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Rotating quantum states

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ABSTRACT

We revisit the definition of rotating thermal states for scalar and fermion fields in unbounded Minkowski space–time. For scalar fields such states are ill-defined everywhere, but for fermion fields an appropriate definition of the vacuum gives thermal states regular inside the speed-of-light surface. For a massless fermion field, we derive analytic expressions for the thermal expectation values of the fermion current and stress–energy tensor. These expressions may provide qualitative insights into the behaviour of thermal rotating states on more complex space–time geometries.

1. Introduction

In the canonical quantisation of a free field, an object of fundamental importance is the vacuum state, from which states containing particles are constructed. For fields of all spins, the process starts by expanding the classical field in terms of an orthonormal basis of field modes, which are split into positive and negative frequency modes. The expansion coefficients are promoted to operators, the expansion coefficients of the positive frequency modes being particle annihilation operators.[1] The vacuum state is defined as the state annihilated by all the particle annihilation operators. The definition of a vacuum state is therefore dependent on how the field modes are split into positive and negative frequency modes. This split is restricted for a quantum scalar field by the fact that positive frequency modes must have positive Klein–Gordon norm. For a quantum fermion field, both positive and negative frequency fermion modes have positive Dirac norm, so the split of the field modes into positive and negative frequency is less constrained compared with the scalar field case. There is therefore more freedom in how the vacuum state is defined for a fermion field, leading to more freedom in how states containing particles are defined.

In this letter we explore this difference between scalar and fermion quantum fields by considering the definition of rotating vacuum and thermal states in Minkowski space. This toy model reveals that there are quantum states which can be defined for a fermion field but which have no analogue for scalar fields.

2. Rotating scalars

We consider Minkowski space in cylindrical coordinates $(t_{\text{Mink}}, \rho, \varphi_{\text{Mink}}, z)$.[2] We wish to define quantum states which are rigidly rotating with angular velocity $\mathbf{\Omega}$. Choosing the z axis of the coordinate system along the angular velocity vector $\mathbf{\Omega}$, the line element of the rotating space–time can be found by making the transformation $\varphi = \varphi_{\text{Mink}} - \Omega t_{\text{Mink}}$, $t = t_{\text{Mink}}$ in the usual Minkowski line element, giving:

$$ds^2 = -\left(1 - \rho^2\Omega^2\right) dt^2 + 2\rho^2 \Omega \, dt \, d\varphi + d\rho^2 + \rho^2 \, d\varphi^2 + dz^2. \tag{1}$$

The Killing vector ∂_t, which defines the co-rotating Hamiltonian $H = i\partial_t$, becomes null on the speed-of-light surface (SOL), defined as the surface where $\rho = \Omega^{-1}$. The Klein–Gordon equation for a scalar field of mass μ on the space–time (1) is:

$$\left[-(H + \Omega L_z)^2 + \frac{L_z^2}{\rho^2} + P_z^2 - \partial_\rho^2 - \frac{\partial_\rho}{\rho} + \mu^2\right]\Phi(x) = 0, \tag{2}$$

where $P_z = -i\partial_z$ and $L_z = -i\partial_\varphi$ are the z components of the momentum and angular momentum operators, respectively. The mode solutions of (2) are:

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[1] The adjoints of the expansion coefficients of the negative frequency modes are also particle annihilation operators. For a real scalar field, these annihilation operators are the same as the expansion coefficients of the positive frequency modes; for a fermion field they are different.

[2] Throughout this paper we use units in which $c = \hbar = k_B = 1$.

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$$f_{\omega km}(x) = \frac{1}{\sqrt{8\pi^2|\omega|}} e^{-i\widetilde{\omega}t + im\varphi + ikz} J_m(q\rho), \tag{3}$$

where $J_m(q\rho)$ is the Bessel function of the first kind of order m, m is the eigenvalue of L_z, k is the eigenvalue of P_z, q is the longitudinal component of the momentum and $\omega = \pm\sqrt{\mu^2 + q^2 + k^2}$ gives the Minkowski energy of the mode. The eigenvalue of the Hamiltonian, $\widetilde{\omega} = \omega - \Omega m$, represents the energy of the mode as seen by a co-rotating observer. It is convenient to introduce the shorthand $j = (\omega_j, k_j, m_j)$ and

$$\delta(j, j') = \delta_{m_j m_{j'}} \delta(k_j - k_{j'}) \frac{\delta(\omega_j - \omega_{j'})}{|\omega_j|}. \tag{4}$$

Using the Klein–Gordon inner product:

$$\langle f, g \rangle = -i \int d^3x \sqrt{-g} \left(f^* \partial^t g - g \partial^t f^*\right), \tag{5}$$

the norm of the modes (3) can be calculated:

$$\langle f_j, f_{j'} \rangle = \frac{\omega_j}{|\omega_j|} \delta(j, j'). \tag{6}$$

As discussed by Letaw and Pfautsch [1], particles must be described by modes with positive norm ($\omega_j > 0$), implying the following expansion for the scalar field operator:

$$\Phi(x) = \sum_{m_j=-\infty}^{\infty} \int_{\mu}^{\infty} \omega_j \, d\omega_j \int_{-p_j}^{p_j} dk_j \left[f_j(x) a_j + f_j^*(x) a_j^\dagger\right], \tag{7}$$

where $p_j = \sqrt{q_j^2 + k_j^2}$ is the Minkowski momentum. The one-particle annihilation and creation operators, a_j and $a_j^\dagger$, satisfy the canonical commutation relations $[a_j, a_{j'}^\dagger] = \delta(j, j')$. The induced vacuum state $|0\rangle$, satisfying $a_j|0\rangle = 0$, coincides with the Minkowski vacuum [1].

At finite inverse temperature $\beta = T^{-1}$, Vilenkin [2] gives the following thermal expectation value (t.e.v.):

$$\langle a_j^\dagger a_{j'} \rangle_\beta = \frac{\delta(j, j')}{e^{\beta\widetilde{\omega}_j} - 1}. \tag{8}$$

The above expression cannot hold when $\widetilde{\omega}_j < 0$ [2], since it would imply that the vacuum expectation value of $a_j^\dagger a_{j'}$, obtained by taking the limit $\beta \to \infty$, is non-zero, contradicting the definition of the vacuum. Furthermore, the divergent behaviour of the thermal weight factor of modes with $\widetilde{\omega}$ close to 0 renders t.e.v.s infinite, causing rotating thermal states for scalar fields to be ill-defined everywhere in the space–time [2,3]. As discussed by [2,3], a resolution to these problems is to enclose the system inside a boundary located inside or on the SOL, restricting wavelengths such that $\widetilde{\omega}$ stays positive for all values of m.

3. Rotating fermions

In the Cartesian gauge [4], a natural frame for the metric (1) can be chosen to be:

$$e_{\hat{t}} = \partial_t - \Omega \partial_\varphi, \qquad e_{\hat{i}} = \partial_i. \tag{9}$$

In the following, hats shall be used to indicate tensor components with respect to the tetrad, i.e. $A^\mu = A^{\hat{\alpha}} e_{\hat{\alpha}}^\mu$. The Dirac equation for fermions of mass μ takes the form:

$$\left[\gamma^{\hat{t}}(H + \Omega M_z) - \boldsymbol{\gamma} \cdot \mathbf{P} - \mu\right]\psi(x) = 0, \tag{10}$$

where the gamma matrices are in the Dirac representation [5] and the covariant derivatives are given by:

$$iD_{\hat{t}} = H + \Omega M_z, \qquad -iD_{\hat{j}} = P_j. \tag{11}$$

The momentum operators P_j and angular momentum operator M_z are:

$$P_j = -i\partial_j, \qquad M_z = -i\partial_\varphi + \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{12}$$

The Dirac equation (10) admits the following solutions:

$$U_{Ekm}^\lambda(x) = \frac{1}{\sqrt{8\pi^2}} e^{-i\widetilde{E}t + ikz} \begin{pmatrix} \sqrt{1 + \frac{\mu}{E}}\, \phi_{Ekm}^\lambda \\ \frac{2\lambda E}{|E|}\sqrt{1 - \frac{\mu}{E}}\, \phi_{Ekm}^\lambda \end{pmatrix}, \tag{13}$$

where the two-spinor ϕ_{Ekm}^λ is defined as:

$$\phi_{Ekm}^\lambda(\rho, \varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \frac{2\lambda k}{p}}\, e^{im\varphi} J_m(q\rho) \\ 2i\lambda\sqrt{1 - \frac{2\lambda k}{p}}\, e^{i(m+1)\varphi} J_{m+1}(q\rho) \end{pmatrix}, \tag{14}$$

where λ is the helicity [4,5], $p = \sqrt{q^2 + k^2}$ is the magnitude of the momentum and $E = \pm\sqrt{p^2 + \mu^2}$ controls the sign of the Minkowski energy of the mode. The eigenvalues of the Hamiltonian are $\widetilde{E} = E - \Omega(m + \frac{1}{2})$, representing, as in the scalar case, the energy seen by a co-rotating observer. The notations $j = (E_j, k_j, m_j, \lambda_j)$ and

$$\delta(j, j') = \delta_{\lambda_j \lambda_{j'}} \delta_{m_j m_{j'}} \delta(k_j - k_{j'}) \frac{\delta(E_j - E_{j'})}{|E_j|} \tag{15}$$

are useful to refer to modes and their norms. The latter can be computed using the Dirac inner product:

$$\langle \psi, \chi \rangle = \int d^3x \sqrt{-g}\, \psi^\dagger(x) \chi(x). \tag{16}$$

It can be shown that $\langle U_j, U_{j'} \rangle = \delta(j, j')$ for all possible labels j, j'. After choosing a suitable definition for particle modes (i.e. a range for the labels in j), the anti-particle modes can be constructed using charge conjugation [4,5]: $V_j = i\gamma^{\hat{2}} U_j^*$. Hence, V_j automatically inherits the same normalisation as U_j, namely: $\langle V_j, V_{j'} \rangle = \delta(j, j')$. Therefore there is no restriction on how the split into particle and anti-particle modes is performed, as long as the charge conjugation symmetry is preserved.

According to Vilenkin [2], the definition of particles for co-rotating observers should be the same as for inertial Minkowski observers, with the field operator written as:

$$\psi_V(x) = \sum_{\lambda_j = \pm\frac{1}{2}} \sum_{m_j=-\infty}^{\infty} \int_{\mu}^{\infty} E_j \, dE_j \int_{-p_j}^{p_j} dk_j \times \left[U_j(x) b_{j;V} + V_j(x) d_{j;V}^\dagger\right]. \tag{17}$$

Vilenkin's quantisation is equivalent to the one suggested by Letaw and Pfautsch [1] for the scalar field, yielding a vacuum state equivalent to the Minkowski vacuum. In contrast, Iyer [6] argues that the modes which represent particles for a co-rotating observer have positive frequency with respect to the co-rotating Hamiltonian, implying the following expression for the field operator:

$$\psi_I(x) = \sum_{\lambda_j = \pm\frac{1}{2}} \sum_{m_j=-\infty}^{\infty} \int_{\widetilde{E}_j > 0, |E_j| > \mu}^{\infty} E_j \, dE_j \int_{-p_j}^{p_j} dk_j \times \left[U_j b_{j;I} + V_j(x) d_{j;I}^\dagger\right], \tag{18}$$

with the integral with respect to E_j running over both positive and negative values of E_j, as long as $\widetilde{E}_j > 0$ and $|E_j| > \mu$. Both quantisation methods lead to the canonical anti-commutation relations $\{b_j, b^\dagger_{j'}\} = \{d_j, d^\dagger_{j'}\} = \delta(j, j')$. The ensuing quantum field theory differs in the two pictures, as the vacuum state corresponding to Iyer's quantisation differs from the Minkowski vacuum. This can be seen by looking at the connection between the Iyer and Vilenkin one-particle operators:

$$b_{j;I} = \begin{cases} b_{j;V} & E_j \text{ is positive}, \\ i^{2m+1} d^\dagger_{\bar{j};V} & E_j \text{ is negative}, \end{cases} \tag{19}$$

and similarly for $d_{j;I}$, where $\bar{j} = (-E_j, -k_j, -m_j - 1, \lambda_j)$. Thus, the Vilenkin vacuum state (i.e. the non-rotating Minkowski vacuum) contains particles as defined according to Iyer's quantisation. Similarly, the Iyer vacuum contains particles as defined according to Vilenkin's quantisation (i.e. relative to the Minkowski vacuum).

Vilenkin [2] also considered rotating thermal states for fermions. In analogy with (8), he gives the following t.e.v.s relative to the Minkowski vacuum [2]:

$$\langle b^\dagger_j b_{j'} \rangle_\beta = \langle d^\dagger_j d_{j'} \rangle_\beta = \frac{\delta(j, j')}{e^{\beta \widetilde{E}_j} + 1}. \tag{20}$$

As in the scalar case, (20) is not valid when $\widetilde{E}_j < 0$ [2]. However, in contrast to the scalar case, the modes with negative $\widetilde{E}_j$ can be eliminated from the set of particle modes by using Iyer's quantisation, without enclosing the system within a boundary inside the SOL. Furthermore, unlike the thermal factor for scalars (8), the Fermi–Dirac density of states factor (20) is regular for all $\widetilde{E}_j$.

Eq. (20) can be used to construct the t.e.v.s of the neutrino charge current operator and of the SET:

$$J^{\hat{\alpha}}(x) = \frac{1}{2}\left[\overline{\psi}, \gamma^{\hat{\alpha}} \frac{1+\gamma^{\hat{5}}}{2} \psi\right], \tag{21a}$$

$$T_{\hat{\alpha}\hat{\sigma}}(x) = -\frac{i}{4}\left\{[\overline{\psi}, \gamma_{(\hat{\alpha}} D_{\hat{\sigma})} \psi] - [\overline{D_{(\hat{\alpha}} \psi} \gamma_{\hat{\sigma})}, \psi]\right\}. \tag{21b}$$

Using the Vilenkin quantisation, we find the following t.e.v.s relative to the Minkowski vacuum:

$$\langle :[\overline{\psi}\psi]_V: \rangle_\beta = -\mu S^+_{000}, \tag{22a}$$

$$\langle :J^{\hat{z}}_V: \rangle_\beta = -\tfrac{1}{2} S^-_{100}, \tag{22b}$$

$$\langle :T_{V;\hat{t}\hat{t}}: \rangle_\beta = S^+_{200}, \tag{22c}$$

$$\langle :T_{V;\hat{\rho}\hat{\rho}}: \rangle_\beta = S^+_{020} - \rho^{-1} S^\times_{011}, \tag{22d}$$

$$\langle :T_{V;\hat{\varphi}\hat{\varphi}}: \rangle_\beta = \rho^{-1} S^\times_{011}, \tag{22e}$$

$$\langle :T_{V;\hat{z}\hat{z}}: \rangle_\beta = S^+_{200} - S^+_{020} - \mu^2 S^+_{000}, \tag{22f}$$

$$\langle :T_{V;\hat{t}\hat{\varphi}}: \rangle_\beta = \tfrac{1}{4}\rho^{-1} S^-_{100} - \tfrac{1}{2}\rho^{-1} S^+_{101} - \tfrac{1}{2} S^\times_{110}, \tag{22g}$$

where the fermion condensate $\langle :[\overline{\psi}\psi]_V: \rangle_\beta$ vanishes when $\mu = 0$. The functions $S^\pm_{abc}$ and $S^\times_{abc}$ introduced above are defined as:

$$S^*_{abc} = \frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} \int_\mu^\infty \frac{dE}{1 + e^{\beta \widetilde{E}}} \int_0^p dk\, E^a q^b \left(m + \tfrac{1}{2}\right)^c J^*_m(q\rho), \tag{23}$$

where $* \in \{+, -, \times\}$ and the functions J^*_m are given by:

$$J^\pm_m(z) \equiv J^2_m(z) \pm J^2_{m+1}(z), \qquad J^\times_m(z) = 2 J_m(z) J_{m+1}(z). \tag{24}$$

Except when $\mu = 0$, numerical integration must be used to calculate S^*_{abc} for arbitrary values of the mass. For massless fermions, the method outlined in Appendix A can be followed to obtain the following exact results ($\varepsilon = 1 - \rho^2 \Omega^2$):

$$\frac{1}{\mu}\langle :[\overline{\psi}\psi]_V: \rangle_\beta = -\frac{1}{6\beta^2 \varepsilon} - \frac{\Omega^2}{8\pi^2 \varepsilon^2}\left(\frac{2}{3} + \frac{\varepsilon}{3}\right), \tag{25a}$$

$$\langle :J^z_V: \rangle_\beta = -\frac{\Omega}{12\beta^2 \varepsilon^2} - \frac{\Omega^3}{48\pi^2 \varepsilon^3}(4 - 3\varepsilon). \tag{25b}$$

To evaluate the massless limit of the t.e.v. of the fermion condensate $\langle :[\overline{\psi}\psi]_V: \rangle_\beta$, the latter was divided by the mass factor in Eq. (22a). In the above, the hat has been dropped from the index of J^z to indicate that the result is with respect to the coordinate basis. For the z component, this coincides with the tetrad component. The t.e.v. of the SET with respect to the coordinate basis has the following components:

$$\langle :T_{V;tt}: \rangle_\beta = \frac{7\pi^2}{60\beta^4 \varepsilon} + \frac{\Omega^2}{8\beta^2 \varepsilon^2}\left(\frac{4}{3} - \frac{1}{3}\varepsilon\right) + \frac{\Omega^4}{64\pi^2 \varepsilon^3}\left(\frac{8}{9} + \frac{56}{45}\varepsilon - \frac{17}{15}\varepsilon^2\right), \tag{25c}$$

$$\langle :T_{V;\rho\rho}: \rangle_\beta = \frac{7\pi^2}{180\beta^4 \varepsilon^2} + \frac{\Omega^2}{24\beta^2 \varepsilon^3}\left(\frac{4}{3} - \frac{1}{3}\varepsilon\right) + \frac{\Omega^4}{192\pi^2 \varepsilon^4}\left(8 - \frac{88}{15}\varepsilon - \frac{17}{15}\varepsilon^2\right), \tag{25d}$$

$$\frac{1}{\rho}\langle :T_{V;\varphi t}: \rangle_\beta = -\rho\Omega\left\{\frac{7\pi^2}{60\beta^4 \varepsilon^2} + \frac{13\Omega^2}{72\beta^2 \varepsilon^3}\left(\frac{16}{13} - \frac{3}{13}\varepsilon\right) + \frac{119\Omega^4}{960\pi^2 \varepsilon^4}\left(\frac{200}{119} - \frac{64}{119}\varepsilon - \frac{1}{7}\varepsilon^2\right)\right\}, \tag{25e}$$

$$\frac{1}{\rho^2}\langle :T_{V;\varphi\varphi}: \rangle_\beta = \frac{7\pi^2}{180\beta^4 \varepsilon^3}(4 - 3\varepsilon) + \frac{\Omega^2}{24\beta^2 \varepsilon^4}(8 - 8\varepsilon + \varepsilon^2) + \frac{\Omega^4}{192\pi^2 \varepsilon^5}\left(64 - \frac{456}{5}\varepsilon + \frac{124}{5}\varepsilon^2 + \frac{17}{5}\varepsilon^3\right), \tag{25f}$$

and $\langle :T_{V;zz}: \rangle_\beta = \langle :T_{V;\rho\rho}: \rangle_\beta$ for any value of the mass μ. The connection between the tetrad and coordinate basis components is made through:

$$T_{tt} = T_{\hat{t}\hat{t}} + 2\rho\Omega T_{\hat{t}\hat{\varphi}} + \rho^2 \Omega^2 T_{\hat{\varphi}\hat{\varphi}}, \tag{26a}$$

$$T_{t\varphi} = \rho T_{\hat{t}\hat{\varphi}} + \rho^2 \Omega T_{\hat{\varphi}\hat{\varphi}}, \tag{26b}$$

$$T_{\varphi\varphi} = \rho^2 T_{\hat{\varphi}\hat{\varphi}}. \tag{26c}$$

The analytic results for $\langle :[\overline{\psi}\psi]_V: \rangle_\beta$, $\langle :J^{\hat{z}}_V: \rangle_\beta$ and $\langle :T_{V;tt}: \rangle_\beta$ given by Eqs. (25) reveal a number of physical features. Firstly, they all contain contributions which are independent of the temperature (equivalently, independent of β). These terms are unphysical as the t.e.v.s should vanish when the temperature is set to zero ($\beta \to \infty$). These temperature-independent terms are generated by modes with $\widetilde{E} < 0$ and arise because the Vilenkin quantisation [2] has been used. If the Iyer quantisation [6] is employed, analytic expressions for the corresponding t.e.v.s for massless fermions $\langle :[\overline{\psi}\psi]_I: \rangle_\beta$, $\langle :J^z_I: \rangle_\beta$ and $\langle :T_{I;\alpha\sigma}: \rangle_\beta$ are found from (25) by subtracting the temperature-independent parts, for example,

$$\langle :J^z_I: \rangle_\beta = -\frac{\Omega}{12\beta^2 \varepsilon^2}. \tag{27}$$

The temperature-independent contributions to Eqs. (25) are the expectation values of the Iyer vacuum relative to the Vilenkin (Minkowski) vacuum. Since these are non-zero and depend on ρ, we see that the Iyer vacuum is not equivalent to, and has fewer symmetries than, the maximally symmetric Vilenkin vacuum. The

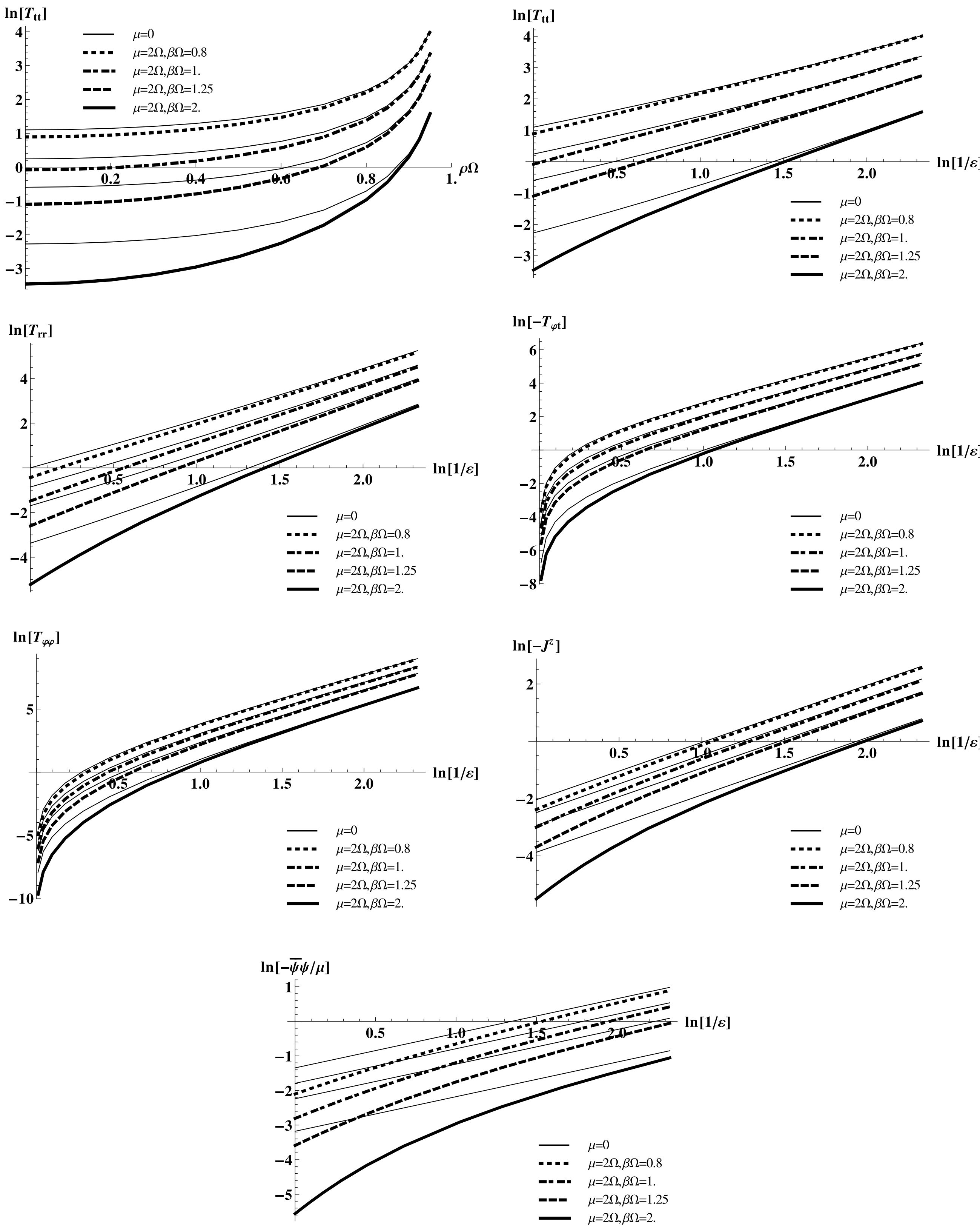

Fig. 1. Plots of the t.e.v.s in Eqs. (22) using Iyer's quantisation for fermions with mass $\mu = 0$ (thin lines) and $\mu = 2\Omega$, at inverse temperatures (from top to bottom) 0.8, 1.0, 1.25 and 2.0, in units of Ω^{-1}. (a) $\langle :T_{I;tt}: \rangle_\beta$ against $\rho\Omega$, (b)–(g) log–log plots of Eqs. (22), showing the polynomial nature of the divergence as the SOL is approached. It can be seen that the massive t.e.v.s diverge at exactly the same rate as in the massless case.

temperature-independent contributions to Eqs. (25) are absent from the t.e.v.s calculated using the Iyer quantisation.

If $\Omega = 0$, there is no rotation and the t.e.v.s (25) reduce to the usual Minkowski t.e.v.s. If $\Omega \neq 0$, on the axis of rotation we have $\varepsilon = 1$ and the expressions in round brackets in (25) evaluate to unity. In this case the t.e.v.s take the form of the Minkowski values plus an Ω-dependent correction. A similar effect is found for rotating thermal states for a scalar field inside a reflecting cylinder [3].

The t.e.v.s (25) are finite as long as $\varepsilon > 0$ but diverge as $\varepsilon \to 0$ and the SOL is approached. Using the Iyer quantisation (when the β-independent terms are absent), the fermion condensate $\langle :[\overline{\psi}\psi]_I: \rangle_\beta$ diverges as ε^{-1}, the current $\langle :J_I^z: \rangle_\beta$ diverges as ε^{-2} and the SET components $\langle :{T_I}^{\alpha}{}_{\sigma}: \rangle_\beta$ diverge as ε^{-4}.

For the case of arbitrary mass, numerical methods can be employed. In Fig. 1 we show the t.e.v.s in Eqs. (22) using the Iyer quantisation for massive fermions. As expected, increasing the mass damps the t.e.v.s. The damping becomes more pronounced as the temperature is decreased (β is increased). It can be seen from the log–log plots that as the SOL is approached, the t.e.v.s in the massive case diverge at exactly the same rate as for massless fermions. The mass contributes corrections which diverge at sub-leading orders, so that, close to the SOL, the quanta behave as if they were massless.

4. Conclusions

We have studied the construction of rotating states for scalar and fermion fields in four-dimensional Minkowski space. Our analysis has demonstrated that the definition of fermion quantum states is less constrained than for scalar fields. This is due to more freedom in how the split into particle and anti-particle modes is performed for fermion fields, since *all* fermion modes have positive norm (the Dirac norm is positive definite but the Klein–Gordon norm is not). We have considered two possible quantisation schemes [2,6] for rotating fermion states, which yield two inequivalent vacuum states. We have computed thermal expectation values (t.e.v.s) for rotating states relative to these two vacua. In the Vilenkin scheme [2], the vacuum state is equivalent to the (non-rotating) Minkowski vacuum, and t.e.v.s for rotating states relative to this vacuum include temperature-independent terms which arise from low energy modes having positive frequency as seen by a non-rotating observer but negative frequency as seen by a rotating observer. As discussed by Vilenkin [2], these low energy modes (and hence the temperature-independent terms in the t.e.v.s) can be removed by enclosing the system within a boundary inside the speed-of-light surface.[3] Using the Iyer scheme [6], the temperature-independent terms in the t.e.v.s are absent, without enclosing the system in a boundary or otherwise modifying the particle spectrum. As discussed in [6], we emphasize that the Iyer vacuum state cannot be defined for a quantum scalar field (which is restricted to the Minkowski vacuum). In addition, while rotating thermal states for scalar fields are ill-defined everywhere on the unbounded space–time [3], we have constructed fermion rotating thermal states which are regular inside the speed-of-light surface.

In this paper we have considered the toy model of rotating states in Minkowski space–time. However, the main physical features extend to curved space–times. For example, recent work on the construction of quantum states for fermion fields on a rotating Kerr black hole [8] has also demonstrated the existence of fermion states which have no analogue for scalar fields. Furthermore, the simplicity of our toy model has enabled us to derive analytic expressions for the thermal expectation values for massless fermions, which could provide qualitative insights into the behaviour of thermal rotating states on more complex space–time geometries, for example, the nature of the divergence as the speed-of-light surface is approached.

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Appendix A. Fermi–Dirac integrals for massless rotating states

To compute the functions S^*_{abc}, defined in Eq. (23), the Fermi–Dirac density of states factor can be expanded about $\Omega = 0$:

$$\frac{1}{1+e^{\beta[E-\Omega(m+\frac{1}{2})]}} = \sum_{n=0}^{\infty} \frac{(-\Omega)^n}{n!} \left(m+\tfrac{1}{2}\right)^n \frac{d^n}{dE^n}\left(\frac{1}{1+e^{\beta E}}\right), \quad \text{(A.1)}$$

leading to:

$$S^*_{abc} = \frac{1}{\pi^2} \sum_{n=0}^{\infty} \frac{(-\Omega)^n}{n!} \int_{\mu}^{\infty} dE\, E^a \frac{d^n}{dE^n}\left(\frac{1}{e^{\beta E}+1}\right) \times \int_0^p dk\, q^b \sum_{m=-\infty}^{\infty} \left(m+\tfrac{1}{2}\right)^{n+c} J^*_m(q\rho). \quad \text{(A.2)}$$

Sum over m

The sum over m in Eq. (A.2) vanishes unless $n+c$ is even for $* = +$ and odd for $* \in \{-, \times\}$. To perform the sum, the following formula can be used to rewrite the product of two Bessel functions as an infinite sum:

$$J_\nu(z) J_\mu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)\Gamma(\mu+k+1)} \times \frac{\Gamma(\nu+\mu+2k+1)}{\Gamma(\nu+\mu+k+1)} \left(\frac{z}{2}\right)^{2k+\nu+\mu}. \quad \text{(A.3)}$$

After the sum over m is performed, the above series terminates after a finite number of terms, as follows:

$$\sum_{m=-\infty}^{\infty} \left(m+\tfrac{1}{2}\right)^{2n} J^+_m(z) = \sum_{j=0}^{n} \frac{2\Gamma(j+\frac{1}{2})}{j!\sqrt{\pi}} s^+_{n,j} z^{2j}, \quad \text{(A.4a)}$$

$$\sum_{m=-\infty}^{\infty} \left(m+\tfrac{1}{2}\right)^{2n+1} J^-_m(z) = \sum_{j=0}^{n} \frac{2\Gamma(j+\frac{1}{2})}{j!\sqrt{\pi}} s^-_{n,j} z^{2j}, \quad \text{(A.4b)}$$

$$\sum_{m=-\infty}^{\infty} \left(m+\tfrac{1}{2}\right)^{2n+1} J^\times_m(z) = \sum_{j=0}^{n} \frac{2\Gamma(j+\frac{1}{2})}{j!\sqrt{\pi}} s^\times_{n,j} z^{2j+1}, \quad \text{(A.4c)}$$

where $s^+_{n,j}$ can be shown to equal:

$$s^+_{n,j} = \frac{1}{(2j+1)!} \lim_{\alpha\to 0} \frac{d^{2n+1}}{d\alpha^{2n+1}} \left(2\sinh\tfrac{\alpha}{2}\right)^{2j+1}. \quad \text{(A.5)}$$

It is clear that $s^+_{n,j}$ vanishes when $j > n$. Using the following properties of J^*_m:

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[3] Enclosing relativistic fermions within a boundary presents some difficulties [2]. We will return to this issue in a future publication [7].

$$\frac{d}{dz}\big(zJ_m^+(z)\big) = (2m+1)J_m^-(z),$$
$$\frac{d}{dz}\big(zJ_m^\times(z)\big) = 2zJ_m^-(z), \tag{A.6}$$

it can be shown that $s_{n,j}^-$ and $s_{n,j}^\times$ are related to $s_{n,j}^+$ through:

$$s_{n,j}^- = \big(j+\tfrac{1}{2}\big)s_{n,j}^+, \qquad s_{n,j}^\times = \frac{j+\frac{1}{2}}{j+1}s_{n,j}^+. \tag{A.7}$$

Hence, $s_{n,j}^* = 0$ for $j > n$. The following values of $s_{n,j}^+$ are important for the calculation of the t.e.v.s in Eqs. (22):

$$s_{j,j}^+ = 1, \tag{A.8}$$
$$s_{j+1,j}^+ = \frac{1}{24}(2j+1)(2j+2)(2j+3), \tag{A.9}$$
$$s_{j+2,j}^+ = \frac{1}{5760}(2j+1)(2j+2)(2j+3)(2j+4) \times (2j+5)(10j+3). \tag{A.10}$$

The integral with respect to k

Following the steps in the previous paragraph, the sum over m involving the Bessel functions in J_m^* is replaced by a sum over j involving powers of q. The integral over k can be computed using:

$$\int_0^p dk\, q^\nu = \frac{\Gamma(\frac{\nu}{2}+1)\sqrt{\pi}}{2\Gamma(\frac{\nu+1}{2}+1)}p^{\nu+1}. \tag{A.11}$$

Analytic expressions in the massless case

Although the calculation of each individual function S_{abc}^* in Eqs. (22) has its own peculiarities, the method is very much the same. To illustrate the method, the simplest case $a = b = c = 0$ shall be considered for the remainder of this section.

After performing the above steps, S_{000}^+ can be brought to the following form:

$$S_{000}^+ = \frac{1}{\pi^2}\sum_{j=0}^\infty \frac{(\rho\Omega)^{2j}}{j+\frac{1}{2}}\sum_{n=0}^\infty \frac{\Omega^{2n}s_{n+j,j}^+}{(2n+2j)!} \times \int_\mu^\infty dE\, p^{2j+1}\frac{d^{2n+2j}}{dE^{2n+2j}}\left(\frac{1}{1+e^{\beta E}}\right). \tag{A.12}$$

In the above, the sums over j and n have been swapped, after which the sum over n was shifted down, i.e. $\sum_{n=0}^\infty\sum_{j=0}^n f_{n,j} \to \sum_{j=0}^\infty\sum_{n=0}^\infty f_{n+j,j}$. While we do not have a method to compute the integral over E in Eq. (A.12) for arbitrary values of the mass μ, in the massless case, $p = E$ and the integration can be done by parts:

$$\int_0^\infty dE\, E^{2j+1}\frac{d^{2n+2j}}{dE^{2n+2j}}\left(\frac{1}{1+e^{\beta E}}\right) = (2j+1)!\begin{cases}\frac{\pi^2}{12\beta^2} & n=0,\\ \frac{1}{2} & n=1,\\ 0 & n>1.\end{cases} \tag{A.13}$$

Then S_{000}^+ takes the form:

$$S_{000}^+ = \sum_{j=0}^\infty (\rho\Omega)^{2j}\left[\frac{1}{6\beta^2} + \frac{\Omega^2}{24\pi^2}(2j+3)\right]. \tag{A.14}$$

The sum over j can be evaluated as a geometric series, giving:

$$S_{000}^+ = \frac{1}{6\beta^2\varepsilon} + \frac{\Omega^2}{8\pi^2\varepsilon^2}\left(\frac{2}{3}+\frac{\varepsilon}{3}\right), \tag{A.15}$$

where $\varepsilon = 1 - \rho^2\Omega^2$.

The above algorithm can be applied to all other terms required for Eqs. (22). For brevity, the individual results shall not be included here, since the terms S_{abc}^* can be inferred from the final results (25), together with the relation

$$S_{110}^\times = \frac{1}{\rho}S_{101}^+ - \frac{1}{2\rho}\frac{d}{d\rho}\big(\rho S_{100}^-\big). \tag{A.16}$$

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