, as figure 9.3 shows, any leaf cutting both E and E' cuts A_1 at an intermediate point.

Hence at least one of the circles E or E' has the property that every leaf which cuts it subsequently A_1. We assume that E' is the circle with this property.
Then E' can be identified with A_1 and it is then clear from figure 9.3 (imagine E' and A_1 joined by a handle) that E bounds a torus with a hole, in M_g, and hence separates M_g. Thus E is the required circle.

It remains to eliminate the case that b_1 and b_2 are the only inward separatrices which cut A_1. Without loss of generality, b_1' and b_2' are the only inward separatrices which cut A_2.
We can also assume that exactly one outward separatrix from each saddle point cuts each circle A_i, since we could

otherwise reverse the orientation of the leaves and repeat the above argument.

Orient the transverse circles A_1 and A_2 so that at any point the pair (tangent to A_i, tangent to $\mathcal{F}$) lies in the orientation of M_g. Denote the points at which separatrices cut the circles A_1, A_2 for the first or last time by the corresponding Greek letter so that, for example, b_2 cuts A_1 at β_2.

The reader is advised to refer constantly to figure 9.4 overleaf whilst reading the following argument.
Without loss of generality every leaf crossing (β_1, β_2) returns to A_1 in the interval (α_1', α_2).
This implies that b_1, b_2, a_1' and a_2 are precisely the separatrices which cut A_1 and b_1', b_2', a_1, a_2' are precisely those which cut A_2.
Since a_1 cuts A_2 it is clear from figure 9.4 and the fact that every inward separatrix which cuts (β_2, β_1) subsequently cuts A_1 that every leaf leaving (β_2, β_1) subsequently cuts A_2 and does so for the first time in the interval (α_2', α_1). Then in fact no inward separatrix cuts (α_2', α_1) since any such separatrix would cut both A_1 and A_2. Since also a_1 never cuts A_1 no leaf cutting (α_2', α_1) ever returns to A_1 and every leaf cutting (α_2', α_1) subsequently cuts A_2.

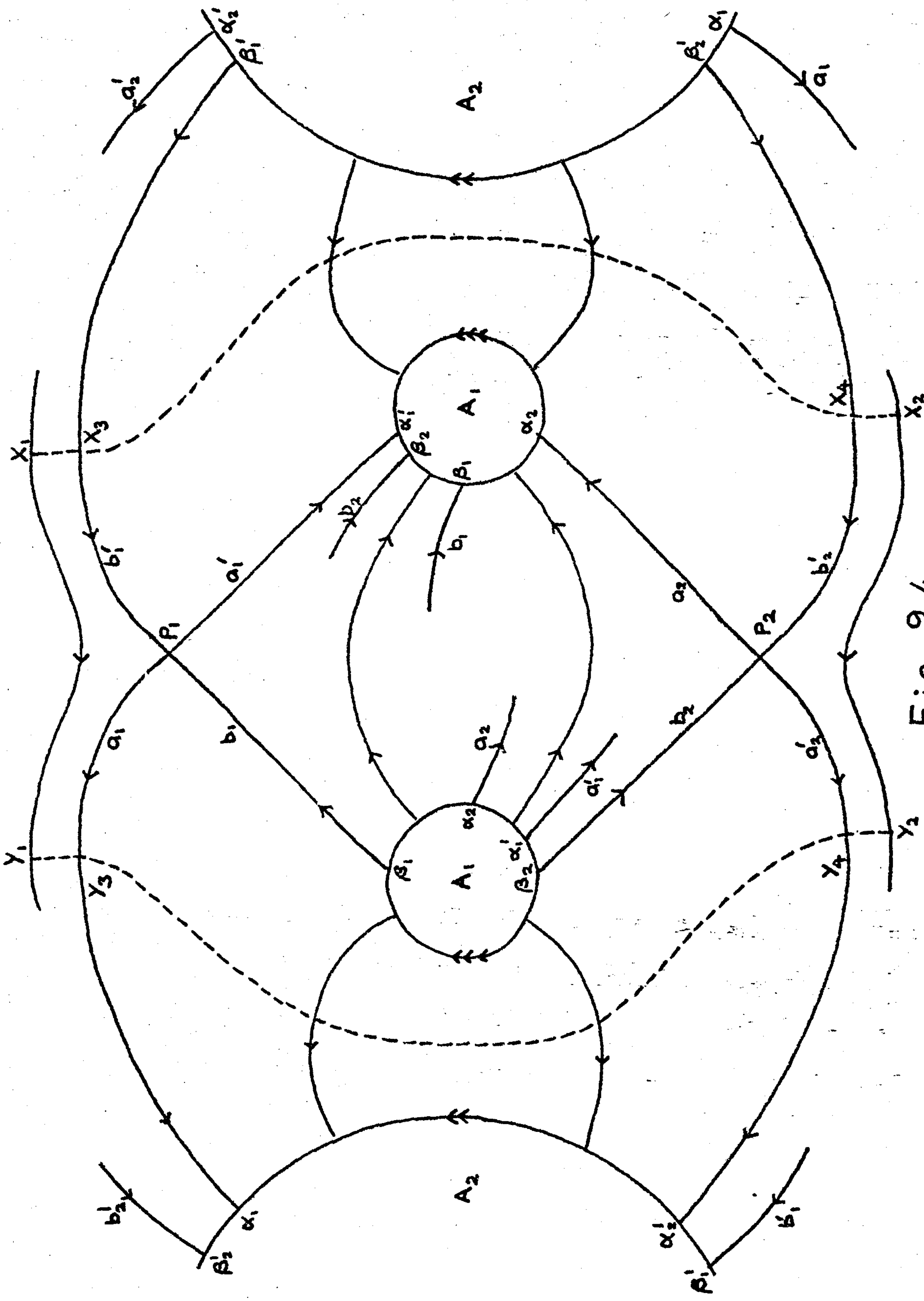


Fig. 9.4

Similar arguments and the configuration of the separatrices show that every leaf cutting A_2 in (β_1', β_2') returns to A_2, no leaf cutting (β_2', β_1') ever returns to A_2 and every leaf cutting (β_2', β_1') subsequently cuts A_1.

Thus: $(\alpha_2, \alpha_1') \subseteq (\beta_1, \beta_2)$

$(\alpha_2', \alpha_1) \subseteq (\beta_1', \beta_2')$.

Hence the points of first or last intersection of separatrices lie on the circles A_1, A_2 in the following order:

$$A_1 : \beta_1 \alpha_2 \alpha_1' \beta_2 \qquad\qquad A_2 : \beta_1' \alpha_2' \alpha_1 \beta_2'$$

as in figure 9.4.

Now choose transverse curves X_1, X_2 and Y_1, Y_2 which start at points X_1, Y_1 on a common leaf in a distinguished neighbourhood of p_1 and which finish at points X_2, Y_2 on a common leaf in a distinguished neighbourhood of p_2 as shown in figure 9.4.

X_1X_2 cuts b_1' at X_3 and b_2' at X_4 in the same distinguished neighbourhoods and Y_1Y_2 cuts a_1 at Y_3 and a_2' at Y_4 in the same distinguished neighbourhoods.

Further every leaf leaving Y_3Y_4 cuts $[\alpha_2', \alpha_1]$ and every leaf arriving at X_3X_4 has cut $[\beta_2', \beta_1']$.

Thus from the preceeding remarks no leaf leaving Y_3Y_4 ever returns to X_3X_4 .

Now cut along the segments of leaf X_1Y_1 & X_2Y_2 and along the transverse curves $X_1X_3X_4X_2$ & $Y_1Y_3Y_2Y_4$. Throw away that part of the manifold containing the saddle points (a torus minus a disc) and glue in a square along the boundary of what remains (also a torus minus a disc) to get a torus.

Foliate this torus by the restriction of $\mathfrak{F}$ outside the square and foliate the square by lines parallel to its sides X_1Y_1 and X_2Y_2 in such a way that the holonomy map from X_1X_3 to Y_1Y_3 and from X_4X_2 to Y_4Y_2 is the same as that for $\mathfrak{F}$. This construction can be carried out so that the resulting (genuine) foliation of the torus is C^r. Denote this foliation of the torus by $\mathfrak{F}'$. Then no leaf of $\mathfrak{F}'$ which cuts Y_3Y_4 ever subsequently cuts X_3X_4. However since $r \geqslant 2$ either every leaf of $\mathfrak{F}'$ is dense or $\mathfrak{F}'$ has a closed leaf. Since $\mathfrak{F}$ had no closed leaf and $\mathfrak{F}'$ has the same leaf structure as $\mathfrak{F}$ outside the square these properties are incompatible.

This proves the lemma.

9.1.3 Description of Morse foliations on M_2

We describe below the geometry of any C^r ($r \geqslant 2$) Morse foliation $\mathfrak{F}$ on M_2, the join of two tori, with no holonomy, no leaf containing more than one saddle point and exactly two non-trivial ω-limit sets (so that in addition $\mathcal{Y}$ has no closed leaf). A typical $\mathcal{Y}$ is shown in figure 9.5 overleaf in which the pairs of transverse circles A_1, A_2 and E have to be identified by suitable diffeomorphisms.

Let E be the transverse circle homologous to zero given by lemma 9.1.2. Since E separates M_2, every non-singular leaf or outward separatrix cutting E has ω-limit set ω_1, every non-singular leaf or inward separatrix cutting E has α-limit set a different set ω_2 and no non-singular leaf or separatrix cuts E more than once. Then when we cut along E, ω_1 and ω_2 lie in different components and hence no non-singular leaf or separatrix in $\omega_1 \cup \omega_2$ cuts E. In addition if p_1 is a saddle point in ω_1, (which must contain one by the theorem of A.J.Schwartz) and p_2 is a saddle point in ω_2, then p_1 and p_2 are distinct & are the only saddle points of $\mathcal{Y}$ (compute the Euler characteristic). Now choose transverse circles A_i $(i = 1,2)$ lying in one or other component (in other words not meeting E) such that $A_1 \cap A_2 = \emptyset$, A_i meets every leaf in ω_i infinitely many times and $A_i \cap \omega_j = \emptyset$ $i \neq j$ as in section 7.2. Now if no inward separatrix cuts E, E is homotopic to A_1, by lemma 9.1.1, which is false since E is homologous to zero and A_1 is not.

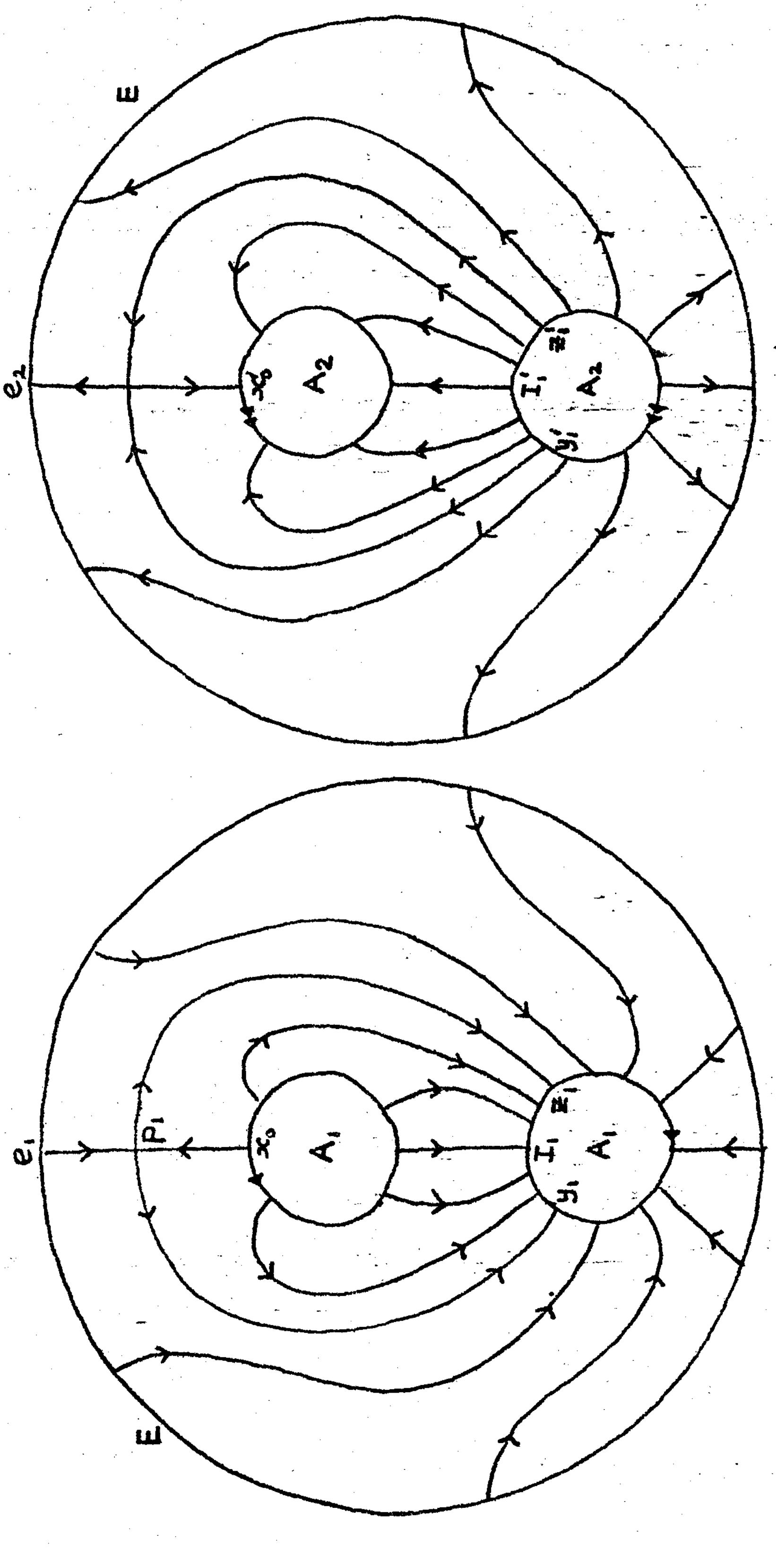


Fig. 9.5

Hence at least one inward separatrix cuts E and does not cut A_1. Now at least one inward separatrix cuts A_1 and lies in ω_1. Since no non-singular leaf or separatrix in $\omega_1 \cup \omega_2$ cuts E there is precisely one inward separatrix cutting A_1 and precisely one inward separatrix cutting E (at e_1 in figure 9.5) and these together are precisely the inward separatrices at p_1.
Similarly precisely one outward separatrix cuts E (at e_2 in figure 9.5) and precisely one outward separatrix cuts A_2 and these are precisely the outward separatrices from p_2.
Let x_0 be the last point of intersection of the unique inward separatrix cutting A_1 with A_1. Let y_1, z_1 be the first points of intersection of the outward separatrices at p_1 with A_1.
From figure 9.5 we see that if a suitable orientation is chosen on A_1 and if $I_1 = [y_1, z_1]$ then the point of first return function f on A_1 is a diffeomorphism

$$f : A_1 \setminus \{x_0\} \longrightarrow A_1 \setminus I_1 .$$

Also every non-singular leaf or outward separatrix crossing E cuts A_1 in the interval $[y_1, z_1]$.

f has the following properties:

(i) f is order preserving .

(ii) $x_0 \in \bigcap_{n \geqslant 0} \operatorname{dom} f^{-n}$ and the points $x_i = f^{-i} x_0$ are all distinct, since $\omega_1 \cap E = \emptyset$ and $\mathcal{F}$ has no closed leaf.

(iii) $I_1 \subseteq \bigcap_{n \geqslant 0} \mathrm{dom} f^n$ and the intervals $I_j = f^{j-1} I_1 = [y_j, z_j]$ are mutually disjoint and contain none of the points x_i. These properties follow from the facts that every leaf cutting (y_1, z_1) has ω-limit set ω_1, y_1 & $z_1 \in \omega_1$ and $\mathcal{F}$ has no closed leaf.

(iv) f has no periodic points since $\mathcal{F}$ has no closed leaf.

(v) $\lim_{x \to x_0^+} D^s f$ and $\lim_{x \to x_0^-} D^s f$ exist for all $s \leqslant r$ and $|Df(x)| \geqslant L > 0$ for some $L \in \mathbb{R}$ and all $x \in \mathrm{dom} f$ by lemma 7.3.2.

(vi) $\omega_1 \cap A_1 = A_1 \setminus \bigcup_{i=1}^{\infty} (y_i, z_i)$ by proposition 8.1.2 and the fact that any non-singular leaf which cuts A_1 in the complement of $\bigcup_{i=1}^{\infty} (y_i, z_i)$ has α and ω-limit set ω_1.

Similarly let x_0' be the first point of intersection of the unique outward separatrix cutting A_2 with A_2 and let y_1' & z_1' be the points of last intersection with A_2 of the inward separatrices at p_2.

Then if $I_1' = [y_1', z_1']$ and g is the point of previous intersection function on A_2, g is a diffeomorphism

$$g : A_2 \setminus \{x_0'\} \longrightarrow A_2 \setminus I_1' .$$

g satisfies the same properties as f.

Every leaf cutting F cuts A_2 for the last time in the interval I_1'.

9.2.1 Diffeomorphisms of the punctured circle.

Let S^1 be the circle and let $x_o \in S^1$ and $I_1 = [y_1, z_1]$ be a proper closed interval in S^1.
Let $f : S^1 \setminus \{x_o\} \longrightarrow S^1 \setminus I_1$ be a C^r $(r \geqslant 2)$ diffeomorphism satisfying the properties (i) to (v) of f in 9.1.3.
If $x \in \bigcap_{n \geqslant 0} \operatorname{dom} f^n$ let $\omega(x)$ be the accumulation set in S^1 of $\{f^n(x) : n \geqslant 0\}$.
Arguing as for diffeomorphisms of the entire circle (see [19] chapter 1) we see that $\omega(x)$ is a perfect, closed, nowhere dense set which is independent of x and invariant under f. The reader is warned that the argument here is non-trivial but since these facts are only required for diffeomorphisms that arise as in 9.1.3 we omit the details.
Suppose also that if $\Omega(f)$ is the accumulation set of every orbit then:
(vi)' $\Omega(f) = S^1 \setminus \bigcup_{i=1}^{\infty} (y_i, z_i)$.
Of course all the diffeomorphisms f arising from Morse foliations of M_2 as in 9.1.3 have all these properties.

By the arguments of lemma 8.3 and the remarks following it there is a unique non-trivial measure μ on S^1 which is invariant under f, has support Ω and satisfies $\mu(S^1) = 1$.
Then the <u>rotation number</u> $\alpha(f)$ of f is an irrational number equal to $\mu(x, f(x))$ for any $x \in \operatorname{dom} f$.

It will be convenient to reinterpret $\alpha(f)$.

This is done as follows.

Let $\pi : \mathbb{R} \longrightarrow S^1$ be the covering map with $\pi 0 = x_o$.

Let $\tilde{I}_1 = [\tilde{y}_1, \tilde{z}_1] \subseteq (0,1)$ be such that $\pi \tilde{I}_1 = I_1$.

Let $F : \mathbb{R} \setminus \mathbb{Z} \longrightarrow \mathbb{R} \setminus \{x \in \mathbb{R} : x+m \in \tilde{I}_1$ some $m \in \mathbb{Z}\}$ be a lift of f (i.e. $f\pi = \pi F$) satisfying:

(a) F is monotone increasing.

(b) $F(0,1) \subseteq (\tilde{z}_1, \tilde{y}_1 + 1)$.

(c) $F(x+1) = F(x) + 1 \quad x \in \mathbb{R} \setminus \mathbb{Z}$.

The graph of a typical F is shown in figure 9.6.

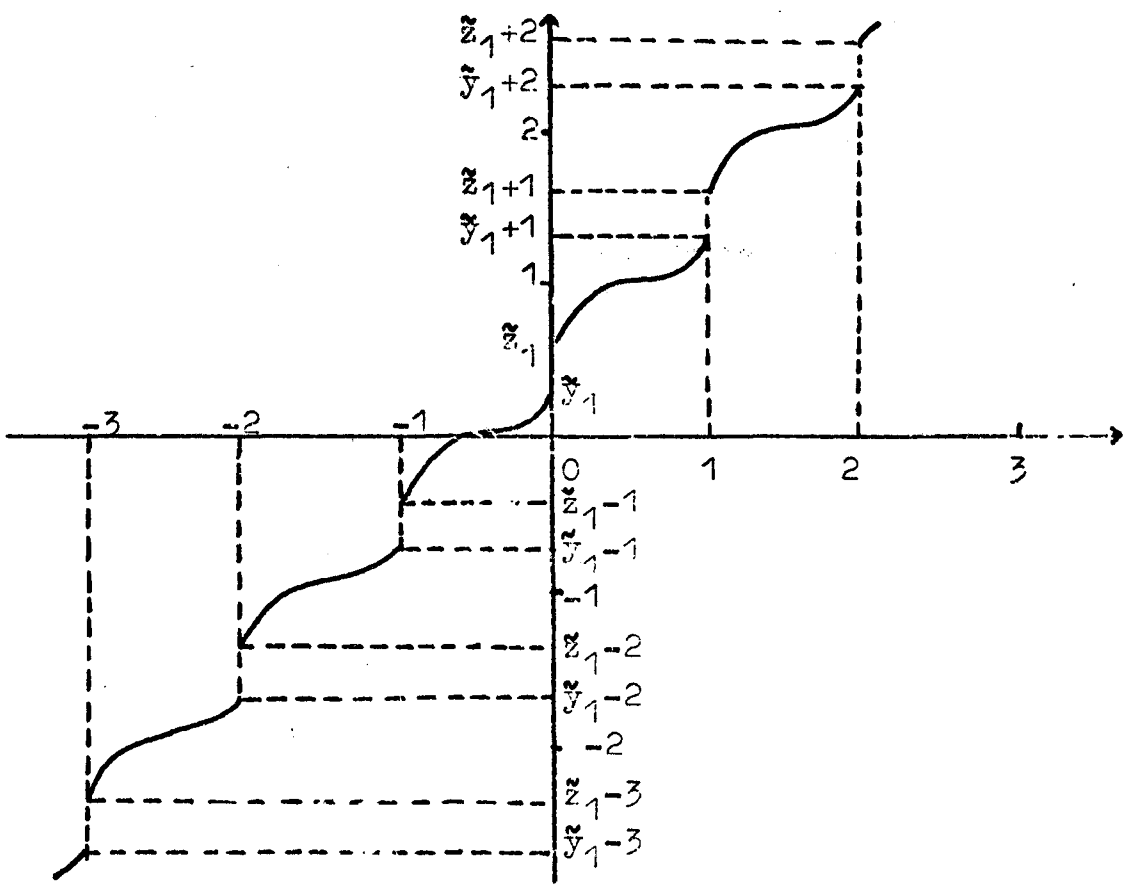


Fig. 9.6

<u>Lemma 9.2.2</u>: Let f,F be as in 9.2.1.

Then if $x \in \bigcap_{n \geq 0} \mathrm{dom} F^n$, $\lim_{n\to\infty} \frac{F^n(x)}{n}$ exists and equals $\alpha(f)$.

<u>Proof</u>: $F^{-1}(1)$ is a well defined point of $(0,1)$ (look at figure 9.6) and by the definition of $\alpha(f)$ and the ergodic theorem: $\alpha(f) = \mu(\pi F^{-1}(1), f\pi F^{-1}(1)) = \mu(\pi F^{-1}(1), x_o)$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi(\{f^i(x)\} \cap (\pi F^{-1}(1), x_o))$$

for almost all x (*).

Let $y \in (0,1)$ be a point such that (*) holds for $x = \pi y$.

Let $p_n = [F^n(y)]$ so that $p_n \leq F^n(y) < p_n + 1$.

Then $\chi(\{f^i(x)\} \cap (\pi F^{-1}(1), x_o)) = 1$

if & only if $p_i + F^{-1}(1) < F^i(y) < 1 + p_i$

if & only if $p_i + 1 < F^{i+1}(y) < 1 + p_i + \tilde{y}_1$ (by applying F to the previous inequality and noting that $\lim_{x\to 0^-} F(x) = \tilde{y}_1$)

if & only if $p_{i+1} = p_i + 1$.

Hence $\chi(\{f^i(x)\} \cap (\pi F^{-1}(1), x_o)) = [F^{i+1}(y)] - [F^i(y)]$.

Hence $\alpha(f) = \lim_{n\to\infty} \frac{[F^n(y)]}{n} = \lim_{n\to\infty} \frac{F^n(y)}{n}$.

We now show that $\lim_{n\to\infty} \frac{F^n(y')}{n}$ is independent of y' for all $y' \in \bigcap_{n \geq 0} \mathrm{dom} F^n$.

Let $G_n(y') = F^n(y') - y'$ for all $y' \in \bigcap_{n \geq 0} \mathrm{dom} F^n$.

Then $F(y'+1) = F(y') + 1$

$\Longrightarrow F^n(y'+1) = F^n(y') + 1$ for all $n \geq 0$

$\Longrightarrow G_n$ is periodic of period 1.

Since F has no periodic points G_n never has an integer value.

From figure 9.6 we see that G_n increases across a discontinuity.

Hence $[G_n(y')]$ is an increasing integer valued function of y' which is periodic of period 1. This means that $[G_n(y')]$ must take a constant integer value p_n.

Then $|F^n(y')-F^n(y'')| \leqslant |(F^n(y')-y')-(F^n(y'')-y'')| + |y'-y''|$

$$\leqslant 1 + |y'-y''| \ .$$

Hence $\lim\limits_{n\to\infty} \dfrac{F^n(y')}{n} = \lim\limits_{n\to\infty} \dfrac{F^n(y'')}{n}$.

Example 9.3: We now construct a diffeomorphism with the properties listed in 9.2.1 and with irrational rotation number α. The lift of this diffeomorphism is piecewise linear.

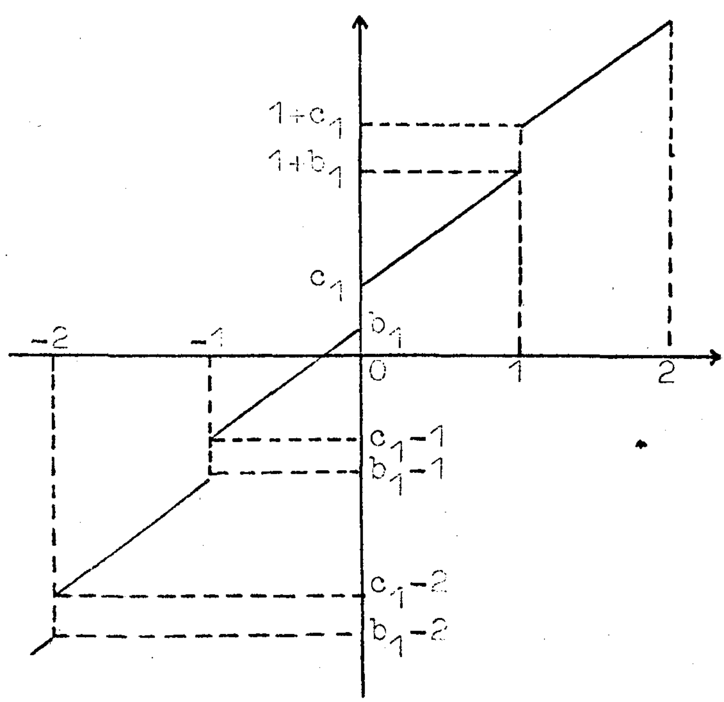


Fig. 9.7

Let α be an irrational number in $(0,1)$ and let $\alpha_m \in (0,1)$ be the number $m\alpha \pmod 1$ for $m \in \mathbb{Z}$.

Suppose $0<\mu<1$.

For $i \leqslant 0$ let

$$a_i = (1-\mu) \sum_{\{m \in \mathbb{Z}: m \geqslant 1\ \alpha_m < \alpha_i\}} \mu^{m-1}$$

For $i > 0$ let

$$b_i = (1-\mu) \sum_{\{m \in \mathbb{Z}: m \geqslant 1\ \alpha_m < \alpha_i\}} \mu^{m-1}$$

$$c_i = (1-\mu) \sum_{\{m \in \mathbb{Z}: m \geqslant 1\ \alpha_m \leqslant \alpha_i\}} \mu^{m-1}$$

Then $\alpha_q < \alpha_p \iff \begin{cases} c_q < b_p & p,q \geqslant 1 \\ c_q < a_p & p \leqslant 0\ ,\ q \geqslant 1 \\ a_q < b_p & p \geqslant 1\ ,\ q \leqslant 0. \end{cases}$

Thus: the intervals $[b_i, c_i]$ are disjoint $(i \geqslant 1) \subseteq (0,1)$;

the points a_i are all distinct$(i \leqslant 0)$ and contained in $[0,1)$;

the points a_i $(i \leqslant 0)$ do not lie in the intervals $[b_j, c_j]$ $(j \geqslant 1)$.

Define $\circledast_{\mu,\alpha} : \mathbb{R}\setminus\mathbb{Z} \longrightarrow \mathbb{R}\setminus\{x \in \mathbb{R} : x+m \in [b_1, c_1] \text{ some } m \in \mathbb{Z}\}$

by $\quad \circledast_{\mu,\alpha}(x) = \mu x + c_1 \qquad x \in (0,1)$

$\quad \circledast_{\mu,\alpha}(x+1) = \circledast_{\mu,\alpha}(x) + 1$, all $x \in \mathbb{R}\setminus\mathbb{Z}$.

Identify S^1 with $[0,1]/_{0=1}$,then the restriction of $\circledast_{\mu,\alpha}$ induces

$$\theta_{\mu,\alpha} : S^1\setminus\{0\} \longrightarrow S^1\setminus[b_1, c_1].$$

Note that $\theta_{\mu,\alpha}(c_i) = \theta_{\mu,\alpha}(b_i) + \mu(c_i - b_i)$

$\qquad\qquad\qquad\qquad = \theta_{\mu,\alpha}(b_i) + c_{i+1} - b_{i+1}.$

We show that $\vartheta_{\mu,\alpha}(b_i) = b_{i+1}$ $(i > 0)$

$\vartheta_{\mu,\alpha}(a_i) = a_{i+1}$ $(i < 0)$.

The proof splits into two cases.

Case 1. $\alpha_i < 1-\alpha$

Then

$$\Theta_{\mu,\alpha}(b_i) = c_i + (1-\mu) \sum_{\{m \,:\, m \geq 1\ \alpha_m < \alpha_i\}} \mu^{m-1}$$

Now in this case

$$\{m : m \geq 1 \ \&\ \alpha_m < \alpha_i\} = \{m : m \geq 1\ \alpha_{m+1} < \alpha_{i+1}\} \setminus \{m : m \geq 1, \alpha_{m+1} < \alpha\}$$

and $\{m : m \geq 1, \alpha_{m+1} < \alpha\} \subseteq \{m : m \geq 1, \alpha_{m+1} < \alpha_{i+1}\}$.

Hence

$$\Theta_{\mu,\alpha}(b_i) = (1-\mu) \sum_{\{m \,:\, m \geq 1, \alpha_m < \alpha\}} \mu^{m-1} + (1-\mu) \sum_{\{n \,:\, n \geq 2\ \alpha_n < \alpha_{i+1}\}} \mu^{n-1}$$

$$- (1-\mu) \sum_{\{n \,:\, n \geq 2, \alpha_n < \alpha\}} \mu^{n-1}$$

$$= (1-\mu) + b_{i+1} - (1-\mu) = b_{i+1}.$$

Case 2. $\alpha_i \geq 1-\alpha$.

Then $\Theta_{\mu,\alpha}(b_i) = c_i + (1-\mu) \sum_{\{m \,:\, m \geq 1, \alpha_m < \alpha_i\}} \mu^{m}$.

Now in this case

$$\{m : m \geq 1\ \alpha_m < \alpha_i\} = \{m : m \geq 1\ \alpha_{m+1} < \alpha_{i+1}\}$$
$$\cup \{m : m \geq 1\ \alpha_{m+1} \geq \alpha\}$$

and $\{m : m \geq 1\ \alpha_{m+1} < \alpha_{i+1}\} \cap \{m : m \geq 1\ \alpha_{m+1} \geq \alpha\} = \emptyset$.

Hence

$$\Theta_{\mu,\alpha}(b_i) = (1-\mu) \sum_{\{m \,:\, m \geq 1\ \alpha_m < \alpha\}} \mu^{m-1} + (1-\mu) \sum_{\{m \,:\, m \geq 2\ \alpha_m < \alpha_{i+1}\}} \mu^{m-1}$$

$$+ (1-\mu) \sum_{\{m \,:\, m \geq 2, \alpha_m \geq \alpha\}} \mu^{m-1}$$

$= 1 + b_{i+1}$.

We now show inductively that $\theta^n_{\mu,\alpha}(b_1) = [(n+1)\alpha] + b_{n+1}$.
This follows easily having noted that $[n\alpha] = n\alpha - \alpha_n$.
Hence the rotation number $\alpha(\theta_{\mu,\alpha}) = \alpha$ for any μ.

Now $\vartheta_{\mu,\alpha}$ is analytic and has constant derivative.
To show that it satisfies our requirements it remains to show that $\Omega(\theta_{\mu,\alpha}) = S^1 \setminus \bigcup_{n \geqslant 1} (b_n, c_n)$.
Otherwise, none of the points $0, a_i, b_i, c_i$ lie in $\Omega(\theta_{\alpha,\mu})$.
Let $S^1 \setminus \Omega(\theta_{\alpha,\mu}) = \bigcup_{i \geqslant 1} (d_i, e_i)$ and suppose $0 \in (d_1, e_1)$.
Now if $[b_1, c_1] \subseteq (d_i, e_i)$
$\theta_{\mu,\alpha}((d_1, e_1) \setminus \{0\}) = (d_i, e_i) \setminus [b_1, c_1]$ since $\vartheta_{\mu,\alpha}$ maps endpoints of maximal complementary intervals to endpoints of maximal complementary intervals.
Since $\vartheta_{\mu,\alpha}$ has no periodic points (for then $\alpha(\theta_{\mu,\alpha})$ would be rational) we may assume $i = 2$.
Then the intervals $\theta^{-n}_{\mu,\alpha}(c_1, d_1)$ are distinct intervals in the complement of $\Omega(\theta_{\mu,\alpha})$, none of which contains $[b_1, c_1]$.
But length $\vartheta^{-n}_{\mu,\alpha}(c_1, d_1) = \mu^{-n}(d_1 - c_1) \longrightarrow \infty$ as $n \longrightarrow \infty$, which is impossible.
Hence $\Omega(\theta_{\mu,\alpha}) = S^1 \setminus \bigcup_{n \geqslant 1} (b_n, c_n)$.

The above example was suggested by a construction of Milnor (see [25]) of an example of a diffeomorphism of S^1

with non-wandering set a perfect, closed, nowhere dense set and without periodic points. Of course, it follows from the work of Denjoy ([4]) that no such diffeomorphism of the entire circle can be C^2, since every C^r $(r \geq 2)$ diffeomorphism of S^1 is C^0-conjugate to a rotation if it has no periodic points.

Below we prove an analogous result for diffeomorphisms

$$f : S^1 \setminus \{x_0\} \longrightarrow S^1 \setminus I_1 .$$

In this case, the diffeomorphisms $\Theta_{\mu,\alpha}$ replace the rotations. Since conjugate diffeomorphisms have the same rotation number, α must be the rotation number of f.

This result will be related to conjugacy of Morse foliations in lemma 10.6.

<u>Proposition 9.4</u>: Let $f : S^1 \setminus \{x_0\} \longrightarrow S^1 \setminus J_1$ be a diffeomorphism with the properties outlined in section 9.2.1. Then given μ with $0 < \mu < 1$, there is a homeomorphism

$$h_\mu : S^1 \longrightarrow S^1$$

such that $\quad h_\mu^{-1} \Theta_{\mu,\alpha(f)} h_\mu | S^1 \setminus \{x_0\} = f$

and $\quad h_\mu | J_1$ is an arbitrary homeomorphism of J_1 onto $[b_1, c_1]$.

<u>Proof</u>: Let F be a lift of f as in 9.2.1 .

We show first that if $n_1, n_2, m_1, m_2 \in \mathbb{Z}$

and $\quad x \in \operatorname{dom} F^{n_1} \cap \operatorname{dom} F^{n_2}$

then $\qquad F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$

if and only if $\quad n_1\alpha(f) + m_1 < n_2\alpha(f) + m_2$.

First note that it follows from the proof of lemma 9.2.2 that for each integer s there is an integer p_s such that

$$p_s < F^s(x) - x < p_s + 1 \text{ for all } x \in \operatorname{dom} F^s.$$

Hence the order of the points $F^{n_1}(x) + m_1$ is independent of x.

Then

$$\begin{aligned}
& F^{n_1}(x) + m_1 && < && F^{n_2}(x) + m_2 \\
\iff & F^{n_1-n_2}(x) + m_1 && < && x + m_2 \\
\iff & F^{n_1-n_2}(x) - x && < && m_2 - m_1 \\
\iff & (F^{p(n_1-n_2)}(x) - x) && < && p(m_2-m_1) \quad \text{for all } p > 0 \\
\iff & \alpha(f)(n_1-n_2) && < && m_2 - m_1 \\
\iff & n_1\alpha(f) + m_1 && < && n_2\alpha(f) + m_2 .
\end{aligned}$$

Now let $\theta_{\mu,\alpha}$, $\Theta_{\mu,\alpha}$ be as in 9.3.

Then since $\Theta_{\mu,\alpha(f)}$ is a lift of $\theta_{\mu,\alpha(f)}$

$$\begin{aligned}
& F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2 \\
\iff & \Theta_{\mu,\alpha(f)}(y) + m_1 < \Theta_{\mu,\alpha(f)}(y) + m_2, \text{ for all suitable } y.
\end{aligned}$$

With the notation of 9.2.1, the above and the fact that

$$\lim_{x\to a_0^-} \theta_{\mu,\alpha(f)}(x) = b_1 \quad , \quad \lim_{x\to a_0^+} \theta_{\mu,\alpha(f)}(x) = c_1$$

imply that $\qquad b_{n_1} + m_1 < b_{n_2} + m_2$

$$\iff \tilde{y}_{n_1} + m_1 < \tilde{y}_{n_2} + m_2$$

and $\quad a_{n_1} + m_1 < a_{n_2} + m_2$

$\Leftrightarrow \quad \tilde{x}_{n_1} + m_1 < \tilde{x}_{n_2} + m_2.$

Let $\quad A = \{a_n+m, b_{n'}+m \ : \ n \geq 0 \ , \ n' > 0 \ , \ m \in \mathbb{Z}\}$

$\quad A' = \{\tilde{x}_n+m, \tilde{x}_{n'}+m \ : \ n \geq 0 \ , \ n' > 0 \ , \ m \in \mathbb{Z}\}.$

Define $H_\mu : A' \longrightarrow A$ by

$$H_\mu(\tilde{x}_n + m) = a_n + m$$

$$H_\mu(\tilde{y}_n + m) = b_n + m.$$

Then H extends to a unique orientation preserving map

$$H_\mu : \pi^{-1}(\Omega(f)) \longrightarrow \pi^{-1}(\Omega(\theta_{\mu,\alpha})), \text{where}$$

$$\pi : \mathbb{R} \longrightarrow S^1 \text{ is the projection.}$$

Further $\quad H_\mu^{-1} F H_\mu(x) = \theta_{\mu,\alpha(f)}(x) \qquad x \in \pi^{-1}(\Omega(f)).$

H_μ maps endpoints of intervals $(\tilde{y}_n, \tilde{z}_n)$ to endpoints of intervals (b_n, c_n).

Now extend H_μ to all of $\mathbb{R}$ by letting $H_\mu | J_1$ be any orientation preserving diffeomorphism of J_1 onto $[b_1, c_1]$ and letting

$$H_\mu(x) = \theta^n_{\mu,\alpha(f)} H_\mu(F^{-n}(x) + m) - m$$

if $x \in [\tilde{y}_{n+1}, \tilde{z}_{n+1}]$ (mod 1) and $[x] = m$.

Then $\quad H_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is an orientation preserving homeomorphism and $H_\mu^{-1} F H_\mu(x) = \theta_{\mu,\alpha(f)}(x)$.

Thus H_μ induces a homeomorphism $h_\mu : S^1 \longrightarrow S^1$ such that $h_\mu^{-1} f h_\mu = \theta_{\mu,\alpha}$.

In [37] the author gives an example of a C^∞ diffeomorphism of the entire circle without periodic points which is not C^1-conjugate to a rotation. Hence it seems likely that the differentiability class of the conjugacy in proposition 9.4 is the best possible for general f. On the other hand in [38] the author proves that for particular rotation numbers diffeomorphisms of the entire circle must be C^∞ conjugate to a rotation.

Chapter 10 Examples of Morse foliations.

10.1.1 In this chapter we shall construct a number of examples of C^r $(r \geq 2)$ Morse foliations without closed leaves.

In order to do this we first generalise the type of point of first return function encountered in chapter 9.
We require C^r order preserving diffeomorphisms

$$f : S^1 \setminus \{x_1, \ldots\ldots, x_u\} \longrightarrow S^1 \setminus I_1 \cup \ldots\ldots \cup I_u$$

satisfying the properties:

(i) $x_1, \ldots\ldots, x_u \in \operatorname{dom} f^{-n}$ $\forall n \geq 0$ and $f^{-n}x_s = x_{s'}$ iff $n=0, s=s'$.

(ii) None of the points $f^{-n}x_s$ $1 \leq s \leq u, n \geq 0$ lies in an interval I_j $1 \leq j \leq u$.

(iii) $I_1, \ldots\ldots, I_u \subseteq \operatorname{dom} f^n$ $\forall$ $n \geq 0$ and $f^n I_s \cap I_{s'} \neq \phi$ iff $s = s', n = 0$.

(iv) f has no periodic points.

(v) For all $s \leq r$ $D^s f$ is bounded and Df is bounded away from 0.

(vi) $\Omega(f)$ is well defined and

$$\Omega(f) = S^1 \setminus \bigcup_{s=0}^{\infty} (f^s I_1 \cup \ldots\ldots \cup f^s I_u).$$

Note also that as in 9.2 we can define the rotation number of f in two ways and it is irrational.
We shall assume $I_j = \left[\lim_{x \to x_j^-} f(x), \lim_{x \to x_j^+} f(x) \right]$.

In the following lemma we show that such diffeomorphisms exist.

<u>Lemma 10.1.2</u>: Let $f : S^1 \setminus \{x_o\} \longrightarrow S^1 \setminus I_1$ be a diffeomorphism as in 9.2.1.

Let u be a positive integer and let

$$I_j = f^{j-1} I_1 \quad 1 \leqslant j \leqslant u \ , \ x_j = f^{j-1} x_o \quad 1 \leqslant j \leqslant u.$$

Then $f^u : S^1 \setminus \{x_1, \ldots, x_u\} \longrightarrow S^1 \setminus I_1 \cup \ldots \cup I_u$ satisfies the conditions of 10.1.1 and has rotation number $u\alpha(f)$ (mod 1).

<u>Proof:</u> Everything except (vi) is obvious.

To prove (vi) we show that $\Omega(f^u) = \Omega(f)$.

Clearly $\Omega(f^u) \subseteq \Omega(f)$.

If $y \in \Omega(f)$, $\exists\, x \in \bigcap_{n \geqslant 0} \mathrm{dom} f^n$ and $n_i \longrightarrow \infty$ such that

$$f^{n_i}(x) \longrightarrow y \text{ as } i \longrightarrow \infty .$$

However for some k, $0 \leqslant k \leqslant u$ we have $n_i = m_i u + k$ for infinitely many i.

Hence we can assume $n_i = m_i u + k$.

Then $(f^u)^{m_i}(f^k(x)) \longrightarrow y$ and hence $y \in \Omega(f)$.

10.2 The Morse foliations $\mathcal{F}_{f,u}$.

Let T_u denote the torus M_1 with u discs removed.

In appendix 4, we construct on T_u a C^r $(r \geqslant 2)$ transversely oriented Morse foliation $\mathcal{F}^+_{f,u}$ transverse to the boundary which has the following properties:

1. Every leaf cutting ∂T_u never returns to ∂T_u.

2. $\mathfrak{F}^{+}_{f,u}$ has exactly u saddle points, no holonomy, no closed leaf and no leaf containing more than one saddle point.

3. Exactly one inward separatrix cuts each component of ∂T_u.

4. There is a closed, non-empty nowhere dense set Ω which meets every transverse interval in a perfect set and in which every leaf is dense such that the ω-limit set of every non-singular leaf or outward separatrix is Ω.

5. There is a transverse circle A_1 not meeting ∂T_u on which the point of first return function is the function f which has the properties outlined in 10.1.1.

The Morse foliation $\mathfrak{F}^{+}_{f,u}$ is sketched in figure 10.1 in a fundamental region of the torus.

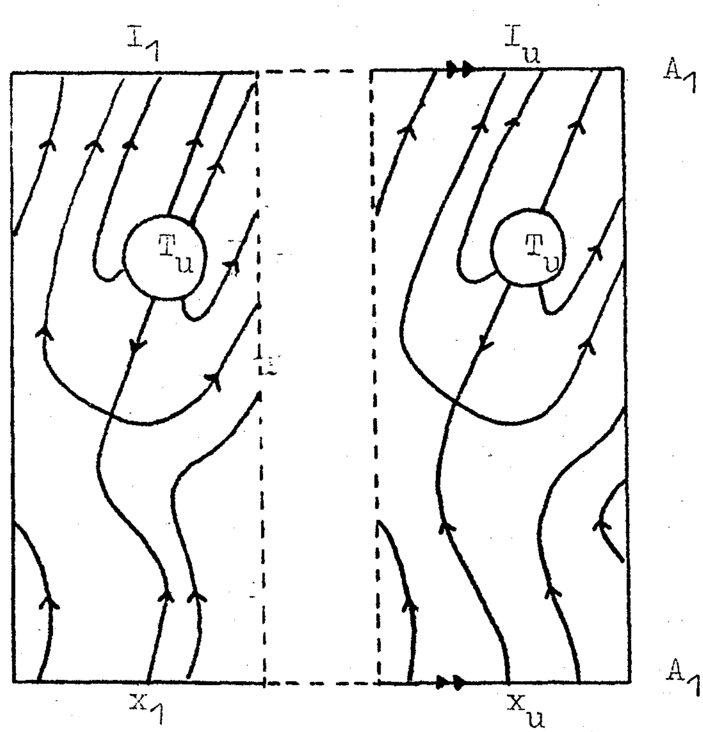


Fig. 10.1

Similarly we have the Morse foliation $\mathfrak{F}^-_{f,u}$ which is $\mathfrak{F}^+_{f,u}$ with the opposite orientation on the leaves.

We shall construct general Morse foliations by gluing together foliations like $\mathfrak{F}^+_{f,u}$, $\mathfrak{F}^-_{g,v}$. To do this in sufficient generality, we need to construct another family of Morse foliations which occurs naturally when one considers Morse foliations on M_2, the join of two tori, with just one limit set.

10.3 The Morse foliation $\mathcal{D}$.

We consider first Morse foliations on M_2 with no closed leaf, no holonomy and exactly one ω-limit set. Let A be a transverse circle cutting this set.

Let p_1, p_2 be the two saddle points and label the inward and outward separatrices as shown in figure 10.2.

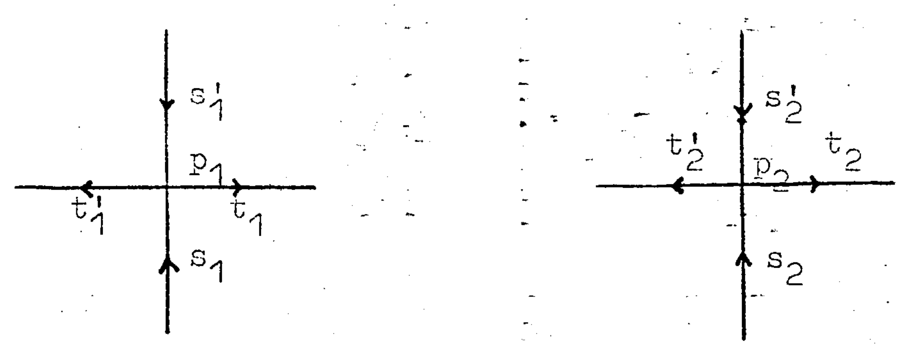


Fig. 10.2

Let T_i, T_i' denote the points at which the separatrices t_i, t_i' first cut A and S_i, S_i' the points at which s_i, s_i' last cut A. Orient A so that the pairs

(tangent to A, tangent to leaf)

lie in the orientation of M_2.

We consider the point of first return function on A. The points T_i, T_i' associated to different saddle points are interlaced around A as are the points S_i, S_i'. For suppose that the interval (S_1, S_1'), say, contains no point S_2 or S_2'.

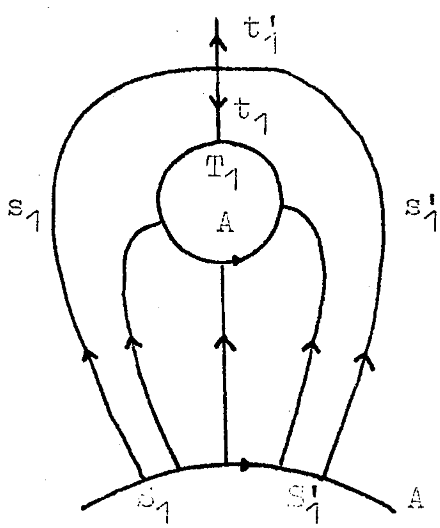


Fig. 10.3

Then every leaf cutting A in the interval (S_1, S_1') returns to A. Further, as can be seen from figure 10.3, the image in A of (S_1, S_1') under the forward holonomy map is $A \setminus T_1$. But since every leaf of $\mathfrak{F}$ cuts A this is clearly absurd. Hence our assertion about the order of the points T, S holds.

Thus we may assume that the points S_i, S'_i appear around A in the order S_1, S_2, S'_1, S'_2 and therefore that the points T_i, T'_i appear around A in the order T_1, T'_2, T'_1, T_2.
Thus the point of first return function

$$f : A\setminus\{S_1, S_2, S'_1, S'_2\} \longrightarrow A\setminus\{T_1, T_2, T'_1, T'_2\}$$

maps intervals

$$\begin{aligned}(S_1, S_2) &\longrightarrow (T_1, T'_2)\\(S_2, S'_1) &\longrightarrow (T_2, T_1)\\(S'_1, S'_2) &\longrightarrow (T'_1, T_2)\\(S'_2, S_1) &\longrightarrow (T'_2, T'_1).\end{aligned}$$

Cutting along A produces a Morse foliation of T_2, the torus with two discs removed, which is transverse to the boundary. Every leaf leaves one boundary component and reaches the other except for four inward separatrices.
If $I_1, \ldots, I_4$ are the four open intervals of one boundary component from which every leaf reaches the other boundary component, appearing in order of the orientation, then their images $J_1, \ldots, J_4$ in the other boundary component appear in the order J_1, J_4, J_3, J_2 .
The foliation in a fundamental region of the universal covering space is shown in figure 10.+.

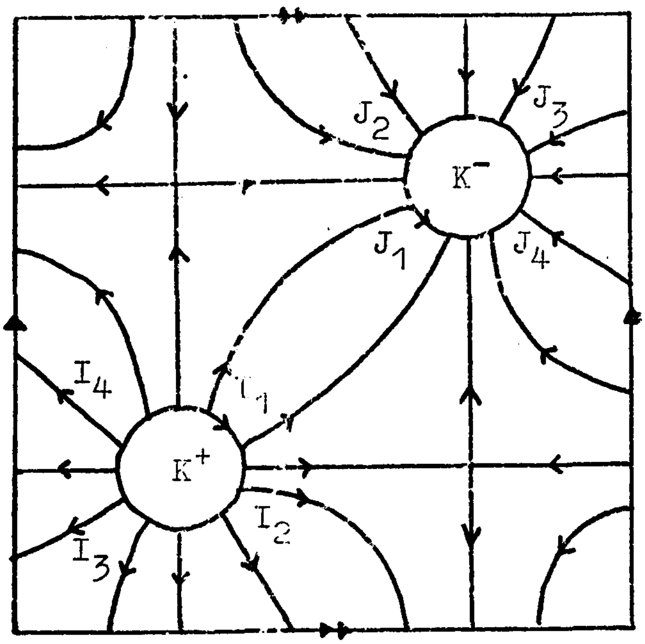


Fig. 10.4

An example of such a C^∞ Morse foliation, $\mathcal{D}$, is constructed in appendix 4.

It is defined on T_2. All leaves leave the boundary component K^+ and return to the boundary component K^-. There are C^∞ embeddings $\iota^\pm : S^1 \longrightarrow K^\pm$ which preserve orientation such that if

$$f : I_1 \cup I_2 \cup I_3 \cup I_4 \longrightarrow J_1 \cup J_2 \cup J_3 \cup J_4$$

is the map given by translating along leaves then

$$g = (\iota^-)^{-1} f \iota^+ : S^1 \setminus \{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\} \longrightarrow S^1 \setminus \{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\}$$

is given by

$$g(x) = \begin{cases} x & x \in (0, \tfrac{1}{4}) \cup (\tfrac{1}{2}, \tfrac{3}{4}) \\ x + \tfrac{1}{2} & x \in (\tfrac{1}{4}, \tfrac{1}{2}) \cup (\tfrac{3}{4}, 1) . \end{cases}$$

10.4 Morse foliations with no dense leaf.

In the preceding sections we constructed three types of Morse foliation:

$\mathfrak{F}^-_{g,v}$: on T_v, the torus minus v discs. $\mathfrak{F}^-_{g,v}$ is a "generalised source" in the sense that a leaf crossing any boundary component of T_v remains for ever in T_v.

$\mathfrak{F}^+_{f,u}$: on T_u. $\mathfrak{F}^+_{f,u}$ is a generalised sink.

$\mathcal{D}$: on T_2. In $\mathcal{D}$ every leaf crossing the boundary component K^+, except for four inward separatrices, arrives at the boundary component K^-. In the Morse foliations constructed in this section $\mathcal{D}$ will always be wandering.

In this section we show how to construct from these components C^r $(r \geqslant 2)$ Morse foliations $\mathfrak{F}$ with no holonomy, no leaf containing more than one saddle point and exactly k $(k \geqslant 2)$ non-trivial limit sets. These will have no dense leaf and order preserving holonomy.

The construction proceeds as follows.
Choose Morse foliations:

$$\mathfrak{F}^+_{f_1,u_1}, \ldots, \mathfrak{F}^+_{f_q,u_q} \quad ; \quad \mathfrak{F}^-_{g_1,v_1}, \ldots, \mathfrak{F}^-_{g_s,v_s}$$

such that

$$\sum_{i=1}^{q} u_i = \sum_{j=1}^{s} v_j = c > 0 .$$

Pair each component of the boundary of T_{u_i} with a boundary component of some T_{v_j}.
Construct c chains of $\mathfrak{D}$'s (possibly of length 0) by inductively gluing the boundary component K^- of $\mathfrak{D}$ to the boundary component K^+ of another copy of $\mathfrak{D}$, taking care not to glue together separatrices.
Then glue each boundary component of T_{u_i} to the remaining component K^- of a chain of $\mathfrak{D}$'s and glue the boundary component K^+ at the other end of this chain to the paired boundary component of T_{v_j}.
The result will always be a Morse foliation of a closed 2-manifold and for suitable choices of the original components and pairing this 2-manifold will be connected. The only constraint on the gluing map is that it should not glue together separatrices.

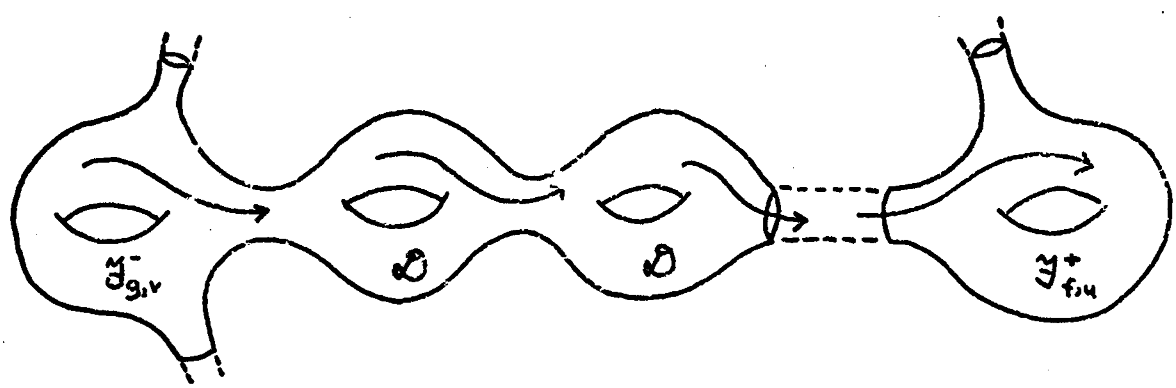


Fig. 10.5

In a Morse foliation obtained like this each $\mathfrak{F}^-_{g,v}$ spews forth v streams of leaf. Each stream flows along a number of $\mathcal{D}$'s and is eventually sucked, together with u-1 other streams, into an $\mathfrak{F}^{+}_{f,u}$, as in figure 10.5.

Suppose that a total of m copies of $\mathcal{D}$ was used in the construction.

Counting up the number of saddle points and using Euler's formula shows that if the resulting manifold is connected then it has genus g where

$$g = m + \sum_{i=1}^{q} u_i + 1 = m + \sum_{j=1}^{s} v_j + 1.$$

Let ω_i be the unique limit set associated to $\mathfrak{F}^{+}_{f_i,u_i}$ and α_j the unique limit set associated to $\mathfrak{F}^{-}_{g_j,v_j}$.

Then the distinct non-trivial limit sets of $\mathfrak{F}$ are:

$$\omega_1, \ldots, \omega_q \text{ and } \alpha_1, \ldots, \alpha_s .$$

Hence the number of distinct limit sets k is equal to r+s.

Further we can choose a basis of 1-forms

$$\eta_1, \ldots, \eta_g \text{ and } \nu_1, \ldots, \nu_g$$

of $H^1(M_g, \mathbb{R})$ with respect to which the asymptotic cycles of $\mathfrak{F}$ are positive multiples of:

$$\begin{cases} \eta_i + \alpha(f_i)\nu_i & 1 \leqslant i \leqslant q \\ \eta_j + \alpha(g_{j-q})\nu_j & q+1 \leqslant j \leqslant q+s \end{cases} ,$$

where α denotes the rotation number.

It follows from 10.6 following that if $g=2$ the asymptotic cycles classify $\mathfrak{Z}$ up to C^0-conjugacy. However this does not remain true for genera greater than 2, even among Morse foliations constructed as above, since there is a counterexample with $g=3, k=2$ and $m=0$ or $m=1$.
On the other hand if, in addition the number of inward and the number of outward separatrices in each limit set and the number of separatrices limiting on each limit set is known, the asymptotic cycles are classifying for our examples.

10.5 Morse foliations with every leaf dense.

We construct Morse foliations with every leaf dense by adjoining g-1 copies of $\mathcal{D}$ to obtain an oriented 2-manifold of genus g.

On a torus, as follows from the results of Denjoy ([4]), every Morse foliation which is C^r $(r \geqslant 2)$ and has no holonomy and no closed leaf has every leaf dense.
I do not know whether the analogous result is true on manifolds of higher genus for Morse foliations with just one ω-limit set.

With the notation of 10.3 let

$$M = \frac{T_2}{\iota^+(x+\alpha) = \iota^-(x)}$$

for some irrational number α.

Then there exists a C^∞ structure on M such that $\mathcal{D}$ defines a C^∞ Morse foliation $\mathcal{D}_\alpha$ on M. Since M is oriented and has genus 2, we can assume that $M = M_2$ by the remarks of 2.1. Note that the irrationality of α implies that $\mathcal{D}_\alpha$ has no closed leaf and that no leaf of $\mathcal{D}_\alpha$ has more than one saddle point on it.

Let $i : S^1 \longrightarrow M : x \longmapsto [\iota^+(x)]$.
Then i is an embedding of S^1 onto a transverse circle A. It follows from the definitions in section 10.3 and the fact that $[\iota^+(x+\alpha)] = [\iota^-(x)]$ in M that with the parametrisation given by i the point of first return function f on A is given by:

$$f(x) = \begin{cases} x+\alpha & 0 < x < \frac{1}{4} \quad \text{or} \quad \frac{1}{2} < x < \frac{3}{4} \\ x+\alpha+\frac{1}{2} & \frac{1}{4} < x < \frac{1}{2} \quad \text{or} \quad \frac{3}{4} < x < 1 \end{cases}.$$

Now it is clear from the construction that A meets the ω-limit set of every leaf.
If no leaf is dense it follows from 7.1.4 that there is an interval I of A such that all iterates of I under f are defined and pairwise disjoint.
On the other hand it is clear from the definition of f that Lebesgue measure is invariant under f. Hence no such interval I can exist.
Hence the Morse foliation $\mathcal{D}_\alpha$ has every leaf dense.

To construct Morse foliations with all leaves dense on a 2-manifold of arbitrary genus $g \geq 2$, we proceed as follows.

Let α be an irrational number with $g\alpha < 1$.

Take $(g-1)$ copies of T_2 : $T_2^{(1)}, \ldots, T_2^{(g-1)}$ with the Morse foliation $\mathcal{D}$ on each of them.

Let $K^{\pm(i)}$ be the corresponding boundary components and let $\iota^{\pm(j)} : S^1 \longrightarrow K^{\pm(j)}$ be embeddings as in 10.4 .

Let M be the manifold obtained by identifying $\iota^{+(j)}(x)$ and $\iota^{-(j+1)}(x+\alpha) \pmod 1$ and $\iota^{+(g-1)}(x)$ with $\iota^{-(1)}(x+\alpha)$.

Then M is an oriented 2-manifold of genus g and hence can be identified with M_g.

Since Lebesgue measure is invariant under the holonomy map it follows that every leaf of the induced foliation is dense.

Finally, we give the promised result on C^0-conjugacy of Morse foliations on M_2 - the join of two tori.

Proposition 10.5: Let $\mathcal{F}$ be a C^r $(r \geq 2)$ Morse foliation on M_2, the join of two tori, with no holonomy, no leaf containing more than one saddle point and with exactly two non-trivial limit sets.

With the notation of lemma 9.1.2 suppose that the point of first return function f has rotation number α and the point of previous intersection function g has rotation number β .

Let $0 < \lambda, \mu < 1$ be arbitrary and let M be the Morse foliated manifold obtained by identifying the boundaries of two copies of T_1, one foliated by $\mathcal{F}^+_{\theta_{\mu,\alpha},1}$ and the other by $\mathcal{F}^-_{\theta_{\lambda,\beta},1}$ (with the notation of proposition 9.4).
Denote the induced Morse foliation by $\mathcal{F}_{\alpha,\beta}$.
Then $\mathcal{F}$ is C^0-conjugate to $\mathcal{F}_{\alpha,\beta}$.
Proof: The meat of the proof is contained in lemma 9.1.2 and proposition 9.4. We sketch the rest of the proof.

$\mathcal{F}_{\alpha,\beta}$ also satisfies the hypotheses of lemma 9.1.2.
Let E be a transverse circle to $\mathcal{F}$ and E' a transverse circle to $\mathcal{F}_{\alpha,\beta}$, homologous to zero. Choose transverse circles A_1, A_2 to $\mathcal{F}$ and A_1', A_2' to $\mathcal{F}_{\alpha,\beta}$ as in the statement of lemma 9.1.2, with diffeomorphisms:

$$\left.\begin{array}{rl} f: & A_1\setminus\{x_0\} \longrightarrow A_1\setminus I_1 \\ \theta_{\mu,\alpha}: & A_1'\setminus\{x_0'\} \longrightarrow A_1'\setminus I_1' \end{array}\right\} \text{points of first return functions}$$

$$\left.\begin{array}{rl} g: & A_2\setminus\{\gamma_0\} \longrightarrow A_2\setminus J_1 \\ \theta_{\lambda,\beta}: & A_2'\setminus\{\gamma_0'\} \longrightarrow A_2'\setminus J_1' \end{array}\right\} \text{points of previous return functions.}$$

Write $I_1 = [y_1, z_1]$, $I_1' = [y_1', z_1']$, $J_1 = [\delta_1, \varepsilon_1]$, $J_1' = [\delta_1', \varepsilon_1']$.
Choose projections

$\rho_i : \mathbb{R} \longrightarrow A_i$ mapping $\mathbb{Z}$ to x_0 $(i=1)$ or γ_0 $(i=2)$

$\rho_i' : \mathbb{R} \longrightarrow A_i'$ mapping $\mathbb{Z}$ to x_0' $(i=1)$ or γ_0' $(i=2)$.

Lift $f, g, \theta_{\mu,\alpha}, \theta_{\lambda,\beta}$ to maps $F, G, \Theta_{\mu,\alpha}, \Theta_{\lambda,\beta}$ with domain $\mathbb{R}\setminus\mathbb{Z}$, as in 9.2.1.

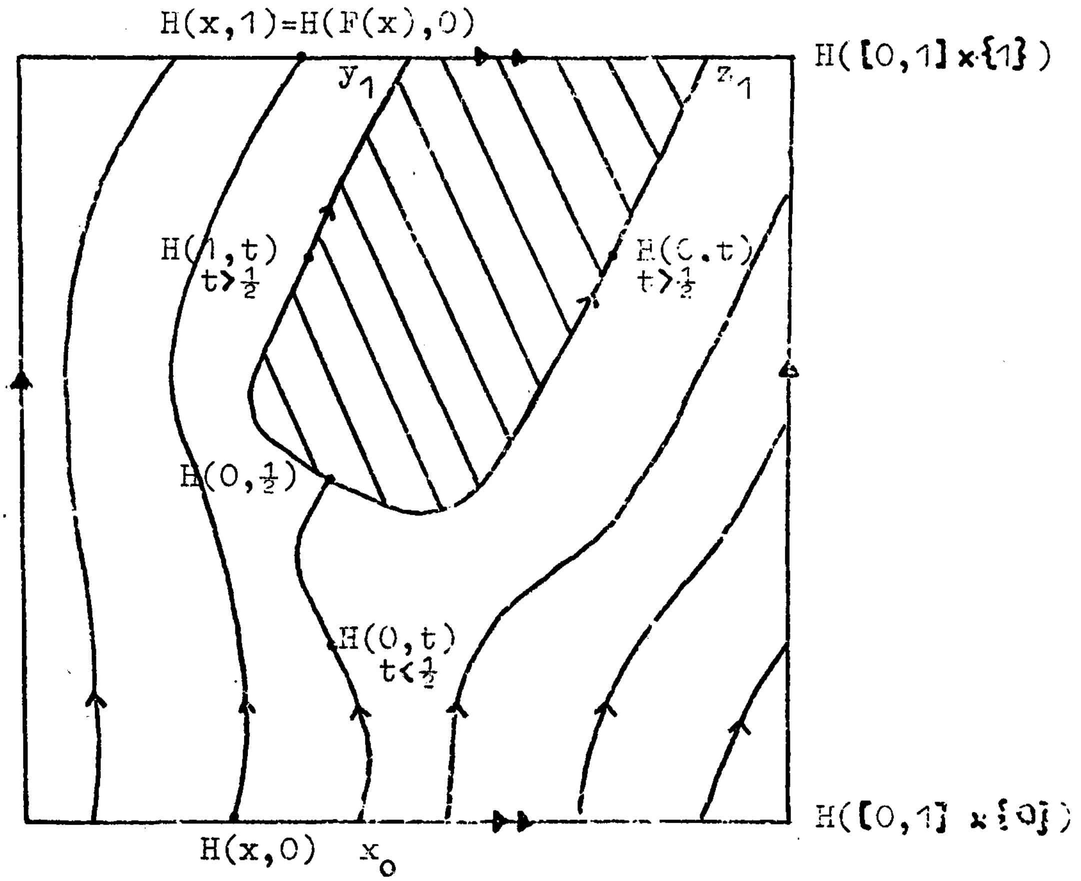


Fig. 10.5

Now use the holonomy lemma in its full C^0-force to construct continuous maps:

$H_i, H_i' : [0,1]\times[0,1] \longrightarrow M_2, M$ (respectively) for $i=1,2$

satisfying the following conditions:

(i) $H_1 | [0,1)\times(0,1)$ is a homeomorphism onto its image and $H_1(x,t) = H_1(y,s)$ if & only if :

a) $t=0, s=1$ and $x=F(y)$ (mod 1) or $y=0$ $x=\lim_{x\to 0^+} F(x)$ or $y=1, x=\lim_{x\to 0^-} F(x)$.

or b) $x=0, y=1$ and $s=t \leq \frac{1}{2}$.

(ii) $H_1(0,0) = H(1,0) = x_0$, $H_1(0,1) = z_1$; $H_1(1,1) = y_1$.

(iii) $H_1(x,0) = \rho_1(x)$.

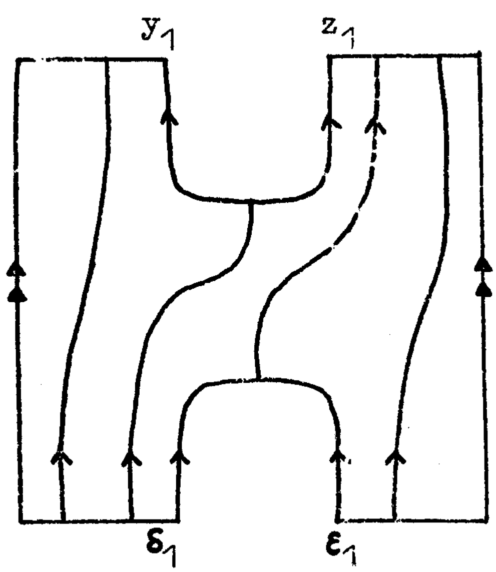
(iv) $H_1(0,\frac{1}{2})$ is a saddle point,

(v) $H_1(x,t)$ lies in a leaf independent of t,

as in figure 10.5.

Let H_2, H_1', H_2' have analogous properties with the appropriate substitutions for F, x_0, y_0, z_0 and ρ_1.

Image of χ :



Identify along $\updownarrow$

Fig. 10.6

Now the closure of $M_2 \setminus H_1([0,1] \times [0,1]) \cup H_2([0,1] \times [0,1])$ is homeomorphic to a cylinder $S^1 \times I$ with boundary components:

$$I_1 \cup H_1([0,1] \times [\tfrac{1}{2},1])$$

$$J_1 \cup H_2([0,1] \times [\tfrac{1}{2},1]).$$

The same is true of $M \setminus H_1'([0,1] \times [0,1]) \cup H_2'([0,1] \times [0,1])$. Let the homeomorphisms be χ, χ' respectively with images as shown in figure 10.6.

Applying the holonomy lemma we can construct a homeomorphism Ψ mapping the image of $\mathcal{X}$ onto the image of $\mathcal{X}'$ and such that:

$$\begin{aligned}
\Psi(H_1(0,t)) &= H_1'(0,t) \qquad t \geq \tfrac{1}{2} \\
\Psi(H_1(1,t)) &= H_1'(1,t) \qquad t \geq \tfrac{1}{2} \\
\Psi(H_2(0,t)) &= H_2'(0,t) \qquad t \geq \tfrac{1}{2} \\
\Psi(H_2(1,t)) &= H_2'(1,t) \qquad t \geq \tfrac{1}{2},
\end{aligned}$$

and the leaves of $\mathcal{J}$ are mapped onto the leaves of $\mathcal{J}_{\alpha,\beta}$ and their orientation preserved.

Now by proposition 9.6 there are homeomorphisms

$$\begin{aligned}
\varphi_1 &: A_1 \longrightarrow A_1' \\
\varphi_2 &: A_2 \longrightarrow A_2'
\end{aligned}$$

preserving orientation and such that:

$$\begin{aligned}
\varphi_1^{-1}\vartheta_{\mu,\kappa}\varphi_1 &= f \qquad \varphi_1|I_1 = \Psi|I_1 \\
\varphi_2^{-1}\vartheta_{\lambda,\beta}\varphi_2 &= g \qquad \varphi_2|J_1 = \Psi|J_1 .
\end{aligned}$$

Lift φ_1, φ_2 to orientation preserving homeomorphisms Φ_1, Φ_2 of $\mathbb{R}$ such that

$$\begin{aligned}
\Phi_1(0) &= 0 = \Phi_2(0) \\
\Theta_{\mu,\alpha} \circ \Phi_1 &= \Phi_1 \circ F \\
\Theta_{\lambda,\beta} \circ \Phi_2 &= \Phi_2 \circ G .
\end{aligned}$$

Now extend Ψ to all of M_2 by defining

$$\begin{aligned}
\Psi H_1(x,t) &= H_1'(\Phi_1(x),t) \\
\Psi H_2(x,t) &= H_2'(\Phi_2(x),t) .
\end{aligned}$$

This completes the lemma.

<u>Corollory</u>: Let $\mathfrak{F}$ and $\mathfrak{F}'$ be C^r $(r > 2)$ Morse foliations on M_2, the oriented 2-manifold of genus 2.

Suppose that $\mathfrak{F}, \mathfrak{F}'$ have no holonomy, no leaf containing more than one saddle point and exactly two non-trivial limit sets.

Suppose that each asymptotic cycle of $\mathfrak{F}$ is a positive multiple of some asymptotic cycle of $\mathfrak{F}'$.

Then $\mathfrak{F}$ and $\mathfrak{F}'$ are C^0-conjugate.

<u>Proof:</u> It follows from the results of paragraph 3.6 that the rotation numbers α, β are determined by the asymptotic cycles.

The fact that care was taken to choose a specific transverse orientation ensures that we can tell which of α and β is associated to which limit set.

Appendix 1.

Lemma 1: Let $\mathcal{F}$ be a C^r Morse foliation on M_g, the oriented 2-manifold of genus g, and let C be a circle leaf of $\mathcal{F}$. Then there is a C^r embedding

$$\Psi : S^1 \times (-1,1) \longrightarrow M_g$$

such that:

(i) $\Psi(S^1 \times \{0\}) = C$

(ii) Any circle leaf meeting the image of Ψ is of the form $(S^1 \times \{t\})$, for some $t \in (-1,1)$.

Proof: Let $\gamma : S^1 \longrightarrow C$ be a C^r diffeomorphism. Identify S^1 with $[0,1]/_{0=1}$ and let

$$\gamma' : [0,1] \longrightarrow C \text{ be the lift of } \gamma.$$

Let $H : [0,1] \times (-1\tfrac{1}{2}, 1\tfrac{1}{2}) \longrightarrow M_g$ be the map determined by the holonomy lemma (2.13) with respect to some transverse vector field.

In particular $H(t,0) = \gamma'(t)$ so that without loss of generality we can suppose that

$$H_1^{-1} H_0((-1,1)) \subseteq (-1\tfrac{1}{2}, 1\tfrac{1}{2})$$

(where $H_t : (-1\tfrac{1}{2}, 1\tfrac{1}{2}) \longrightarrow M_g : x \longmapsto H(t,x)$).

Let $\varphi : [0,1] \longrightarrow [0,1]$ be a smooth function equal to 0 on a neighbourhood of 0 and 1 on a neighbourhood of 1.

Define $K : [0,1] \times (-1,1) \longrightarrow (-1\tfrac{1}{2}, 1\tfrac{1}{2})$ by

$$K(t,v) = (1-\varphi(t))v + \varphi(t)H_1^{-1}(v) .$$

K is C^r and $K(t,v) = v$ t near 0, $K(t,v) = H_1^{-1}(v)$ t near 1.

Define $\Psi' : [0,1] \times (-1,1) \longrightarrow M_g$ by

$$\Psi'(t,v) = H(t,\dot{K}(t,v)).$$

Then Ψ' projects to the required map

$$\Psi : S^1 \times (-1,1) \longrightarrow M_g.$$

Lemma 2: Let $B_1 = \{x \in \mathbb{R}^2 : \|x\| < 1\}$, and let $\mathcal{F}, M_g$ be as in lemma 1.

Let $\varphi : B_1 \longrightarrow M_g$ be a C^r embedding

and $\Psi : S^1 \times (-1,1) \longrightarrow M_g$ be a C^r embedding such that

$$\Psi(S^1 \times (-1,1)) \cap \varphi(B_1) = \Psi(S^1 \times (-1,-\tfrac{1}{2})).$$

Then there is a C^r embedding

$$\varphi' : B_1 \longrightarrow M_g$$

with $\varphi'(B_1) = \varphi(B_1) \cup \Psi(S^1 \times (-1,\tfrac{1}{2}))$.

Proof: Let $\rho : (-1,1) \longrightarrow (-1,1)$ be a C^r orientation preserving diffeomorphism with $\rho(-\tfrac{1}{2}) = \tfrac{1}{2}$, and ρ equal to the identity map near ± 1.

Then let
$$\varphi'(x) = \begin{cases} x & x \notin \varphi^{-1}\Psi(S^1 \times (-1,1)) \\ \Psi(\mathrm{id} \times \rho)\Psi^{-1}\varphi(x) & x \in \varphi^{-1}\Psi(S^1 \times (-1,1)). \end{cases}$$

Then φ' is the required C^r embedding.

Lemma 3: Let $\mathcal{F}, M_g$ be as in lemma 1.

Let C be a circle leaf of $\mathcal{F}$, and suppose that there is a one-sided neighbourhood of C containing no circle leaf except C.

Then there is a C^r embedding

$$\Psi : S^1 \times (-1,1) \longrightarrow M_g$$

such that (i) $\Psi(S^1 \times \{-\frac{1}{2}\}) = C$

(ii) All the circles $\Psi(S^1 \times \{t\})$ with $0 \leq t \leq \frac{1}{2}$

are transverse to $\mathfrak{F}$.

Proof: The proof of this lemma is similar to that of lemma 1.

For we may choose the transverse interval in the definition of H so that $H_1(v) \neq v \;\; v \in (0, 1\frac{1}{2})$

Then choosing φ to be a diffeomorphism with

$$\varphi(0) = 0 \; , \; \varphi(1) = 1$$
$$\frac{d\varphi^k}{dt}(0) = \frac{d\varphi^k}{dt}(1) \qquad k > 0$$

and after reparametrisation of the second factor in $S^1 \times (-1,1)$ we obtain the required embedding.

Lemma 4: Let $\rho : S^1 \times (-1,1) \longrightarrow S^1 \times (-1,1)$ be a C^r orientation preserving diffeomorphism which maps circles $S^1 \times \{t\}$ to circles $S^1 \times \{\eta(t)\}$ with ρ orientation preserving. Then there is a real number $\varepsilon > 0$ and a diffeomorphism

$$\mu : S^1 \times (-1,1) \longrightarrow S^1 \times (-1,1)$$

which has all the properties of ρ and satisfying:

(i) $\mu | S^1 \times (-1, -\varepsilon)$ = identity map

(ii) $\mu | S^1 \times (\varepsilon, 1) = \rho | S^1 \times (\varepsilon, 1)$.

Proof: ρ is of the form

$$\rho(\theta, t) = (\nu(\theta, t), \eta(t))$$

where $\eta : (-1,1) \longrightarrow (-1,1)$ is an orientation preserving C^r diffeomorphism.

Let $\eta' : (-1,1) \longrightarrow (-1,1)$ be an orientation preserving C^r diffeomorphism with

$$\eta' \mid (-1,-\varepsilon) = \text{identity}$$
$$\eta' \mid (\varepsilon,1) = \eta \mid (\varepsilon,1) \quad ;$$

and $\varphi : (-1,1) \longrightarrow [0,1]$ a C^r map satisfying

$$\varphi(x) = \begin{cases} 1 & x \in (-1,-\varepsilon) \\ 0 & x \in (\varepsilon,1) \, . \end{cases}$$

Then define μ by

$$\mu(\theta,t) = (\varphi(t)\theta + (1-\varphi(t))\nu(\theta,t), \eta'(t)).$$

Then μ is the required diffeomorphism.

<u>Lemma 5</u>: Let $\rho : [0,1] \times [0,1] \longrightarrow [0,1] \times [0,1]$ be a C^r orientation preserving diffeomorphism which maps lines $[0,1] \times \{x\}$ to lines $[0,1] \times \eta(x)$ where η is orientation preserving.

Then there is a real number $\varepsilon > 0$ and a diffeomorphism

$$\mu : [0,1] \times [0,1] \longrightarrow [0,1] \times [0,1]$$

having all the properties of ρ and satisfying

(i) $\mu \mid [0,1] \times [0,\varepsilon]$ = identity map

(ii) $\mu \mid [0,1] \times [1-\varepsilon,1] = \rho \mid [0,1] \times [1-\varepsilon,1]$.

<u>Proof:</u> Similar to that of lemma 4.

Appendix 2.

Construction of $\mathcal{B}^+$.

In this example the square is foliated by lines x = constant except in a neighbourhood of $\overline{D}_c$ - where c is the centre. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^∞ bump function with the following properties:

(i) $\varphi(x) \geqslant 0 \ \forall x \in \mathbb{R}$

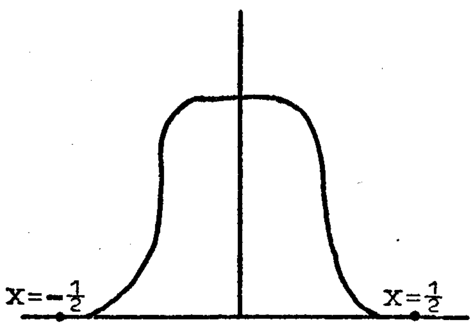
(ii) $\varphi(x) = \varphi(-x)$

(iii) $\varphi(x) = 0 \quad x \notin (-\frac{1}{2}, \frac{1}{2})$

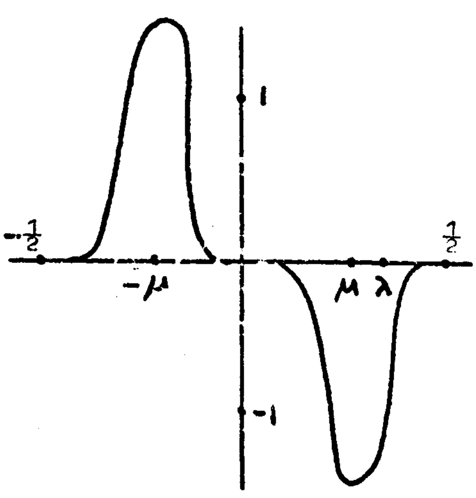
(iv) $\varphi(x) = 1$ on a small neighbourhood of 0

(v) φ monotone increasing on $(-\infty, 0)$ and monotone decreasing on $(0, \infty)$

(vi) φ' monotone increasing on $(-\infty, -\mu)$, (μ, ∞) and monotone decreasing on $(-\mu, \mu)$.



Graph of φ



Graph of φ'

Fig. A2.1

Choose $\lambda > 0$ such that $\varphi'(\lambda) < 0$, $\varphi''(\lambda) > 0$

Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x,y) \longmapsto x\varphi'(\lambda) - \varphi(\sqrt{(x^2+y^2)})$.

Then f is C^∞ and (x,y) is a critical point of f

if & only if $\quad \varphi'(\lambda) = \varphi'(|x|)\operatorname{sign}x \qquad (1)$

$$y = 0 .$$

The Hessian of f at such a point is

$$\begin{pmatrix} \varphi''(|x|) , & 0 \\ 0 , & -\varphi'(|x|)/|x| \end{pmatrix}$$

Now $\varphi'(\lambda) < 0$ implies that (1) can only be satisfied if $x > 0$.
Figure A2.1 shows that $\varphi'(|x|) = \varphi'(\lambda)$ at precisely two values $x = \lambda$ and $x = \lambda'$ with $0 < \lambda' < \lambda$.
Computing the Hessian we see that $(\lambda, 0)$ is a centre and $(\lambda', 0)$ a saddle point.
The Morse foliation determined on $(-1,1) \times (-1,1)$ by the level curves of f has the properties stated in section 4.3 and is the required Morse foliation $\mathcal{B}^+$.

Construction of $\mathcal{E}^+$.

This example is constructed as follows.
Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^∞ function with $\varphi = 0$ on a neighbourhood of 0 and $\varphi(x) = 1$ for $|x| > 1-\varepsilon$, where $0 < \varepsilon < 1$.
Suppose that $\varphi(\mathbb{R}) \subseteq [0,1]$.
Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x,y) = N\sin 2\pi y - \varphi(\sin 2\pi y)\cos 2\pi x \sin 2\pi y$$

for $N \in \mathbb{R}$ sufficiently large.

Then the critical points of f are the points with

$$y \equiv \tfrac{1}{4}, \tfrac{3}{4} \pmod 1 \quad \text{and} \quad x \equiv 0, \tfrac{1}{2} \pmod 1.$$

The Hessian at such a point is

$$\begin{pmatrix} -4\pi^2 \cos 2\pi x \sin 2\pi y, & 0, \\ 0, & -4\pi^2 (N \sin 2\pi y + \sin 2\pi y \cos 2\pi x) \end{pmatrix}$$

with sign of the determinant the same as that of $N\cos 2\pi x$ as N is large.

Thus the points $x \equiv 0 \pmod 1$ are centres and the points $x \equiv \tfrac{1}{2}$ are saddle points.

The induced foliation on the half torus $0 < y < \tfrac{1}{2}$ is $\mathcal{C}^+$ or $\mathcal{C}^-$ according to the transverse orientation chosen.

Appendix 3.

Lemma 1: Let $f : (-1,1) \times (-1,1) \longrightarrow (-1,1) \times (-1,1)$ be a C^r $(1 \leq r \leq \infty)$ diffeomorphism which agrees with the identity map on a neighbourhood of the boundary of $(-1,1) \times (-1,1)$.
Then f is C^r isotopic to the identity through diffeomorphisms which agree with the identity on a neighbourhood of the boundary of $(-1,1) \times (-1,1)$.
Proof: A proof can be found in [42] or [44].

Lemma 2: Every orientation preserving C^r (1 r) diffeomorphism of the 2-sphere is C^r-isotopic to the identity map.
Proof: In [44] it is proved that SO(3) is a strong deformation retract of the space of diffeomorphisms of S^2.
Since SO(3) is path connected this implies the result.

The result also follows from lemma 1 by showing that every diffeomorphism of S^2 is isotopic to one which agrees with the identity on some disc.
To see that this is true let f be a diffeomorphism of S^2.
Taking an isotopy through rotations we can assume that f fixes a point. Stereographic projection then gives:

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 .$$

Taking another isotopy we can assume that $g(0) = 0$.

Lemma 8.1 of [41] gives a diffeomorphism

$$g_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

which agrees with g on the disc of radius $\frac{1}{2}$, with the identity map outside the unit disc and which is isotopic to the identity through diffeomorphisms which agree with the identity map outside the unit disc.

Then g is isotopic to gg_0^{-1} which agrees with the identity map on the disc of radius $\frac{1}{2}$.

The result follows.

Appendix 4

The Morse foliations $\mathfrak{F}^+_{f,u}$.

The construction is in two parts.
First we construct a Morse foliation on an annulus with u holes which depends only on the domain of f - this is the left-hand threequarters of the diagram in figure A4.2. Then on a second annulus we construct a flow which when adjoined to the first Morse foliation gives the required Morse foliation with point of first return function f - this is the right-hand quarter of the diagram in figure A4.2.

In appendix 2 we constructed a Morse foliation $\mathfrak{D}^+$ on $(-1,1)\times(-1,1)$ as the level surfaces of a Morse function f. Then the flow of the vector field

$$\frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y}$$

is everywhere transverse to $\mathfrak{D}^+$ (see figure A4.1).

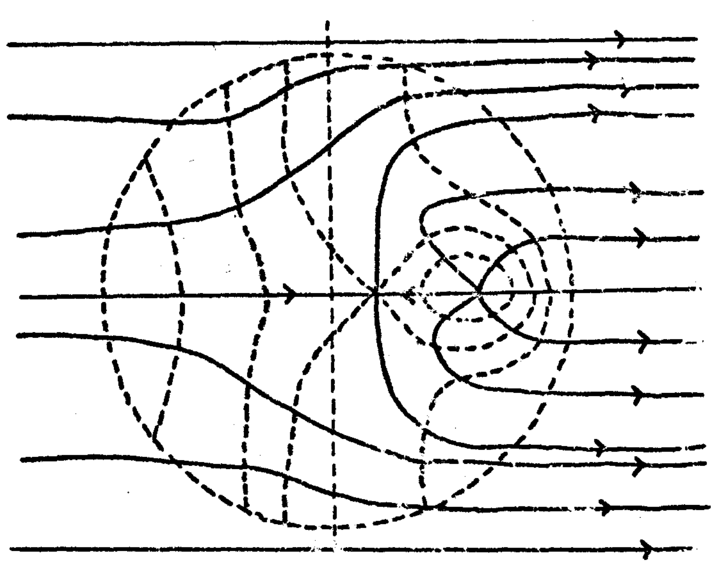


Fig. A4.1

Furthermore, outside the circle centre the origin, of radius $\frac{1}{2}$, the flow lines are lines y = constant.
Orient the transverse flow so that a pair
(tangent to transverse flow, tangent to $\mathcal{B}^+$) lies in the orientation of $\mathbb{R}^2$.

Denote this flow by $\mathcal{G}$, and suppose that it is defined on $[-1,1] \times [-1,1]$.

Suppose $f : S^1 \setminus \{x_1,\ldots,x_u\} \longrightarrow S^1 \setminus I_1 \cup \ldots \cup I_u$ is a diffeomorphism satisfying the properties given in 10.1.1.
Identify S^1 with $[0,1]/_{0=1}$ and choose representatives of the points $x_1,\ldots,x_u$ in $[0,1]$ with $0 < x_1 < \ldots < x_u < 1$.
We wish to define a flow on $[0,\frac{3}{4}] \times [0,1]$, as shown in figure A4.2.
Choose $\varepsilon_1,\ldots,\varepsilon_u > 0$ such that the closed intervals $[x_i-\varepsilon_i, x_i+\varepsilon_i]$ are disjoint and define

$$f_i : [-1,1] \times [-1,1] \longrightarrow [0,\tfrac{3}{4}] \times [x_i-\varepsilon_i, x_i+\varepsilon_i] \text{ by}$$

$$f_i(x,y) = ((3/8)(x+1), \varepsilon_i y + x_i).$$

Choose the flow on $[0,\frac{3}{4}] \times [x_i-\varepsilon_i, x_i+\varepsilon_i]$ to be $f_i^{-1*}\mathcal{G}$ and extend this by lines y=constant to a flow on $[0,\frac{3}{4}] \times [0,1]$ which agrees with the flow given by lines y = constant near the boundary.
It thus determines a well-defined flow $\mathcal{H}$ on $[0,\frac{3}{4}] \times S^1$.

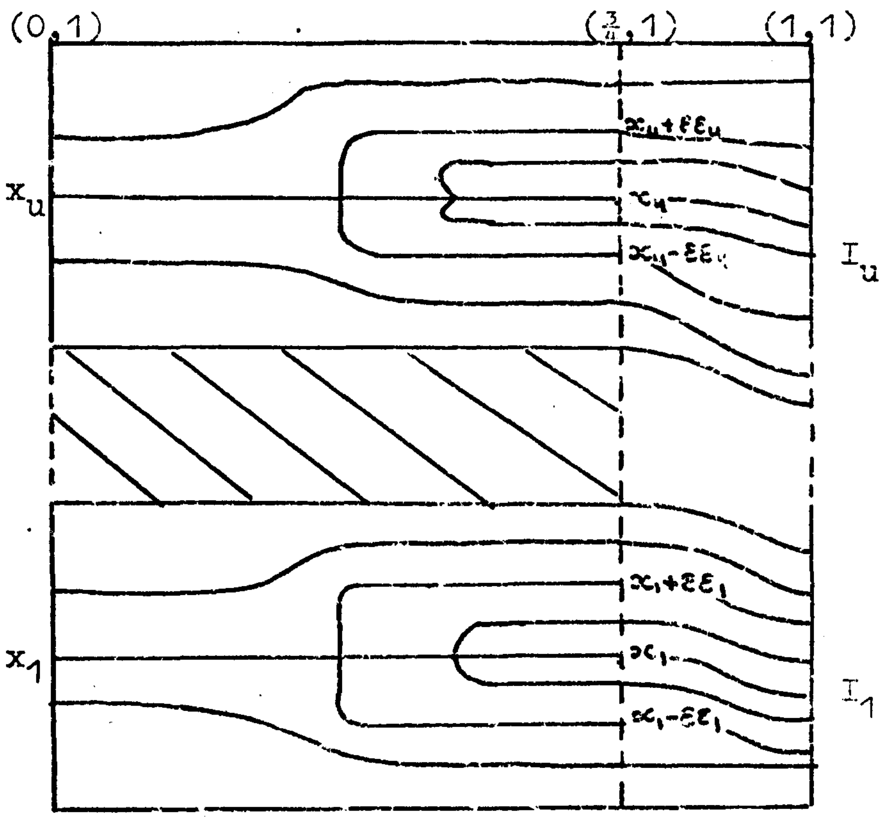


Fig. A4.2

Now in $\mathcal{G}$, the holonomy map is a diffeomorphism

$$g : \{-1\} \times ([-1,1]\setminus\{0\}) \longrightarrow \{1\} \times ([-1,1]\setminus[-\varepsilon,\varepsilon]).$$

Thus in $\mathcal{H}$, the holonomy map is a diffeomorphism

$$h : \{0\} \times S^1\setminus\{x_1,\dots,x_u\} \longrightarrow \{\tfrac{3}{4}\} \times S^1\setminus J_1 \cup \dots \cup J_u$$

where $J_k = [x_k-\varepsilon\varepsilon_k, x_k+\varepsilon\varepsilon_k]$.

Further : $\lim\limits_{x\to x_i^-} h(0,x) = x_i-\varepsilon\varepsilon_i$, $\lim\limits_{x\to x_i^+} h(0,x)=x_i+\varepsilon\varepsilon_i$.

$\mathcal{H}$ has u saddle points and u sources.

For each i, there is a source such that one flow line emanating from it is a separatrix and the other flow lines eventually cut $\{\frac{3}{4}\} \times (x_i - \varepsilon\varepsilon_i, x_i + \varepsilon\varepsilon_i)$.

This completes the first part of the proof.

Now identify S^1 with $\{0\} \times S^1$ and $\{\frac{3}{4}\} \times S^1$ in $[0, \frac{3}{4}] \times S^1$.

Consider the map

$$\rho : S^1 \setminus J_1 \cup \dots \cup J_u \longrightarrow S^1 \setminus I_1 \cup \dots \cup I_u$$

defined by $\rho(x) = fg^{-1}(x)$.

Then ρ extends to a C^r diffeomorphism (which we also call ρ) of S^1.

Again identifying S^1 with $[0,1]/_{0=1}$, choose an isotopy

$$\Phi_t : S^1 \longrightarrow S^1 \qquad \tfrac{3}{4} \leqslant t \leqslant 1$$

such that Φ_t is the identity near $t = \frac{3}{4}$ and ρ near $t = 1$.

Φ determines a flow on $[\frac{3}{4}, 1] \times S^1$ and adjoining this to $\mathcal{H}$ determines a flow on $[0,1] \times S^1$ which is C^r and such that the holonomy map from $\{0\} \times S^1$ to $\{1\} \times S^1$ is given by f.

Now identifying $\{0\} \times S^1$ and $\{1\} \times S^1$ determines a flow on the torus $S^1 \times S^1$.

Let A be the circle corresponding to $\{0\} \times S^1$ and remove small discs whose boundary circles are transverse to the flow from the sources.

This gives the required flow $\mathfrak{F}^+_{f,u}$ on T_u.

The Morse foliation $\mathcal{D}$ on T_2

We construct a C^∞ Morse foliation on T_2, the torus with two discs removed, which has the properties outlined at the end of section 10.3.

Let $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x,y) \longmapsto -N\sin 2\pi y + \Psi(\sin 2\pi y)\sin 2\pi y \sin 2\pi x$ where $N \in \mathbb{R}$ is a large positive number and Ψ is a function on $\mathbb{R}$ with range $[-1,1]$ and

$$\Psi(x) = \begin{cases} 0 & x \text{ near } 0 \\ 1 & x \geqslant 1-\varepsilon \\ -1 & x \leqslant -1+\varepsilon \end{cases},$$

$$\Psi(x) = -\Psi(-x).$$

φ is a Morse function with saddle points at points $(\frac{1}{4}+m, \frac{3}{4}+n)$ and $(\frac{3}{4}+m, \frac{1}{4}+n)$ $m,n \in \mathbb{Z}$ and centres at points $(\frac{1}{4}+m, \frac{1}{4}+n)$ and $(\frac{3}{4}+m, \frac{3}{4}+n)$. The level surfaces of φ define a Morse foliation of the torus.

The vector field $(\partial\varphi/\partial x, \partial\varphi/\partial y)$ has a flow which is everywhere transverse to the Morse foliation defined by φ. It has sources at the points $(\frac{1}{4}+m, \frac{1}{4}+n)$ $m,n \in \mathbb{Z}$ sinks at the points $(\frac{3}{4}+m, \frac{3}{4}+n)$ $m,n \in \mathbb{Z}$ and saddle points at points $(\frac{1}{4}+m, \frac{3}{4}+n)$ and $(\frac{3}{4}+m, \frac{1}{4}+n)$ $m,n \in \mathbb{Z}$.
Projecting onto the torus defines a flow $\mathcal{D}'$ on the torus everywhere transverse to the flow defined by φ.

Now remove from the source and sink a small neighbourhood bounded by a flow line of the flow defined by φ.
This defines a flow $\mathcal{D}$ on T_2 as shown in figure A4.3.

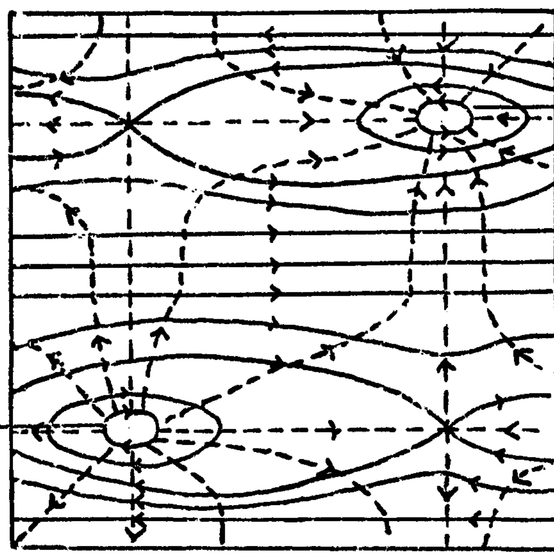


Fig. A4.3

Orient the flow as shown in figure A4.3 and let the boundaries of T_2 be $\varphi^{-1}(a)$ and $\varphi^{-1}(-a)$ for some $a > 0$.
Let K^+ be the component of T_2 from which all leaves depart and let K^- be the other boundary component.
Now φ has the symmetries:

$$\varphi(x,\tfrac{1}{4}+y) = \varphi(x,\tfrac{1}{4}-y) ,$$
$$\varphi(x,\tfrac{3}{4}+y) = \varphi(x,\tfrac{3}{4}-y) ,$$
$$\varphi(x+\tfrac{1}{2},y+\tfrac{1}{2}) = \varphi(x,y).$$

Hence we may choose C^∞ embeddings

$$\iota^{\pm} : S^1 \longrightarrow K^{\pm}$$

with the properties required in 10.3.
This completes the construction.

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PART II

NATURAL DIFFERENTIAL OPERATORS ON RIEMANNIAN MANIFOLDS AND REPRESENTATIONS OF THE ORTHOGONAL AND SPECIAL ORTHOGONAL GROUPS

In his paper "The Foundation of the General Theory of Relativity" ([3]) published in 1916, A.Einstein remarked that on a Lorentz manifold (M,g), the only covariant tensors of order 2 which depend in any local co-ordinate system only on the metric tensor and its first two derivatives and which depend linearly on the second derivative, are linear combinations of the tensors gR and $R_{ij}dx^i \otimes dx^j$ where R is the scalar curvature and $R_{ij}dx^i \otimes dx^j$ is the Ricci curvature. In an appendix to [10] H.Weyl proves that R is the only function with these properties.

More recently in [6], P.Gilkey investigated, in a similar vein, forms on Riemannian manifolds and his results are important tools in the proof of the index theorem given by Atiyah, Bott and Patodi in [2]. In [5] D.B.A.Epstein introduces the concept of natural tensor field on Riemannian manifolds. His paper was a major catalyst in the production of this one and should preferably be read before it.

The purpose of this paper is twofold. Firstly it is to study natural tensor fields on Riemannian and oriented Riemannian manifolds. Maintaining the spirit of the earlier results I shall impose a regularity condition on natural tensor fields, which leads to their complete classification as a space of homomorphisms, between certain representation spaces for the general linear group, which are equivariant under the action of the orthogonal or special orthogonal group. The second reason for writing this paper is to define and investigate the notion of natural differential operator in an analogous fashion. It turns out that this problem reduces to the study of natural tensor fields.

Ishall only give results on Riemannian manifolds. However P.Gilkey has now extended the Gilkey theorem (c.f. [2] paragraph 2) to apply to manifolds with an indefinite metric ([8]).
All manifolds, all functions between manifolds and all tensor fields in this paper are C^∞.

1 Preliminaries.

(1.1) We shall be concerned with functors E from the category of vector spaces and homomorphisms to itself (see e.g. [4]). With such a functor we shall assume given:
(i) A monomorphism of functors $i_E: E \longrightarrow T^r$ where T^r is the rth tensor power functor for some r.
(ii) For each ordered basis (v_i) of a vector space V an ordered basis $E(v_i)$ of EV.

(iii) Given vector spaces V,W and an isomorphism $\varphi \in \mathrm{Hom}(V,W)$; $E(\varphi v_i) = E\varphi(E(v_i))$.

The rank of E is r.

(1.2) An inner product b on a vector space V induces an innerproduct b on T^rV and hence on EV, which we denote by Eb. Thus EV is a representation space for $GL(V)$ and $O(V,b)$ with

$$E(O(V,b)) \subseteq O(EV,Eb).$$

We denote $(EV)^*$ by E^*V, then $GL(V)$ acts on E^*V via

$$\varphi \longmapsto (E\varphi^{-1})^* \ , \text{ for } \varphi \in GL(V).$$

With this action $O(V,b)$ acts on E^*V as a subgroup of $O(E^*V,E^*b)$.

If (v_i) is an ordered basis of V and $E(v_i) = (w_j)$ take the ordered basis $E(v^i)$ of E^*V to be the ordered basis (w^k) where $w^k(w_j) = \delta^k_j$.

(1.3) Given a Riemannian manifold (M,g), afunctor E as in (1.1) induces Riemannian vector bundles $(EM,Eg),(E^*M,E^*g)$ over M with connection induced from the Levi-Civita connection.

These constructions are functorial and determine subfunctors of the rth tensor power of the tangent bundle and cotangent bundle respectively.

Further it follows from (1.1) (ii) that given any local co-ordinate system x, there are determined unique ordered local bases of sections $E(\partial/\partial x^i)$, $E(dx^i)$ for EM, E^*M respectively.

(1.4) We describe in detail certain functors with the properties required in (1.1), which will be needed later. Given a vector space V, S_r (Symmetric group of degree r) acts on T^rV in the usual way.

(i) The functor S^r.

$S^rV = \{v \in T^rV : v = \sigma v \text{ all } \sigma \in S_r\}$.

Let $\dim V = n$ and let (v_i) be an ordered basis for V.

For each r-tuple of integers $(i_1, \ldots\ldots, i_r)$ $1 \leqslant i_1 \leqslant \ldots\ldots \leqslant i_r \leqslant n$ let $v_{i_1 \ldots\ldots i_r} = \sum v_{j_1} \otimes \ldots\ldots \otimes v_{j_r}$ where summation takes place over all distinct r-tuples $(j_1, \ldots\ldots, j_r)$ which are rearrangements of $(i_1, \ldots\ldots, i_r)$.

We let $S^r(v_i) = (v_{i_1 \ldots\ldots i_r})$ ordered by lexicographical ordering on $(i_1, \ldots\ldots, i_r)$.

(ii) The functors Y_r, $r \geqslant 2$.

Let T be the Young tableau with r squares in the first row and two in the second.

Let the first r - 2 positive integers (starting at 1) be arranged in T in increasing order down the columns from left to right.

Let I_k denote the kth column in this arrangement. For each integer m in I_{k+1} let $P_{k,m}$ denote the set of permutations $\sigma \in S_{r+2}$ which fix every integer except those in $I_k \cup \{m\}$, and which preserve the order of those in I_k.

$Y_r V = \{ v \in T^{r+2}V : \sum_{\sigma \in P_{k,m}} \mathcal{E}(\sigma)\sigma v = 0 \ 1 \leqslant k \leqslant r-1, \ m \in I_{k+1}, v+(i,j)v=0 \quad i,j\in$

For each ordered r+2-tuple of integers $(i_1, \dots, i_{r+2})$

with $1 \leqslant i_k \leqslant n$, $i_1 < i_2$; $i_3 < i_4$; $i_2 \leqslant i_4$;

$i_1 \leqslant i_3 \leqslant i_5 \leqslant i_6 \leqslant \dots \leqslant i_{r+2}$

Let $v_{i_1 \dots i_r} = \sum_{\sigma} (v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} - v_{i_{\sigma(2)}} \otimes v_{i_{\sigma(1)}}) \otimes$

$(v_{i_{\sigma(3)}} \otimes v_{i_{\sigma(4)}} - v_{i_{\sigma(4)}} \otimes v_{i_{\sigma(3)}}) \otimes (v_{i_{\sigma(5)}} \otimes \dots \otimes v_{i_{\sigma(r+2)}})$

where σ runs over all permutations in S_{r+2} which preserve the sets $\{2,4\}$, $\{1,3,5,6,\dots,r+2\}$ and lead to distinct $(r+2)$-tuples $(i_{\sigma(1)}, \dots, i_{\sigma(r+2)})$.

Then $Y_r(v_i) = (v_{i_1 \dots i_{r+2}})$ ordered by lexicographical ordering on $(i_1, \dots, i_{r+2})$.

(iii) If E_1, E_2 are functors as in (1.1) so is $E_1 \otimes E_2$ with lexicographical ordering taken for the basis.

(1.5) It is well known that for any vector space V, $Y_r V$ is an irreducible representation space for $GL(V)$.

For each $r \geqslant 2$ define $GL(V)$ maps:

$\alpha_r : S^2(V) \otimes S^r(V) \longrightarrow Y_r(V)$

$\beta_r : Y_r(V) \longrightarrow S^2(V) \otimes S^r(V)$ by:

$\alpha_r(\varepsilon_{i_1 \dots i_{r+2}} v_{i_1} \otimes \dots \otimes v_{i_{r+2}}) =$

$$(g_{i_1 i_4 i_3 i_2 i_5 \dots i_{r+2}} - g_{i_1 i_3 i_2 i_4 i_5 \dots i_{r+2}} - g_{i_4 i_2 i_3 i_1 i_5 \dots i_{r+2}}$$
$$+ g_{i_3 i_2 i_1 i_4 i_5 \dots i_{r+2}}) v_{i_1} \otimes \dots \otimes v_{i_{r+2}} .$$
$$\beta_r (R_{i_1 \dots i_{r+2}} v_{i_1} \otimes \dots \otimes v_{i_{r+2}}) =$$
$$-\frac{(r-1)}{(r+1)!} \sum_{\sigma} R_{i_1 i_{\sigma(3)} i_2 i_{\sigma(4)} i_{\sigma(5)} \dots i_{\sigma(r+2)}} v_{i_1} \otimes \dots \otimes v_{i_r}$$

where (v_i) is an ordered basis of V, the summation convention is used and the sum runs over all permutations σ of $\{3,\dots,r+2\}$. α_r and β_r do not depend on the basis (v_i) chosen.

Note that α_r and β_r satisfy :

a) $\alpha_r \beta_r = \mathrm{id}$

b) If $\sum_{\sigma \in S_{r+1}} g_{i_1 i_{\sigma(2)} \dots i_{\sigma(r+2)}} = 0$ (where S_{r+1} is the group of permutations of $\{2,\dots,r+2\}$) then

$$\beta_r \alpha_r (g_{i_1 \dots i_{r+2}} v_{i_1} \otimes \dots \otimes v_{i_{r+2}}) = g_{i_1 \dots i_{r+2}} v_{i_1} \otimes \dots \otimes v_{i_{r+2}} .$$

Note that every element in the image of β_r satisfies this symmetry condition.

(1.6) The maps α_r, β_r in (1.5) determine GL(V) maps :

$$\alpha : \bigoplus_{r \geq 2} S^2(V) \otimes S^r(V) \longrightarrow \bigoplus_{r \geq 2} Y_r(V)$$

$$\beta : \bigoplus_{r \geq 2} Y_r(V) \longrightarrow \bigoplus_{r \geq 2} S^2(V) \otimes S^r(V)$$ such that $\alpha\beta = \mathrm{id}$.

Let (e_i) be the standard basis of $\mathbb{R}^n$, where $\mathbb{R}$ denotes the real numbers, with dual basis (e^i).

Let $W \in \bigoplus_{r \geqslant 2} Y_r{}^*\mathbb{R}^n$ and suppose that the component of $\beta(W)$ in $S^{2*}\mathbb{R}^n \otimes S^{r*}\mathbb{R}^n$ is the tensor $g_{i_1 \dots i_{r+2}} e^{i_1} \otimes \dots \otimes e^{i_{r+2}}$.

Let $g(W)_{ij}$ be the real valued functions on $\mathbb{R}^n$ $(1 \leqslant i,j \leqslant n)$ defined by $g(W)_{ij}(x) = \delta_{ij} + \sum_{r=2}^{\infty} g_{ijk_1 \dots k_r} x^{k_1} \dots x^{k_r}$

(the superscripts denoting co-ordinates and not powers). It follows from paragraph 2 of [5] that these functions determine a Riemannian metric on a neighbourhood $U(W)$ of 0 in $\mathbb{R}^n$.

The oriented Riemannian manifold $(U(W), g(W))$ has the inclusion chart as a normal co-ordinate chart at the origin. It is oriented by the usual orientation on $\mathbb{R}^n$. Conversely it is also shown in paragraph 2 of [5] that given a Riemannian manifold (M,g) there exist tensors $W^r \in Y_r{}^*M$ $(r \geqslant 2)$, obtained from contracting tensor products of no higher than the (r-2)th covariant derivative of the curvature tensor, such that in any normal co-ordinate system at $p \in M$ the coefficients of $\beta_r(W^r(p))$ are the rth partial derivatives of the metric.

2 The classification theorem.

(2.1) Natural tensor fields on Riemannian manifolds are introduced by D.B.A.Epstein in [5].

We extend the definition to the case of oriented Riemannian manifolds.

DEFINITION: Let E,F be functors as in (1.1).

A <u>natural tensor field t on Riemannian manifolds</u> (respectively <u>oriented Riemannian manifolds</u>) of type (E,F) assigns to each Riemannian manifold (resp. oriented Riemannian manifold) (M,g) a tensor field

$$t(M,g) \in C^{\infty}(EM \otimes F^{*}M)$$

such that if $f : M' \longrightarrow M$ is a diffeomorphism (resp. orientation preserving diffeomorphism) onto an open submanifold then

$$f^{*}t(M,g) = t(M',f^{*}g).$$

(2.2) Epstein has pointed out in [5] that the problem of classifying all natural tensor fields is a complicated one.

However there is a natural concept of regularity for such tensor fields which was essentially introduced by Atiyah, Bott and Patodi in [2] paragraph 2.

A natural tensor field t, of type (E,F), on Riemannian (respectively oriented Riemannian) manifolds is <u>regular</u> if given (M,g), a Riemannian (resp. oriented Riemannian) manifold, and a local co-ordinate chart x on $U \subseteq M$, then the coefficients of $t(M,g)$ with respect to the local basis $E \otimes F(\partial/\partial x^{i} \otimes dx^{j})$ are given by universal polynomials in g_{ij},

$\partial^{|\alpha|} g_{ij}/\partial x^{\alpha}$ (α a multi-index, $|\alpha| \leq N$ large) and $(\det g_{ij})^{-1}$ (or $(\det g_{ij})^{-\frac{1}{2}}$ in the oriented case).

A justification of this definition is given by Atiyah, Bott and Patodi in [2] para. 2 for the unoriented case. On the other hand, the space of oriented Riemannian structures on a vector space V is naturally identified with $GL(n,\mathbb{R})/SO(n,\mathbb{R})$ $n = \dim V$. It is well known and is shown in the appendix (A.2) that any rational function f on $GL(n,\mathbb{R})$ invariant under the action of $SO(n,\mathbb{R})$ is of form: $f(A) = F(AA^t) + (\det A)^{-1}G(AA^t)$ $A \in GL(n,\mathbb{R})$ where

$F,G : SM(n,\mathbb{R}) \longrightarrow \mathbb{R}$ are rational functions on the space of symmetric matrices.

Since the identification of $GL(n,\mathbb{R})/SO(n,\mathbb{R})$ with the space of oriented Riemannian structures is given by :

$$[A] \in GL(n,\mathbb{R})/SO(n,\mathbb{R}) \longrightarrow (AA^t, \mathrm{sign}(\det A)),$$

the corresponding identification of rings of rational functions shows that it is natural to regard $\mathbb{R}[g_{ij},(\det g_{ij})^{-\frac{1}{2}}]$ as the ring of functions on the space of oriented Riemannian structures.

REMARK: In applications to the Index Theorem, polynomial dependence on $(\det g_{ij})^{-\frac{1}{2}}$ appears explicitly even in the unoriented case although this was overlooked in the original proof in [2], see [1]. In fact it follows from [5] theorem 5.2 that even if we merely demand that the coefficients of our tensor field be given by universal polynomials in $\partial g_{ij}/\partial x^{\alpha}$ (α a multi-index, $1 \leqslant |\alpha| \leqslant N$ large) with coefficients functions of the g_{ij} (not necessarily continuous) then the tensor field is regular (polynomial in the terminology of [5]).

(2.3) An important class of natural tensor fields is those which are homogeneous (see [5] paras. 6&7).
A natural tensor field t is homogeneous of weight k if
$t(M,\lambda^2 g) = \lambda^k t(M,g)$, all real numbers λ.
Note that g itself is homogeneous of weight 2 and that the tensors W^r $(r \geqslant 2)$ introduced in (1.3) are also homogeneous of weight 2.
If t has weight k and is of type (E,F) with rankE = a. rankF = b then t has normalised weight w = b-a-k.
t has maximal weight if w = 0.

(2.4) Before proceding with the main theorem, we need the following crucial lemma:
LEMMA: Let $(V,\langle\, , \rangle)$ be an oriented inner product space. Then:

(i) The vector space $\mathrm{Hom}_{O(V)}(\overset{k}{\otimes} V , \mathbb{R})$ is zero if k is odd, and if k is even is spanned by elements of the form :

$$v_1 \otimes \cdots \otimes v_k \longmapsto \langle v_{\Pi(1)}, v_{\Pi(2)}\rangle \cdots \langle v_{\Pi(k-1)}, v_{\Pi(k)}\rangle$$
where $\Pi \in S_k$.

(ii) The vector space $\mathrm{Hom}_{SO(V)}(\overset{k}{\otimes} V, \mathbb{R})$ is equal to $\mathrm{Hom}_{O(V)}(\overset{k}{\otimes} V,\mathbb{R})$ except, when k-n is non negative and even, any linear combination of maps :

$$v_1 \otimes \dots \otimes v_k \longmapsto$$
$$\sum_{\mu\in S_n} \varepsilon(\mu) v_{\pi(1)\mu(1)} \cdots v_{\pi(n)\mu(n)} \langle v_{\pi(n+1)}, v_{\pi(n+2)}\rangle \cdots \langle v_{\pi(k-1)}, v_{\pi(k)}\rangle$$

where $\pi\in S_k$, and $v_j = \sum_{l=1}^{n} v_{jl} e_l$ $((e_1,\dots,e_n)$ a positively oriented basis); also lies in $\mathrm{Hom}_{SO(V)}(\overset{k}{\otimes} V,\mathbb{R})$.

PROOF: (i) is proved in [2] ,appendix 1.

(ii) is proved in [11] p.64. A proof is also given in the appendix (A.1) to this paper.

(2.5) The theorem we shall prove in this section tells us that every natural regular tensor field on Riemannian manifolds is polynomial in the sense of Epstein [5] para. [5]. However it goes further than this. It follows from the theorem and the theory of representations, that the space of homogeneous natural regular tensor fields of some fixed weight is finite dimensional and that the problem of calculating it reduces to a problem in representation theory.

In the oriented case, in addition to the usual polynomial tensor fields, tensor fields of the form:

$$\sum_{\pi\in S_n} (\det g_{ij})^{\frac{1}{2}} g^{\pi(1)\cdot} \cdots g^{\pi(n)\cdot} g^{\cdot\cdot} \cdots g^{\cdot\cdot} g_{\cdot\cdot} \cdots g_{\cdot\cdot} \overset{\varepsilon_1}{\nabla} R_{\cdot\cdot\cdot\cdot} \cdots \overset{\varepsilon_r}{\nabla} R_{\cdot\cdot\cdot\cdot}$$

where R is the Riemann-Christoffel tensor $R_{ijkl}dx^i \otimes dx^j \otimes dx^k \otimes dx^l$, where the dots indicate contraction or summation with a local basis, are allowed.

Equivalently we introduce tensor fields of the form:

$$\sum_{\pi\in S_n} (\det g_{ij})^{\frac{1}{2}} g^{\pi(1)\cdot}\cdots g^{\pi(n)\cdot} g^{\cdot\cdot}\cdots g^{\cdot\cdot} g_{\cdot\cdot}\cdots g_{\cdot\cdot} W^{\varepsilon_1}_{\cdot\cdot\,\cdot\cdot}\cdots W^{\varepsilon_r}_{\cdot\cdot}\,.$$

THEOREM: There is a bijection between natural regular tensor fields on Riemannian manifolds (respectively oriented Riemannian manifolds) of type (E,F) ($\operatorname{rank}E = a$, $\operatorname{rank}F = b$) and equivariant $O(n)$ (resp. $SO(n)$) homomorphisms :

$$\varphi:\ \mathbb{R}\oplus\bigoplus_{i=1}^{\infty}\ \bigoplus_{2\leq r_1<\dots<r_i}\ \bigoplus_{s=1}^{\infty}\ \bigoplus_{\substack{j_1,\dots,j_i\geq 1\\ j_1+\dots+j_i=s}} S^{j_1}(Y_{r_1}{}^*\mathbb{R}^n)\otimes\dots\otimes S^{j_i}(Y_{r_i}{}^*\mathbb{R}^n)$$

$$\longrightarrow E\mathbb{R}^n\otimes F^*\mathbb{R}^n$$

which vanish except on a finite number of direct summands.

Further:

(i) There are no such tensor fields which are homogeneous of normalised weight $w<0$, or $w = 1$.

(ii) The tensor fields which are homogeneous of maximal weight correspond bijectively to $O(n)$ (resp. $SO(n)$) maps:

$$\varphi:\ \mathbb{R}\longrightarrow E\mathbb{R}^n\otimes F^*\mathbb{R}^n\,.$$

(iii) The tensor fields, homogeneous of normalised weight $w\geq 2$, correspond bijectively to $O(n)$ (respectively $SO(n)$) maps φ_w

$$: \bigoplus_{i=1}^{N_w} \bigoplus_{2\leq r_1<\ldots<r_i} \bigoplus_{\substack{j_1,\ldots,j_i\geq 1\\ r_1j_1+\ldots+r_ij_i=w}} S^{j_1}(Y_{r_1}{}^*\mathbb{R}^n)\otimes\ldots\otimes S^{j_i}(Y_{r_i}{}^*\mathbb{R}^n)$$

$$\longrightarrow E\mathbb{R}^n\otimes F^*\mathbb{R}^n$$

where $N_w = [\frac{1}{2}(-3+(9+8w)^{\frac{1}{2}})]$.

PROOF: The proof is given in the oriented case. The unoriented case is slightly simpler.

So let t be a natural regular tensor field on oriented Riemannian manifolds. Define :

$$\Psi_t : \mathbb{R}\oplus\bigoplus_{r\geq 2} Y_r^*\mathbb{R}^n \longrightarrow E\mathbb{R}^n\otimes F^*\mathbb{R}^n \text{ by } W\longmapsto t(U(W),g(W))(0)$$

identifying the fibre of $EU(W)\otimes F^*U(W)$ at 0 with $E\mathbb{R}^n\otimes F^*\mathbb{R}^n$ via the canonical basis determined by the inclusion chart. Now let $a\in SO(n)$.

Then the expansion of $g_{ij}(aW)$ at 0 in the normal co-ordinate chart determined by a is the same as that of $g_{ij}(W)$ with respect to the inclusion chart. Since the coefficients of t are given by universal polynomials, the coefficients of $t(U(aW),g(aW))(0)$ with respect to the basis of $E\mathbb{R}^n\otimes F^*\mathbb{R}^n$ obtained by applying a to the standard basis, are the same as those of $t(U(W),g(W))(0)$ with respect to the standard basis.

Thus Ψ_t is an equivariant polynomial map vanishing except on a finite number of direct summands.

Complete polarisation determines φ_t.

Conversely, suppose an equivariant $SO(n)$ map φ is given. Let (M,g) be an oriented Riemannian manifold, and let $p \in M$. Then there is a natural identification of T_pM with $\mathbb{R}^n$ which is well defined up to composition with elements of $SO(n)$.

Since φ is equivariant under the action of $SO(n)$, φ determines a unique $SO(T_pM,g(p))$ map $\varphi(M,g)(p)$

$$: \mathbb{R} \oplus \bigoplus_{i=2}^{\infty} \bigoplus_{2\leqslant r_1<\dots<r_i} \bigoplus_{s=1}^{\infty} \bigoplus_{\substack{j_1,\dots,j_i\geqslant 1\\ j_1+\dots+j_i=s}} S_p^{j_1}(Y^*_{r_1}M)\otimes\dots\otimes S_p^{j_i}(Y^*_{r_i}M)$$

$$\longrightarrow E_pM\otimes F^*_pM \,,$$

vanishing except on a finite number of summands.

Define $t_\varphi(M,g)(p) =$

$$\varphi(M,g)(p)(1\oplus \bigoplus_{i=2}^{N} \bigoplus_{2\leqslant r_1<\dots<r_i} \bigoplus_{s=1}^{\infty} \bigoplus_{\substack{j_1,\dots,j_i\geqslant 1\\ j_1+\dots+j_i=s}} W^{r_1}(p)^{j_1}\otimes\dots\otimes W^{r_i}(p)^{j_i}$$

with N large.

It follows from (2.4) that t_φ is determined in the required way by universal polynomials.

Since the whole construction is functorial, t_φ is the required natural tensor field.

Clearly $\varphi_{t_\varphi} = \varphi$.

Conversely it follows from (1.5) and (2.4) that $t_{\varphi_t} = t$.

For the last part of the theorem consider SO(n) maps :

$$S^{j_1}(Y^*_{r_1}\mathbb{R}^n)\otimes\dots\otimes S^{j_i}(Y^*_{r_i}\mathbb{R}^n)\longrightarrow E\mathbb{R}^n\otimes F^*\mathbb{R}^n$$

with $j_k \geqslant 1$; $2\leqslant r_1 < \dots\dots < r_i$.

These are determined by SO(n) maps :

$$S^{j_1}(Y^*_{r_1}\mathbb{R}^n)\otimes\dots\dots\otimes S^{j_i}(Y^*_{r_i}\mathbb{R}^n)\otimes \overset{a}{\otimes}\,\mathbb{R}^{n^*}\otimes\overset{b}{\otimes}\,\mathbb{R}^n\longrightarrow\mathbb{R}$$

and hence the component natural tensor fields thus obtained are of two types :

$$1)\overbrace{g^{\cdot\cdot}\dots g^{\cdot\cdot}}^{u\text{ times}}\overbrace{g_{\cdot\cdot}\dots g_{\cdot\cdot}}^{l\text{ times}}\overbrace{W^{r_1-2}_{\cdot\cdot\;\cdot}\dots W^{r_1-2}_{\cdot\cdot\;\cdot}}^{j_1\text{ times}}\dots\dots\overbrace{W^{r_i-2}_{\cdot\cdot\;\cdot}\dots W^{r_i-2}_{\cdot\cdot\;\cdot}}^{j_i\text{ times}} = P$$

where there are c contractions and summation is over all indices except a upper and b lower.

$$2)\;\sum_{\pi\in S_n}(\det g_{ij})^{\frac{1}{2}}g^{\pi(1)\cdot}\dots g^{\pi(n)\cdot}\;P$$

where there are c contractions and summation is over all indices except a upper and b lower.

In case 1):equating weights gives $b-a-w = -2u+2l+2\sum_{k=1}^{i} j_k$

counting indices gives $a = 2u-c$

$$b = 2l+\sum_{k=1}^{i} r_k j_k + 2\sum_{k=1}^{i} j_k - c$$

whence $w = \sum_{k=1}^{i} r_k j_k$.

In case 2):equating weights gives $b-a-w = n+2l+2\sum_{k=1}^{i} j_k-2u-2n$

counting indices gives $a = 2u+n-c$

$$b = 2l+\sum_{k=1}^{i} r_k j_k+2\sum_{k=1}^{i} j_k-c$$

$$\text{whence } w = \sum_{k=1}^{i} r_k j_k \;.$$

That w cannot equal 1 follows from $r_k \geqslant 2$ all k.

Finally the computation of N_w is left to the reader.

REMARK: For future reference we note that in the unoriented case all natural regular homogeneous tensor fields have even weight.

(2.6) Finally in this paragraph we extend the Gilkey theorem ([1] para. 2) to the oriented case.

Recall that $* : C^\infty(\overset{r}{\Lambda}\, T^*M) \longrightarrow C^\infty(\overset{n-r}{\Lambda}\, T^*M)$ is defined by

$$\omega' \wedge *\omega = \langle \omega', \omega \rangle \underline{v}$$

where (M^n,g) is an oriented Riemannian manifold, ω' any r-form and $\underline{v}$ is the orientation form given in a positively oriented local co-ordinate system by $(\det g_{ij})^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n$. Further * maps natural regular r-forms on <u>oriented</u> Riemannian n manifolds to natural regular n-r forms on oriented Riemannian n manifolds.

COROLLORY: The natural regular homogeneous r forms on oriented Riemannian manifolds (M^n,g) of weight k are linear combinations of forms of two types :

1) Natural regular r-forms ω on Riemannian manifolds, homogeneous of weight k. For k=0 these are precisely the Pontrjagin r-forms.

2) The forms $*\omega$ where ω is a natural regular n-r form on Riemannian manifolds, homogeneous of weight k+n-2r.

In particular the conformal (weight 0) n-forms are sums of :

a) The Pontrjagin n-forms.

b) The forms $f(\det g_{ij})^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n$ where f is a natural regular function on Riemannian manifolds, homogeneous of weight $-n$.

Thus if n is odd, it follows from (2.5) remark that there are no conformal natural regular n-forms.

PROOF: [2] para.2, (2.5) above and the fact that * adds $n-2r$ to the weight of a homogeneous r-form.

REMARK: P.Gilkey has recently proved ([7]) the following result which was originally conjectured by I.M.Singer.

"Let ω be a natural regular n-form on oriented Riemannian n-manifolds such that for each n-manifold M $I(M) = \int_M \omega(M,g)$ is independent of the metric. Then there is a real number c, a natural regular (n-1)-form ρ and a Pontrjagin n-form η such that

$$\omega = d\rho + cE_n + \eta \text{ where } E_n \text{ is the Euler class."}$$

Certainly ω has to be conformal, for if we write $\omega = \sum_{i \geq 0} \omega_i$ where ω_i is homogeneous of weight i, then for all real numbers λ

$$I(M) = \sum_{i \geq 0} \int_M \omega_i(M,g) = \sum_{i \geq 0} \int_M \omega_i(M,\lambda g) = \sum_{i \geq 0} \lambda^i \int_M \omega_i(M,g).$$

Hence $\int_M \omega_i(M,g) = 0$ $i>0$ and ω has to be of type a) or b).

3 Natural Differential Operators.

(3.1) For a review of differential operators, we refer the reader to R.S.Palais [9] . Before making our definitions, however, there are some notions which we would like to recall explicitly.

(3.2) Let ξ, η be C^∞ vector bundles over a smooth manifold M, with $C^\infty(\xi)$ the space of C^∞ sections of ξ.
We let $\mathrm{Diff}_k(\xi,\eta)$ denote the space of differential operators of order $\leqslant k$ from $C^\infty(\xi)$ to $C^\infty(\eta)$.
Let $S^k(\xi)$ denote the k-fold symmetric tensor power of ξ with itself and let

$$S^k : \overset{k}{\otimes}\xi \longrightarrow S^k(\xi)$$

be the map characterised by

$$S^k(v_1 \otimes \dots \otimes v_k) = (k!)^{-1} \sum_{\pi \in S_k} v_{\pi(1)} \otimes \dots \otimes v_{\pi(k)}$$

where $v_i \in \xi_x$ some $x \in M$.
Then we have the symbol exact sequence :

$$0 \longrightarrow \mathrm{Diff}_{k-1}(\xi,\eta) \xrightarrow{i} \mathrm{Diff}_k(\xi,\eta) \xrightarrow{\gamma_k} \mathrm{Hom}(S^k(T^*M)\otimes\xi,\eta) \longrightarrow 0$$

where i is inclusion and γ_k is characterised by

$$\gamma_k(D)(S^k(v_1 \otimes \dots \otimes v_k)\otimes e) = (k!)^{-1} D(g_1 \dots g_k s)(x)$$

where $g_i \in C^\infty(M)$, $g_i(x) = 0$, $d_x g_i = v_i \in T^*M$, $s \in C^\infty(\xi)$
$s(x) = e \in \xi_x$.

(3.3) Let E,F be functors as in (1.1)

Recall that given a Riemannian manifold (M,g) there is a unique torsion free connection ∇ on TM satisfying $\nabla g = 0$. This the Levi-Civita connection. ∇ induces a connection ∇ on $EM \otimes F^*M$ in a natural way.

Define differential operators

$$D_k : C^\infty(EM \otimes F^*M) \longrightarrow C^\infty(S^k(T^*M) \otimes EM \otimes F^*M)$$

by taking the composition:

$$C^\infty(EM \otimes F^*M) \xrightarrow{\nabla^k} C^\infty(\overset{k}{\otimes}\, T^*M \otimes EM \otimes F^*M) \xrightarrow{S^k \otimes 1} C^\infty(S^k(T^*M) \otimes EM \otimes F^*M).$$

Then $\mathcal{V}_k(D_k) \in \mathrm{Hom}(S^k(T^*M) \otimes EM \otimes FM, S^k(T^*M) \otimes EM \otimes F^*M)$ is the identity map.

(3.4) DEFINITION: Let E,F,G,H be functors as in (1.1).

A natural differential operator of type (E,F,G,H) on Riemannian (resp. oriented Riemannian) manifolds assigns to each Riemannian (resp. oriented Riemannian) manifold (M,g) a differential operator

$$D(M,g) : EM \otimes F^*M \longrightarrow GM \otimes H^*M$$

such that if $f : M \longrightarrow M'$ is a diffeomorphism onto an open submanifold (resp. orientation preserving diffeomorphism onto an open submanifold) then

$$D(M',f^*g) = f^*(D(M,g)) .$$

(3.5) Let (M,g) be a Riemannian manifold and let x be a local co-ordinate system on $U \subseteq M$.

Then x determines local bases of sections $(e_\alpha)_{\alpha\in A}, (f^\beta)_{\beta\in B}, (g_\gamma)_{\gamma\in C}, (h^\delta)_{\delta\in D}$ for EM, F^*M, GM, H^*M as in (1.3).

Let $D : EM \otimes F^*M \longrightarrow GM \otimes H^*M$ be a differential operator of order $\leqslant k$.

Then locally we may write

$$D(s^\alpha_\beta\, e_\alpha \otimes f^\beta\,) = \sum_{r=0}^{k} a_{\alpha\delta}^{\beta\gamma i_1\ldots i_r} \frac{\partial^r s^\alpha_\beta}{\partial x^{i_1}\ldots\partial x^{i_r}}\, g_\gamma \otimes h^\delta$$

using the summation convention, where the functions $a_{\alpha\delta}^{\beta\gamma i_1\ldots i_r}$ are symmetric in $i_1,\ldots,i_r$ $(2\leqslant r\leqslant k)$.

We refer to the local functions $a_{\alpha\delta}^{\beta\gamma i_1\ldots i_r}$ as the coefficients of D with respect to the co-ordinate system x.

Note that locally:

$$\gamma_k(D)(v^\alpha_{\beta i_1\ldots i_r} dx^{i_1}\otimes\ldots\otimes dx^{i_r}\otimes e_\alpha\otimes f^\beta) = a_{\alpha\delta}^{\beta\gamma i_1\ldots i_r} v^\alpha_{\beta i_1\ldots i_r} g_\gamma \otimes h^\delta.$$

(3.6) A natural differential operator D on Riemannian manifolds (resp. oriented Riemannian manifolds) is <u>regular</u> if the coefficients of $D(M,g)$ in any local co-ordinate system are given by universal polynomials in g_{ij}, $\partial^{|\alpha|} g_{ij}/\partial x^\alpha$ (α a multi-index $|\alpha|\leqslant N$, N large) and $(\det g_{ij})^{-1}$ (resp. $(\det g_{ij})^{-\frac{1}{2}}$).

The operators D_k introduced in (3.3) are examples of such operators.

Note also that natural bundle maps and natural tensor fields correspond bijectively, and are therefore classified by (2.5). Our main theorem says that in fact this classification also works for natural differential operators.

(3.7) THEOREM: Let D be a natural differential operator of type (E,F,G,H) and order $\leqslant k$.
Then there are unique natural bundle maps
$t_r : C^\infty(S^r(T^*M)\otimes EM \otimes F^*M) \longrightarrow C^\infty(GM \otimes H^*M) \quad (0 \leqslant r \leqslant k)$
such that $D = \sum_{r=0}^{k} t_r D_r$.
The t_r are regular if & only if D is .
PROOF: The result is proved by induction on k and is clear for $k = 0$.
Suppose that the result has been proved for operators of order k-1 and let D have order k.
Then $\gamma_k(D)$ is a natural bundle map which is regular if D is and $\gamma_k(D)D_k$ is a natural differential operator of order k.
Since $\gamma_k(D-\gamma_k(D)D_k) = 0$ by the remark at the end of (3.3), $D - \gamma_k(D)D_k$ is a natural differential operator of order k-1, regular if D is.
Setting $t_k = \gamma_k(D)$, the result follows by induction.

4 Examples

(4.1) Let D be a natural regular differential operator on Riemannian manifolds of order k and type (E,F,G,H) with the ranks of E,F,G,H equal to a,b,c,d respectively. D is determined by natural regular bundle maps:

$$t_r : C^\infty(S^r(T^*M)\otimes EM\otimes F^*M) \longrightarrow C^\infty(GM\otimes H^*M)$$

It follows from the general theory in (2.5) that if t_r is homogeneous of weight w_r then :

1) $w_r \leqslant a+d-b-c-r$

2) If a monomial appears in t_r involving exactly j_l terms W^{ε_l} (equivalently $\nabla^{\varepsilon_l - 2} R$) $1 \leqslant l \leqslant i$, $2 \leqslant \varepsilon_l < \varepsilon_{l+1}$ then

$$a+d = b+c+r+w_r+\sum_{l=1}^{i} j_i\varepsilon_i \quad .$$

Thus $t_r = 0$ if $w_r > a+d-b-c-r$, $w_r = a+d-b-c-r-1$ or w_r odd (by (2.5)).

We say that D has <u>maximal weight</u> if it is homogeneous of weight a+d-b-c-k .

The homogeneous natural regular differential operators of maximal weight are of some interest since any differential operator between vector bundles over $\mathbb{R}^n$ which is the evaluation of a natural regular operator is a sum of these.

(4.2) Hence if D in (4.1) has order 1 and maximal weight then $D = \sigma\circ\nabla$ where σ is a natural bundle map and ∇ the

Levi-Civita connection. Thus operators of maximal weight and order 1 correspond bijectively with bundle maps.
It follows that the Levi-Civita connection on $EM \otimes F^*M$ is the unique connection of maximal weight, which is in this case weight 0 (c.f. Epstein [5] 5.6).
Similarly the exterior derivative on forms and its adjoint are unique of maximal weight, in this case weights 0 and -2 respectively, up to multiplication by constants.
Finally note that it follows from (4.1) that there are no natural vector fields homogeneous of weight greater than -4.

(4.3) Having seen that the Levi-Civita connection is unique of maximal weight, we move on to consider the Laplacian on forms. Again we consider the unoriented case. The situation is not as simple as in the order 1 case, but we can say the following.
Let $\sigma_1, \sigma_2 : C^\infty(\overset{p}{\Lambda} T^*M) \longrightarrow C^\infty(\overset{p}{\Lambda} T^*M)$ be the bundle maps defined by :

$$\sigma_1(dx^{i_1} \wedge \dots \wedge dx^{i_r}) =$$

$$\sum_{s=1}^{r} \sum_{j=1}^{n} R_j{}^{i_s} dx^{i_1} \wedge \dots \wedge dx^{i_{s-1}} \wedge dx^j \wedge dx^{i_{s+1}} \wedge \dots \wedge dx^{i_r}$$

$$\sigma_2(dx^{i_1} \wedge \dots \wedge dx^{i_r}) = \sum_{1 \leq s < t \leq r} \sum_{j,k=1}^{n} R^{i_s i_t}{}_{jk} dx^{i_1} \wedge \dots \wedge dx^{i_{s-1}} \wedge dx^j \wedge dx^{i_{s+1}} \wedge \dots dx^{i_{t-1}} \wedge dx^k \wedge dx^{i_{t+1}} \wedge \dots \wedge dx^{i_r} ,$$

where $R_i{}^j$ is the Ricci tensor, $R^{ij}{}_{kl}$ the curvature tensor with second index raised and dim $M = n$.

Then σ_1 and σ_2 are self adjoint.

Let R be the scalar curvature, d the exterior derivative, d^* its adjoint and Δ the Laplacian.

Let $D : C^\infty(\overset{p}{\Lambda} T^*M) \longrightarrow C^\infty(\overset{p}{\Lambda} T^*M)$ be a natural regular differential operator of maximal weight, in this case -2, and order 2. Then:

1)If $p = 0$ or n $\quad D = a + cR$

2)If $p = 1$ or $n-1$ $D = a_1 dd^* + a_2 d^*d + b\sigma_1 + cR$

3)If $2 \leqslant p \leqslant n-2$ $\quad D = a_1 dd^* + a_2 d^*d + b_1\sigma_1 + b_2\sigma_2 + cR$

where $a, a_1, a_2, b, b_1, b_2, c$ are uniquely determined constants.

Further if D is self-adjoint $a_1 = a_2 = a$ so that :

2) If $p=1$ or $n-1$ $D = a \quad + b\sigma_1 + cR$.

3) If $2 \leqslant p \leqslant n-2$ $\quad D = a \quad + b_1\sigma_1 + b_2\sigma_2 + cR$.

Appendix

(A.1) LEMMA: The vector space $\mathrm{Hom}_{SO_n}(\overset{k}{\otimes}\mathbb{R}^n,\mathbb{R})$ is equal to $\mathrm{Hom}_{O_n}(\overset{k}{\otimes}\mathbb{R}^n,\mathbb{R})$ except that if k-n is non-negative and even then any linear combination of maps :

$$v_1\otimes\ldots\otimes v_k \longmapsto$$
$$\sum_{\mu\in S_n} \varepsilon(\mu) v_{\pi(1)\mu(1)}\cdots v_{\pi(n)\mu(n)}\langle v_{\pi(n+1)},v_{\pi(n+2)}\rangle\cdots\langle v_{\pi(k-1)},v_{\pi(k)}\rangle$$

where $\pi\in S_k$, ε denotes the sign of a permutation, $\mathbb{R}$ denotes the real numbers and $v_j = \sum_{l=1}^{n} v_{jl}e_l$ with respect to the standard basis of $\mathbb{R}^n$ is permitted.

PROOF: Note that $\mathbb{Z}_2 = O_n/SO_n$ acts on $\mathrm{Hom}_{SO_n}(\overset{k}{\otimes}\mathbb{R}^n,\mathbb{R})$ splitting it as the direct sum of $\mathrm{Hom}_{O_n}(\overset{k}{\otimes}\mathbb{R}^n,\mathbb{R})$ and the -1 eigenspace Λ.

If $f\in\Lambda$ define $\tilde{f}\in\mathrm{Hom}_{O_n}(\overset{n+k}{\otimes}\mathbb{R}^n,\mathbb{R})$ by

$$\tilde{f}(v_1\otimes\ldots\otimes v_{k+n}) = f(v_1\otimes\ldots\otimes v_k)\sum_{\mu\in S_n}\varepsilon(\mu)v_{k+1\,\mu(1)}\cdots v_{k+n\,\mu(n)}.$$

Then $\tilde{f}(v_1\otimes\ldots\otimes v_k\otimes e_1\ldots\otimes e_n) = f(v_1\otimes\ldots\otimes v_k)$.

Hence $f(v_1\otimes\ldots\otimes v_k) =$

$$\sum_{\pi\in S_k} c_\pi v_{\pi(1)1}\cdots v_{\pi(n)n}\langle v_{\pi(n+1)},v_{\pi(n+2)}\rangle\cdots\langle v_{\pi(k-1)},v_{\pi(k)}\rangle,$$

some constants c_π, if k-n is positive and even and is zero otherwise.

But $\mu\in S_n$ determines an element $\mu\in O_n$ of determinant $\varepsilon(\mu)$ by permuting co-ordinates.

Thus $f(v_1\otimes\ldots\otimes v_k) = \mathcal{E}(\mu)f(\mu v_1\otimes\ldots\otimes\mu v_k)$

$$= \sum_{\pi\in S_k} \mathcal{E}(\mu)c_\pi v_{\pi(1)\mu(1)}\cdots v_{\pi(n)\mu(n)}\langle v_{\pi(n+1)}, v_{\pi(n+2)}\rangle\cdots$$
$$\cdots\langle v_{\pi(k-1)}, v_{\pi(k)}\rangle.$$

So that since $f \in \Lambda$

$$f(v_1\otimes\ldots\otimes v_k) = (n!)^{-1}\sum_{\pi\in S_k} c_\pi \sum_{\mu\in S_n} \mathcal{E}(\mu)v_{\pi(1)\mu(1)}\cdots v_{\pi(n)\mu(n)}$$
$$\langle v_{\pi(n+1)}, v_{\pi(n+2)}\rangle\cdots\langle v_{\pi(k-1)}, v_{\pi(k)}\rangle$$

if k-n is positive and even, and is zero otherwise.

(A.2) LEMMA: Any rational function f on $GL(n,\mathbb{R})$ invariant under the action of $SO(n,\mathbb{R})$ by right multiplication is of form: $f(A) = F(AA^t) + (\det A)^{-1}G(AA^t)$ $A \in GL(n,\mathbb{R})$ where $F,G : SM(n,\mathbb{R}) \longrightarrow \mathbb{R}$ are rational functions on the space of symmetric matrices.

PROOF: Consider the space of rational functions

$$f : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}$$

invariant under right multiplication by elements of SO_n. Again $\mathbb{Z}_2 = O_n/SO_n$ acts on this space, splitting it as the direct sum of the O_n invariant maps and the -1 eigenspace Λ. If $f \in \Lambda$ then $h : GL(n,\mathbb{R}) \longrightarrow \mathbb{R} : X \longmapsto (\det X)f(X)$ is O_n invariant and hence $f(X) = (\det X)^{-1}h(X)$.

The required result then follows from Appendix 1 of [2].

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