# STOCHASTIC DYNAMICAL SYSTEMS

# AND

# PROCESSES WITH DISCONTINUOUS SAMPLE PATHS

by

Stephen John Rogerson

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# ABSTRACT

In Chapter 1 we use a Poisson stochastic measure to establish a method of localizing, and a change of chart formula for, a class of stochastic differential equations with discontinuous sample paths. This is based on Gikhman and Skorohod [4].

In Chapter 2 we use essentially the method of Elworthy [2], to construct a unique, maximal solution to a stochastic differential equation defined on a manifold M.

Chapter 3 establishes some properties of solutions of the equation. In particular if M is compact, then the solutions have infinite explosion time. We evaluate the infinitesimal generator of the process. By defining stochastic development of α-stable processes on the tangent space, we produce a process on the manifold which, as is shown in Section 6, is not α-stable on M.

# INTRODUCTION

This work is an attempt to generalize the stochastic calculus of manifolds to include the case of discontinuous sample paths.

Unlike Stroock [9], we are interested in pathwise uniqueness and so must use other techniques than solving the "Martingale problem", which guarantees uniqueness of distribution only.

We use the concept of a Poisson stochastic measure (see Gikhman and Skorohod [4]) to provide the necessary extension.

In Chapter 1, we first review the local theory of stochastic differential equations, as set down in Gikhman and Skorohod, with minor changes (Sections 1 to 4). In Section 5 we specialize to the differentiable coefficient case and use the change of variables formula: Theorem 1.3.1, together with a stopping time, constructed in Section 5, to produce a change of chart formula: Theorem 1.5.4. This is stated in "Stratonovich" form, as we are interested in the integrals behaving well under change of chart.

In Chapter 2 we formulate and prove the existence and uniqueness of a unique maximal solution to a stochastic differential equation. This is done using substantially the techniques of Chapter 7 of Elworthy [2].

We have, however, some problems caused by the lack of continuity of sample paths. In particular, we are obliged to change the definition of a process being affirmed as a solution, saying that a solution must be affirmed *by all* regular localizations rather than, *there exists* a cover of the manifold M, by regular localization which affirm M. The point is that we may otherwise lose uniqueness of solutions (see Remark after Lemma 2.2.2).

In fact we need the notion of a big atlas (definition at the beginning of Section 2) i.e. an atlas of charts A such that any pair of points, of the manifold M, are contained in a chart of A. We observe that we may always do this. This is so that if x(t, ) is a solution, then x(t-, ) and x(t+, ) are always contained in a single chart. Using this we prove global uniqueness of solutions.

We proceed to construct a solution of the stochastic differential equation and show that this solution is the required one. This differs from Elworthy [2], but is formally equivalent.

In Chapter 3 we establish some properties of the solutions. In Section 1 we prove a theorem which implies that if M is compact, then the solutions of stochastic differential equations on M have infinite explosion time. In Section 3 we show the solutions are Markov processes and find their infinitesimal operators. Section 5 generalizes the construction of Brownian motion in Elworthy [2], in that we place on every tangent space of the frame bundles of M, a process with independent increments related to one another by parallel translation, and solve the resulting equation.

In the special case of (symmetrical) $\alpha$-stable processes, "something goes wrong", i.e. the infinitesimal generator of the solution process is not what one might expect. Molchanov [7] shows how to construct an $\alpha$-stable process on a (Riemannian) manifold by subordinating Brownian motion, producing a process having infinitesimal generator $-(\Delta/2)^{\alpha/2}$. In Section 6 of Chapter 3 we produce an example to show that this is not the infinitesimal generator of the solution process constructed earlier. In fact it is clear that they never will be equal unless M is flat.

It is not clear why this should be so, except to observe that we are dealing with *global* operators, as opposed to local ones.

On the sphere, the example of Section 6, it is possible to work out the eigenvalues of the infinitesimal generator of the solution process and, by looking at the first three-hundred, it seems that they converge to the 'correct' value. Explicitly for $\alpha = 1$, let $\mu_n$, $\lambda_n$ be the eigenvalues (for the eigenfunctions $P_n \langle x,e\rangle$, $P_n$-Legendre polynomials) for the infinitesimal operator of the solution process and $-(-\Delta/2)^{\frac{1}{2}}$, then we find

$$\lambda_n - \mu_n \approx \frac{(-1)^n}{4n+2}$$

although no proof is available.

It is clear that to resolve this problem we need some information about the geometry of the manifold. However, in the case above there seems no obvious way of obtaining this.

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## STANDING ASSUMPTIONS AND NOTATION

1. We assume a fixed probability base throughout, i.e. the collection $(\Omega, F, P, \{F_t\}_{t \le T})$ where $\Omega$ is a set, $F$ is a $\sigma$-algebra of subsets of $\Omega$ and P is a probability measure on $(\Omega, F)$. Also $T \subset R$ is an interval, either $[a,b]$ or $[a,\infty)$ and $F_t \subset F$ are a collection of sub-$\sigma$-algebras of $F$ such that if $s,t \subset T$ and $s < t$, then $F_s \subset F_t$.

We assume in addition that the $F_t$'s are complete with respect to $F$, i.e. that $\forall\ N \in F$ with $P(N) \equiv 0$ and $\forall\ A \in F_t$, we have that $N \Delta A \in F_t$.

2. We call a map $\xi : T \times \Omega \to X$, where X is some measure space, adapted if $\xi(t,\omega)$ is $F_t$-measured $\forall\ t \in T$.

3. $C^{n-}(A, \mathbb{R}^q)$, where $A \subset \mathbb{R}^p$ is an open set, is the collection of n times differentiable functions with locally Lipschitz n-th differential.

# CHAPTER 1 - STOCHASTIC DIFFERENTIAL EQUATIONS IN VECTOR SPACES

## 1. Stochastic Integration:

The results of this section can be found in Chapter 1, Part II of [4].

Given a finite dimensional vector space V, define the set $\Pi_0(L,C,g)$ to be the set of random functions $\alpha:[a,b] \times V \times [0,\epsilon) \times \Omega \to V$, subject to the following conditions:

(1) $|E\{\alpha(t,x,h)|F_t\}| \le Lh$;

(2) $E\{|\alpha(t,x,h)|^2|F_t\} \le Lh$;

(3) $|E\{\alpha(t,x,h) - \alpha(t,y,h)|F_t\} \le c\,|x-y|h$;

(4) $E\{|\alpha(t,x,h) - \alpha(t,y,h)|^2\ |F_t\} \le c|x-y|^2h$;

(5) $|E\{\alpha(t,x,h_1 + h_2) - \alpha(t,x,h_1) - \alpha(t+h_1,x,h_2)\quad |F_{t+h_1}\}|$

$$\le Lh_2g(h_1) \quad \text{and}$$

(6) $E\{|\alpha(t,x,h_1 + h_2) - \alpha(t,x,h_1) - \alpha(t+h_1,x,h_2)|^2|F_{t+h_1}\}$

$$\le Lh_2g(h_1)$$

where L and C are positive constants and g is a non-random, non-decreasing function and $\lim_{h\to 0} g(h) = 0$.

Let $H_c$ be the space of function $\xi:[a,b] \times \Omega \to V$ such that:

(i) $\xi(.,\omega)$ is Borel measurable,

(ii) $\xi(t,.)$ is $F_t$-measurable

and $E(|\xi(t+h) - \xi(t)|^2) \to 0$ as $h \to 0$,

with the norm $||\xi|| = \sup_{a \le t \le b} [E|\xi(t)|^2]^{\frac{1}{2}}$.

Given $\xi \in H_c$, we may define $\xi_1$ by

$$\xi_1(t) = \xi_a + \int_a^t \alpha(s,\xi(s),ds), \quad (\xi_a\text{-}F_a\text{-measurable}),$$

to be the mean square limit of $\xi_1^\delta$ as $\delta \to 0$, where

$$\xi_1^\delta(t_{k+1}) = \xi_a + \sum_{k=0}^{n-1} \alpha(t_k, \xi(t_k), t_{k+1} - t_k)$$

We have $\xi_1 \in H_c$.

Theorem 1.1.1.

Suppose $E|\xi_a|^2 < \infty$, $\xi_a$ is $F_a$-measurable and

$$\alpha \in \Pi_0(L,C,g),$$

then there exists a unique $\xi \in H_c$ such that

$$(1) \qquad \xi(t) = \xi_a + \int_a^t \alpha(s,\xi(s),ds).$$

Proof

This is essentially Theorem 1 of Section 3, Chapter 1, Part II of [4]. □

Remark

The method is standard, i.e. using the Lipschitz conditions (3) and (4) for $\alpha$, we show that the operation defined above has a unique fixed point in $H_c$.

Theorem 1.1.2.

Let $\xi$ be as in Theorem 1.1.1 and suppose that $\alpha(t,x,.)$ is right-continuous with left-hand limits, then there exists a separable version of $\xi$ which is also right-continuous with left-hand limits.

Proof

This is essentially Theorem 1, Section 2, Chapter 1, Part II of [4]. □

Remark

If $\xi(t)$ is as in Theorem 1 then we will often say that $\xi(t)$ is the solution of the stochastic differential equation

$$d\xi(t) = \alpha(t,\xi(t),dt), \text{ given } \xi(a) = \xi_a.$$

Definition 1.1.1.

An Euler approximate solution of $d\xi = \alpha(t,\xi(t),dt)$, given initial condition $\xi_a$, is a stochastic process $\xi_\delta(t)$, satisfying

$$\xi_\delta(0) = \xi_a,\ \xi_\delta(t) = \xi_\delta(t_k) + \alpha(t_k,\xi_\delta(t_k),t-t_k)$$

for $t_k \le t \le t_{k+1}$, $k = 0,\ldots,n-1$, where $\delta = \{a = t_0,t_1,\ldots,t_n = b\}$ is some partition of $[a,b]$. We write $|\delta| = \max_k |t_{k+1}-t_k|$.

Theorem 1.1.3.

If $E|\xi_a|^2 < \infty$ ; and $\alpha \in \Pi_0$, then for any $\epsilon > 0$, there exists an $\epsilon_0$ such that for $|\delta| < \epsilon_0$,

$$E\{\sup_{a\le t\le b} |\xi(t) - \xi_\delta(t)|^2\} \le \epsilon.$$

Furthermore, $\lim_{|\delta|\to 0} E\{|\xi(t) - \xi_\delta(t)|^2\} = 0.$

Proof

This is Lemma 2, and Corollary 1, of Section 3, Part 2 of [4]. □

2. Poisson Stochastic Measures and Integration

Definition 1.2.1.

A random variable $\nu$ on a probability space $(\Omega,P,F)$ has a Poisson distribution with parameter $\Pi \in [0,\infty)$ if

$$P((\omega)|\nu(\omega) = n) = \frac{\Pi^n}{n!} e^{-\Pi}, \; n \in N \cup \{0\}.$$

We have $E(\nu) = \Pi$

and $Var(\nu) = \Pi$.

Definition 1.2.2.

If X is a separable, locally compact, Hausdorff space and $\mathcal{B}$ its σ-algebra of Borel sets, then a *Poisson measure*, with *parameter measure* Π, is an assignment, to every relatively compact set $A \in \mathcal{B}$ of a random variable $\nu(A)$ having a Poisson distribution, with parameter $\Pi(A)$, which satisfies the following two conditions:

(1) If $A_1$, $A_2$ are disjoint and relatively compact, then $\nu(A_1)$ and $\nu(A_2)$ are independent;

(2) If $A_i \in \mathcal{B}$ for $i \in \mathbb{N}$, the $A_i$ are disjoint and $\cup A_i$ is relatively compact, then $\nu(\bigcup_i A_i) = \sum_i \nu(A_i)$, (it follows that $\Pi(\bigcup_i A_i) = \sum_i \Pi(A_i)$, hence we may take Π to be a measure).

Definition 1.2.3.

If in Definition 1.2.2 we take $X = [a,b] \times (V\setminus\{0\})$ where V is a finite dimensional vector space, then we say that ν *is a Poisson stochastic measure* on X, with parameter measure Π if:

(1) $E(\nu(\Delta\times A)) = |\Delta|\Pi(A)$ (where $\Delta \in \mathcal{B}[a,b]$ $A \in \mathcal{B}(V\backslash\{0\})$, and $|\cdot|$ is Lebesgue measure on $[a,b]$);

(2) $\nu([a,t]\times A)$ is $\mathcal{F}_t$-measurable for any $t$ and for any $A \in \mathcal{B}(V\backslash\{0\})$ and

(3) $\nu((t,t+h) \times A))$ is independent of $\mathcal{F}_t$ for any $h > 0$ $A \in \mathcal{B}(V\backslash\{0\})$.

Remark

We may have $\Pi(\{x|\ |x| < \epsilon\}) = \infty$ or $\Pi(\{x|\ |x| > \epsilon\}) = \infty$ for any $\epsilon > 0$. The former will be the case if

$$\Pi(du) = \frac{c_{n,\alpha}\, du}{|u|^{n+\alpha}},$$ where $n = \dim V$, $\alpha \in (0,2)$ and $c_{n,\alpha}$ a constant,

corresponding to the case of a symmetric $\alpha$-stable process.

To integrate with respect to $\nu$, we first define the auxiliary process $\tilde{\nu}$, by

$$\tilde{\nu}(\Delta \times A) = \nu(\Delta \times A) - |\Delta|\Pi(A).$$

Then $E(\tilde{\nu}(\Delta \times A)) = 0$
$\mathrm{Var}\,(\tilde{\nu}(\Delta \times A)) = |\Delta|\Pi(A)$.

Suppose that an adapted random function

$$\phi : [a,b] \times V \times \Omega \to W \quad \text{(where } W \text{ is a vector space)}$$

is simple in the sense that

$$\phi(t,x) = \phi_k(x), \text{ for } t \in [t_k,t_{k+1}).$$

$$\phi_k(x) = \sum_i \alpha_{ik} \chi_{A_{ik}}(x), \quad A_{ik} \in \mathcal{B}(V)$$

$$\alpha_{ik} : \Omega \to W \qquad \text{is } \mathcal{F}_{t_k}\text{-measurable.}$$

If $\int E|\phi_k(x)|^2 \Pi(dx) < \infty$

we may define $\int \int_a^b \phi(t,x)\tilde{\nu}(dt,dx) = \sum_{ik} \alpha_{ik} \tilde{\nu}([t_k,t_{k+1}) \times A_{ik})$.

We have

$$E \int \int^b \phi(t,x)\tilde{\nu}(dt,dx) = 0$$

and $\mathrm{Var} \int \int_a \phi(t,x)\tilde{\nu}(dt,dx) = \int \int_a^b E(|\phi(t,x)|^2) \, dt\Pi(du).$

We may, by standard techniques [4], extend the definition of the integral to the set of adapted random functions $\phi(t,y)$, such that

$$\int \int_a^b |\phi(t,y)|^2 \, dt\Pi(dy) < \infty \qquad \text{a.e.}$$

If also $\int \int_a^b |\phi(t,y)| dt\Pi(dy) < \infty$ a.e.,

then we may define

$$\int \int_a^b \phi(t,y)\nu(dt,dy)$$

$$= \int \int_a^b \phi(t,y)\tilde{\nu}(dt,dy) + \int \int_a^b \phi(t,y) \, dt\Pi(dy).$$

## 3. The Generalized Itô Formula

Theorem 1.3.1.

Suppose we have:

(i) finite dimensional vector spaces U,V and W;

(ii) (a) A Weiner process w adapted to $F_t$ on W, and

(b) A Poisson stochastic measure on U, ν, adapted to $F_t$ and independent of w, with parameter measure Π;

(iii) $\alpha : [a,b] \times \Omega \to V$

$\beta : [a,b] \times \Omega \to L(W,V)$ adapted to $F_t$

$\gamma : [a,b] \times U \times \Omega \to V$

Such that $E\int_a^b |\alpha(t)|dt < \infty$,

$E\int_a^b |\beta(t)|^2 dt < \infty$,

$\gamma(..)(\omega)$ is a Borel function

and $\int_a^b \int |\gamma(t,u)|^2 \Pi(du)dt < \infty$; and

(iv) $g: [a,b] \times V \to V$ such that if $\frac{\partial}{\partial t}$ and D denote differentiati[on] with respect to the first and second variable respectively, then $D^2 g$ and $\frac{\partial g}{\partial t}$ exist and are continuous

$$\int_a^b \int |g(t,x + \gamma(t,u)) - g(t,x)|^2 \Pi(du)dt < \infty ,$$

and

$$\int_a^b \int |g(t,x+\gamma(t,u))-g(t,x)-D_2 g(t,x)(\gamma(t,u))|\Pi(du)dt < \infty.$$

Then if $\xi(t) = \xi_a + \int_a^t \alpha(s)ds + \int_a^t \beta(s)dw(s) + \int\int_a^t \gamma(s,u)\tilde{\nu}(dt,dy)$

where $\xi_a$ is $F_a$-measurable, it follows that

$$g(\xi(t)) = g(\xi_a) + \int_a^t L(g)(s,\xi(s))ds + \int_a^t Dg(s,\xi(s))\beta(s)dw(s)$$

$$+ \iint_a^b [g(s,\xi(s) + \gamma(s,u)) - g(s,\xi(s))]\tilde{\nu}\,(ds,du),$$

where $L(g)(\xi(s),s) \;=\; \frac{\partial g}{\partial t}(S,\xi(s)) + D_2g(s,\xi(s))\alpha(s)$

$$+ \frac{1}{2}\operatorname{Tr}(D_2^2 g(s,\xi(s))(\beta(s)\beta^*(s))$$

$$+ \int [g(s,\xi(s)+\gamma(s,u)) - g(s,\xi(s)) - D_2 g(s,\xi(s))(\gamma(s,u))]\Pi(du$$

Proof.

This is Theorem 2 §7, Chapter 2, of Part 2 of [4]. □

Remark

The terms involving $\alpha$ and $\beta$ are standard. To give an idea of how the 'Poisson component' transforms, suppose $S_n = \sum_{i=1}^{n} a_i$, then $g(S_n) = g(\sum_{i=1}^{n} a_i)$

$$= \sum_{r=1}^{n} (g(\sum_i^r a_i) - g(\sum_i^{r-1} a_i)) + g(0).$$

This is the essence of the form of the formula, the term involving the differential results from using $\tilde{\nu}$, as opposed to $\nu$.

## 4. Stochastic Differential Equations

Theorem 1.4.1.

Suppose we have U, V, W, w and $\nu$ as in Theorem 1.3.1, and

$$a:[a,b] \times V \to V,$$

$$b:[a,b] \times V \to L(W,V)$$

and $$c:[a,b] \times V \times U \to V$$

satisfying:

$$|a(t,x)|^2 + |b(t,x)|^2 + \int |c(t,x)-x|^2 \Pi(du) \leq L;$$

$$|a(t,x)-a(t,y)|^2 + |b(t,x)-b(t,y)|^2 + \int |c(t,x,u)-x-c(t,y,u)+y|^2 \Pi(du$$

$$\leq c|x-y|^2$$

and $|a(t+h,x) - a(t,x)|^2 + |b(t+h,x) - b(t,x)|^2$

$$+ \int |c(t+h, x,u)-c(t,x,u)|^2 \Pi(du) \leq Lg(h),$$

where $g:\mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ and $g(h) \to 0$ as $h \to 0$.

Then given $\xi_a$ - $F_a$-measurable, with $E(|\xi_a|^2) < \infty$,

there exists a unique solution to the stochastic differential equation

2) $$d\xi(t) = a(t,\xi(t))dt + b(t,\xi(t))dw(t)$$

$$+ \int c(t,\xi(t),u) - \xi(t)\tilde{\nu}(dt,du)$$

with $\xi(a) = \xi_a$.

Such a solution will possess a separable version with left-hand limits and which is right-continuous.

Proof.

Define $\alpha(t,x,h) = a(t,x)h + b(t,x)(w(t+h) - w(t))$

$$+ \int c(t,x,u)-x\ \tilde{\nu}([t,t+h),du)$$

Then (1) $E\{\alpha(t,x,h)|F_t\} \le |a(t,x)|h \le L^{\frac{1}{2}}h$

(2) $E\{|\alpha(t,u,h)|^2|F_t\} \le |a(t,x)|^2 h^2 + Tr(b(t,x)b^*(t,x))h$

$$+ \int |c(t,x,u)-x|^2\Pi(du)h$$

$$\le Lh \text{ (for small h)}$$

(3) $|E\{\alpha(t,x,h) - \alpha(t,y,h) \mid F_t\}|$

$$=|E\{a(t,x)h-a(t,y)h + (b(t,x)-b(t,y))(w(t+h)-w(t))$$

$$+ \int c(t,x,u)-x-c(t,y,u) + y\tilde{\nu}([t,t+h),du)\}|$$

$$\le |a(t,x)-a(t,y)|h \le L^{\frac{1}{2}}h.$$

(4) $E\{|\alpha(t,x,h)-\alpha(t,y,h)|^2|F_t\}$

$$\le |a(t,x)-a(t,y)|^2h^2 + |Tr((b(t,x)-b(t,y))(b^*(t,x)-b^*(t,y)))$$

$$+ \int |c(t,x,u)-x-c(t,y,u) + y|^2\Pi(du)h$$

$$\le Ch \quad \text{(small h)}$$

(5) $|E\{\alpha(t,x,h_1+h_2) - \alpha(t,x,h_1) - \alpha(t+h_1,x,h_2)|F_{t+h_1}\}|$

$$= |a(t,x) - a(t+h_1,x)|h_2 \leq L^{\frac{1}{2}} g^{\frac{1}{2}}(h_1)h_2$$

(6) $E\{|\alpha(t,x,h_1+h_2) - a(t,x_1,h_1) - \alpha(t+h_1,x_1h_2)|^2 \; F_{t+h_1}\}$

$$\leq |a(t,x) - a(t+h_1,x)|^2 h_2^2 + |Tr(b(t,x)b^*(t,x))$$

$$- Tr\; b(t+h_1,x)b^*(t+h_1,x)|h_2$$

$$+ \int |c(t,x,u)-c(t+h_1,x,u)|^2 \Pi(du)h_2$$

$$\leq Lg\; (h_1)h_2.$$

So there exists L', C', g' such that $\alpha(t,x,y) \in \Pi_o$ (L',c',g') and by Theorem 1.1.1 there exists a unique solution $\xi \in H_c$ of (2).

The regularity properties follow from Theorem 1.1.2. □

5. Stochastic Dynamical Systems in Vector Spaces.

In this section we will fix the following:

(i) finite dimensional vector spaces U,V,W;

(ii) a  a $C^{1-}$ bounded vector field on V,

b  a $C^{2-}$ bounded map $b:V \to L(W,V)$

and c  a $C^{2-}$ map $c:V \times U \to V$ such that $c(x,u)-x$ is bounded,
$D_2c(x,0)$ is bounded and $c(x,0)=x$ and

(iii) a Wiener process w, and a Poisson stochastic measure ν, both adapted to $\{F_t\}$ and independent on each other with the parameter measure Π satisfying

$$\int \frac{|u|^2}{1+|u|^2}\,\Pi(du) < \infty.$$

Theorem 1.5.1.

Given $\xi_a$, $F_a$-measurable with $E(|\xi_a|^2) < \infty$, then there exists a unique solution, $\xi$, to the stochastic differential equation:

$$d\xi(t) = a(\xi(t))dt + b(\xi(t))dw(t) + \int c(\xi(t),u)-\xi(t)\tilde{\nu}(dt,dw)$$

given initial condition $\xi_a$.

Proof.

We verify the conditions of Theorem 1.4.1. We have $c(x,u)-x = D_2c(x,0)(u) + O(|u|^2)$, by the assumptions on c, so

$$\int |c(x,u)-x|^2\Pi(du) = \int_{|u|\le\epsilon} |c(x,u)-x|^2\Pi(du) + \int_{|u|\ge\epsilon} |c(x,u)-x|^2\Pi(du))$$

$$\le \int_{|u|\le\epsilon} A_0|u|^2\Pi(du) + \int_{|u|>\epsilon} A_1\Pi(du),$$ (where $A_0$, $A_1$ and $\epsilon$ are positive constants

$< \infty$,

by assumption on $\Pi$.

Also a and b are bounded, so

$$|a(t,x)|^2 + |b(t,x)|^2 + \int|c(x,u)-x|^2\Pi(du) < L, \text{ some } L.$$

The Lipschitz condition follows similarly, observing that a and b are Lipschitz.

Since a, b and c are independent of time, the third condition is trivial. □

Proposition 1.5.2.

Suppose that g is a bounded $C^2$ map from V to V.

If $\xi(t)$ is as in Theorem 1.5.1 then

$$g(\xi(t)) = g(\xi_a) + \int_a^t Dg(\xi(s))a(\xi(s))ds$$

$$+ \frac{1}{2} \int_a^t Tr(D^2 g(\xi(s))(b(\xi(s)),b(\xi(s))))ds$$

$$+ \int_a^t \int [g(c(\xi(s),u))-g(\xi(s))-Dg(\xi(s))(c(\xi(s),u)$$

$$\xi(s))]\Pi(du)ds$$

$$+ \int_a^t Dg(\xi(s))(b(\xi(s)))dw(s)$$

$$+ \int \int_a^t g(c(\xi(s),u)) - g(\xi(s)) \tilde{\nu}(ds,dw).$$

Proof.

This is a restatement of Theorem 1.3.1. We verify the conditions.

The conditions of $\alpha,\beta,\gamma$, imposed in part (iii), follow from the boundedness conditions of a, b and c respectively. $g(c(x,u)) - g(x) = D_2(g\circ c)(x,0)(u) + O(|u|^2)$, in a neighbourhood of zero and bounded outside this neighbourhood so the conditions on g follow from arguments similar to that in the previous theorem.

The proposition follows. □

We wish to use the above "change of variable" formula, to create a "change of chart" formula. To this end it is useful to have an invariant formulation, such as, in the continuous case, is afforded by the Stratonovich integral. To this end we introduce the following objects.

Definition 1.5.1.

In the following formulas the left-hand sides are defined by the right-hand sides, whenever they make sense:

$$(1)\qquad \int_0^t b(x(s))d\bar{w}(s) = \int_0^t b(x(s))dw(s) + \frac{Tr}{2}\int_0^t Db(x(s))b(x(s))ds$$

$$(2)\qquad \iint_0^t c(x(s),u) - x(s)\bar{\nu}(ds,du) = \iint_0^t c(x(s),u) - x(s)\tilde{\nu}(ds,du)$$

$$+ \iint_0^t c(x(s),u) - x(s) - D_2c(x(s),0)\left(\frac{u}{1+|u|^2}\right) \Pi(du)ds.$$

Remarks:

1) is identical to the standard Stratonovich integral. Note that the second integral exists, for suitable processes $x(s)$, since $c$ is $C^{2-}$ and is bounded together with its first derivatives.

Note also that the function $\frac{u}{1+|u|^2}$ appearing in (2) is to some extent arbitrary, more precisely we could use any bounded function $\mu(u)$ such that $\mu$ is differentiable in a neighbourhood of the origin with $D\mu(0) = 1$. Later when we consider manifolds, it will be found necessary to consider the case in which we are free to assign a and b outside a set A and c outside a set $D \subset V \times U$. It is hence necessary to show that the solutions corresponding to different assignemtns are equal at least up to some stopping time. The choice of stopping time is crucial. The first exit time of the process from A, for example, is generally not adequate.

Definition 1.5.2.

Given bounded open subsets $A_0$, $A_1$ and A of V, with $\bar{A}_0 \subset A_1$, $\bar{A}_1 \subset A$ and open neighbourhoods $D_1$ and D of $A_1 \times \{0\}$ and $A \times \{0\}$ respectively such that $\bar{D}_1 \subset D$, write $F = \{A_0, A_1, A, D_1, D\}$. Then define $\tau(F)$, by

$$\tau(F) = \inf\{t \,|\, (x(t), J(t)) \notin A_0 \times (-\tfrac{1}{2}, \tfrac{1}{2})\}$$

where $J(t) = \int\int_0^t (1 - \chi_{D_1}(x(s),u))\nu(ds,du)$

and x is a right-continuous process with left-hand limits.

Remarks: If $\tau(A_0)$ is the first exit time of $x(t)$, from $A_0$, then $\tau(F) \leq \tau(A_0)$.

(2) For $t < \tau(F)$, we have that $(x(s),u) \in D_1$, for all u with $|u| < \epsilon$, for some positive $\epsilon$. So the integrand of J(t) misses a neighbourhood of zero and hence J is well defined. We note also that J(t) is an increasing positive integer valued process.

Lemma 1.5.3.

Let x(t) be a solution to the S.D.E.

$$dx(t) = a(x(t)dt + b(x(t))d\bar{\omega}(dt) + \quad c(x(t),u) - x(t)\bar{\nu}(dt,du)$$

$\forall\ \epsilon > 0, \exists\ \gamma$ st $\forall$ partition of $[0,T], \delta$, with $|\delta| < \gamma$

$t < \tau(F,\omega) \Rightarrow t < \tau^{\delta}(F,\omega)$, for $\omega \in \Omega^{\epsilon}$, where $P(\Omega_{\epsilon}) > \epsilon$, and

$$\tau^{\delta}(F) = \inf\{t|[x^{\delta}(t), \sum_k \int\int_{t_k \wedge t}^{t_{k+1}\wedge t} (1-\chi_D(\chi^{\sigma}(s),u)\nu(ds,du)] \notin A_1 \times (-\tfrac{1}{2},\tfrac{1}{2})\}.$$

Proof

$\forall\ \epsilon_1 > 0, \exists\ \gamma$ st $\forall \delta$ with $|\delta| < \gamma$,

$\sup_{0<s<t} |x^{\sigma}(s,\omega) - x(s,\omega)| < \epsilon_1$, for $\omega \notin \Omega^{\epsilon_1}$, where

$P(\Omega^{\epsilon_1}) < \epsilon_1$. This follows from Theorem 1.1.3. Now since $\bar{D}_1 \subset D$,

$$(x(s),u) \in A_0 \times B(0,R) \cap D_1 \Rightarrow (x^{\delta}(s),u) \in A_1 \times B(0,R) \cap D,$$

for $\epsilon_1$, small enough.

Also $\forall\ \epsilon_2 > 0, \quad \exists$ R st

$$\sup_{0<s<T} \Big| \sum_k \int_{|u|<R} \int_{t_k \wedge s}^{t_{k+1} \wedge s} (1-\chi_D(x\,(s),u))\nu(ds,du) - \int\int_{t_k \wedge s}^{t_{k+1} \wedge s} (1-\chi_D(x^\delta(s),u))\nu(ds,du) \Big| < \tfrac{1}{2}$$

except for set $\Omega^{\epsilon_2}$ with $P(\Omega^{\epsilon_2}) < \epsilon_2$.

Now $\forall\ \epsilon > 0$, choose $\epsilon_1$ and $\epsilon_2$ such that

$\epsilon_1 + \epsilon_2 < \epsilon$, and we are done. □

<u>Theorem 1.5.4</u>.

Let $\xi_i(t)$ be the solution to the S.D.E.

(1) $d\xi_i(t) = a_i(\xi(t))dt + b_i(\xi(t))d\bar{w}(dt)$

$+ \int c_i(\xi(t),u) - \xi(t)\bar{\nu}(dt,du)$ with initial conditions,

$\xi_i(0) = \xi_i^o$, for $i = 1,2$.

Suppose that $g:V \to V$ is $C^2$ and, together with its derivatives, is bounded;

(2) that $A$ is an open set and $D \subset A \times U$ is an open neighbourhood of $A \times \{0\}$;

(3) $g|A$ is a diffeomorphism;

(4) $Dg(x)a_1(x) = a_2(g(x))$, $x \in A$;

(5) $Dg(x)b_1(x) = b_2(g(x))$, $x \in A$;

(6) $g(c_1(x,u)) = c_2(g(x),u)$, $(x,u) \in D$.

If $F = \{A_o, A_1, A, D_1, D\}$ is as in Definition 1.5.2. and $G = \{B_o, B_1, B, E_1, E\}$, is the image of F under g, (i.e. $B_o = g(A_o)$, etc and $E_1 = \{(x,u) | x \in B_1, (g^{-1}(x), u) \in D_1\}$ and similarly for E), and if $\xi_2^o = g(\xi_1^o)$ for $\xi_1^o \in A$ and $\xi_2^o \in B$ then

$\xi_2(t) = g(\xi_1(t))$ a.e. for $t < \tau(F)$ and $\tau(F) = \tau(G)$ a.e.

Proof

We write $\eta(t) = g(\xi(t))$, by Proposition 1.5.2, $\eta(t)$ is the stochastic line integral $\eta(t) = g(\xi_1^o) + \int_o^t \beta(s, \xi_1(s), ds)$, where

$$\beta(s,\xi_1(s),t-s) = [Dg(\xi_1(s))a_1(\xi_1(s)) + \frac{Tr}{2} D^2 g(\xi_1(s))(b_1(\xi_1(s), b_1(\xi_1(s))$$

$$+ \int g(c_1(\xi_1(s),u)) - g(\xi_1(s)) - D_2 g(\xi_1(s))(c_1(\xi_1(s),u) - \xi_1(s))\Pi(du)](t-s)$$

$$+ Dg(\xi_1(s))(Db(\xi_1(s))(b(\xi_1(s)) \ (t-s)$$

$$+ (t-s\ ) \int g_2(Dg(\xi_1(s))(c(\xi_1(s),u) - \xi_1(s) - D_2 c(\xi_1(s),u)(\frac{u}{1+|u|^2})\ \Pi(du)$$

$$+ Dg(\xi_1(s))(b_1(\xi_1(s)))(w(t) - w(s))$$

$$+ \iint_s^t g(c(\xi_1(s),u)) - g(\xi_1(s))\tilde{\nu}(ds,du).$$

Write, for a partition $\delta = \{0 = t_o < t_1 \ .. \ < t_k = T\}$

$$\eta_1^\delta(t) = \eta_1^\delta(t_k) + \beta(t_k, \xi_1(t_k), t - t_k), \quad t \in [t_{k_1} \; t_{k+1}]$$

$$\eta_1^\delta(t_o) = g(\xi_1^o) \quad \text{and}$$

$$\eta^\delta(t) = \eta^\delta(t_k) + \beta(t_k, \xi_1^\delta(t_k), t - t_k), \quad t \in [t_k, t_{k+1}],$$

where $\xi_1^\delta(t)$ is the Euler approximate solution to $\xi_1(t)$.

Now $E(\sup_t |\eta(t) - \eta^\delta(t)|^2)$

$$\leq E(\sup_t |\eta(t) - \eta^\delta(t)|^2) + E(\sup_t |\eta_1^\delta(t) - \eta^\delta(t)|^2).$$

By Lemma 4 Section 2 of Part 2 of [4], $\forall \epsilon_1 > 0$, $\exists \gamma$ s.t. $\forall \delta$ with $|\delta| < \epsilon_o$,

$$E(\sup_t |\eta(t) - \eta_1^\delta(t)|^2) < \epsilon_1$$

Also by the remark following Lemma 2, of Section 2 of Part 2 of [4]

$$\forall \epsilon_2 > 0, \exists \gamma \;, \text{ st } \quad \forall \delta \quad \text{with} |\delta| < \epsilon_2$$

$$E(\sup_{0 \leq t \leq T} |\eta_1^\delta(t) - \eta^\delta(t)|^2) \leq C \int_o^T E|\xi_1^\delta(t) - \xi(t)|^2 \, dt$$

$$\leq CT \; E(\sup_{0 \leq t \leq T} |\xi_1^\delta(t) - \xi(t)|^2 < \epsilon_2$$

So $\forall \epsilon > 0$, we choose $\epsilon_1 + \epsilon_2 < \epsilon$.

Assume that $t < \tau^{\delta}(F)$, and that $\eta^{\delta}(t_k) = \xi_2^{\delta}(t_k)$, then for $t \in [t_k, t_{k+1}]$,

$$\eta^{\delta}(t) = \eta^{\delta}(t_k) + \beta(s, \xi_1^{\delta}(t_k), t - t_k).$$

Let $I_k$ denote the k-th term of $\beta(s,x,h)$, then

$$I_1 = Dg(\xi_1^{\delta}(t_k)a_1(\xi_1^{\delta}(t_k)) = a_2(g(\xi_1(t_k))\ (t - t_k)$$

$$= a_2(\xi_2^{\delta}(t_k))(t - t_k).$$

$$I_2 + I_4 = \frac{Tr}{2}\ [(D^2g(\xi_1(t_k))(b(\xi_1(t_k),\ b(\xi_1(t_k))$$

$$+ Dg(\xi_1^{\delta}(t_k))(Db(\xi_1^{\delta}(t_k)b(\xi_1^{\delta}(t_k)))](t - t_k).$$

Now $D_x^2 g(x)(b_1(x), b_1(x)) + D_x g(x)(D_1 b(x) b_1(x))$

$$= D_x(Dg(x)b_1(x))b(x)$$

$$= D_y(b_2(y))Dg(x)b_1(x)$$

$= D_y(b_2(y))b_2(y)$. It follows that

$$I_2 + I_4 = D(b_2(\xi_2(t_k)))b_2(\xi_2(t_k))\ (t - t_k).$$

$$I_6 = Dg(\xi_1(t_k))(b_1(\xi_1(t_k))(w(t) - w(t_k))$$

$$= b_2(\xi_2(t_k))(w(t) - w(t_k))\ .$$

Now $I_3 + I_5 + I_7 = \int g(c_1(\xi_1(t_k),u)) - g(\xi,(t_k))$

$$- D_2 g(\xi_1(t_k))(c_1(\xi_1(t_k),u) - \xi_1(t_k))\Pi(du)(t - t_k)$$

$$+ \int Dg(\xi_1(t_k))[c_1(\xi_1(t_k),u) - \xi_1(t_k)$$

$$- D_2 c_1(\xi_1(t_k),0)\ (\frac{u}{1+|u|^2})]\ \Pi(du)(t - t_k)$$

$$+ \int\int_{t_k}^{t} g(c_1(\xi_1(t_k),u)) - g(\xi_1(t_k))\tilde{\nu}(ds,du).$$

Combining the first two terms and expanding the last we get

$$I_3 + I_5 + I_7 = \int g(c_1(\xi_1(t_k),u) - g(\xi_1(t_k))$$

$$- Dg(\xi_1(t_k))D_2 c(\xi_1(t_k),0)(\frac{u}{1+|u|^2})\ \Pi(du)\ (t - t_k)$$

$$\int\int_{t_k}^{t} \chi_D(\xi_1(t_k),u)[g(c_1(\xi_1(t_k),u)) - g(\xi_1(t_k))]\tilde{\nu}(ds,du)$$

$$\int\int_{t_u}^{t} (1 - \chi_D(\xi_1(t_k),u))[g(c_1(\xi_1(t_k),u)) - g(\xi_1(t_k))]\nu(ds,du)$$

$$- \int (1 - \chi_D(\xi_1(t_k),u))[g(c_1(\xi_1(t_k),u)) - g(\xi_1(t_k))]\Pi(du)\ (t - t_k)$$

where the last two terms come from the definition of $\nu$ in terms of $\tilde{\nu}$.

Combining the first and last integrals we get

$$I_3 + I_5 + I_4 = \int \chi_D(\xi_1(t_k),u)[g(c_1(\xi_1(t_k),u)) - g(\xi_1(t_k))]$$

$$- Dg(\xi_1(t_k))D_2c(\xi_1(t_k),0)(\frac{u}{1+|u|^2})\Pi(du)$$

$$+ \iint_{t_k}^{t} \chi_D(\xi_1(t_k),u)[g(c_1(\xi_1(t_k),u)) - g(\xi_1(t_k))]\tilde{\nu}(ds,du)$$

$$\iint_{t_k}^{t} (1-\chi_D(\xi_1(t_k),u))[g(c_1(\xi_1(t_k),u)) - c_1(\xi_1(t_k))]\nu(ds,du).$$

From the definition of D and $\tau^\delta(F)$, the last term is zero. It follows that

$$I_3 + I_5 + I_7 = \int \chi_D(\xi_1(t_k),u)[\xi_2(\xi_2(t_k),u) - \xi_2(t_k)]$$

$$- D_2c_2(\xi_2(t_k),0)(\frac{u}{1+|u|^2})\ \Pi(du)$$

$$+ \int\int_{t_k}^{t_k} \chi_D(\xi_1(t_k),u)[c_2(\xi_2(t_k),u) - \xi_2(t_k)]\tilde{\nu}(ds,du)$$

$$+ \int\int_{t_k}^{t} (1-\chi_D(\xi_1(t_k),u)[c_2(\xi_2(t_k),u) - \xi_2(t_k)]\nu(ds,du).$$

We reverse the above procedure to obtain

$$I_3 + I_5 + I_7 = \int c_2(\xi_2(t_k),u) - \xi_2(t_k)-D_2c_2(\xi_2(t_k),0)(\frac{u}{1+|u|^2})\Pi(du)\ (t-t_k)$$

$$+ \int\int_{t_k}^{t} c_2(\xi_2(t_k),u) - \xi_2(t_k)\tilde{\nu}(ds,du)$$

So $\beta(t_k,\xi_1(t_k),t - t_k) = \alpha_2(t_k,\xi_2(t_k), t - t_k)$, and so

$$\eta^\delta(t_{k+1}) = \xi_2(t_{k+1}), \quad \text{for } t_{k+1} < \tau^\delta(T).$$

The result follows now by Lemma 1.5.3 and taking the limit as $|\delta| \to 0$. □

Definition 1.5.3.

We refer to $(a,b,c,w,\nu)$ as a stochastic dynamical system (SDS) on V.

Theorem 1.5.5.

Let $(w_i,\nu_i)$, for $i = 1,2$, be a pair consisting of a Wiener process $w_i$ and a Poisson stochastic measure $\nu_i$ as at the beginning of this section. Also suppose that $\Pi_1(du) = \Pi_2(du)$, i.e. that the parameter measures of $\nu_1$ and $\nu_2$ are the same. Let $\xi_i$ be the solution to

$$d\xi_i = a(\xi_i)dt + b(\xi_i)d\bar{w}\ (dt) + \int c(\xi_i,u)-\xi_i\ \nu(dt,du)$$

with initial condition $\xi_i(a) = \xi_{ia}$.

If $P(\xi_{1a} \in A) = P(\xi_{2a} \in A)\quad \forall\ A \in B(V)$, then

$P(\xi_1(t) \in A_1,\ \xi_1(t_2) \in A_2,\ldots,\xi_1(t_n) \in A_n) =$

$P(\xi_2(t_1) \in A_1,\xi_2(t_2) \in A_2,\ldots,\xi_2(t_n) \in A_n)$ where the $A_i \in B(V)$, and $t_1 < t_2,\ldots, < t_n$.

Proof

It is clear that the finite dimensional distributions of Euler approximations to $\xi_1$ and $\xi_2$ (with the same partition) will be equal. The result follows since Theorem 1.1.3 implies that the finite dimensional distributions of the Euler approximations tend to those of the solution.

# CHAPTER 2 - STOCHASTIC DYNAMICAL SYSTEMS ON MANIFOLDS

## 1. Preliminaries

Definition 2.1.1.

Suppose that M is a $C^3$ Hausdorff locally compact manifold based on $\mathbb{R}^n$ and we have:

(i) a a $C^{1-}$ vector field on M;

(ii) b a $C^{2-}$ section of Hom($\underline{\mathbb{R}}^m$,TM), where $\underline{\mathbb{R}}^m$ is the trivial bundle;

(iii) c a $C^{2-}$ map from D to M where D is an open neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}^p$, and satisfies the following maximality condition: if a sequence of points $(x_i,u_i)$ of D tends to $(x,u) \notin D$, then $c(x_i,u_i)$ eventually leaves any compact subset of $M$.

(iv) a Wiener process w adapted to $\{F_t\}$; and

(v) a Poisson stochastic measure ν, adapted to $\{F_t\}$ and independent of w, such that its parameter measure Π satisfies

$$\int \frac{|u|^2}{1+|u|^2}\, \Pi(du) < \infty.$$

Then we say that *(a,b,c,w,ν) form a stochastic dynamical system on M* (SDS).

Remark

The condition in part (iii) is a maximality condition in the following sense: suppose $D \subset D'$ and c is defined on $D'$, then D is open in $D'$. Suppose $\{(m_i, u_i)\}$ is a sequence of points in D with limit (m,u) in D', by maximality we have $(m,u) \in D$, and so the only way D' can be an extension of D, is if it has a whole new component, which also satisfies the maximality condition: We note that if $D = M \times \mathbb{R}^n$, then D is automatically maximal.

Remark 2

Objects such as c can, and often will, be constructed from objects such as b as follows. Suppose b is a section of Hom $(\mathbb{R}^p, TM)$. Let $\alpha(m,u,t)$ be the integral curve of the vector field b(u) such that $\alpha(m,u,0) = m$. If $D = \{(m,u) | \alpha(m,u,1)$ is defined$\}$ and $c(m,u) = \alpha(m,u,1)$, it is easy to see that D is maximal.

In the following, we construct an example which the domain D of c is much larger than the maximal domain of b.

Example 1

Let $M = \mathbb{R}/\mathbb{Z}$ and $m = p = 1$. Let $b(u) = u$ and $c(m,u) = m + u \bmod 1$. Here we have $\alpha(m,u,1) = c(m,u)$, and $D = M \times \mathbb{R}$. If we remove one point, say 0, from M and consider it as (0,1) we still have $b(u) = u$, and $\alpha(m,u,1) = c(m,u)$, but now only on $D_1 = \{(m,u) \in (0,1) \times \mathbb{R} | m + u \in (0,1)\}$.

If we consider c separately, however, we see that we may sensibly define c on

$$D_\ell = \{(m,u) \in M\backslash\{0\} \times \mathbb{R} \mid m + u \not\equiv 0 \bmod 1\}$$

clearly a much larger domain.

We sum up the situation in the following informal diagram, which fails to commute.

$$\begin{array}{ccc} \text{b's} & \xrightarrow{\text{integrate}} & \text{c's} \\ \text{restrict}\ \downarrow & & \downarrow\ \text{restrict} \\ \text{b's} & \longrightarrow & \text{c's} \end{array}$$

<u>Definition 2.1.2</u>.

If $\xi:\Omega \to [a,b]$ is a stopping time, we define $\Omega_t = \{\omega \in \Omega \mid t < \xi(\omega)\}$.

<u>Definition 2.1.3</u>.

We say that a stochastic process $x:[a,\xi) \times \ \to M$ is admissible if

(i) it is adapted to $\{F_t\}$,

(ii) a.a. sample paths are right-continuous with left-hand limits.

Two admissible processes $x_i:[a,\xi_i) \times \Omega \to M$, $i = 1$, will be called equivalent( $\sim$ ) if $\xi_1 = \xi_2$ a.e., and

$$x_1(\omega)\ [a,t] = x_2(\omega)|[a,t], \text{ a.e. for } \omega \in \Omega^1_t \cap \Omega^2_t .$$

Definition 2.1.4.

We say that $\Lambda = \{(\phi,U),U_o,U_1,\lambda'\}$ is a regular localization (r.ℓ.) for $(a,b,c,w,\nu)$ if:

(i) $(\phi,U)$ is a chart such that $\phi(U) = W$, say, is bounded ;

(ii) $U_o \subset U_1 \subset U$ are open and if $\phi(U_o) = W_o$, $\phi(U_1) = W_1$, then $\bar{W}_o \subset W_1$ and $\bar{W}_1 \subset W$;

(iii) $\lambda: V \to [0,1]$ satisfies supp $\lambda \subset W$, $\lambda|W_1 \equiv 1$ and

(iv) if, $a_\Lambda$, $b_\Lambda$, $c_\Lambda$ are defined by

$$a_\Lambda : \mathbb{R}^n \to \mathbb{R}^n$$

$$v' \to \lambda(v)\phi_*(a)(v)$$

$$b_\Lambda : \mathbb{R}^n \to L(\mathbb{R}^m,\mathbb{R}^n)$$

$$v \to \lambda(v)\phi_*(b)(v)$$

and $c_\Lambda : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$

$$(v,u) \to \lambda(\phi(c(\phi^{-1}(v),u)))\lambda(v)(\phi(c(\phi^{-1}(v),u))-v) + v,$$

$$[\phi_*(X)v = T_{\phi^{-1}(v)} \phi \circ X(\phi^{-1}(v)), \text{ where } X = a \text{ or } b]$$

then $a_\Lambda$, $b_\Lambda$ and $D_2 c_\Lambda(x,v)$ are globally Lipschitz and $c_\Lambda$ is $C^{2-}$.

It is here that the maximality condition on $c$ is used. More precisely if $(x_i,u_i) \in D$ is a sequence of points with $\lim(x_i,u_i) = (x,u) \notin D$, $x_i \in U$ and $c(x_i,u_i) \in U$, then for any compact set $K$ with $U_1 \subset K \subset U$, $c(x_i,u)$ eventually lies outside $K$. Hence we can choose $\lambda$ such that $c_\Lambda(\phi(x_i),u) - \phi(x_i) \to 0$ showing that $c_\Lambda$ is continuous. A similar argument using the derivatives of $c$ shows that we can find a $\lambda$ making $c_\Lambda$ $C^{2-}$.

Notation

Suppose that $x:[T_o,\xi) \times \Omega \to M$ is admissible and $\Lambda$ is r.1 for $(a,b,c,w,\nu)$. Then for $t \in [T_o,T_1)$, set

$$\Omega_{t_o}^{\Lambda_o} = \{\omega \in \Omega | t_o < \xi(\omega) \text{ and } x(\omega,t_o) \in U_o\}, \text{ so}$$

$$\Omega_{t_o}^{\Lambda_o} \in F_{t_o}.$$

Let $x_{t_o}^{\Lambda} : [t_o,T_1) \times \Omega_{t_o}^{\Lambda} \to \mathbb{R}^n$ be the uniquely defined solution to the stochastic differential equation

$$y(t) = \phi o x(t_o) + \int_{t_o}^{t} a_\Lambda(y(s))ds + \int_{t_o}^{t} b_\Lambda(y(s))d\bar{w}(s)$$

$$+ \int\int_{t_o}^{t} c_\Lambda(y(s),u)-y(s)\bar{\nu}(ds,du)$$

and let $\tau_{t_o}^{\Lambda} = \tau(F)$, where $F=\{U_o,U_2,U_1,D_2,D_1\}$, $\bar{U}_o \subset U_2$, $\bar{U}_2 \subset U_1$ and $D_i = \{(x,u)| x \in U_i, c(x,u) \in U_i\}$ for $i = 1,2$.

Definition 2.1.5.

Let $(a,b,c,w,\nu)$ be an S.D.S. on M and $\Lambda$ an r.1. of it, suppose $x:[a,\xi) \times \Omega \to M$ is admissible. We say $\Lambda$ *affirms* $x$ if for each $t_o \in [T_o,T_1)$

$$\theta o x|[t_o,\tau_{t_o}^{\Lambda} \wedge \xi) \times \Omega_{t_o}^{\Lambda}, \sim y|[t_o,\tau(U_o)\wedge \xi) \times \Omega_{t_o}^{\Lambda}$$

where $\theta:M \to \mathbb{R}^p$

is some, not necessarily continuous extension of $\phi$.

Definition 2.1.6.

Let $x:[T_0,\xi) \times \Omega \rightarrow$ be admissible. Then k is said to be *a locally regular solution of the stochastic differential equation*

$$dx(t) = a(x(t))dt + b(x(t))d\bar{w}(t) + \int c(x(t),u)-x(t)\bar{\nu}(dt,du)$$

if every r.1 of $(a,b,c,w,\nu)$ affirms x.

Remark

This statement makes even less sense than usual since the last term contains manifold valued things. However, we regard it as a global representation of what goes on in charts.

2. Uniqueness and Extension

We shall take a different approach to that of Elworthy [2] in that we shall construct a solution and show that it has the required properties. In fact the two approaches are formally equivalent, the difference being mainly one of emphasis.

Definition 2.2.1

We say an Atlas $A$ consists of big charts (by abuse, $A$ is a big atlas) if $\forall\ x,y \in M$, $\exists (\phi,U) \in A$ s.t. $x,y \in U$.

We use big atlases to cover the paths of an admissible process as shown in the following lemma.

Lemma 2.2.1.

Let $A$ be a countable big atlas, and let $x:[a,\xi) \times \Omega \to M$ be an admissible process. If we define $L^i[t_j,t_k]$ as

$$L^i[t_j,t_k] = [t_j,t_k] \times \{\omega | \; x([t_j,t_k],\omega) \in U^i\}$$

then $\bigcup_{ijk} L^i[t_j,t_k] = [a,\xi) \times \Omega$, up to a set of measure zero, where the $t_i$ form a countable dense subset of $[a,\infty)$ and contain a.

Proof

We can suppose that $\xi > a$ on $\Omega$.

Since a.a. sample paths are right continuous, for any V, a nbd of x(a) there exists a nbd. of a, $[a,\varepsilon)$ say, such that $x([a,\varepsilon),\omega) \in V$.

So for a.a. $\omega$ there exists i and j such that $(s,\omega) \in L^i[a,t_j]$ for $s \in [a,t_j]$.

For a.a. $\omega$ and $s \in [a,\xi(\omega))$ and for any neighbourhoods $V^+$ and $V^-$ of $x(s,\omega)$ and $x(s^-,\omega)$ respectively there exists $\epsilon > 0$ such that $x((s-\epsilon,s+\epsilon)) \in V^+ \cup V^-$.

So for a.a. $\omega$ there exists i, j and k with $(\tau,\omega) \in L^i[t_j,t_k]$ for all $\tau \in [t_j,t_k]$.

The lemma follows. □

Lemma 2.2.2. (Global Uniqueness)

Suppose that $x_i:[T_0,\xi_i) \times \Omega \to M$, $i = 1,2$ are locally regular solutions of

$$dx(t) = a(x(t))dt + b(x(t)d\bar{w}(t) + \int c(x(t),u)-x(t)\bar{\nu}(dt,du)$$

with $x_1(T_o) = x_2(T_o)$ a.e.

Then $x_1 \mid [T_o,\xi_1 \wedge \xi_2) \times \Omega \sim x_2 \mid [T_o,\xi_1 \wedge \xi_2) \times \Omega$.

Proof

Let $\{\Lambda^i\}$ be a countable collection of r.l's of (a,b,c,w,ν) such that $\{\phi^j \mid U_o^j, U_o^j\}$ forms a big atlas and let T' be a countable dense subset of $[T_o,T_1)$

Define $E(t) = \{\omega \mid t < \xi_1 \wedge \xi_2,\ x_1(t,\omega) = x_2(t,\omega)\}$,

so $E(t) \in F_t$. For $t_o$, $t_1 \in T'$, let

$$\Lambda^j[t_o,t_1] = \{\omega\epsilon E(t_o) \mid x_1(\omega,t_o) \in U_o^j, t_1 < \xi_1 \wedge \xi_2 \wedge \tau_{1t_o}^{\Lambda^j} \wedge \tau_{2t_o}^{\Lambda^j}\},$$

we have $\Lambda^j[t_o,t_1] \in F_t$ .

By hypothesis $\exists\, Z^j[t_o,t_1] \in F_t$ of measure zero such that for $\omega \in \Lambda^j[t_o,t_1] \setminus Z^j[t_o,t_1]$

$$\theta^i \circ x_1(\omega) \mid [t_o,t_1] = \tilde{x}_j(\omega) \mid [t_o,t_1] = \theta^j \circ x_2(\omega) \mid [t_o,t_1],$$

where $\theta^i$ and $\theta^j$ are extensions of $\phi^i$ and $\phi^j$, and

$\tilde{x}_1 : [t_o,b) \times E(t_o) \to \mathbb{R}^n$ is a solution to

$$y(t) = \theta^j \circ x_1(t_o) + \int_{t_o}^{t} a_\Lambda(x_j)(s))ds + \int_{t_o}^{t} b_\Lambda(y(s))d\bar{w}(s)$$

$$+ \int \int_{t_o}^{t} c(y(s),u)-y(s)\bar{\nu}(ds,du).$$

Suppose that $\omega$ satisfies $t < \xi_1(\omega) \wedge \xi_2(\omega)$ and $x_1(\omega)|[T_0,t]$ is right-continuous with left-hand limits. Suppose $\exists\, s \in [T_0,t)$ such that

$$x_1(\omega)|[T_0,s) = x_2(\omega)|[T_0,s).$$

Then, $\exists$ j with $\omega \in \Lambda^j[t_0,t_1]$ for some $t_0,t_1 \in T'$ such that $T_0 \leq t_0 \leq s \leq t_1 \leq t$, since $\{(\phi^j|U_0^j,U_0^j)\}$ forms a big atlas. It follows that if $\omega \in \Lambda^j[t_0,t_1]$ we have

$$x_1(\omega)|[T_0,t_1] = x_2(\omega)|[T_0,t_1] \qquad \text{a.e.}$$

and hence that $x_1|[T_0,t] = x_2|[T_0,t]$ a.s. for $t < \xi_1 \wedge \xi_2$. □

Remark

If we had not used a big atlas in the proof then we could not have deduced the result.

For example, suppose $M = \mathbb{R}$, $a = b$, $a = b = 0, c(m,u) = m + u$ and $\nu$ is such that its parameter measure satisfies, $\Pi\{(\pm 1)\} = 1$, $\Pi(A) = 0$ if $A \cap (\{-1\} \cup \{+1\}) = \emptyset$.
Then $x(t) = \nu(t,1) - \nu(t,-1)$ is a solution of the stochastic differential equation, induced by $(c,\nu)$. If we take an atlas of r.I's of $\mathbb{R}$ such that $\mathrm{diam}|U_0| < 1$, then we cannot get information beyond the first jump time, so for example, $-x(t)$ is also affirmed as a solution by this cover.

Lemma 2.2.3.

Suppose that $x_i:[a_i,\xi) \times \Omega \to M$, for $i = 1,2$, are l.r. solutions to

$$dx(t) = a(x(t))dt + b(x(t))d\bar{w}(t) + \int c(x(t)u)-x(t)\bar{\nu}(dt,du)$$

given $x(a_i) = x_{a_i}$, a.s. for $i = 1,2$, with $a_1 \leq a_2$.

If we define:

(i) $x':[a_1,\xi') \times \Omega \to M$

where $\xi' = \begin{cases} \xi_1 \text{ if } \xi_1 \leq a_2 \text{ or } a_2 \leq \xi_2 \leq \xi_1 \\ \xi_2 \text{ if } \xi_2 > a_2 \text{ and } a_2 \leq \xi_1 \leq \xi_2 \end{cases}$

and $x'(t) = \begin{cases} x_1(t) \quad a_1 \leq t < \xi_1(\omega) \leq a_2 \\ \text{or} \quad a_2 \leq t \leq \xi' \quad \xi_2 \geq \xi_1 \\ x_2(t) \text{ if } \xi_2 > \xi_1 \end{cases}$

and

(ii) if $a_1 = a_2$ define $x:[a_1,\xi) \times \Omega \to M$

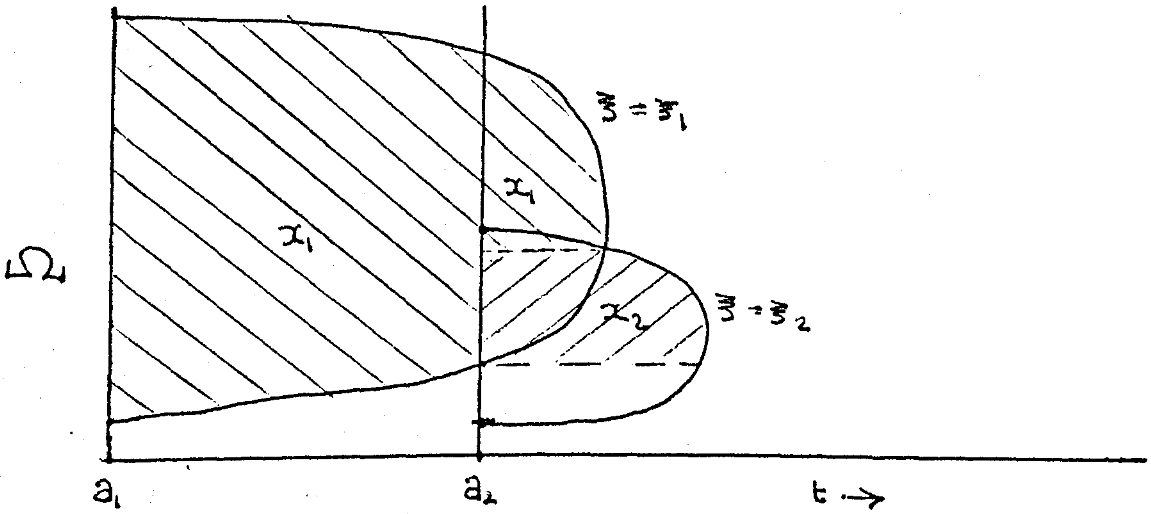
where $\xi = \xi_1 \vee \xi_2$

by $x(t) = \begin{cases} x_1(t) \quad \xi_1 \geq \xi_2 \\ x_2(t) \quad \xi_2 > \xi_1 \end{cases}$

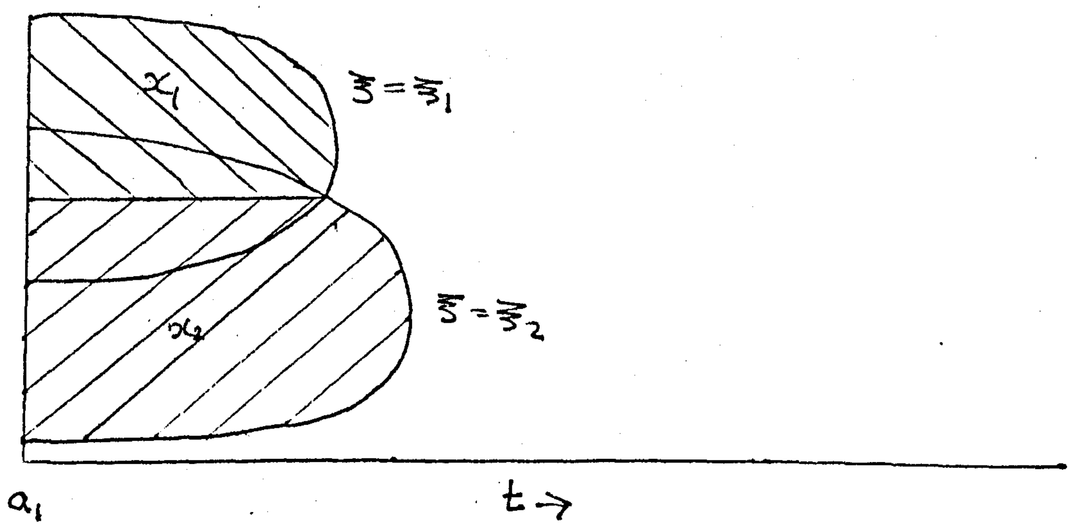
then $x':[a_1,\xi')$, and $x:[a_1,\xi)$, are admissible processes, and are solutions to the stochastic differential equation.

These definitions are made clear by the following diagrams:

(i)




(ii)



Proof

In case (ii) the proof follows if $\{\omega | \xi_1 > a_1\} \cap \{w | \xi_2 > a_1\} = \emptyset$, since then for any l.r. $\Lambda$,

$$[a_1, \tau^{\Lambda}_{t_o \wedge}(\xi_1 \vee \xi_2)) \times \Omega^{\Lambda}_{t_o} = [a_1, \tau^{\Lambda}_{t_o} \wedge \xi_1) \times \Omega^{\Lambda}_{t_o}$$
$$\bigcup_{\text{disjoint}} [a_1, \tau^{\Lambda}_{t_o} \wedge \xi_2) \times \Omega^{\Lambda}_{t_o}.$$

Hence if $x_1$ and $x_2$ are affirmed by $\Lambda$, so is x.

Otherwise put
$$\xi_1' = \begin{cases} \xi_1 & \text{if } \xi_1 \geq \xi_2 \\ a & \xi_2 \geq \xi_1 \end{cases}$$
$$\xi_2' = \begin{cases} \xi_2 & \xi_2 > \xi_1 \\ a & \xi_1 \geq \xi_2 \end{cases}$$

Then $\xi_1' \vee \xi_2' = \xi_1 \vee \xi_2$ and

$x_i | [a_1, \xi_i') \times \Omega$ is a l.r. solution for $i = 1,2$. The proof follows.

In case (i) Put
$$\xi_1' = \begin{cases} \xi_1 & \text{if } \xi_1 \leq a_2 \\ & \text{or } \xi_2 \leq \xi_1 \\ a_1 & \xi_2 > \xi_1 \end{cases}$$
$$\xi_2' = \begin{cases} \xi_2 & \text{if } \xi_2 > \xi_1 \\ a_1 & \text{otherwise.} \end{cases}$$

Again we have $\xi_1 \vee \xi_2 = \xi_1' \vee \xi_2'$, and given a r.l. $\Lambda$

$$[t_o, \tau^\Lambda_{t_o} \wedge (\xi_1 \vee \xi_2)) \times \Omega^\Lambda_{t_o} = [t_o, \tau^\Lambda_{t_o} \wedge \xi_1' ) \times \Omega^\Lambda_{t_o}$$

$$\bigcup_{\text{disjoint}} [t_o, \tau^\Lambda_{t_o} \wedge \xi_2^2) \times \Omega^\Lambda_{t_o}$$

Now $x | [t_o, \tau^\Lambda_{t_o} \wedge \xi_1' ) \times \Omega^\Lambda_{t_o} = x_1 | [t_o, \tau^\Lambda_{t_o} \wedge \xi_1' ) \times \Omega^\Lambda_{t_o}$

and if $t_o \geq a_2$,

$$x | [t_o, \tau^\Lambda_{t_o} \wedge \xi_1' ) \times \Omega^\Lambda_{t_o} = x_2 | [t_o, \tau^\Lambda_{t_o} \wedge \xi_2' ) \times \Omega^\Lambda_{t_o}.$$

In the remaining case the result follows from the existence and uniqueness of solutions.

The proof follows. □

## 3. Construction of Solutions

In this section we shall construct a solution and show that it is maximal in a strong sense.

**Lemma 2.3.1.**

Given the equations $dx(t) = a(x(t))dt + b(x(t))d\bar{w}(t) + \int c(x(t),u) - x(t)\tilde{\nu}(dt,du)$ together with $x(T_o) = x_{T_o}$, $F_{T_o}$-measurable, and $\Lambda$ a r.1. $\Lambda = \{(\phi,U),U_o,U_1,\lambda\}$. Define $x:[T_o,\xi)\times \Omega \to M$, where $\xi = \tau^\Lambda_{T_o}$, by

$x = \phi^{-1}x_\Lambda : [a, \tau^\Lambda_{T_o}) \times \Omega \to M$, where $x_\Lambda(T_o) = \theta . x_{T_o}$.

Then x(t) is a solution to the above equation.

Proof

We need to show that if $\Lambda'$ is any other r.l. then $\Lambda'$ affirms x.

Without loss of generality we can suppose that

$U_o' \subset U_o$, $U_1' \subset U_1$, $U' \subset U$ and $\lambda' \leq \lambda$.

Let g be a bounded extension to $\mathbb{R}^n$ of the change of chart map $\phi' \circ \phi^{-1}$

We have $x_\Lambda(t) = x_\Lambda(a) + \int_{T_o}^t a_\Lambda(x_\Lambda(s))ds + \int_{T_o}^t b_\Lambda(x_\Lambda(s))d\bar{w}(s)$

$$+ \int \int_{T_o}^t c_\Lambda(x_\Lambda(s),u) - x_\Lambda(s)\tilde{\nu}(ds,du).$$

Writing $y(t) = g(x_\Lambda(t))$, then by Theorem 1.5.5,

$$g(x_\Lambda(t)) = g(x_\Lambda(T_o)) + \int_{T_o}^t a'(y(s))ds + \int_{T_o}^t b'(y(s))d\bar{w}(s)$$

$$+ \iint_{T_o}^t c'(y(s),u) - y(s)\tilde{\nu}(ds,du),$$

for $t < \tau^\Lambda_{T_o}$, where
$$a' = Dg(g^{-1}(y))a(g^{-1}(y)),\ y \in U_1'$$
$$b' = Dg(g^{-1}(y))b(g^{-1}(y)),\ y \in U_1'$$
$$c'(y,u) = gc(g^{-1}(y),u)\ (y,u) \in D(U_1', U_1')$$

and are suitably smooth bounded functions.

Now by definition of $a'_\Lambda$ , $b'_\Lambda$ , $c'_\Lambda$ we see

$$\left.\begin{aligned} a'_\Lambda &= a' \\ b'_\Lambda &= b' \\ c'_\Lambda &= c' \end{aligned}\right. \qquad \text{on } U'_1 \text{ or } D\,(U'_1, U'_1) \text{ respectively}$$

So $\theta' \circ x|[T_o, \tau^{\Lambda'}_{T_o}) \sim x_{\Lambda'}|[T_o, \tau^{\Lambda}_{T_o})$, by Theorem 1.5.3,

and we are done.

Now let $\{\Lambda_i\}$ be a countable collection of l.r's such that $\{U^i_o\}$ cover M, and let T' be a countable dense subset of $[T_o, T_1)$ including $T_o$.

Let $\{t_{i_j}, \Lambda_{k_j}\}_{j \in \mathbb{N}_o}$ be an ordering of $T' \times \{\Lambda_i\}$, such that $t_{i_o} = T_o$.

Define $y_o : [T_o, \xi_o) \times \Omega \to M$, where $\xi_o = \tau^{\Lambda_{i_o}}_{T_o}$ by $y_o = \phi^{i_o}\, x_{\Lambda_{i_o}}(t)$.

By Lemma 2.3.1, $y_o$ is a solution.

Now given $y_{n-1}$: $[T_o, \xi_{n-1}) \times \Omega \to M$, define

$$y_n : [T_o, \xi_n) \times \Omega \to M, \text{ with } \xi_n \geq \xi_{n-1} \text{ by using Lemma 2.2.3.}$$

Explicitly we put $y_{n-1} = x_1$, $a_1 = T_o$, $a_2 = t_{i_n}$ and

$$x_2 = (\phi^{i_n})^{-1} \circ x_{\Lambda_{i_n}} : [t_{i_n}, \tau^{\Lambda^{i_n}}_{t_{i_n}}) \times \Omega \to M$$

($x_2$ is a solution by Lemma 2.3.1).

Then define $y_n$ to be $x$ or $x'$, $\xi_n$ to be $\xi$ or $\xi'$ according to whether $t_{i_n} = T_o$ or not.

By Lemma 2.2.3,

$$y_n : [T_o, \xi_n) \times \Omega \to M \text{ is a solution.}$$

Finally let $\xi = \sup\limits_n \xi_n$ and define

$$y : [T_o, \xi) \times \Omega \to M, \text{ by } y(t) = y_n(t)$$

if $t < \xi_n \le \xi$.

<u>Lemma 2.3.2.</u>

Let $y : [T_o, \xi) \times \Omega \to M$ be the solution just constructed and let $x : [T_o, \eta) \times \Omega \to M$ be any other solution. If the $\{U_o^i, \phi^i | U_o^i\}_{i \in \mathbb{N}}$ form a big atlas, then, $\eta \le \xi$ a.e. and

$$x(t,\omega) = y(t,\omega) \text{ for } t < \eta(\omega).$$

<u>Proof</u>

The second assertion follows from Lemma 2.2.3. Suppose the Lemma is false, then we can assume without loss of generality that $\eta > \xi$.

By Lemma 2.2.1. $\bigcup\limits_{ijk} L^i[t_j, t_k] = [T_o, \eta) \times \Omega$ and so

for a.a. $\omega$, $\exists$ $i$, $t_j$, $t_k$ such that $t_j < \xi(\omega) < t_k$ and $(\xi(\omega), \omega) \in L^i[t_j, t_k]$.

However we have $t_j \in T'$ and so $L^i[t_j,t_k] \subset [t_j,t_k] \times \Omega_{t_o}^{\Lambda_i}$ and we must have included this in one of our inductive steps, so we have a contradiction, and $\eta \le \xi$. □

Collecting together the results of the previous sections we have:

<u>Theorem 2.2.3</u>.

Given the stochastic dynamical system $(a,b,c,w,\nu)$, and initial condition $x_a$, $F_a$-measurable. There exists a unique solution $x:[T_o,\xi) \times \Omega \to M$, of the corresponding S.D.E. which is maximal in the sense that if $y:[T_o,\eta) \times \Omega \to M$ is any other solution, then $\eta \le \xi$ a.e.

CHAPTER 3

## 1. Submanifolds and the Completeness Theorem for Compact Manifolds

Proposition 3.1.1.

Let $(a_1, b_1, c_1, w,\nu)$ and $(a_2, b_2, c_2, w,\nu)$ be stochastic dynamical systems on the manifolds $M_1$ and $M_2$ respectively and suppose that $h:M_1 \to M_2$ is a $C^3$-diffeomorphism of $M_1$ onto an open subset of $M_2$ such that,

$$h_*(a_1) = a_2|h(M_1), \quad h_*(b_1) = b_2|h(M_1) \text{ and}$$
$$h o c_1(h^{-1}(\cdot),\cdot) = c_2|D(h(M),h(M)).$$

If $x_1 : [T_o,\xi) \times \Omega \to M_1$ is a locally regular solution of

$$(1) \quad dx_1(t) = a_1(x_1(t))dt + b_1(x_1(t))d\bar{w}(t) + \int c_1(x_1(t),u) - x_1(t)\bar{\nu}(dt,du),$$

then $x_2 = h o x_1 : [T_o,\xi) \times \Omega \to M_2$ is a locally regular solution to (1) with the index 2 replacing 1, throughout.

Proof

Choose an ascending sequence of open sets $\{U_i\}$ of $h(M_1)$ such that $\bar{U}_i \subset U_{i+1}$, $U_i \nearrow h(M_1)$. Let $\tau_i$ be the first exit time of x from $h^{-1}(U_i)$, set $\xi_i = \tau_i \wedge \xi$, $x_1^i = x|[T_o,\xi_i) \times \Omega$ and

$x_2^i = h \circ x_1^i$. Each $x_2^i$ is a solution of the second equation, since it is affirmed by all regular localizations supported in $U_{i+1}$. Since $\xi = \sup \xi_i$, by uniqueness we have that $x_2$ is also a solution, as required, c.f. Lemma 2.2.3. □

Theorem 3.1.2.

Let $j: M_1 \to M_2$ be a $C^3$-embedding of a manifold $M_1$ into the manifold $M_2$. Suppose the stochastic dynamical system $(a_2, b_2, c_2, w, \nu)$ on $M_2$ has $a_2|j(M)$ and $b_2|j(M)$ tangent to $j(M)$ and $c_2 \mid j(M_1) \times \mathbb{R}^p \cap D \subset j(M_1)$. This induces the stochastic dynamical system $(a_1, b_1, c_1, w, \nu)$ on $M_1$ given by $j_*(a_1) = a_2|j(M_1)$, $i_*(b_1) = b_2|j(M_1)$ and $j \circ c_1(j^{-1}, \cdot) = c_2(\cdot,\cdot)|j(M_1) \times \mathbb{R}^p \cap D$. Let $x_i : [T_0, \xi_i) \times \Omega \to M_i$ a locally regular solution to

$$\text{(i)} \qquad dx_i(t) = a_i(x_i(t))dt + b_i(x_i(t))d\bar{w}(dt) + \int c_i(x_i(t),u) - x_i(t)\bar{\nu}(dt,du) \qquad i = 1,2.$$

Then $x' = j \circ x_1 : [T_0, \xi) \times \Omega \to M_2$ is locally regular solution of (2). Conversely if $j$ is a closed embedding then any locally regular solution of (2), with $x_2(T_0)(\Omega) \subset j(M_1)$ is equivalent to one of the form $j \circ x_1$, where $x_1$ is a solution of (1).

Proof

We first suppose that $j$ is a closed embedding. Choose a cover of $M_2$ by a countable family of local regularizations $\{\Lambda_2^k\}$, such that $\{(\phi|U_0^k, U_0^k)\}$ forms a big atlas of $M_2$, and such that if $M_1 \cap U_0^k \neq \emptyset$, then,

$$\phi_k^o(U) = W = W' \times W'' \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$$

$$\phi^k(U_o) = W_o = W_o' \times W_o'' \subset \mathbb{R}^r \times \mathbb{R}^{n-r},$$

$$\phi^k(U_1) = W_1 = W_1' \times W_1'' \subset \mathbb{R}^r \times \mathbb{R}^{n-r},$$

and $\phi^k(M_1 \cap U) = W' \times \{0\} \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$

where $r = \dim M_1$.

Then if $M_1 \cap U_o^k \neq \emptyset$

$\Lambda_2^k$ restricts to a local regularization $\Lambda_1^k$ of $(a_1, b_1, c_1, w, \nu)$, with

$$a_{2\Lambda_2^k}|\mathbb{R}^r = a_{1\Lambda_1^k},$$

$$b_{2\Lambda_2^k}|\mathbb{R}^r = b_{1\Lambda_1^k},$$

and $S_{\Lambda_2^k}|\mathbb{R}^r \times \mathbb{R}^p = c_{1\Lambda_1^k}$.

It follows that any solution to the stochastic integral equation

$$y(t) = y(t_o) + \int_{t_o}^t a_{i\Lambda}(y\ (s))ds + \int_{t_o}^t b_{i\Lambda}(y\ (s))d\bar{w}(s)$$

$$+ \int\int_{t_o}^t c_{i\Lambda}(y\ (s),u) - y\ (s)\bar{\nu}(ds,du),$$

with $i = 1$ is also a solution with $i = 2$, and vice-versa if $y(t_o) \in \mathbb{R}^1 \times \{0\}$.

First suppose that each $x_i:[T_o,\xi_i) \times \Omega \to M_i$ is maximal. Each $x_i$ is then equivalent to a constructed solution. Let T' be a countable dense subset of $[T_o,\infty)$ and take an ordering of $T' \times \{\Lambda_2^k\}$ with $t_{\ell 1} = a$, giving a corresponding ordering of $T' \times \{\Lambda_1^k\}$. Let $x_i^n:[T_o,\xi_i^n) \times \Omega \to M_i$ be the n-th construct. Assume that $j o x_1^n:[T_o,\xi_1^n) \times \Omega \simeq x_2^n \; [T_o,\xi_2^n) \times \Omega$. From the above and the method of construction it will follow that

$$j o x_1^{n+1}:[T_o,\xi_1^{n+1}) \times \Omega \simeq x_2^{n+1}:[T_o,\xi_1^{n+1}) \times \Omega.$$

The theorem will follow in this case by taking the limit as $n \to \infty$, in particular $\xi_1 = \xi_2$ a.e. write $\xi_1 = \xi$.

If $x_i$ is not maximal, then $\xi_i = \xi \wedge \eta_i$ a.e. for some stopping time $\eta_i \leq \xi$, and the theorem follows.

In general j is locally closed, so take a sequence of compact sets $U_i$ such that $\bar{U}_\ell \subset U_{\ell+1}$ and $U_\ell \nearrow M_1$. Let $V_\ell$ be a sequence of open subsets of $M_2$ such that $V_\ell \cap j(M_1) = j(U_\ell)$. Let $\tau^\ell = \xi_1 \wedge \tau(U_\ell)$, where $\tau(U_\ell)$ is the first exit time of $x_i$ from $U_\ell$. Now each $j|U_\ell$ is a closed embedding and so $j o x_i \; [T_o,\tau_\ell) \times \Omega \simeq x_2:[T_o,\tau_\ell) \times \Omega$ by the first part of the proof. the theorem follows since $\xi_1 = \sup_\ell \tau_\ell$.

Corollary 3.1.3.

If M is compact then the explosion time $\xi$ satisfies $\xi \equiv \infty$, for any SDE on M.

Proof

Let $i$ be a $C^3$ embedding of M into $\mathbb{R}^s$ for some s.
We write $M_1 = M$ and identify M with $i(M)$.

Let $\alpha : V \to M$ be a $C^3$-normal bundle to TM in $\mathbb{R}^s|M$. Using the exponential map, this gives a bounded neighbourhood U of M in $\mathbb{R}^s$ with $C^3$ retraction map $\beta : U \to M$, such that, $D\alpha(x) : \mathbb{R}^s \to T_{\beta\alpha(x)}M$ is surjective. For each $x \in U$, let $S_x$ be the orthogonal complement of Ker $D\alpha(x)$ in $\mathbb{R}^n$, then $D\beta(x)|S_x$ is an isomorphism onto $T_{\beta(x)}M$.

Define

$$a_o(x) = (D\beta(x)|S_x)^{-1}\, a(\beta(x)), \qquad x \in U$$

$$b_o(x) = (D\beta(x)|S_x)^{-1}\, b(\beta(x)), \qquad x \in U$$

$$c_o(x,u) = c(\beta(x),u) + \beta(x) - x, \qquad (x,u) \in U \times \mathbb{R}^p$$

[We observe that $D = M \times \mathbb{R}^p$, since M is compact].
Choose a $C^2$-function $\lambda : \mathbb{R}^s \to [0,1]$, with supp $\lambda \subset U$, and $\lambda(M) \equiv 1$,
define

$$\tilde{a}(x) = \lambda(x)\, a_o(x)$$

$$\tilde{b}(x) = \lambda(x) b_o(x)$$

$$\tilde{c}(x,u) = \lambda(c_o(x,u))\lambda(x)(c_o(x,u) - x) + x.$$

Then $(\tilde{a}, \tilde{b}, \tilde{c}, w, \nu)$ is an SDS on $\mathbb{R}^s$, which satisfies the conditions of Theorem 1.5.1.
Since $x(T_o)(\Omega) \subset M$, and M is compact, $E(|x(T_o)|^2) < \infty$, if we let $x(t)$ be the maximal solution of the stochastic differential equation corresponding to $(\tilde{a}, \tilde{b}, \tilde{c}, w, \nu)$, then we have $\xi \equiv \infty$, and by Theorem 3.1.2, it follows that the solution on M has $\xi \equiv \infty$ a.e.

## 2. Stochastic Dynamical Systems on $\mathbb{R}^n$.

Lemma 3.2.1.

For a complete separable metric space M and a finite measure space $(\Omega,\mathcal{F},\mu)$, suppose $y \in L^0(\Omega,\mathcal{F},D([a,b],M))$. Then there exists for any $\epsilon > 0$, a compact subset $K_\epsilon$ of M, and an $\Omega_\epsilon \in \mathcal{F}$, such that:

(i) $\mu(\Omega_\epsilon) \geq \mu(\Omega) - \epsilon$, and

(ii) for all $\omega \in \Omega_\epsilon$ and $t \in [a,b]$, $y(\omega)(t) \in K_\epsilon$.

Proof.

Since $D([a,b],M)$ is a complete separable metric space, the measure $\nu = y(\mu)$, induced on it by y is tight. Thus for $\epsilon > 0$, there is a compact subset of $D([a,b],M)$, $D_\epsilon$, with $\nu(D_\epsilon) > \mu(\Omega) - \epsilon$. By Theorem 6.2 [8] $\{f(t) \in M;\ f \in D_\epsilon,\ t \in [a,b]\}$, is compact in M. Let $K_\epsilon$ denote this subset and $\Omega_\epsilon = y^{-1}(D_\epsilon)$. □

Theorem 3.2.2.

Let $(a,b,c,w,\nu)$ be a stochastic dynamical system on $\mathbb{R}^n$. Let $x:[T_0,\xi) \times \Omega \to \mathbb{R}^n$ be a locally regular solution to

$$dx(t) = a(x(t))dt + b(x(t))d\bar{w}(dt) + \int c(x(t),u)-x(t)\bar{\nu}(dt,du).$$

Then for any $T_1 > T_0$, $x|[T_0,\xi \wedge T_1)$, satisfies the stochastic integral equation

$$x(t) = x(a) + \int_a^t a(x(s))ds + \int_a^t b(x(s))d\bar{w}(ds)$$

$$+ \int\int_a^t c(x(s),u) - x(s)\bar{\nu}(ds,du).$$

Proof

We restrict $\Omega$ to $\Omega_{T_1}$, and apply the previous lemma to construct, for any $\epsilon > 0$, a compact $K_\epsilon \in \mathbb{R}^n$, and $\Omega_\epsilon \subset \Omega_{T_1}$, such that $P(\Omega_{T_1} \backslash \Omega_\epsilon) < \epsilon$, and $x(t,\omega) \in K_\epsilon$, for any $(t,\omega) \in [T_0,T_1] \times \Omega_\epsilon$. We have a regular localization $\Lambda_\epsilon = \{(U,\phi),U_0,U_1,\lambda\}$, for $(a,b,c,w,\nu)$, with $K_\epsilon \subset U_0$ and $\phi:U_1 \to \mathbb{R}^n$ the inclusion, by definition $\Lambda_\epsilon$ affirms x. Since $\tau_{T_0}^{\Lambda^\epsilon}(\omega) \geq T_1$, for a.a. $\omega \in \Omega_\epsilon$, this implies that $x|[T_0,T_1] \times \Omega_\epsilon$ satisfies the integral equation.

Since $\epsilon$ was arbitrary, the result follows. □

3. Solutions as Markov Processes

From now on we assume that we can find an embedding $i:M \to \mathbb{R}^q$ such that the coefficients a,b and c can be extended to be globally Lipschitz. This includes for example the case when M is compact.

Theorem 3.3.1

Suppose that $(a,b,c,w,\nu)$ is a stochastic dynamical system and that $x:[a,\infty) \times \Omega \to M$ is the maximal locally regular solution to

(1) $dx(t) = a(x(t))dt + b(x(t))d\bar{w}(t) + \int c(x(t),u)-x(t)\bar{\nu}(dt,du).$

Then x is a Markov process with transition probabilities given by $P(s,x_o,t,A) = P(x_{s,x_o}(t) \in A)$ where $x_{s,x_o}(t)$ is the solution of (1) with $x_{s,x_o}(s) = x_o$.

Proof

We take an embedding $i:M \to \mathbb{R}^q$ such that the coefficients can be extended to be globally Lipschitz. The theorem then follows from Theorem 1 of Section 9 of Part 2 of [4]. □

Proposition 3.3.2.

If $(a,b,c,w,\nu)$ is a stochastic dynamical system on a manifold M with infinite explosion time, for any $F_{T_o}$-measurable starting distribution, we have

$$f(x(t)) = f(x(a)) + \int_{T_o}^{t} df(a(x(s)))ds + \frac{1}{2}\,\mathrm{Tr}\int_{T_o}^{t} d(\,df(b(x(s))))(b(x(s)))ds$$

$$+ \int_{T_o}^{t}\int f(c(x(s),u)) - f(x) - df\,(T_2\ c(x,0)\ (\frac{u}{1+|u|^2})\Pi(du)ds$$

$$+ \int_{T_o}^{t} df(b(x(s)))dw(s) + \int\int_{T_o}^{t} f(c(x(s),u)) - f(x(s))\tilde{\nu}(ds,du),$$

where: (1) $\mathrm{Tr}\ d(df(b(x)))(b(x)) = \sum_i d(df(b(x)(e_i)))(b(x)(e_i)),$

for any orthonormal basis $e_i$ of $\mathbb{R}^m$;

(2) $T_2\, c(x,0):\mathbb{R}^p \to T_xM$ is the derivative of $c(x,\cdot):\mathbb{R}^p \to M$, evaluated at zero in $\mathbb{R}^p$, and

(3) $f$ is $C^2$ and is bounded together with its derivatives.

<u>Proof</u>

By Lemma 3.2.1, there exists $\Omega_\epsilon \subset \Omega$, with $P(\Omega_\epsilon) > 1-\epsilon$ and $K_\epsilon \subset M$, compact such that $x(t,\omega) \in K_\epsilon$, $\forall\ \omega \in \Omega_\epsilon$, $(t \in [T_0,T_1])$.

Take an embedding $i$, of $M$ into $\mathbb{R}^s$ for some $s$, Since $i$ is continuous $i(K_\epsilon)$ is compact. Let $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{f}$ be extensions of $i_*(a)$, $i_*(b)$, $i_c\ (i^{-1},\cdot)$ and $foi^{-1}$, to a compact neighbourhood of $i(K_\epsilon)$.

$$\text{If } \tilde{x}(t) = \tilde{x}(a) + \int_{T_0}^{t} \tilde{a}(\tilde{x}(s))ds + \int_{T_0}^{t} \tilde{b}(\tilde{x}(s))d\bar{w}(s)$$

$$+ \iint_{T_0}^{t} \tilde{c}(\tilde{x}(s),u) - \tilde{x}(s)d\bar{\nu}(ds,du), \tilde{x}(a) = i o x(a),$$

then by Theorem 3.1.2 for $\omega \in \Omega_\epsilon$, $\tilde{x}(t) = iox(t)$ a.e. where $x(t)$ is the solution to the corresponding stochastic differential equation on $M$.

From the definitions of $\bar{w}$ and $\bar{\nu}$, we obtain

$$\tilde{x}(t) = \tilde{x}(T_0) + \int_{T_0}^{t} \tilde{a}(\tilde{x}(s))ds + \int_{T_0}^{t} \frac{1}{2} \operatorname{Tr} D\tilde{b}(\tilde{x}(s))(\tilde{b}(\tilde{x}(s)))ds$$

$$+ \int_a^t \int \tilde{c}(\tilde{x}(s),u) - \tilde{x}(s) - D_2\tilde{c}(\tilde{x}(s),0)\ \left(\frac{u}{1+|u|^2}\right)\Pi\ (du)ds$$

$$+ \int_{T_o}^{t} \tilde{b}(\tilde{x}(s))dw(s) + \iint_{T_o}^{t} \tilde{c}(\tilde{x}(s),u) - \tilde{x}(s)\tilde{\nu}(ds,du).$$

Now, by the change of variables formula:

$$\tilde{f}(\tilde{x}(t)) = \tilde{f}(\tilde{x}(T_o)) + \int_{T_o}^{t} D\tilde{f}(\tilde{x}(s))(\tilde{a}(\tilde{x}(s)))ds$$

$$+ \int_{T_o}^{t}\int Df(\tilde{x}(s))[\tilde{c}(\tilde{x}(s),u) - \tilde{x}(s) - D_2\tilde{c}(\tilde{x}(s),0)(\frac{u}{1+|u|^2})]\Pi(du,ds)$$

$$+ \int_{T_o}^{t} \frac{1}{2} \text{Tr } D^2 \tilde{f}(\tilde{x}(s))(\tilde{b}(\tilde{x}(s)),\tilde{b}(\tilde{x}(s)))ds$$

$$+ \int_{T_o}^{t}\int \tilde{f}(\tilde{c}(\tilde{x}(s),u)) - \tilde{f}(\tilde{x}(s)) - D\tilde{f}(\tilde{x}(s))(\tilde{c}(x(s),u) - \tilde{x}(s))\Pi(du)ds$$

$$+ \int_{T_o}^{t} D\tilde{f}(\tilde{x}(s))\tilde{b}(\tilde{x}(s))dw(s)$$

$$+ \int\int_{T_o}^{t} \tilde{f}(\tilde{c}(\tilde{x}(s),u)) - \tilde{f}(\tilde{x}(s))\tilde{\nu}(ds,du)$$

$$\tilde{f}(\tilde{x}(t)) = \tilde{f}(\tilde{x}(a)) + \int_{T_o}^{t} D\tilde{f}(\tilde{x}(s))(\tilde{a}(\tilde{x}(s))ds$$

$$+ \frac{1}{2} \text{Tr} \int_{T_o}^{t} D(D\tilde{f}(\tilde{x}(s))\tilde{b}(\tilde{x}(s)))(\tilde{b}(\tilde{x}(s)))ds$$

$$+ \int_{T_o}^{t}\int \tilde{f}(\tilde{c}(\tilde{x}(s),u)) - \tilde{f}(\tilde{x}(s)) - D\tilde{f}(x(s))(D_2\tilde{c}(x(s),0))(\frac{u}{1+|u|^2})\Pi(du)ds$$

$$+ \int_{T_o}^{t} D\tilde{f}(\tilde{x}(s))\tilde{b}(\tilde{x}(s))dw(s) + \int\int_{T_o}^{t} \tilde{f}(\tilde{c}(\tilde{x}(s),u)) - \tilde{f}(\tilde{x}(s))\tilde{\nu}(ds,du).$$

The result follows by observing that

$$\tilde{f}(\tilde{x}(t)) = f(x(t)), \ \tilde{a}(\tilde{x}(s)) = a(x(s)) \text{ etc.}$$

and since $\epsilon$ was arbitrary the formula is valid for a.e. $\omega$. □

Corollary 3.3.3.

Given a linear connection on M, with covariant derivative $\nabla$,

$$f(x(t)) = f(x_{T_o}) + \int_{T_o}^{t} df(a,(x(s)))ds + \int_{T_o}^{t} \frac{1}{2} \mathrm{Tr}[\nabla df(b,b)+\nabla_b b)f]x(s)ds$$

$$+ \int_{T_o}^{t}\int f(c(x(s),u)) - f(x(s)) - df\ (T_2c(x(s),0))\ (\frac{u}{1+|u|^2})\ \Pi(du)ds$$

$$+ \int_{T_o}^{t} df(b(x(s)))dw(s) + \int\int_{a}^{t} f(c(x(s),u)) - f(x(s))\tilde{\nu}(ds,du)$$

where (1) $\mathrm{Tr}\ \nabla^2 f(b(x),b(x)) = \sum_i \nabla^2 f(b(x)e_i,(x)e_i)$

(2) $\mathrm{Tr}\ \nabla_{b(x)}b(x) = \sum_i \nabla_{b(x)e_i}b(x)e_i$

and $e_i$ forms an orthonormal basis of $\mathbb{R}^n$.

Proof

This is immediate from the theorem, and the fact that $\nabla^2 f(X,X) = \nabla_X{}^2 f - \nabla_X Xf$. □

Theorem 3.3.4.

If $(a,b,c,w,\nu)$, and f are as in Proposition 3.3.2, then the infinitesimal operator is given by

$$Af(x) = \lim_{t\to 0} E\left(\frac{f(x_t)-f(x)}{t-a}\right) = df(a(x)) + \frac{1}{2}\mathrm{Tr}\ d(df(b(x)))(b(x))$$

$$+ \int f(c(x,u)) - f(x) - df(T_2c(x,0))(\frac{u}{1+|u|^2})\ \Pi(du)$$

where x is the solution of the stochastic differential equation corresponding to $(a,b,c,w,\nu)$, with $x(a) = x$.

Proof

From Proposition 3.3.2 using the fact that $E(\int_0^t h(s)dw(s)) = 0$ and $E(\int_0^t W(s,u)\tilde{\nu}(ds,du)) = 0$, where h and W are suitably integrable and adapted functions, we have $Af(x) = \lim_{t\to T_0} \frac{1}{t-T_0} E[\int_{T_0}^t df(a(x(s))ds +$

$$\int_{T_0}^t \frac{1}{2} \operatorname{Tr} d(df(b(x(s))(b(x(s))ds$$

$$+ \int_{T_0}^t \int f(c(x(s),u) - f(x) - df(T_2c(x(s),0))(\frac{u}{1+|u|^2}) \Pi(du)ds]$$

$$= df(a(x)) + \frac{1}{2} \operatorname{Tr} d(df(b))(b)$$

$$+ \int f(c(x,u)) - f(x) - df(T_2c(x,0))(\frac{u}{1+|u|^2}) \Pi(du).$$

(since f and its derivatives are bounded). □

Corollary 3.3.5.

Given a connection on M,

$$Af(x) = a(f) + \frac{1}{2} \operatorname{Tr} \nabla df(b(x)\ b(x)) + \frac{1}{2}(\operatorname{Tr} \nabla_b b(f)(x)$$

$$+ \int f(c(x,u)) - f(x) - df(T_2c(x,0))(\frac{u}{1+|u|^2}) \Pi(du).$$

Proof.

Immediate.

## 4. Processes with independent increments

Let M be $\mathbb{R}^n$, a and b be constant, and $c(x,u) = x+u$. Then we construct the maximal solution

$$x:[0,\xi) \times \Omega \to \mathbb{R}^n, \text{ to}$$

$$dx(t) = a\,dt + b\,d\bar{w}(t) + \int u\,\bar{\nu}\,(dt,du).$$

Now if $\int |u|^2 \Pi(du) < \infty$, then $(a,b,c,w,\nu)$ satisfy the conditions of Theorem 1.5.1 and we would be able to conclude that $\xi \equiv \infty$ by Theorem 1.1.1. In the general case we need the following proposition.

Proposition 3.4.1.

If a, b, c, and x are as above, then $\xi \equiv \infty$.

Proof

Define the stopping times, $\tau_R$, by

$\tau_R = \inf\,\{t \,|\, \nu([0,t], \widetilde{B(0;R)}) \neq 0\}$, where $\widetilde{B(0;R)}$ is the complement of the ball of radius R centred at 0.

Since $E(\nu([0,t], \widetilde{B(0;R)})) \to 0$ as $R \to \infty$ we have $\tau_R \nearrow \infty$. Define $\nu_R([t_1,t_2],A) = \nu([t_1,t_2],A \cap B(0;R))$, then we have that the integral equation

$$y(t) = \int_0^t a\, ds + \int_0^t b\, d\bar{w}(s) + \int \int_0^t u\, \tilde{\nu}_R(ds,du)$$

has a unique solution, say $x_R(t):[0,\infty) \times \Omega \to R^n$. Now define

$$x(t) = x_R(t), \text{ if } t < \tau_R$$

and we are done.

Proposition 3.4.2.

If x is as above, then x is a process with independent increments.

Proof

This is immediate since if $s_1 < s_2 < s_3 < s_4$ then $x(s_2) - x(s_1)$ is $F_{s_2}$-measurable, and $x(s_4) - x(s_3)$, depends only on $w(r) - w(s_3)$, $s_3 \leq r \leq t$, and $\nu((s,r),du)$, $s_3 \leq r \leq t$, and by definition, these are independent of $F_s$, for $s_2 < s < s_3$. □

Remark

Any homogeneous process with independent increments, at least up to distribution, can be constructed in this way as a glance at the Levy-Khinchine formula immediately shows [5] Chapter 3 page 154.

In a sense, in the general case, we place a process with independent increments in each tangent space, corresponding in ordinary differential equations to a tangent vector at each point on the manifold. This interpretation is only loose, however, due to the character of $(c,\nu)$ term, which, in some sense, represents a process on the manifold at every point.

If, however, c is related to a geometric object, as for example in the next section we may make this interpretation.

5. G-bundles, connections and stochastic development.

Following Elworthy [2] we are going to represent a class of processes on GM, where GM is a sub-bundle of LM, the bundle of frame fields on M, with structure group $G \subset GL(n)$.

A connection of GM is a splitting of TGM, into 'horizontal' and 'vertical' spaces.

$$TGM = HTGM \oplus VTGM.$$

If $\pi : GM \to M$ is the projection, then

$$T\pi | HTGM \text{ is an isomorphism onto each fibre of } TM.$$

We define $X : \underline{\mathbb{R}}^n \to TM$ ($\underline{\mathbb{R}}^n = \mathbb{R}^n \times M$, the trivial bundle) as follows. Each element $g \in GM$, naturally defines a map

$$g : \mathbb{R}^n \to T_{\pi(g)}M, \text{ so we define } X \text{ by}$$

$$X(g,u) = ((T\pi)|HTGM)^{-1} g(u).$$

Suppose that on a tangent space $T_{x_o}M$, we have:

(1) $a_o \in T_{x_o}M$;

(2) $b_o \in L(V, T_{x_o}M)$ ($V \subset T_{x_o}M$, a subspace),

(3) a Wiener process $w$ on $V$, and

(4) A Poisson stochastic measure $\nu$, on $T_{x_o}M$ independent of $w$ and such that the associated parameter measure $\Pi$ satisfies

$$\int \frac{|u|^2}{1+|u|^2}\,\Pi(du) < \infty.$$

(Note that these elements together give us a process with independent increments on $T_{x_o}M$ given by

$$Z(t) = a_o(t) + b_o(\bar{w}(t)) + \int\int_o^t u\,\tilde{\nu}(ds,du).$$

What we shall do is to transfer this process onto every tangent space $T_gGM$; of $GM$).

Given $g_o \in GM$, with $\pi(g_o) = x_o$ define,

(1) $a_{g_o}(g) = X(g, g_o^{-1}(a_o))$;

(2) $b_{g_o}(g)e = X(g, g_o^{-1} o b_o(e))$;

(3) $c_{g_o}(g,u) = \alpha_{g_o}(g,u,1)$, where $\frac{d}{dt}\alpha_{g_o}(g,u,t) = X(\alpha_{g_o}(g,u,t), g_o^{-1}u)$, $\alpha_{g_o}(g,u,0) = g$, and $D_{g_o} = \{(g,u) \in GM \times T_{x_o}M \mid \alpha_{g_o}(g,u,1)$ is defined$\}$.

In this way we produce a stochastic dynamical system

$(a_{g_o}, b_{g_o}, c_{g_o}, w, \nu)$ on $GM$.

Take a maximal solution of

$$dg(t) = a_{g_o}(g(t))dt + b_{g_o}(g(t))d\bar{w}(t) + \int c_{g_o}(g(t),u)-g(t)\bar{\nu}(dt,du$$

with $g(a) = g_o$.

Define $\tau_{g_o}:[a,\xi) \times \Omega \to M$ by

$\tau_{g_o} = \pi o g$. This is called the stochastic development of $Z(t)$.

<u>Proposition 3.5.1</u>.

$\tau_{g_o}$ is independent of the choice of $g_o$ in $\pi^{-1}(x_o)$.

<u>Proof</u>

Suppose $h_o \in \pi^{-1}(x_o)$ also, then there exists a $\gamma \in G$ with $R_\gamma(g_o) = h_o$, where $R_\gamma$ denotes the right action of $G$ on $GM$:

$$R_\gamma(g) = g \circ \gamma, \; g \in GM.$$

By the invariance of connections, for each $g \in GM$,

$$X(g o\gamma)(e) = T_g R_\gamma \circ X(g, \gamma(e)).$$

Thus $\quad a_{h_o}(g o\gamma) = X(g o\gamma, h_o^{-1} a_o)$

$$= T_g R_\gamma o X(g, \gamma(h_o^{-1} a_o))$$

$$= T_g R_\gamma o X(g, g_o^{-1} a_o)$$

$$= T_g R_\gamma \, a_{g_o}(g) \; .$$

Similarly

$$b_{h_o}(g\circ\gamma) = T_g R_\gamma\, b_{g_o}(g)$$

and, from the definition of $\alpha_{g_o}(g,u,t)$,

$$c_{h_o}(g\circ\gamma,u) = R_\gamma c_{g_o}(g,u), \quad \text{and}$$

$$D_{h_o} = R_\gamma D_{g_o} = \{(g,u) \mid ((R_\gamma^{-1})(g),u) \in D_{g_o}\}.$$

It follows from Proposition 3.1.1 that

$$h(t) = g(t)\cdot\gamma\colon [0,\xi) \times \Omega \to GM \quad \text{is a maximal solution of}$$

$$dh(t) = a_{h_o}(h(t))dt + b_{h_o}(h(t))d\bar{w}(t) + \int c(h(t),u) - h(t)\bar{\nu}(dt,du)$$

and since $\pi\circ h(t) = \pi\circ g(t)$, the result follows. □

<u>Examples</u>

(1) If $a_o = 0$, $U = 0$, for $U \in B(T_{x_o}M)$, $b_o\colon T_{x_o}M \to T_{x_o}M$, is the identity and $G = O(n)$, then $\tau$ reduces to Brownian motion on $M$, as defined in Elworthy [2].

(2) Suppose $G = O(n)$, $a = 0$ and $V = 0$, so b is trivial, and suppose that the parameter measure $\Pi$, of $\nu$ is

$$\Pi(du) = \frac{c_{n,\alpha}\, du}{|u|^{n+\alpha}}, \quad 0 < \alpha < 2,$$

where $\frac{2^{\alpha/2}}{c_{n,\alpha}} = \int 1 - \cos \langle\xi,u\rangle \quad \frac{du}{|u|^{n+\alpha}}$, where $|\xi| = 1$.

Note that $c_{n,\alpha}$ is independent of $\xi$.

We shall call $\tau$ (starting from $x_o$, say) a geometric $\alpha$-stable process on M. This term will be explained in Section 6.

Theorem 3.5.2.

If $f:M \to \mathbb{R}$ is $C^2$ and, together with its derivatives, is bounded, then

$$f(x(t)) = f(x) + \int_o^t df(g(s)g_o^{-1}a_o)ds$$

$$+ \tfrac{1}{2}\int_o^t \mathrm{Tr}\ \nabla df(g(s)g_o^{-1}b_o, g(s)g_o^{-1}b_o)ds$$

$$+ \int_o^t \int f(\exp_{x(s)}(g(s)g_o^{-1}u)) - f(x(s)) - df(g(s)g_o^{-1})\left(\frac{u}{1+|u|^2}\right)\Pi(du)ds$$

$$+ \int_o^t df(g(s))g_o^{-1}b_o)dw(s)$$

$$+ \int_o^t \int f(\exp_{x(s)}(g(s)g_o^{-1}u)) - f(x(s))\tilde{\nu}(ds,du).$$

Proof'

Define $\tilde{f}:O(M) \to \mathbb{R}$ by $\tilde{f}(g) = f(\pi(g))$, and choose $g_o \in \pi^{-1}(x)$.

Then from Proposition 3.2.3,

$$\tilde f(g(t)) = \tilde f(g_o) + \int_o^t d\tilde f(a_{g_o}(g(s))ds + \int_o^t \tfrac{1}{2}\,\mathrm{Tr}\, d(d\tilde f b_{g_o}(g(s)))b_{g_o}(g(s))$$

$$+ \int_o^t \int \tilde f(c_{g_o}(g(s),u)) - \tilde f(g(s)) - d\tilde f(T_2 c_{g_o}(g(s),0)(\frac{u}{1+|u|^2})\ \Pi(du)ds$$

$$+ \int_o^t d\tilde f(b_{g_o}(g(s)))dw(s) + \iint_o^t \tilde f(c_{g_o}(g(s),u)) - \tilde f(g(s))\tilde\nu(ds,du).$$

Now writing $(T\pi)^{-1}$ for $(T\pi|HTGM)^{-1}$, we have:

$$a_{g_o}(g) = (T\pi)^{-1} g o g_o^{-1}(a_o);$$

$$b_{g_o}(g,u) = (T\pi)^{-1} g o g_o^{-1}(b_o(u)),\ \text{and}\ d\tilde f = df o T\pi.$$

It follows that:

(1) $d\tilde f(a_{g_o}(g)) = df(g o g_o^{-1}(a_o))$ and

(2) $d\tilde f(b_{g_o}(g,u)) = df(g o g_o^{-1}(b_o(u)).$

From the definition it follows that the curves $\beta(v,t) = \pi\ o\ \alpha_{g_o}(g,v,t)$, are geodesics.

We have, at $t = 0$,

$$d(d\tilde f(b_{g_o}(g)))b_{g_o}(g) = \frac{d^2}{dt^2}\,[f o \pi o \alpha_{g_o}(g, b_{g_o}(g,u),t)]$$

$$= \frac{d^2}{dt^2}\,[f o \beta(b_{g_o}(g,u),t)]$$

$$= \frac{d}{dt}\,[df(\frac{d\beta}{dt}(b_{g_o}(g,u),t))]$$

$$= \nabla\, df\,(\frac{d\beta}{dt}(b_{g_o}(g,u)), \frac{d\beta}{dt}(b_{g_o}(g,u))),$$

since $\beta$ is a geodesic.

Now $\frac{d\beta}{dt}(t,v) = gog_o^{-1}v$, at $t = 0$ and so we have,

$$(3) \quad d(d\tilde{f}(b_{g_o}(g)))b_{g_o}(g) = \nabla df(gog_o^{-1}b_o, gog_o^{-1}b_o).$$

We also have that, at $t = 0$,

$$d\tilde{f}(T_2 c_{g_o}(x,0)v) = \frac{d}{dt}(fo\pi o\alpha_{g_o}(g,v,1))$$

$$= \frac{d}{dt}(fo\pi o\alpha_{g_o}(g,v,t))$$

$$= d \quad (\beta(v,0)).$$

Observing that

$$\beta(v,t) = \exp_{\pi(g)} gog_o^{-1}v, \quad \text{gives}$$

$$(4) \quad d\tilde{f}(T_2 c_{g_o}(q,0))(\frac{u}{1+|u|^2}) = df(\exp_{\pi(g)}(gog_o^{-1}(\frac{u}{1+|u|^2}))).$$

(1), (2) (3) and (4) give the result. □

## 6. Invariance By Holonomy

To avoid complications with explosion we assume from now on that M is compact.

Suppose that M is complete Riemannian manifold, so we have a reduction to OM. We can make a further reduction to $P(g_o)$, the holonomy bundle of M, through the point $g_o \in OM$ with the structure group $\Phi(g_o)$. Now $P(g_o)$ is a closed sub-manifold of OM, and the induced stochastic differential equation on $P(g_o)$ is the restriction of the equation induced on OM, by the Theorem 3.1.2 the unique solution, $g:[a,\xi) \times \Omega \to P(u_o)$, of

$$dg(t) = a_{g_o}(g(t))dt + b_{g_o}(g(t))d\bar{w}(t) + \int c(g(t),u)-g(t)\bar{\nu}(dt,du)$$

is hence equal to the solution of the corresponding equation on $O(M)$. Let $\Phi(x_o)$ be the holonomy group at $x_o = \pi(g_o)$ and suppose that $\gamma(a_o) = a_o$, for any $\gamma \in \Phi(x_o)$. Define $a(y)$ to be the vector field on M defined by $a(y) = \tau^y_{x_o} a_o$ where $\tau^y_{x_o}$ is the parallel translatation operator along a curve $\alpha$ such that $\alpha(0) = x_o$ and $\alpha(1) = y$. Since $\gamma(a_o) = a_o$ for any $\gamma \in \Phi(x_o)$, $a(y)$ is well defined. We will also require that $b_o$ is a $\Phi(x_o)$-map, i.e. $\gamma b_o(v) = b_o(\gamma(v))$, for any $\gamma \in \Phi(x_o)$ and $v \in V \subset T_{x_o}M$ (which implies that V is an irreducible $\Phi(x_o)$-space). We also suppose that $\nu$ is such that its parameter measure, $\Pi$, satisfies $\Pi(U) = \Pi(\gamma U)$, for $\gamma \in \Phi(x_o)$ and $U \in \mathcal{B}(T_{x_o}M)$.

By $g_{s,g}(t)$ we denote the unique solution of

$$dg(t) = a_{g_o}(g)dt + b_{g_o}(g(t))d\bar{w}(dt) + \int c(g(t),u)-g(t)\bar{\nu}(dt,du)$$

given $g_{s,g}(s) = g$. Then by Theorem 3.3.1, $g_o, g_o$ is a Markov process with transition probabilities $\tilde{P}(s,g,t,\Gamma)$, given by $\tilde{P}(s,g,t,\Gamma) = P(g_{s,g}(t) \in \Gamma)$, for $\Gamma \in \mathcal{B}(P(g_o))$, with corresponding infinitesimal generator $\tilde{A}$.

<u>Theorem 3.6.1</u>.

Under the above conditions on $a_o$, $b_o$ and $\nu$, $\tau_{x_o}$ is a Markov process, with transition probabilities given by

$P(s,x,t,\Gamma) = P(s,g,t,\Pi^{-1}(\Gamma))$, where $\Gamma \in \mathcal{B}(M)$ and $\Pi(g) = x$.

Also the infinitesimal generator A is given by

$Af(x) = \tilde{A}\tilde{f}(g)$, where f is defined by

$\tilde{f}(g) = f(\pi(g))$.

<u>Proof</u>

We first show that $P(s,x,t,\Gamma)$ is well defined. Let $g(t) = g_{s,g}(t)$ and $h(t) = g_{s,h}(t)$, where $\pi(g) = \pi(h) = x$.

Write $\Psi(g_o) = \mathrm{Ad}g_o\, \Phi(g_o)$, i.e. if $\delta \in \Phi(g_o)$ then

$$\gamma \in \Psi(g_o) \text{ if } g_o\ \gamma_{g_o}^{-1} = \delta_o \quad \text{we put } \delta = \gamma_{g_o}.$$

Now for $\gamma \in \Psi(g_o)$

$$\begin{aligned} a_{g_o}(g\rho\gamma) &= X(g\circ\gamma, g_o^{-1}a_o), \\ &= T_gR_\gamma\ X(g,\gamma_{g_o}^{-1}\ a_o), \\ &= T_gR_\gamma\ X(g,g_o^{-1}\gamma_{g_o}a_o), \\ &= T_gR_\gamma\ a_{g_o}(g), \quad \text{by assumption.} \end{aligned}$$

Similarly $b_{g_o}(g\circ\gamma,u) = T_gR_\gamma\ b_{g_o}(g,\gamma_{g_o}u)$

and $c_{g_o}(g_o\gamma,u) = R_\gamma\ c(g,\gamma_{g_o}u)$.

We now define w' and $\nu'$ by

$$w'(t) = \gamma_{g_o}w(t), \text{ and } \nu'(dt,du) = \nu(dt,\gamma_{g_o}^{-1}u).$$

It follows that w' and $\nu'$ are a Weiner process and a Poisson stochastic measure satisfying our standing assumptions.

It follows that $R_\gamma\, g(t)$ satisfies

$$d(R_\gamma(g(t))) = T_g R_\gamma a_{g_o}(g(t))dt + T_g R_\gamma b_{g_o}(g)d\bar{w}'(t)$$

$$+ \int R_\gamma c_{g_o}(g(t),u) - R_\gamma g(t)\bar{\nu}'(dt,du),$$

with initial condition $R_\gamma g(s) = g o \gamma = h$.

Since w' and $\nu'$ have the same finite dimensional distributions as w and $\nu$, by the assumptions on $\Pi$, it follows from Theorem 1.5.5 that h(t) and $g(t) o \gamma$ have the same finite dimensional distributions.

We hence have the chain of equalities,

$$\tilde{P}(s,h,t,\pi^{-1}(\Gamma)) = P(s,g o \gamma,t,\pi^{-1}(\Gamma))$$

$$= P(s,g,t,R_\gamma^{-1}\pi^{-1}(\Gamma))$$

$$= P(s,g,t,\pi^{-1}(\Gamma)).$$

$P(s,x,t,\Gamma)$ is hence well defined.

The theorem follows now by applying Theorem 10.13 of [1], having made the observation of Jørgenson [6] that the Theorem 10.13 of [1] is true under weaker hypothesis than that $\pi$ maps Borel sets into Borel sets. Explicitly we need only the following: if $\bar{F}:P(g_o) \to \mathbb{R}$ is $\mathcal{B}(P(g_o))$-measureable, and is constant on fibres, then the function, $F:M \to \mathbb{R}$, defined by $F(\pi(g)) = \bar{F}(y)$ is $\mathcal{B}(M)$-measurable. This is immediate, and the Theorem follows. □

Theorem 3.6.2.

Under the above conditions,

$$f(x(t)) = f(x_o) + \int_o^t df(a(x(s))ds$$

$$+ \tfrac{1}{2}\int_o^t \mathrm{Tr}\ \nabla df(\ (g(s)og_o^{-1}b_o),\ (g(s)og_o^{-1}b_o))ds$$

$$+ \int_o^t \int f(\exp_{x(s)}u) - f(x(s)) - df\ (\frac{u}{1+|u|^2})\ \Pi'(du)ds$$

$$+ \int_o^t df(g(s)g_o^{-1})(dw(s))$$

$$+ \int \int_o^t f(\exp_{x(s)}(g(s)g_o^{-1}u)) - f(x(s))\tilde{\nu}(ds,du),$$

where by abuse we denote by $\Pi'$, the measure defined on $T_yM$ for any y, by $\Pi'(U) = \Pi(g_o og(s)^{-1}(U))$. This is well defined, by assumption.

Proof

This is immediate from Theorem 3.5.2 and the restrictions on a, b and $\nu_o$. □

Theorem 3.6.3.

Given $f:M \to \mathbb{R}$, $C^2$ and bounded together with its derivatives, then

$$Af(x_o) = df(a_o(x_o)) + \tfrac{1}{2}\,\mathrm{Tr}\,\nabla d(f(b_o'(x_o), b_o'(x_o))$$

$$+ \int f(\exp_{x_o} u) - f(x_o) - df(\frac{u}{1+|u|^2})\ \Pi(du)$$

where $b_o'(y)(u)$ denotes the parallel transport of $b_o(u)$ along a curve joining $x_o$ and $y$.

Proof

This follows from Theorems 3.6.1 and 3.6.2. □

## 7. Brownian motion and (symmetric)α-stable processes

If we start with Brownian motion in an n-dimensional vector space V then we may, essentially by example 1 of Section 5, construct Brownian motion on an n-dimensional (Riemannian)-manifold M (the construction of [2]).

In a vector space we have the notion of (symmetrical)α-stable processes $0 < \alpha \leq 2$. These are characterized by having characteristic functions $J(t,w) = \exp\{-t \frac{|w|^\alpha}{2^{\alpha/2}}\}$.

Lemma 3.7.1.

If V is an n-dimensional vector space with a Poisson stochastic measure $\nu$ defined on it, then $Z_\alpha(t) = \int\int_0^t u\,\tilde{\nu}(ds,du)$ is a symmetric α-stable process for $0 < \alpha < 2$,

if the parameter measure $\Pi$ satisfies

$$\Pi(du) = \frac{c_{n,\alpha}\,du}{|u|^{n+\alpha}}\ , \text{ where } \frac{2^{-\alpha/2}}{c_{n,\alpha}} = \int \cos\langle u,\xi\rangle - 1\ \frac{du}{|u|^{n+\alpha}},\ |\xi| = 1.$$

Proof

$Z_\alpha(t)$ is well defined since it is a process with independent increments and so the solution of the corresponding S.D.E. has infinite explosion time.

Let $f:V \to \mathbb{C}$ be $u \to \exp i \langle u,w\rangle$.

Then by Proposition 3.3.2.

$$f(Z_\alpha)(t) = f(0) + \int_0^t \int f(Z_\alpha(s)+u) - f(Z_\alpha(s) - df(Z_\alpha(s)(\frac{u}{1+|u|^2})\Pi(du)ds$$

$$+ \int \int_0^t f(Z_\alpha(s) + u) - f(Z_\alpha(s))\tilde{\nu}(ds,du).$$

Now

$$f(Z_\alpha(s)+u) = \exp\{i \quad \langle u,w\rangle\} f(Z_\alpha(s)) \quad \text{and}$$

$$df(Z_\alpha(s))(\frac{u}{1+|u|^2}) = \frac{i\langle u,w\rangle}{1+|u|^2} f(Z_\alpha(s)).$$

We take expectations of both sides and use Fubini's theorem to obtain

$$J(t,w) = J(0,w) + \int_0^t \int J(s,w)[\exp i\langle u,w\rangle -1-i \frac{\langle u,w\rangle}{1+|u|^2} \frac{c_{n,\alpha} du.ds}{|u|^{n+\alpha}}$$

$$= J(0,w) + \int_0^t \int J(s,w) [\cos \langle u,w\rangle - 1] \frac{c_{n,\alpha} du.ds}{|u|^{n+\alpha}} .$$

By changing variables,

$$J(t,w) = 1 - \frac{|w|^\alpha}{2^{\alpha/2}} \int_0^t J(s,w)ds$$

i.e. $$J(t,w) = \exp - \frac{t|w|^\alpha}{2^{\alpha/2}} .$$

Now by example 2 of Section 5, we construct a process we have called geometric $\alpha$-stable on M. That is we take an $\alpha$-stable process on a tangent space $T_xM$ and transfer this process to an $\alpha$-stable process on each $T_gHO(M)$, and relate this to a process on the manifold, via a stochastic differential equation.

By the process of subordination we may relate Brownian motion on vector spaces to $\alpha$-stable processes. Explicitly let $T_{\alpha/2}(t)$ have stable distribution of index $\alpha/2$ on $[0,\infty)$ (See [3] and [5] for the definitions and a fuller discussion), and if $Z(t)$ is Brownian motion on V, then $Z(T_{\alpha/2}(t))$ has an $\alpha$-stable distribution. We can do the same on a manifold, i.e. if $x(t)$ is Brownian motion on M, then we can construct $\bar{x}(t) = x(T_{\alpha/2}(t))$, and we will call the resulting process spectral-$\alpha$-stable.

The above statement is justified by the fact that the infinitesimal operator $\bar{A}_\alpha$ of $\bar{x}_\alpha$ satisfies

$$A_\alpha f(x) = -\left(-\frac{\Delta}{2}\right)^{\alpha/2} f(x)$$

which is true for the vector space case also. The proof of this is outlined in [7], the right-hand side being defined in terms of the spectral decomposition of $\Delta$ (in vector spaces by Fourier transforms). Now, by Theorem 3.3.4, with the case of Example 2 of Section 3 with $M = \mathbb{R}^n$, we have

$$Af(x) = \int f(x+u) - f(x) - df(x)\left(\frac{u}{1+|u|^2}\right) \Pi(du).$$

If we take Fourier transforms of both sides we see, in a similar manner to the proof of Lemma 3.7.1, that

$$Af(x) = -(-\frac{\Delta}{2})^{\alpha/2} f(x).$$

For a general manifold M this is no longer true.

Theorem 3.7.2.

If $x_a(t)$ is a geometric $\alpha$-stable process on a manifold M, then the infinitesimal operator A of the process is given by

$$Af(x) = -(-\frac{\Delta_x}{2})^{\alpha/2} \tilde{f}_x(0),$$

where $\tilde{f}_x : T_xM \to \mathbb{R}$ is defined by

$$u \to f(\exp_x u),$$

and $\Delta_x$ is the Laplacian on $T_xM$, with metric induced on $T(T_xM)$, by parallel translation of $g_x$, the Riemannian metric evaluated at x, after identifying $T_xM$ with $T_o(T_xM)$.

Proof

$$Af(x) = \int f(\exp_x u) - f(x) - df(x)(\frac{u}{1+|u|^2})\Pi(du)$$

$$= \tfrac{1}{2}\int f(\exp_x u) - 2f(x) + f(\exp_x -u)\Pi(du)$$

$$= \tfrac{1}{2}\int \tilde{f}_x(u) - 2\tilde{f}_x(0) + \tilde{f}_x(-u)\Pi(du).$$

So $Af(x) = -(-\frac{\Delta_x}{2})^{\alpha/2} \tilde{f}_x(0)$, since $\tilde{f}$ is just a map from a vector space to $\mathbb{R}$. □

It is not true, in general, that

$$Af(x) = -(-\Delta/2)^{\alpha/2} f(x)$$

if M is not flat, as the following example shows.

Proposition 3.7.3.

Let $f_n(x) = P_n(\langle x,e\rangle)$ where $P_n$ are Legendre polynomials, and e is a constant element of $\mathbb{R}^3$ with $|e| = 1$. (We will take e along the "z-axis"). If $M = S^2$, considered as the unit sphere in $\mathbb{R}^3$, then

$$Af_n(x) + \sqrt{\pi}\,\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(\frac{1+\alpha}{2})}\left(\sum_{s=0}^{[n/2]} a_s^n\,(n-2s)^{\alpha}\right) f_n(x) = 0,$$

where the $a_s^n$ are defined by

$$P_n(\cos\theta) = \sum_{s=0}^{[n/2]} a_s^n \cos(n-2s)\theta.$$

Proof

If $x = (0,0,1)$, $u \in T_{(0,0,1)}S^2$. Write $u = (r\sin\theta,\ r\cos\theta)$ with respect to some orthonormal basis in $T_{(0,0,1)}S^2$. We then have

$$\exp_{(0,0,1)} u = (\sin r \sin\theta,\ \sin r\cos\theta,\ \cos r),$$

up to a rotation in the tangent space. For a general x, we have

$$\exp_x u = \Theta \exp_{(0,0,1)} u', \text{ where } \Theta \in SO(3),\ |u| = |u'|, \text{ and}$$

$$\Theta((0,0,1)) = x.$$

Now $P_n(\langle x,e\rangle)$ is invariant under rotations about e, so we can suppose that

$$\Theta = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

It follows that

$$\langle \exp_x u, e\rangle = \cos\phi \cos r + \sin\phi \sin\theta \sin r.$$

Now

(1) $P_n(\cos\phi \cos r + \sin\phi \sin\theta \sin r)$

$$= P_n(\cos\phi)\, P_n(\cos r) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\phi) P_n^m(\cos r) \cos n\theta$$

(See [10] page 328).

To compute $Af_n$, we write $\Pi$ in terms of polar coordinates.

$$\frac{c_{2,\alpha}\, du}{|u|^{2+\alpha}} = \frac{c_{2,\alpha}\, dr d\theta}{r^{1+\alpha}} .$$

We observe that if we integrate (1) with respect to $\theta$, all the "$P_n^m$" terms vanish and we are left with

$$\int_0^{2\pi} P_n(\langle \exp_x u, e\rangle) d\theta = 2\pi P_n(\cos\phi) P_n(\cos r).$$

We note that, for any suitable f,

$$Af(x) = \int f(\exp_x u) - f(x) - df(x)\ \left(\frac{u}{1+|u|^2}\right) \Pi(du)$$

$$= \frac{1}{2} \int f(\exp_x u) - 2f(x) + f(\exp_x -u)\Pi(du)$$

since $\Pi(A) = \Pi(-A)$.

This together with (1) gives

$$Af_n(x) = 2\pi P_n(\cos\phi) \int_0^\infty P_n(\cos r) - P_n(1) \frac{c_{2,\alpha}\, dr}{r^{1+\alpha}} .$$

Now $P_n(\cos r) = \sum_{s=0}^{[n/2]} a_s^n \cos (n-2s)r$

where the $a_s^n$ are known constants (see [10] page 303).

So $Af_n(x) = 2\pi P_n(\cos\phi) \int_0^\infty \sum_{s=0}^{[n/2]} a_s^n(\cos(n-2s)r-1) \frac{c_{2,\alpha}\, dr}{r^{1+\alpha}}$

($P_n(1) = 1$ which implies $\Sigma a_s^n = 1$)

and $\int_0^\infty \cos(n-2s)r-1 \frac{c_{2,\alpha}\, dr}{r^{1+\alpha}} = (n-2s)^\alpha \int_0^\infty \cos r-1 \frac{c_{2,\alpha} dr}{r^{1+\alpha}}$ ,

now $\frac{1}{c_{2,\alpha}} = 2^{\alpha/2} \int 1-\cos\langle x,u\rangle \frac{du}{|u|^{2+\alpha}}$ $(|x| = 1)$,

$$= 2^{\alpha/2} \cdot \int_0^{2\pi} |\cos\theta|^\alpha\, d\theta \int_0^\infty 1-\cos r \frac{dr}{r^{1+\alpha}},$$

$$= 2^{\alpha/2}\ 2.\ \sqrt{\pi}\ \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(1+\frac{\alpha}{2})} \int_0^\infty 1 - \cos r \frac{dr}{r^{1+\alpha}} .$$

So $c_{2,\alpha}\int_0^\infty \cos r - 1 \frac{dr}{r^{1+\alpha}} = \frac{-\Gamma(1+\alpha/2)}{2^{\alpha/2}.2\sqrt{\pi}.\Gamma(\frac{1+\alpha}{2})}$,

which implies that

$$\int_0^\infty \cos(n-2s)r-1 \frac{c_{2,\alpha}\, dr}{r^{1+\alpha}} = -\frac{\Gamma(1+\alpha/2)\ (n-2s)^\alpha}{2^{\alpha/2}.2.\sqrt{\pi}.\Gamma(\frac{1+\alpha}{2})}$$

and in turn implies that

$$Af_n(x) + \frac{\sqrt{\pi}}{2^{\alpha/2}} \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(\frac{1+\alpha}{2})} \left(\sum_{s=0}^{[n/2]} a_s^n\ (n-2s)^\alpha\right) f_n(x) = 0. \qquad \square$$

Now $f_n$ is an eigenfunction of the Laplacian on $S^2$, with eigenvalue $-n(n+1)$ and by definition, we have

$$(-(-\Delta/2)^{\alpha/2} f_n)(x) + \left(\frac{n(n+1)}{2}\right)^{\alpha/2} f_n(x) = 0.$$

So if we put $n = 1, \alpha = 1$ in the two formulae and compare, we have

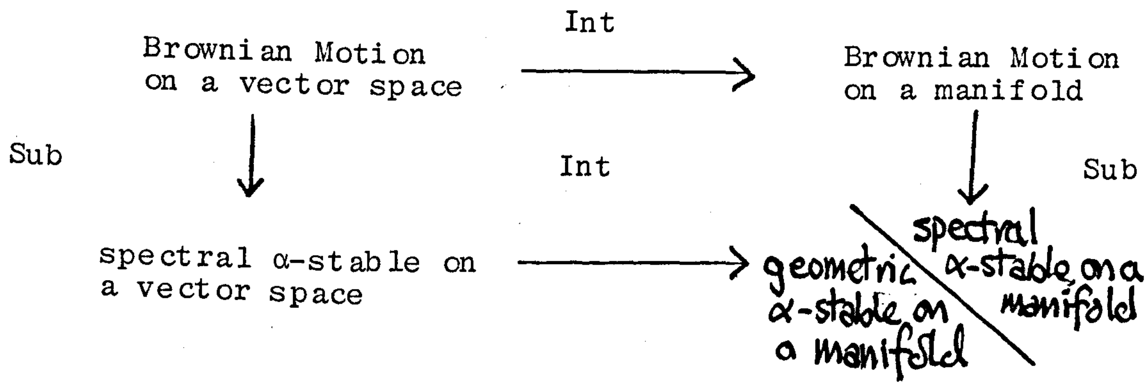
$$Af_1(x) + \frac{\pi}{2\sqrt{2}} f_1(x) = 0$$

and $(-(-\Delta/2)^{\frac{1}{2}} f_1)x + f_1(x) = 0$, so $A \neq -(-\Delta/2)^{\alpha/2}$.

If we let $\alpha \to 2$ in the formula for the eigenvalues of A, we find that

$$\frac{\sqrt{\pi}\,\Gamma(1+\frac{\alpha}{2})}{2^{\alpha/2}\,\Gamma(\frac{1+\alpha}{2})} \sum_{s=0}^{[n/2]} a_s^n\ (n-2s)^\alpha \to \frac{n(n+1)}{2}\ .$$

This is because $\frac{d^2 P_n}{dr^2} = - \sum_{s=0}^{[n/2]} a_s^n (n-2s)^2 \cos(n-2s)r$ and the result now follows by simple manipulation of Legendre polynomials. We summarize the above in the following diagram, which does not commute.




The vertical arrows correspond to subordination, and the horizontal to solving a stochastic differential equation on a manifold.

The construction fails to give the "correct" answer because of the global nature of the infinitesimal operator. It would seem that the measure Π ought to be related, in some way, to the geometry of the manifold. Unfortunately, it is not clear in which way to proceed.

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