Lyapunov Exponents for Certain Stochastic Flows.

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For my Mother and Father

and Sarah,

With Love.

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Declarations

I declare that the work contained in this thesis is original unless stated otherwise.

Much of the work that constitutes Chapter 2 was first published as [20] in the Proceedings of the Workshop on Lyapunov Exponents mentioned above; in particular first use of the Girsanov method of Proposition 2.2.3 was given here. Certain parts of Chapters 3 and 4 are also due to appear as [21] in the Proceedings of the Symposium on Stochastic Mechanics also mentioned above. In [21] the time dependent situation is examined whereas in this thesis only stationary processes are considered.

Summary

This thesis examines the asymptotic behaviour of solution flows of certain stochastic differential equations utilising the theory of Lyapunov exponents. The approach is taken on two fronts. Initially flows are considered on compact manifolds that arise from embedding the manifold in some Euclidean space - the Gradient Brownian flow. In this case the existence of the Lyapunov exponents is known. Results are obtained for the sum of the exponents - which has the geometrical interpretation as the exponential rate of change of volume under the action of the flow - and for the largest exponent on generalised Clifford Tori and convex hypersurfaces.

The situation on non-compact manifolds is then considered - where the existence of the exponents is uncertain due to the fact that the existence of a finite invariant measure is not guaranteed. However, by considering a stochastic mechanical system this problem is overcome and conditions for existence are obtained for both the Lyapunov spectrum and the sum of the exponents. Some examples are then considered in the non-compact case.

Finally in the Appendix a computational method for calculating the largest Lyapunov exponent on a hypersurface is considered.

CHAPTER 1.

1.1. Introduction

Our main aim in this thesis is to examine the asymptotic behaviour of certain solution flows of stochastic dynamical systems on manifolds. This requires the theory of Lyapunov exponents, a background history of which is discussed in [A]. A breakthrough was made in A. Carverhill's Ph.D. thesis [13], forming [14] and [15], in which the existence of such exponents for stochastic dynamical systems on compact manifolds was examined and the existence of local and global stable manifolds, this closely following the work of Ruelle, [56], for deterministic systems. The theory has been developed extensively in recent years, see in particular [A] and the works of Arnold [2], Baxendale [5], Carverhill [14 and 15], Carverhill and Elworthy [18], and Kifer [40]. In this thesis we continue this work on two fronts. In Chapter 2 we consider the stochastic flow arising from embedding a compact manifold in some higher dimensional Euclidean space - the gradient Brownian flow. For such stochastic systems we consider the sum of the Lyapunov exponents, this has the geometric interpretation as the almost sure exponential rate at which the flow changes volume. We also obtain other results on the Lyapunov spectrum, in particular examining the Lyapunov stability of convex surfaces.

In Chapter 3 we consider stochastic dynamical systems on non-compact manifolds. The existence of a Lyapunov spectrum in this case is uncertain as there is no guarantee that the process will possess a finite invariant measure - an essential requirement. However by considering a certain type of S.D.E. which has the form of a ground state stochastic mechanical system (see e.g. Nelson [48]) the existence of the finite invariant measure is ensured and hence, under certain regularity conditions, the existence of a Lyapunov spectrum. This is first achieved on $\mathbb{R}^n$ and then extended to more general systems on complete Riemannian manifolds. The approach taken is basically that of Carverhill in [13]. The theory is also developed for higher energy levels of the stochastic mechanical system, (the time dependent case being dealt with in [21]). For ground state systems on manifolds examples are considered concerning deterministic systems under a small stochastic perturbation (analogous to Carverhill's section 4.3, [13]). Here the invariant measure concentrates on hyperbolic fixed points of the deterministic system.

In Chapter 4 examples are considered using the theory developed in Chapter 3. In particular the spectrum is examined for a stochastic model of the ground state of the Hydrogen atom, originally developed by Lewis and Truman in [44].

In the Appendix we consider the problem of calculating the top Lyapunov exponent using computational methods, considering Brownian motion on an ellipsoid of revolution in $\mathbb{R}^3$ as a particular example.

1.2. Preliminary Results

Throughout we shall consider a stochastic dynamical system on a complete Riemannian manifold M of dimension n of the form

$$dx_t = \sum_{i=1}^{m} X_i(x_t)\circ dB_t^i + A(x_t)dt \qquad (1.1)$$

where $X_i(\cdot) \in \mathbb{L}(\mathbb{R}^m; T_x M)$, $1 \le i \le m$, $B_t \in BM(\mathbb{R}^m)$ and A is some vector field on M . Here $\circ$ denotes that the system is taken in the Stratonovich sense. We shall assume that the process $\{x_t ; t \ge 0\}$ is stochastically complete (i.e. non-explosive). Under conditions that we will employ this assumption will be a natural consequence.

Denote the underlying probability space for $B_t \in BM(\mathbb{R}^m)$ by $(\Omega, F, \mathbb{P})$, taking Ω to be the set of continuous paths in $\mathbb{R}^m$ starting from the origin. Denote the time shift by time s on Ω by θ_s where θ_s is defined by

$$(\theta_s(\omega))(t) = \omega(t+s) - \omega(s) .$$

Under the completeness assumption, system (1.1) has a measurable solution flow $\xi_t(\omega) : M \to M$, $t \ge 0$ such that

$$d\xi_t(\omega)(x) = \sum_{i=1}^{m} X_i(\xi_t(\omega)(x)) \circ dB_t^i + A(\xi_t(\omega)(x))dt \qquad (1.2)$$

$$\xi_0(\omega)(x) = x .$$

(See e.g. [17] or [32]).

In the case M is compact $\{\xi_t(\omega) ; t \ge 0\}$ is a flow of diffeomorphisms of M .

The following result is immediate from the time homogeneity of the Brownian motion and the a.s. uniqueness of the flow.

Lemma 1.2.1

If the flow of system (1.1) exists then for each $s \geq 0$ we have a.s. that

$$\xi_t(\theta_s(\omega))\xi_s(\omega) = \xi_{s+t}(\omega) \qquad \forall\, t \geq 0 .$$

Proof

See Carverhill and Elworthy [17]. □

In [56], Ruelle considers an abstract probability space (M,Σ,ρ), a measure preserving map $\tau : M \to M$ and a measurable map

$$T : M \to \mathbb{L}(\mathbb{R}^m;\mathbb{R}^m) .$$

As in Carverhill we shall apply the results of Ruelle's sections 1-5 by considering the product measure space $(M\times\Omega,\ \mathcal{B}(M) \otimes \mathcal{F}\ ,\ \rho \otimes \mathbb{P})$ and the map $\Phi_s : M \times \Omega \to M \times \Omega$ $(s > 0)$ defined by

$$\Phi_s(x,\omega) = (\xi_s(\omega)(x)\ ,\ \theta_s(\omega)) \tag{1.3}$$

corresponding to Ruelle's τ. We now consider the following result which will prove important in the following chapters.

Proposition 1.2.2

For any $s > 0$, the map $\Phi_s(x,\omega)$ preserves the measure $\rho \otimes \mathbb{P}$ on $M \times \Omega$.

Proof

As in Carverhill [13], it suffices to show that for any $B \in \mathcal{B}(M)$, $A \in \mathcal{F}$, the set $\Phi_s^{-1}(B\times A)$ has measure $\rho(B)\mathbb{P}(A)$. Since

$$\begin{aligned}\rho \otimes \mathbb{P}(\Phi_s^{-1}(B\times A)) &= \rho\otimes\mathbb{P}\{(x,\omega)\ ;\ \xi_s(\omega)x \in B\ ,\ \theta_s(\omega) \in A\}\\ &= \rho\otimes\mathbb{P}\{(x,\omega)\ ;\ \xi_s(\omega)x \in B\}.\rho\otimes\mathbb{P}\{(x,\omega);\theta_s(\omega)\in A|\xi_s(\omega)(x)\in B\}\\ &= \left(\int_{x\in M} p_s(x,B)d\rho(x)\right).\rho \otimes \mathbb{P}\{(x,\omega);\theta_s(\omega)\in A\}\end{aligned}$$

as the events of the conditional probability are independent. Hence

$$\rho \otimes \mathbb{P}(\Phi_s^{-1}(B\times A)) = \rho(B).\mathbb{P}(A)$$

as required. □

We shall also frequently use the following.

Proposition 1.2.3 (Strong Law of Large Numbers (special case) or the Ergodic theorem).

Suppose that the stochastic dynamical system (1.1) is nondegenerate with unique invariant measure ρ on M. Then for $\rho \otimes \mathbb{P}$ - almost every $(x,\omega) \in M \times \Omega$ we have

$$\frac{1}{t}\int_0^t g(\xi_s(\omega)(x))ds \to \int_M g(y)d\rho(y) \qquad \text{as} \quad t \to \infty$$

for any $g \in C(M;\mathbb{R})$.

Proof

See for example Yosida [65], Chapter 13, or Doob [26]. □

As in Carverhill [13], it will be more convenient to consider the S.D.E.'s that we shall be dealing with as systems defined on a flat space. In order to do this we have the following:

Lemma 1.2.4

Consider the stochastic system (1.1) with measurable solution flow $\xi_t(\omega) : M \to M$. If M is embedded in some $\mathbb{R}^m$ $(m>n)$, the system (1.1) can be extended to give a system on $\mathbb{R}^m$ with measurable solution flow $\tilde{\xi}_t(\omega) : \mathbb{R}^m \to \mathbb{R}^m$, $t \geq 0$ given by

$$d\tilde{\xi}_t(\omega)(x) = \sum_{i=1}^{m} \tilde{X}_i(\tilde{\xi}_t(\omega)(x)) \circ dB_t^i + \tilde{A}(\tilde{\xi}_t(\omega)(x))dt . \qquad (1.4)$$

Proof

(Similar to proof of theorem 2.1 (1) in Carverhill, [13].) Consider a continuous map $\tau : M \to \mathbb{R}(>0)$ such that the set $M_\tau^x = \{M_{\tau(x)}; x \in M\}$ of points in $\mathbb{R}^m$ less than a distance $\tau(x)$ from M at x forms a (pseudo-tubular) neighbourhood of M. Then for any $y \in M_\tau^x$ the nearest point $z \in M$ to y is unique by the definition of a tubular neighbourhood, and the line yz is perpendicular to M. Also for any other $p \in M$, if yp is perpendicular to M then $d(y,p) > \tau(x)$. Take a smooth bump function $f : M \times \mathbb{R} \to \mathbb{R}\ (\geqslant 0)$, supported on the set $\{(x,r); |r| < \tau(x)\}$ and such that $f(A) = 1$ where $A = \{(x,r); |r| < \frac{1}{2}\tau(x)\}$. Then for any $y \in \mathbb{R}^m$, if $y \notin M_\tau^x$ set $\tilde{X}_i(y) = 0$ $(1 \leq i \leq m)$, $\tilde{A}(y) = 0$, otherwise take the nearest point z to y in M and set

$$\tilde{X}_i(y)e = f(x,|z-y|)X_i(z)e \qquad 1 \leq i \leq m$$

$$\tilde{A}(y) = f(x,|z-y|)A(z) .$$

Clearly $\tilde{X}_i$ $(1 \le i \le m)$ and $\tilde{A}$ are just as smooth as X_i $(1 \le i \le m)$ and A and are supported on an open domain $\tilde{M}$, say, in $\mathbb{R}^m$, giving system (1.4) with solution flow $\tilde{\xi}_t(\omega) : \mathbb{R}^m \to \mathbb{R}^m$. □

We now give a general hypothesis that ensures the existence, almost surely, of a Lyapunov spectrum for (1.1).

We note first that as a measurable flow $\{\xi_t(\omega)(x) \; ; \; t \ge 0\}$ exists for (1.1) that the derivative process of $\xi_t(\omega)(x)$ with respect to $x \in M$ exists in probability to give a process $T\xi_t(\omega)(x)(v)$ for each tangent vector v at x (see for example [32]).

Hypothesis 1.2.5

For the stochastic dynamical system (1.1) where the measurable solution flow $\{\xi_t(\omega)(x) \; ; \; t \ge 0\}$ has finite invariant probability measure ρ, suppose that

$$\sup_{t \in [0,T]} \log^+ ||T\xi_t(\omega)(x)^{\pm 1}||_{op} \in L^1(M \times \Omega, \rho \otimes \mathbb{P}) \tag{1.5}$$

where $\log^+ f = \max\{0, f\}$, $T\xi_t(\omega)(x)$ is the derivative of $\xi_t(\omega)(x)$ in probability and $||\cdot||_{op}$ denotes the operator norm on $GL(\mathbb{R}^m)$.

Note that under the above hypothesis we have that

$$\sup_{t \in [0,T]} \log^+ ||T(\xi_{T-t}(\theta_t(\omega)))(\xi_t(\omega)(x))||_{op} \in L^1(M \times \Omega, \rho \otimes \mathbb{P}) \tag{1.6}$$

since $$T(\xi_{T-t}(\theta_t(\omega)))(\xi_t(\omega)(x)) = (T\xi_T(\omega)(x)) \circ (T\xi_t(\omega)(x)^{-1}) \; . \tag{1.7}$$

We then have the following major result.

Theorem 1.2.6 (cf Ruelle [56], Theorem 1.6 and introduction to section 6, and Carverhill [13], Theorem 2.1)

Suppose that Hypothesis 1.2.5 is satisfied. For a stochastic dynamical system (1.1) on a complete Riemannian manifold M, choose a measurable version of the flow $\xi_t(\omega) : M \to M$, $t \geq 0$. Then there exists a set $\Gamma \subset M \times \Omega$ of full $\rho \otimes \mathbb{P}$ measure such that for each $(x,\omega) \in \Gamma$ we have a Lyapunov spectrum

$$\lambda^r_{(x,\omega)} < \lambda^{r-1}_{(x,\omega)} < \dots < \lambda^1_{(x,\omega)} \qquad (r \leq n)$$

and associated filtration

$$\{0\} = V^{r+1}_{(x,\omega)} \subset V^r_{(x,\omega)} \subset \dots \subset V^1_{(x,\omega)} = T_xM$$

such that if $v \in V^i_{(x,\omega)} \setminus V^{i+1}_{(x,\omega)}$ then

$$\frac{1}{t} \log||T\xi_t(\omega)(x)v|| \to \lambda^i_{(x,\omega)} \quad \text{as} \quad t \to \infty .$$

Here $||\cdot||$ denotes the Riemannian metric on T_xM for which (1.5) holds.

Proof

(As in Carverhill [13], proof of theorem 2.1)

As in lemma 1.2.4 embed M in some $\mathbb{R}^m$ $(m>n)$ and consider the solution flow $\tilde{\xi}_t(\omega) : \mathbb{R}^m \to \mathbb{R}^m$ of the extended system (1.4). In this way the tangent spaces of M can be identified in a Borel measurable way.

We prove the result first for discrete time increments of length T.

Consider the time shift map

$$\tilde{\Phi}_T : \mathbb{R}^m \times \Omega \to \mathbb{R}^m \times \Omega$$

as defined in (1.3). By Proposition 1.2.2, $\tilde{\Phi}_T$ preserves the measure $\rho \otimes \mathbb{P}$ on $\mathbb{R}^m \times \Omega$ and in fact $\tilde{\Phi}_T$ is ergodic with respect to $\rho \otimes \mathbb{P}$ (see [13] Appendix B). Consider also the linear map $G_o : \mathbb{R}^m \times \Omega \to GL(\mathbb{R}^m)$ defined by

$$G_o(x,\omega) = D\tilde{\xi}_T(\omega)(x)$$

and put

$$G_n(x,\omega) = G_o(\tilde{\Phi}_{nT}(x,\omega)) .$$

So

$$G_n(x,\omega) = D(\tilde{\xi}_T(\theta_{nT}(\omega)))\tilde{\xi}_{nT}(\omega)(x)$$

and by lemma 1.2.1, (1.7) and the definition of the time shift θ_t

$$G_n(x,\omega) = (D\tilde{\xi}_T(\theta_{nT}(\omega))(\tilde{\xi}_{nT}(\omega)(x)))\circ(D\tilde{\xi}_{nT}(\omega)(x))$$

$$= D\tilde{\xi}_{(n+1)T}(\omega)(x)\circ(D\tilde{\xi}_{nT}(\omega)(x))^{-1}\circ(D\tilde{\xi}_{nT}(\omega)(x)) .$$

Thus by the Chain rule we have that the map

$$G^n(x,\omega) = G_{n-1}(x,\omega) \circ \ldots\ldots \circ G_o(x,\omega) = D\tilde{\xi}_{nT}(\omega)(x) .$$

Now by Hypothesis 1.2.5 we have that

$$\int_{\tilde{M}} \int_{\Omega} \log^{+} ||G_0(x,\omega)||_{op} \, d\mathbb{P}d\rho < \infty \quad . \tag{1.8}$$

For each $q = 1,\ldots,m$ consider $||G_0(x,\omega)^{\wedge q}||$ where $\wedge q$ denotes the q^{th} exterior power. Consider then the linear map

$$(G_0(x,\omega)^{*} G_0(x,\omega))^{\frac{1}{2}}$$

which has eigenvalues $t_1 \leq \ldots \leq t_m$, say. We then have

$$||G_0(x,\omega)^{\wedge q}|| = \prod_{p=m-q+1}^{m} t_p$$

and hence

$$\log^{+} ||G_0(x,\omega)^{\wedge q}|| \leq \sum_{p=m-q+1}^{m} \log^{+} t_p \quad .$$

Also, for each p

$$t_p \leq ||G_0(x,\omega)||^2 \quad .$$

Hence by (1.8), for each $q = 1,\ldots,m$

$$\int_{\tilde{M}} \int_{\Omega} \log^{+} ||G_0(x,\omega)^{\wedge q}|| \, d\mathbb{P}\rho(dx) < \infty$$

and by Kingman's Subadditive ergodic theorem (Ruelle, Theorem 1.1) applied to $\log||G^n(x,\omega)^{\wedge q}||$ we have that $1/n \log||G^n(x,\omega)^{\wedge q}||$ tends to a limit a.s. which is invariant under the map $\tilde{\Phi}_T$. Also by Birkhoff's Ergodic

Theorem, $1/n \sum_{i=1}^{n-1} \log^{+}||G_i(x,\omega)||$ tends to a limit a.s. Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log||G_n(x,\omega)|| \le 0 \qquad \text{a.s..}$$

We then apply Ruelle's Proposition 1.3 for each (x,ω) a.s. where $G_n(x,\omega)$ corresponds to Ruelle's T_n, from which we can deduce the existence a.s. of a spectrum

$$\lambda^{r}_{(x,\omega)} < \ldots\ldots < \lambda^{1}_{(x,\omega)}$$

and filtration

$$\{0\} = V^{r+1}_{(x,\omega)} \subset V^{r}_{(x,\omega)} \ldots\ldots \subset V^{1}_{(x,\omega)} \equiv \mathbb{R}^m$$

such that if $v \in V^{i}_{(x,\omega)} \setminus V^{i+1}_{(x,\omega)}$ then

$$\frac{1}{nT} \log||G^n(x,\omega)|| \to \lambda^{i}_{(x,\omega)} \qquad \text{as} \quad n \to \infty \quad .$$

The discrete time version of the theorem follows.

To extend to the full continuous time result, we continue as in Ruelle's Appendix B. For each s, $0 < s < t$ we have by lemma 1.2.1 that

$$\tilde{\xi}_t(\theta_s(\omega))(\tilde{\xi}_s(\omega)) = \tilde{\xi}_{s+t}(\omega) \quad .$$

Then a.s. independently of n,t we have

$$\tilde{\xi}_t(\omega) = \tilde{\xi}_{t-nT}(\theta_{nT}(\omega)).\tilde{\xi}_{nT}(\omega) \tag{1.9}$$

$$\tilde{\xi}_{(n+1)T}(\omega) = \tilde{\xi}_{(n+1)T-t}(\theta_t(\omega)).\tilde{\xi}_t(\omega) \tag{1.10}$$

for all n and all $t \in [nT,(n+1)T]$. Hence a.s. we have by (1.9) and (1.10) that

$$\log||D\tilde{\xi}_t(\omega)(x)v|| \le \log||D(\tilde{\xi}_{t-nT}(\theta_{nT}(\omega)))\tilde{\xi}_{nT}(\omega)x|| + \log||D\tilde{\xi}_{nT}(\omega)(x)v||$$

and

$$\log||D\tilde{\xi}_t(\omega)(x)v|| \ge \log||D\tilde{\xi}_{(n+1)T}(\omega)(x)v|| - \log||D(\tilde{\xi}_{(n+1)T-t}(\theta_t(\omega)))(\xi_t(\omega)(x))||$$

for all $t \in [nT,(n+1)T]$, $x,v \in \mathbb{R}^m$.

Thus, if we set

$$\phi_1(x,\omega) = \sup_{t\in[0,T]} \log||D\tilde{\xi}_t(\omega)(x)||$$

$$\phi_2(x,\omega) = \sup_{t\in[0,T]} \log||D(\tilde{\xi}_{T-t}(\theta_t(\omega)))\tilde{\xi}_t(\omega)(x)|| \quad .$$

Then we have a.s. independently of n, x, v, that

$$\log||D\tilde{\xi}_{(n+1)T}(\omega)(x)v|| - \phi_2(\tilde{\Phi}_{nT}(x,\omega)) \le \log||D\tilde{\xi}_t(\omega)(x)v||$$

$$\le \log||D\tilde{\xi}_{nT}(\omega)(x)v|| + \phi_1(\tilde{\Phi}_{nT}(x,\omega))$$

for all $t \in [nT,(n+1)T]$. Thus ϕ_1 and ϕ_2 correspond to the functions (B.1) and (B.2) in Ruelle's Appendix B. By Hypothesis 1.2.5 and (1.6) ϕ_1 and ϕ_2 are $\rho \otimes \mathbb{P}$ integrable and are clearly non-negative.

Therefore by Birkhoff's Ergodic theorem

$$\frac{1}{n}\,\phi_i(\tilde{\Phi}_{nT}(x,\omega)) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for a.e.} \quad (x,\omega)\ .$$

So for these $(x,\omega) \in \mathbb{R}^m \times \Omega$

$$\lim_{n\to\infty} \frac{1}{nT} \log||D\tilde{\xi}_{nT}(\omega)(x)v|| = \lim_{t\to\infty} \frac{1}{t} \log||D\tilde{\xi}_t(\omega)(x)v||$$

and the continuous time result follows. □

Remarks 1.2.7

(i) It is clear that for the above result to hold it is only required that a measurable solution flow $\{\xi_t(\omega)(x)\ ;\ t \geq 0\}$ of (1.1) exists and hence its derivative $T\xi_t(\omega)(x)(v)$ exists in probability; strong completeness (i.e. such that $\xi_t(\omega) : M \to M$ is continuous a.s.) and the existence of a flow of diffeomorphisms is not required. Completeness is assured by the existence of the finite invariant probability measure ρ (see for example [64]).

(ii) If M is compact then we have Carverhill's Theorem 2.1 where $||\cdot||$ in Theorem 1.2.6 is any Riemannian metric on T_xM.

(iii) From the proof we see that the Lyapunov spectrum is a.s. invariant under the time shift Φ_t. Indeed in Carverhill's Appendix B it is shown that Φ_t is ergodic with respect to the measure $\rho \otimes \mathbb{P}$ for a non-degenerate system (1.1) and for such a system the Lyapunov spectrum is a.s. constant, independent of (x,ω) a.s..

So for a non-degenerate system the statement of Theorem 1.2.6 could be rephrased so that the spectrum and associated filtration can be written as

$$\lambda^r < \ldots\ldots < \lambda^1 \qquad (r \leq n)$$

and

$$\{0\} = V^{r+1}_{(x,\omega)} \subset V^r_{(x,\omega)} \subset \ldots\ldots \subset V^1_{(x,\omega)} \equiv T_xM$$

and if $v \in V^i_{(x,\omega)} \setminus V^{i+1}_{(x,\omega)}$ then

$$\lim_{t\to\infty} \frac{1}{t} \log||T\xi_t(\omega)(x)v|| = \lambda^i$$

where all the λ^i's $(1 \leq i \leq r)$ are independent of (x,ω) a.s..

In this case we can also consider the weighted sum of the exponents given by

$$\begin{aligned} \lambda_\Sigma &= \sum_{j=1}^{r} \dim(V^j_{(x,\omega)} / V^{j+1}_{(x,\omega)}).\lambda^j \\ &= \lim_{t\to\infty} \frac{1}{t} \log|\det T\xi_t(\omega)(x)| \end{aligned} \tag{1.11}$$

1.3. Formulae for the Lyapunov Exponents

In [14], Carverhill obtains a formula for the Lyapunov exponents, in particular this formula picks out the leading (largest) exponent λ^1. This is given by

$$\lim_{t\to\infty} \frac{1}{t} \log||T\xi_t(\omega)(x)v|| = \int_{v\in SM} g(v)d\nu \tag{1.12}$$

where SM denotes the sphere bundle to M and ν is an invariant probability measure on SM . The function $g : SM \to \mathbb{R}$ is defined by

$$g(v) = g_A(v) + \tfrac{1}{2} \sum_{i=1}^{m} dg_{X_i}(\tilde{X}(v)) \tag{1.13}$$

where

$$g_A(v) = \langle \nabla A(v), v \rangle$$

and

$$g_{X_i}(v) = \langle \nabla X_i(v), v \rangle \qquad \tilde{X}_i = \tilde{X}(\cdot, e_i), \text{ where } \tilde{X}_i \equiv X_i\big|_{SM} .$$

D. Elworthy has reformulated (1.12) to give the following

$$\begin{aligned}\lambda_{(x,\omega)} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \{&\langle \eta_s(\omega)v, \nabla(A + \tfrac{1}{2} \sum_{i=1}^{m} \nabla X_i(X_i))(\eta_s(\omega)v)\rangle \\ &+ \tfrac{1}{2} \sum_{i=1}^{m} \{|\nabla X_i(\eta_s(\omega)v)|^2 - 2\langle \nabla X_i(\eta_s(\omega)v), \eta_s(\omega)v\rangle^2 \\ &- \langle R(X_i, \eta_s(\omega)v)\eta_s(\omega)v, X_i\rangle\}\}ds \end{aligned} \tag{1.14}$$

where $\eta_s(\omega) = \dfrac{T\xi_s(\omega)(x)}{||T\xi_s(\omega)(x)||}$ is the sphere bundle flow and $R(\cdot,\cdot)$ is the curvature tensor (in the notation of Kobayashi and Nomizu, [41]). In particular when (1.1) determines a Brownian motion on M then $A + \tfrac{1}{2} \sum_{i=1}^{m} \nabla X_i(X_i) \equiv 0$ and we can choose orthonormal co-ordinates so that $X_1, \ldots, X_m$ form a frame at x , whence

$$\sum_{i=1}^{m} \langle R(X_i, \eta_s(\omega)v)\eta_s(\omega)v, X_i\rangle = \mathrm{Ric}(\eta_s(\omega)v, \eta_s(\omega)v)$$

where Ric $(\cdot,\cdot)$ denotes the Ricci tensor. Then (1.14) becomes

$$\lambda_{(x,\omega)} = \lim_{t\to\infty} \frac{1}{2t} \int_0^t \{ \sum_{i=1}^m [|\nabla X_i(\eta_s(\omega)v)|^2 - 2<\nabla X_i(\eta_s(\omega)v),\eta_s(\omega)v>^2$$

$$- \mathrm{Ric}(\eta_s(\omega)v,\eta_s(\omega)v)]\}ds \quad . \quad (1.15)$$

In the special case that M is embedded in some $\mathbb{R}^m$ and the system (1.1) is obtained from the embedding (i.e. the gradient Brownian flow) then (1.15) becomes (see for example [16])

$$\lambda_{(x,\omega)} = \lim_{t\to\infty} \frac{1}{2t}\int_0^t \{|\alpha_x(\eta_s(\omega)v,\cdot)|^2 - 2|\alpha_x(\eta_s(\omega)v,\eta_s(\omega)v)|^2 - \mathrm{Ric}(\eta_s(\omega)v,\eta_s(\omega)v)\}ds \quad (1.16)$$

where $\alpha_x : T_xM \times T_xM \to T_xM^\perp$ denotes the second fundamental form of M.

We also have the following formula for the determinant which enables us to examine λ_Σ, the sum of the Lyapunov exponents. This was first shown to us in a private communication by P. Baxendale and basically follows from Itô's formula. We have for system (1.1)

$$\log|\det T\xi_t(\omega)(x)| = \int_0^t \sum_{i=1}^m \mathrm{div}\, X_i(\xi_s(\omega)(x))dB_s^i + \int_0^t \mathrm{div}\, A(\xi_s(\omega)(x))ds$$

$$+ \tfrac{1}{2} \int_0^t \sum_{i=1}^m <\nabla \mathrm{div}\, X_i(\xi_s(\omega)(x)),X_i(\xi_s(\omega)(x))>ds \qquad (1.17)$$

which on compact M yields, by (1.11) and Proposition 1.2.3

$$\lambda_\Sigma = \lim_{t\to\infty} \frac{1}{t} \log|\det T\xi_t(\omega)(x)|$$

$$= \int_M \operatorname{div} A(x)\rho(dx) + \tfrac{1}{2} \int_M \sum_{i=1}^{m} \langle \nabla \operatorname{div} X_i(x), X_i(x)\rangle \rho(dx) \tag{1.18}$$

where ρ is the unique invariant measure on M. We note here that the advantage of (1.18) is that knowledge is only required of the invariant measure ρ on M and not that of any derivative system as required for (1.12) - thus making λ_Σ more accessible to actual calculation.

1.4. Stationary Stochastic Mechanics

For the following see Nelson [48].

On a Riemannian manifold M^n consider the Schrödinger operator

$$-\tfrac{1}{2}\Delta + V, \tag{1.19}$$

where Δ is the Laplace-Beltrami operator and V is some potential function $V : M \to \mathbb{R}$. Let E_0 be the lowest eigenvalue (ground state) with corresponding normalized eigenfunction $\phi_0 : M \to \mathbb{R}\ (> 0)$ such that

$$(-\tfrac{1}{2}\Delta + V)\phi_0 = E_0\phi_0 . \tag{1.20}$$

V is assumed to be sufficiently regular such that $\nabla\phi_0 \in L^2(M,dx)$ where dx denotes the Riemannian volume element.

Consider the renormalization procedure $\phi_0^{-1}(-\tfrac{1}{2}\Delta + V)(\phi_0\cdot)$ this yields the operator

$$\tfrac{1}{2}\,\Delta + A(\cdot).\nabla \tag{1.21}$$

where $A(x) = \frac{1}{2}\nabla\log|\phi_0(x)|^2$. Under suitable regularity conditions there exists a stationary Markov process with generator (1.21) which has invariant distribution

$$\rho(dx) = |\phi_0(x)|^2 dx \; . \tag{1.22}$$

Such a process can be represented as the solution of a stochastic differential equation

$$dx_t = X(x_t)\circ dB_t + Z(x_t)dt + A(x_t)dt \tag{1.23}$$

provided the solutions to the Stratonovich equation

$$dy_t = X(y_t)\circ dB_t + Z(x_t)dt$$

form a Brownian motion on M . Here Z is a vector field on M and for each $y \in M$, $X(y) : R^m \to T_yM$ maps $\mathbb{R}^m$, some m , linearly into T_yM . The system (1.23) is called a ground state stochastic mechanical system and its solution $\{x_t;t \geq 0\}$ is called a ground state process. The Kinetic energy of the stochastic mechanical particle satisfying (1.23) is given by

$$K.E. = \tfrac{1}{2}\int_M |\nabla\phi_0(x)|^2 \, dx \; . \tag{1.24}$$

By the existence of the finite invariant measure (1.22) the system is complete, but not necessarily strongly complete ($\equiv$ strictly conservative), so solutions starting from some point $x_0 \in M$ continue for all time a.s..

For the time dependent case, (1.19) becomes

$$i \frac{\partial \psi_t(x)}{\partial t} = (-\tfrac{1}{2} \Delta + V(x))\psi_t(x)$$

and (1.22) is then given by

$$\rho_t(x) = |\psi_t(x)|^2 dx .$$

This situation is more complicated, but the work of the following chapters has been considered for such cases in [21].

CHAPTER 2.

2.1. Average Lyapunov Exponents and Gradient Brownian Flows

Throughout this chapter we assume that M is a smooth, compact, n-dimensional Riemannian manifold. A standard way of obtaining Brownian motion on M is to isometrically embed it in some Euclidean space $\mathbb{R}^m$. If $f : M \to \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$ is the embedding map then we can consider the vector fields

$$X_i = \nabla f_i \qquad 1 \leq i \leq m .$$

This is equivalent to considering the map $X : M \times \mathbb{R}^m \to TM$ of (1.23) being given by

$$X(x) = \text{orthogonal projection of } \mathbb{R}^m \text{ onto } T_x M .$$

The vector fields X_i $(1 \leq i \leq m)$ give rise to the stochastic differential equation

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i \tag{2.1}$$

on M, where $\circ$ denotes that the equation is taken in the Stratonovich sense. As shown in [32], (pg.253), this equation has infinitesimal generator $A = \frac{1}{2}\Delta$, that is each solution is a Brownian motion on M.

Definition 2.1.1

A solution flow of a gradient stochastic differential equation of the form given by (2.1) is called a gradient Brownian flow.

We shall consider the asymptotic behaviour of such a gradient Brownian flow by considering λ and λ_Σ, the top exponent and sum of the Lyapunov exponents associated with such a process.

We first consider the sum of the exponents λ_Σ. The following gives a formula for λ_Σ.

Theorem 2.1.2

For a gradient stochastic system on a compact n-dimensional Riemannian manifold M, the sum of the Lyapunov exponents λ_Σ is given by the formula

$$\lambda_\Sigma = \frac{-n^2}{2(\text{Vol } M)} \int_M |\underline{H}(x)|^2 dx \tag{2.2}$$

where $\underline{H}(\cdot)$ denotes the mean curvature vector of M.

Proof

By (1.11)

$$\lambda_\Sigma = \lim_{t\to\infty} \frac{1}{t} \log|\det D\xi_t(\omega)(x)| \tag{2.3}$$

where $\xi_t(\omega)(x)$ denotes the solution flow of equation (2.1). For a gradient Brownian system of the form (2.1), $A \equiv 0$ and (1.17) reduces to

$$\log|\det D\xi_t(\omega)(x)| = \int_0^t \sum_{i=1}^m \text{div } X_i(\xi_s(\omega)(x)) dB_s^i$$
$$+ \tfrac{1}{2}\int_0^t \sum_{i=1}^m \langle\nabla\text{div } X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle ds\,. \tag{2.4}$$

We first show that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i = 0 \qquad \text{a.s..}$$

For this we shall show that $\int_0^t \sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i = \phi_t(\omega)(x)$, say,

is a time changed Brownian motion. By the Itô formula

$$\phi_t^2(\omega)(x) = 2\int_0^t \phi_s(\omega)(x).\sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i$$

$$+ \int_0^t \sum_{i=1}^m (\operatorname{div} X_i(\xi_s(\omega)(x)))^2 ds \ .$$

Define $\sigma : [0,\infty) \times \Omega \to [0,\infty)$ by

$$\sigma(t,\omega) = \operatorname*{Min}_s \{\int_0^s \sum_{i=1}^m (\operatorname{div} X_i(\xi_r(\omega)(x)))^2 dr = t\}$$

if such an s exists, or ∞ otherwise. (Note that since the integrand is non-negative a.s., $\sigma(t,\omega)$ is non-decreasing a.s.). So

$$\int_0^{\sigma(t,\omega)} (\sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i)^2 = t$$

and $\phi^2_{\sigma(t,\omega)}(\omega)(x)-t$ is a martingale. Hence by Levy's characterization $\phi_{\sigma(t,\omega)}(\omega)(x)$ is a Brownian motion, $\hat{B}_t(\omega)$ say. (See e.g. [32] pg.80-84).

Hence

$$\phi_t(\omega)(x) = \hat{B}_{\rho(t,\omega)}(\omega)$$

where

$$\rho(t,\omega) = \int_0^t \sum_{i=1}^m (\operatorname{div} X_i(\xi_s(\omega)(x)))^2 ds \quad .$$

Since M is compact, the integrand $\sum_{i=1}^m (\operatorname{div} X_i(x))^2$ is bounded $\forall\, x \in M$, by K, say. Hence $\rho(t,\omega) \le Kt$ a.s. Also $\frac{1}{t}\hat{B}_{Kt}(\omega) \to 0$ a.s. as $t \to \infty$ by Hinčin's Law (see Mckean [45] pg.9), therefore

$$\lim_{t\to\infty} \frac{1}{t}\,\phi_t(\omega)(x) \;\left(\equiv \frac{1}{t}\,\hat{B}_{\rho(t,\omega)}(\omega)\text{a.s.}\right) = 0 \qquad \text{a.s.}$$

as required.

For the second integral in (2.4), by the Strong Law of Large Numbers and the fact that the unique invariant measure on M is given by $\rho(dx) = (\operatorname{Vol} M)^{-1}dx$ we have that

$$\lambda_\Sigma = \frac{1}{2(\operatorname{Vol} M)} \int_M \sum_{i=1}^m \langle \nabla \operatorname{div} X_i(x), X_i(x)\rangle dx \;.$$

Since $-\operatorname{div}$ and ∇ are adjoints, integrating by parts gives

$$= -\frac{1}{2(\operatorname{Vol} M)} \int_M \sum_{i=1}^m (\operatorname{div} X_i(x))^2 dx$$

$$= -\frac{1}{2(\operatorname{Vol} M)} \int_M (\operatorname{trace} \alpha_x)^2\, dx$$

where $\alpha_x : T_xM \times T_xM \to T_xM^\perp \subset \mathbb{R}^m$, $x \in M$, denotes the second fundamental form, [41]. Thus since $\underline{H}(x) = \frac{1}{n}$ (trace α_x)

$$\lambda_\Sigma = - \frac{n^2}{2(\mathrm{Vol}\ M)} \int_M |\underline{H}(x)|^2 dx$$

and hence the required result. □

We remark that since $f : M \to \mathbb{R}^m$ is an isometry (2.2) can be written as

$$\lambda_\Sigma = - \frac{1}{2(\mathrm{Vol}\ M)} \int_M ||\tau_f||^2\, dx$$

where τ denotes the tension field of Eells and Lemaire, [31].

Examples 2.1.3

By (2.2) it is clear to see that the leading and only exponent for an embedded 1-dimensional manifold is given by

$$\lambda = \frac{-1}{2(\mathrm{Vol}\ M)} \int_M K^2(x) dx$$

where $K(\cdot)$ denotes the curvature of the plane curve, i.e. $K(x) = 1/r(x)$ where $r(\cdot)$ is the radius of curvature.

(i) Consider the standard embedding of S^1 in $\mathbb{R}^2$ given by

$$f(x) = (\cos x, \sin x)\ .$$

Equation (2.1) has the form

$$dx_t = -\sin x_t \circ dB_t^1 + \cos x_t \circ dB_t^2\ .$$

Clearly $K(x) = 1$, $\forall\, x \in S^1$ and hence $\lambda = -\frac{1}{2}$.

(ii) Consider S^1 isometrically embedded in $\mathbb{R}^2$ as an ellipse, the embedding given by

$$f(\theta) = (a\sin\theta, b\cos\theta) \qquad a \geq b > 0 .$$

Then

$$K(\theta) = \frac{ab}{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}$$

the volume element

$$dx = (a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}d\theta$$

and the length of the ellipse is given by $4a\, E(k)$ where $E(k)$ is the complete elliptic integral of the second kind, [58] , defined by

$$E(k) = \int_0^{\pi/2} (1 - k^2\sin^2\theta)^{\frac{1}{2}}d\theta$$

and $k^2 = \dfrac{a^2-b^2}{a^2}$.

So

$$\lambda = \frac{-b^2}{8a^4E(k)} \int_0^{2\pi} \frac{d\theta}{(1 - k^2\sin^2\theta)^{5/2}}$$

$$= \frac{-(a^2+b^2)}{3a^2b^2} + \frac{F(k)}{E(k)} \cdot \frac{1}{6a^2}$$

where $F(k)$ denotes the complete elliptic integral of the first kind

[58], defined by

$$F(k) = \int_0^{\pi/2} \frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}} \quad ,$$

k as before. We now examine λ for various values of a and b. Clearly $\lambda = -\frac{1}{2}$ when $a = b$ (i.e. the circle S^1).

Suppose that the length of the perimeter of the ellipse is kept fixed, i.e. $\ell = 4aE(k)$, ℓ constant. Then

$$\lambda = -\frac{1}{3a^2} - \frac{1}{3b^2} + \frac{F(k)}{\ell} . \frac{2}{3a} \quad .$$

Since $a \geq b$ and the length of the perimeter is kept fixed, as a increases b must decrease and tend to zero. Clearly as a increases $-1/3a^2$ remains finite and small. We therefore need to examine $-\frac{1}{3b^2} + \frac{F(k)}{\ell} . \frac{2}{3a}$ for small b.

By obtaining the maximum value of the continuous function

$$h(\theta) = (1 - k^2\sin^2\theta)^{-\frac{1}{2}} \qquad \theta \in [0,\pi/2], \quad k^2 < 1$$

we can find an upper bound for $F(k)$, namely

$$\int_0^{\pi/2} \frac{d}{(1-k^2\sin^2\theta)^{\frac{1}{2}}} \leq \frac{\pi}{2} \cdot \frac{1}{(1-k^2)^{\frac{1}{2}}} = \frac{\pi a}{2b} \quad .$$

Hence

$$\lim_{b\to 0} -\frac{1}{3b^2} + \frac{F(k)}{\ell} . \frac{2}{3a} \leq \lim_{b\to 0} -\frac{1}{3b^2} + \frac{\pi a}{2b} . \frac{2}{3a\ell}$$

$$= \lim_{b\to 0} \frac{-\ell + b\pi}{3b^2\ell} \quad .$$

For small enough b , $\ell > b\pi$, hence

$$\lim_{b\to 0} -\frac{1}{3b^2} + \frac{F(k)}{\ell} \cdot \frac{2}{3a} = -\infty$$

and $\lambda \to -\infty$ as a increases and b approaches zero. This implies increasing stability of the system as the eccentricity of the ellipse increases, ($k^2 = e^2$ where e is the eccentricity).

In [19] it was also shown that for S^1 isometrically embedded in $\mathbb{R}^2$ the induced gradient Brownian flow satisfies:

$$\lambda \le \text{the leading eigenvalue of } \tfrac{1}{2}\Delta \quad (\text{i.e. } -\tfrac{1}{2}) .$$

(iii) Consider the standard embedding of S^n in $\mathbb{R}^{n+1}$. For this embedding it is well known (see e.g. [41]) that $H(x) = 1, \forall x \in S^n$. Hence by (2.2)

$$\lambda_\Sigma = -\frac{n^2}{2} . \tag{2.5}$$

We note in [16], Elworthy showed that for this embedding the top Lyapunov exponent $\lambda = -n/2$ and hence by (2.5) all the exponents for $S^n \subset \mathbb{R}^{n+1}$ are equal.

(iv) Consider the torus of revolution, or ε-tube, embedded in $\mathbb{R}^3$. The embedding is given by (see for example [49])

$$f(u,v) = ((1+\varepsilon \cos v)\cos u,\ (1+\varepsilon \cos v)\sin u, \varepsilon \sin v) \quad 0 < \varepsilon < 1 .$$

Then the elements of the first and second fundamental forms are

$$E = (1 + \varepsilon \cos v)^2 \qquad n = \varepsilon$$

$$F = 0 \qquad m = 0$$

$$G = \varepsilon^2 \qquad l = (1 + \varepsilon \cos v)\cos v\ .$$

Hence the mean curvature is given by

$$H = \tfrac{1}{2}\left[\frac{1}{\varepsilon} + \frac{\cos v}{(1+\varepsilon\cos v)}\right]$$

and the unique invariant measure on $\underset{\sim}{M}_r$ is given by

$$dx = \varepsilon\,\frac{(1 + \varepsilon \cos v)}{4\pi^2\varepsilon}\,dudv\quad .$$

Thus

$$\lambda_\Sigma = -\,2\int_0^{2\pi}\!\!\int_0^{2\pi}\frac{(1+2\varepsilon\cos v)^2}{4\varepsilon^2(1+\varepsilon\cos v)^2}\cdot\varepsilon\,\frac{(1+\varepsilon\cos v)}{4\pi^2\varepsilon}\,dudv$$

$$= -\,\frac{1}{8\pi\varepsilon^2}\int_0^{2\pi}\frac{(1+2\varepsilon\cos v)^2}{(1+\varepsilon\cos v)}\,dv$$

which by residue theory yields

$$\lambda_\Sigma = -\,\frac{1}{2\varepsilon^2\sqrt{1-\varepsilon^2}}\quad .$$

We see from this that λ_Σ tends to $-\infty$ as ε tends to 0 or 1 and also λ_Σ has a maximum value when $\varepsilon^2 = 2/3$, this being $\lambda_\Sigma^{max} = -3\sqrt{3}/4$.

<u>Remark</u> 2.1.4

We note that it is clear from the formula for λ_Σ given in Theorem 2.1.2 that $\lambda_\Sigma \leq 0$. The above examples all in fact have $\lambda_\Sigma < 0$. The following well known result from differential geometry confirms this fact.

Theorem 2.1.5

There are no compact minimal submanifolds of Euclidean space.

Proof

See for example Willmore [63]. □

Proposition 2.1.6

For a gradient Brownian flow we have $\lambda_\Sigma < 0$ a.s..

Proof

Since

$$\lambda_\Sigma = -\frac{n^2}{2}\int_M |\underline{H}(x)|^2 \frac{dx}{(\mathrm{Vol}\ M)} \tag{2.6}$$

and the unique invariant measure $\rho(dx) = (\mathrm{Vol}\ M)^{-1}dx$ has positive density the integral in (2.6) is zero a.s. iff the integrand $|\underline{H}(\cdot)|^2$ is zero a.e..

By Theorem 2.1.5

$$P\{\underline{H}(x) = 0 \quad \text{a.e.}\} = 0$$

hence $\lambda_\Sigma < 0$ a.s.. □

Remark 2.1.7

P. Baxendale has since proved by entropy arguments that $\lambda_\Sigma < 0$

for more general processes on compact manifolds, (see [5]).

Having shown that λ_Σ is strictly negative using results of Reilly [52] and Takahashi [61], we are able to obtain an upper bound for λ_Σ .

<u>Theorem</u> 2.1.8

For any compact M embedded in $\mathbb{R}^m$ $(m > n)$ the induced gradient Brownian flow satisfies

$$\lambda_\Sigma \le n\mu$$

where μ is the leading (first negative) eigenvalue of $\frac{1}{2}\Delta$ (i.e. $\frac{1}{2}\Delta f = \mu f$) , and equality holds if and only if M is embedded as a minimal submanifold of some hypersphere in $\mathbb{R}^m$ of radius $(n/2\,|\mu|)^{\frac{1}{2}}$.

<u>Proof</u>

By Reilly's Theorem A, [52], with $r = 1$ we have

$$-\frac{n}{2(\mathrm{Vol}\ M)}\int_M |\underline{H}(x)|^2 dx \le \mu \quad .$$

Thus by Theorem 2.1.2 and the formula (2.2) for λ_Σ

$$\lambda_\Sigma \le n\mu$$

as required. The fact that equality holds if and only if M is embedded as a minimal submanifold of some hypersphere of radius $(n/2|\mu|)^{\frac{1}{2}}$ follows from Takahashi's result ([41] note 14 or [61]). $\square$

Remark 2.1.9

An alternative proof of this result is given in [16].

Example 2.1.10

Consider the embedding $f : S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \to \mathbb{R}^4$ (the Clifford Torus) given by

$$f(u,v) = (1/\sqrt{2}\cos u,\ 1/\sqrt{2}\sin u,\ 1/\sqrt{2}\cos v,\ 1/\sqrt{2}\sin v)\ .$$

Then $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ is a minimal submanifold of $S^3 \subset \mathbb{R}^4$ (see, e.g. [63]), $\mu = -1$ and $(n/2|\mu|)^{\frac{1}{2}} = 1$. Hence by Theorem 2.1.8 for this f, $\lambda_\Sigma = -2$.

Suppose now that the second fundamental form α_x defined in Theorem 2.1.2 is given in local co-ordinates by the symmetric matrix (h_{ij}). In [23] Cheng and Yau give a result which, given certain conditions on the curvature properties of the manifold M, provides a lower bound for λ_Σ on a hypersurface.

Theorem 2.1.11

For a gradient Brownian flow on a compact hypersurface M with positive scalar curvature R and on which the form $nH\delta_{ij} - h_{ij}$ is positive semi-definite, then

$$\lambda_\Sigma \geq -\tfrac{1}{2}\sup_M \left\{ -\frac{n^2\Delta R}{\min\limits_{i\neq j} R_{ijij}} + n^2R \right\} \tag{2.7}$$

where R_{ijkl}, is the Riemann curvature tensor computed relative to an orthonormal frame. There is equality in (2.7) if and only if M is embedded as a hypersphere in $\mathbb{R}^{n+1}$.

Proof

Since for a hypersurface, by (2.2)

$$\lambda_\Sigma = -\frac{n^2}{2(\mathrm{Vol}\ M)}\int_M H(x)^2 dx$$

we have

$$\lambda_\Sigma \geq -\tfrac{1}{2}\sup_M n^2H^2 \geq -\tfrac{1}{2}\sup_M\left\{-\frac{n^2\Delta R}{\min\limits_{i\neq j} R_{ijij}} + n^2R\right\} \tag{2.8}$$

by Cheng and Yau (Theorem of Section 3) [23]. The fact that we have equality throughout in (2.8) if and only if M is embedded as a hypersphere in $\mathbb{R}^{n+1}$ again follows from the results of Reilly [52], [53] and Takahashi [61]. Note also that for a hypersphere of radius r , $R = 1/r^2$ = constant, and each extreme of (2.8) is given by $-n^2/2r^2$. □

Remark 2.1.12

We note that if $n = 2$ the condition that the form $(2H\delta_{ij} - h_{ij})$ is positive semi-definite gives $h_{11}h_{22}-(h_{12})^2 \geq 0$ which is just the condition that M bounds a convex domain (see e.g. [41]). Indeed more generally the condition that $(nH\delta_{ij} - h_{ij})$ is positive definite implies that M is a strictly convex hypersurface.

We also have the following lower bound for surfaces of higher codimension.

Theorem 2.1.13

For $n > 2$, the gradient Brownian flow on

(1) the Veronese surface in $\mathbb{R}^5$, $(p = 3)$

(2) the generalised Clifford torus in $\mathbb{R}^{n+2}$ and

(3) the n-sphere in $\mathbb{R}^{n+p}$ $(p > 1)$

satisfy

$$\lambda_\Sigma \geq - \frac{n}{2(\text{Vol } M)} \left(\frac{2p-3}{p-1}\right)\left[(n-1)\left(\frac{2p-3}{p-1}\right) - 1\right]^{-1} . \int_M R \, dx$$

where R denotes the scalar curvature of M .

Proof

By a Theorem of Chen [22] we have for the above manifolds that

$$R \geq \frac{n(p-1)}{(2p-3)} \, |\underline{H}(x)|^2 \{(n-1)\left(\frac{2p-3}{p-1}\right) - 1\} \tag{2.9}$$

and by the formula (2.2) for λ_Σ the result follows. We note that in [22] Chen states that these manifolds are the only closed pseudo-umbilical submanifolds M^n of $\mathbb{R}^{n+p}$ $(p > 1)$ with mean curvature nowhere zero that satisfy (2.9). □

2.2 The p^{th} Moment of the Determinant

Analagous to the work of L. Arnold (see e.g. [2]) on p^{th} moment exponents, we now consider for a nonlinear system, the p^{th} moment of the determinant. We define the function

$$s(p;x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{|\det D\xi_t(\omega)(x)|^p\} \qquad p \in \mathbb{R}\ ,\ x \in M\ . \quad (2.10)$$

We now show some elementary properties of the function $s(p;x)$ and shall obtain a link between sample and moment stability. Initially we consider a process satisfying the stochastic differential equation

$$d\xi_t(\omega)(x) = \sum_{i=1}^{m} X_i(\xi_t(\omega)(x))\circ dB_t^i + A(\xi_t(\omega)(x))dt \qquad (2.11)$$

for suitably smooth vector fields X_i $(1 \le i \le m)$ and A on M. We recall that for such an S.D.E. by (1.17) we have the following formula for the determinant

$$|\det D\xi_t(\omega)(x)| = \exp\{\int_0^t \sum_{i=1}^{m} \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i + \int_0^t \operatorname{div} A(\xi_s(\omega)(x))ds$$

$$+ \tfrac{1}{2}\int_0^t \sum_{i=1}^{m} \langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle ds\}. \quad (2.12)$$

Proposition 2.2.1

For a process satisfying the S.D.E. (2.11) the p^{th} moment of the determinant is given by

$$s(p;x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{M_t \exp[p \int_0^t \operatorname{div} A(\xi_s(\omega)(x))ds$$

$$+ \frac{p}{2} \int_0^t \sum_{i=1}^{m} \langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x))\ ,\ X_i(\xi_s(\omega)(x))\rangle ds$$

$$+ \frac{p^2}{2} \int_0^t \sum_{i=1}^{m} |\operatorname{div} X_i(\xi_s(\omega)(x))|^2 ds]\} \quad p \in \mathbb{R},\ x \in M \qquad (2.13)$$

where M_t is a bounded martingale.

Proof

Using the Girsanov Theorem (see for example [32], pg.258) we can dispense with the stochastic integral in (2.12). Set

$$W(u) = \sum_{i=1}^{m} X_i(u).(p \operatorname{div} X_i(u)) \qquad u \in M , \quad p \in \mathbb{R} ,$$

and

$$M_t = \exp\Big[\int_0^t p \sum_{i=1}^{m} \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i - \tfrac{1}{2} \int_0^t p^2 \sum_{i=1}^{m} |\operatorname{div} X_i(\xi_s(\omega)(x))|^2 \, ds\Big] .$$

Then substitute for M_t in (2.12) which together with (2.10) yields the required result. □

The following gives elementary results for $s(p;x)$ which are similar to those for $g(p;x)$ obtained by Arnold in [2].

Proposition 2.2.2

$s(p;x)$ is a finite function which is convex in p such that $\forall\, p \in \mathbb{R}$

(i) $|s(p;x)| \le |p|(K_1+K_2) + p^2K_3$ $\qquad K_1, K_2$ and K_3 are constants,

(ii) $s(p;x) \ge p\lambda_\Sigma$,

(iii) $\dfrac{s(p;x)}{p}$ is increasing in p ,

(iv) $s'(0^-;x) \le \lambda_\Sigma \le s'(0^+;x)$.

Proof

(i) By Proposition 2.2.1

$$|s(p;x)| \le \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{|M_t| \exp[|p| \int_0^t |\mathrm{div}\, A(\xi_s(\omega)(x))| ds$$

$$+ \frac{|p|}{2} \int_0^t \sum_{i=1}^m |<\nabla \mathrm{div}\, X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))>| ds$$

$$+ \frac{p^2}{2} \int_0^t \sum_{i=1}^m |\mathrm{div}\, X_i(\xi_s(\omega)(x))|^2 ds]\}$$

$$\le \lim_{t\to\infty} \frac{1}{t} \log K_0 + \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[|p|K_1 t + |p|K_2 t$$

$$+ p^2 K_3 t]\}$$

where $K_0 = \max\{M_t\}$ is finite since M_t is a bounded martingale and

$$K_1 = \sup_{s\in[0,t]} |\mathrm{div}\, A(\xi_s(\omega)(x))|$$

$$K_2 = \tfrac{1}{2} \sup_{s\in[0,t]} \sum_{i=1}^m |\nabla\, \mathrm{div}\, X_i(\xi_s(\omega)(x))| |X_i(\xi_s(\omega)(x))|$$

$$K_3 = \tfrac{1}{2} \sup_{s\in[0,t]} \sum_{i=1}^m |\mathrm{div}\, X_i(\xi_s(\omega)(x))|^2$$

are all finite since M is compact. Thus

$$|s(p;x)| \le |p|(K_1 + K_2) + p^2 K_3$$

as required. The convexity of $s(p;x)$ follows immediately from the convexity of $\log \mathbb{E}\{|X|^p\}$.

(ii) By considering the expression (2.12) for $|\det D\xi_t(\omega)(x)|$, by (2.10)

$$s(p;x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[p(\int_0^t \sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i + \int_0^t \operatorname{div} A(\xi_s(\omega)(x))ds$$
$$+ \tfrac{1}{2}\int_0^t \sum_{i=1}^m \langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle ds)]\}. \qquad (2.14)$$

Jensen's inequality applied to $\exp X$ yields

$$\mathbb{E}\{\exp[p(\int_0^t \sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i + \int_0^t \operatorname{div} A(\xi_s(\omega)(x))ds$$
$$+ \tfrac{1}{2}\int_0^t \sum_{i=1}^m \langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle ds)]\}$$
$$\geq \exp[pt\,\mathbb{E}\{\tfrac{1}{t}(\int_0^t \sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i + \int_0^t \operatorname{div} A(\xi_s(\omega)(x))ds$$
$$+ \tfrac{1}{2}\int_0^t \sum_{i=1}^m \langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle ds)\}] .$$

Taking $\lim_{t\to\infty} \frac{1}{t}\log$ on both sides and applying the Dominated Convergence Theorem on the right hand side gives

$$s(p;x) \geq p\,\lambda_\Sigma \ .$$

(iii) Again using expression (2.14) for $s(p;x)$

$$\frac{s(p;x)}{p} = \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}\{\exp[p(\int_0^t \sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i$$
$$+ \int_0^t \operatorname{div} A(\xi_s(\omega)(x))ds$$

$$+ \tfrac{1}{2} \int_0^t \sum_{i=1}^m \langle \nabla \mathrm{div}\ X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x)) \rangle ds)]\}$$

$$= \lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}\{\exp[p(\int_0^t \sum_{i=1}^m \mathrm{div}\ X_i(\xi_s(\omega)(x)) dB_s^i + \int_0^t \mathrm{div}\ A(\xi_s(\omega)(x)) ds$$

$$+ \tfrac{1}{2} \int_0^t \sum_{i=1}^m \langle \nabla \mathrm{div}\ X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x)) \rangle ds)]\})^{1/p}$$

and monotonicity follows from the monotonicity of $\mathbb{E}\{|X|^p\}^{1/p}$ for $p > 0$. For $p < 0$ we consider $\mathbb{E}\{(1/|X|)^{-p}\}^{-1/p}$.

(iv) Follows from (ii), the existence of the one-sided derivatives being assured by convexity. □

We now consider the gradient Brownian flow and examine $s(p;x)$ for this particular case.

Proposition 2.2.3

For a gradient Brownian system (2.1) the p^{th} moment of the determinant is given by

$$s(p;x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[- \frac{pn^2}{2} \int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds$$

$$+ \frac{p^2 n^2}{2} \int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds]\} .$$

Proof

For the gradient Brownian flow $A \equiv 0$ and $\sum_{i=1}^m \nabla X_i(X_i) = 0$. Also for $e \in \mathbb{R}^m$, let X^e denote the vector field $X(\cdot)^e$. When e has norm one, $\operatorname{div} X^e(x)$ is just the component of the trace of the second fundamental form α_x in the direction e, since

$$\operatorname{div} X^e = \operatorname{trace} \nabla X^e = \langle(\operatorname{trace} \nabla^2 f), e\rangle .$$

In particular it vanishes for e tangent to M at x, while $X^e(x)$ vanishes for e normal at x. Thus $\sum_{i=1}^m (\operatorname{div} X_i)X_i \equiv 0$ for the gradient Brownian system. Hence in the proof of Proposition 2.2.1 $W(u) = 0$ and by the Girsanov Theorem, since $\mathbb{E}\{M_t\} = 1$, (2.13) becomes

$$s(p;x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[\frac{p}{2} \int_0^t \sum_{i=1}^m \langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle ds$$
$$+ \frac{p^2}{2} \int_0^t \sum_{i=1}^m |\operatorname{div} X_i(\xi_s(\omega)(x))|^2 ds]\} .$$

Taking divergences in the first integral and by the proof of Theorem 2.1.2 this becomes

$$s(p;x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[-\frac{p}{2} \int_0^t (\operatorname{trace} \alpha_{\xi_s(\omega)(x)})^2 ds$$
$$+ \frac{p^2}{2} \int_0^t (\operatorname{trace} \alpha_{\xi_s(\omega)(x)})^2 ds]\}$$
$$= \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[-\frac{pn^2}{2} \int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds$$
$$+ \frac{p^2 n^2}{2} \int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds]\} \qquad (2.15)$$

as required. □

Lemma 2.2.4

For the gradient Brownian system (2.1) consider the perturbed operator

$$A_p = \tfrac{1}{2}\Delta - \tfrac{1}{2}\, pn^2 |\underline{H}(\cdot)|^2 + \frac{p^2n^2}{2}|\underline{H}(\cdot)|^2 \tag{2.16}$$

then A_p is for each $p \in \mathbb{R}$ the generator of a strongly continuous semigroup on $C(M;\mathbb{R})$

$$S_t(p)f(y) = \mathbb{E}_y\{(\exp[-\frac{pn^2}{2}\int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds + \frac{p^2n^2}{2}\int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds])f(y)\} \tag{2.17}$$

Proof

The generator of the gradient Brownian system (2.1) is $A = \tfrac{1}{2}\Delta$. Since M is compact $n^2|\underline{H}(x)|^2$ is bounded $\forall\, x \in M$. The perturbation of a generator by a bounded operator $(-\frac{pn^2}{2}|\underline{H}(x)|^2 + \frac{p^2n^2}{2}|\underline{H}(x)|^2)$ is again a generator and (2.17) is just the Feynman-Kac formula (see e.g. [32] pg.259). □

Remark 2.2.5

The perturbed operator A_p is self-adjoint over $L^2(M;\mathbb{R})$.
We now use the following standard result in analytic perturbation theory of linear operators on Hilbert space, following this preliminary lemma:

Lemma 2.2.6

The leading eigenvalue of the perturbed operator A_p, (2.16), is given by

$$\mu(p) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[p(p-1)\frac{n^2}{2} \int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds]\} . \qquad (2.18)$$

Proof

Consider the eigenfunction $h : M \to \mathbb{R}(>0)$ such that $A_p h = \mu(p)h$ and the modified Doob-h-transform semigroup $\{S_t^h \; ; \; t > 0\}$

$$S_t^h f(x) = \frac{1}{h(x)} \; e^{\mu(p)t} \, S_t(hf)(x)$$

and its differential generator A_p^h where

$$A_p^h(f(x)) = \frac{1}{h(x)} \; A_p(hf)(x) - \mu f(x)$$

with diffusion $\{\xi_t^h(\omega)(x) \; ; \; t \geq 0\}$ (see for example [34]). Then

$$A_p^h f(x) = A_o f(x) + \langle \nabla \log h(x), \nabla f(x) \rangle \quad .$$

Considering this set up we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[p(p-1) \frac{n^2}{2} \int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds]\}$$

$$= \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp(-\tfrac{1}{2} \int_0^t \Delta \log h(\xi_s(\omega)(x))ds) .$$

$$\cdot \exp\left(\tfrac{1}{2}\int_0^t \Delta \log\, h(\xi_s(\omega)(x))ds + p(p-1)\,\frac{n^2}{2}\int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds\right)\}$$

$$= \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp(-\tfrac{1}{2}\int_0^t \Delta \log\, h(\xi_s^h(\omega)(x))ds - \tfrac{1}{2}\int_0^t |\nabla \log\, h(\xi_s^h(\omega)(x))|^2 ds).e^{\mu(p)t}\}$$

which after a simplification using Itô's formula for $\log h(z_t)$ gives

$$= \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\frac{h(x)}{h(\xi_t^h(\omega)(x))} \cdot M_t \cdot e^{\mu(p)t}\} \tag{2.19}$$

where

$$M_t = \exp\{\int_0^t \nabla \log\, h(\xi_s^h(\omega)(x))dB_s - \tfrac{1}{2}\int_0^t |\nabla \log\, h(\xi_s^h(\omega)(x))|^2 ds\} \quad .$$

Thus by the Girsanov Theorem (see [32] pg.258) and the fact that on compact M, h is uniformly bounded, i.e. $0 < K_1 \le h(x) \le K_2$ for all $x \in M$, (2.19) yields

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{K_1}{K_2} + \mu(p) \le \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\{\exp[p(p-1)\frac{n^2}{2}\int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds]\}$$

$$\le \lim_{t\to\infty} \frac{1}{t} \log \frac{K_2}{K_1} + \mu(p)$$

and since K_1, K_2 are constant the result follows. □

<u>Theorem</u> 2.2.7 (Kato - Rellich)

Suppose that $A(p)$ is a bounded operator defined on a Hilbert space H which is a power series in p

$$A(p) = A_0 + pA_1 + p^2A_2 + \ldots\ldots$$

convergent in a neighbourhood of $p = 0$. Suppose that for real p , p small, $A(p)$ is self-adjoint on H . Suppose that μ is an eigenvalue of finite multiplicity h of the operator $A(0) = A_0 = A$ and suppose there exist positive numbers d_1, d_2 such that the spectrum of A in the open interval $\mu - d_1 < k < \mu + d_2$ consists exactly of the point eigenvalue μ (some $k \in \mathbb{R}$) . Then there exist ordinary power series

$$\mu_1(p) , \ldots , \mu_h(p)$$

and power series in Hilbert-Space

$$\phi_1(p) , \ldots , \phi_h(p)$$

all convergent in a neighbourhood of $p = 0$ which satisfy the condition: The element $\phi_i(p)$ is an eigenelement of $A(p)$ belonging to the eigenvalue $\mu_i(p)$, i.e.

$$A(p)\phi_i(p) = \mu_i(p)\phi_i(p) \qquad 1 \le i \le h .$$

Furthermore $\mu_i(0) = \mu$ $(1 \le i \le h)$; and for real p the eigenelements form an orthonormal set, i.e. for real p

$$\langle\phi_i(p), \phi_j(p)\rangle_H = \delta_{ij} \qquad 1 \le i,j \le h .$$

Proof

See Rellich [54],pg.57 → 64. □

Using this result we have the following.

Theorem 2.2.8

For the underlying gradient Brownian system (2.1) consider the perturbed self-adjoint operator A_p on $L^2(M;\mathbb{R})$ given in Lemma 2.2.4, A_p has lowest eigenvalue $\mu(p)$ of multiplicity one and corresponding eigenfunction $\phi(p)$. Both $\mu(p)$ and $\phi(p)$ have convergent power series representations in a neighbourhood of $p = 0$ and

$$\mu'(0) = \lambda_\Sigma \quad .$$

In particular $\mu(p)$ and $s(p;x)$ coincide and $s(p)$ is independent of $x \in M$.

Proof

Since $A = A_0 = \frac{1}{2}\Delta$ is self-adjoint on $L^2(M;\mathbb{R})$ (see for example [50]) and has eigenvalue $\mu_0 (=0)$ of multiplicity 1, by the Kato-Rellich Theorem 2.2.7 there exist power series

$$\mu(p) = \mu_0 + p\mu_1 + p^2\mu_2 + \ldots. \tag{2.20}$$

and

$$\phi(p) = \phi_0 + p\phi_1 + p^2\phi_2 + \ldots.. \tag{2.21}$$

convergent in a neighbourhood of $p = 0$ which satisfy

$$A(p)\phi(p) = \mu(p)\phi(p) \ . \tag{2.22}$$

Substituting in for these power series in (2.22) yields on the left hand side

$$\begin{aligned}(A_0 + pA_1 + p^2A_2)(\phi_0 + p\phi_1 + p^2\phi_2 + \ldots.) &= A_0\phi_0 + p(A_0\phi_1 + A_1\phi_0) \\ &\quad + p^2(A_2\phi_0 + A_1\phi_1 + A_0\phi_2) \\ &\quad + \ldots\ldots \end{aligned} \tag{2.23}$$

and on the right hand side

$$(\mu_0 + p\mu_1 + p^2\mu_2 + \ldots)(\phi_0 + p\phi_1 + p^2\phi_2 + \ldots.)$$

$$= \mu_0\phi_0 + p(\mu_1\phi_0 + \mu_0\phi_1) + p^2(\mu_2\phi_0 + \mu_1\phi_1 + \mu_0\phi_2) + \ldots.$$

Equating coefficients of powers of p up to p^2 gives

$$A_0\phi_0 - \mu_0\phi_0 = 0 \tag{2.24}$$

$$A_0\phi_1 - \mu_0\phi_1 = \mu_1\phi_0 - A_1\phi_0 \tag{2.25}$$

$$A_0\phi_2 - \mu_0\phi_2 = \mu_2\phi_0 - A_2\phi_0 + \mu_1\phi_1 - A_1\phi_1 . \tag{2.26}$$

Then taking the L^2 inner-product of ϕ_0 with (2.25) gives

$$\langle\phi_0, A_0\phi_1 - \mu_0\phi_1\rangle = \langle\phi_0, \mu_1\phi_0 - A_1\phi_0\rangle .$$

Since the ϕ_i are orthonormal and $\mu_0 = 0$

$$0 = \mu_1 - \langle\phi_0, A_1\phi_0\rangle .$$

In particular

$$\mu_1 = \mu'(0) = \langle \phi_0, A_1 \phi_0 \rangle$$

and since $A_1 = -\frac{n^2}{2}|\underline{H}(\cdot)|^2$, $\phi_0 = (\text{Vol } M)^{-\frac{1}{2}}$,

$$\mu'(0) = -\frac{n^2}{2(\text{Vol } M)} \int_M |\underline{H}(x)|^2 \, dx$$

$$= \lambda_\Sigma$$

by Theorem 2.1.2. We know by Proposition 2.2.2 (iv) that $s'(0;x) = \lambda_\Sigma$. The fact that $\mu(p)$ and $s(p)$ coincide follows from Lemma 2.2.6 and also Donsker and Varadhan's extensions of Kac's result for Brownian motion, (see in particular [25]).

Clearly since $\mu(p)$ and $s(p)$ coincide $s(p)$ is independent of $x \in M$. Analycity of $s(p)$ follows from the analycity of $\mu(p)$. □

Proposition 2.2.9

For a gradient Brownian system $s(p)$ has the following properties:-

(i) $s(p) = 0$ at $p = 0$ and $p = 1$, $\forall\, n$.

(ii) $s'(1) = -s'(0) = -\lambda_\Sigma$.

(iii) $s(p)$ is symmetric about the point $p = \frac{1}{2}$.

Proof

(i) From (2.15) $s(p) = 0$ if

$$\frac{n^2}{2} p(p-1) \int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds = 0 \quad \text{for all } \omega . \tag{2.27}$$

By Theorem 2.1.5 $H(\xi_s(\omega)(x)) \neq 0$ a.s.. Hence (2.27) is zero if $p = 0$ or $p = 1$.

(ii) and (iii). By Theorem 2.2.8

$$A(p)\phi(p) = s(p)\phi(p) \tag{2.28}$$

where $A(p)$, $s(p)$ and $\phi(p)$ have convergent power series expansions in a neighbourhood of $p = 0$. Differentiating (2.28) with respect to p we obtain

$$A'(p)\phi(p) + A(p)\phi'(p) = s'(p)\phi(p) + s(p)\phi'(p) \quad . \tag{2.29}$$

Taking the inner product on both sides of this with $\phi(p)$ and assuming

$$\langle\phi(p),\phi(p)\rangle = 1 \tag{2.30}$$

gives

$$\langle\phi(p),A'(p)\phi(p)\rangle + \langle\phi(p),A(p)\phi'(p)\rangle$$
$$= s'(p)\langle\phi(p),\phi(p)\rangle + s(p)\langle\phi(p),\phi'(p)\rangle \ .$$

Then by (2.23) since $A_2 = -A_1$ we obtain

$$(1-2p)\langle\phi(p),A_1\phi(p)\rangle + \langle\phi(p),A(p)\phi'(p)\rangle$$
$$= s'(p) + s(p)\langle\phi(p),\phi'(p)\rangle$$

and since $A(p)$ is self adjoint and satisfies (2.28) we have

$$s'(p) = (1-2p)\langle\phi(p),A_1\phi(p)\rangle \quad .$$

Thus with this general formula for $s'(p)$

at $p = 0$ $\quad s'(0) = \langle \phi_0, A_1\phi_0 \rangle = \lambda_\Sigma$

at $p = 1$ $\quad s'(1) = -\langle \phi_0, A_1\phi_0 \rangle = -\lambda_\Sigma$ $\quad$ (since $\phi(0) = \phi(1)$)

at $p = \frac{1}{2}$ $\quad s'(\frac{1}{2}) = 0$.

Also

$$A(p) = A_0 + [(p-\tfrac{1}{2})^2 - \tfrac{1}{4}]A_2$$

thus $A(p)$ and hence $s(p)$ are symmetric about $p = \frac{1}{2}$. $\square$

Remark 2.2.10

We note here that the proof of (ii) and (iii) above was obtained independently by L. Arnold.

We now use the following to obtain a Central Limit theorem for λ_Σ and in particular to obtain further information on the expansion of $s(p)$ at $p = 0$.

Theorem 2.2.11

For a smooth function f on a compact manifold M such that $\mathbb{E}_\rho\{f\} = \int_M f d\rho = 0$ and $f \in L^2(M,\rho)$, the equation

$$A_0 u = f$$

has a solution $A_0^{-1} f = u \in L^2(M,\rho)$ and

$$\frac{1}{\sqrt{t}} \int_0^t f(\xi_s(\omega)(x))ds$$

converges in distribution as $t \to \infty$ to a Gaussian random variable with mean 0 and variance σ^2 given by

$$\sigma^2(f) = -2\langle f,u\rangle = -2\langle f,A_0^{-1}f\rangle = -2\int_M fu\, d\rho \quad .$$

Proof

See Bhattacharya [8]. □

From (2.16) in our case $A_0 = \frac{1}{2}\Delta$.

Theorem 2.2.12

For a gradient Brownian system on M as $t \to \infty$

$$\frac{1}{\sqrt{t}}\int_0^t (A_1 - \lambda_\Sigma)1.\, ds \quad \Rightarrow \quad N(0,\sigma^2)$$

in distribution, where $\sigma^2 = -2\langle (A_1-\lambda_\Sigma)\phi_0, (\frac{1}{2}\Delta)^{-1}(A_1 - \lambda_\Sigma)\phi_0\rangle$ and

$$s''(0) = \sigma^2 - 2\lambda_\Sigma \quad .$$

Proof

Assuming that $\phi(p)$ has norm one, from (2.21) we obtain

$$\langle\phi_0,\phi_1\rangle = 0 \; . \tag{2.31}$$

From the expansion (2.20) for $s(p)$ $(= \mu(p))$ we have that

$$s''(0) = 2\mu_2$$

which, by taking the inner product of ϕ_0 with (2.26), using (2.31) and the fact that $A_2 = -A_1$, gives

$$s''(0) = 2(-\langle\phi_0, A_1\phi_0\rangle + \langle\phi_0, A_1\phi_1\rangle)$$

$$= 2(\langle\phi_0, A_1\phi_1\rangle - \lambda_\Sigma) \quad . \tag{2.32}$$

But by (2.25) since $\mu_0 = 0$, $\mu_1 = \lambda_\Sigma$

$$\phi_1 = (\tfrac{1}{2}\Delta)^{-1}\ (-A_1 + \lambda_\Sigma)\phi_0 \quad .$$

As A_1 is self adjoint

$$s''(0) = -2(\langle A_1\phi_0, (\tfrac{1}{2}\Delta)^{-1}(A_1-\lambda_\Sigma)\phi_0\rangle + \lambda_\Sigma\langle\phi_0,\phi_0\rangle)$$

$$= -2(\langle A_1\phi_0, (\tfrac{1}{2}\Delta)^{-1}(A_1-\lambda_\Sigma)\phi_0\rangle - \lambda_\Sigma\langle\phi_0,\phi_1\rangle + \lambda_\Sigma\langle\phi_0,\phi_1\rangle$$

$$+ \lambda_\Sigma\langle\phi_0,\phi_0\rangle)$$

$$= -2(\langle(A_1-\lambda_\Sigma)\phi_0, (\tfrac{1}{2}\Delta)^{-1}(A_1-\lambda_\Sigma)\phi_0\rangle + \lambda_\Sigma) \quad .$$

So

$$s''(0) = \sigma^2 - 2\lambda_\Sigma$$

as required. □

Remark 2.2.13

We note that Theorem 2.2.12 can also be rephrased in a slightly different context.

Let $\theta = A_1 - \lambda_\Sigma$ and assume that $\phi_0 \equiv 1$. Then $G = -(\tfrac{1}{2}\Delta)^{-1}$ is the Greens function for $A_0^{-1} = (\tfrac{1}{2}\Delta)^{-1}$ on M and

$$Gf(x) = \int_M g(x,y)f(y)dy$$

where $g(\cdot,\cdot)$ denotes the Greens kernel (see e.g. Brosamler, [11]). Then Theorem 2.1.12 can be expressed as:

Theorem 2.2.12 (a)

For a gradient Brownian system on M, as $t \to \infty$

$$\frac{1}{\sqrt{t}} \int_0^t \theta(x_s)ds \quad \Rightarrow \quad N(0,\sigma^2)$$

in distribution, where

$$\sigma^2 = 2\langle\theta(x),\, G\theta(x)\rangle$$

$$= 2 \int_{M\times M} \theta(x)g(x,y)\theta(y)dxdy$$

and

$$s''(0) = \sigma^2 - 2\lambda_\Sigma \quad .$$

Remark 2.2.14

(i) We note that Theorem 2.2.12 was also proved simultaneously, independently and by a different method by L. Arnold in a private communication.

(ii) This result shows that

$$s(p) = \lambda_\Sigma p + \left(\frac{\sigma^2}{2} - \lambda_\Sigma\right)p^2 + 0(p^3) \qquad \text{as} \quad p \to 0 \ .$$

(iii) By the symmetry of $s(p)$

$$s''(1) = -\sigma^2 + 2\lambda_\Sigma \quad .$$

Hence

$$s(p) = -\lambda_\Sigma p + \left(-\frac{\sigma^2}{2} + \lambda_\Sigma\right)p^2 + 0(p^3) \qquad \text{as} \quad p \to 1 \ .$$

We therefore have a complete description of $s(p)$ up to order 2 near the origin for a gradient Brownian flow. We now consider its asymptotic behaviour as p tends to infinity.

D. Elworthy observed the similarity between the operator (2.16) and that considered by B. Simon in [57]. We have the following:

Lemma 2.2.15

The perturbed operator (2.16) of Lemma 2.2.4 can be written as an operator of the form

$$-\Delta + pg(\cdot) + p^2 h(\cdot) \tag{2.33}$$

where (i) $g(\cdot)$ and $h(\cdot)$ are C^∞ ,

(ii) $g(\cdot)$ is bounded below and $h(x) \geq 0 \quad \forall\, x \in M$.

Proof

For the operator (2.16) we have that

$$\tfrac{1}{2}\{\Delta f - pn^2|\underline{H}(\cdot)|^2 f + p^2 n^2|\underline{H}(\cdot)|^2 f\} = s(p)f \tag{2.34}$$

for some eigenfunction $f \in L^2(M;\mathbb{R})$ corresponding to the leading eigenvalue $s(p)$. From both sides of (2.34) subtract the operator $p^2\beta$ where $\beta = \max_M\{n^2|\underline{H}(\cdot)|^2\}$. This yields

$$\{-\Delta + pn^2|\underline{H}(\cdot)|^2 + p^2(\beta - n^2|\underline{H}(\cdot)|^2)\}f = (p^2\beta - 2s(p))f \tag{2.35}$$

which is just (2.33) with

$$g(x) = n^2|\underline{H}(x)|^2 \quad \text{and} \quad h(x) = \beta - n^2|\underline{H}(x)|^2 .$$

For (i) g and h are clearly C^∞ .

For (ii) g is bounded below by zero and $h \geq 0$ by the definition of β . □

The operator (2.33) is that considered in [57], (i) and (ii) of Lemma 2.2.15 are two of the four hypotheses placed on the functions g and h by Simon. In order to satisfy the remaining hypotheses we now make the following assumption on the geometry of the manifold M .

Assumption 2.2.16

We now assume that the global maxima of the length of the mean curvature vector $\underline{H}(\cdot)$ of the compact manifold M are non-degenerate.

Lemma 2.2.17

For g,h defined as in Lemma 2.2.15 under the Assumption 2.2.16 we have that

(i) h has a finite number of zeros $\{x^{(a)}\}_{a=1}^k$,

(ii) at each zero $x^{(a)}$ the matrix

$$A_{ij}^{(a)} = \frac{1}{2} \frac{\partial^2 h}{\partial x_i \partial x_j} (x^{(a)})$$

is strictly positive definite.

Proof

(i) Follows from Milnor [47] page 8.

(ii) Clear by assumption 2.2.16. □

We then have the following result concerning the asymptotic behaviour of $\frac{s(p)}{p}$ as $p \to \infty$.

Theorem 2.2.18

For the gradient Brownian flow on an embedded compact manifold M whose absolute maxima of the mean curvature vector $\underline{H}(\cdot)$ are non-degenerate we have that

$$\lim_{p\to\infty} \frac{(p^2\beta - 2s(p))}{p} = e_1$$

or

$$s(p) = \tfrac{1}{2} p^2\beta - \tfrac{1}{2} pe_1 + o(p)$$

as $p \to \infty$, where e_1 is the leading eigenvalue of the harmonic oscillator $\bigoplus_{a=1}^{k} K^a$ where each K^a acts on $L^2(\mathbb{R}^n)$ and is given by

$$K^a = -\Delta + g(x^{(a)}) + \sum_{i,j} A_{ij}^{(a)} x_i x_j .$$

So

$$e_1 = \inf \bigcup_{a=1}^{k} \{g(x^{(a)}) + \sum_{i=1}^{n} (2n_i+1)\omega_i^{(a)} ; n_i = 0,1,2,\ldots\}$$

where $(\omega_i^{(a)})^2$ are the eigenvalues of $A^{(a)}$.

Proof

By Lemma 2.2.15 and Lemma 2.2.17 and under the assumption on the mean curvature vector $\underline{H}(\cdot)$ of M, the four hypotheses given in Simon [57] are satisfied. The theorem is then just a consequence of Simon's Theorem 1.1, the proof of which is given in [57]. $\square$

Example 2.2.19

Consider S^1 embedded in $\mathbb{R}^2$ as an ellipse. The embedding is defined by

$$f(\theta) = (a\sin\theta, b\cos\theta) \qquad a > b .$$

Then

$$H(\theta) = ab(a^2\cos^2\theta + b^2\sin^2\theta)^{-3/2}$$

and this has absolute maxima at $\theta = \pi/2$ and $\theta = 3\pi/2$. At these points

$$\beta = \frac{a^2}{b^4} \text{ and } A_{ij}^{(\pi/2,3\pi/2)} = \frac{3a^2(a^2-b^2)}{b^6} .$$

So

$$K^{\pi/2,3\pi/2} = -\frac{d^2}{d\theta^2} + \frac{a^2}{b^4} + \frac{3a^2(a^2-b^2)\theta^2}{b^6}$$

which has eigenvalues

$$\frac{a^2}{b^4} + (2n+1)\frac{a}{b^3}\sqrt{3(a^2-b^2)} \qquad n = 0,1,2,\ldots .$$

Then by Theorem 2.2.18 we have that

$$\lim_{p\to\infty} \frac{(p^2a^2/b^4 - 2s(p))}{p} = \frac{a^2}{b^4} + \frac{a}{b^3}\sqrt{3(a^2-b^2)}$$

$$= \frac{a^2}{b^3}\left[\frac{1}{b} + \sqrt{3}e\right]$$

where e denotes the eccentricity of the ellipse.

Remark 2.2.20

Subsequent to [20], Baxendale and Stroock, [7], have provided results on $s(p)$ for more general processes.

2.3 A Result of Bougerol and Counterexamples for Surfaces of Higher Codimension

In a recent paper by P. Bougerol [10] he obtains the following important result using Lie theoretic criteria.

Proposition 2.3.1

If M is a compact hypersurface in $\mathbb{R}^{n+1}$ then all of the Lyapunov exponents corresponding to the gradient Brownian flow on M are distinct unless M is a sphere.

Proof

Bougerol, [10], Proposition 7.7. □

Using this result, and the results of Carverhill [13] and Baxendale [4] it may be possible to completely determine the Lyapunov spectrum of a compact hypersurface. Indeed in [6], Baxendale has obtained the complete stable manifold structure for an n-sphere embedded in $\mathbb{R}^{n+1}$, the case where all the Lyapunov exponents are equal.

The following result, however, shows that Bougerol's result does not extend to manifolds of higher codimension. We also furnish an example of a manifold with codimension 2 for which all the exponents are equal, and extend this to manifolds of higher codimension.

Theorem 2.3.2

If a compact manifold M is embedded isometrically in $\mathbb{R}^m$ $(m>n)$ and the second fundamental form of M, represented in terms of local co-ordinates by the matrix (h_{ij}), is diagonal with $h_{11} = h_{22} = \dots = h_{nn}$ then all of the Lyapunov exponents associated with the gradient Brownian flow on M are equal.

Proof

By D. Elworthy's reformulation of Carverhill's formula for the top Lyapunov exponent we have by (1.16) and [16] (equation (12)) that

$$\lambda^1 = \lim_{t\to\infty} \frac{1}{t}\int_0^t \{|\alpha_x(n_s,\cdot)|^2 - |\alpha_x(n_s,n_s)|^2 - \tfrac{1}{2}\langle\alpha_x(n_s,n_s),\text{trace }\alpha_x\rangle\}ds$$

where $||n_s|| = 1$. If the second fundamental form α_x is represented in co-ordinates by the matrix (h_{ij}) and this satisfies the conditions given above then

$$\lambda^1 = \lim_{t\to\infty} \frac{1}{t}\int_0^t \{h_{11}^2 \sup_{||\omega_s||=1} |\langle n_s,\omega_s\rangle|^2 - h_{11}^2 - \frac{n}{2} h_{11}^2\}(m-n)\, ds$$

Using the Cauchy-Schwarz inequality we have

$$\lambda^1 \le \lim_{t\to\infty} \frac{1}{t} \int_0^t h_{11}^2 \{ \sup_{||\omega_s||=1} ||\eta_s||^2 ||\omega_s||^2 - (1 + \frac{n}{2}) \} (m-n)\, ds$$

$$= \lim_{t\to\infty} \frac{1}{t} \int_0^t - \frac{nh_{11}^2 (m-n)}{2} ds$$

which by the Strong Law of Large Numbers yields

$$\lambda^1 \le \int_M - \frac{nh_{11}^2 (m-n)}{2} \frac{dx}{(\mathrm{Vol}\ M)} = - \frac{n}{2(\mathrm{Vol}\ M)} \int_M |\underline{H}(x)|^2 dx \quad .$$

So

$$\lambda^1 \le \frac{1}{n} \lambda_\Sigma$$

and as λ^1 is the largest exponent we must have $n\lambda^1 = \lambda_\Sigma$ as required. $\square$

Example 2.3.3

Consider the Clifford Torus in $\mathbb{R}^4$ given by the isometric embedding $f : S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \to \mathbb{R}^4$ defined by

$$f(u,v) = (1/\sqrt{2} \cos \sqrt{2}u,\ 1/\sqrt{2} \sin \sqrt{2}u,\ 1/\sqrt{2} \cos \sqrt{2}v,\ 1/\sqrt{2} \sin \sqrt{2}v) \ . \tag{2.36}$$

By example 2.1.10 we know that for this embedding $\lambda_\Sigma = -2$. We shall now calculate the second fundamental form. The first partial derivatives are given by

$$f_u = (-\sin \sqrt{2}u,\ \cos \sqrt{2}u,\ 0,\ 0)$$

$$f_v = (0,0,\ - \sin \sqrt{2}v,\ \cos \sqrt{2}v) \ .$$

Hence the elements of the first fundamental form are $E = G = 1$, $F = 0$ and the volume element is given by

$$ds^2 = du^2 + dv^2 = F'(du^2+dv^2) .$$

The second derivatives are given by

$$f_{uu} = (-\sqrt{2}\cos\sqrt{2}u, -\sqrt{2}\sin\sqrt{2}u, 0, 0)$$

$$f_{uv} = (0, 0, 0, 0)$$

$$f_{vv} = (0, 0, -\sqrt{2}\cos\sqrt{2}v, -\sqrt{2}\sin\sqrt{2}v) .$$

The elements of the second fundamental form are given by, (see [9])

$$\begin{aligned} h_{ij} &= \frac{1}{F'} \quad f \wedge f_u \wedge f_v \wedge f_{ij} \\ &= \frac{1}{F'} \begin{vmatrix} f \\ f_u \\ f_v \\ f_{ij} \end{vmatrix} \end{aligned} \tag{2.37}$$

where f_{ij} denotes the second partial derivative with respect to the variables i and j and (2.37) is the determinant. So, as $F' = 1$

$$h_{11} = \begin{vmatrix} 1/\sqrt{2}\cos\sqrt{2}u & 1/\sqrt{2}\sin\sqrt{2}u & 1/\sqrt{2}\cos\sqrt{2}v & 1/\sqrt{2}\sin\sqrt{2}v \\ -\sin\sqrt{2}u & \cos\sqrt{2}u & 0 & 0 \\ 0 & 0 & -\sin\sqrt{2}v & \cos\sqrt{2}v \\ -\sqrt{2}\cos\sqrt{2}u & -\sqrt{2}\sin\sqrt{2}u & 0 & 0 \end{vmatrix}$$

and by rearrangement within the determinant

$$h_{11} = \begin{vmatrix} -\cos\sqrt{2}u & -\sin\sqrt{2}u & 0 & 0 \\ -\sin\sqrt{2}u & \cos\sqrt{2}u & 0 & 0 \\ 0 & 0 & -\sin\sqrt{2}v & \cos\sqrt{2}v \\ 0 & 0 & \cos\sqrt{2}v & \sin\sqrt{2}v \end{vmatrix}$$

so

$$h_{11} = 1 .$$

Clearly $h_{12} = h_{21} = 0$ since $f_{uv} = (0,0,0,0)$ and $h_{22} = 1$ as it is identical in form to h_{11} after rearrangement. Thus for this embedding the second fundamental form satisfies the conditions of Theorem 2.3.2 and hence all the Lyapunov exponents associated with the gradient Brownian flow on the Clifford Torus are equal. Since $\lambda_\Sigma = -2$ we therefore have $\lambda^1 = \lambda^2 = -1$.

This clearly extends to embeddings of the form

$$f : \underbrace{S^1(r) \times S^1(r) \times \ldots\ldots \times S^1(r)}_{n} \rightarrow \mathbb{R}^{2n} \tag{2.38}$$

defined by

$$f(x_1,\ldots\ldots,x_n) = (r\cos x_1, r\sin x_1, \ldots\ldots\ldots\ldots, r\cos x_n, r\sin x_n) .$$

As above, for such an embedding the elements of the second fundamental form are given by $h_{11} = h_{22} = \ldots\ldots = h_{nn} =$ constant and $h_{ij} = 0$, $i \neq j$, $1 \leq i,j \leq n$. Thus by Theorem 2.3.2 for such an embedding $\lambda_\Sigma = n\lambda^1$.

Remark 2.3.4

The gradient stochastic system arising from the isometric embedding

f defined by (2.38) is given by

$$d(x_t^1,\ldots,x_t^n) = \sum_{i=1}^{2n} X_i(x_t^1,\ldots,x_t^n) \circ dB_t^i$$

where $B_t \in BM(\mathbb{R}^{2n})$ and the vector fields X_i, $1 \le i \le 2n$, are given by

$$X_1 = (-1/r \sin x_1, 0,\ldots\ldots,0) \;,\quad X_2 = (1/r \cos x_1, 0,\ldots\ldots,0)$$

$$\ldots\ldots \qquad , \qquad \ldots\ldots$$

$$X_{2n-1} = (0,0,\ldots\ldots,-1/r \sin x_n) \;,\; X_{2n} = (0,0,\ldots\ldots,1/r \cos x_n) \;.$$

Let L denote the Lie subalgebra of $C^\infty(TM)$ generated by $X_1,\ldots,X_{2n}$. Then if we define

$$L(x) = \{X(x) \;;\; X \in L\} \subset T_xM$$

and

$$M_x = \{\nabla X(x) - \tfrac{1}{n}\mathrm{tr}(\nabla X(x))I \;;\; X \in L \;,\; X(x) = 0\}$$

it is clear from the form of $X_1,\ldots,X_{2n}$ above that $L \equiv L(x) = T_xM$ $\forall\, x \in M$ and $\mathbf{M}_x$ acts irreducibly on T_xM. Thus by Baxendale [5], Theorem 7.5, condition (A), we have that $n\lambda^1 = \lambda_\Sigma$, that is all the exponents for this process are equal.

2.4 Conditions for Negativity of the top Lyapunov Exponent for the Gradient Brownian Flow on a Compact Convex Hypersurface

Let M be an n-dimensional compact manifold isometrically embedded in $\mathbb{R}^{n+1}$. Throughout this section our main assumption is that M is uniformly convex. This is so if all the eigenvalues of the second fundamental form (h_{ij}) of M are strictly positive, that is there is some $\varepsilon > 0$ such that the inequality

$$h_{ij} \geq \varepsilon H g_{ij} \tag{2.39}$$

holds everywhere on M (see for example [37]). Here H again denotes the mean curvature and (g_{ij}) is the metric on M induced by the embedding.

We shall obtain a condition on ε in (2.39) such that the top Lyapunov exponent associated with the gradient Brownian flow on M is negative. This condition is obtained directly from Elworthy's reformulation of Carverhill's formula for the top exponent.

Theorem 2.4.1

If M is embedded isometrically in $\mathbb{R}^{n+1}$ $(n \geq 2)$ as a uniformly convex hypersurface and $\varepsilon > \frac{1}{2}$ then the top Lyapunov exponent associated with the gradient Brownian flow on M is strictly negative.

Proof

By Elworthy's reformulation of Carverhill's formula for the top Lyapunov exponent associated with the gradient Brownian flow on M (1.16), we have

$$\lambda^1 = \lim_{t\to\infty}\frac{1}{t}\left\{\int_0^t \tfrac{1}{2}|\alpha_x(n_s,\cdot)|^2 ds - \int_0^t |\alpha_x(n_s,n_s)|^2 ds \right.$$
$$\left. - \tfrac{1}{2}\int_0^t \mathrm{Ric}(n_s,n_s)ds\right\} \tag{2.40}$$

where α denotes the second fundamental form

$$\alpha_x : T_xM \times T_xM \to T_xM^{\perp} \subset \mathbb{R}^{n+1} \quad ,$$

$Ric(\cdot,\cdot)$ denotes the Ricci tensor and η_s is the sphere bundle process $\eta_s = v_s/|v_s|$. Note that $T_xM^{\perp}$ is one-dimensional in this particular situation. Since

$$Ric(v,v) = -|\alpha_x(v,\cdot)|^2 + \langle\alpha_x(v,v) , nH_x\rangle \qquad v \in T_xM$$

substituting for $|\alpha_x(\eta_s,\cdot)|$ in (2.40) gives

$$\lambda^1 = \lim_{t\to\infty} \frac{1}{t}\int_0^t \{-Ric(\eta_s,\eta_s)-|\alpha_x(\eta_s,\eta_s)|^2 + \tfrac{1}{2}\langle\alpha_x(\eta_s,\eta_s),nH_x\rangle\}ds \quad . \tag{2.41}$$

In local co-ordinates the integrand in (2.41) is given by

$$I(v) = - \left(\sum_{i,j=1}^{n} v_i R_{ij} v_j\right) - \left(\sum_{i,j=1}^{n} v_i h_{ij} v_j\right)^2 + \tfrac{1}{2} \sum_{i,j=1}^{n} v_i h_{ij} v_j . nH \tag{2.42}$$

where $\sum_{i=1}^{n} v_i^2 = 1$,

$$R_{ij} = nHh_{ij} - \sum_{\ell,k=1}^{n} h_{i\ell} g^{\ell k} h_{kj}$$

and R_{ij} is the Ricci curvature tensor. Since we are working locally we can assume that $g_{ij} = \delta_{ij}$ and

$$h_{ij} = \begin{bmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{bmatrix}$$

where $k_1,\ldots,k_n$ denote the principal curvatures in the directions $e_1,\ldots,e_n$ respectively. Hence (2.42) becomes

$$I(v) = -(\sum_{i,j=1}^{n} v_i[(k_1+\ldots+k_n)h_{ij} - (h_{i1}h_{1j}+\ldots\ldots+h_{in}h_{nj})]v_j)$$

$$- (k_1v_1^2+\ldots.+k_nv_n^2)^2 + \tfrac{1}{2}(k_1+\ldots.+k_n)(k_1v_1^2+\ldots.+k_nv_n^2) \; .$$

As $h_{ij} = k_i\delta_{ij}$, this gives

$$I(v) = -([(k_1+\ldots+k_n)k_1-k_1^2]v_1^2 +\ldots\ldots+ [(k_1+\ldots.+k_n)k_n-k_n^2]v_n^2)$$

$$-(k_1v_1^2+\ldots+k_nv_n^2)^2 + \tfrac{1}{2}(k_1+\ldots.+k_n)(k_1v_1^2+\ldots+k_nv_n^2)$$

$$= -(k_1(k_2+\ldots+k_n)v_1^2 +\ldots. + k_n(k_1+\ldots.+k_{n-1})v_n^2)$$

$$-(k_1v_1^2+\ldots\ldots+k_nv_n^2)^2 + \tfrac{1}{2}(k_1+\ldots+k_n)(k_1v_1^2+\ldots\ldots+k_nv_n^2) \; .$$

Then by the assumption of convexity we have

$$\begin{aligned} k_1 &\geq \varepsilon \, \frac{(k_1+ \ldots\ldots + k_n)}{n} \\ &\vdots \\ k_n &\geq \varepsilon \, \frac{(k_1+\ldots\ldots + k_n)}{n} \; . \end{aligned} \tag{2.43}$$

Thus

$$I(v) \leq -(n-1)\varepsilon(\frac{k_1+\ldots+k_n}{n})(\sum_{i=1}^{n} k_iv_i^2) - [(k_1v_1^2+\ldots+k_nv_n^2)$$

$$- \tfrac{1}{2}(k_1+\ldots+k_n)](\sum_{i=1}^{n} k_iv_i^2)$$

$$\leq [-(n-1)\varepsilon(\frac{k_1+\ldots+k_n}{n}) - \varepsilon(\frac{k_1+\ldots+k_n}{n}) + \tfrac{1}{2}(k_1+\ldots+k_n)](\sum_{i=1}^{n} k_i v_i^2)$$

hence

$$I(v) \leq (-\varepsilon + \tfrac{1}{2})nH\ (\sum_{i=1}^{n} k_i v_i^2)$$

and since $k_i > 0$, $1 \leq i \leq n$, $I(v)$ is strictly negative if $\varepsilon > \frac{1}{2}$. □

Example 2.4.2

Consider the ellipsoid of revolution embedded in $\mathbb{R}^3$. The embedding is given by

$$f(u,v) = (a\ \sin u\ \cos v,\ a\ \sin u\ \sin v,\ c\ \cos u) \qquad c > a > 0\ .$$

The elements of the first and second fundamental form are given by

$$E = a^2\cos^2 u + c^2\sin^2 u \qquad \ell = ac\ /\ (a^2\cos^2 u + c^2\sin^2 u)^{\frac{1}{2}}$$

$$F = 0 \qquad m = 0$$

$$G = a^2\sin^2 u \qquad n = ac\ \sin^2 u\ /\ (a^2\cos^2 u + c^2\sin^2 u)^{\frac{1}{2}}.$$

Since f defines a surface of revolution we have that the principal curvatures are given by, (see e.g. [49]),

$$k_1 = \frac{\ell}{E} = \frac{ac}{(a^2\cos^2 u + c^2\sin^2 u)^{3/2}}$$

$$k_2 = \frac{n}{G} = \frac{c}{a(a^2\cos^2 u + c^2\sin^2 u)^{1/2}}\ .$$

In Theorem 2.4.1 for the top exponent for the gradient Brownian flow to be negative we require that

$$k_1 > \frac{1}{4}(k_1 + k_2) \tag{2.44}$$

and

$$k_2 > \frac{1}{4}(k_1 + k_2) \quad . \tag{2.45}$$

So (2.44) becomes

$$\frac{ac}{(a^2\cos^2 u + c^2\sin^2 u)^{3/2}} > \frac{1}{4}\left(\frac{ac}{(a^2\cos^2 u + c^2\sin^2 u)^{3/2}} + \frac{c}{a(a^2\cos^2 u + c^2\sin^2 u)^{1/2}}\right)$$

which reduces to

$$3a^2 > a^2\cos^2 u + c^2\sin^2 u \ . \tag{2.46}$$

The maximum value that the right hand side of (2.46) can take since $c > a$ occurs when $u = \pi/2$, then we require that

$$3a^2 > c^2 \quad .$$

Also (2.45) becomes

$$\frac{c}{a(a^2\cos^2 u + c^2\sin^2 u)^{1/2}} > \frac{1}{4}\left(\frac{ac}{(a^2\cos^2 u + c^2\sin^2 u)^{3/2}} + \frac{c}{a(a^2\cos^2 u + c^2\sin^2 u)^{1/2}}\right)$$

which reduces to

$$a^2\cos^2 u + c^2\sin^2 u > \frac{a^2}{3} \tag{2.47}$$

and since the minimum value that the left hand side of (2.47) can take occurs when $u = 0$, we require that

$$a^2 > \frac{a^2}{3}$$

which is clearly always true. Thus the Theorem implies that the top exponent is strictly negative provided that the major and minor axes of lengths c and a respectively satisfy

$$3a^2 > c^2 > a^2 .$$

It is clear from this example that the criterion for negativity of the top exponent given in Theorem 2.4.1 is somewhat crude. Indeed we are able to improve on this in the two-dimensional case with the following:

Proposition 2.4.3

If M^2 is embedded isometrically in $\mathbb{R}^3$ as a uniformly convex hypersurface and the principal curvatures $k_2 \geq k_1 > 0$ satisfy

$$(7 + 4\sqrt{3})k_1 > k_2 \geq k_1$$

then the top Lyapunov exponent associated with the gradient Brownian flow on M^2 is strictly negative.

Proof

For such a two-dimensional hypersurface (2.42) becomes

$$I(v) = k_1^2v_1^2+k_2^2v_2^2 - (k_1v_1^2+k_2v_2^2)^2 - \tfrac{1}{2}(k_1+k_2)(k_1v_1^2+k_2v_2^2) \tag{2.48}$$

where $v_1^2 + v_2^2 = 1$. Assuming $k_2 \geq k_1 > 0$ then dividing through on both sides by k_1^2 , setting $k = k_2/k_1$ and substituting for v_1^2 , (2.48) becomes

$$\tilde{I}(v) = \frac{I(v)}{k_1^2} = 1 + (k^2-1)v_2^2 - (1+(k-1)v_2^2)^2 - \tfrac{1}{2}(1+k)(1+(k-1)v_2^2)$$

$$= \tfrac{1}{2}(1-k) + \tfrac{1}{2}(k+1)(k-1)v_2^2 - (1+(k-1)v_2^2)^2 \ . \qquad (2.49)$$

If $k = 1$ (i.e. $k_2 = k_1$) then $\tilde{I}(v) = -1$. Also if $v_2^2 = 0$, $\tilde{I}(v) = -\tfrac{1}{2}(1+k) < 0$ and if $v_2^2 = 1$, $\tilde{I}(v) = -\frac{k}{2}(1+k) < 0$. So we must find the maximum value of $\tilde{I}(v)$ to see when (2.49) is positive. Differentiating (2.49) with respect to v_2^2 to find the maximum we obtain

$$\frac{d}{dv_2^2}\tilde{I}(v) = (k-1)\left[\frac{k+1}{2} - 2(1 + (k-1)v_2^2)\right] \ . \qquad (2.50)$$

Assuming that we are away from the umbilic points (i.e. when $k = 1$) since $\tilde{I}(v) = -1$ at such points, then (2.50) is zero if and only if

$$v_2^2 = \frac{k-3}{4(k-1)} \ . \qquad (2.51)$$

This only makes sense if $k \geq 3$, indeed if $1 \leq k < 3$, $\tilde{I}(v)$ is negative. So for $k \geq 3$, substitute (2.51) into (2.50) to give

$$\tilde{I}(v) = \tfrac{1}{2}(1-k) + \tfrac{1}{2}(k+1)(k-1)\frac{(k-3)}{4(k-1)} - \left(1 + (k-1)\frac{(k-3)}{4(k-1)}\right)^2$$

$$= \frac{1}{16}(k^2 - 14k + 1) \ . \qquad (2.52)$$

Now $k^2-14k + 1 \geq 0$ if $k \geq 7 + 4\sqrt{3}$ (since we must have $k \geq 3$) . Thus if

$$k < 7 + 4\sqrt{3}$$

then $\tilde{I}(v)$, and hence $I(v)$, is strictly negative, hence the negativity of the top Lyapunov exponent. □

Example 2.4.4

Again consider the ellipsoid of revolution of example 2.4.2. For this example $k_2 \geq k_1$ and hence the top Lyapunov exponent is negative if

$$\frac{k_2}{k_1} < 7 + 4\sqrt{3}$$

which by the values for k_1 and k_2 given in example 2.4.2 yields

$$a^2\cos^2u + c^2\sin^2u < (7 + 4\sqrt{3})a^2 \quad .$$

The maximum value that the left hand side can obtain occurs when $u = \pi/2$, then we must have

$$c^2 < (7 + 4\sqrt{3})a^2 \quad .$$

So for negativity of the top exponent we require c and a to lie within the range

$$(7 + 4\sqrt{3})a^2 > c^2 > a^2$$

which is clearly an improvement on the range obtained in example 2.4.2.

2.5 Lyapunov Exponents for Processes with Gradient Drifts

Consider the process

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + A(x_t)dt \tag{2.53}$$

on a compact n-dimensional manifold M, where $A = \nabla f$ for some function $f : M \to \mathbb{R}$, $X_i(x) : \mathbb{R}^m \to T_x M$, $1 \le i \le m$, and $B_t = (B_t^1, \ldots, B_t^m) \in BM(\mathbb{R}^m)$. Since M is compact all derivatives of X_i $(1 \le i \le m)$ and A are bounded and there exists a solution flow of (2.53) $\xi_t(\omega) : M \to M$, $t \ge 0$ such that

$$d\xi_t(\omega)(x) = \sum_{i=1}^{m} X_i(\xi_t(\omega)(x)) \circ dB_t^i + A(\xi_t(\omega)(x))dt \tag{2.54}$$

$$\xi_0(\omega)(x) = x .$$

The following is well known:

Lemma 2.5.1

The process of system (2.54) has finite invariant measure

$$\rho(dx) = \text{const. } e^{2f(x)}dx . \tag{2.55}$$

Proof

The differential generator of (2.54) is given by

$$\tfrac{1}{2}\Delta + A(\cdot).\nabla$$

and since $A = \nabla f$ for some f the adjoint operator is given by

$$A^* = \tfrac{1}{2}\Delta - \nabla f.\nabla$$

which has solution ρ of $A^*\rho = 0$ given by $\rho = \text{const}.e^{2f(x)}$. □

Carverhill's Theorem 2.1, [13], tells us that, for a process satisfying (2.54) on a compact manifold M with invariant probability measure (2.55), a Lyapunov spectrum exists a.s.. Also since the system has an elliptic differential generator the spectrum is almost surely constant and the sum of the exponents λ_Σ is given by

$$\lambda_\Sigma = \lim_{t\to\infty} \frac{1}{t} \log|\det D\xi_t(\omega)(x)| \quad .$$

As in the case without drift, it is difficult to calculate the top exponent as this requires knowledge of the invariant measure of the process on the sphere bundle to M . However knowing the invariant measure on M enables us to look at λ_Σ .

<u>Proposition</u> 2.5.2

For the system (2.54)

$$\lambda_\Sigma = c\int_M \sum_{i=1}^m <\nabla \text{div } X_i(x), X_i(x)> e^{2f(x)}dx + c\int_M \Delta f(x)e^{2f(x)}dx$$

where $c = (\int_M e^{2f(x)}dx)^{-1}$.

<u>Proof</u>

This follows by the formula (1.18) for λ_Σ , Lemma 2.5.1 and the fact that

$$\operatorname{div} A(x) = \operatorname{div}(\nabla f(x)) = \Delta f(x) \quad . \qquad \square$$

Corollary 2.5.3

If M is isometrically embedded in some $\mathbb{R}^m$ then for system (2.54)

$$\lambda_\Sigma = -\frac{cn^2}{2}\int_M |\underline{H}(x)|^2 e^{2f(x)}dx + c\int_M \Delta f(x)e^{2f(x)}dx \ . \tag{2.56}$$

Proof

Follows from the above Proposition and Theorem 2.1.2. $\square$

Remark 2.5.4

We note here that (2.56) may also be written as

$$\lambda_\Sigma = \frac{-c}{2}\int_M ||\tau_F||^2 \, e^{2f(x)}dx + c\int_M \Delta f(x)e^{2f(x)}dx$$

where $||\tau_F||$ is the tension field of Eells and Lemaire [31]. The integrand of the first integral, that is $\frac{1}{2}||\tau_F||^2 \, e^{2f(x)}$, is called a Monge-Ampère density and in this case is also called the total tension density for isometries, (see [30] section 4, page 22). Indeed the Euler-Lagrange operator of the first integral is given by a Monge-Ampère equation of the form

$$\det|\nabla^2 F(x)| = f \tag{2.57}$$

(see for example [29] example 6). This thus provides a link between the Schrödinger operator associated with the diffusion process (2.53) and Monge-Ampère equations.

Examples 2.5.5

(i) Consider the gradient Brownian flow on the unit circle S^1 (cf. example 2.1.3 (i)) with gradient drift $A(x) = -\sin x$, so

$$dx_t = -\sin x_t \, dB_t^1 + \cos x_t \, dB_t^2 - \sin x_t \, dt.$$

Then by (2.56) the leading and only exponent for this process is given by

$$\lambda = -\frac{c}{2}\int_{S^1} 1^2 . e^{2\cos x} dx - c\int_{S^1} \cos x \, e^{2\cos x} dx$$

where $c = \left(\int_{S^1} e^{2\cos x} dx\right)^{-1}$. So

$$\lambda = -\tfrac{1}{2} - c\int_0^{2\pi} \cos x \, e^{2\cos x} dx .$$

Which by considering the Laurents series for $e^{2\cos x}$ (see for example, Duncan [27]) gives

$$\lambda = -\tfrac{1}{2} - 2\pi c\left(\sum_{j=0}^{\infty} \frac{1}{(j+1)!j!}\right)$$

which is clearly negative. Note that this is also true if $A(x) = \sin x$. We also note that for the original drift the potential function of the ground state eigenvalue problem (see section 1.4) is given by

$$V(x) = \frac{\Delta\phi_0(x)}{2\phi_0(x)} = \tfrac{1}{2}\,(\sin^2 x - \cos x) \tag{2.58}$$

where $\phi_0(x) = e^{\cos x}$.

(ii) Again consider the gradient Brownian flow on the unit circle S^1 with gradient drift given by $A(x) = x \mod(2\pi)$. Then, as above, the leading and only exponent is given by

$$\lambda = -\tfrac{1}{2} + c\int_0^{2\pi} e^{x^2}\,dx$$

where $c = (\int_0^{2\pi} e^{x^2} dx)^{-1}$. So

$$\lambda = \tfrac{1}{2}$$

and we have a positive exponent. The potential function of the ground state eigenvalue problem as in (2.58) is given by

$$V(x) = \tfrac{1}{2}(x^2+1) \quad .$$

We thus have the opposite of the harmonic oscillator on S^1 . In the harmonic oscillator case the drift $A(x) = -x \mod(2\pi)$ and $\lambda = -3/2$.

CHAPTER 3.

3.1. The Existence of Lyapunov Exponents for Stochastic Mechanical Systems on Non-Compact Manifolds (Introduction)

Throughout this chapter we shall consider M^n as a finite n-dimensional, simply connected, complete Riemannian manifold. We shall consider the following Stratonovich stochastic differential equation on M,

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + A(x_t)dt \tag{3.1}$$

where $X_i \in \mathbb{L}(\mathbb{R}^m; T_xM)$, $1 \le i \le m$, $B_t \in BM(\mathbb{R}^m)$ and A is a vector field of the form

$$A(x) = \tfrac{1}{2}\nabla \log|\phi_0(x)|^2 .$$

Here ϕ_0 is the normalised square integrable eigenfunction corresponding to the ground state eigenvalue E_0 of the Schrödinger operator

$$(-\tfrac{1}{2}\Delta + V)\phi_0 = E_0\phi_0$$

for $V : M \to \mathbb{R}$ a sufficiently regular potential such that $\nabla\phi_0 \in L^2(M,dx)$ where dx denotes the volume element of M. We assume throughout this section that (3.1) has the form:- Brownian motion + drift A , so that the Stratonovich correction term of (3.1) is zero, i.e. $\frac{1}{2}\sum_{i=1}^{m} \nabla X_i(X_i) = 0$. Thus the Itô and Stratonovich forms of (3.1) are equivalent. By this assumption the differential generator of (3.1) is given by

$$\tfrac{1}{2}\Delta + A(\cdot).\nabla \quad ,$$

so the system is non-degenerate and in fact elliptic. It also follows from section 1.4 and for example Nelson, [48], that the process x_t has a finite invariant probability measure

$$\rho(dx) = |\phi_0(x)|^2 \, dx \quad . \tag{3.2}$$

We shall call (3.1) a ground state stochastic mechanical system. We shall also assume the existence of a measurable solution flow of (3.1) denoted by $\xi_t(\omega) : M \to M$, $t \geq 0$ such that

$$d\xi_t(\omega)(x) = \sum_{i=1}^{m} X_i(\xi_t(\omega)(x)) \circ dB_t^i + A(\xi_t(\omega)(x))dt$$

$$\xi_0(\omega)(x) = x \ .$$

We shall initially consider such a stochastic mechanical system on $\mathbb{R}^n$ where the coefficient of the noise is spatially homogeneous and show that the Lyapunov exponents associated with such a system exist "naturally", mainly due to the existence of the finite invariant probability measure (3.2). We shall then extend this result to general non-compact manifolds by placing stronger conditions on the vector fields X_i, $1 \leq i \leq m$, and A and the geometric properties of the underlying manifold. In both cases we shall consider the existence of λ_Σ , the sum of the exponents, and obtain conditions for the existence of stable manifolds as given by Ruelle

[56] and the stochastic extension of Ruelle's results by Carverhill [13]. Our approach is basically that considered by Carverhill in [13].

3.2 The Existence of Lyapunov Exponents for Stochastic Mechanical Systems on $\mathbb{R}^n$.

In this section $M = \mathbb{R}^n$ and we consider the system

$$dx_t = dB_t + A(x_t)dt \qquad \text{on } \mathbb{R}^n \tag{3.3}$$

analagous to system (3.1) of section 3.1. In this case the coefficient of the noise term is spatially homogeneous. In conjunction with the stochastic mechanical interpretation of this system, the process x_t is termed as the "ground state process". Here $B_t \in BM(\mathbb{R}^n)$ and $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$A(x) = \tfrac{1}{2}\, \nabla \log|\phi_0(x)|^2 \; . \tag{3.4}$$

The invariant measure ρ for the process x_t is given by

$$\rho(dx) = |\phi_0(x)|^2 dx \tag{3.5}$$

where dx denotes the Lebesgue measure on $\mathbb{R}^n$. Assuming suitable conditions on the vector field A (for example globally Lipschitz, see Kunita [43]), there exists a stochastic flow of diffeomorphisms which is a solution of equation (3.3), that is a map $\xi_t(\omega) : \mathbb{R}^n \to \mathbb{R}^n$, $(t \geq 0)$, such that $\xi_0(\omega)(x) = x$ and

$$d\xi_t(\omega)(x) = dB_t + A(\xi_t(\omega)(x))dt \; . \tag{3.6}$$

We shall however only require that the solution flow is measurable.

We now consider the existence of the Lyapunov exponents. Despite the lack of compactness because of the stochastic mechanical nature of the system the process has finite invariant measure (3.5) which is in fact a probability measure. Before we can apply the results of Ruelle [56] and Carverhill [13] (i.e. Theorem 1.2.6 of Chapter 1) we have to check that Hypothesis 1.2.5 is satisfied for the stochastic mechanical process defined by (3.6). We consider the following:

Lemma 3.2.1

For a ground state stochastic mechanical system (3.6), for any $t \in [0,T]$ we have $\sup_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)||_{op}$ and $\sup_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)^{-1}||_{op} \in L^1(\mathbb{R}^n \times \Omega, \rho \otimes \mathbb{P})$. Here $||\cdot||_{op}$ denotes the operator norm on $GL(\mathbb{R}^n;\mathbb{R}^n)$.

Proof

We shall first show that $\sup_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)||_{op} \in L^1(\mathbb{R}^n\times\Omega,\rho \otimes \mathbb{P})$.

The derivative process associated with equation (3.6) is given by

$$d(D\xi_t(\omega)(x)v) = DA(\xi_t(\omega)(x))(D\xi_t(\omega)(x)v)dt\ , \quad v \in \mathbb{R}^n . \tag{3.7}$$

By Itô's formula

$$\log||D\xi_t(\omega)(x)v||^2 = \log||v||^2+2\int_0^t \frac{\langle D\xi_s(\omega)(x)v, DA(\xi_s(\omega)(x))(D\xi_s(\omega)(x)v)\rangle}{\langle D\xi_s(\omega)(x)v,\ D\xi_s(\omega)(x)v\rangle} ds$$

and since sup log ≡ log sup

$$\sup_{||v||=1} \log||D\xi_t(\omega)(x)v|| = \log \sup_{||v||=1} ||D\xi_t(\omega)(x)v||$$

$$= \log||D\xi_t(\omega)(x)||_{op} \quad .$$

Hence

$$\log||D\xi_t(\omega)(x)||_{op} \leq \int_0^t \sup_{||v||=1} \frac{<D\xi_s(\omega)(x)v,\ DA(\xi_s(\omega)(x))(D\xi_s(\omega)(x)v)>}{<D\xi_s(\omega)(x)v,\ D\xi_s(\omega)(x)v>} ds. \quad (3.8)$$

Now, since $\log^+ f = \max\{0,f\}$,

$$\int_{\mathbb{R}^n} \int_\Omega \sup_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)||_{op} \ d\mathbb{P}\rho(dx)$$

$$\leq \int_{\mathbb{R}^n} \int_\Omega \sup_{t\in[0,T]} |\log||D\xi_t(\omega)(x)||_{op}| \ d\mathbb{P}\rho(dx)$$

which by (3.8)

$$\leq \int_{\mathbb{R}^n}\int_\Omega \sup_{t\in[0,T]}\int_0^t \sup_{||v||=1} \left| \frac{<D\xi_s(\omega)(x)v,DA(\xi_s(\omega)(x))(D\xi_s(\omega)(x)v)>}{<D\xi_s(\omega)(x)v,\ D\xi_s(\omega)(x)v>} \right| ds \ d\mathbb{P}\rho(dx).$$

The Cauchy-Schwarz inequality yields

$$\leq \int_{\mathbb{R}^n} \int_\Omega \sup_{t\in[0,T]} \int_0^t \sup_{||v||=1} ||DA(\xi_s(\omega)(x))||ds \ d\mathbb{P}\rho(dx)$$

$$= \int_{\mathbb{R}^n} \int_\Omega \sup_{t\in[0,T]} \int_0^t ||DA(\xi_s(\omega)(x))|| \ ds \ d\mathbb{P}\rho(dx) \quad (3.9)$$

$$\leq \int_{\mathbb{R}^n} \int_\Omega \int_0^T ||DA(\xi_s(\omega)(x))|| \ ds \ d\mathbb{P}\rho(dx)$$

since the integrand in (3.9) is positive. By Fubini's Theorem and the Φ_t-invariance of Proposition 1.2.2 we have

$$\int_{\mathbb{R}^n}\int_{\Omega} \sup_{t\in[0,T]} \log^+ ||D\xi_t(\omega)(x)||_{op}\, d\mathbb{P}\rho(dx) \le T\int_{\mathbb{R}^n} ||DA(x)||\rho(dx)$$

and by (3.5) the right hand side becomes

$$= T\int_{\mathbb{R}^n} ||DA(x)||\; |\phi_0(x)|^2\, dx\ .$$

As $A(x) = \frac{1}{2}\nabla\log|\phi_0(x)|^2$

$$DA(x) = \frac{1}{|\phi_0(x)|^2}\begin{bmatrix} \phi_0 \dfrac{\partial^2\phi_0}{\partial x_1^2} - \left(\dfrac{\partial\phi_0}{\partial x_1}\right)^2 & \phi_0 \dfrac{\partial^2\phi_0}{\partial x_2\partial x_1} - \dfrac{\partial\phi_0}{\partial x_1}\cdot\dfrac{\partial\phi_0}{\partial x_2} & \cdots\cdots \\ & \ddots & \\ & & \phi_0 \dfrac{\partial^2\phi_0}{\partial x_n^2} - \left(\dfrac{\partial\phi_0}{\partial x_n}\right)^2 \end{bmatrix}$$

$$= \frac{1}{|\phi_0(x)|^2}\cdot(\hat{A}_{ij})\ , \qquad \text{say.}$$

Each entry of the matrix $\hat{A}_{ij}$ is given by

$$\phi_0\frac{\partial^2\phi_0}{\partial x_i\partial x_j} - \frac{\partial\phi_0}{\partial x_i}\cdot\frac{\partial\phi_0}{\partial x_j}\ , \qquad 1 \le i,j \le n\ .$$

If $i = j$ then $\left|\dfrac{\partial\phi_0}{\partial x_i}\right|^2$ is integrable over $\mathbb{R}^n$ by the finite energy assumption and for $i \neq j$, by the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}^n} \left|\frac{\partial\phi_0}{\partial x_i}\frac{\partial\phi_0}{\partial x_j}\right| dx \le \left(\int_{\mathbb{R}^n}\left|\frac{\partial\phi_0}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n}\left|\frac{\partial\phi_0}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} < \infty$$

again by the finite energy assumption.

Also $\phi_0\frac{\partial^2\phi_0}{\partial x_i\,\partial x_j}$, $(1 \le i,j \le n)$ is integrable over $\mathbb{R}^n$ since

$$\left\langle|\phi_0|, \left|\frac{\partial^2\phi_0}{\partial x_i\partial x_j}\right|\right\rangle_0 = \left|\left|\phi_0\cdot\frac{\partial^2\phi_0}{\partial x_i\partial x_j}\right|\right|_{L^1} \le ||\phi_0||_{+1}\left|\left|\frac{\partial^2\phi_0}{\partial x_i\partial x_j}\right|\right|_{-1} < \infty$$

where the inner product and norms are those of the Sobolev spaces $H^0(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ and $H^{-1}(\mathbb{R}^n)$ respectively. Hence, if we consider the norm

$$||\hat{A}_{ij}|| = \sum_{i,j=1}^{n} |\hat{a}_{ij}|$$

then

$$\int_{\mathbb{R}^n}\int_\Omega \sup_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)||_{op}\ dP_\rho(dx) < \infty$$

as required.

In order to consider the integrability of $\sup_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)^{-1}||_{op}$ over $\mathbb{R}^n \times \Omega$, note that by Itô's formula for the linear map $D\xi_t(\omega)(x)^{-1} \in GL(\mathbb{R}^n;\mathbb{R}^n)$

$$D\xi_t(\omega)(x)^{-1} = -\int_0^t D\xi_s(\omega)(x)^{-1}DA(\xi_s(\omega)(x))(D\xi_s(\omega)(x)^{-1})(D\xi_s(\omega)(x)\quad)ds$$

$$= -\int_0^t D\xi_s(\omega)(x)^{-1}DA(\xi_s(\omega)(x)).(Id)ds\ . \qquad (3.10)$$

Since $GL(\mathbb{R}^n;\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^{n^2}$ consider (3.10) as an equation on $\mathbb{R}^{n^2}$. Again by Itô's formula we have

$$\log||D\xi_t(\omega)(x)^{-1}||_{\mathbb{R}^{n^2}} = \int_0^t \frac{\langle D\xi_s(\omega)(x)^{-1}, D\xi_s(\omega)(x)^{-1} DA(\xi_s(\omega)(x))\rangle_{\mathbb{R}^{n^2}}}{\langle D\xi_s(\omega)(x)^{-1}, D\xi_s(\omega)(x)^{-1}\rangle_{\mathbb{R}^{n^2}}} \, ds \ .$$

Hence, by the Cauchy-Schwarz inequality

$$|\log||D\xi_t(\omega)(x)^{-1}||_{\mathbb{R}^{n^2}}| \le \int_0^t \frac{||D\xi_s(\omega)(x)^{-1} DA(\xi_s(\omega)(x))||_{\mathbb{R}^{n^2}}}{||D\xi_s(\omega)(x)^{-1}||_{\mathbb{R}^{n^2}}} \, ds$$

and since all norms on finite dimensional Euclidean space are equivalent $\exists\ \alpha,\beta > 0$ such that

$$\alpha||f||_{op} \le ||f||_{\mathbb{R}^{n^2}} \le \beta||f||_{op} \ .$$

Thus

$$|\log||D\xi_t(\omega)(x)^{-1}||_{op}| \le \int_0^t \frac{\beta}{\alpha}\, ||DA(\xi_s(\omega)(x))||\, ds \ + \ \text{const.}$$

and as above $\sup\limits_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)^{-1}||_{op} \in L^1(\mathbb{R}^n \times \Omega,\ \rho \otimes \mathbb{P})$. □

We also need the following lemma.

Lemma 3.2.2

For a stochastic mechanical system (3.6) on $\mathbb{R}^n$, for any $t \in [0,T]$, then

$$\sup_{t\in[0,T]} \log||D(\xi_{T-t}(\theta_t(\omega)))(\xi_t(\omega)(x))||_{op} \in L^1(\mathbb{R}^n\times\Omega,\rho \otimes \mathbb{P}) . \qquad (3.11)$$

Proof

Note that (3.11) is a.s. not negative. Now from (1.7) we have

$$D(\xi_{T-t}(\theta_t(\omega)))(\xi_t(\omega)(x)) = D\xi_T(\omega)(x)\circ(D\xi_t(\omega)(x)^{-1})$$

so that

$$\sup_{t\in[0,T]} \log||D(\xi_{T-t}(\theta_t(\omega)))(\xi_t(\omega)(x))||_{op}$$

$$\le \sup_{t\in[0,T]} \log^+||D\xi_T(\omega)(x)||_{op} + \sup_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)^{-1}||_{op}$$

and both terms on the right hand side are integrable over $\mathbb{R}^n \times \Omega$ by Lemma 3.2.1, hence the result. □

We are now in a position to give the following result.

Theorem 3.2.3

For a ground state stochastic mechanical system of the form (3.6) on $\mathbb{R}^n$ there exists a.s. a Lyapunov spectrum and associated filtration of $\mathbb{R}^n$.

Proof

Hypothesis 1.2.5 is satisfied by Lemma 3.2.1. The proof then follows that of Theorem 1.2.6. □

Remarks 3.2.4

(i) It is clear from Remark 1.2.7 that the solution flow $\{\xi_t(\omega)\ ;\ t \geq 0\}$ need only be measurable for the above result to hold.

(ii) Since the ground state stochastic mechanical system is elliptic by Remark 1.2.7 the Lyapunov spectrum is a.s. constant, that is the spectrum is independent of (x,ω) a.s.

The importance of this remark will become apparent when we consider the limit $\frac{1}{t}\log|\det D\xi_t(\omega)(x)|$ as $t \to \infty$ and show that this yields the sum of the exponents.

(iii) It is clear from Lemma 3.2.1 and Ruelle ([56]) that $-\infty$ does not belong to the Lyapunov spectrum for such a stochastic mechanical system.

We note that Theorem 1.2.6 can also be rephrased in the following way.

Theorem 3.2.5 (cf. Baxendale [4])

For a ground state stochastic mechanical system (3.6) on $\mathbb{R}^n$, for $\rho \otimes \mathbf{P}$ almost all $(x,\omega) \in M \times \Omega$

$$(D\xi_t(\omega)(x)^* D\xi_t(\omega)(x))^{1/2t} \to \Lambda(x,\omega) \qquad \text{as} \quad t \to \infty$$

where $\Lambda(x,\omega)$ is a random linear map on $\mathbb{R}^n$ with non-random eigenvalues

$$0 < e^{\lambda^1} \leq e^{\lambda^2} \leq \ldots. \leq e^{\lambda^n}$$

where $\lambda^1 \ldots \lambda^n$ are the Lyapunov exponents for the stochastic flow. The eigenvalues are non-random because they depend only on the remote future of the stochastic flow, whereas $\Lambda(x,\omega)$ is in general random because the eigenspaces corresponding to distinct eigenvalues depend upon the entire evolution of the stochastic flow.

Using this and Remark3.2.4(ii) we have the following corollary of Theorem 3.2.3.

Corollary 3.2.6

For a ground state stochastic mechanical system on $\mathbb{R}^n$. For $\rho \otimes \mathbb{P}$-almost all $(x,\omega) \in \mathbb{R}^n \times \Omega$ we have

$$\lim_{t\to\infty} \frac{1}{t} \log|\det D\xi_t(\omega)(x)| = \lambda^1 + \ldots.. + \lambda^n = \lambda_\Sigma \quad .$$

Proof

By Theorem 3.2.5 we have that

$$\lim_{t\to\infty} \frac{1}{t} \log|\det(D\xi_t(\omega)(x)^* D\xi_t(\omega)(x))^{\frac{1}{2}}|$$

$$= \lim_{t\to\infty} \frac{1}{t} \log|\det D\xi_t(\omega)(x)|$$

$$= \sum_{i=1}^{n} \lambda^i = \lambda_\Sigma \quad . \qquad \square$$

We now have the following result, conjectured first by Elworthy in [16].

Theorem 3.2.7

For a ground state stochastic mechanical system (3.6) on $\mathbb{R}^n$ we have

λ_Σ = - const. × (Kinetic energy of the stochastic mechanical particle).

Proof

By Corollary 3.2.6 the sum of the Lyapunov exponents

$$\lambda_\Sigma = \lim_{t\to\infty} \frac{1}{t} \log|\det D\xi_t(\omega)(x)|$$

by (1.18)

$$= \int_{\mathbb{R}^n} \operatorname{div} A(x)\rho(dx) \tag{3.12}$$

$$= \int_{\mathbb{R}^n} \frac{1}{\phi_0(x)^2}\{ \sum_{i=1}^{n} (\phi_0 \cdot \frac{\partial^2 \phi_0}{\partial x_i^2} - (\frac{\partial \phi_0}{\partial x_i})^2)\}|\phi_0(x)|^2 dx \quad .$$

Considering the first term and integrating by parts gives

$$\int_{\mathbb{R}} \phi_0(x)\cdot\frac{\partial^2\phi_0(x)}{\partial x_i^2} dx_i = [\phi_0(x) \frac{\partial\phi_0(x)}{\partial x_i}]_{-\infty}^{\infty} - \int_{\mathbb{R}} (\frac{\partial\phi_0(x)}{\partial x_i})^2 dx_i$$

and since $\phi_0 \in L^2(\mathbb{R}^n, dx)$

$$\lambda_\Sigma = - 2 \int_{\mathbb{R}^n} \sum_{i=1}^{n} (\frac{\partial\phi_0(x)}{\partial x_i})^2 dx \quad .$$

which is a negative constant × (the Kinetic energy of the stochastic mechancial particle), see e.g. [48]. □

Remark 3.2.8

That $\lim_{t\to\infty} \frac{1}{t} \log|\det D\xi_t(\omega)(x)|$ is given by the Kinetic energy of the stochastic mechanical particle was first given by Elworthy in [16], but we have actually shown that the limit considered by Elworthy is indeed the sum of the exponents. We also note that λ_Σ has the interpretation that it gives the rate of change of measure under the action of the measurable flow (see for example [21]).

Examples 3.2.9

(i) Consider the process on $\mathbb{R}$ given by

$$dx_t = dB_t - x_t^3 dt$$

where B_t is a one-dimensional Brownian motion. The process is strongly complete; since it possesses a finite invariant measure it is complete and completeness and strong completeness are equivalent in one dimension. However the flow is not surjective onto $\mathbb{R}$, (see e.g. the test given by Elworthy [33]), and hence a C^∞ flow exists but not a flow of diffeomorphisms. Thus if ξ_t denotes the flow

$$d\xi_t(\omega)(x) = dB_t - \xi_t(\omega)(x)^3 dt \; .$$

For this stochastic mechanical process $A(x) = -x^3 = \nabla(-x^4/4)$ and the invariant measure (normalised) is given by

$$\rho(dx) = \frac{e^{-x^4/2} 2^{3/4}}{\Gamma(1/4)} dx \tag{3.13}$$

where $\Gamma(\cdot)$ is the gamma function

$$\Gamma(n) = \int_0^\infty u^{n-1} e^{-u} \, du \; .$$

For this process we have

$$\int_{\mathbb{R}} ||DA(x)||\rho(dx) = \int_{-\infty}^{\infty} 2^{3/4}.3x^2 . \frac{e^{-x^4/2}}{\Gamma(1/4)}\, dx$$

$$= \frac{3\sqrt{2}\Gamma(3/4)}{\Gamma(1/4)} < \infty \tag{3.14}$$

hence, as in Lemma 3.2.1, the integrability of $\sup\limits_{t\in[0,T]} \log^+||D\xi_t(\omega)(x)^{\pm 1}||$.

Since there is only one exponent for this 1-dimensional process we have by (3.12) that

$$\lambda = \int_{-\infty}^{\infty} \operatorname{div} A(x)\rho(dx) = - \int_{-\infty}^{\infty} \frac{3x^2 e^{-x^4/2}}{\Gamma(1/4)}\, dx \quad .$$

So by (3.14)

$$\lambda = - \frac{3\sqrt{2}\Gamma(3/4)}{\Gamma(1/4)}$$

and the process is Lyapunov stable. (cf. Carverhill [13] Chapter 2).

(ii) Consider the harmonic oscillator on $\mathbb{R}^n$ given by

$$-\tfrac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n} \frac{\mu_i^2 x_1^2}{2}$$

where $\mu_n \le \mu_{n-1} \le \ldots\ldots \le \mu_1 < 0$ and $\mu_i \in \mathbb{R}$ $(1 \le i \le n)$.

Associated with this operator is the ground state process given by

$$d(x_1\ldots,x_n)_t = dB_t + (\mu_1 x_1,\ldots,\mu_n x_n)_t dt \qquad \text{on } \mathbb{R}^n . \tag{3.15}$$

For this process $A(x) = (\mu_1 x_1,\ldots\ldots,\mu_n x_n)$ is globally Lipschitz, hence the process is strongly complete (see for example Kunita [43]) and the solution flow exists for all time and is surjective. Thus take a version of the flow such that

$$d\xi_t(\omega)(x) = dB_t + A(\xi_t(\omega)(x))dt$$

and this process has invariant measure

$$\rho(dx) = \frac{1}{(|\mu_1\cdots\cdots\mu_n|)^{-\frac{1}{2}}\,(\pi)^{n/2}}\, e^{\mu_1 x_1^2 + \cdots + \mu_n x_n^2} dx \ .$$

The derivative process associated with the ground state process (3.15) is given by

$$dv_t = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix} v_t\, dt \qquad\qquad v_t \in \mathbb{R}^n \ . \tag{3.16}$$

This is just a deterministic equation with unique solution

$$v_t = \begin{bmatrix} e^{\mu_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\mu_n t} \end{bmatrix} v_0$$

given $v_0 \in \mathbb{R}^n$. Since $v_t = D\xi_t(\omega)(x)v_0$, by Theorem 3.2.5

$$(D\xi_t(\omega)(x)^* D\xi_t(\omega)(x))^{1/2t} \to \Lambda(x,\omega)$$

where in this case $\Lambda(x,\omega)$ is the matrix $\begin{bmatrix} e^{\mu_1} & & 0 \\ & \ddots & \\ 0 & & e^{\mu_n} \end{bmatrix}$ and $\mu_1,\ldots \mu_n$

are the Lyapunov exponents for the stochastic flow. We therefore also have a filtration of $\mathbb{R}^n$

$$\{0\} = V^{n+1}_{(x,\omega)} \subset V^{n}_{(x,\omega)} \cdots\cdots \subset V^{1}_{(x,\omega)} = \mathbb{R}^n \quad .$$

This is just an Ornstein-Uhlenbeck process on $\mathbb{R}^n$. In particular if $\mu_1 = \ldots\ldots = \mu_n$ (= 1 say) then all the exponents are the same and the filtration is given by

$$\{0\} = V^{2}_{(x,\omega)} \subset V^{1}_{(x,\omega)} = \mathbb{R}^n$$

i.e. the whole of $\mathbb{R}^n$. Also the sum of the exponents for (3.15) is given by

$$\lambda_\Sigma = \int_{\mathbb{R}^n} \operatorname{div} A(x)\rho(dx)$$

$$= \mu_1 + \ldots\ldots + \mu_n \quad .$$

3.3 Stable Manifold Theorems on $\mathbb{R}^n$ for Stochastic Mechanical Systems

We now consider, for a stochastic mechanical system on $\mathbb{R}^n$ of the form (3.6), the existence of stable manifolds. In this case we have to insist on the existence of the flow although it does not necessarily need to be a flow of diffeomorphisms onto $\mathbb{R}^n$. Again the work is analogous to that of Carverhill [13], Chapter 2 and his extension of the work of Ruelle [56].

Given the previous regularity estimates of Lemmas 3.2.1 and 3.2.2 we have the following:

<u>Theorem</u> 3.3.1 (Local Stable Manifold Theorem, cf. Carverhill [13], Theorem 2.2.1 and Ruelle [56], Theorem 6.1).

For a stochastic mechanical system (3.6) on $\mathbb{R}^n$ with a C^k $(k \geq 2)$ stochastic flow $\xi_t(\omega)$, $t \geq 0$, with bounded derivatives up to order k take $\bar{\lambda} < 0$ such that $\bar{\lambda}$ is a.s. disjoint from the Lyapunov spectrum. Then we have a set $\Gamma^{\bar{\lambda}} \subset \Gamma$ of full $\rho \otimes \mathbb{P}$ measure and measurable functions $\alpha, \beta, \gamma : \Gamma^{\bar{\lambda}} \to \mathbb{R}^{>0}$ such that if we denote by $\nu^{\bar{\lambda}}_{(x,\omega)}(\alpha(x,\omega))$ the set $\{y \in \bar{B}(x,\alpha(x,\omega))\ ;\ d(\xi_t(\omega)y, \xi_t(\omega)(x)) \leq \beta(x,\omega)e^{\bar{\lambda}t}$ for all $t \geq 0\}$ where $\bar{B}(x,\alpha)$ denotes the closed ball at $x \in \mathbb{R}^n$ of radius α. Then:

(a) $\nu^{\bar{\lambda}}_{(x,\omega)}(\alpha(x,\omega))$ is a C^k submanifold of $\bar{B}(x,\alpha(x,\omega))$ which is tangent at x to $V^{i+1}_{(x,\omega)}$ where i is such that $\lambda^{(i+1)}_{(x,\omega)} < \bar{\lambda} < \lambda^{(i)}_{(x,\omega)}$.

(b) If $y,z \in \nu^{\bar{\lambda}}_{(x,\omega)}(\alpha(x,\omega))$ then

$$d(\xi_t(\omega)y, \xi_t(\omega)z) \leq \gamma(x,\omega)d(y,z)e^{\bar{\lambda}t}\ .$$

(b') If $\hat{\lambda} < \bar{\lambda}$ and $[\hat{\lambda},\bar{\lambda}]$ is disjoint from the spectrum for all $(x,\omega) \in \Gamma^{\bar{\lambda}}$ then there exists a measurable map $\gamma' : \Gamma^{\bar{\lambda}} \to \mathbb{R}^{>0}$ such that if $y,z \in \nu^{\bar{\lambda}}_{(x,\omega)}(\alpha(x,\omega))$ then

$$d(\xi_t(\omega)y, \xi_t(\omega)z) \leq \gamma'(x,\omega).d(y,z)e^{\hat{\lambda}t}\ .$$

<u>Proof</u>

Exactly as in Carverhill ([13], Theorem 2.2.1). Again since we are working on $\mathbb{R}^n$ the tangent space $T\mathbb{R}^n \cong \mathbb{R}^n$ can be identified in a Borel measurable way. Also if $F_0(x,\omega) : \overline{B(T)} \to \mathbb{R}^n$, where $\overline{B(T)}$ denotes the closed unit ball centred at the origin, is defined by

$$F_0(x,\omega)y = \xi_T(\omega)(x+y) - \xi_T(\omega)(x) \tag{3.17}$$

for the discrete time interval of length T (as in Theorem 1.2.6), then the regularity condition required by Ruelle's Theorem 5.1 is

$$\int_{\mathbb{R}^n} \int_{\Omega} ||F_0(x,\omega)||_{C^k} \, d\mathbb{P} \, \rho(dx) < \infty \tag{3.18}$$

where

$$||F_0(x,\omega)||_{C^k} = \sup_{y \in \overline{B(T)}} \{ \sum_{|\alpha| \le k} || \frac{\partial^{|\alpha|}}{\partial y^\alpha} F_0(x,\omega)y|| \} .$$

Clearly since the flow is C^k all of these derivatives are bounded over $\mathbb{R}^n$ (see for example [38]) and hence (3.18) holds.

Also as in Remark 3.2.4 (iii) for such a stochastic mechanical system, by lemma 3.2.1 the Lyapunov spectrum of $G_0(x,\omega)$ (cf. $G_0(x,\omega)$ of Theorem 1.2.6) does not contain $-\infty$. Thus for discrete time increments of length T we have a version of Ruelle's Theorem 5.1.

The extension to continuous time follows as in Carverhill Theorem 2.2.1 using lemmas 3.2.1 and 3.2.2. □

<u>Theorem</u> 3.3.2 (Full Stable Manifold Theorem, cf. Carverhill, Theorem 2.2.2 and Ruelle, Theorem 6.3).

Consider the stochastic mechanical system (3.6) on $\mathbb{R}^n$ with a C^k $(k \ge 2)$

with bounded derivatives up to order k

stochastic flow $\xi_t(\omega)$, $t \geq 0$. By remark 3.2.4(ii) the Lyapunov spectrum for this system is a.s. constant. Let $\lambda^r < \ldots . < \lambda^{r-q}$ be the strictly negative Lyapunov numbers. Then there is a set Γ_1 in $\Gamma \subset \mathbb{R}^n \times \Omega$ of full measure and such that for each $(x,\omega) \in \Gamma_1$ we have:

For each $p = r-q,\ldots.,r$ the set

$$V^{(p)}_{(x,\omega)} = \{y \in \mathbb{R}^n; \overline{\lim_{t\to\infty}} \frac{1}{t} \log d(\xi_t(\omega)x,\xi_t(\omega)y) \leq \lambda^p\}$$

is the image of $V^{(p)}_{(x,\omega)}$ by a C^{k-1} immersion which is tangent to the identity at x . Thus $V^{(p)}_{(x,\omega)}$ is locally a C^{k-1} submanifold of $\mathbb{R}^n$.

Proof

As in Carverhill [13] Theorem 2.2.2. □

Examples 3.3.3

Consider the system of example 3.2.9 (ii), the Ornstein-Uhlenbeck process. Then all of the Lyapunov exponents are negative for this process and we have the cases

(i) if $\mu_1 = \mu_2 = \ldots . = \mu_n$

then the filtration is given by $\{0\} = V^2_{(x,\omega)} \subset V^1_{(x,\omega)} = \mathbb{R}^n$

and the stable manifold $V^{(1)}_{(x,\omega)}$ is the whole of $\mathbb{R}^n$;

(ii) suppose $n = 2$ and $\mu_1 > \mu_2$, then the filtration is given by

$$\{0\} = V^3_{(x,\omega)} \subset V^2_{(x,\omega)} \subset V^1_{(x,\omega)} = \mathbb{R}^2$$

and the stable manifold corresponding to the exponent μ_2 is one dimensional and that corresponding to μ_1 is the whole of $\mathbb{R}^2$.

This can be compared with the compact case ([13]) where Carverhill shows that the stable manifold is not the whole of M.

3.4 The Existence of Lyapunov Exponents for Stochastic Mechanical Systems on Non-Compact Manifolds

We now consider a stochastic mechanical system on a simply connected, complete, n-dimensional Riemannian manifold M. This system takes the form

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + A(x_t)dt \tag{3.19}$$

where $X_i(x) \in \mathbb{L}(\mathbb{R}^m;T_xM)$, $1 \le i \le m$, $B_t \in BM(\mathbb{R}^m)$ and A is a vector field of the form

$$A(x) = \tfrac{1}{2} \nabla \log|\phi_0(x)|^2 . \tag{3.20}$$

Here once again ϕ_0 is the normalised square integrable eigenfunction corresponding to the ground state of a Schrödinger operator of the form (1.19). We assume that the Stratonovich correction term $\frac{1}{2} \sum_{i=1}^{m} \nabla X_i(X_i) = 0$, so the differential generator of the ground state process (3.19) is given by

$$\tfrac{1}{2}\Delta + A(\cdot).\nabla \tag{3.21}$$

and the process x_t has invariant measure

$$\rho(dx) = |\phi_0(x)|^2 \, dx \tag{3.22}$$

where dx denotes the Riemannian volume element.

A problem when dealing with solutions of stochastic differential equations on non-compact manifolds is that of explosion of solutions, however since for all the processes that we shall consider we have a finite invariant measure (3.22) these processes will be complete i.e. non-explosive.

Consider the situation of lemma 1.2.4, for such a system we have

Lemma 3.4.1

For a ground state stochastic mechanical system (3.19) on M, if $\nabla X_i \in L^2(M,\rho(dx))$ for $1 \le i \le m$ and the form $\mathrm{Ric}-\nabla A$ is uniformly bounded over M, then for any $t \in [0,T]$

$$\sup_{t\in[0,T]} \log^+||D\tilde{\xi}_t(\omega)(x)^{\pm 1}||_{op} \in L^1(\tilde{M} \times \Omega, \rho \otimes \mathbb{P})$$

where $||\cdot||_{op}$ denotes the operator norm on $GL(\mathbb{R}^m)$.

Proof

We shall first show that $\sup_{t\in[0,T]} \log^+||D\tilde{\xi}_t(\omega)(x)||_{op} \in L^1(\tilde{M} \times \Omega, \rho \otimes \mathbb{P})$.

By Carverhill's formula (1.12) and its reformulation by Elworthy (1.14) we have for the process $\xi_t(\omega)(x)$ on M that

$$\log||T\xi_t(\omega)(x)v|| = \int_0^t \sum_{i=1}^m <\eta_s(\omega)v, \nabla X_i(\eta_s(\omega)v)>dB_s^i + \int_0^t <\eta_s(\omega)v, \nabla A(\eta_s(\omega)v)>ds$$

$$+ \tfrac{1}{2}\int_0^t \{ \sum_{i=1}^m ||\nabla X_i(\eta_s(\omega)v)||^2 - 2 \sum_{i=1}^m <\nabla X_i(\eta_s(\omega)v), \eta_s(\omega)v>^2$$

$$- \mathrm{Ric}(\eta_s(\omega)v, \eta_s(\omega)v)\}ds \ . \tag{3.23}$$

Since $\mathrm{Ric}-\nabla A$ is bounded from below, by $-C$, say, for some $C > 0$, we have

$$\log||T\xi_t(\omega)(x)v|| \le \int_0^t \sum_{i=1}^m <\eta_s(\omega)v, \nabla X_i(\eta_s(\omega)v)>dB_s^i$$

$$+ \tfrac{1}{2}\int_0^t \{ \sum_{i=1}^m ||\nabla X_i(\eta_s(\omega)v)||^2 - 2 \sum_{i=1}^m <\nabla X_i(\eta_s(\omega)v), \eta_s(\omega)v>^2 + \bar{C}\}ds$$

where $-\bar{C}$ is a lower bound for $\mathrm{Ric}-2\nabla A$. Thus

$$\int_M\int_\Omega \sup_{t\in[0,T]} \log^+||T\xi_t(\omega)(x)||_{op} \ d\mathbb{P}\rho(dx)$$

$$\le \int_M\int_\Omega \sup_{t\in[0,T]} |\log||T\xi_t(\omega)(x)||_{op}| \ d\mathbb{P}\rho(dx)$$

$$\le \int_M\int_\Omega \sup_{t\in[0,T]} [|\int_0^t \sup_{||v||=1} \sum_{i=1}^m <\eta_s(\omega)v, \nabla X_i(\eta_s(\omega)v)>dB_s^i|$$

$$+ \tfrac{1}{2}\int_0^t \sup_{||v||=1} \{ \sum_{i=1}^m |||\nabla X_i(\eta_s(\omega)v)||^2| +$$

$$+ 2 \sum_{i=1}^m |<\nabla X_i(\eta_s(\omega)v), \eta_s(\omega)v>^2| + \bar{C}\}ds] \ d\mathbb{P}\rho(dx)$$

$$\leq \int_M \beta \int_0^T \mathbb{E}\{ \sup_{||v||=1} | \sum_{i=1}^m <\eta_s(\omega)v, \nabla X_i(\eta_s(\omega)v)>|^2\} ds \ \rho(dx)$$

$$+ \int_M\int_\Omega \tfrac{1}{2} \int_0^T \{ \sum_{i=1}^m |\nabla X_i|^2 + 2 \sum_{i=1}^m |\nabla X_i|^2 + \bar{C}\} ds \ d\mathbb{P}\rho(dx)$$

for some constant β by the Birkholder-Davis-Grundy inequality, the positivity of the integrand in the second integral and since $||\eta_s(\omega)v|| = 1$. Then by the Φ_t-invariance, after application of Fubini's theorem in the second integral

$$\int_M \int_\Omega \sup_{t\in[0,T]} \log^+||T\xi_t(\omega)(x)||_{op} \ d\mathbb{P}\rho(dx)$$

$$\leq \beta T(\int_M [\sum_{i=1}^m |\nabla X_i|^2 + 2 \sum_{\substack{i\neq j \\ i,j=1}}^m |\nabla X_i||\nabla X_j|]\rho(dx))$$

$$+ \frac{3T}{2} \int_M \sum_{i=1}^m |\nabla X_i|^2 \ \rho(dx) + \frac{\bar{C}}{2} .$$

Each term on the right hand side is finite since $|\nabla X_i| \in L^2(M,\rho(dx))$, $1 \leq i \leq m$, the cross terms being finite by the Cauchy-Schwarz inequality.

Clearly all these quantities can be extended from M to $\tilde{M} \subset \mathbb{R}^m$ and hence the result.

The result for the inverse follows since by Itô's formula

$$\log||T\xi_t(\omega)(x)^{-1}v|| = \int_0^t - \sum_{i=1}^m <\eta_s(\omega)v, (\eta_s(\omega)v)\nabla X_i> dB_s^i$$

$$+ \tfrac{1}{2}\int_0^t \{ \sum_{i=1}^m ||\nabla X_i(\eta_s(\omega)(v))||^2 - 2 \sum_{i=1}^m <(\eta_s(\omega)v)\nabla X_i, \eta_s(\omega)v>^2$$

$$+ \mathrm{Ric}(\eta_s(\omega)v, \eta_s(\omega)v) - 2\langle \eta_s(\omega)v, (\eta_s(\omega)v)\nabla A\rangle\}ds$$

and using the fact that $\mathrm{Ric}-\nabla A \le C$ some $C > 0$, by taking absolute values the result then follows in the same manner as above. □

Corollary 3.4.2

For a ground state stochastic mechanical system (3.19) on M embedded in some $\mathbb{R}^m$ $(m > n)$ where the system is obtained from the embedding (i.e. gradient Brownian flow + drift A) then if the second fundamental form $\alpha_x \in L^2(M,\rho(dx))$ and $\nabla A \in L^1(M,\rho(dx))$ for any $t \in [0,T]$

$$\sup_{t\in[0,T]} \log^+||D\tilde{\xi}_t(\omega)(x)^{\pm 1}||_{op} \in L^1(\tilde{M} \times \Omega, \rho \otimes \mathbb{P}) .$$

Proof

For the system (3.19) obtained from the embedding $M \to \mathbb{R}^m$, (3.23) becomes

$$\log||T\xi_t(\omega)(x)v|| = \int_0^t \sum_{i=1}^m \langle \eta_s(\omega)v, \nabla X_i(\eta_s(\omega)v)\rangle dB_s^i + \int_0^t \langle \eta_s(\omega)v, \nabla A(\eta_s(\omega)v)\rangle ds$$

$$+ \tfrac{1}{2}\int_0^t \{|\alpha_x(\eta_s(\omega)v,\cdot)|^2 - 2|\alpha_x(\eta_s(\omega)v,\eta_s(\omega)v)|^2$$

$$- \mathrm{Ric}(\eta_s(\omega)v,\eta_s(\omega)v)\}ds .$$

The result follows in the same manner as lemma 3.4.1 using the fact that by the definition of the Ricci tensor in Theorem 2.4.1 if $\alpha_x \in L^2(M,\rho(dx))$ then $\mathrm{Ric} \in L^1(M,\rho(dx))$. □

Lemma 3.4.3

For a ground state stochastic mechanical system (3.19) on M such that $\nabla X_i \in L^2(M,\rho(dx))$ for $1 \le i \le m$ and the form $Ric-\nabla A$ is uniformly bounded over M, then for all $t \in [0,T]$

$$\sup_{t\in[0,T]} ||D(\tilde{\xi}_{T-t}(\theta_t(\omega)))(\tilde{\xi}_t(\omega)(x))|| \in L^1(\tilde{M} \times \Omega, \rho(dx) \otimes \mathbb{P}) .$$

Proof

Follows from (1.7), lemma 3.2.2 and the integrability of the quantities in lemma 3.4.1. □

Using these results we now have a Multiplicative Ergodic Theorem for a stochastic mechanical process on a non-compact, complete, simply connected manifold M.

Theorem 3.4.4

Given a ground state stochastic mechanical system (3.19) on M such that $\nabla X_i \in L^2(M,\rho(dx))$ and $Ric-\nabla A$ is uniformly bounded then there exists a.s. a Lyapunov spectrum and associated filtration of the tangent space TM.

Proof

By lemma 3.4.1, Hypothesis 1.2.5 is satisfied. The proof then follows that of Theorem 1.2.6. □

Remarks 3.4.5

(i) As in lemma 1.2.4, by considering the system (3.19) as one defined on $\tilde{M} \subset \mathbb{R}^m$ we are able to identify the tangent spaces of M in a Borel measurable way, thus simplifying the approach originally taken by Ruelle in [56].

(ii) As for Theorem 1.2.6 the above result still holds if only a measurable solution flow $\{\xi_t(\omega) ; t \geq 0\}$ of (3.19) exists.

(iii) As the stochastic mechanical system (3.19) is nondegenerate, in fact elliptic, as in Remark 1.2.7 (iii) the spectrum is a.s. constant, independent of $(x,\omega) \in M \times \Omega$.

We now consider the existence of λ_Σ for such a stochastic mechanical process (3.19) on M . We shall first require certain regularity results, analagous to the conditions of Hypothesis 1.2.5.

<u>Lemma</u> 3.4.6

For a ground state stochastic mechanical system (3.19) on M , under the conditions of lemma 3.4.1 we have for any $t \in [0,T]$

$$\sup_{t\in[0,T]} \log^+|\det T\xi_t(\omega)(x)^{\pm 1}| \in L^1(M \times \Omega, \rho \otimes \mathbb{P}) \quad .$$

<u>Proof</u>

For any $A \in GL(\mathbb{R}^n)$

$$|\det A| \leq n^2||A||^n$$

which implies that

$$\log|\det A| \leq 2\log n + n \log||A|| \ .$$

So if $A = T\xi_t(\omega)(x)^{\pm 1}$ the result follows from lemma 3.4.1 since the first term on the right hand side is constant. □

We are however able to weaken the conditions of the above lemma, consider the following:

Lemma 3.4.7

For a ground state stochastic mechanical system (3.19) on M, if $\operatorname{div} X_i \in L^2(M,\rho(dx))$, $1 \le i \le m$, then for any $t \in [0,T]$

$$\sup_{t\in[0,T]} \log^+|\det T\xi_t(\omega)(x)^{\pm 1}| \in L^1(M \times \Omega, \rho \otimes \mathbf{P}) \ .$$

Proof

By (1.17) for the stochastic mechanical system (3.19) on M

$$\log|\det T\xi_t(\omega)(x)| = \int_0^t \sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i + \int_0^t \operatorname{div} A(\xi_s(\omega)(x))ds$$

$$+ \tfrac{1}{2}\int_0^t \sum_{i=1}^m \langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle ds \ . \qquad (3.24)$$

Hence

$$\int_M\int_\Omega \sup_{t\in[0,T]} \log^+|\det T\xi_t(\omega)(x)| \, d\mathbb{P}\rho(dx) \le \int_M\int_\Omega \sup_{t\in[0,T]} |\log|\det T\xi_t(\omega)(x)|| d\mathbb{P}\rho(dx)$$

$$\le \int_M\int_\Omega \sup_{t\in[0,T]} \left|\int_0^t \sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))dB_s^i\right| d\mathbb{P}\rho(dx)$$

$$+ \int_M\int_\Omega \sup_{t\in[0,T]} \Big[\int_0^t |\operatorname{div} A(\xi_s(\omega)(x))|\,ds$$

$$+ \tfrac{1}{2}\int_0^t \sum_{i=1}^m |\langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle|\,ds\Big]\, d\mathbb{P}\rho(dx)$$

$$\leq \gamma\int_M \int_0^T \mathbb{E}\{|\sum_{i=1}^m \operatorname{div} X_i(\xi_s(\omega)(x))|^2 ds\}\rho(dx)$$

$$+ \int_M \int_\Omega \Big[\int_0^T |\operatorname{div} A(\xi_s(\omega)(x))|\,ds$$

$$+ \tfrac{1}{2}\int_0^T \sum_{i=1}^m |\langle\nabla \operatorname{div} X_i(\xi_s(\omega)(x)), X_i(\xi_s(\omega)(x))\rangle|\,ds\Big]\, d\mathbb{P}\rho(dx)$$

for some constant γ by the Birkholder-Davis-Grundy inequality for the first integral and by the positivity of the integrands of the remaining integrals. Then by the Φ_t-invariance, after application of Fubini's theorem, we have

$$\int_M\int_\Omega \sup_{t\in[0,T]} \log^+|\det T\xi_t(\omega)(x)|\, d\mathbb{P}\rho(dx)$$

$$\leq \gamma T \int_M |\sum_{i=1}^m \operatorname{div} X_i(x)|^2\rho(dx) + T\int_M |\operatorname{div} A(x)|\rho(dx)$$

$$+ \frac{T}{2}\int_M \sum_{i=1}^m |\langle\nabla \operatorname{div} X_i(x), X_i(x)\rangle|\rho(dx) \quad .$$

The first integral is finite since $\operatorname{div} X_i \in L^2(M,\rho(dx))$, the cross terms $|\mathrm{div}X_i|.|\mathrm{div}X_j|$ being finite by use of the Cauchy-Schwarz inequality. The second integral is finite as $\mathrm{div}A(x) = \mathrm{trace}\ \nabla A(x)$ and since $A(x) = \frac{1}{2}\nabla\log|\phi_0(x)|^2$

$$|\mathrm{div}A(x)| = |\tfrac{1}{2}\Delta(\log|\phi_0(x)|^2)|$$

$$= \frac{|\langle\phi_0,\Delta\phi_0\rangle - |\nabla\phi_0|^2|}{|\phi_0|^2} .$$

This is integrable with respect to $\rho(dx)$ by the finite energy assumption and the fact that ϕ_0 and $\Delta\phi_0 \in L^2(M,dx)$ after using the Cauchy-Schwarz inequality (or equivalently using the fact that div and $-\nabla$ are formal adjoints). The third integral is finite, again by using the fact that div and $-\nabla$ are adjoints and then by assumption.

The result follows for the inverse $T\xi_t(\omega)(x)^{-1}$ as

$$\det T\xi_t(\omega)(x)^{-1} = \int_0^t -(\det T\xi_t(\omega)(x)^{-1})\sum_{i=1}^m \mathrm{div}X_i(\xi_s(\omega)(x))dB_s^i$$

$$- \int_0^t(\det T\xi_t(\omega)(x)^{-1})\mathrm{div}A(\xi_s(\omega)(x))ds$$

$$+ \tfrac{1}{2}\int_0^t(\det T\xi_t(\omega)(x)^{-1})\sum_{i=1}^m(\mathrm{div}X_i(\xi_s(\omega)(x))^2 ds .$$

Using Itô's formula for $\log(\det T\xi_t(\omega)(x)^{-1})$ and taking absolute values gives the required result as above. $\square$

Remark 3.4.8

We note that the integrability of $\operatorname{div}A(x)$

$$\int_M |\operatorname{div}A(x)|\rho(dx) = 2\int_M |\nabla\phi_0(x)|^2 \, dx$$

is a "natural" consequence of the stochastic mechanical hypothesis that $\phi_0 \in L^2(M,dx)$ is the normalised eigenfunction of the ground state $-\frac{1}{2}\Delta\phi_0 + V\phi_0 = E_0\phi_0$ and the finite energy assumption.

Corollary 3.4.9

For a ground state stochastic mechanical system (3.19) on M embedded in some $\mathbb{R}^m$ $(m>n)$ where the system is obtained from the embedding, if the mean curvature vector $\underline{H}(\cdot) \in L^2(M,\rho(dx))$ then for any $t \in [0,T]$

$$\sup_{t\in[0,T]} \log^+|\det T\xi_t(\omega)(x)^{\pm 1}| \in L^1(M \times \Omega, \rho \otimes P) .$$

Proof

For the system obtained from the embedding $M \to \mathbb{R}^m$ (3.24) becomes

$$\log|\det T\xi_t(\omega)(x)| = \int_0^t \sum_{i=1}^m \operatorname{div}X_i(\xi_s(\omega)(x))dB_s^i + \int_0^t \operatorname{div}A(\xi_s(\omega)(x))ds$$

$$-\frac{n^2}{2}\int_0^t |\underline{H}(\xi_s(\omega)(x))|^2 ds$$

and the result follows in the same manner as lemma 3.4.7. $\square$

Lemma 3.4.10

For a ground state stochastic mechanical system (3.19) on M, if the conditions of lemma 3.4.7 are satisfied then for any $t \in [0,T]$

$$\sup_{t\in[0,T]} \log|\det T(\xi_{T-t}(\theta_t(\omega)))(\xi_t(\omega)(x))| \in L^1(M \times \Omega, \rho \otimes \mathbb{P}) .$$

Proof

This is true since

$$\sup_{t\in[0,T]} \log|\det T(\xi_{T-t}(\theta_t(\omega)))(\xi_t(\omega)(x))|$$

$$\leq \sup_{t\in[0,T]} \log^+|\det T\xi_T(\omega)(x)| + \sup_{t\in[0,T]} \log^+|\det T\xi_t(\omega)(x)^{-1}|$$

and both terms on the right hand side are integrable over $M \times \Omega$ by lemma 3.4.7. □

Theorem 3.4.11

For a ground state stochastic mechanical system (3.19) on M, then under the conditions of lemma 3.4.7 there exists a set $\Gamma \subset M \times \Omega$ of full $\rho \otimes \mathbb{P}$ measure such that for each $(x,\omega) \in \Gamma$ the limit

$$\frac{1}{t} \log|\det T\xi_t(\omega)(x)| \quad \text{exists as} \quad t \to \infty .$$

Denote the limit by $\bar{\lambda}_\Sigma$, then if the conditions of lemma 3.4.1 are also satisfied we have $\bar{\lambda}_\Sigma = \lambda_\Sigma$, the sum of the exponents.

Proof

We again first prove the result for discrete time intervals of length T, say. Consider the map $H_0 : M \times \Omega \to \mathbb{R}$ defined by

$$H_0(x,\omega) = \det T\xi_T(\omega)(x)$$

and set

$$\begin{aligned} H_n(x,\omega) &= H_0(\Phi_{nT}(x,\omega)) \\ &= \det T\,(\xi_T(\theta_{nT}(\omega)))\xi_{nT}(\omega)(x) \\ &= \det(T\xi_{(n+1)T}(\omega)(x).T\xi_{nT}(\omega)(x)^{-1})T\xi_{nT}(\omega)(x) \end{aligned}$$

and by the Chain rule and the fact that $\det AB = \det A.\det B$ we have

$$H^n(x,\omega) = H_{n-1}(x,\omega)\circ\ldots\ldots\circ H_0(x,\omega) = \det T\xi_{nT}(\omega)(x) \quad .$$

Now consider

$$\begin{aligned} \log|H^n(x,\omega)| &= \log|\det T\xi_{nT}(\omega)(x)| \\ &= \sum_{i=0}^{n-1} \log|H_i(x,\omega)| \quad . \end{aligned}$$

Now by lemma 3.4.7

$$\int_M\int_\Omega \log||H_0(x,\omega)||\, d\mathbb{P}\rho(dx) < \infty \quad .$$

So by the Birkhoff ergodic theorem

$$\frac{1}{n}\sum_{i=0}^{n-1} \log||H_i(x,\omega)|| \text{ tends to a limit a.s. as } n \to \infty$$

and is some function $h^* \in L^1(M \times \Omega, \rho \otimes \mathbb{P})$. In fact since $\rho \otimes \mathbb{P}(\Gamma) = 1$

$$\int_M \int_\Omega h^*(x,\omega)\, d\mathbb{P}\rho(dx) = \int_M \int_\Omega \log|\det T\xi_T(\omega)(x)|\, d\mathbb{P}\rho(dx)$$

and in particular

$$\frac{1}{n}\sum_{i=0}^{n-1} \log|H_i(x,\omega)| \underset{n\to\infty}{\to} \int_M \int_\Omega \log|\det T\xi_T(\omega)(x)|\, d\mathbb{P}\rho(dx) \quad .$$

To extend to the continuous time result we again use the fact that

$$\xi_t(\omega) = \xi_{t-nT}(\theta_{nT}(\omega)).\ \xi_{nT}(\omega)$$

$$\xi_{(n+1)T}(\omega) = \xi_{(n+1)T-t}(\theta_t(\omega)).\xi_t(\omega)$$

for all n and all $t \in [nT,(n+1)T]$. Thus if we set

$$\phi_1(x,\omega) = \sup_{t\in[0,T]} \log|\det T\xi_t(\omega)(x)|$$

and

$$\phi_2(x,\omega) = \sup_{t\in[0,T]} \log|\det T(\xi_{T-t}(\theta_t(\omega)))\xi_t(\omega)(x)| \ .$$

Then as in the proof of Theorem 1.2.6 we have a.s. that

$$\log|\det T\xi_{(n+1)T}(\omega)(x)| - \phi_2(\Phi_{nT}(x,\omega)) \leq \log|\det T\xi_t(\omega)(x)|$$

$$\leq \log|\det T\xi_{nT}(\omega)(x)| + \phi_1(\Phi_{nT}(x,\omega))$$

for all $t \in [nT,(n+1)T]$. By Lemma 3.4.7 and Lemma 3.4.10 ϕ_1 and ϕ_2 are $\rho \otimes \mathbb{P}$ integrable and clearly non-negative. Therefore by Birkhoff's ergodic theorem $1/n\, \phi_i(\Phi_{nT}(x,\omega)) \to 0$ as $n \to \infty$ for a.e. (x,ω). So for these $(x,\omega) \in M \times \Omega$

$$\lim_{n\to\infty} \frac{1}{nT} \log|\det T\xi_{nT}(\omega)(x)| = \lim_{t\to\infty} \frac{1}{t} \log|\det T\xi_t(\omega)(x)|$$

and the continuous time result follows.

The fact that

$$\lim_{t\to\infty} \frac{1}{t} \log|\det T\xi_t(\omega)(x)| = \lambda_\Sigma$$

follows from the fact that the system is elliptic and hence that the Lyapunov spectrum is a.s. constant and also using the manifold versions of Theorem 3.2.5 and Corollary 3.2.6. □

Remark 3.4.12

(i) As for Theorem 1.2.6 the above result holds if the system is not strongly complete, indeed it still works if the system is complete and only a partial or measurable flow exists.

(ii) The above result also provides a method for proving that

$\lim\limits_{t\to\infty} \frac{1}{t} \log|\det D\xi_t(\omega)(x)| = \lambda_\Sigma$ for a ground state process on $\mathbb{R}^n$ as in section 3.2.

We now have the following:

Theorem 3.4.13

If (3.19) is a ground state stochastic mechanical system on a complete Riemannian manifold M embedded in $\mathbb{R}^m$ $(m>n)$ of constant mean curvature and the system is obtained from the embedding then $\bar{\lambda}_\Sigma$ exists a.s.

Proof

As in Corollary 3.4.9 if the system (3.19) is obtained from the embedding $M \to \mathbb{R}^m$ then the solution flow $\xi_t(\omega) : M \to M$ is the gradient Brownian flow with drift A and

$$\sum_{i=1}^{m} |\operatorname{div} X_i(x)|^2 = n^2 \underline{H}(x)^2 = C \text{ , some constant.}$$

So $\underline{H}(\cdot) \in L^2(M,\rho(dx))$ and the conditions of Corollary 3.4.9 and Theorem 3.4.11 are satisfied so $\bar{\lambda}_\Sigma$ exists a.s.. Thus by the Strong Law of Large Numbers and (1.18)

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log|\det T\xi_t(\omega)(x)| &= \bar{\lambda}_\Sigma \\ &= \int_M \operatorname{div} A(x)\rho(dx) + C \text{ .} \qquad (3.25) \ \square \end{aligned}$$

Corollary 3.4.14

Under the same conditions as Theorem 3.4.13, if M is also minimally embedded then $\bar{\lambda}_\Sigma$ exists a.s. and

$\bar{\lambda}_\Sigma = -$ const. (the Kinetic energy of the stochastic mechanical particle).

Proof

If M is embedded minimally in $\mathbb{R}^m$ then $\underline{H}(x) \equiv 0$ for all $x \in M$. Thus $\bar{\lambda}_\Sigma$ exists a.s. and by (3.25)

$$\bar{\lambda}_\Sigma = \int_M \operatorname{div} A(x)\rho(dx)$$

$$= \int_M \operatorname{div}(\tfrac{1}{2}\nabla \log|\phi_0(x)|^2)|\phi_0(x)|^2 dx$$

$$= \int_M \langle\phi_0(x), \Delta\phi_0(x)\rangle - |\nabla\phi_0(x)|^2 dx$$

which by taking divergences yields

$$\bar{\lambda}_\Sigma = -2\int_M |\nabla\phi_0(x)|^2 \, dx$$

and hence the result. □

Corollary 3.4.15

If (3.19) is a ground state stochastic mechanical system defined on the orthonormal frame bundle $O(M)$ then $\bar{\lambda}_\Sigma$ exists and is given by

$\bar{\lambda}_\Sigma = -$ const. (the Kinetic energy of the stochastic mechanical particle).

Proof

For the canonical system (see for example [16], section 4)

$$d\xi_t(\omega)(u) = \sum_{i=1}^{m} X_i(\xi_t(\omega)(u)) \circ dB_t^i + A(\xi_t(\omega)(u))dt$$

$$\xi_0(\omega)(u) = u \qquad\qquad u \in O(M)$$

we have $\operatorname{div} X_i \equiv 0$. The proof then follows that of Corollary 3.4.14. □

Examples 3.4.16

(i) Consider a process defined on a parabola embedded isometrically in $\mathbb{R}^2$ by the embedding $f : \mathbb{R} \to \mathbb{R}^2$ where

$$f(x) = (ax^2, 2ax) \qquad\qquad a > 0$$

with the Riemannian metric

$$g = 4a^2(x^2+1) .$$

Take the stochastic mechanical system given by

$$dx_t = \frac{x}{2a(x^2+1)} \circ dB_t^1 + \frac{1}{2a(x^2+1)} \circ dB_t^2 - \frac{x}{4a^2(x^2+1)} dt$$

where the drift vector $A(x) = \nabla(-\frac{x^2}{2})$. For this embedding the curvature of the parabola $K(x) = 1/\rho(x)$ where $\rho(\cdot)$ is the radius of curvature. $K(x)$ is given by

$$K^2(x) = \frac{1}{4a^2(x^2+1)^3} .$$

This is clearly bounded for all $x \in \mathbb{R}$ and hence lies in $L^2(\mathbb{R},\rho(dx))$. Also by (3.22) the invariant probability measure for this process is given by $ce^{-x^2}d\underline{x}$, where $d\underline{x}$ denotes the Riemannian volume element given by

$$d\underline{x} = 2a\sqrt{x^2+1}\ dx ,$$

and $$c = [\int_{-\infty}^{\infty} 2ae^{-x^2}(x^2+1)^{\frac{1}{2}}dx]^{-1} .$$

So by (1.18) the leading and only exponent is given by

$$\begin{aligned}\lambda &= \int_{\mathbb{R}} -\frac{K^2(x)}{2}\rho(dx) + \int_{\mathbb{R}} \operatorname{div} A(x)\rho(dx) \\ &= \int_{-\infty}^{\infty} -\frac{c}{4a}\frac{e^{-x^2}}{(x^2+1)^{5/2}}\,dx + \int_{-\infty}^{\infty} -\frac{c}{2a}\frac{e^{-x^2}}{(x^2+1)^{3/2}}\,dx \qquad (3.26)\\ &= -\frac{c}{4a}\int_{-\infty}^{\infty}[\frac{1}{(x^2+1)^{5/2}} + \frac{2}{(x^2+1)^{3/2}}]\,e^{-x^2}\,dx\end{aligned}$$

which is given by (see Gradshteyn and Ryzhik, [36])

$$\lambda = -\frac{c}{4a}\Gamma(\tfrac{1}{2})\sum_{k=0}^{\infty}[\binom{3/2+k}{k} + 2\binom{\frac{1}{2}+k}{k}]\frac{\Gamma(\frac{1}{2}+k)}{\Gamma(\frac{1}{2})}$$

where $(\because)$ denotes the Binomial coefficient.

Analagous to the work of Carverhill ([13]sections 3.1 and 4.3) we consider (3.19) as a deterministic system under the action of a small stochastic perturbation. So, for small ε , (3.19) is now given by

$$dx_t^\varepsilon = \sqrt{\varepsilon} \sum_{i=1}^{2} X_i(x_t^\varepsilon) \circ dB_t^i + A(x_t^\varepsilon)dt \tag{3.27}$$

which has differential generator $\frac{\varepsilon\Delta}{2} + A.\nabla$. This is equivalent to considering (3.19) as

$$dx_{t'} = \sum_{i=1}^{2} X_i(x_{t'}) \circ dB_{t'}^i + \frac{1}{\varepsilon} A(x_{t'})dt' \tag{3.28}$$

where (3.28) is just (3.27) under the time change $t = \frac{t'}{\varepsilon}$. For (3.28) the invariant probability measure is given by $\rho^\varepsilon(dx) = ce^{-x^2/\varepsilon}d\underline{x}$ where $c = [\int_{-\infty}^{\infty} 2ae^{-x^2/\varepsilon}(x^2+1)^{\frac{1}{2}}dx]^{-1}$ and the exponent for (3.27) is is given by

$$\lambda^\varepsilon = \lim_{t\to\infty} \frac{1}{t} \log||v_t^\varepsilon|| = \lim_{t'\to\infty} \frac{\varepsilon}{t'} \log||v_{t'}|| = \varepsilon \lim_{t'\to\infty} \frac{1}{t'} \log||v_{t'}|| \quad .$$

From (3.26) and (3.28)

$$\lim_{t'\to\infty} \frac{1}{t'} \log||v_{t'}|| = \int_{-\infty}^{\infty} - \frac{c}{4a} \frac{e^{-x^2/\varepsilon}}{(x^2+1)^{5/2}}dx + \int_{-\infty}^{\infty} - \frac{c}{2a\varepsilon} \frac{e^{-x^2/\varepsilon}}{(x^2+1)^{3/2}} dx \ . \tag{3.29}$$

Since, by the definition of the δ-function

$$f(0) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{-\infty}^{\infty} f(x)e^{-x^2/\varepsilon} dx$$

(3.29) becomes

$$\lim_{t'\to\infty} \frac{1}{t'} \log||v_{t'}|| = -\frac{1}{8a^2} - \frac{1}{4a^2\varepsilon} \quad .$$

Thus

$$\lambda^{\varepsilon} = \varepsilon \lim_{t'\to\infty} \frac{1}{t'} \log||v_{t'}|| = -\frac{\varepsilon}{8a^2} - \frac{1}{4a^2}$$

and

$$\lim_{\varepsilon\to 0} \lambda^{\varepsilon} = -\frac{1}{4a^2}$$

which is also the value of $\nabla A(0) = \operatorname{div} A(0)$. This concurs with the results of Carverhill, [13], for the compact manifold case, namely that as ε tends to zero, the invariant measure ρ^{ε} concentrates on the hyperbolic attracting fixed point of the flow $\xi_t(\omega)$ - the fixed point being the origin.

(ii) Consider the process defined on a catenoid isometrically embedded in $\mathbb{R}^3$ where the embedding $f : M \to \mathbb{R}^3$ is defined by

$$f(u,v) = (u, \cosh u \cos v , \cosh u \sin v)$$

with the Riemannian metric

$$g_{ij} = \begin{bmatrix} \cosh^2 u & 0 \\ 0 & \cosh^2 u \end{bmatrix} .$$

Take the stochastic mechanical system given by

$$d(u_t,v_t) = \sum_{i=1}^{3} X_i(u_t,v_t)\circ dB_t^i - \left(\frac{u_t}{\cosh^2 u_t} , \frac{v_t}{\cosh^2 u_t} \right) dt \tag{3.30}$$

where $X_i = \nabla f_i$,$(1 \le i \le 3)$, and $A(u,v) = \nabla(- \frac{(u^2+v^2)}{2})$. It is well known that such a surface is minimal in $\mathbb{R}^3$ and hence $H \equiv 0$ for all $(u,v) \in \mathbb{R} \times S^1$. Also the Gaussian (Ricci) curvature is given by

$$K = - \frac{1}{\cosh^4 u}$$

which clearly lies within the range $-1 \le K < 0$ with minimum value on the circle $u = 0$. Also $\nabla A(u,v)$ is given by

$$\nabla A(u,v) = \begin{bmatrix} \frac{u\sinh u}{\cosh^3 u} - \frac{1}{\cosh^2 u} & - \frac{v\sinh u}{\cosh^3 u} \\ \frac{2v\sinh u}{\cosh^2 u} - \frac{v\sinh u}{\cosh^3 u} & - \frac{1}{\cosh^2 u} - \frac{u\sinh u}{\cosh^3 u} \end{bmatrix}$$

which is uniformly bounded. Hence by Theorem 3.4.13 and (3.25)

$$\lambda_\Sigma = c\int_{-\infty}^{\infty} \int_0^{2\pi} \operatorname{div} A(u,v)\, e^{-(u^2+v^2)} \cosh^2 u \, dvdu$$

where $$c = \left(\int_{-\infty}^{\infty}\int_0^{2\pi} e^{-(u^2+v^2)} \cosh^2 u \, dvdu\right)^{-1} .$$

Thus

$$\lambda_\Sigma = c\int_{-\infty}^{\infty}\int_0^{2\pi} - \frac{2}{\cosh^2 u} \, . \, e^{-(u^2+v^2)} \cosh^2 u \; dvdu$$

$$= - 2c\int_{-\infty}^{\infty}\int_0^{2\pi} e^{-(u^2+v^2)} \; dvdu$$

$$= - c \sqrt{\pi}[\sqrt{\pi} - \Gamma(\tfrac{1}{2},4\pi^2)] < 0$$

where

$$\Gamma(\tfrac{1}{2},4\pi^2) = \int_{4\pi^2}^{\infty} \frac{e^{-u}}{u^{\frac{1}{2}}} du \; .$$

As in example (i) we consider (3.30) as a deterministic system under a small stochastic perturbation (i.e. (3.30) is of the form (3.27)). The only hyperbolic attracting fixed point of the flow $\xi_t(\omega)(u,v)$ of (3.30) is the origin, i.e. at $(0,0)$, $A(0,0) = 0$. Also

$$\lambda_\Sigma = \varepsilon \lim_{t\to\infty} \frac{1}{t} \log|\det v_t| = - 2 = \text{trace } \nabla A(0,0) \; .$$

We now have justification in giving the following extensions of the results given in Carverhill [13] sections 3.1 and 4.3.

Consider, on a non-compact manifold M, a stochastic perturbation of a deterministic system, i.e.

$$dx_t^\varepsilon = \sqrt{\varepsilon} \sum_{i=1}^{m} X_i(x_t^\varepsilon)\circ dB_t^i + A(x_t^\varepsilon)dt \tag{3.31}$$

where

$$A(x) = \tfrac{1}{2} \nabla \log |\phi_0(x)|^2$$

as in (3.19). For (3.31) the invariant probability measure is given by

$$\rho^\varepsilon(dx) = |\phi_0(x)|^2 \, dx \quad .$$

We assume that as ε tends to zero ρ^ε concentrates on a finite set of hyperbolic stable fixed points of the flow $\xi_t(\omega)(x)$ of (3.31), such a situation exists from the evidence of examples 3.4.16. We also assume the conditions of Theorem 3.4.4 and Theorem 3.4.11 so that for (3.31) the associated Lyapunov spectrum exists a.s.. We then have the following:

Proposition 3.4.17

If the flow $\xi_t(\omega)$ of (3.31) has a hyperbolic attracting fixed point x_0 then the invariant measure $\rho^\varepsilon(dx)$ tends weakly to δ_{x_0} as $\varepsilon \to 0$. Also as $\varepsilon \to 0$ the sum of the Lyapunov exponents

$$\lambda_\Sigma^\varepsilon = \text{trace} \quad \nabla A(x_0) \quad .$$

Proof

See Carverhill, [13], lemma 3.1 and Proposition 3.1.2. □

The following is also true for the non-compact case - it is Carverhill's Theorem 4.3.

Theorem 3.4.18

For the system (3.31) above. If, as ε tends to zero the invariant measure ρ^{ε} concentrates on a finite set $x_0,\dots,x_p$ of hyperbolic attracting fixed points of $\xi_t(\omega)$, then if $\varepsilon > 0$ is sufficiently small the system (3.31) is Lyapunov stable, i.e. for a.e. (x,ω), $\sup_{v \in S_xM} \{\lambda^{\varepsilon}_{(v,\omega)}\}$ is strictly negative where

$$\lambda^{\varepsilon}_{(v,\omega)} = \lim_{t\to\infty} \frac{1}{t} \log||T\xi^{\varepsilon}_t(\omega)v|| \quad .$$

Remark 3.4.19

If the flow (measurable or otherwise) of the stochastic mechanical system (3.19) on M is C^k $(k \geq 2)$ and the conditions of Theorem 3.4.4. are satisfied (namely $\nabla X_i \in L^2(M,\rho(dx))$ and $\mathrm{Ric} - \nabla A$ is uniformly bounded) then for the negative part of the Lyapunov spectrum associated with (3.19) we have analogues of Theorems 3.3.1 and 3.3.2 for a stochastic mechanical process on M, that is the existence of local and global stable manifolds.

3.5 The Existence of λ and λ_{Σ} for Stochastic Mechanical Processes Corresponding to Higher Energy Levels

In this section we again consider the differential operator

$$-\tfrac{1}{2}\Delta + V \tag{3.32}$$

acting on functions $f : \mathbb{R}^n \to \mathbb{R}$ with sufficiently regular potential

$V : \mathbb{R}^n \to \mathbb{R}$. We now consider higher energy levels above the ground state, that is eigenvalues E_p , $p = 1,2,3 \ldots\ldots$ of the operator (3.32) and the corresponding p^2 orthogonal, normalised eigenfunctions ϕ_p such that

$$(-\tfrac{1}{2}\Delta + V)\phi_p = E_p\phi_p \quad .$$

Again since V is sufficiently regular, each ϕ_p is smooth and of finite energy (i.e. $\nabla\phi_p \in L^2(\mathbb{R}^n,dx)$) . As in section 1.4 associated with this operator is the diffusion process

$$dx_t = dB_t + A(x_t)dt \tag{3.33}$$

where $B_t \in BM(\mathbb{R}^n)$ and

$$A(x) = \tfrac{1}{2}\nabla\log|\phi_p(x)|^2 \quad . \tag{3.34}$$

It is well known that for these higher energy levels, unlike the ground state E_0 , that the corresponding eigenfunctions ϕ_p have zeros, called nodes, and we can consider the set

$$N_{\phi_p} = \{x \in \mathbb{R}^n \; ; \; \phi_p(x) = 0\}$$

consisting of the nodes of ϕ_p . Under certain assumptions on V (see e.g. [1]) it can be shown that if the corresponding stochastic mechanical process $\xi_t(\omega)$ starts at some point $x \notin N_{\phi_p}$ then the trajectory of the process never reaches N_{ϕ_p} . It is also known that (see e.g. [50]) any solution of (3.33) is strictly positive if $p = 0$, i.e. the ground state. Thus for any E_p $(p > 0)$, ϕ_p is orthogonal to ϕ_0 and hence N_{ϕ_p} is non-void and divides $\mathbb{R}^n$ into finitely many disjoint connected regions Γ_i such that

$$\mathbb{R}^n = N_{\phi_p} \cup \bigcup_i \Gamma_i \, . \tag{3.35}$$

Also if $\xi_t(\omega)$ starts at $x \in \Gamma_i$ some i then $\xi_t(\omega)$ never reaches N_{ϕ_p} and remains within Γ_i for all time.

By [21] (Proposition 1D) we know that a measurable solution flow of (3.33) exists and its derivative $D\xi_t(\omega)(x)$ exists at least in the L^0 sense. We shall show the existence of a Lyapunov spectrum and sum of exponents.

As in Theorem 1.2.6 there is a measurable map (time shift)

$$\Phi_t : \Gamma_i \times \Omega \to \Gamma_i \times \Omega$$

defined by

$$\Phi_t(x,\omega) = (\tilde{\xi}_t(x,\omega), \theta_t\omega)$$

where $\theta_t : \Omega \to \Omega$ is the shift and $\tilde{\xi}_t(\omega)$ is a measurable solution flow for (3.33) off the nodal set N_{ϕ_p} . Again, as in section 1.4, the process has invariant measure

$$\rho(dx) = |\phi_p(x)|^2 dx$$

on $\mathbb{R}^n$ and the semigroup of transformations Φ_t preserves the measure $\rho(dx)$. We again require the following lemma

Lemma 3.5.1

For the stochastic mechanical system (3.33) on $\mathbb{R}^n \backslash N_{\phi_p}$ we have for any $t \in [0,T]$

$$\sup_{t\in[0,T]} \log^+ || D\tilde{\xi}_t(\omega)(x)^{\pm 1} ||_{op} \in L^1((\mathbb{R}^n \backslash N_{\phi_p}) \times \Omega, \rho \otimes \mathbb{P}) .$$

Proof

Exactly as in lemma 3.2.1 since $\tilde{\xi}_t(\omega)$ is a measurable flow off the nodal set N_{ϕ_p} . Also see [21]. □

Theorem 3.5.2

For the stochastic mechanical system (3.33) defined on $\mathbb{R}^n \backslash N_{\phi_p}$ there exists a.s. a Lyapunov spectrum and associated filtration of the connected component of $\mathbb{R}^n$ in which the solution flow remains.

Proof

By lemma 3.5.1, Hypothesis 1.2.5 is satisfied. The proof follows from that of Theorem 1.2.6. □

Remark 3.5.3

(i) The system is elliptic on $\mathbb{R}^n \backslash N_{\phi_p}$ and hence the spectrum is a.s. constant on the connected component of $\mathbb{R}^n$ in which $\tilde{\xi}_t(\omega)$ remains.

(ii) By Carverhill's formula (1.12) for the system defined on $\mathbb{R}^n \backslash N_{\phi_p}$, the top exponent is given by

$$\lambda^1 = \int_{\Gamma_i \times S^{n-1}} g(\eta_s(\omega)v)\mu(d(x,v/|v|))$$

where $g(\cdot)$ is given by (1.13), and μ is the invariant probability measure on the sphere bundle to $\mathbb{R}^n \backslash N_{\phi_p}$ normalised on $\Gamma_i \times S^{n-1}$, the sphere bundle over the connected component Γ_i of $\mathbb{R}^n$ in which the process is restricted.

To consider the sum of the exponents λ_Σ, we have the following lemma:

Lemma 3.5.4

For a stochastic mechanical system (3.33) on $\mathbb{R}^n \backslash N_{\phi_p}$ we have for any $t \in [0,T]$

$$\sup_{t \in [0,T]} \log^+ |\det D\tilde{\xi}_t(\omega)(x)^{\pm 1}| \in L^1((\mathbb{R}^n \backslash N_{\phi_p}) \times \Omega, \rho \otimes \mathbb{P}) .$$

Proof

As in lemma 3.4.6. $\square$

This enables us to give the following:

Theorem 3.5.5

For the stochastic mechanical system (3.33) defined on $\mathbb{R}^n \backslash N_{\phi_p}$ the sum of the exponents, λ_Σ exists a.s..

Proof

By lemma 3.5.4, the analogue of Hypothesis 1.2.5 is satisfied. The proof then follows that of Theorem 3.4.11 or Corollary 3.2.6. □

We now have the following:

Proposition 3.5.6

For a stochastic mechanical system (3.33) on $\mathbb{R}^n \backslash N_{\phi_p}$

$$\lambda_\Sigma = \int_{\Gamma_i} \operatorname{div} A(x) \, \frac{\rho(dx)}{|\Gamma_i|}$$

where

$$|\Gamma_i| = \int_{\Gamma_i} |\phi_p(x)|^2 \, dx$$

and

$$\lambda_\Sigma = - \text{ const. (Kinetic Energy of the stochastic mechanical particle attained within the connected region } \Gamma_i) .$$

Proof

Since

$$\lambda_\Sigma = \lim_{t\to\infty} \frac{1}{t} \log|\det D\tilde{\xi}_t(\omega)(x)| = \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{div}A(\tilde{\xi}_s(\omega)(x))ds$$

and if $x \in \Gamma_i \subset \mathbb{R}^n$ then $\tilde{\xi}_t(\omega)(x) \in \Gamma_i \quad \forall\ t \geq 0$. Thus by the Strong Law of Large Numbers

$$\lambda_\Sigma = \int_{\Gamma_i} \operatorname{div} A(x)\ \rho'(dx)$$

where $\rho'(dx)$ is the invariant probability measure on Γ_i , thus $\rho'(dx) = \frac{\rho(dx)}{|\Gamma_i|}$ where $|\Gamma_i|$ is given above.

Now since $A = \frac{1}{2}\nabla\log|\phi_p(x)|^2$

$$\lambda_\Sigma = \frac{1}{|\Gamma_i|} \int_{\Gamma_i} \operatorname{div}\tfrac{1}{2}\nabla(\log|\phi_p(x)|^2)|\phi_p(x)|^2\ dx \quad .$$

The vector field $\operatorname{div}A(x)\cdot|\phi_p(x)|^2$ on $\mathbb{R}^n\backslash N_{\phi_p}$ is just the restriction of the gradient of $|\phi_p(x)|^2$ which has N_{ϕ_p} as the set on which this attains its absolute minimum. The vector field is therefore complete on $\mathbb{R}^n\backslash N_{\phi_p}$ if it is complete on $\mathbb{R}^n$, but this is assured by the assumption on the original potential V . We thus have

$$\lambda_\Sigma = \frac{1}{|\Gamma_i|} \int_{\Gamma_i} \{<\phi_p(x),\Delta\phi_p(x)> - |\nabla\phi_p(x)|^2\}dx \tag{3.36}$$

and taking divergences in the first integrand (by completeness)

$$\lambda_\Sigma = -\frac{2}{|\Gamma_i|}\int_{\Gamma_i} |\nabla\phi_p(x)|^2\, dx$$

$$= -\frac{4}{|\Gamma_i|} \times (\text{K.E. attained by the particle in } \Gamma_i)\ . \qquad \square$$

Remark 3.5.7

We note here that for the ground state process, Theorem 3.2.7 shows us that λ_Σ, which represents the almost sure exponential rate of change of volume under the flow (see for example [21]), is given by some negative constant times the Kinetic energy of the particle over $\mathbb{R}^n$. The above result is clearly analagous for stochastic mechanical processes corresponding to higher energy levels, but this does not correspond to standard concepts in quantum mechanics.

Example 3.5.8

Consider the harmonic oscillator on $\mathbb{R}$ given by

$$-\tfrac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}\ . \qquad (3.37)$$

Then the eigenvalue problem is given by

$$\left(-\tfrac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}\right)\phi_p(x) = \lambda_p\phi_p(x) \qquad (3.38)$$

where the eigenvalues λ_p are given by

$$\lambda_p = \frac{(2p+1)}{2} \qquad p = 0,1,2,\ldots..$$

with corresponding eigenfunctions

$$\phi_p(x) = e^{-x^2/2} H_p(x)$$

where $H_p(x)$ are the Hermite Polynomials given by

$$H_p(x) = (-1)^p e^{x^2} \frac{d^p}{dx^p} (e^{-x^2}) \quad .$$

The stochastic mechanical diffusion process associated with the operator (3.37) is given by

$$dx_t = dB_t + A(x_t)dt$$

where

$$A(x) = \tfrac{1}{2}\nabla \log|\phi_p(x)|^2 \quad .$$

We consider the first few energy levels associated with the operator (3.38).

(i) The Ground State $p = 0$

$\lambda_0 = \frac{1}{2}$ and the normalised eigenfunction is given by

$\phi_0(x) = \dfrac{e^{-x^2/2}}{(\sqrt{\pi})^{\frac{1}{2}}}$ which is clearly never zero.

Thus by (3.36)

$$\lambda_\Sigma = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \phi_0(x)\phi_0''(x) - \phi_0'(x)^2 \, dx = - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \phi'(x)^2 \, dx$$

since, from ϕ_0 given above, $\phi_0(x) \to 0$ as $x \to \pm \infty$.

So

$$\lambda_\Sigma = - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx$$

and the Kinetic energy is given by

$$\text{K.E.} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx \, .$$

(ii) $p = 1$

Then $\lambda_1 = 3/2$ and the unnormalised eigenfunction is $\phi_1(x) = 2xe^{-x^2/2}$ which has a zero at $x = 0$. Thus $N_{\phi_1} = \{0\}$ and $\mathbb{R}$ is divided up as

$$\mathbb{R} = \underbrace{(-\infty,0)}_{\Gamma_1} \cup \{0\} \cup \underbrace{(0,\infty)}_{\Gamma_2} \, .$$

Then

$$\lambda_\Sigma^{\Gamma_i} = \frac{1}{2c} \int_0^{\infty} -4 \, (1-x^2)^2 e^{-x^2} \, dx \qquad \text{for } i = 1,2 \, ,$$

where $c = \int_{-\infty}^{\infty} 4x^2 e^{-x^2} dx$ and $|\Gamma_i| = \frac{1}{2}$. Also the Kinetic energy

$$\text{K.E.}^{\Gamma_i} = \frac{1}{2c}\int_0^\infty 4(1-x^2)^2\, e^{-x^2}\,dx \qquad \text{for } i = 1,2 .$$

(iii) $p = 2$

Then $\lambda_2 = 5/2$ and the unnormalised eigenfunction is $\phi_2(x) = (4x^2-2)e^{-x^2/2}$. This has zeros at $x = \pm 1/\sqrt{2}$, and $\mathbb{R}$ is divided up as

$$\mathbb{R} \equiv \underbrace{(-\infty,-1/\sqrt{2})}_{\Gamma_1} \cup \{-1/\sqrt{2}\} \cup \underbrace{(-1/\sqrt{2},1/\sqrt{2})}_{\Gamma_2} \cup \{1/\sqrt{2}\} \cup \underbrace{(1/\sqrt{2},\infty)}_{\Gamma_3} .$$

Then

$$\lambda_\Sigma^{\Gamma_1,\Gamma_3} = -\frac{2}{c|\Gamma_{1,3}|}\int_{1/\sqrt{2}}^\infty (10x-4x^3)^2 e^{-x^2}\,dx$$

where $$c = \int_{-\infty}^\infty (4x^2-2)^2 e^{-x^2}\,dx \quad \text{and} \quad |\Gamma_{1,3}| = \int_{1/\sqrt{2}}^\infty \frac{\phi_2^2(x)}{c}\,dx$$

and

$$\lambda_\Sigma^{\Gamma_2} = -\frac{2}{c|\Gamma_2|}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (10x-4x^3)^2 e^{-x^2}\,dx$$

where $$|\Gamma_2| = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{\phi_2^2(x)}{c}\,dx = 2\int_0^{1/\sqrt{2}} \frac{(4x^2-2)^2}{c} e^{-x^2}\,dx .$$

Also the Kinetic energies are given by

$$\text{K.E.}^{\Gamma_1,\Gamma_3} = \frac{1}{2c}\int_{1/\sqrt{2}}^{\infty} (10x-4x^3)^2\, e^{-x^2}\, dx$$

$$\text{K.E.}^{\Gamma_2} = \frac{1}{2c}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (10x-4x^3)^2\, e^{-x^2}\, dx\ .$$

Remark 3.5.9

The situation of excited states for the time dependent case (i.e. when the drift vector A also depends on t and the process is no longer stationary) has been examined in [21].

From the evidence of the above results and those of sections 3.2 and 3.4 we have the following abstract result:

Theorem 3.5.10

For an Itô stochastic system given by

$$dx_t = \sum_{i=1}^{m} X_i(x_t)dB_t^i + A(x_t)dt \tag{3.39}$$

defined on some connected region $U \subset \mathbb{R}^m$ where $X_i \in \mathbb{L}(\mathbb{R}^m;T_xU)$, $1 \le i \le m$, and A is a vector field on U. Suppose that the measurable solution flow $\{\xi_t(\omega)(x)\ ;\ t \ge 0\}$ has finite invariant probability measure ρ then if

$$\int_U \left(\sum_{i=1}^{m} ||DX_i(x)||^2 + ||DA(x)||\right)\rho(dx) < \infty \tag{3.40}$$

then there exists a.s. a Lyapunov spectrum for (3.39) with associated filtration of the tangent space to U.

Proof

If (3.40) holds, by lemmas 3.4.1 and 3.5.1, Hypothesis 1.2.5 is satisfied. The existence of the Lyapunov spectrum then follows from Theorem 1.2.6. □

We also have the following Corollary.

Corollary 3.5.11

For a stochastic mechanical system of the form (3.39) defined on an open connected region $U \subset \mathbb{R}^m$ where the coefficient of the noise is spatially homogeneous, there exists a.s. a Lyapunov spectrum and associated filtration of the tangent space to U .

Proof

Since the coefficient of the noise is spatially homogeneous (i.e. constant) and the system is of stochastic mechanical form, by lemmas 3.2.1 and 3.5.1, condition (3.40) is satisfied and the system has a finite invariant probability measure of the form (1.22). Hence the result. □

Corollary 3.5.12

For system (3.39), if instead of (3.40) we have

$$\int_U \Big(\sum_{i=1}^{m} ||\mathrm{div}\ X_i(x)||^2 + ||\mathrm{div}\ A(x)|| \Big) \rho(dx) < \infty \tag{3.41}$$

then λ_Σ exists a.s..

Proof

If (3.41) holds, then by lemma 3.4.7 and Theorem 3.4.11, λ_Σ exists a.s.. □

Remark 3.5.13

By Corollary 3.2.6, λ_Σ exists a.s. for a system of the form given in Corollary 3.5.11.

3.6 The Existence of "Natural" Exponents on a Complete Riemannian Manifold

In this section M is an n-dimensional complete Riemannian manifold. We shall follow closely the work of Sullivan in [60]. The Laplacian of M is by definition naturally linked with the geometry of M. We shall consider the L^2-spectrum of Δ and the invariant quantity $\mu_0(M)$ which separates the L^2-spectrum from the positive spectrum of Δ, that is the set of μ for which there is a positive μ-harmonic function ϕ, such that $(\Delta-\mu)\phi = 0$. Thus a positive μ_0-harmonic function which is square integrable (and complete) generalizes the constant function of a complete finite volume manifold. Then using ϕ_0, if it exists, we may renormalise manifolds, formerly of infinite volume, so that they have finite volume and if we consider, for example, Brownian motion on such a manifold then under this renormalisation the process considered will also have a finite invariant measure. We would then hope to consider the existence of a Lyapunov spectrum associated with such a process.

By the Rayleigh-Ritz argument [60],

$$\mu_0 = -\inf_{\phi\in C_0^\infty(M)} \{\int_M |\nabla\phi|^2 / \int_M |\phi|^2\} .$$

Then the L^2 spectrum of Δ is contained in the interval $(-\infty,\mu_0]$ (see Sullivan's Theorem 2).

Example 3.6.1

For $M = \mathbb{R}$, $\mu_0 = 0$, the functions $e^{i\alpha x}$ for $\alpha \in \mathbb{R}$ are α^2-harmonic functions and $\{e^{-i\alpha x}\}$ are virtual L^2 eigenfunctions belonging to $-\alpha^2$ as a continuous spectrum.

Consider

$$(\tfrac{1}{2}\Delta - \mu_0)\phi_0 = 0 \qquad \text{on } M \tag{3.42}$$

where ϕ_0 is square-integrable over M, then given Brownian motion on M we can add to this a drift term $\frac{1}{2}\nabla\log|\phi_0(x)|^2$. This "biased" random motion (the "ϕ_0-process") has differential generator

$$\tfrac{1}{2}\Delta + \tfrac{1}{2}\nabla\log|\phi_0|^2.\nabla \quad . \tag{3.43}$$

Note here the similarity between the above and the stochastic mechanics discussed in Chapters 1 and 3. In this case however ϕ_0 is related naturally to the geometry of M and not to some potential function on M. As before (Section 1.4) the process associated with (3.43) preserves the finite measure $|\phi_0(x)|^2dx$. Thus under certain regularity conditions on the way in which the noise is introduced and if ϕ_0 exists then we have the following:

Theorem 3.6.2

Given that the square integrable function ϕ_0 exists for (3.42) then under the conditions of Theorem 3.4.4 where $A = \frac{1}{2}\nabla\log|\phi_0|^2$ we have a "natural" Lyapunov spectrum associated with the ϕ_0-process.

Proof

As Theorem 3.4.4. □

We now, as in Sullivan [60], restrict attention to Hyperbolic space H^{n+1} - the unique, complete, simply-connected (n+1)-dimensional manifold of constant negative curvature. Let Γ be any discrete group of hyperbolic isometries. If Γ has no torsion then H^{n+1}/Γ is a complete Riemannian manifold with constant negative curvature.

Definitions 3.6.3

(i) The limit set of Γ is the set of cluster points in S^n of any Γ orbit in H^{n+1}.

(ii) Γ is geometrically finite without cusps if Γ has a finite sided fundamental domain in H^{n+1} which does not touch the limit set.

Theorem 3.6.4 (Sullivan)

H^{n+1}/Γ has a square integrable positive μ_0-harmonic function if and only if $D > n/2$ where D is the Hausdorff dimension of the limit set and if Γ is geometrically finite then $\mu_0 = D(D-n)$ if $D > n/2$.

Proof

Sullivan's Theorem 7,[60]. □

Corollary 3.6.5

Under the conditions of Theorem 3.4.4, if $M = H^{n+1}/\Gamma$ and $D > n/2$ then the ϕ_0-process has a.s. a Lyapunov spectrum.

Remark 3.6.6

We note here that in particular if $n = 1$, then any group of isometries of the hyperbolic plane H^2 is a union of geometrically finite groups, thus $\mu_0(H^2) = D(D-1)$ if $D > \frac{1}{2}$.

CHAPTER 4.

4.1 The Ground State Process for the Hydrogen Atom

In [44] Truman and Lewis discuss the stochastic mechanics of a model of the ground state of the Hydrogen atom. The model is defined by the ground state of the Schrödinger operator

$$- \frac{\hbar^2}{2m} \Delta\phi_0(x) - \frac{Ze^2}{||x||} \phi_0(x) = E_0\phi_0(x)$$

where the particle has mass m , $\hbar = h/2\pi$ is Planck's constant and the nucleus has charge Ze . Here $\phi_0(x) : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}$ is the ground state with corresponding ground state energy E_0 , (in Gaussian units $E_0 = -\hbar^2/2ma^2$ where $a = \hbar^2/me^2Z$ is the Bohr radius.) The corresponding ground state process is given by

$$dx_t = \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} dB_t - \frac{\hbar}{ma} \frac{x_t}{||x_t||} dt \tag{4.1}$$

defined on $\mathbb{R}^3\backslash\{0\}$ and $B_t \in BM(\mathbb{R}^3)$. In [44] Truman and Lewis discussed first hitting times of the radial process corresponding to (4.1). We shall examine the long time behaviour of the process x_t using the theory developed in Chapter 3 and consider the Lyapunov exponents associated with the process x_t .

The drift vector field in (4.1) is given by

$$A(x) = - \frac{\hbar}{ma} \frac{x}{||x||} = - \frac{\hbar}{ma} \nabla(||x||) \tag{4.2}$$

and the process has finite invariant probability measure given by

$$\rho(dx) = \frac{1}{\pi a^3} e^{-2||x||/a}\, dx . \tag{4.3}$$

Here we are considering a process on $\mathbb{R}^3\backslash\{0\}$ which is slightly different from that considered in Chapter 3, Section 2, on $\mathbb{R}^n$. The system (4.1) is complete on $\mathbb{R}^3\backslash\{0\}$ (see [21]), that is its trajectories from any point in $\mathbb{R}^3\backslash\{0\}$ almost surely never hit the origin. However it is not strongly complete; the drift vector field is clearly not globally Lipschitz over $\mathbb{R}^3\backslash\{0\}$. There consequently is no smooth version of the solution flow $\xi_t(\omega) : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}^3\backslash\{0\}$ of system (4.1). However we have the following:

Proposition 4.1.1

There is no smooth flow $\xi_t(\omega) : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}^3\backslash\{0\}$ for system (4.1) defined for almost all $\omega \in \Omega$. However there does exist, for almost all ω, a flow

$$\tilde{\xi}_t(\omega) : \mathbb{R}^3 \to \mathbb{R}^3 \qquad t \geq 0 .$$

which is globally Lipschitz and such that $\{\tilde{\xi}_t(\omega)(x) ; t \geq 0\}$ solves (4.1) with initial point x, for each $x \in \mathbb{R}^3\backslash\{0\}$ where any trajectories reach 0 they are replaced by ones constantly at x.

Proof

As in [21].

From (4.1) if $x_0 \in \mathbb{R}^3\backslash\{0\}$ with $||x_0|| = \delta$, say, then

$$||x_t - B_t|| \le \delta + t/a \qquad\qquad t \ge 0 \ .$$

Now there is a positive probability that $||B_t|| > \delta + t/a$. Thus if a flow existed it must have positive probability of mapping a sphere of radius δ about 0 into some disc which does not contain 0; this is topologically impossible for a continuous map.

To obtain the Lipschitz flow on $\mathbb{R}^3$ let $x_0, y_0 \in \mathbb{R}^3\backslash\{0\}$ with x_t, y_t, $t \ge 0$ the corresponding solutions to (4.1). From (4.1) using Itô's formula, then by the Cauchy-Schwarz inequality we have

$$||x_t - y_t|| = ||x_0 - y_0|| + \frac{2\mathcal{K}}{ma}\int_0^t \left(\frac{\langle x_s, y_s\rangle}{||x_s||\,||y_s||} - 1\right)ds \le ||x_0 - y_0|| \quad \text{for all } t \ .$$

The smooth partial flow $\xi_t(\omega)$ defined on a dense open subset of $\mathbb{R}^3\backslash\{0\}$ by [21], Proposition 1D, therefore has a Lipschitz extension $\tilde{\xi}_t(\omega)$ as required. □

These maps $\tilde{\xi}_t(\omega)$, $t \ge 0$, need not be diffeomorphisms but we are able to give the existence of the Lyapunov spectrum and λ_Σ for (4.1).

For notational convenience we shall write

$$x_t = \tilde{\xi}_t(\omega)(x) \qquad \text{and} \qquad v_t = D\tilde{\xi}_t(\omega)(x)(v)$$

for the solution flow and derivative flow.

Theorem 4.1.2

For the ground state process given by (4.1) there exists a Lyapunov spectrum a.s..

Proof

The derivative process $\{v_t \; ; \; t \geq 0\}$ satisfies

$$dv_t = DA(x_t)(v_t)dt$$

$$= -\frac{\hbar}{ma}\left(\frac{v_t}{||x_t||} - \frac{\langle x_t, v_t\rangle x_t}{||x_t||^3}\right) dt \quad . \tag{4.4}$$

By condition (3.40) of Theorem 3.5.10 for the existence of the exponents we only need to consider the integrability of $||DA(x)||$ over $\mathbb{R}^3\backslash\{0\}$. Thus as

$$DA(x_1,x_2,x_3) = -\frac{\hbar}{ma||x||^3}\begin{bmatrix} x_2^2+x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2+x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_1^2+x_2^2 \end{bmatrix}$$

then

$$||DA(x)|| = \frac{\sqrt{2}\hbar}{ma} \cdot \frac{1}{||x||}$$

and since $\phi_0(x) = (a^{-3}/\pi)^{\frac{1}{2}} \, e^{-||x||/a}$

$$\int_{\mathbb{R}^3\backslash\{0\}} ||DA(x)||\rho(dx) = \int_{\mathbb{R}^3\backslash\{0\}} \frac{\sqrt{2}\hbar}{\pi ma^4} \; \frac{e^{-2||x||/a}}{||x||} \, dx$$

which by changing to polar co-ordinates yields

$$\int_{\mathbb{R}^3\setminus\{0\}} ||DA(x)||\rho(dx) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\sqrt{2}\hbar}{\pi m a^4} \frac{r^2 \sin\theta}{r} e^{-2r/a} \, dr d\theta d\phi$$

$$= \sqrt{2}\hbar/ma^2 < \infty \quad .$$

Thus the Lyapunov spectrum for (4.1) exists. Note also that

$$\text{trace } DA(x) = - \frac{2\hbar}{ma||x||}$$

which is also clearly integrable over $\mathbb{R}^3\setminus\{0\}$ and hence by condition (3.41) of Corollary 3.5.12 the sum of the exponents λ_Σ also exists a.s.. □

We also have the following:

<u>Proposition</u> 4.1.3

For the ground state process given by (4.1)

$$\lambda_\Sigma = - \text{ const. (mean Kinetic Energy of the particle).}$$

<u>Proof</u>

By (1.18)

$$\lambda_\Sigma = \int_{\mathbb{R}^3\setminus\{0\}} \text{div } A(x)\rho(dx)$$

$$= - \frac{2\hbar}{\pi m a^4} \int_{\mathbb{R}^3\setminus\{0\}} \frac{e^{-2||x||/a}}{||x||} \, dx$$

which by changing to spherical polar co-ordinates yields

$$\lambda_\Sigma = -\frac{2\hbar}{\pi m a^4}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{1}{r}\,r^2 \sin\theta\, e^{-2r/a}\,dr\,d\theta\,d\phi$$

$$= -\frac{2\hbar}{ma^2} \quad .$$

The Mean Kinetic energy for the Hydrogen atom process is given by

$$\text{K.E.} = \tfrac{1}{2}\int_{\mathbf{R}^3\setminus\{0\}} |\nabla\phi_0(x)|^2 dx = \frac{1}{2a^2} \quad .$$

Hence

$$\lambda_\Sigma = -\frac{4\hbar}{m} \times (\text{mean K.E. of the particle}). \qquad \square$$

We thus have that $\lambda_\Sigma < 0$ and from evidence given in the form of computer graphics by Durran and Truman [28] it would appear that the process is Lyapunov stable; that is the top exponent $\lambda^1 < 0$. We shall now show this by the following series of results.

Lemma 4.1.4

For the ground state process given by (4.1), $\lambda^1 \leq 0$.

Proof

By Itô's formula acting on the derivative process $\{v_t \; ; \; t \geq 0\}$ given by (4.4) we have

$$\log||v_t|| = \log||v_0|| + \frac{\hbar}{ma}\int_0^t \frac{1}{||x_s||}\left(\frac{\langle x_s, v_s\rangle^2}{||x_s||^2||v_s||^2} - 1\right)ds \qquad (4.5)$$

$$\leq \log \|v_0\| \tag{4.6}$$

by the Cauchy-Schwarz inequality. Thus

$$\lambda^1 = \lim_{t\to\infty} \frac{1}{t} \log \|v_t\| \leq \lim_{t\to\infty} \frac{1}{t} \log \|v_0\| = 0 \quad . \qquad \square$$

<u>Remark</u> 4.1.5

We note that by the Cauchy-Schwarz inequality we have equality in (4.6) if and only if x_s and v_s are collinear, that is $v_s = \alpha x_s$ for some $\alpha \neq 0$.

To overcome this we have the following series of results.

<u>Proposition</u> 4.1.6

The radial process $\|x_t\|$ satisfies

$$d\|x_t\| = \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} db_t + \left(\frac{\hbar}{m} \cdot \frac{1}{\|x_t\|} - \frac{\hbar}{ma}\right)dt \tag{4.7}$$

where $b_t \in BM(\mathbb{R})$ and the process has finite invariant probability measure

$$\rho(dr) = \text{const. } r^2 e^{-2r/a} \, dr \tag{4.8}$$

on $(0,\infty)$.

<u>Proof</u>

As in Lewis and Truman [44]. Using Itô's formula we get (4.7) with

$$db_t = \frac{\langle x_t, dB_t \rangle}{\|x_t\|}$$

which by the martingale characterization of Brownian motion yields $b_t \in BM(\mathbb{R})$ since $d\langle b\rangle_t = \langle \frac{x_t}{||x_t||}, \frac{x_t}{||x_t||}\rangle dt = dt$. The finite invariant probability measure arises from the fact that the drift term in (4.7) is of gradient form. □

Proposition 4.1.7

The process $w_t = \langle \frac{x_t}{||x_t||}, \frac{v_t}{||v_t||}\rangle$ satisfies

$$dw_t = \frac{(1-w_t^2)^{\frac{1}{2}}}{||x_t||} d\tilde{b}_t - (\frac{w_t}{||x_t||}[w_t^2 - 1 + \frac{1}{||x_t||}])dt \tag{4.9}$$

where $\tilde{b}_t \in BM(R)$ and is independent of the Brownian motion b_t of Proposition 4.1.6.

Proof

Writing $w_t = \langle \eta_t, \theta_t\rangle = \langle \frac{x_t}{||x_t||}, \frac{v_t}{||v_t||}\rangle$, by Itô's formula, setting $\cancel{k} = m = 1$ for convenience, we have

$$d\eta_t = \frac{1}{||x_t||} dB_t - \frac{\eta_t}{||x_t||} db_t - \frac{\eta_t}{||x_t||^2} dt$$

and

$$d\theta_t = \frac{1}{||x_t||}(\langle\eta_t,\theta_t\rangle\eta_t - \langle\eta_t,\theta_t\rangle^2\theta_t)dt .$$

Whence, again using Itô's formula

$$d\langle\eta_t,\theta_t\rangle = \frac{1}{||x_t||}\,d\hat{b}_t - \frac{1}{||x_t||}\langle\eta_t,\theta_t\rangle db_t - \Big(\frac{\langle\eta_t,\theta_t\rangle}{||x_t||}[\langle\eta_t,\theta_t\rangle^2 - 1 + \frac{1}{||x_t||}]\Big)dt \tag{4.10}$$

where

$$d\hat{b}_t = \langle\theta_t, dB_t\rangle$$

which again by the martingale characterization of Brownian motion gives $\hat{b}_t \in BM(\mathbb{R})$ since $d\langle\hat{b}\rangle_t = \langle\theta_t,\theta_t\rangle dt = dt$.

We note here that $\langle db_t, \frac{1}{||x_t||}\,d\hat{b}_t - \frac{1}{||x_t||}\langle\eta_t,\theta_t\rangle db_t\rangle = 0$. So considering the martingale term of (4.10) and calling this M_t , say, then

$$\begin{aligned} d\langle M\rangle_t &= \Big(\frac{1}{||x_t||}\langle\theta_t - \langle\eta_t,\theta_t\rangle\eta_t\ ,\ dB_t\rangle\Big)^2 \\ &= \frac{(1-w_t^2)}{||x_t||^2}\,dt \ . \end{aligned}$$

Thus (4.10) can be written as (4.9) with $\tilde{b}_t$ independent of b_t . □

<u>Proposition</u> 4.1.8

Writing $r_t = ||x_t||$, the coupled process (r_t,w_t) has an invariant probability measure μ on $\mathbb{R}^+ \times [-1,1]$. Moreover the set $G = \mathbb{R}^+ \times \{-1,1\}$ is non-attainable, i.e.

$$\mathbb{P}\{(r_t,w_t) \in G \text{ for some } t > 0\} = 0 ,$$

and μ has no support on G .

Proof

Since $[-1,1]$ is compact, by the Markov-Kakutani fixed point theorem there exists a probability measure μ on $\mathbb{R}^+ \times [-1,1]$ invariant for the coupled Markov process (r_t,w_t), $t \geq 0$, which projects onto the measure (4.8) on $\mathbb{R}^+$, (see Crauel, [24]).

For the non-attainability of G, note that in equation (4.9) for w_t, when $w_t = \pm 1$ the noise term vanishes leaving a drift $\mp \frac{1}{r_t^2}$ respectively. This drift vector points into $\mathbb{R}^+ \times (-1,1)$ away from G (or ∂G) which is equivalent to the Fichera drift at ∂G of the coupled process $\{(r_t,w_t) ; t \geq 0\}$ pointing into the exterior of G. Thus by Friedman's Theorem 4.1 ([35], Vol. I, section 9.4) the set G is non-attainable, and any process started on G will not reach G again. This implies that the transition probabilities

$$p_t((r,\pm 1), \mathbb{R}^+ \times (-1,1)) = 1 \qquad t > 0$$

which implies that μ is not supported on $\mathbb{R}^+ \times \{-1,1\}$. $\square$

Remark 4.1.9

We note that the Markov process $\{(r_t,w_t) ; t \geq 0\}$ on $\mathbb{R}^+ \times [-1,1]$ has differential generator

$$\tfrac{1}{2} \frac{\partial^2}{\partial r^2} + \tfrac{1}{2} \frac{(1-w^2)}{r^2} \frac{\partial^2}{\partial w^2} + (\frac{1}{r} - 1) \frac{\partial}{\partial r} - \frac{w}{r} (w^2 - 1 + \frac{1}{r}) \frac{\partial}{\partial w} \tag{4.11}$$

which is elliptic on $\mathbb{R}^+ \times (-1,1)$.

This equation does however appear extremely difficult to solve.

Theorem 4.1.10

For the ground state process given by (4.1)

$$\lambda^1 < 0 \quad . \tag{4.12}$$

Proof

By lemma 4.1.4 and equation (4.5) we have

$$\lambda^1 = \lim_{t\to\infty} \frac{1}{t} \frac{\hbar}{ma} \int_0^t \frac{1}{||x_s||} \left(\frac{\langle x_s, v_s\rangle^2}{||x_s||^2 ||v_s||^2} - 1 \right) ds$$

which by the Strong Law of Large Numbers and Proposition 4.1.8 yields

$$\lambda^1 = \frac{\hbar}{ma} \int_{\mathbb{R}^+\times[-1,1]} \frac{1}{r} (w^2 - 1)\, \mu(dr,dw) \quad . \tag{4.13}$$

This clearly shows that $\lambda^1 < 0$ unless μ has support in the set $\{(r,w)\,;\ w = \pm 1\} \equiv \{(|x|, \langle \frac{x}{|x|}, \frac{v}{|v|}\rangle)\ ;\ x = \alpha v \text{, some } \alpha \neq 0\}$ which by Proposition 4.1.8 is impossible. Also this set contains no subset invariant under the sphere bundle flow which follows from (4.5) since $x = \alpha v$ implies that v_t is constant for all $t > 0$. □

Remark 4.1.11

(i) The differential generator (4.11) can be obtained from that for the process $\{(x_t, v_t/|v_t|) ; t \geq 0\}$ on the sphere bundle $\mathbb{R}^3\backslash\{0\} \times S^2$ by considering the transformation $(r,w) = (||x||, \langle \frac{x}{||x||}, \frac{v}{||v||} \rangle)$. Hence the link between the above proof and that given for the same result in [21] where invariant measures on $\mathbb{R}^3\backslash\{0\} \times S^2$ were used.

(ii) By Theorem 4.1.10 it might be conjectured that stable manifolds exist for (4.1). By (4.12) it is clear that the solution flow is C^1 (since $||D\xi_t(\omega)(x)|| < 1$ a.s.). However the flow does not appear to be C^2 due to singularities on approaching the origin and we are unable to apply Theorems 3.3.1 and 3.3.2. It might be hoped, however, that the flow is $C^{1+\alpha}$ $(0 < \alpha < 1)$ to apply Ruelle's original results (see [56]) or that a slightly weaker version of Ruelle's results could be obtained.

4.2 The First Energy Level above the Ground State for the Hydrogen Atom Model

Analagous to section 3.5 we now consider an excited state for the Hydrogen atom model, namely the first excited state. We thus consider the eigenvalue problem

$$-\frac{\hbar^2}{2m}\Delta\phi_p(x) - \frac{Ze^2}{||x||}\phi_p(x) = E_p\phi_p(x) \qquad p = 1,2,3\ldots$$

where for notational convenience the ground state is given by $p = 1$ and $p = 2$ yields the first excited state. As in section 4.1 the associated diffusion process is given by

$$dx_t = \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} dB_t + A(x_t)dt \tag{4.14}$$

defined on $\mathbb{R}^3\backslash\{0\}$ where

$$A(x) = \tfrac{1}{2}\nabla\log|\phi_p(x)|^2$$

with invariant probability measure for x_t given by

$$\rho(dx) = |\phi_p(x)|^2\, dx\ .$$

From Messiah [46] (Appendix B, Section 3), the first excited state has energy eigenvalue $E_2 = \dfrac{-\hbar^2}{8ma^2}$ and corresponding to this eigenvalue there are four linearly independent eigenfunctions given in terms of spherical polar co-ordinates by, for $r > 0$, $0 \le \theta \le \pi$, $0 \le \Phi \le 2\pi$

$$\phi_{2,0,0}(r,\theta,\Phi) = \frac{a^{-3/2}}{\sqrt{2\pi}} \cdot \frac{1}{4}\, e^{-r/2a}\left(2 - \frac{r}{a}\right) \tag{4.15}$$

$$\phi_{2,1,0}(r,\theta,\Phi) = \frac{a^{-3/2}}{\sqrt{2\pi}} \cdot \frac{1}{4}\frac{r}{a}\, e^{-r/2a} \cos\theta \tag{4.16}$$

$$\phi_{2,1,-1}(r,\theta,\Phi) = \frac{a^{-3/2}}{\sqrt{\pi}} \cdot \frac{1}{8} \cdot \frac{r}{a}\, e^{-r/2a} \sin\theta\, e^{-i\Phi} \tag{4.17}$$

$$\phi_{2,1,1}(r,\theta,\Phi) = -\frac{a^{-3/2}}{\sqrt{\pi}} \cdot \frac{1}{8} \cdot \frac{r}{a}\, a^{-r/2a} \sin\theta\, e^{i\Phi}\ . \tag{4.18}$$

For each of these eigenstates we have the following:

Theorem 4.2.1

For each $\phi_{2\ell m}(r,\theta,\Phi)$ given above $(\ell = 0,1,\ m = -\ell,..,\ell)$ the associated diffusion process given by (4.14) has a Lyapunov spectrum.

Proof

For the connected region of $\mathbb{R}^3\backslash\{0\}$ in which the flow remains for each $\phi_{2\ell m}$ by the change of variable formula for integrals and corollary 3.5.11

$$\int_{(\mathbb{R}^3\backslash\{0\})\backslash N_{\phi_{2\ell m}}} ||DA(r,\theta,\Phi)||\,r^2\sin\theta\,|\phi_{2\ell m}(r,\theta,\Phi)|^2 dr d\theta d\Phi =$$

$$\int_{(\mathbb{R}^3\backslash\{0\})\backslash N_{\phi_{2\ell m}}} ||DA(x)||\ |\phi_{2\ell m}(x)|^2 dx < \infty.$$

Hence the existence of the Lyapunov spectrum for each $\phi_{2\ell m}$. In particular λ_Σ exists for each $\phi_{2\ell m}$. □

We now examine λ_Σ for the first excited states $\phi_{2,0,0}$, $\phi_{2,1,0}$, $\phi_{2,1,-1}$ and $\phi_{2,1,1}$.

Proposition 4.2.2

$\phi_{2,0,0}$ is the only one of the four first excited states which has different values for λ_Σ on the connected regions off the nodal set on which the trajectories of the associated diffusion process are restricted.

Proof

$\phi_{2,1,-1}$ and $\phi_{2,1,1}$ both have zeros at $\theta = 0$ and $\theta = \pi$ and the nodal set for both of these eigenfunctions is the z-axis. So the connected region off the nodal set is $(\mathbb{R}^3\backslash\{0\})\backslash\{\text{the z-axis}\}$ and by Proposition 3.5.6 for these eigenstates λ_Σ is associated with the Kinetic energy of the particle attained in this connected region.

$\phi_{2,1,0}$ has a zero at $\theta = \pi/2$ and this divides $\mathbb{R}^3\backslash\{0\}$ into two symmetrically equal regions (in fact $\mathbb{R}^3\backslash\{0\}$ is cut in half by the x-y plane). Thus for $\phi_{2,1,0}$ the Kinetic energy attained in each of these regions by the particle is the same (i.e. $\lambda_\Sigma^{z>0} = \lambda_\Sigma^{z<0}$) .

$\phi_{2,0,0}$ has a zero at $r = 2a$. Thus $\mathbb{R}^3\backslash\{0\}$ is split into two disjoint connected regions separated by, and not including, the surface of the sphere $S_0^2(2a)$, i.e.

$$\mathbb{R}^3\backslash\{0\} = S_0^2(2a) \cup (S_0^2(2a))^c$$

$(S_0^2 = S^2\backslash\{0\})$ and where c denotes the complement. Thus

$$\lambda_\Sigma^{S_0^2(2a)} = \int_{S_0^2(2a)} \operatorname{div}(\tfrac{1}{2}\nabla\log|\phi_{2,1,0}(r,\theta,\Phi)|^2)\rho^{S_0^2(2a)}(dr,d\theta,d\Phi)$$

$$= \frac{3-e^2}{a^2(e^2-7)}$$

where e is the exponential, and

$$\lambda_\Sigma^{(S_0^2(2a))^c} = \int_{\mathbb{R}^3\backslash\{0\}} \operatorname{div}(\tfrac{1}{2}\nabla\log|\phi_{2,1,0}(r,\theta,\Phi)|^2)\rho^{(\mathbb{R}^3\backslash\{0\})\backslash S_0^2(2a)}(dr,d\theta,d\Phi)$$

$$- \int_{S_0^2(2a)} \operatorname{div}(\tfrac{1}{2}\nabla \log|\phi_{2,1,0}(r,\theta,\Phi)|^2)\rho^{(\mathbb{R}^3\backslash\{0\})\backslash S_0^2(2a)}(dr,d\theta,d\Phi)$$

$$= - \frac{(3+e^2)}{14a^2} .$$

It can easily be verified that

$$\lambda_\Sigma^{S_0^2(2a)} > \lambda_\Sigma^{(\mathbb{R}^3\backslash\{0\})\backslash S_0^2(2a)}$$

and hence the exponential rate at which the flow changes volume (or measure, see e.g. [21]) is greater within the sphere of radius $2a$ (a = Bohr radius) than that outside. □

Remark 4.2.3

Note the above result and observe the computer simulations of the excited states $\phi_{2\ell m}$ obtained by Durran and Truman in [28].

4.3 An example on $\mathbb{R}^2\backslash\{0\}$

Consider the process on $\mathbb{R}^2\backslash\{0\}$ given by

$$d(x_t^1,x_t^2) = \frac{1}{||x_t||}\begin{bmatrix} x_t^1 & -x_t^2 \\ x_t^2 & x_t^1 \end{bmatrix}\begin{bmatrix} dB_t^1 \\ dB_t^2 \end{bmatrix} - (x_t^1,x_t^2)dt . \qquad (4.19)$$

Carverhill considered this process, without the drift, in [12] and showed that his system is strongly complete and gives Brownian motion on $\mathbb{R}^2\backslash\{0\}$. Thus a smooth flow exists for the system without drift. By adding the above (inward) drift which is of gradient form the process has a finite invariant measure and we have the following:

Proposition 4.3.1

The system (4.19) is strongly complete.

Proof

As in [12].

Let $\xi_t(\omega)(x)$ denote the solution flow of (4.19). Since the process has a finite invariant measure the system is complete (i.e. non-explosive). Denote by $M_t(\omega)$ the open set $\{x \in \mathbb{R}^2\backslash\{0\}$; $t < \tau(x,\omega) = \infty\}$ where $\tau(x,\omega)$ denotes the explosion time map (∞ in this case). Then $\xi_t(\omega)$ is continuous a.s. on $M_t(\omega)$. Note, also that system (4.19) is G invariant, i.e.

$$\xi_t(\omega)(gx) = g\xi_t(\omega)(x) \qquad \text{for all} \quad g \in G \text{ , } x \in \mathbb{R}^2\backslash\{0\} \tag{4.20}$$

where G is the group of rotations of $\mathbb{R}^2\backslash\{0\}$ about the origin. Also the action of $\xi_t(\omega)$ on a circle S_R of radius R centre the origin is to alter its radius but not its centre. Also $\mathbb{P}\{\xi_t(\omega)(x) = 0$, some $t \geq 0\} = 0$. Take $x \in S_R$ then by (4.20)

$$\mathbb{P}\{\text{radius of } \ \xi_t(\omega)(S_R) = 0 \ \text{ for some } \ t\} = 0 \quad .$$

Now take $y \in \mathbb{R}^2\backslash\{0\}$, $||y|| > R$. Since $\xi_t(\omega)$ is G invariant and injective on $M_t(\omega)$, $||\xi_t(\omega)(y)|| > ||\xi_t(\omega)(x)||$ for all t , because $\xi_t(\omega)(y)$ cannot pass through $\xi_t(\omega)(S_R)$, so $y \in M_t(\omega)$.

Taking the sequence $R = \{1, 1/2, 1/3, \ldots\}$ we can deduce that the reachable set $M_t(\omega)$ under $\xi_t(\omega)$ is the whole of $\mathbb{R}^2\backslash\{0\}$ and

$\xi_t(\omega)$ is continuous on $\mathbb{R}^2\backslash\{0\}$ a.s.; $\xi_t(\omega)$ does not however possess the diffeomorphism property. □

We note also that the differential generator of the process is given by

$$\tfrac{1}{2}\Delta - (x^1,x^2).\nabla$$

which is the same as that of the Ornstein-Uhlenbeck system of example 3.2.9 (ii). We therefore have a diffusion process defined in a different manner to that of example 3.2.9 (ii) and we shall examine the existence of a Lyapunov spectrum for this system.

Proposition 4.3.2

Hypothesis 1.2.5 is not satisfied for system (4.19) on $\mathbb{R}^2\backslash\{0\}$.

Proof

By Theorem 3.5.10 we need to check the integrability of $||DX(x)||^2$ and $||DA(x)||$. Clearly

$$DA(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is uniformly bounded over $\mathbb{R}^2\backslash\{0\}$. Now (4.19) is of the form

$$dx_t = X(x_t)dB_t + A(x_t)dt$$

where $X(x) : \mathbb{R}^2 \to \mathbb{R}^2$. Thus for $e = (e_1,e_2) \in \mathbb{R}^2$

$$X(x)e = 1/||x|| \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$= 1/||x||\ (x_1e_1-x_2e_2,\ x_2e_1+x_1e_2)$$

and

$$D_x X(x)e = \begin{bmatrix} \dfrac{x_2^2e_1 + x_1x_2e_2}{||x||^3} & \dfrac{-x_1^2e_2 - x_1x_2e_1}{||x||^3} \\ \dfrac{-x_1x_2e_1 + x_2^2e_2}{||x||^3} & \dfrac{x_1^2e_1 - x_1x_2e_2}{||x||^3} \end{bmatrix} .$$

Therefore, since $e_ie_j = \delta_{ij}$

$$||D_x X(x)||^2 = \frac{2}{||x||^2} .$$

The invariant measure is given by

$$\rho(dx) = \frac{1}{\pi} e^{-||x||^2} dx = \frac{1}{\pi} re^{-r^2} drd\theta$$

in polar co-ordinates. Hence

$$\int_{\mathbf{R}^2\setminus\{0\}} ||DX(x)||^2\rho(dx) = \frac{1}{\pi}\int_0^{2\pi}\int_0^\infty \frac{2}{r^2} re^{-r^2} drd\theta = 2\Gamma(0)$$

which diverges. So the integrability requirements of Hypothesis 1.2.5 are not satisfied and we cannot apply Theorem 1.2.6 to determine the existence of a Lyapunov spectrum for (4.19).

We note also that

$$\text{div } X(x)e = \frac{e_1}{||x||}$$

hence $||\text{div } X(x)||^2 = 1/||x||^2$ which by the above arguments is also not integrable over $(\mathbb{R}^2\backslash\{0\}, \rho(dx))$. We are therefore also unable to apply corollary 3.5.12 to determine the existence of λ_Σ . □

Remark 4.3.3

This again confirms the remark made in [16] (page 60), that the Lyapunov exponents are not determined by the infinitesimal generator of the process but depend upon the stochastic differential equation itself. In particular they depend upon the way in which the noise is introduced into the system.

We also have the following:

Proposition 4.3.4

The radial process $||x_t||$ of system (4.19) is given by

$$d||x_t|| = dB_t^1 + \left(\frac{1}{2||x_t||} - ||x_t|| \right)dt \tag{4.21}$$

and the angular process $\hat{X}(x_t) = x_t/||x_t||$ is given by

$$d\hat{X}(x_t) = \left(- \frac{x_t^2}{||x_t||} , \frac{x_t^1}{||x_t||} \right) dB_t^2 - \frac{1}{2||x_t||} \hat{X}(x_t)dt . \tag{4.22}$$

Proof

By Itô's formula for both (4.21) and (4.22). □

Note also that for the radial process we have an invariant measure

$$\rho(dr) = cre^{-r^2}dr$$

where c is a normalising constant. Then, despite the existence of a "radial" flow and the fact that (4.21) has a spatially homogeneous coefficient of the noise, the integral

$$\int_0^\infty ||DA(r)||\rho(dr) = \int_0^\infty |-\frac{1}{2r^2} - 1| \, cre^{-r^2} \, dr$$

$$= \int_0^\infty (\frac{1}{2r^2} + 1)cre^{-r^2} dr = \frac{c}{4}\,\Gamma(0) + \frac{c}{2} \to \infty \qquad (4.23)$$

since $\Gamma(0)$ diverges. So neither condition (3.40) nor condition (3.41) are satisfied and we are unable to apply corollary 3.5.12 to determine the existence of an exponent for (4.21).

Indeed by a simple geometrical argument it seems unlikely that a Lyapunov spectrum will exist for (4.19). Suppose $x_0 \in S_R$ for some $R > 0$ (S_R as in Proposition 4.3.1), then $gx_0 \in S_R$ also for some $g \in G$. Since the action of $\xi_t(\omega)$ is just to alter the radius of S_R and not its centre we have that $g\xi_t(\omega)(x_0) = \xi_t(\omega)(gx_0)$ and $\xi_t(\omega)(x_0)$, $t > 0$, still lie on the same circle, but under the action of $||\xi_t(\omega)(x_0)||$ this circle fluctuates (i.e. the radial process is just a Bessel process with drift). Thus

$$d(\xi_t(\omega)(x_0),\ \xi_t(\omega)(gx_0))$$

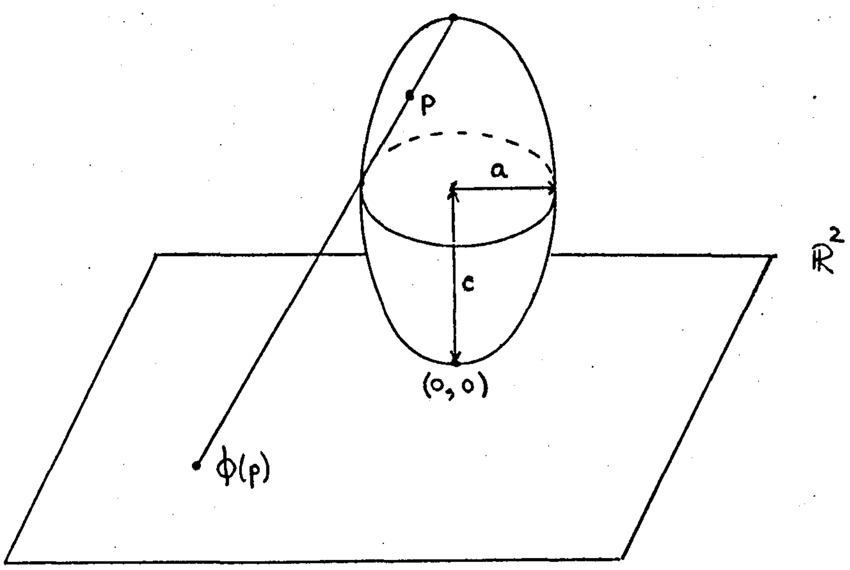
continuously increases and decreases and despite the existence of the smooth flow there appears little hope of any stable manifold or Lyapunov spectrum. In fact from Proposition 4.3.2 and (4.21) we see that $||D\xi_t(\omega)(x)||$ and $||D\phi_t(\omega)(r)||$, where $\phi_t(\omega)$ denotes the radial flow, are of the form: $\frac{\text{constant}}{||\xi_t(\omega)(x)||^2}$, and by (4.23) we are unable to make the transition from discrete to continuous time as in Theorem 1.2.6 to define a "radial exponent". However the discrete time limit does exist and equals $-\infty$.

APPENDIX A.

Computation of λ^1

In this section, prompted by the computer simulations of Brownian motion on embedded surfaces obtained by Durran and Truman, [28], we consider Brownian motion on an ellipsoid of revolution in $\mathbb{R}^3$ and attempt by computer analysis to calculate the top Lyapunov exponent associated with this system for various values of the minor and major axes (i.e. by increasing the eccentricity of the ellipsoidal nature of the surface). To obtain the stochastic system on the ellipsoid we first take the ellipsoid of revolution in $\mathbb{R}^3$ and consider the stereographic projection onto the plane $\mathbb{R}^2$ where the origin in $\mathbb{R}^2$ is the lower antipodal point of the ellipsoid (see figure (1) below and compare with Spivak [59], Vol. 4, pages 6-11).

Fig. (1)



If the major and minor axes are of lengths c and a respectively then the stereographic projection is given by $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\phi(p_1, p_2, p_3) = \left(\frac{2cp_1}{2c-p_3} , \frac{2cp_2}{2c-p_3} \right) .$$

By considering the inverse of this map we have an embedding $f : \mathbb{R}^2 \to \mathbb{R}^3$ which is defined by

$$f(y_1, y_2) = \left(\frac{y_1}{1 + \frac{1}{4a^2} \sum_{i=1}^{2} y_i^2} , \frac{y_2}{1 + \frac{1}{4a^2} \sum_{i=1}^{2} y_i^2} , \frac{\frac{c}{2a^2} \cdot \sum_{i=1}^{2} y_i^2}{1 + \frac{1}{4a^2} \sum_{i=1}^{2} y_i^2} \right) \tag{A.1}$$

As in Chapter 2 we shall derive the gradient Brownian flow on this ellipsoid of revolution via the embedding map (A.1). The gradient Brownian flow is the solution of the S.D.E.

$$dx_t = \sum_{i=1}^{3} X_i(x_t) \circ dB_t^i \tag{A.2}$$

where $X_i = \nabla f_i$; here ∇ is taken with respect to the induced Riemannian metric, and the f_i's $(1 \le i \le 3)$ are the co-ordinate functions of (A.1). Associated with (A.2) is the derivative equation

$$dv_t = \sum_{i=1}^{3} \nabla X_i(x_t) v_t \circ dB_t^i \tag{A.3}$$

and using linear approximations of solutions to (A.2) and (A.3) we shall attempt to determine the value of the top Lyapunov exponent, which by Elworthy's reformulation of Carverhill's formula (1.16) is given by

$$\lambda^1 = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\theta_s)ds \tag{A.4}$$

where $\theta_t = (\theta_{1_t}, \theta_{2_t}) = \left(\frac{v_{1_t}}{||v_t||}, \frac{v_{2_t}}{||v_t||}\right)$

and

$$f(\theta_t) = \tfrac{1}{2}\{\sum_{i=1}^{3}(||\nabla X_i(\theta_t)||^2 - 2\langle\nabla X_i(\theta_t),\theta_t\rangle^2) - K(y_t)\}\ , \tag{A.5}$$

here $K(\cdot)$ denotes the Gaussian curvature of the ellipsoid.

In Chapter 2, Section 2.4, examples 2.4.2 and 2.4.4, we obtained bounds on c and a such that the top Lyapunov exponent is strictly negative (hence Lyapunov stability). This was for the embedding $f : S^1 \times S^1 \to \mathbb{R}^3$. Despite the lack of compactness of the embedded space in (A.1) the system (A.2) will still yield Brownian motion on the ellipsoid of revolution minus its upper antipodal point (since the stereographic projection maps this point to infinity). The omission of this point makes no difference since, with probability one, Brownian motion started from another point will not hit this particular point (it is non-attainable, see e.g. Mckean [45], Friedman [35]).

By considering the noise in (A.2) and (A.3) as a set of randomly generated piecewise linear controls it is possible to formulate and solve these S.D.E.'s on a computer and achieve some estimate of the convergence or divergence of the limit of the integral in (A.4) - though whether this represents the true limit is open to question.

We shall need the following formulae arising from (A.1) in order to evaluate (A.2), (A.3) and (A.5). The computer program variable associated with each formula is given in italics.

For $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by (A.1), throughout $\Psi = 1 + \frac{1}{4a^2}\sum_{i=1}^{2} y_i^2$

$$f_{y_1} = \left(\frac{1 - \frac{y_1^2}{4a^2} + \frac{y_2^2}{4a^2}}{\Psi^2} , \frac{\frac{-y_1y_2}{2a^2}}{\Psi^2} , \frac{\frac{cy_1}{a^2}}{\Psi^2} \right) \qquad (LA,\ LC,\ LE)$$

$$f_{y_2} = \left(\frac{\frac{-y_1y_2}{2a^2}}{\Psi^2} , \frac{1 + \frac{y_2^2}{4a^2} - \frac{y_2^2}{4a^2}}{\Psi^2} , \frac{\frac{cy_2}{a^2}}{\Psi^2} \right) . \qquad (LB,\ LD,\ LF)$$

Hence the elements of the first fundamental form are given by

$$E = \frac{1}{\Psi^4}\left((1 - \frac{y_1^2}{4a^2} + \frac{y_2^2}{4a^2})^2 + \frac{y_1^2y_2^2}{4a^4} + \frac{cy_1^2}{a^4} \right) = \frac{1}{16a^4\Psi^4}\ f_1(y_1,y_2) \qquad (E)$$

$$F = \frac{1}{\Psi^4}\left(\frac{c^2-a^2}{a^4}\right) y_1y_2 \qquad (F)$$

$$G = \frac{1}{\Psi^4}\left((1 + \frac{y_1^2}{4a^2} - \frac{y_2^2}{4a^2})^2 + \frac{y_1^2y_2^2}{4a^4} + \frac{c^2y_2^2}{a^4} \right) = \frac{1}{16a^4\Psi^4}\ f_2(y_1,y_2) \qquad (G)$$

Then

$$f_{y_1y_1} = \frac{1}{\Psi^3}\left(-\frac{y_1\Psi}{2a^2} - \frac{y_1}{a^2}\left(1-\frac{y_1^2}{4a^2} + \frac{y_2^2}{4a^2}\right),\ \frac{y_2}{2a^2}\left(-\Psi + \frac{y_1^2}{a^2}\right),\ -\frac{c}{a^2}\left(-\Psi + \frac{y_1^2}{a^2}\right)\right) \quad (L1, L2, L3)$$

$$f_{y_1y_2} = \frac{1}{\Psi^3}\left(\frac{y_2\Psi}{2a^2} - \frac{y_2}{a^2}\left(1-\frac{y_1^2}{4a^2} + \frac{y_2^2}{4a^2}\right),\ \frac{y_1}{2a^2}\left(-\Psi + \frac{y_2^2}{a^2}\right),\ \frac{-cy_1y_2}{a^4}\right) \quad (L4, L5, L6)$$

$$f_{y_2y_2} = \frac{1}{\Psi^3}\left(\frac{y_1}{2a^2}\left(-\Psi + \frac{y_2^2}{a^2}\right),\ -\frac{y_2\Psi}{2a^2} - \frac{y_2}{a^2}\left(1+\frac{y_1^2}{4a^2} - \frac{y_2^2}{4a^2}\right),\ -\frac{c}{a^2}\left(-\Psi + \frac{y_2^2}{a^2}\right)\right) \quad (L7, L8, L9)$$

The normal vector to the ellipsoid of revolution is given by

$$f_{y_1} \wedge f_{y_2} = \frac{1}{\Psi^4}\left(-\frac{cy_1y_2^2}{2a^4} - \frac{cy_1}{a^2}\left(1+\frac{y_1^2}{4a^2} - \frac{y_2^2}{4a^2}\right),\ -\frac{cy_2}{a^2}\left(1-\frac{y_1^2}{4a^2} + \frac{y_2^2}{4a^2}\right) - \frac{cy_1^2y_2}{2a^4},\right.$$
$$\left.\left(1-\frac{y_1^2}{4a^2} + \frac{y_2^2}{4a^2}\right)\left(1+\frac{y_1^2}{4a^2} - \frac{y_2^2}{4a^2}\right) - \frac{y_1^2y_2^2}{4a^4}\right)$$

$$= (N1, N2, N3)\ .$$

Thus the unit normal is given by

$$\underline{N} = \frac{1}{\sqrt{EG - F^2}}\,(N_1, N_2, N_3)\ .$$

The elements of the second fundamental form are given by

$$\ell = \underline{N} \cdot f_{y_1y_1}$$

$$m = \underline{N} \cdot f_{y_1y_2}$$

$$n = \underline{N} \cdot f_{y_2y_2}$$

and hence the Gaussian curvature

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{c^2}{a^4\Psi^8(EG - F^2)^2} \quad . \qquad (M7)$$

The vector fields X_i $(1 \le i \le 3)$ in (A.2) are given by

$$X_i = \nabla f_i = g^{ij} \frac{\partial f_i}{\partial y_j} \qquad (A.6)$$

where (g^{ij}) is the inverse matrix of the first fundamental form given by

$$(g^{ij}) = \frac{1}{(EG - F^2)} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \quad .$$

Thus

$$g^{11} = \frac{G}{EG-F^2} \quad , \quad g^{12} = g^{21} = \frac{-F}{EG-F^2} \quad , \quad g^{22} = \frac{E}{EG-F^2} \quad . \qquad (A.7)$$

We shall also need the following:

$$\frac{\partial g_{11}}{\partial y_1} = \frac{\partial E}{\partial y_1} = \frac{y_1}{16a^6\Psi^5} [(4a^2+y_1^2+y_2^2)(y_1^2+y_2^2+4(2c^2-a^2)) - 2(16a^4+8a^2y_1^2+8y_2^2(2c^2-a^2) + y_1^4 + y_2^4 +2y_1^2y_2^2)] \qquad (G1)$$

$$\frac{\partial g_{11}}{\partial y_2} = \frac{\partial E}{\partial y_2} = \frac{y_2}{16a^6\Psi^5} [(4a^2+y_1^2+y_2^2)^2 - 2(16a^4+y_1^4+y_2^4+2y_1^2y_2^2+8y_1^2(2c^2-a^2) + 8a^2y_2^2)] \qquad (G2)$$

$$\frac{\partial g_{12}}{\partial y_1} = \frac{\partial F}{\partial y_1} = \frac{(c^2-a^2)}{4a^6\Psi^5} y_2 \;(4a^2 - 7y_1^2 + y_2^2) \tag{G3}$$

$$\frac{\partial g_{12}}{\partial y_2} = \frac{\partial F}{\partial y_2} = \frac{(c^2-a^2)}{4a^6\Psi^5} y_1 \;(4a^2 - 7y_2^2 + y_1^2) \tag{G4}$$

$$\frac{\partial g_{22}}{\partial y_1} = \frac{\partial G}{\partial y_1} = \frac{y_1}{16a^6\Psi^5}\left[(4a^2+y_1^2+y_2^2)^2 - 2(16a^4+8a^2y_1^2+8y_2^2(2c^2-a^2)+y_1^4+y_2^4+2y_1^2y_2^2)\right] \tag{G5}$$

$$\begin{aligned}\frac{\partial g_{22}}{\partial y_2} = \frac{\partial G}{\partial y_2} = \frac{y_2}{16a^6\Psi^5} \big[&(4a^2+y_1^2+y_2^2)(y_1^2+y_2^2+4(2c^2-a^2)) \\ &- 2(16a^4+8a^2y_1^2+8y_2^2(2c^2-a^2)+y_1^4+y_2^4+2y_1^2y_2^2)\big] \;.\end{aligned} \tag{G6}$$

For equation (A.3) we shall need to calculate ∇X_i $(1 \le i \le 3)$. Since if $X = (x_1, x_2)$

$$\nabla X = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} + \Gamma^1_{11}x_1 + \Gamma^1_{12}x_2 & \dfrac{\partial x_1}{\partial y_2} + \Gamma^1_{21}x_1 + \Gamma^1_{22}x_2 \\[2ex] \dfrac{\partial x_2}{\partial y_1} + \Gamma^2_{11}x_1 + \Gamma^2_{12}x_2 & \dfrac{\partial x_2}{\partial y_2} + \Gamma^2_{21}x_1 + \Gamma^2_{22}x_2 \end{bmatrix}$$

where Γ^k_{ij} are the Christoffel symbols which can be calculated from (A.7) and *G1* to *G6*. Also by (A.6) for each X_i $(1 \le i \le 3)$

$$\frac{\partial x_1}{\partial y_1} = \frac{1}{(EG-F^2)^2}\{[\frac{\partial G}{\partial y_1}\frac{\partial f_i}{\partial y_1} + G\frac{\partial^2 f_i}{\partial y_1^2} - \frac{\partial F}{\partial y_1}\frac{\partial f_i}{\partial y_2} - F\frac{\partial^2 f_i}{\partial y_1 \partial y_2}](EG-F^2)$$

$$- (G\frac{\partial f_i}{\partial y_1} - F\frac{\partial f_i}{\partial y_2})[\frac{\partial E}{\partial y_1}G + E\frac{\partial G}{\partial y_1} - 2\frac{\partial F}{\partial y_1}F]\} \qquad \textit{(H1)}$$

$$\frac{\partial x_1}{\partial y_2} = \frac{1}{(EG-F^2)^2}\{[\frac{\partial G}{\partial y_2}\frac{\partial f_i}{\partial y_1} + G\frac{\partial^2 f_i}{\partial y_2 \partial y_1} - \frac{\partial F}{\partial y_2}\frac{\partial f_i}{\partial y_2} - F\frac{\partial^2 f_i}{\partial y_2^2}](EG-F^2)$$

$$- (G\frac{\partial f_i}{\partial y_1} - F\frac{\partial f_i}{\partial y_2})[\frac{\partial E}{\partial y_2}G + E\frac{\partial G}{\partial y_2} - 2\frac{\partial F}{\partial y_2}F]\} \qquad \textit{(H2)}$$

and similar formulae follow for $\frac{\partial x_2}{\partial y_1}$ and $\frac{\partial x_2}{\partial y_2}$. All of these (12 in total) can be calculated from *LA* to *LF*, *L1* to *L9*, *E*, *F* and *G* and *G1* to *G6* to give *H1* to *H9*, *HA*, *HB* and *HC*.

Using these formulae we shall attempt to evaluate the limit (A.4) in the following manner:

Consider $\frac{1}{t}\int_0^t f(\theta_s)ds$ as the sum

$$\frac{1}{t}\sum_{n=0}^{N} f(\theta_{n\delta t})\delta t$$

where $t = N.\delta t$. So this becomes

$$\frac{1}{N} \sum_{n=0}^{N} f(\theta_{n\delta t})$$

and hence the approximation to the ergodic limit as N increases.

Below is a listing of the program used. The noise is introduced in a standard computational manner, (see for example [55]). The program was written in the main by Neil Watling on his own COMMODORE 128 personal computer, for which the author owes his grateful thanks. Following the listing are the results obtained from runs of the program where the lengths of the major and minor axes c and a are varied, the time increment δt (*DT*) is altered and the number of time intervals over which the limit is evaluated is varied.

```
1 REM************************************
2 REM****    STOCHASTIC MODEL II    ****
3 REM*********(29/5/87)****************
10 GOSUB1000
20 GOSUB900
25 TRAP1100
30 GOSUB100
40 GOSUB1200
50 END
100 REM*** CALCULATION!!
110 FORT=1TON
112 GOSUB1500
115 Y1=Y(1,T-1):Y2=Y(2,T-1)
120 C2=2*C0*Y1:C3=2*C0*Y2:C5=4*Y1*Y1:C6=4*Y2*Y2:C7=Y1*Y2:C8=C5
/4:C9=C6/4
125 A1=C0+C8+C9:A2=C0-C8+C9:A3=C0+C8-C9:A4=C0-C8-C9
130 B2=(A2*A2)+(4*C7*C7)+(4*C*C*C5):B3=(A3*A3)+(4*C7*C7)+(4*C*
C*C6)
135 D=A1/C0:D0=2*C4*C5*C6:D1=1/(C0*C0*D*D*D*D):D2=A4-8*C*C
140 D3=(D1*Y1)/(A*A*D):D4=(D1*Y2)/(A*A*D):D7=(Y1*D)/(A*A):D8=(
Y2*D)/(A*A)
142 REM*** FIRST FUNDAMENTAL FORM
145 E=D1*B2:F=16*C4*C7*D1:G=D1*B3
147 REM*** DERIVATIVES OF FIRST FUNDMENTAL FORM
150 G1=D3*((A1*(-D2))-(2*B2)):G2=D4*((A1*A1)-(2*B2)):G3=4*C4*D
4*(A2-(3*C5/2))
155 G6=D4*((A1*(-D2))-(2*B3)):G5=D3*((A1*A1)-(2*B3)):G4=4*C4*D
3*(A3-(3*C6/2))
160 E0=D*D:DF=SQR(E*G-F*F):DG=C0*D/2:DH=DF*DF
165 D9=1/(E0*E0*2):DA=C1*D1*D:DC=D7*D9:DD=((C5/C0)-D):DE=((C6/
C0)-D)
167 REM*** SECOND DERIVATIVES OF EMBEDDING
170 L1=-D1*D7*C0*(DG+A2)
175 L2=D8*D9*DD
180 L3=(-DA)*DD
185 L4=D8*D1*C0*(DG-A2)
190 L5=DC*DE
195 L6=-16*C*D*D1*C7
200 L7=L5
205 L8=-D1*D8*C0*(DG+A3)
210 L9=(-DA)*DE
212 REM*** FIRST PARTIAL DERIVATIVES OF EMBEDDING
215 LA=A2/(C0*E0)
220 LB=-(2*C7)/(C0*E0)
225 LC=LB
230 LD=A3/(C0*E0)
235 LE=4*C*Y1/(C0*E0)
240 LF=4*C*Y2/(C0*E0)
242 REM*** COORDINATES OF GRADIENT VECTOR FIELDS
243 REM***                 (X1,X2,X3)
250 U1=(G*LA-F*LB)/DH:U2=(E*LB-F*LA)/DH
255 W1=(G*LC-F*LD)/DH:W2=(E*LD-F*LC)/DH
260 Z1=(G*LE-F*LF)/DH:Z2=(E*LF-F*LE)/DH
265 E1=G1*G+E*G5-2*G3*F:E2=G2*G+E*G6-2*G4*F
267 REM*** PARTIAL DERIVATIVES OF COORDINATES
268 REM***          OF GRADIENT VECTOR FIELDS
270 H1=((G5*LA+G*L1-G3*LB-F*L4)-U1*E1)/DH
275 H2=((G6*LA+G*L4-G4*LB-F*L7)-U1*E2)/DH
```

```
280 H3=((G1*LB+E*L4-G3*LA-F*L1)-U2*E1)/DH
285 H4=((G2*LB+E*L7-G4*LA-F*L4)-U2*E2)/DH
290 H5=((G5*LC+G*L2-G3*LD-F*L5)-W1*E1)/DH
295 H6=((G6*LC+G*L5-G4*LD-F*L8)-W1*E2)/DH
300 H7=((G1*LD+E*L5-G3*LC-F*L2)-W2*E1)/DH
305 H8=((G2*LD+E*L8-G4*LC-F*L5)-W2*E2)/DH
310 H9=((G5*LE+G*L3-G3*LF-F*L6)-Z1*E1)/DH
315 HA=((G6*LE+G*L6-G4*LF-F*L9)-Z1*E2)/DH
320 HB=((G1*LF+E*L6-G3*LE-F*L3)-Z2*E1)/DH
325 HC=((G2*LF+E*L9-G4*LE-F*L6)-Z2*E2)/DH
327 REM*** CHRISTOFFEL SYMBOLS
330 K1=(G1*G-(2*G3-G2)*F)/(2*DH)
335 K2=(G2*G-G5*F)/(2*DH)
340 K3=((2*G4-G5)*G-G6*F)/(2*DH)
345 K4=((2*G3-G2)*E-G1*F)/(2*DH)
350 K5=(G5*E-G2*F)/(2*DH)
355 K6=(G6*E-(2*G4-G5)*F)/(2*DH)
390 REM*** LINEAR APPROXIMATION OF SOLUTION TO
391 REM***              S.D.E ON ELLIPSOID
400 Y(1,T)=Y(1,T-1)+(U1*N1+W1*N2+Z1*N3)
405 Y(2,T)=Y(2,T-1)+(U2*N1+W2*N2+Z2*N3)
408 REM*** INTIAL CONDITIONS FOR DERIVATIVE EQUATION
410 IFT=1THENV(1,0)=(Y(1,1)-Y(1,0))/DT:V(2,0)=(Y(2,1)-Y(2,0))/
DT
415 J1=V(1,T-1):J2=V(2,T-1):GOSUB800
417 REM*** LINEAR APPROXIMATION OF SOLUTION TO
418 REM***              DERIVATIVE EQUATION
420 V(1,T)=V(1,T-1)+(F1*N1+F2*N2+F3*N3)
425 V(2,T)=V(2,T-1)+(F4*N1+F5*N2+F6*N3)
427 REM*** NORMALIZE SOLUTION TO GIVE PROCESS
428 REM***              ON THE SPHERE BUNDLE
430 NM=SQR(J1*J1+J2*J2)
435 U(1,T)=V(1,T-1)/NM:U(2,T)=V(2,T-1)/NM
440 J1=U(1,T):J2=U(2,T):GOSUB800
460 REM*** GAUSSIAN CURVATURE
470 M7=(32*C*C*D1*D9)/(DH*DH)
475 REM*** NORMS OF COVARIANT DERIVATIVES FOR INTEGRAND
480 P1=(F1*F1*E)+(2*F1*F4*F)+(F4*F4*G)
485 P2=(F2*F2*E)+(2*F2*F5*F)+(F5*F5*G)
490 P3=(F3*F3*E)+(2*F3*F6*F)+(F6*F6*G)
495 P4=(F1*U(1,T)*E)+(F4*U(1,T)*F)+(F1*U(2,T)*F)+(F4*U(2,T)*G)
500 P5=(F2*U(1,T)*E)+(F5*U(1,T)*F)+(F2*U(2,T)*F)+(F5*U(2,T)*G)
505 P6=(F3*U(1,T)*E)+(F6*U(1,T)*F)+(F3*U(2,T)*F)+(F6*U(2,T)*G)
507 P7=P1+P2+P3:P8=(P4*P4)+(P5*P5)+(P6*P6)
508 REM*** INTEGRAND
510 I(T)=(P7-2*P8-M7)/2
515 GOSUB700
520 NEXT
550 RETURN
700 REM*** APPROXIMATION OF INTEGRAND AND ERGODIC LIMIT
710 I=I+I(T)
720 IT=I/T
725 OPEN4,3
730 PRINT#4,"AT TIME INTERVAL";T
740 PRINT#4,"Y(1,";T;")  :- ";Y(1,T),"Y(2,";T;")  :- ";Y(2,T)
750 PRINT#4,"V(1,";T;")  :- ";V(1,T),"V(2,";T;")  :- ";V(2,T)
760 PRINT#4,"U(1,";T;")  :- ";U(1,T),"U(2,";T;")  :- ";U(2,T)
765 PRINT#4,"CURVATURE :- ";M7
770 PRINT#4,"I(";T;")  :- ";I(T)
780 PRINT#4,"INTEGRAL APPROX.";IT
790 PRINT#4,CHR$(13)
```

```
792 CLOSE4
795 RETURN
800 REM*** COVARIANT DERIVATIVES OF GRADIENT VECTOR FIELDS
810 F1=((H1+(K1*U1)+(K2*U2))*J1)+((H2+(K2*U1)+(K3*U2))*J2)
820 F2=((H5+(K1*W1)+(K2*W2))*J1)+((H6+(K2*W1)+(K3*W2))*J2)
830 F3=((H9+(K1*Z1)+(K2*Z2))*J1)+((HA+(K2*Z1)+(K3*Z2))*J2)
840 F4=((H3+(K4*U1)+(K5*U2))*J1)+((H4+(K5*U1)+(K6*U2))*J2)
850 F5=((H7+(K4*W1)+(K5*W2))*J1)+((H8+(K5*W1)+(K6*W2))*J2)
860 F6=((HB+(K4*Z1)+(K5*Z2))*J1)+((HC+(K5*Z1)+(K6*Z2))*J2)
870 RETURN
900 REM*** INITIALIZATION
905 DIMY(2,N),V(2,N),U(2,N),I(N)
910 DEFFNA(X)=SQR(-2*LOG(X))
915 DEFFNB(X)=2*π*X
940 Q3=5*(RND(0)-.5):Q4=5*(RND(0)-.5):IFQ3=0ORQ4=0THEN940
950 Y(1,0)=Q3:Y(2,0)=Q4
955 C0=4*A*A:C1=4*C*C0:C4=C*C-A*A
960 T=0:DT=6E-2:I=0
990 RETURN
1000 REM*** INPUT ROUTINE
1010 PRINT"(CLR)":INPUT"A,C";A,C
1020 IFC<ATHEN1010
1030 INPUT"NUMBER OF TIME PERIODS";N
1040 PRINT"SCREEN OR PRINTER ?"
1050 GETA$:IFA$=""THEN1050
1060 DV=3
1070 IFA$="P"THENDV=4
1075 PRINT"O.K."
1080 RETURN
1099 REM*** ERROR ROUTINE
1100 IFER=15THENA$="OVERFLOW AT TIME ":GOTO1115
1105 IFER=20THENA$="DIVISION BY ZERO AT TIME ":GOTO1115
1110 PRINTERR$(ER);EL:GOTO50
1115 OPEN4,DV
1120 PRINT#4,A$;T
1125 CLOSE4
1130 RESUME 550
1200 REM*** VARIABLE DETAILS
1205 OPEN4,DV
1210 PRINT#4,"GENERAL VARIABLES :-"
1215 PRINT#4," NUMBER OF TIME INTERVALS ";N
1220 PRINT#4," A = ";A,"C = ";C
1225 PRINT#4," STARTING POINT :-  Y(1,0),Y(2,0)"
1230 PRINT#4,Y(1,0),Y(2,0)
1235 PRINT#4," STARTING VELOCITY :-  V(1,0),V(2,0)"
1240 PRINT#4,V(1,0),V(2,0)
1245 PRINT#4," TIME INTERVAL :-",DT
1250 PRINT#4," FIRST INTEGRAL APPROXIMATION :-",I(1)
1255 PRINT#4," FINAL INTEGRAL APPROXIMATION :-",IT
1260 CLOSE4
1500 REM*** NOISE TERM
1510 Q1=RND(0):IFQ1=0THEN1510
1520 Q2=RND(0):IFQ2=0THEN1520
1530 R1=FNA(Q1):R2=FNA(Q2)
1540 S1=FNB(RND(0)):S2=FNB(RND(0))
1550 N1=SQR(DT)*R1*COS(S1)
1560 N2=SQR(DT)*R1*SIN(S1)
1570 N3=SQR(DT)*R2*SIN(S2)
1580 RETURN
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1              C =  1
 STARTING POINT :-  Y(1,0),Y(2,0)
 1.07431948            -.0194376707
 STARTING VELOCITY :-  V(1,0),V(2,0)
 242.48835            -17.9879908
 TIME INTERVAL :-            6E-05
 FIRST INTEGRAL APPROXIMATION :-           -.561544228
 FINAL INTEGRAL APPROXIMATION :-           -.589452908
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1              C =  2
 STARTING POINT :-  Y(1,0),Y(2,0)
 .0977763533           -2.4022311
 STARTING VELOCITY :-  V(1,0),V(2,0)
-224.240831            101.444886
 TIME INTERVAL :-            6E-05
 FIRST INTEGRAL APPROXIMATION :-           -.0872294592
 FINAL INTEGRAL APPROXIMATION :-           -.0848872563
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1              C =  5
 STARTING POINT :-  Y(1,0),Y(2,0)
-1.9920668             1.56261176
 STARTING VELOCITY :-  V(1,0),V(2,0)
-340.074658           -499.676587
 TIME INTERVAL :-            6E-05
 FIRST INTEGRAL APPROXIMATION :-           .0322762875
 FINAL INTEGRAL APPROXIMATION :-           .03153557
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1              C =  10
 STARTING POINT :-  Y(1,0),Y(2,0)
-.156155825           -1.26944482
 STARTING VELOCITY :-  V(1,0),V(2,0)
 238.420689           -42.6014498
 TIME INTERVAL :-            6E-05
 FIRST INTEGRAL APPROXIMATION :-           -.0104983336
 FINAL INTEGRAL APPROXIMATION :-           -.0127566191
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1              C =  15
 STARTING POINT :-  Y(1,0),Y(2,0)
 .839964151          -.273325443
 STARTING VELOCITY :-  V(1,0),V(2,0)
 22.5074395             124.485077
 TIME INTERVAL :-              6E-05
 FIRST INTEGRAL APPROXIMATION :-            -.274621375
 FINAL INTEGRAL APPROXIMATION :-            -.269859061
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1              C =  100
 STARTING POINT :-  Y(1,0),Y(2,0)
-.605390966             1.48452014
 STARTING VELOCITY :-  V(1,0),V(2,0)
 163.893122             70.5759973
 TIME INTERVAL :-              6E-05
 FIRST INTEGRAL APPROXIMATION :-            .0502010158
 FINAL INTEGRAL APPROXIMATION :-            .0497175526
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  1000
 A =  1              C =  1
 STARTING POINT :-  Y(1,0),Y(2,0)
-1.11313939             .273516476
 STARTING VELOCITY :-  V(1,0),V(2,0)
 39.4534475          -123.914273
 TIME INTERVAL :-              6E-05
 FIRST INTEGRAL APPROXIMATION :-            -.537750595
 FINAL INTEGRAL APPROXIMATION :-            -.529715427
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  1000
 A =  1              C =  2
 STARTING POINT :-  Y(1,0),Y(2,0)
 1.79695994          -2.01161891
 STARTING VELOCITY :-  V(1,0),V(2,0)
-5.18265491          -195.760777
 TIME INTERVAL :-              6E-05
 FIRST INTEGRAL APPROXIMATION :-            -.118288204
 FINAL INTEGRAL APPROXIMATION :-            -.124956291
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  1000
 A =  1              C =  5
 STARTING POINT :-  Y(1,0),Y(2,0)
 .937792063             -1.26930058
 STARTING VELOCITY :-  V(1,0),V(2,0)
 265.877275             213.349215
 TIME INTERVAL :-              6E-05
 FIRST INTEGRAL APPROXIMATION :-              .0255964501
 FINAL INTEGRAL APPROXIMATION :-              .025456772
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  1000
 A =  1              C =  10
 STARTING POINT :-  Y(1,0),Y(2,0)
-.80060333              -2.05059022
 STARTING VELOCITY :-  V(1,0),V(2,0)
 608.947155             -205.64332
 TIME INTERVAL :-              6E-05
 FIRST INTEGRAL APPROXIMATION :-              .0556664844
 FINAL INTEGRAL APPROXIMATION :-              .051734661
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  1000
 A =  1              C =  15
 STARTING POINT :-  Y(1,0),Y(2,0)
 1.75791174             -1.46472991
 STARTING VELOCITY :-  V(1,0),V(2,0)
-147.144046             -257.859969
 TIME INTERVAL :-              6E-05
 FIRST INTEGRAL APPROXIMATION :-              .0544448985
 FINAL INTEGRAL APPROXIMATION :-              .0428694572
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  1000
 A =  1              C =  100
 STARTING POINT :-  Y(1,0),Y(2,0)
-2.12880761              2.32433051
 STARTING VELOCITY :-  V(1,0),V(2,0)
-163.948024             -159.055817
 TIME INTERVAL :-              6E-05
 FIRST INTEGRAL APPROXIMATION :-              .0419186189
 FINAL INTEGRAL APPROXIMATION :-              .0390429723
```

```
DIVISION BY ZERO AT TIME  15
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1               C =  1
 STARTING POINT :-  Y(1,0),Y(2,0)
-2.12882459            2.03139842
 STARTING VELOCITY :-  V(1,0),V(2,0)
 20.5251469           -1.9941103
 TIME INTERVAL :-              .06
 FIRST INTEGRAL APPROXIMATION :-              -.460044385
 FINAL INTEGRAL APPROXIMATION :-              -.469596834
```

```
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1               C =  2
 STARTING POINT :-  Y(1,0),Y(2,0)
 1.26962125            1.05476916
 STARTING VELOCITY :-  V(1,0),V(2,0)
-4.77922491            1.42556714
 TIME INTERVAL :-              .06
 FIRST INTEGRAL APPROXIMATION :-              -.120409754
 FINAL INTEGRAL APPROXIMATION :-              -.211931778
```

```
DIVISION BY ZERO AT TIME  36
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1               C =  5
 STARTING POINT :-  Y(1,0),Y(2,0)
-1.54282659            .156382918
 STARTING VELOCITY :-  V(1,0),V(2,0)
-.271211417            13.8461709
 TIME INTERVAL :-              .06
 FIRST INTEGRAL APPROXIMATION :-              .0201166184
 FINAL INTEGRAL APPROXIMATION :-              -.986515392
```

```
DIVISION BY ZERO AT TIME  44
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1               C =  10
 STARTING POINT :-  Y(1,0),Y(2,0)
-2.460832             -1.46474719
 STARTING VELOCITY :-  V(1,0),V(2,0)
 8.95973007           -19.5644572
 TIME INTERVAL :-              .06
 FIRST INTEGRAL APPROXIMATION :-              .0406869269
 FINAL INTEGRAL APPROXIMATION :-              -1.93860695
```

```
DIVISION BY ZERO AT TIME  77
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1             C =  15
 STARTING POINT :-  Y(1,0),Y(2,0)
-.488201678            .0196793675
 STARTING VELOCITY :-  V(1,0),V(2,0)
 .896788287           6.89067129
 TIME INTERVAL :-             .06
 FIRST INTEGRAL APPROXIMATION :-              -1.67296568
 FINAL INTEGRAL APPROXIMATION :-              -1.54103055


DIVISION BY ZERO AT TIME  44
GENERAL VARIABLES :-
 NUMBER OF TIME INTERVALS  100
 A =  1            C =  100
 STARTING POINT :-  Y(1,0),Y(2,0)
-2.40226686           -2.08969951
 STARTING VELOCITY :-  V(1,0),V(2,0)
 1.2497664           -1.53009598
 TIME INTERVAL :-             .06
 FIRST INTEGRAL APPROXIMATION :-              .0413001268
 FINAL INTEGRAL APPROXIMATION :-             -8.34299817
```

We see from the above results that as the ratio of the major and minor axes c and a increases the limit considered decreases (and becomes increasingly negative); the division by zero denotes a very large negative result. Compare this with the result obtained in example 2.1.3 (ii) where for the 1-dimensional ellipse the associated Lyapunov exponent tends to $-\infty$ as the eccentricity of the ellipse increases.

We also note that the above limits are negative (or approach negativity). If this limit truly represents the top Lyapunov exponent then we conjecture from the results of section 2.4 that the invariant measure on the sphere bundle required for (1.12) has little or no support on the regions where the integrand (2.48) is positive, these regions lying away from the umbilic points. Our overall conjecture being that the Lyapunov spectrum for a convex hypersurface embedded in Euclidean space is almost surely negative.

APPENDIX B.

A Non-Attainability Result

Following Remark 4.1.5 we have the following direct non-attainability result for the ground state system (4.1) associated with the Hydrogen atom model of Lewis and Truman [44].

Proposition

For the ground state process x_t given by (4.1) the submanifold $M = \{(x,\alpha x)\ ;\ \alpha \neq 0\}$ included in $(\mathbb{R}^3\backslash\{0\}) \times (\mathbb{R}^3\backslash\{0\})$ is non-attainable.

Proof

We shall follow closely the method used in Friedman, Vol. II, Chapter 11 [35]. Note that M is closed in $(\mathbb{R}^3\backslash\{0\}) \times (\mathbb{R}^3\backslash\{0\})$. Clearly M is a 4-dimensional submanifold of $(\mathbb{R}^3\backslash\{0\}) \times (\mathbb{R}^3\backslash\{0\})$. We can therefore assume that there exist 2 linearly independent vectors in $(\mathbb{R}^3\backslash\{0\}) \times (\mathbb{R}^3\backslash\{0\})$ which are normal to M. Let these be denoted by

$$\underline{N}_1 = (\underline{\omega}_1,\underline{\omega}_2)$$

$$\underline{N}_2 = (\underline{\omega}_3,\underline{\omega}_4)\ .$$

The equation for the process (x_t,v_t) on $(\mathbb{R}^3\backslash\{0\}) \times (\mathbb{R}^3\backslash\{0\})$ is given by

$$d(x_t,v_t) = \left(\frac{\cancel{h}}{m}\right)^{\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dB_t^1 \\ dB_t^2 \\ dB_t^3 \\ dB_t^4 \\ dB_t^5 \\ dB_t^6 \end{bmatrix}$$

$$- \frac{\cancel{h}}{ma} \left(\frac{x_t}{||x_t||}, \frac{v_t}{||x_t||} - \frac{\langle x_t,v_t\rangle x_t}{||x_t||^3}\right) dt$$

where $B_t = (B_t^1,\ldots,B_t^6) \in BM(\mathbb{R}^6)$. This equation is of the form

$$dY_t = \sigma(Y_t)dB_t + b(Y_t)dt$$

as required by Friedman. We now verify that (4.1) satisfies Friedman's condition (A1):

(i) $||\sigma(x)|| + ||b(x)|| \leq C(1 + ||x||)$

for some constant C , for all $x \in (\mathbb{R}^3\backslash\{0\}) \times (\mathbb{R}^3\backslash\{0\})$.

This is clearly true choosing $C = 3$.

(ii) For any $R > 0$, $\exists\, C_R > 0$ such that

$$||\sigma(x) - \sigma(y)|| + ||b(x) - b(y)|| \leq C_R||x-y|| \quad .$$

If $||x|| < R$, $||y|| < R$.

Since σ is independent of $x \in (\mathbb{R}^3\backslash\{0\}) \times (\mathbb{R}^3\backslash\{0\})$, $||\sigma(x)-\sigma(y)|| = 0$. So we need to verify that for some $C_R > 0$

$$||b(x,v) - b(y,\omega)|| \le C_R||(x,v) - (y,\omega)||$$

if $||x|| < R$, $||y|| < R$.

From (4.1)

$$||b(x,v) - b(y,\omega)||^2 = \frac{\kappa^2}{m^2a^2}\{|| \frac{y}{||y||} - \frac{x}{||x||}||^2 + ||\frac{<x,v>x}{||x||^3} - \frac{v}{||x||} - \frac{<y,\omega>y}{||y||^3} + \frac{\omega}{||y||}||^2\} \tag{A.8}$$

and this must be less than or equal to

$$C_R^2 (||x - y||^2_{\mathbb{R}^3} + ||v - \omega||^2_{\mathbb{R}^3}) \tag{A.9}$$

for some $C_R > 0$. Let $D_R = (\frac{C_R ma}{\kappa})^2$. We first consider the first terms in the brackets of expressions (A.8) and (A.9). So we require that

$$2 - \frac{2<x,y>}{||x||||y||} \le D_R(||x||^2 - 2<x,y> + ||y||^2)$$

which gives

$$2<x,y>\{D_R - \frac{1}{||x||||y||}\} \le D_R(||x||^2 + ||y||^2) - 2$$

or

$$<x,y> \le \frac{||x||||y||(D_R(||x||^2 + ||y||^2) - 2)}{2(D_R||x||||y|| - 1)}$$

which by the Cauchy-Schwarz inequality is true if

$$\frac{D_R\ (||x||^2 + ||y||^2) - 2}{2\ (D_R||x||\,||y|| - 1)} \geq 1 \quad ,$$

that is

$$D_R(||x||^2 + ||y||^2) - 2 \geq 2\ D_R||x||\,||y|| - 2$$

yielding

$$D_R(||x|| - ||y||)^2 \geq 0$$

which is clearly true for all $R > 0$.

We now consider the second terms in the brackets of (A.8) and (A.9). We then require that for some $D_R > 0$

$$\{\frac{||v||^2}{||x||^2} - \frac{2<v,\omega>}{||x||\,||y||} - \frac{2<v,x>^2}{||x||^4} + \frac{2<v,y><y,\omega>}{||x||\,||y||^3} + \frac{||\omega||^2}{||y||^2} + \frac{2<x,v><x,\omega>}{||y||\,||x||^3} - \frac{2<y,\omega>^2}{||y||^4}$$

$$+ \frac{<x,v>}{||x||^4} - \frac{2<x,v><y,\omega><x,y>}{||x||^3||y||^3} + \frac{<y,\omega>}{||y||^4}\} \leq D_R(||v||^2 - 2<v,\omega> + ||\omega||^2)$$

$$\text{if} \quad ||x||^2 + ||v||^2 < R^2 \quad \text{and} \quad ||y||^2 + ||\omega||^2 < R^2 \ . \tag{A.10}$$

Using the Cauchy-Schwarz inequality this reduces to

$$\frac{<x,v>}{||x||^4}(1 - <x,v>) + \frac{<y,\omega>}{||y||^4}(1 - <y,\omega>) \leq D_R(||v||^2 - 2<v,\omega> + ||\omega||^2) \ .$$

Rearranging this as

$$0 \le D_R||x||^4||y||^4(||v-\omega||^2) + ||y||^4<x,v>(<x,v> - 1) + ||y||^4<y,\omega>(<y,\omega>-1) \tag{A.11}$$

and as $<x,v> \le \frac{||x||^2+||v||^2}{2}$ and $<y,\omega> \le \frac{||y||^2+||\omega||^2}{2}$, using (A.10) above a $D_R > 0$ can always be found such that (A.11) holds.

As condition (A.1) is satisfied, following Friedman we consider the matrix

$$a(x) = \sigma(x)\sigma^*(x) \quad .$$

By (4.1) this is given by

$$a(x) = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

where I_3 is the 3×3 identity matrix. We now consider the matrix (α_{ij}), $1 \le i,j \le 2$, where

$$\alpha_{ij} = <a(x)\underline{N}_i, \underline{N}_j>_{\mathbb{R}^6} \quad .$$

Thus

$$\alpha_{ij} = \begin{bmatrix} ||\underline{\omega}_1||^2 & <\underline{\omega}_1, \underline{\omega}_3>_{\mathbb{R}^3} \\ <\underline{\omega}_3, \underline{\omega}_1>_{\mathbb{R}^3} & ||\underline{\omega}_3||^2 \end{bmatrix} \quad .$$

So

$$\det(\alpha_{ij}) = ||\underline{\omega}_1||^2||\underline{\omega}_3||^2 - \langle\underline{\omega}_1,\underline{\omega}_3\rangle^2 \geq 0$$

by the Cauchy-Schwarz inequality and equality holds if and only if $\underline{\omega}_3 = \beta\underline{\omega}_1$ for some $\beta \neq 0$. (Note, $\underline{\omega}_1$ and $\underline{\omega}_3$ are non-zero since they belong to $\mathbb{R}^3\backslash\{0\}$). We now show that $\underline{\omega}_3 \neq \beta\underline{\omega}_1$ for any $\beta \neq 0$.

Consider the tangent space to $M = \{(x_t,\alpha x_t) ; \alpha \neq 0\}$. Tangent vectors are given by

$$\frac{d}{dt}(\sigma_t,\alpha_t\sigma_t)\Big|_{t=0} = (\dot{\sigma}_t,\dot{\alpha}_t\sigma_t + \alpha_t\dot{\sigma}_t)\Big|_{t=0}$$

where

$$(\sigma_0,\alpha_0\sigma_0) = (x,\alpha x) .$$

So

$$T_{(x,\lambda x)}M = \{(u,\alpha u+\gamma x) \in \mathbb{R}^3 \times \mathbb{R}^3; \ u \in \mathbb{R}^3 , \ \gamma \in \mathbb{R}\} . \qquad (A.12)$$

Now suppose $\underline{\omega}_3 = \beta\underline{\omega}_1$ some $\beta \neq 0$, then $\underline{N}_1 = (\underline{\omega}_1,\underline{\omega}_2)$ and $\underline{N}_2 = (\beta\underline{\omega}_1,\omega_4)$. If such vectors are normal to M then by (A.12)

$$\langle(\underline{\omega}_1,\underline{\omega}_2),(u,\alpha u+\gamma x)\rangle_{\mathbb{R}^6} = \langle\underline{\omega}_1,u\rangle_{\mathbb{R}^3} + \langle\underline{\omega}_2,\alpha u+\gamma x\rangle_{\mathbb{R}^3} = 0 \qquad (A.13)$$

and

$$\langle(\beta\underline{\omega}_1,\underline{\omega}_4),(u,\alpha u+\gamma x)\rangle_{\mathbb{R}^6} = \beta\langle\underline{\omega}_1,u\rangle_{\mathbb{R}^3} + \langle\underline{\omega}_4,\alpha u+\gamma x\rangle_{\mathbb{R}^3} = 0 . \qquad (A.14)$$

Multiplying (A.13) by $\beta \neq 0$ and subtracting yields

$$\langle \underline{\omega}_4 - \beta\underline{\omega}_2 \, , \, \alpha u + \gamma x \rangle = 0 \quad .$$

This is true if either

(i) $\underline{\omega}_4 - \beta\underline{\omega}_2 = 0$ which implies $(\underline{\omega}_3, \underline{\omega}_4) = \beta(\underline{\omega}_1, \underline{\omega}_2)$ which contradicts $\underline{N}_1$ and $\underline{N}_2$ being linearly independent

or

(ii) $\alpha u + \gamma x = 0$ for all $u \in \mathbb{R}^3$, $\gamma \in \mathbb{R}$, which is not necessarily so,

or

(iii) $\underline{\omega}_4 - \beta\underline{\omega}_2$ is orthogonal to $\alpha u + \gamma x$ for all $u \in \mathbb{R}^3$, $\gamma \in \mathbb{R}$. But

$$\underline{N}_2 - \beta\underline{N}_1 = (0, \underline{\omega}_4 - \beta\underline{\omega}_2) \tag{A.15}$$

and this vector cannot lie in the normal space since its first co-ordinate is zero and hence $\underline{N}_2 - \beta\underline{N}_1 \notin (\mathbb{R}^3\backslash\{0\}) \times (\mathbb{R}^3\backslash\{0\})$. Also since $\underline{N}_1$ and $\underline{N}_2$ are linearly independent (i) implies $\underline{\omega}_4 - \beta\underline{\omega}_2$ must be non-zero, and some $u \in \mathbb{R}^3$ and $\gamma \in \mathbb{R}$ can be found such that $\langle \underline{\omega}_4 - \beta\underline{\omega}_2, \alpha u + \gamma x \rangle \neq 0$.

Thus $\underline{\omega}_3 \neq \beta\underline{\omega}_1$ for any $\beta \neq 0$ and hence $\det \alpha_{ij} > 0$. Thus (α_{ij}) has rank 2. We can then apply Friedman's Theorem 4.2, Chapter 11, which concludes that

$$\mathbb{P}\{(x_t, v_t) \in M \text{ for some } t > 0\} = 0 \quad ,$$

that is M is non-attainable. □

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