FUZZY TOPOLOGICAL SPACES

by

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ABSTRACT

(1) We define normality for fuzzy topological spaces, define a fuzzy unit interval, and prove a Urysohn type lemma.

(2) We define uniformities on fuzzy lattices, and characterise uniformizability in terms of complete regularity.

(3) We define the product of a collection of fuzzy topological spaces. We define compactness and connectedness, and show that the product is compact (connected) iff each factor space is.

(4) We place normality and complete regularity within a coherent hierarchy of separation and regularity axioms. We prove the usual implications, and the usual theorems about compactness and products.

(5) We give alternative definitions of uniformities and pseudometrics, and show a compact R_1 space has a unique uniformity.

1. INTRODUCTION

1. History

Over ten years ago, Lotfi Zadeh wrote a paper on "fuzzy sets" [16]. The basic idea was to rigourously introduce sets to represent "ill defined" concepts such as "the set of very large positive integers." This was done by allowing points to have other degrees of inclusion besides "in" and "not in," eg: maybe 10 is ¼ in the set of very large positive integers, 100 is ½ in, and 1,000,000 9/10 in. This gave the idea of a fuzzy set on X as a map $A: X \to L$, where

L is elements of L represent the "degree of membership" of a point in the set A. L was first assumed to be the unit interval, and then a complete, completely distributive lattive with order reversing involution $'$. Since then, the concept of fuzziness has produced over 500 papers, of which about twenty or so have been on fuzzy topological spaces (five of which are mine). The most comprehensive bibliography so far produced is one by B.R. Gaines and L.J. Kohout [17].

2. Fuzzy Topology

The concept of fuzzy topological spaces was first introduced by Chang [1], and then later developed by Goguen [2], Lowen [4,5,6],Mesequer and Sols [7], Warren [8,9,10], Wong [11,12,13,14,15]and myself. Their papers develop the concept of fuzzy topology; they also develop the concepts of compactness, products, and uniformities , but in a very different manner from mine. I have always demanded that fuzzy topology should be a generalization of ordinary topology as done on the lattice of subsets of a set. The definitions of products and uniformities so far given have failed to do this. The definition of compactness so far given does not seem to satisfy the intuition upon which compactness is usually based, and is unable to prove a lot of the standard results. There have also been some "applications" of fuzzy topologies.

3. The development of this thesis

This thesis is a collection of five papers written by myself between November 1973 and June 1976. The first paper developed the concept of normality and the fuzzy unit interval The second paper developed the concept of uniformities, and in the process defines complete regularity. However, there was one major gap. I could not prove that compactness implied a unique uniformity. This was because I had the wrong definition of compactness, and the wrong definition of a product space (I still had the "right" definition of a uniformity, though, since it was not defined in terms of products). Thus I produced another paper on products and compactness which brought me closer to proving compactness implies a unique uniformity. I finally managed to show uniqueness. In the meantime, between writing the first paper on uniformities, and the paper on products, I had produced a paper on separation and regularity axioms. This paper placed normality and complete regularity in a coherent hierarchy of regularity axioms. However, there were several faults. I had the wrong definitions for products and compactness, hence limiting the theorems I could prove. I had the wrong definition of T_0 to allow development of compactification theory in the futur Also, at that stage I was only just developing my "pointless" stance. Hence I have rewritten this paper using the new definitions of T_0, compactness, and products.

Producing this thesis in parts has had its faults. Definitions have varied slightly from paper to paper (eg: the fuzzy unit interval) depending upon what seemed more convenient at the time. Notation has also varied slightly (eg: interchanging $\wedge$ and $\cap$, $\vee$ and $\cup$ caused no problems until products occurred, but more rigourous conventions had to be developed from then on. Also, the "inverse" of a uniformity was renamed the "reflection" later on.). For details, see Appendix.

Finally, I would like to mention the help I have had in producing these papers. I was introduced to papers of Chang and Goguen by Ivan Reilly, who suggested I may be able to produce something on uniformities. As a consequence, I have made the paper on separation axioms a joint paper, as a sign of appreciation, in spite of the fact that he took no part in the research. Also, I would like to mention the fact that my co-student Ralph Fox suggested that the property

$$D^r(U) \leq V' \quad \text{iff} \quad D(V) \leq U' \quad \text{for}$$

elements of a (non-fuzzy) uniformity may be of some use. It certainly was.

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2. NORMALITY IN FUZZY TOPOLOGICAL SPACES

1. Introduction

In this paper we extend the notion of normality in topological spaces to fuzzy topological spaces, as introduced in [1].

Normality is one of the few separation axioms which can be defined purely in terms of the properties of the open and closed sets* (i.e. with no mention of points). We characterise normality in terms of a 'Urysohn' type lemma, and in the process construct a fuzzy topological space which plays the important role in fuzzy topological spaces that the unit interval plays in ordinary topological spaces.

2. Preliminaries

Suppose we consider a set X. If we identify a subset A of X with its characteristic function $\chi_A : X \to \{0,1\}$, then we may consider the value of A (i.e. χ_A) at a point x in X as the degree to which x is a member of A. When we replace $\{0,1\}$ by a more general lattice we obtain what is called a fuzzy set. More precisely: let $(L, \leq, ')$

be a completely distributive lattice with order reversing involution $'$, then an L-fuzzy set on X is a map $A : X \to L$. Allowing a more general lattice than $\{0,1\}$ allows us more degrees of membership in a set than ' is a member' and 'is not a member'. The most important example may be when L is the collection of subsets of a set with a probability space structure.

Throughout this paper we shall consider an arbitrary but fixed lattice $(L, \leq, ')$ of the above type. We define the union, intersection and complement of fuzzy sets as follows:

$$(\cup_i A_i)(x) = \vee_i A_i(x) \qquad x \in X.$$
$$(\cap_i A_i)(x) = \wedge_i A_i(x) \qquad x \in X.$$
$$A'(x) = A(x)' \qquad x \in X.$$

We define a fuzzy topological space as a pair (X,τ) where X is a set and τ is a collection of L-fuzzy sets closed under arbitrary union and finite intersection. A set is called open if it is in τ, and closed if its complement is in τ. If (X,τ_1) and (Y,τ_2) are fuzzy topological spaces, then a map $f : X \to Y$ is said to be continuous if for every τ_2 open set U, $f^{-1}(U) \in \tau_1$, where $f^{-1}(U)(x) = U(f(x))$ for $x \in X$. The interior and closure of fuzzy sets is defined in the obvious way (see Chang [1]).

3. Normality and the fuzzy unit interval

Definition 1

A fuzzy topological space is normal if for every closed set K and open set U such that $K \subseteq U$, there exists a set V such that $K \subseteq V^0 \subseteq \overline{V} \subseteq U$.

It turns out that the fuzzy topological space we need to prove the equivalent of Urysohn's lemma is the following:

Definition 2

The fuzzy unit interval $[0,1](L)$ is the set of all monotonic decreasing maps $\lambda : \mathbb{R} \to L$ satisfying:

(1) $\lambda(t) = 1$ for $t < 0$, $t \in \mathbb{R}$

(2) $\lambda(t) = 0$ for $t > 1$, $t \in \mathbb{R}$

after the identification of $\lambda : \mathbb{R} \to L$ and $\mu : \mathbb{R} \to L$ if for every $t \in \mathbb{R}$ $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ (where $\lambda(t-) = \inf_{s<t} \lambda(s) = \lim_{s \uparrow t} \lambda(s)$ etc.).

We may define a partial ordering on $[0,1](L)$ by $\lambda \leq \mu$ if for every $t \in \mathbb{R}$ $\lambda(t-) \leq \mu(t-)$ and $\lambda(t+) \leq \mu(t+)$. We may embed the unit interval in the fuzzy unit interval by identifying $r \in [0,1]$ with the map $R : \mathbb{R} \to L$ where $R(t) = 1$ for $t < r$ and $R(t) = 0$ for $t > r$.

We define a fuzzy topology on $[0,1](L)$ by taking as a sub-base $\{L_t, R_t | t \in \mathbb{R}\}$ where we define

$$L_t(\lambda) = \lambda(t-)'$$

$$R_t(\lambda) = \lambda(t+)$$

This topology is called the usual topology for $[0,1](L)$. $\{L_t | t \in \mathbb{R}\}$ and $\{R_t | t \in \mathbb{R}\}$ are called the left and right hand topology respectively.

Note that these really are topologies and that $[0,1](L)$ and its topology reduces to $[0,1]$ and its usual topology for $L = \{0,1\}$.

Theorem 1. (Urysohn's lemma)

A fuzzy topological space (X,τ) is normal if and only if for every closed set K and open set U such that $K \subseteq U$, there exists a continuous function $f : X \to [0,1](L)$ such that for every $x \in X$

$$K(x) \le f(x)(1-) \le f(x)(0+) \le U(x).$$

Proof

($\Leftarrow$) Since

$$K(x) \le f(x)(1-) \le f(x)(0+) \le U(x)$$

we have that for any $t \in (0,1)$

$$K(x) \le f(x)(t+) \le f(x)(t-) \le U(x).$$

Now $f^{-1}(L_t')(x) = f(x)(t-)$ and $f^{-1}(R_t)(x) = f(x)(t+)$. Since f is continuous we have $f^{-1}(L_t')$ is closed and $f^{-1}(R_t)$ is open (see [1]). Hence

$$K \subseteq f^{-1}(R_t) \subseteq f^{-1}(L_t') \subseteq U,$$

that is (X,τ) is normal.

($\Rightarrow$) Conversely;

construct $\{V_r | r \in (0,1)\}$ so that $K \subseteq V_r \subseteq U$ and $r < s$ implies $\overline{V}_s \subseteq V_r^0$ (see for example Pervin). Define $f(x)(t) = V_t(x)$. Clearly

$$K(x) \leq f(x)(1-) \leq f(x)(0+) \leq U(x).$$

$$\text{Now } f^{-1}(R_t) = \bigcup_{r>t} (V_r) = \bigcup_{r>t} V_r^0 \text{ is open}$$

$$\text{and } f^{-1}(L_t') = \bigcap_{r<t} (V_r) = \bigcap_{r<t} \overline{V}_r \text{ is closed.}$$

Hence f is continuous.

We note that perfect normality also has a natural generalisation to fuzzy topological spaces.

Definition 3

A fuzzy topological space is perfectly normal if for every closed set K and open set U such that $K \subseteq U$, there exists a continuous function $f : X \to [0,1](L)$ such that for every x in X

$$K(x) = f(x)(1-) \leq f(x)(0+) = U(x).$$

Theorem 2

A fuzzy topological space is perfectly normal if and only if it is normal and every closed set is a countable intersection of open sets.

The proof is a trivial consequence of Theorem 1 and a generalisation of the usual topological proof.

4. Properties of the fuzzy unit interval.

We now describe some of the properties of the fuzzy unit interval. In particular we show that under certain lattice conditions the fuzzy topology of the fuzzy unit interval is like the topology of the ordinary unit interval.

Theorem 3

Let $(L, \leq, ')$ be a completely distributive lattice with orthocomplement. Then there exists a natural 1-1 correspondence between the open sets in the usual topology for $[0,1]$ and the open sets in the fuzzy topology for $[0,1](L)$ which preserves arbitrary unions and finite intersections.

Proof

(1) Each open set in $[0,1]$ can be written uniquely as a union of nonempty disjoint open intervals $\cup_i(a_i,b_i)$, (allowing $a_i \in [0,1] \cup \{-\infty\}$ and $b_i \in [0,1] \cup \{+\infty\}$ and interpreting as $(-\infty,b)$ as $[0,b)$ etc). Define a map ϕ from the topology of $[0,1]$ to the topology of $[0,1](L)$ by

$$\phi(\cup_i(a_i,b_i))(\lambda) = \vee_i(\lambda(a_i+) \wedge \lambda(b_i-)').$$

ϕ is obviously well defined.

(2) ϕ is 1-1.

If R is an element of $[0,1](L)$ representing $r \in [0,1]$, then $\phi(U)(R) = 1$ if $r \in U$ and $\phi(U)(R) = 0$ if $r \notin U$. But $U \neq V$ implies there exists $r \in [0,1]$ which is in one of U or V but not the other, and thus $\phi(U)(R) \neq \phi(V)(R)$. Hence $\phi(U) \neq \phi(V)$.

(3) $\phi(U \cap V) = \phi(U) \cap \phi(V)$.

Let $U = \bigcup(a_i,b_i)$ and $V = \bigcup(c_j,d_j)$. Then $U \cap V = \bigcup_{i,j}[(a_i,b_i) \cap (c_j,d_j)]$ and each of $(a_i,b_i) \cap (c_j,d_j)$ is disjoint. Hence

$$[\phi(U) \cap \phi(V)](\lambda) = \vee_{i,j}[\lambda(a_i+) \wedge \lambda(b_i-)' \wedge \lambda(c_j+) \wedge \lambda(d_j-)']$$

$$= \vee_{i,j}[\lambda(a_i \vee c_j+) \wedge \lambda(b_i \wedge d_j-)']$$

$$= \phi\ (U \cap V)\ (\lambda).$$

Since if $(a_i,b_i) \cap (c_j,d_j) = \phi$ then $b_i \wedge d_j \leq a_i \vee c_j$ and thus $\lambda(a_i \vee c_j+) \wedge \lambda(b_i \wedge d_j-)' = 0$. Otherwise $(a_i,b_i) \cap (c_j,d_j) = (a_i \vee c_j, b_i \vee d_j)$.

(4) $\phi(\vee_i U_i) = \bigcup_i \phi(U_i)$.

Firstly in an orthocomplemented distributive lattice L, if $\alpha, \beta, \gamma \in L$ and $\alpha \geq \beta \geq \gamma$ then

$$(\alpha \wedge \beta') \vee (\beta \wedge \gamma') = \alpha \wedge \gamma'.$$

This is because

$$(\alpha \wedge \beta') \vee (\beta \wedge \gamma') = (\alpha \vee \beta) \wedge (\alpha \vee \gamma') \wedge (\beta \vee \beta') \wedge (\beta' \vee \gamma')$$

$$= \alpha \wedge (\alpha \vee \gamma') \wedge \gamma'$$

$$= \alpha \wedge \gamma'.$$

Now let $U = (a,b)$, $V = (c,d)$. Then $\phi(U \cup V) = \phi(U) \cup \phi(V)$. This is trivially true if $U \cap V = \phi$ or $U \subseteq V$, so without loss of generality $a < c < b < d$. Let e be such that $c < e < b$. Then

$$[\phi(U) \cup \phi(V)](\lambda) = [\lambda(a+) \wedge \lambda(b-)'] \vee [\lambda(c+) \wedge \lambda(d-)']$$

$$\geq [\lambda(a+) \wedge \lambda(e)'] \vee [\lambda(e) \wedge \lambda(d-)']$$

$$= \lambda(a+) \wedge \lambda(d-)'$$

$$= \phi(U \cup V)\ (\lambda).$$

The other inequality is trivial.

Hence by simple properties of the real line we obtain $\phi(\bigcup_{i=1}^{n} U_i) = \bigcup_{i=1}^{n} \phi(U_i)$ for U_i an open interval.

Now to prove the general case. Let U_i be any open set in $[0,1]$. Consider any open interval (a,b) in $\bigcup_i U_i$, and closed interval $[c,d] \subseteq (a,b)$. By the compactness of $[c,d]$ we may find a finite subcover of open intervals, each contained in some U_i. By the result for finite unions of intervals we have $\phi(c,d) \subseteq \bigcup_i \phi(U_i)$. Let c converge down to a, and d converge up to b. Complete distributivity implies that $\lambda(c+) \wedge \lambda(d-)'$ converges up to $\lambda(a+) \wedge \lambda(b-)'$ for $\lambda \in [0,1](L)$. Hence $\phi(a,b) \subseteq \bigcup_i \phi(U_i)$, and by the definition of ϕ we have $\phi(\bigcup U_i) \subseteq \bigcup_i \phi(U_i)$. The opposite inclusion is trivial.

(5) ϕ is onto.

The set $\{\phi(a,b) \mid a,b \in [0,1] \cup \{+\infty,-\infty\}, a < b\}$ forms a basis for the open sets of $[0,1](L)$, and $\{(a,b) \mid a,b \in [0,1] \cup \{+\infty,-\infty\}\ a < b\}$ forms a basis for the open sets of $[0,1]$. Hence by (4) $\{\phi(U) \mid U \text{ open in } [0,1]\}$ is the topology of $[0,1](L)$.

<u>Corollary 4</u>

Let L be a completely distributive lattice with orthocomplement. Then any statement properly phrased purely in terms of open and closed sets which is true for $[0,1]$ is also true for $[0,1](L)$.

<u>Corollary 5</u>

Let L be a completely distributive lattice with orthocomplement.

Then [0,1](L) is perfectly normal. It is compact in the sense of [1], [2] and [4]. Is is also connected in the sense that if U is a fuzzy set which is both open and closed then $U = \phi$ or $U = [0,1](L)$.

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3. UNIFORMITIES ON FUZZY TOPOLOGICAL SPACES, PART I

1. Introduction

In [3] we generalised normality to fuzzy topological spaces as introduced in [1], and characterised it by a sort of Urysohn's lemma. In the process we constructed an interesting fuzzy topological space, the fuzzy unit interval.

In this paper we generalise the notions of quasi-uniformities and uniformities on topological spaces to fuzzy topological spaces. We prove theorems corresponding to many of the usual theorems. In particular we show that every fuzzy topological space is quasi-uniformizable. The fuzzy unit interval plays an essential part in a characterisation of uniformizability in terms of a type of complete regularity. To achieve this we construct a natural uniformity on the fuzzy unit interval

2. Preliminaries

Throughout this paper $(L,\leq,')$ will be a completely distributive lattice with order reversing involution $'$.

An L-fuzzy set on a set X is any map $A : X \to L$. We interpret L as a set of truth values, and A(x) as the degree of membership of x in the fuzzy set A. When L is the lattice $\{0,1\}$ then the collection of fuzzy sets corresponds to the characteristic functions of ordinary sets.

We define the union, intersection and complement of fuzzy sets as follows:

$$(\cup_i A_i) (x) = \vee_i A_i (x) \quad \text{for} \quad x \in X$$

$$(\cap_i A_i) (x) = \wedge_i A_i (x) \quad \text{for} \quad x \in X$$

$$A'(x) = A(x)' \quad \text{for} \quad x \in X$$

We define a fuzzy topological space as a pair (X,τ) where $\tau \subseteq L^X$ (all maps from X to L) and τ is closed under arbitrary unions and finite intersections. A set is called open if it is in τ, and closed if its complement is in τ. If (X,τ_1) and (Y,τ_2) are fuzzy topological spaces, then a map $f:X \to Y$ is said to be continuous if for every τ_2 open set U, $f^{-1}(U) \in \tau_1$, where $f^{-1}(U)(x) = U(f(x))$ for $x \in X$. The interior and closure of fuzzy sets are defined in the obvious way(see [1]).

3. Quasi-Uniformities

Consider a quasi-uniformity on X in the usual topological sense. An element D is a subset of $X \times X$.

We may define $D : 2^X \to 2^X$ by $D(V) = \{y|x \in V \text{ and } (x,y) \in D\}$. It is obvious that $V \subseteq D(V)$ and $D(\cup V_\lambda) = \cup D(V_\lambda)$ for V and V_λ in 2^X. Conversely, given $D : 2^X \to 2^X$ satisfying $V \subseteq D(V)$ and $D(\cup V_\lambda) = \cup D(V_\lambda)$ for V and V_λ in 2^X, we may define $D \subseteq X \times X$ such that D contains the diagonal by $D = \{(x,y) \mid y \in D(\{x\})\}$. Thus in defining a quasi-uniformity for a fuzzy topology, we take our basic elements of the quasi-uniformity to be elements of the set $\mathcal{Q}$ of maps $D : L^X \to L^X$ which satisfy:

(A1) $\quad V \subseteq D(V) \quad$ for $\quad V \in L^X$.

(A2) $\quad D(\cup V_\lambda) = \cup D(V_\lambda) \quad$ for $\quad V_\lambda \in L^X$.

Before we define what we mean by a quasi-uniformity we need some preliminary results.

Lemma 1

Suppose L is a completely distributive lattice and $\alpha \in L$. Then there exists a set $B \subseteq L$ such that $\sup B = \alpha$ and if $A \subseteq L$ satisfies $\sup A = \alpha$ then for every $\beta \in B$ there exists $\gamma \in A$ such that $\beta \leq \gamma$.

Proof

Consider all possible sets $A \subseteq L$ such that $\sup A = \alpha$. Index these sets $\{A_j | j \in J\}$ and index the elements in the sets by $A_j = \{\alpha_{ij} \mid i \in I_j\}$.

Consider $\quad B = \{\wedge_{j\in J}\alpha_{i(j)j} \mid i \in \Pi I_j\}$.

Then $\quad \text{Sup } B = \vee_{i\in\Pi I_j}(\wedge_{j\in J}\alpha_{i(j)j})$

$$= \wedge_{j\in J}(\vee_{i(j)\in I_j}\alpha_{i(j)j})$$

$$= \wedge_{j\in J}\ \alpha$$

$$= \alpha .$$

Also $\wedge_{j\in J}\alpha_{i(j)j} \le \alpha_{i(k)k}$ for $k \in J$. Thus for every $\beta \in B$ there exists $\gamma \in A_k$ such that $\beta \le \gamma$ (for any $k \in J$).

<u>Lemma 2</u>

Suppose L is a completely distributive lattice and $f: L \to L$ satisfies

(a1) $\alpha \le f(\alpha)$ for $\alpha \in L$

(a2′) $\alpha \le \beta$ implies $f(\alpha) \le f(\beta)$ for $\alpha,\beta\in L$.

Then $f^* : L \to L$ defined by

$$f^*(\alpha) = \wedge_{\sup \Gamma = \alpha}(\vee_{\gamma\in\Gamma}f(\gamma))$$

is the greatest $g : L \to L$ which takes values less than or equal to f and satisfies

(a1) $\alpha \le g(\alpha)$ for $\alpha \in L$

(a2) $g(\vee_i\alpha_i) = \vee_i g(\alpha_i)$ for $\alpha_i \in L$.

Also $f^*(\alpha) = \vee_{\beta\in B}f(\beta)$ for B as in lemma 1.

Proof

Clearly f^* satisfies (a1) and (a2´). Also $f^*(\alpha) \leq f(\alpha)$ for $\alpha \in L$.

Choose B as in lemma 1. If $\sup \Gamma = \alpha$ then for every $\beta \in B$ there exists $\gamma \in \Gamma$ such that $\beta \leq B$.

Hence $\quad \vee_{\beta\in B} f(\beta) \leq \vee_{\gamma\in\Gamma} f(\gamma)$

which implies $f^*(\alpha) = \vee_{\beta\in B} f(\beta)$.

Suppose $\vee_i \alpha_i = \alpha$. Then we may find B_i such that $\sup B_i = \alpha_i$ and

$$f^*(\alpha_i) = \vee_{\beta_i \in B_i} f(\beta_i)$$

so $$\vee_i f^*(\alpha_i) = \vee_i \vee_{\beta_i \in B_i} f(\beta_i)$$

$$\geq f^*(\alpha) \text{ since } \sup \vee_i B_i = \alpha .$$

That f^* is the greatest such g is obvious.

Definition 1

Let $f_1 : L \to L$ and $f_2 : L \to L$ satisfy (a1) and (a2) (as in lemma 2). Let $g : L \to L$ be defined by $g(\alpha) = f_1(\alpha) \wedge f_2(\alpha)$ (so g satisfies (a1) and (a2´)). Then we define $f_1 \wedge f_2 : L \to L$ by $f_1 \wedge f_2 = g^*$ (so $f_1 \wedge f_2$ satisfies (a1) and (a2)) .

Lemma 3

Suppose $f_1 : L \to L$ and $f_2 : L \to L$ satisfy (a1) and (a2) . Then

$$(f_1 \wedge f_2)(\alpha) = \bigwedge_{\alpha_1 \vee \alpha_2 = \alpha} (f_1(\alpha_1) \vee f_2(\alpha_2)) .$$

Proof

$$\begin{aligned}(f_1 \wedge f_2)(\alpha) &= \bigwedge_{\sup\Gamma = \alpha} (\vee_{\gamma\in\Gamma}(f_1(\gamma) \wedge f_2(\gamma))) \\ &= \bigwedge_{\sup\Gamma = \alpha} (\wedge_{A\subseteq\Gamma}([\vee_{\gamma\in A} f_1(\gamma)] \vee [\vee_{\gamma\notin A} f_2(\gamma)])) \\ &= \bigwedge_{\sup\Gamma = \alpha} (\bigwedge_{A\subseteq\Gamma} (f_1(\bigvee_{\gamma\in A} \gamma) \vee f_2(\bigvee_{\gamma\notin A} \gamma))) \\ &= \bigwedge_{\alpha_1 \vee \alpha_2 = \alpha} (f_1(\alpha_1) \vee f_2(\alpha_2)).\end{aligned}$$

Note that L^X is a completely distributive lattice if L is. Hence lemmas 1,2, and 3 may be applied with L replaced by L^X. Thus (a1) and (a2) are now the conditions (A1) and(A2) . For $D : L^X \to L^X$ and $E : L^X \to L^X$ we denote $D \wedge E$ by $D \cap E$. We say $D \subseteq E$ if $D(V) \subseteq E(V)$ for every $V \in L^X$. We define $D \circ E$ by composition of functions. We are now in a position to define a quasi-uniformity.

Definition 2

A (fuzzy) quasi-uniformity on a set X is a subset $\mathcal{D}$ of $\mathcal{Q}$ (the set of all maps satisfying (A1) and (A2)) such that:

(Q1) $\mathcal{D} \neq \phi$

(Q2) $D \in \mathcal{D}$ and $D \subseteq E \in \mathcal{Q}$ implies $E \in \mathcal{D}$.

(Q3) $D \in \mathcal{D}$ and $E \in \mathcal{D}$ implies $D \wedge E \in \mathcal{D}$.

(Q4) $D \in \mathcal{D}$ implies there exists $E \in \mathcal{D}$ such that $E \circ E \subseteq D$.

Note that this definition agrees with the usual definition for $L = \{0,1\}$. Note that (Q3) may be replaced by

(Q3´) $D_1 \in \mathcal{D}$ and $D_2 \in \mathcal{D}$ implies there exists $D \in \mathcal{D}$ such that $D \subseteq D_1$ and $D \subseteq D_2$.

Also note that any subset $\mathcal{B}$ of $\mathcal{Q}$ which satisfies (Q4) generates a fuzzy quasi uniformity in the sense that the collection of all $D \in \mathcal{Q}$ which contain a finite intersection of elements of $\mathcal{B}$ is a quasi-uniformity. Such a set $\mathcal{B}$ is called a sub-basis for the quasi-uniformity generated. If $\mathcal{B}$ also satisfies (Q3´) then $\mathcal{B}$ is called a basis.

Before we define the Fuzzy topology generated by a quasi-uniformity, we state the following trivial proposition.

Proposition 4

Suppose a map $i : L \to L$ satisfies the interior axioms:

(I1) $i(X) = X$

(I2) $i(V) \subseteq V$ for $V \in L$

(I3) $i(i(V) = i(V)$ for $V \in L$.

(I4) $i(V \cap W) = i(V) \cap i(W)$ for $V,W \in L$.

Then $\tau = \{ V \in L^X \mid i(V) = V \}$ is a fuzzy topology and $i(V) = \mathrm{Int}(V)$.

Definition 3

Let $(X,\mathcal{D})$ be a quasi-uniformity. Define $\mathrm{Int}\colon L^X \to L^X$ by $\mathrm{Int}(V) = \bigcup\{ U \in L^X \mid D(U) \subseteq V \text{ for some } D \in \mathcal{D} \}$.

Proposition 5

Int satisfies the interior axioms.

Proof

(I1) and (I2) are trivially satisfied.

(I3) is satisfied since

If U and V are fuzzy sets and $D \in \mathcal{D}$ is such that $D(U) \subseteq V$, then we can find $E \in D$ such that $E \circ E \subseteq D$. So in particular $E(E\ U) \subseteq V$. Thus $E(U) \subseteq \mathrm{Int}(V)$, which implies $U \subseteq \mathrm{Int}(\mathrm{Int}\ V))$. Hence $\mathrm{Int}(V) \subseteq \mathrm{Int}(\mathrm{Int}(V))$, and since the other inclusion follows by (I2) we have $\mathrm{Int}(V) = \mathrm{Int}(\mathrm{Int}(V))$.

(I4) follows by (Q3).

Definition 4

The fuzzy topology generated by D is the fuzzy topology generated by Int.

Hence in particular we note that $D(U)$ is a neighbourhood of U in the topology generated by D.

Lemma 6

Let (X,τ) be a fuzzy topological space. Suppose $D_i \in \mathcal{Q}$ and $D_i(U)$ is a neighbourhood of U for any fuzzy set U $(i=1,2)$. Then $(D_1 \cap D_2)(U)$ is a neighbourhood of U for any fuzzy set U.

Proof

By Lemmas 1,2 and 3

$$(D_1 \cap D_2)(U) \quad = \quad \cup_j (D_1(U_j) \cap D_2(U_j))$$

for some fuzzy sets U_j whose union is U. But $D_i(U_j)$ is a neighbourhood of U_j $(i=1,2)$ and hence $D_1(U_j) \cap D_2(U_j)$ is a neighbourhood of U_j. So there exists open sets W_j such that $U_j \subseteq W_j \subseteq D_1(U_j) \cap D_2(U_j)$. Hence $W = \cup_j W_j$ is open and $U \subseteq W \subseteq \cup_j (D_1(U_j) \cap D_2(U_j))$.

Theorem 7

Every fuzzy topology is fuzzy quasi-uniformizable.

Proof

Let (X,τ) be a fuzzy topological space. Let G be any open set in τ.

Define $D_G(V) = X$ for $V \not\subseteq G$

$= G$ for $V \subseteq G$

So $D_G \circ D_G = D_G$. Thus by Proposition 6 $\{ D_G \mid G \in \tau\}$ forms a sub-base for a quasi-uniformity which generates the topology.

We may define quasi-uniform continuity between quasi-uniform spaces.

Definition 5

Let $(X,\mathcal{D})$ and $(Y,\mathcal{E})$ be quasi-uniform spaces. A map $f : X \to Y$ is said to be quasi-uniformly continuous if for every $E \in \mathcal{E}$, there exists a $D \in \mathcal{D}$ such that $D \subseteq f^{-1}(E)$. That is, for $V \in L^X$, $D(V) \subseteq f^{-1}E(f(V))$.

Proposition 8

Every quasi-uniformly continuous function is continuous in the induced fuzzy topologies.

Proof

Let $f : X \to Y$ be quasi-uniformly continuous. Consider an open set V in the fuzzy topology generated by $\mathcal{E}$. So $V = \cup \{ U \mid E(U) \subseteq V$ for some $E \in \mathcal{E}\}$.

If $E(U) \subseteq V$ then there exists $D \in \mathcal{D}$ such that

$Df^{-1}(U) \subseteq f^{-1}(E(ff^{-1}(U)) \subseteq f^{-1}(E(U)) \subseteq f^{-1}(V)$ (Ref[1]).

So $f^{-1}(U) \subseteq \mathrm{Int}\, f^{-1}(V)$, and hence

$\cup\{ f^{-1}(U) \mid E(U) \subseteq V \text{ for some } E \in \mathcal{E} \} \subseteq \mathrm{Int}(f^{-1}(V))$.

But $f^{-1}(\cup U_\lambda) = \cup f^{-1}(U_\lambda)$ and hence $f^{-1}(V) \subseteq \mathrm{Int}(f^{-1}(V))$.

That is $f^{-1}(V)$ is open, which is the definition of continuity.

We now prove a theorem corresponding to the characterisation of quasi-pseudo metrizability in terms of quasi-uniformities. We effectively define quasi-pseudo metrizablity in terms of a special sort of base for a quasi-uniformity. There is a way of converting this quasi-uniformity into a map satisfying the triangle inequality from $X \times X$ to a monoid, but this is no more than a notational change. The description in terms of this special base appears to be more intuitively pleasing at the moment and so we leave it like this.

<u>Theorem 9</u> (Quasi-pseudo metrization).

Let $(X,\mathcal{D})$ be a quasi-uniformity. Then $\mathcal{D}$ has a base $\{D_r \mid r \in \mathbb{R}, r > 0\}$ such that $D_r \circ D_s \subseteq D_{r+s}$ for r and s positive reals if and only if $\mathcal{D}$ has a countable base.

<u>Proof</u>

($\Rightarrow$) is trivial

($\Leftarrow$) conversely;

suppose $\mathcal{D}$ has a countable base $\{ U_n | n = 1,2,3,\ldots\}$. We may rechoose $\{U_n\}$ such that $U_n \circ U_n \circ U_n \subseteq U_{n-1}$ (see for example Pervin [4]).

Define $\phi_\varepsilon \in \mathcal{D}$ for $\varepsilon > 0$ by $\phi_\varepsilon = U_n$ if $2^{-n} \leq \varepsilon \leq 2^{-(n-1)}$ and $\phi_\varepsilon(V) = X$ if $1 \leq \varepsilon$ (so that $\phi_\varepsilon \circ \phi_\varepsilon \circ \phi_\varepsilon \subseteq \phi_{2\varepsilon}$).

$$\text{Define } D_\varepsilon = \bigcup_{\sum_{i=1}^{k} \varepsilon_i = \varepsilon} \phi_{\varepsilon_1} \circ \cdots \circ \phi_{\varepsilon_k} .$$

D_ε is obviously in $\mathcal{Q}$ since ϕ_ε is.

Now (1) $\phi_\varepsilon \subseteq D_\varepsilon$ trivially.

(2) $D_\varepsilon \subseteq \phi_{2\varepsilon}$ since:

If $k = 1$ $\phi_{\varepsilon_1} \circ \ldots \circ \phi_{\varepsilon_k} \subseteq \phi_{2\varepsilon}$ trivially. Assume that $k > 1$ and if $\ell < k$ and $\varepsilon > o$ then $\phi_{\varepsilon_1} \circ \ldots \circ \phi_{\varepsilon_\ell} \subseteq \phi_{2\varepsilon}$ (where $\varepsilon_1 + \ldots + \varepsilon_\ell = \varepsilon$). Consider $\phi_{\varepsilon_1} \circ \ldots \circ \phi_{\varepsilon_k}$ (where $\varepsilon_1 + \ldots + \varepsilon_k = \varepsilon$). Choose the largest j such that $\varepsilon_1 + \ldots + \varepsilon_j \leq \frac{1}{2}\varepsilon$. Thus $\varepsilon_{j+2} + \ldots + \varepsilon_k \leq \frac{1}{2}\varepsilon$.

By induction
$$\begin{aligned} \phi_{\varepsilon_1} \circ \ldots \circ \phi_{\varepsilon_j} &\subseteq \phi_\varepsilon \\ \phi_{\varepsilon_{j+1}} &\subseteq \phi_\varepsilon \\ \phi_{\varepsilon_{j+2}} \circ \ldots \circ \phi_{\varepsilon_k} &\subseteq \phi_\varepsilon \end{aligned}$$

Hence
$$\begin{aligned} \phi_{\varepsilon_1} \circ \ldots \circ \phi_{\varepsilon_k} &\subseteq \phi_\varepsilon \circ \phi_\varepsilon \circ \phi_\varepsilon \\ &\subseteq \phi_{2\varepsilon} . \end{aligned}$$

Hence $\{ D_\varepsilon | \varepsilon > o \}$ generates the same quasi-uniformity as $\{\phi_\varepsilon\}$ and hence as $\{U_n\}$. The family $\{D_\varepsilon | \varepsilon > o \}$ obviously satisfy $D_r \circ D_s \subseteq D_{r+s}$.

4. Uniformities

Consider a uniformity in the usual topological sense. We define D^{-1} by $(x,y) \in D^{-1}$ if and only if $(y,x) \in D$.

Equivalent statements are:

$y \in D^{-1}(\{x\})$ if and only if $x \in D(\{y\})$

$y \notin D^{-1}(\{x\})$ if and only if $x \notin D(\{y\})$

$D^{-1}(\{x\}) \subseteq \{y\}'$ if and only if $D(\{y\}) \subseteq \{x\}'$

$D^{-1}(V) \subseteq U'$ if and only if $D(U) \subseteq V'$.

This suggests the following.

Definition 6

Let L be a completely distributive lattice with order reversing involution $'$. Let $f : L \to L$ satisfy (a1) and (a2). Then we define $f^{-1} : L \to L$ by $f^{-1}(\alpha) = \inf\{\beta \mid f(\beta') \leq \alpha'\}$.

Proposition 10

(1) $f(\alpha) \leq \beta$ if and only if $f^{-1}(\beta') \leq \alpha'$.

(2) f^{-1} satisfies (a1) and (a2)

(3) $(f^{-1})^{-1} = f$.

(4) $f \leq g$ if and only if $f^{-1} \leq g^{-1}$.

(5) $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Proof

(1) $(\Rightarrow)$ is trivial

$(\Leftarrow)$ holds since if $f^{-1}(\beta') \subseteq \alpha'$

then $f(\alpha) \leq f(f^{-1}(\beta')')$

$= f(\vee\{\gamma | f(\gamma) \leq \beta\})$

$= \vee\{f(\gamma) | f(\gamma) \leq \beta\}$

$\leq \beta$

(2) (a1) is trivially satisfied.

(a2)is true since

$$\begin{aligned} \vee_i f^{-1}(\alpha_i) \subseteq \beta \Leftrightarrow f^{-1}(\alpha_i) &\subseteq \beta \text{ for all } i . \\ \Leftrightarrow f(\beta') &\subseteq \alpha_i' \text{ for all } i \\ \Leftrightarrow f(\beta') &\subseteq (\vee_i\alpha_i)' \\ \Leftrightarrow f^{-1}(\vee_i\alpha_i) &\subseteq \beta . \end{aligned}$$

hence $f^{-1}(\vee_i\alpha_i) = \vee_i f^{-1}(\alpha_i)$.

(3),(4) and (5) follow by a similar argument.

Proposition 11

$$(f_1 \wedge f_2)^{-1} = f_1^{-1} \wedge f_2^{-1} .$$

Proof

Let $\Delta \subseteq L$ be such that $\sup \Delta = \alpha$ and if $\sup \Gamma = \alpha$ then for every $\delta \in \Delta$ there exists $\gamma \in \Gamma$ such that $\delta \leq \gamma$ (as in lemma 1).

Then $(f_1 \wedge f_2)^{-1}(\alpha)$

$$= \wedge\{\beta \mid (f_1 \wedge f_2)(\beta') \leq \alpha'\}$$

$$= \wedge\{\beta \mid \bigwedge_{\beta_1 \wedge \beta_2 = \beta} (f_1(\beta_1') \vee f_2(\beta_2')) \leq \alpha'\}$$

$$= \wedge\{\beta \mid \forall \delta \in \Delta\ ,\ \exists \beta_1, \beta_2 \text{ such that } \beta_1 \wedge \beta_2 = \beta \text{ and } f_1(\beta_1') \vee f_2(\beta_2') \leq \delta'\}$$

$$= \bigvee_{\delta \in \Delta} (\wedge\{\beta \mid \exists \beta_1, \beta_2 \text{ such that } \beta_1 \wedge \beta_2 = \beta \text{ and } f_1(\beta_1') \leq \delta',\ f_2(\beta_2') \leq \delta'\})$$

$$= \bigvee_{\delta \in \Delta} (\wedge\{\beta_1 \wedge \beta_2 \mid f_1(\beta_1') \leq \delta' \text{and } f_2(\beta_2) \leq \delta'\})$$

$$= \bigvee_{\delta \in \Delta} (f_1^{-1}(\delta) \wedge f_2^{-1}(\delta))$$

$$= (f_1^{-1} \wedge f_2^{-1})(\alpha) \quad \text{(by lemma 3)}$$

Now if L is a completely distributive lattice with order reversing involution $'$, then so is L . Hence for every D:L $\to$ L in $\mathcal{Q}$ we can define D^{-1} as in definition 6. Note that if $\mathcal{D}$ is a quasi-uniformity on X then so is $\mathcal{D}^{-1} = \{D^{-1} \mid D \in \mathcal{D}\}$. Also note that if $D \in \mathcal{Q}$ then $(D \cap D^{-1})^{-1} = D \cap D^{-1}$, that is $D \cap D^{-1}$ is symmetric.

We are now able to define uniform spaces.

<u>Definition 7.</u>

A quasi-uniformity $\mathcal{D}$ is a uniformity if it also satisfies

(Q5) $D \in \mathcal{D}$ implies $D^{-1} \in \mathcal{D}$

or equivalently

(Q5$'$) $\mathcal{D}$ has a base of symmetric elements.

We introduced the fuzzy unit interval in [3]. It is defined as follows.

Definition 8

The fuzzy unit interval $[0,1](L)$ is the set of all monotonic decreasing maps $\lambda : \mathbb{R} \to L$ for which

$$\lambda(t) = 1 \quad \text{for} \quad t < 0$$

$$\lambda(t) = 0 \quad \text{for} \quad t > 0$$

after the identification of $\lambda : \mathbb{R} \to L$ and $\mu : \mathbb{R} \to L$ if

$$\lambda(t-) = \mu(t-) \quad \text{for} \quad t \in \mathbb{R}$$

and $$\lambda(t+) = \mu(t+) \quad \text{for} \quad t \in \mathbb{R}$$

(where $\lambda(t+) = \sup\limits_{s>t} \lambda(s)$ etc).

We define a fuzzy topology on $[0,1](L)$ as the topology generated by the sub-base $\{L_t, R_t \mid t \in \mathbb{R}\}$

where $L_t : [0,1](L) \to L$

and $R_t : [0,1](L) \to L$

are defined by $L_t(\lambda) = \lambda(t-)$

$$R_t(\lambda) = \lambda(t+) .$$

The fuzzy topology $\{R_t \mid t \in \mathbb{R}\}$ is called the right hand topology.

We can construct a uniform structure on $[0,1](L)$ as follows.

Definition 9

We define $B_\varepsilon : L^X \to L^X$ by

$$B_\varepsilon(U) = R_{t-\varepsilon} \text{ where } t \text{ is the greatest } s \in \mathbb{R} \text{ such that } U \subseteq L_s' .$$

$$= \bigcap \{R_{s-\varepsilon} \mid U \subseteq L_s' \} .$$

Proposition 12

(1) B_ε satisfies (A1) and(A2).

(2) $B_\varepsilon^{-1} = \bigcap \{L_{s+\varepsilon} \mid U \subseteq R_s' \}$

(3) $B_\varepsilon \circ B_\delta \subseteq B_{\varepsilon+\delta}$

(so in particular $B_\varepsilon \circ B_\varepsilon \subseteq B_{2\varepsilon}$).

Proof

(1) (A1) is true since $L_s' \subseteq R_{s-\varepsilon}$

(A2) is true since

$$\begin{aligned} U_\lambda \subseteq L_{s_\lambda}' &\Rightarrow \forall\delta>0 \quad U_\lambda \subseteq R_{s_\lambda - \delta} \\ &\Rightarrow \forall\delta>0 \quad \bigcup U_\lambda \subseteq \bigcup_\lambda R_{s_\lambda - \delta} \\ &\Rightarrow \forall\delta>0 \quad \bigcup U_\lambda \subseteq R_{\wedge s_\lambda - \delta} \\ &\Rightarrow \quad \bigcup U_\lambda \subseteq L_{\wedge s_\lambda}' \\ &\Rightarrow B_\varepsilon(\bigcup U_\lambda) \subseteq R_{\wedge s_\lambda - \varepsilon} \\ &\qquad\qquad = \bigcup_\lambda B_\varepsilon(U_\lambda) \end{aligned}$$

and the other inclusion is trivial.

(2) $$B_\varepsilon^{-1}(U) = \bigcap \{V \mid B_\varepsilon(V') \subseteq U'\}$$

$$= \bigcap \{L_t \mid B_\varepsilon(L_t') \subseteq U'\}$$

$$= \bigcap\{ L_t \mid R_{t-\varepsilon} \subseteq U' \}$$

$$= \bigcap\{ L_{t+\varepsilon} \mid U \subseteq R_t' \}$$

$$= L_{t+\varepsilon}, \text{ where } t \text{ is the smallest } s \text{ such that } U \subseteq R_s' .$$

$$(3) \quad B_\varepsilon(B_\delta(U)) = B_\varepsilon(R_{t-\delta}) \text{ where } t \text{ is the greatest } s \text{ such that } U \subseteq L_s' .$$

$$= R_{t-\delta-\varepsilon}$$

$$= B_{\varepsilon+\delta}(U).$$

Corollary 13

The set $\{B_\varepsilon \mid \varepsilon>0\}$ is a basis for a quasi-uniformity which generates the right hand topology.

Proof

Every open set in the topology generated by $\{B_\varepsilon\}$ is open in the usual right hand topology since it is a union of sets of the form $B_\varepsilon(U)(=R_t$ for some $t)$.

Conversely R_t is in the topology generated by $\{B_\varepsilon\}$ since $B_\varepsilon(L_{t+\varepsilon}) = R_t$ and $\bigcup_{\varepsilon>0} L_{t+\varepsilon} = R_t$.

Corollary 14

The set $\{B_\varepsilon, B_\varepsilon^{-1} \mid \varepsilon>0 \}$ is a sub-basis for a uniformity on $[0,1](L)$. The topology generated by the uniformity is the usual (fuzzy) topology.

This uniformity is called the usual uniformity for the usual fuzzy topology on $[0,1](L)$.

We are now in a position to characterise uniformizability.

Theorem 15

Let $(X,\mathcal{D})$ be a uniform space and let $D \in \mathcal{D}$. Suppose $D(U) \subseteq V$. Then there exists a uniformly continuous function $f : X \to [0,1](L)$. such that

$$U(x) \leq f(x)(1-) \leq f(x)(0+) \leq V(x) \text{ for } x \in X .$$

Proof

Construct fuzzy sets $\{A_r | r \in \mathbb{R}\}$

such that (1) $A_r = X$ for $r < 0$

(2) $A_r = \phi$ for $r > 1$

(3) $A_o = V$

(4) $A_1 = U$

and symmetric elements $\{D_\varepsilon | \varepsilon > 0\}$ of the uniformity such that

$$D_\varepsilon(A_r) \subseteq A_{r-\varepsilon} \text{ for } r \in \mathbb{R} .$$

Since D_ε is symmetric we have

$$D_\varepsilon (A_r') \subseteq A'_{r+\varepsilon}$$

Now define $f : X \to [0,1](L)$ by $f(x)(r) = A_r(x)$.

Clearly f is well defined and satisfies

$$U(x) \le f(x)(1-) \le f(x)(0+) \le V(x) \text{ for } x \in X \ .$$

Hence we only need to show f is uniformly continuous. Clearly

$f^{-1}(R_t) = \bigcup_{s>t} A_s$ and $f^{-1}(L_t') = \bigcap_{s<t} A_s$

Hence $D_\varepsilon(f^{-1}(L_t')) \subseteq D_\varepsilon(A_{t-\delta})$ for any $\delta > 0$

$$\begin{aligned} &\subseteq A_{t-\delta-\varepsilon} \\ &\subseteq \bigcup_{s>t} A_{s-2\delta-\varepsilon} \\ &\subseteq f^{-1}(B_{\varepsilon+2\delta}(L_t')). \end{aligned}$$

Letting $\delta = \frac{1}{2}\varepsilon$ we have $D_\varepsilon \subseteq f^{-1}(B_{2\varepsilon})$. Similarly $D_\varepsilon \subseteq f^{-1}(B_{2\varepsilon}^{-1})$ and so f is uniformly continuous.

Corollary 16

Let $(X,\mathcal{D})$ be a uniform space and U be an open set in the fuzzy topology generated by U. Then there exists a collection $\{W_\lambda\}$ of sets such that $\cup W_\lambda = U$ and continuous functions $f_\lambda : X \to [0,1](L)$ such that

$$W_\lambda(x) \le f_\lambda(x)(1-) \le f_\lambda(x)(0+) \le U(x) \text{ for } x \in X \ .$$

Proof

Since U is open then $U = \cup\{W \mid D(W) \subseteq U\}$.

Apply the previous theorem.

Note

The condition that for any open set U there exist a collection of sets $\{W_\lambda\}$ and continuous functions as above is equivalent to complete regularity for the topological case (since $\{W_\lambda\}$ might as well be all singletons contained in U).

We use this as our definition of complete regularity for fuzzy topological spaces.

We now show the converse to corollary 16 is true.

Theorem 17

Suppose (X,τ) is a completely regular fuzzy topological space. Then (X,τ) is uniformizable.

Proof

The set $\{f^{-1}(B_\varepsilon),\ f^{-1}(B_\varepsilon^{-1}) \mid f : X \to [0,1](L) \text{ is continuous and } \varepsilon>0\}$ forms a sub-base for a uniformity $\mathcal{D}$ since

(1) $f^{-1}(B_\varepsilon) \circ f^{-1}(B_\delta) \subseteq f^{-1}(B_{\varepsilon+\delta})$

(2) $f^{-1}(B_\varepsilon^{-1}) = f^{-1}(B_\varepsilon)^{-1}$.

The uniformity induces the topology since

(1) Any continuous function $f : X \to [0,1](L)$ is uniformly continuous in $(X,\mathcal{D})$ and so is continuous in the topology generated by $\mathcal{D}$ and so the topology induced by $\mathcal{D}$ is coarser that τ .

(2) However, suppose U is open in τ . We may find $\{W_\lambda\}$ such that $\cup W_\lambda = U$ and $f_\lambda : X \to [0,1](L)$ such that

$$W_\lambda(x) \leq f_\lambda(x)(1-) \leq f_\lambda(x)(0+) \leq U(x) \ .$$

Hence in particular $f_\lambda^{-1}(B_{\frac{1}{2}})\ (W_\lambda) \subseteq U$ and so $W_\lambda \subseteq \mathrm{Int}(U)$ in the topology generated by $\mathcal{D}$. Thus U is open in the topology generated by $\mathcal{D}$.

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4. PRODUCTS OF FUZZY TOPOLOGICAL SPACES

1. Introduction

In this paper we continue with the same philosophical attitude developed in our previous papers. That is: we try and develop "pointless" definitions for properties and structures which depend purely upon the lattice structure of the collection of fuzzy sets, and not upon its decomposition into the form L^X (where X is an ordinary set). We then try to extract the essence of the usual topological theorems, and generalise their proofs.

A definition of the product of a collection of fuzzy topological spaces has already appeared in the literature [3],[13]. However this is a "pointed" definition, and hence does not fit within our "pointless" framework. Hence we give a new "pointless" definition, which is in fact a category theoretic definition. We prove various classical theorems including the "Tychonov product theorem." When proving this theorem, we use a stronger definition than the usual compactness definition [11] of "every open cover of the whole space has a finite subcover." We say that to be compact, every open cover of any closed set must have a finite subcover. This has the advantage that in some sense it agrees more with the usual meaning of compactness, as defined by its applications (for example in proving Hausdorf and compact implies T_4). We hope these new definitions of products and compactness will turn out to be much more useful than the old definitions in mimicking the usual theorems about uniformities.

2. Preliminaries

Throughout this paper a fuzzy lattice shall be a complete, completely distributive lattice $(L, \leq, ')$ with order reversing involution $'$. The elements of the lattice shall be called fuzzy sets. A fuzzy topology τ on L is a subset of L closed under finite infima, and arbitrary suprema. The elements of τ are called open sets, and their complements closed sets.

Fuzzy lattices form a category. The objects are fuzzy lattices. The morphisms from L_1 to L_2 are maps from L_2 to L_1 which preserve $\wedge$, $\vee$, and $'$. If the morphism is called f, then we call the map from L_2 to L_1, f^{-1} (since we want f to correspond to a map $f : X_1 \to X_2$, and f^{-1} to correspond to $f^{-1}: \mathcal{P}(X_2) \to \mathcal{P}(X_1)$). We can also define a map $f: L_1 \to L_2$ by $f(a) = \inf \{ b \in L_2 : a \leq f^{-1}($

This map preserves suprema, but may not preserve infima or complements.

If we have two fuzzy topological spaces and a morphism $f : (L_1, \tau_1) \to (L_2, \tau_2)$ between them, then we say f is continuous if $u \in \tau_2 \Rightarrow f^{-1}(u) \in \tau_1$.

If we have a family of morphisms $f_\lambda : L \to L_\lambda$, where L_λ has a topology τ_λ, then the topology induced on L is the topology generated by $\{ f_\lambda^{-1}(u_\lambda) : \lambda \in \Lambda, u_\lambda \in \tau_\lambda \}$. It is the smallest topology making $\{ f_\lambda \}$ continuous.

3. Products of fuzzy lattices

Definition 1 (Product of lattices).

Suppose $\{ L_\lambda : \lambda \in \Lambda \}$ is a family of fuzzy lattices.

Then we define $L = \otimes_\lambda L_\lambda$, the product of $\{ L_\lambda : \lambda \in \Lambda \}$ as follows.

The elements of L are the subsets A of $\Pi_\lambda L_\lambda^n$ (where $L_\lambda^n = L_\lambda - \{0\}$) which satisfy:

(P 1) $a \le b$, $b \in A \Rightarrow a \in A$ (giving $\Pi_\lambda L_\lambda^n$ the obvious partial ordering).

(P 2) If $B_\lambda \subseteq L_\lambda^n$, and $\Pi_\lambda B_\lambda \subseteq A$, then $b \in A$, where $b_\lambda = \sup B_\lambda$.

The ordering on L is the ordering of set inclusion.

We define the complement as follows:

If $B \in L$, then $B' = \{ x : (\forall y \in B)(\exists \lambda \in \Lambda)(x_\lambda \le y_\lambda') \}$.

<u>Proposition 1</u>

L is a fuzzy lattice.

<u>Proof</u>

(1) L is obviously a complete lattice with $\wedge$ equal to $\cap$, and with $\vee$ equal to $\cup$ followed by closure under (P 2).

(2) Complete distributivity of L follows from the fact that if B_i is closed under (P 1) and generates $A_i \in L$ under closure by (P 2), then $\cup_i B_i$ generates $\vee_i A_i$, and $\cap B_i$ generates $\wedge_i A_i$. Thus $A \wedge (\vee_i B_i) = \vee_i (A \wedge B_i)$ is generated by $A \cap (\cup_i B_i) = \cup_i (A \cap B_i)$. Similarly for the other case.

(3) Complement interchanges $\vee$ and $\wedge$ since:

(a)
$$\begin{aligned}(\vee_i A_i)' &= \{ x : (\forall y \in \vee A_i)(\exists \lambda \in \Lambda)(x_\lambda \le y_\lambda') \} \\ &\doteq \{ x : (\forall y \in \cup A_i)(\exists \lambda \in \Lambda)(x_\lambda \le y_\lambda') \} \\ &= \cap_i \{ x : (\forall y \in A_i)(\exists \lambda \in \Lambda)(x_\lambda \le y_\lambda') \} \\ &\doteq \wedge_i A_i'\end{aligned}$$

(b)
$$(\wedge_i A_i)' = \bigcap_{y \in \wedge_i A_i} \bigcup_{\lambda \in \Lambda} \{x : x_\lambda \le y_\lambda'\} = \bigcap_{y \in \Pi_i A_i} \bigcup_{\lambda \in \Lambda} \{x : x_\lambda \le \vee_i y_{i\lambda}'\}$$

Now $\{x: x_\lambda \le \vee_i y_{i\lambda}'\}$ is generated by $\cup_i \{x: x_\lambda \le y_{i\lambda}'\}$ so

$(\wedge_i A_i)'$ is generated by $\displaystyle\bigcap_{y\in\Pi A_i} \bigcup_{\lambda\in\Lambda} \bigcup_{i\in I} \{x: x_\lambda \le y_{i\lambda}'\}$

$$= \bigcup_i \bigcap_{y_i\in A_i} \bigcup_\lambda \{x: x_\lambda \le y_{i\lambda}'\} = \cup_i A_i'.$$

Hence $(\wedge_i A_i)' = \vee_i A_i'$.

(c) Now $A'' = A$ for $A=\{x: x_\lambda \le a\}$ where $\lambda \in \Lambda$ and $a \in L_\lambda^n$. Also elements of this form generate all elements of L under arbitrary $\wedge$ and $\vee$. Hence $A'' = A$ for any $A \in L$.

<u>Definition 2</u> (Projection maps)

Define fuzzy lattice morphisms $\pi_\lambda: L \to L_\lambda$ by

$\pi_\lambda^{-1}(y) = \{x \in L : x_\lambda \le y\}$. Thus $\pi_\lambda(A) = \vee\{x_\lambda : x \in A\}$.

<u>Proposition 2</u>

$(\pi_\lambda : L \to L_\lambda)_{\lambda\in\Lambda}$ is a product in the category theoretic sense.

<u>Proof</u>

That the π_λ are lattice morphisms is obvious (in fact we discovered what the definition of the complement should be by assuming the π_λ were lattice morphisms).

Suppose $f_\lambda : M \to L_\lambda$ are lattice morphisms, then define $f: M \to L$ by

$$f^{-1}(A) = \vee\{\wedge_\lambda f_\lambda^{-1}(x_\lambda) : x \in A\}.$$

This is a lattice morphism, and makes the diagram

$$\begin{tikzcd} & L \arrow[d, "\pi_\lambda"] \\ M \arrow[ur, "f"] \arrow[r, "f_\lambda"'] & L_\lambda \end{tikzcd}$$

commute.

The morphism f is unique since $\{ \pi_\lambda^{-1}(y) : \lambda \in \Lambda , y \in L_\lambda \}$ generates L under arbitrary $\wedge$, $\vee$.

Proposition 3

$\otimes L_\lambda$ does not depend upon labelling or bracketing (eg:

$L_1 \otimes L_2 \cong L_2 \otimes L_1$, and $(L_1 \otimes L_2) \otimes L_3 \cong L_1 \otimes L_2 \otimes L_3$).

Proof

That the product does not depend upon any labelling (or order) is obvious.

That the product does not depend upon bracketing is obvious, once one realises precisely what the procedure for taking suprema is. The correspondence

$$\otimes_{\lambda \in \Lambda_1 \cup \Lambda_2} L_\lambda \leftrightarrow (\otimes_{\lambda \in \Lambda_1} L_\lambda) \otimes (\otimes_{\lambda \in \Lambda_2} L_\lambda) \quad \text{is by}$$

$$\prod_{\lambda \in \Lambda_1 \cup \Lambda_2} L_\lambda^n \leftrightarrow (\prod_{\lambda \in \Lambda_1} L_\lambda^n) \times (\prod_{\lambda \in \Lambda_2} L_\lambda^n)$$

being induced on subsets. We only need to realise that taking suprema for elements of $\prod_{\lambda \in \Lambda_1 \cup \Lambda_2} L_\lambda^n$, as in (P 2) can be factored into taking suprema in each component, and then taking suprema over the pairs.

Proposition 4

Suppose $L_\lambda \cong \mathcal{P}(X_\lambda)$ where X_λ is an ordinary set.

Then $\otimes_\lambda L_\lambda \cong \mathcal{P}(\Pi_\lambda X_\lambda)$.

Proof

The isomorphism $\otimes_\lambda L_\lambda \leftrightarrow \mathcal{P}(\Pi_\lambda X_\lambda)$ is

$A \in \otimes_\lambda L_\lambda \rightarrow \cup \{ a : a \in A \} \subseteq \Pi_\lambda X_\lambda$, or its inverse

$B \subseteq \Pi_\lambda X_\lambda \to \{ a : a \subseteq B,$ and a is a "box" $= \Pi_\lambda a_\lambda$, where

$\phi \neq a_\lambda \subseteq X_\lambda \}$.

These maps are bijective inverses since the properties (P 1) and (P 2) are precisely the properties to ensure that all "box" sets included in a subset of $\Pi_\lambda X_\lambda$ are listed in the corresponding element of $\otimes_\lambda L_\lambda$.

The fact that the maps are bijective and order preserving is sufficient to imply they preserve $\wedge$ and $\vee$. They preserve the complement since they preserve it for sets of the form $\pi_\lambda^{-1}(a)$ where $a \subseteq X_\lambda$, and these sets generate the whole lattice.

Proposition 5

Suppose $L_\lambda \cong M_\lambda^{X_\lambda}$ where M_λ is a fuzzy lattice, and X_λ is an ordinary set. Then $\otimes_\lambda L_\lambda \cong (\otimes_\lambda M_\lambda)^{(\Pi X_\lambda)}$.

(ie: if $L_\lambda \cong$ fuzzy sets of a set X_λ with respect to the fuzzy lattice M_λ then

$\otimes_\lambda L_\lambda \cong$ fuzzy sets of ΠX_λ with respect to the fuzzy lattice $\otimes_\lambda M_\lambda$).

This allows us to relate the work in this paper to our previous papers, and other papers, in which fuzzy sets are maps from a set X into a fuzzy lattice L.

Proof

The proof is not dissimilar from the proof of the previous proposition, and is omitted for the sake of brevity.

However, the isomorphism is the following:

$\varphi : \Pi X_\lambda \to \otimes_\lambda M_\lambda$ corresponds to

$$\{ \psi \in \Pi_\lambda (M_\lambda^{X_\lambda}) : \forall x \in \Pi_\lambda X_\lambda , (\psi_\lambda (x_\lambda))_{\lambda \in \Lambda} \in \varphi(x) \}.$$

Proposition 6 (Characterisation of products)

$L_1 \otimes L_2$ is lattice isomorphic to the lattice of supremum preserving maps $\varphi : L_1 \to L_2$, with order $\varphi \leq \psi$ iff $(\forall a \in L_1)(\varphi(a) \leq \psi(a))$.

The isomorphism is θ: $A \leftrightarrow \varphi_A$ where

$\varphi_A(a) = \wedge \{ b' : (a, b) \in A' \}$ for $a \neq 0$, and $\varphi(0) = 0$.

(So $\varphi_A(a) \leq b' \Leftrightarrow (a, b) \in A'$) (Note: We must have complements on the lattices, since otherwise there is no isomorphism. For example if $L = \mathcal{P}(X) \cup \{1\}$, with the usual partial ordering on $\mathcal{P}(X)$, and with $A \leq 1$ for $A \subset X$. Then $L \otimes L$ has n^2 minimal elements > 0, but there are only n minimal elements in the lattice of sup preserving maps φ: $L \to L$ (where X has n elements).).

Proof

The map φ_A is obviously sup preserving, since $(\forall i, (a_i, b) \in A') \Leftrightarrow (\vee a_i, b) \in A'$.

The map θ is also obviously a bijection, since we may recover A from $A' = \{(a, b) : \varphi_A(a) \leq b'\}$. (This set is in $L_1 \otimes L_2$, since $(\forall i,j\ \varphi(a_i) \leq b') \Leftrightarrow (\varphi(\vee a_i) \leq \wedge b_j')$,

The map θ is also order preserving. This is sufficient to show that we have a lattice isomorphism.

Proposition 7

Consider the "reflection" isomorphism

$r: L_1 \otimes L_2 \to L_2 \otimes L_1$, defined by $r(A) = \{(b,a) : (a,b) \in A\}$.

The correspondence between products and sup preserving maps defined in the previous proposition induces a "reflection" map between sup preserving maps $\varphi: L_1 \to L_2$, and sup preserving maps $\psi: L_2 \to L_1$ by $\varphi \to \varphi^r$ where:

$$\varphi^r(b) = \wedge \{ a : \varphi(a') \leq b' \}.$$

Proof

As in Proposition 10 of our paper on uniformities.

Note:

Propositions 5 and 6 offer a relationship between products and uniformities, as defined in our previous paper. We hope to develop this later. We do however feel that it is now necessary to change the notation from D^{-1} to D^r in our previous paper on uniformities, to avoid confusion with the multiple use of "inverse" that now occurs. Thus this map "r" is precisely the same as the map "inverse of an element of Q," defined in [5].

Definition 3 (Product topology)

Suppose $(L_\lambda , \tau_\lambda)$ are fuzzy topological spaces. The product topology on $\otimes_\lambda L_\lambda$ is the topology induced by the projection morphisms π_λ .

Proposition 8

$\otimes_\lambda L_\lambda$ with the product topology is a category theoretic product in the category of fuzzy topological spaces (with continuous maps as morphisms).

4. Compactness and Products

Definition 4 (Open covers and filters)

Suppose (L, τ) is a fuzzy topological space.

(1) An open cover $\mathfrak{U}$ of a fuzzy set K is a collection $\mathfrak{U}$ of open fuzzy sets such that $K \leq \vee\, \mathfrak{U}$.

(2) Suppose $\mathfrak{A}$ is a subset of L closed under finite infima, then an $\mathfrak{A}$ – filter $\mathcal{F}$ relative to a fuzzy set U is a non-empty subset $\mathcal{F}$ of $\mathfrak{A}$, which satisfies:

(F 1) $F_1 \in \mathcal{F},\ F_1 \leq F_2 \in \mathfrak{A} \Rightarrow F_2 \in \mathcal{F}$.

(F 2) $F_1 \in \mathcal{F},\ F_2 \in \mathcal{F} \Rightarrow F_1 \wedge F_2 \in \mathcal{F}$.

(F 3) $F \leq U \Rightarrow F \notin \mathcal{F}$.

(If $\mathfrak{A} = L$, we just say $\mathcal{F}$ is a filter. If $\mathfrak{A}$ = closed sets, we say $\mathcal{F}$ is a closed-filter, etc.).

(3) An $\mathfrak{A}$– filter pair $(\mathcal{F}, \mathfrak{U})$ is a pair of subsets $\mathcal{F}, \mathfrak{U} \subseteq L$, such that

(FP 1) $\mathcal{F} \subseteq \mathfrak{A}$, and $\mathcal{F}$ satisfies (F 1) and (F 2).

(FP 2) $\mathfrak{U}' = \{ U' : U \in \mathfrak{U} \} \subseteq \mathfrak{A}$, and $\mathfrak{U}'$ satisfies (F 1) and (F 2).

(FP 3) If $F \in \mathcal{F}$ and $U \in \mathfrak{U}$, then $F \nleq U$.

(4) We partially order $\mathfrak{A}$– filters by inclusion, and $\mathfrak{A}$– filter pairs by inclusion for each component. Thus

Zorn's Lemma implies every $\mathfrak{a}$- filter relative to U ($\mathfrak{a}$- filter pair resp.) is contained in a maximal $\mathfrak{a}$- filter relative to U ($\mathfrak{a}$- filter pair resp.).

(5) A subset $\mathcal{F} \subseteq \mathfrak{a}$ is said to satisfy the F.I.P. relative to a fuzzy set U if $F_1, \ldots, F_n \in \mathcal{F} \Rightarrow F_1 \wedge \cdots \wedge F_n \nleq U$.

Thus every subset $\mathcal{F}$ of $\mathfrak{a}$ which satisfies the F.I.P. relative to U is contained in an $\mathfrak{a}$- filter relative to U.

(6) The cluster set of a filter $\mathcal{F}$ is $\wedge \{ \bar{F} : F \in \mathcal{F} \}$.

Proposition 9 (Equivalent definitions of compactness).

Suppose (L, τ) is a fuzzy topological space. The following are equivalent:

(1) Every open cover $\mathfrak{U}$ of a closed set F has a finite subcover.

(2) Every collection of closed sets $\mathcal{F}$ satisfying the F.I.P. relative to an open set U has $\wedge \mathcal{F} \nleq U$.

(3) Every closed-filter $\mathcal{F}$ relative to an open set U satisfies $\wedge \mathcal{F} \nleq U$.

(4) Every maximal closed-filter $\mathcal{F}$ relative to an open set U satisfies $\wedge \mathcal{F} \nleq U$.

(5) Every closed-filter pair $(\mathcal{F}, \mathfrak{U})$ satisfies $\wedge \mathcal{F} \nleq \vee \mathfrak{U}$.

(6) Every maximal closed-filter pair $(\mathcal{F}, \mathfrak{U})$ satisfies $\wedge \mathcal{F} \nleq \vee \mathfrak{U}$.

Proof

The equivalence of (1), (2), (3), (4) is similar to the standard proof. The equivalence of (5) and (6) is obvious. The equivalence of (3) and (5) is proved as follows:

(3) $\Rightarrow$ (5). Consider $\mathcal{F}$ as a closed filter relative to U, for each $U \in \mathfrak{U}$. Thus $\wedge \mathcal{F} \nleq U$. Then consider the complement of $\mathfrak{U}$ relative to the complement of $\wedge \mathcal{F}$.

(5) $\Rightarrow$ (3). Since $(\mathcal{F}, \{U\})$ is a closed-filter pair.

Definition 5 (Compactness)

We say (L, τ) is compact if it satisfies any of the equivalent statements in the previous proposition.

Proposition 10

Suppose $f : L \to M$ is a continuous surjection (as a morphism in the category of fuzzy topological spaces. That is: f^{-1} is 1-1 as a map). Suppose L is compact, then so is M.

Proof

Essentially the standard proof.

Lemma 11

Suppose $(\mathcal{F}, \mathfrak{U})$ is a maximal $\mathfrak{a}$-filter pair. Suppose $F_1, \ldots, F_n \in \mathfrak{a}$, and $F_1 \vee F_2 \vee \cdots \vee F_n \in \mathcal{F}$, then at least one of $F_i \in \mathcal{F}$.

Proof

If $F \in \mathcal{F}$, $U \in \mathfrak{U}$, then $F \wedge (F_1 \vee \cdots \vee F_n) = \vee_i (F \wedge F_i) \nleq U$ so for some i $F \wedge F_i \nleq U$. It is not hard to show that there is an i which works for all F and U. Hence we may generate a larger filter $\mathcal{F}$ by including F_i in the filter. But $\mathcal{F}$ is maximal. Hence $F_i \in \mathcal{F}$.

Lemma 12

Suppose $(\mathcal{F}, \mathfrak{U})$ is a maximal closed-filter pair in $L = \otimes_\lambda L_\lambda$. Then $\wedge \mathcal{F} = \wedge \mathcal{F}^b$ and $\vee \mathfrak{U} = \vee \mathfrak{U}^b$, where $\mathcal{F}^b$ is the filter of sets in $\mathcal{F}$ of the form $\Pi_\lambda F_\lambda$, with each F_λ closed, and $\mathfrak{U}^b$ is the dual filter of sets in $\mathfrak{U}$ of the form $(\Pi_\lambda U'_\lambda)'$ with each U_λ open (ie: the complement of sets of the form $\Pi_\lambda F_\lambda$, F_λ closed.).

Proof

Every closed set F is the infimum of a family of closed sets which are the finite supremum $F_1 \vee \ldots \vee F_n$ of sets of the form $\pi_\lambda^{-1}(K_\lambda)$ where K_λ is a closed fuzzy set in L_λ. By the previous lemma, if $F \in \mathcal{F}$, then one of the $F_i \in \mathcal{F}$. Hence $\wedge \mathcal{F} \leq \wedge \mathcal{F}^b \leq F$. Taking all $F \in \mathcal{F}$ gives $\wedge \mathcal{F} = \wedge \mathcal{F}^b$. By duality $\vee \mathfrak{U} = \vee \mathfrak{U}^b$.

Theorem 13 (Tychonov product theorem)

$\otimes_\lambda L_\lambda$ is compact iff L_λ is compact for each $\lambda \in \Lambda$.

Proof

($\Rightarrow$). Follows by Proposition 10, since $\pi_\lambda : L \to L_\lambda$ is continuous.

($\Leftarrow$). We take the "maximal closed-filter pair" characterisation of compactness.

Suppose $(\mathcal{F}, \mathfrak{U})$ is a maximal closed-filter pair. Then by Lemma 12, $\wedge \mathcal{F} = \wedge \mathcal{F}^b$, and $\vee \mathfrak{U} = \vee \mathfrak{U}^b$, where $\mathcal{F}^b$ is the filter of sets in $\mathcal{F}$ of the form $\Pi_\lambda F_\lambda$, with each F_λ closed, and $\mathfrak{U}^b$ is the dual filter of sets in $\mathfrak{U}$ whose complements are of the form $\Pi_\lambda F_\lambda$, with each F_λ closed.

Hence $\mathcal{F}_\lambda = \{ F_\lambda : F \in \mathcal{F}^b \}$ is a closed-filter in L_λ.

Similarly, if we write each $U \in \mathfrak{U}^b$ in the form $U = (\Pi_\lambda U'_\lambda)'$, then $\mathfrak{U}_\lambda = \{ U_\lambda : U \in \mathfrak{U}^b \}$ is a dual closed-filter in L_λ.

Morever since each $F \in \mathcal{F}^b$, $U \in \mathfrak{U}^b$ satisfy $F \nleq U$, we have that $\forall \lambda \in \Lambda$, $F_\lambda \nleq U_\lambda$ (see the definition of complement for a product space). Hence $\forall \lambda \in \Lambda$, $(\mathcal{F}_\lambda, \mathfrak{U}_\lambda)$ is a closed-filter pair.

Since each L_λ is compact, $\wedge \mathcal{F}_\lambda \nleq \vee \mathfrak{U}_\lambda$. Hence, as before,

$\Pi_\lambda(\wedge \mathcal{F}_\lambda) \nleq (\Pi_\lambda(\vee \mathfrak{U}_\lambda)')'$. But the left hand side is just p59
$\wedge \mathcal{F}$, and the right is just $\vee \mathfrak{U}$. Hence $\wedge \mathcal{F} \nleq \vee \mathfrak{U}$, and so $\otimes_\lambda L_\lambda$ is compact.

5. Connectedness and Products

Definition 6

We say a fuzzy topological space (L, τ) is disconnected if there exists a fuzzy set $\neq 0, 1$ which is both open and closed.

(L, τ) is said to be connected if it is not disconnected.

In [7, Lemma 17] we have proved the following:

Lemma 14

Suppose $U \leq \otimes_\lambda L_\lambda$ is open and $0 \neq \Pi_\lambda A_\lambda \leq U$. Suppose moreover that $0 \neq B_\lambda \in L_\lambda$ is atomic-like with respect to A_λ, for $\lambda \neq \lambda_\circ$ (ie: for any $\mathcal{C} \subseteq L_\lambda$ such that $\sup \mathcal{C} = A_\lambda$, then there exists $C \in \mathcal{C}$ such that $B_\lambda \leq C$). Then the largest box set $V \leq U$ such that $V_\lambda = B_\lambda$ for $\lambda \neq \lambda_\circ$ has $V_{\lambda_\circ}$ open.

Theorem 15

$L = \otimes_\lambda L_\lambda$ is connected iff L_λ is connected for each $\lambda \in \Lambda$.

Proof

$(\Rightarrow)$ Suppose there exists $\lambda \in \Lambda$, and $U_\lambda \in L_\lambda, \neq 0,1$, such that U_λ is both open and closed. Then $\pi_\lambda^{-1}(U_\lambda)$ is both open and closed. Hence L_λ disconnected for some $\lambda \in \Lambda$ implies L is disconnected.

($\Leftarrow$) Suppose $U \neq 0,1$ is a set in L which is both open and closed. Choose a maximal box set $0 \neq \Pi A_\lambda \leq U$. Then $A_{\lambda_\circ} \neq 1$ for some $\lambda_\circ$. Choose $B_\lambda \neq 0$ atomic-like with respect to A_λ such that the largest box set $V \leq U$ such that $V_\lambda = B_\lambda$ for $\lambda \neq \lambda_\circ$ satisfies $V_{\lambda_\circ} \neq 1$ (We can do this, since otherwise $A_{\lambda_\circ} = 1$). Then $V_{\lambda_\circ}$ is closed (since $V \leq U$, U closed implies $\bar{V} \leq U$), and $V_{\lambda_\circ}$ is open, by the lemma. Hence $L_{\lambda_\circ}$ is disconnected. Hence L disconnected implies L_λ is disconnected for some $\lambda \in \Lambda$.

6. Separation Axioms and Products.

In our paper on fuzzy separation axioms [7] we have shown:

Theorem 16

$L = \otimes_\lambda L_\lambda$ satisfies any of the following separation/Regularity axioms iff L_λ does for each $\lambda \in \Lambda$:

R_0 , R_1 , Regular, completely regular.

T_0, T_1, T_2, T_3, $T_{3\frac{1}{2}}$.

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5. SEPARATION AXIOMS IN FUZZY TOPOLOGICAL SPACES

(Version 2)

Following the introduction of the notion of fuzzy sets in the classical paper of Zadeh [14], several papers have considered the general theory of fuzzy topological spaces, as introduced by Chang [1]. In our previous papers we have developed the concepts of Normality and the fuzzy unit interval [4], uniformities and complete regularity [5], a system of regularity axioms [7], and finally we have developed new (and we feel more satisfactory) definitions of product spaces and compactness [6] which differ from those previously defined [3], [11], [13]. In this modified version of [7] we place the regularity axioms into what we believe is a coherent picture. The main differences between this and our previous version are that we have now modified the definition of T_0 (and hence of T_i for $i > 0$) (in preparation for a paper on compactifications) and we have utilized our new definitions of compactness and products to enable us to prove better results.

For an explanation of terms and notation not defined here, see our paper on products [6], and also our other papers [4] and [5].

2. Preliminaries

Throughout this paper L is a fuzzy lattice, ie: L is a complete, completely distributive lattice with order reversing involution $'$, and τ is a topology on L (ie: a subset of L closed under arbitrary suprema and finite infima). We shall frequently omit the word "fuzzy" when talking about fuzzy sets (elements of L). $\overline{A}$ shall denote the closure of A, and A° the interior. This paper differs from its predecessor [7] in that all definitions shall be pointless definitions in the sense that they depend only upon the lattice structure of L, and not on any decomposition of the form M^X where M is a fuzzy lattice, and X is a set. They shall, as before, be generalisations of the standard topological definitions.

3′. T_0

Definition ("Generates")

We say $\mathfrak{U} \subseteq L$ generates $\mathcal{V} \subseteq L$ if $\mathcal{V}$ is the smallest subset of L containing $\mathfrak{U}$, and closed under arbitrary $\vee$ and $\wedge$.

Proposition 1

The collection of fuzzy sets generated by $\mathfrak{U}$ is the collection of fuzzy sets of the form

$$A = \bigvee_{i \in I} \bigwedge_{j \in J_i} U_{ij} \quad \text{where } \{U_{ij} : i \in I,\ j \in J_i\} \subseteq \mathfrak{U}.$$

Proof

By complete distributivity of L.

Definition: T_0

(L, τ) is T_0 if the open and closed sets generate L.

Equivalently, if every fuzzy set $A \in L$ is written in the form

$A = \bigvee_i \bigwedge_{j \in J_i} U_{ij}$ where U_{ij} is an open or closed set.

Notation

Let L_τ = the collection of fuzzy sets generated by τ = open sets

Let L_σ = the collection of fuzzy sets generated by σ = closed sets.

Let $L_{\tau\sigma}$ = the collection of fuzzy sets generated by $\tau \cup \sigma$ the sets which are open or closed.

Proposition 2

Suppose $\{ (L_\lambda , \tau_\lambda) : \lambda \in \Lambda \}$ is a collection of fuzzy, topological spaces.

Then $L = \otimes_\lambda L_\lambda$ is T_0 iff L_λ is T_0 for each $\lambda \in \Lambda$.

Proof

($\Leftarrow$) Suppose each L_λ is T_0. Then each element of the form $\pi_\lambda^{-1}(A)$ for $A \in L_\lambda$ is generated by sets of the form $\pi_\lambda^{-1}(K)$ where K is an open or closed set in L_λ. But the sets of the form $\pi_\lambda^{-1}(A)$ for $A \in L_\lambda$, $\lambda \in \Lambda$, generate all the elements of L, which proves the result.

($\Rightarrow$) Suppose L is T_0, ie: fuzzy sets of the form

$\pi_\lambda^{-1}(A)$ for $A \in (L_\lambda)_{\tau\sigma}$ generate L. Thus

every fuzzy set K is a supremum of "box" sets of the form $\Pi_\lambda U_\lambda$ where $U_\lambda \in (L_\lambda)_{\tau\sigma}$.

Consider in particular the case where $K = \pi_\lambda^{-1}(A)$ for a given $\lambda \in \Lambda$. Then $K = \vee_i U_i$ where

$U_{i\lambda} \in (L_\lambda)_{\tau\sigma}$, and so $A = \pi_\lambda(K) = \vee_i \pi(U_i) = \vee_i U_{i\lambda} \in (L_\lambda)_{\tau\sigma}$

Hence $(L_\lambda, \tau_\lambda)$ is T_0.

4. R_0 and T_1 (Ref [2] for the standard topological definitions).

Definitions: R_0, T_1

(L, τ) is R_0 if every open set is a supremum of closed sets.

(L, τ) is T_1 if R_0 and T_0.

Proposition 3

(a) R_0 is equivalent to:

(1) Every fuzzy set in $L_{\tau\sigma}$ is a supremum of closed sets.

(2) $L_\tau = L_\sigma = L_{\tau\sigma}$.

(b) T_1 is equivalent to:

(1) Every fuzzy set is a supremum of closed sets.

(2) $L_\tau = L_\sigma = L$.

Proof Trivial.

Proposition 4

$T_1 \Rightarrow T_0$.

Proposition 5

(a) $\otimes_\lambda L_\lambda$ is R_0 $\Leftrightarrow$ L_λ is R_0 for each λ.

(b) $\otimes_\lambda L_\lambda$ is T_1 $\Leftrightarrow$ L_λ is T_1 for each λ.

Proof

(a) ($\Leftarrow$) As in Proposition 2, each element of $\pi_\lambda^{-1}(\tau)$ is a supremum of closed sets, and $\{ \pi_\lambda^{-1}(\tau) : \lambda \in \Lambda \}$ generates the topology, so the closed sets generate the open sets.

($\Rightarrow$) Again, as in Proposition 2, each open set is the supremum of "box" sets of the form $\Pi_\lambda U_\lambda$ where $U_\lambda \in (L_\lambda)_{\tau\sigma}$. Again, taking projections gives the result.

(b) follows from (a) and Proposition 2.

5. R_1 and T_2.

Definition: R_1 , T_2

(L, τ) is R_1 if every open set U can be written in the form:

$$U = \vee_i \wedge_{j \in J_i} U_{ij} = \vee_i \wedge_{j \in J_i} \overline{U}_{ij} \quad \text{where}$$

the U_{ij} are open sets. (L, τ) is T_2 if R_1 and T_0.

In lemma 1 of [5] we proved that for any $A \in L$, there exists $\mathcal{B} \subseteq L$ such that $\sup \mathcal{B} = A$ and for any $\mathcal{C} \subseteq L$ such that $\sup \mathcal{C} = A$, we have $(\forall B \in \mathcal{B})(\exists C \in \mathcal{C}) | (B \leq C)$. We call such a set $\mathcal{B}$ an atomic-like decomposition of A.

Lemma 6

If $\mathcal{B} \subseteq L = \otimes_\lambda L_\lambda$ is an atomic-like decomposition of $\Pi_\lambda A_\lambda$, then $\mathcal{B}_\lambda = \{ \pi_\lambda(B) : B \in \mathcal{B} \}$ is an atomic-like decomposition of $A_\lambda \neq 0$.

Conversely, if $\mathcal{B}_\lambda$ is an atomic-like decomposition of A_λ, then $\Pi_\lambda \mathcal{B}_\lambda = \{ \Pi_\lambda B_\lambda : B_\lambda \in \mathcal{B}_\lambda \}$ is an atomic-like decomposition of $\Pi_\lambda A_\lambda$.

<u>Proof</u>

The first part is true, since if $\mathcal{C}_{\lambda_\circ} \subseteq L_{\lambda_\circ}$ satisfies $\sup \mathcal{C}_{\lambda_\circ} = A_{\lambda_\circ}$, then $\mathcal{C}$ satisfies $\sup \mathcal{C} = \Pi_\lambda A_\lambda$, where $\mathcal{C}$ is the collection of boxes $\Pi_\lambda C_\lambda$ such that $C_{\lambda_\circ} \in \mathcal{C}_{\lambda_\circ}$, and $C_\lambda = A_\lambda$ for $\lambda \neq \lambda_\circ$.

Hence, for each $B \in \mathcal{B}$, there is a $C \in \mathcal{C}$ such that $B \leq C$, which implies $\pi_{\lambda_\circ}(B) \leq C_{\lambda_\circ}$, as desired.

Now to prove the "converse." Take $B_\lambda \in \mathcal{B}_\lambda$.
Then $D_\lambda = \sup \{ D \leq A_\lambda : B_\lambda \nleq D \}$ satisfies $D_\lambda \neq A_\lambda$. Suppose $\mathcal{C} \subseteq L$ satisfies $\sup \mathcal{C} = \Pi A_\lambda$, although $\Pi_\lambda B_\lambda \nleq C$ for all $C \in \mathcal{C}$.

Without loss of generality, $\mathcal{C}$ is a collection of box sets, and so for any $C \in \mathcal{C}$ there exists a λ such that $B_\lambda \nleq C_\lambda$.
Then $C_\lambda \leq D_\lambda$. But the collection D of box sets $\Pi_\lambda C_\lambda \leq \Pi_\lambda A_\lambda$ such that $C_\lambda \leq D_\lambda$ for some λ is a set in $\otimes_\lambda L_\lambda$ (since it is closed under (P 1) and (P 2) – the defining conditions for elements of $\otimes_\lambda L_\lambda$ in [6]). (Note: we hope there is no confusion caused by our identifying $\Pi_\lambda C_\lambda$ with $\{ \Pi_\lambda K_\lambda : K_\lambda \leq C_\lambda$ for each $\lambda \in \Lambda \}$ – ie by identifying a set with its collection of box subsets).

Hence $\sup \mathcal{C} \leq D \neq \Pi A_\lambda$, a contradiction. Thus there is a C such that $\Pi_\lambda B_\lambda \leq C$.

Also, of course, $\sup \Pi_\lambda B_\lambda = \Pi_\lambda A_\lambda$ iff $\sup B_\lambda = A_\lambda$.

This completes the proof.

Note:

If $\{A_m\}$ is an atomic-like decomposition of U, and $\{B_n\}$ is a dual atomic-like decomposition of U (ie: $\{B_n'\}$ is an atomic-like decomposition of U') then the existence of open sets U_{ij} such that $U = \bigvee_i \bigwedge_{j \in J_i} U_{ij} = \bigvee_i \bigwedge_{j \in J_i} \overline{U}_{ij}$ is equivalent to saying that for each A_m, B_n there is an open set V_{mn} such that $A_m \leq V_{mn} \leq \overline{V}_{mn} \leq B_n$. The proof of this is essentially trivial.

Proposition 7

(a) R_1 is equivalent to:

(1) Every fuzzy set U in $L_{\tau\sigma}$ can be written in the form

$$U = \bigvee_i \bigwedge_{j \in J_i} U_{ij} = \bigvee_i \bigwedge_{j \in J_i} \overline{U}_{ij} \quad \text{where the } U_{ij}$$

are open.

(2) The smallest fuzzy set in $(L \otimes L)_{\tau\sigma}$ bigger than the diagonal Δ is closed.

(Δ is the fuzzy set in $L \otimes L$ which corresponds to the sup preserving map $\Delta : L \to L$, with $\Delta(A) = A$ for $A \in L$, under the isomorphism

$\theta : L \otimes L \longleftrightarrow \{$ sup preserving maps $\varphi : L \to L \}$

defined in [6, Prop. 6]. That is:

$$\Delta' = \{ (A, B) : A, B \subset L, \text{ and } A \leq B' \}) .$$

(b) T_2 is equivalent to:

(1) Every fuzzy set can be written in the form given above (in the definition of R_1).

(2) The diagonal Δ is closed.

<u>Proof</u>

The equivalence of R_1 (T_2) with (a) (1) ((b) (1) resp.) is trivial. We shall only bother to prove the equivalence of T_2 with (b) (2), since the equivalence of R_1 with (a) (1) is merely a technical variation.

($\Rightarrow$) We shall show Δ' is open. Consider $A \in L$. Then by T_2, $A = \vee_i \wedge_j U_{ij} = \vee_i \wedge_j \overline{U}_{ij}$ where U_{ij} open. Now $(U_{ij}, \overline{U}_{ij}')$ represents an open set. Moreover:

$(A, A') \leq \vee_{ij} (U_{ij} , \overline{U}_{ij}') \leq \Delta'$. The second inequality is by definition of Δ'. The first follows since:

$$\begin{aligned}
\vee_{ij} (U_{ij}, \overline{U}_{ij}') &\geq \vee_{ij} (\wedge_k U_{ik} , \overline{U}_{ij}') \\
&= \vee_i (\wedge_j U_{ij}, (\wedge_j \overline{U}_{ij})') \\
&\geq \vee_i (\wedge_j U_{ij}, \wedge_l (\wedge_j \overline{U}_{lj})') \\
&= (\vee_i \wedge_j U_{ij}, (\vee_i \wedge_j \overline{U}_{ij})') \\
&= (A, A') .
\end{aligned}$$

Hence $(A, A') \leq \operatorname{Int} \Delta'$. But

$\Delta' = \sup \{ (A, A') : A \in L \}$. Hence Δ' is open.

$(\Leftarrow)$ Suppose Δ' is open, and suppose A is a fuzzy set. Then we can obtain an "atomic-like" decomposition for A in the form $A = \vee_i A_i$, and a dual "atomic-like" decomposition in the form $A = \wedge_j B_j$ (by Lemma 1 in [6]).

By Lemma 6,

$\{(A_i, B_j')\}$ is an "atomic-like" decomposition of (A, A'). Since Δ' is open, it is the supremum of open "boxes," and since $\{(A_i, B_j')\}$ is atomic-like, one of these open boxes must be bigger than (A_i, B_j'). Hence $(A_i, B_j') \leq (U_{ij}, F_{ij}') \leq \Delta'$,

where U_{ij} is open, and F_{ij} is closed. So

$A_i \leq U_{ij} \leq F_{ij} \leq B_j$. Now $A = \vee A_i = \wedge B_j$,

so $A = \vee_i A_i \leq \vee_i \wedge_j U_{ij} \leq \wedge_i \vee_j F_{ij} \leq \wedge B_j = A$

which proves the result.

<u>Note:</u>

We do not yet know whether there is a characterisation of T_2 in terms of "uniqueness of limits" for filters. Indeed we do not yet even have a definition of convergence for a filter.

Proposition 8

$$R_1 \not\Rightarrow R_0 , \quad T_2 \not\Rightarrow T_1 .$$

Proof Trivial.

Proposition 9

$$L = \otimes_\lambda L_\lambda \text{ is } R_1 \iff L_\lambda \text{ is } R_1 \text{ for each } \lambda .$$

$$L = \otimes_\lambda L_\lambda \text{ is } T_2 \iff L_\lambda \text{ is } T_2 \text{ for each } \lambda .$$

Proof

The second statement follows immediately from the first and Proposition 2. Hence we shall only prove the first.

($\Leftarrow$) Suppose each L_λ is R_1. Thus any open set of the form $\pi_\lambda^{-1}(U)$ where U is open in L_λ can be written in the desired form. But these sets generate all open sets. Hence L is R_1.

($\Rightarrow$) Suppose $U = \pi_{\lambda_\circ}^{-1}(U_{\lambda_\circ})$ where $U_{\lambda_\circ} \in \tau_{\lambda_\circ}$.

Let $\mathcal{a}_\lambda = \{ A_{\lambda i} : i \in I \}$ be an atomic-like decomposition of $1 \in L_\lambda$. Then there exist $i, j \in I$ such that $A_{\lambda i} \not\leq A_{\lambda j}'$. This is because if $A_{\lambda i} \leq A_{\lambda j}'$ for every $i, j \in I$ then $\vee_i A_{\lambda i} \leq \wedge A_{\lambda j}'$, ie $1 \leq 0$, which is a contradiction (we may assume that L_λ has more than one element, since otherwise the proposition is trivial). Hence choose $B_\lambda, C_\lambda \in \mathcal{a}_\lambda$ for $\lambda \neq \lambda_\circ$, such that $B_\lambda \nleq C'_\lambda$.

Suppose $B_{\lambda_\circ}$ is an element of an atomic-like decomposition of $U_{\lambda_\circ}$, and $C'_{\lambda_\circ}$ is an element of a dual atomic-like decomposition of $U_{\lambda_\circ}$. Then by Lemma 6 and its adjoining note, we may find an open set V (without loss of generality V is an open box) such that

$$\Pi_\lambda B_\lambda \leq V \leq \bar{V} \leq (\Pi_\lambda C'_\lambda)'. \text{ Then,}$$

using the definition of complement in [6] we have that $\bar{V}_\lambda \leq C'_\lambda$ for some $\lambda \in \Lambda$. But $B_\lambda \leq \bar{V}_\lambda \leq C'_\lambda$ is false for $\lambda \neq \lambda_\circ$ (since $B_\lambda \nleq C'_\lambda$) and hence $\bar{V}_{\lambda_\circ} \leq C'_{\lambda_\circ}$, ie $B_{\lambda_\circ} \leq V_{\lambda_\circ} \leq \bar{V}_{\lambda_\circ} \leq C'_{\lambda_\circ}$.

This is sufficient to prove that $L_{\lambda_\circ}$ is R_1 (since $B_{\lambda_\circ}$, $C_{\lambda_\circ}$ are arbitrary).

<u>Proposition 10</u>

Suppose (L, τ) is T_2, and suppose A is a fuzzy set in L for which every open cover has a finite subcover. (ie. A is "compact" in a rather weak sense). Then A is a closed fuzzy set. (Compare with [6 , section 4].).

<u>Proof</u>

Let $A = \wedge_i \vee_{j \in J_i} U_{ij} = \wedge_i \vee_{j \in J_i} \bar{U}_{ij}$. Then

$\{ U_{ij} : j \in J_i \}$ is an open cover of A. Hence there is a finite subcover $\{ U_{ij_1}, \dots, U_{ij_n} \}$, so that

$A \leq U_{ij_1} \vee \dots \vee U_{ij_n} \leq \bar{U}_{ij_1} \vee \dots \vee \bar{U}_{ij_n}$. The right hand side is a closed set, which we shall call K_i, say.

Thus $A \leq K_i \leq \vee_{j \in J_i} \bar{U}_{ij}$, which implies

$$A \leq \wedge_i K_i \leq \wedge_i \vee_j \bar{U}_{ij} = A. \text{ Hence}$$

$$A = \wedge_i K_i \text{ is closed.}$$

Proposition 11

Suppose (L, τ) is T_2. Suppose $K \leq U$, and K, U' are "compact" in the sense that every open cover has a finite subcover. Then there exists a fuzzy set V such that $K \leq V^\circ \leq \bar{V} \leq U$.

Proof

Write $K = \wedge_i \vee_j U_{ij} = \wedge_i \vee_j \bar{U}_{ij}$. Then for each i,

$K \leq \vee_j U_{ij} \leq \vee_j \bar{U}_{ij}$. Taking a finite subcover, we get

$K \leq V_i \leq \bar{V}_i \leq \vee_j \bar{U}_{ij}$, where $V_i = U_{ij_1} \vee \dots \vee U_{ij_n}$ is the union of the finite subcover. Then $K = \wedge V_i = \wedge \bar{V}_i \leq U$.

Hence $\{ \bar{V}'_i \}$ is a finite subcover of U'. Thus taking a finite $\wedge$ of $\{ \overline{V}_i \}$ gives us V such that $K \leq V^\circ \leq \overline{V} \leq U$.

6. Regular and T_3

Definition: R, T_3

(L, τ) is Regular (R) if every open set U is a supremum of open sets whose closure is less than U.

(L, τ) is T_3 if Regular and T_0.

Proposition 12

Regular $\Rightarrow R_1$

$T_3 \Rightarrow T_2$.

Proposition 13

$L = \otimes_\lambda L_\lambda$ is Regular $\Leftrightarrow$ each L_λ is regular.

$L = \otimes_\lambda L_\lambda$ is T_3 $\Leftrightarrow$ each L_λ is T_3.

Proof

The second statement obviously follows from the first. Hence we only prove the first.

($\Leftarrow$) Suppose each L_λ is regular. Every open set in L is a supremum of open "box" type sets of the form ΠU_λ where U_λ is an open set in L_λ and all but a finite number of the U_λ equal 1. We may write $U_\lambda = \vee_{i_\lambda \in I_\lambda} U_{\lambda,i_\lambda}$ where $\bar{U}_{\lambda,i} \leq U_\lambda$. (where we choose $U_{\lambda,i} = 1$ if $U_\lambda = 1$). Hence let $U_i = \Pi_\lambda U_{\lambda,i_\lambda}$, where $i \in I = \Pi_\lambda I_\lambda$. Then $\vee_i U_i = U$, and $\bar{U}_i \leq \Pi_\lambda \bar{U}_{\lambda,i_\lambda} \leq U$, which proves the result for "box" type sets. Taking the supremum of box type sets gives the result.

($\Rightarrow$) Suppose L is Regular. Then consider $U = \pi_\lambda^{-1}(U_\lambda)$ for a given $U_\lambda \in \tau_\lambda$. Then U is a supremum of open sets whose closure is less than or equal to U. We may assume these open sets are box sets. It is not too hard to show (using the definition of "complement" in [6]) that $\overline{\Pi_\lambda F_\lambda} = \Pi \bar{F}_\lambda$. Hence for a box set U_i, $\overline{\pi_\lambda(U_i)} \leq U_\lambda$ iff $\bar{U}_i \leq U$. So $\{\pi_\lambda(U_i)\}$ is a suitable collection of open sets to demonstrate the regularity of L_λ.

7. Completely Regular and $T_{3\frac{1}{2}}$

Definition: CR, $T_{3\frac{1}{2}}$

(L, τ) is completely regular if for every open set U, there exist families of sets $\{ U_{it} : i \in I, t \in [0,1] \}$ such that $\vee_i U_{i0} = U$ and $t < s \Rightarrow \overline{U}_t \leq U_s^{\circ} \leq U$.

(L, τ) is $T_{3\frac{1}{2}}$ if completely regular and T_0.

The fuzzy unit interval

For technical reasons, in future papers, we shall slightly modify the definition of the fuzzy unit interval $[0,1](L)$, from the definitions given in [4] and [5]. We shall now say the fuzzy unit interval $[0,1](L)$ is the set of all monotonic increasing maps $\lambda : \mathbb{R} \to L$ for which

$$\lambda(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

after equivalence classing $\lambda, \mu : \mathbb{R} \to L$ if $\lambda(t+) = \mu(t+)$, $\lambda(t-) = \mu(t-)$ for $t \in \mathbb{R}$. We define an order on $[0,1](L)$ by $\lambda \leq \mu$ if $\mu(t-) \leq \lambda(t-)$, and $\mu(t+) \leq \lambda(t+)$ for $t \in \mathbb{R}$.

We define $L_t : [0,1](L) \to L$ and $R_t : [0,1](L) \to L$ by

$L_t(\lambda) = \lambda(t-)$, $R_t(\lambda) = \lambda(t+)'$.

L_t and R_t generate the open sets in the lattice of fuzzy sets of $[0,1](L)$ (a fuzzy set is a map $\varphi : [0,1](L) \to L$).

Thus the correspondence between the new definition of $[0,1](L)$ and the old definition is that we have replaced $\lambda : \mathbb{R} \to L$ by $\lambda' : \mathbb{R} \to L$ (where $\lambda'(t) = \lambda(t)'$).

Proposition 14

If (L^X, τ) is a fuzzy topological space (where L is a fuzzy lattice, and X is an ordinary set), then complete regularity is equivalent to:

"For every open set $U \in \tau$, there exist continuous functions $f_i : X \to [0,1](L)$ such that $\sup_i f_i^{-1}(R_0') = \sup_i f_i^{-1}(L_1) = U$ " (ie $\sup_i f_i(x)(0+) = \sup_i f_i(x)(1-)=$ for each $x \in X$). (Note that here we are dealing with ordinary functions as used in [4] and [5], which of course induce morphisms in the sense of [6].).

Proposition 15

Complete regularity is equivalent to the following:

1. (L, τ) is uniformizable.
2. Consider all order preserving maps $f : [0,1] \to L$ such that $f(t-) \in \tau$, and $f(t+) \in \tau'$. Then τ is the weakest of all topologies τ^* such that $f(t-) \in \tau^*$, and $f(t+) \in \tau^*$, for all f as above.

Proof

The equivalence with 1. has been proved in [5]. The equivalence with 2. is essentially trivial.

Proposition 16

Completely Regular $\Rightarrow$ Regular.

$$T_{3\frac{1}{2}} \Rightarrow T_3 .$$

Lemma 17

Suppose U is an open set, and $\Pi_\lambda A_\lambda$ is a box subset of U. Suppose B_λ is atomic-like with respect to A_λ for $\lambda \neq \lambda_\circ$ (ie: for any $\mathcal{C}$ such that $\sup \mathcal{C} = A_\lambda$, $\exists\, C \in \mathcal{C}$ such that $B_\lambda \leq C$). Then the largest box set $V \leq U$ such that $V_\lambda = B_\lambda$ for $\lambda \neq \lambda_\circ$ has $V_{\lambda_\circ}$ open.

Proof

Suppose $B_{\lambda_\circ}$ is atomic like with respect to $V_{\lambda_\circ}$. Then $\Pi_\lambda B_\lambda$ is atomic-like with respect to U, and hence since U is a supremum of open box sets, there exists an open box W such that $\Pi_\lambda B_\lambda \leq W$. Hence $B_{\lambda_\circ} \leq W_{\lambda_\circ} \leq V_{\lambda_\circ}$. But the supremum of all possible $B_{\lambda_\circ}$ is $V_{\lambda_\circ}$. Hence $\sup B_{\lambda_\circ} = \sup W_{\lambda_\circ} = V_{\lambda_\circ}$. That is, $V_{\lambda_\circ}$ is open.

Proposition 18

$L = \otimes_\lambda L_\lambda$ is completely regular $\Leftrightarrow$ L_λ is completely regular for each λ. (Similarly for $T_{3\frac{1}{2}}$)

Proof

($\Leftarrow$) Similar to the previous proofs on products.

($\Rightarrow$) Suppose we have a family $\{ U_t : t \in [0,1] \}$ of open sets in L such that $t < s$ implies $\overline{U}_t \leq U_s$. Suppose also that $0 \neq \Pi_\lambda A_\lambda \leq U_0$. Then choose $B_\lambda \neq 0$, for $\lambda \neq \lambda_\circ$, atomic-like with respect to A_λ. Let V_t be the largest box set such that

$V_t \leq U_t$ and $V_{t\lambda} = B_\lambda$ for $\lambda \neq \lambda_\circ$. Then by Lemma 17, $W_t = V_{t\lambda_\circ}$ is open. Moreover $\overline{V}_t \leq \overline{U}_t$, since $V_t \leq U_t$, and hence $\overline{V}_{t\lambda_\circ} \leq V_{s\lambda_\circ}$ for $t < s$ (since V_t is a box set, and $\overline{U}_t \leq U_s$). Now since $W_t = V_{t\lambda_\circ}$, we have $\overline{W}_t \leq W_s$ for $t < s$.

This completes most of the technicalities of the proof. Suppose L is completely regular, and suppose

$U = \pi_{\lambda_\circ}^{-1}(U_{\lambda_\circ})$, where $U_{\lambda_\circ} \in \tau_{\lambda_\circ}$. Then there exist box sets $A_i = \Pi_\lambda A_{i\lambda}$ such that $\sup A_i = U$, and families of open sets $\{ U_{it} : t \in [0,1] \}$ such that $A_i \leq U_{it} \leq U$, and $t < s \Rightarrow \overline{U}_{it} \leq U_{is}$.

Applying the previous paragraph gives us families of open sets $\{ W_{it} : t \in [0,1] \}$ such that $A_{i\lambda_\circ} \leq W_{it} \leq U_{\lambda_\circ}$, and $t < s \Rightarrow \overline{W}_{it} \leq W_{is}$. Since $U_{\lambda_\circ}$ is arbitrary and $\sup A_{i\lambda_\circ} = U_{\lambda_\circ}$, we have proved complete regularity for $L_{\lambda_\circ}$.

8. Normality and T_4

Definition: N, T_4 (Ref [4])

(L, τ) is normal if for any closed set K and open set U such that $K \leq U$, there exists a set V such that $K \leq V^\circ \leq \overline{V} \leq U$. (L, τ) is T_4 if normal $+ T_0 + R_0$.

In [4] we proved:

Proposition 19

If (L^X, τ) is a fuzzy topological space where L is a fuzzy lattice, and X is an ordinary set. Then Normality is equivalent to:

"For every closed set K and open set U in L^X such that $K \leq U$, there exists a continuous function $f\colon X \to [0,1](L)$ such that $K \leq f^{-1}(R_0{}') \leq f^{-1}(L_1) \leq U$.

(ie: $K(x) \leq f(x)(0+) \leq f(x)(1-) \leq U(x)$ for any $x \in X$).

The "pointless" version of this is:

Proposition 20

(L, τ) is normal iff for every closed set K, and open set U in L such that $K \leq U$, there exist $\{ V_t : t \in [0,1] \}$ such that $V_\circ = K$, $V_1 = U$ and $(t < s \Rightarrow \overline{V}_t \leq V_s{}^\circ)$.

Proposition 21

Normal + R_0 $\Rightarrow$ Completely Regular.

$$T_4 \Rightarrow T_{3\frac{1}{2}} .$$

Proof By Proposition 20.

Proposition 22

Normality is not productive.

Proof By the usual topological counterexample.

Proposition 23

Suppose L is compact in the sense that every open cover of a closed set has a finite subcover. Then if L is R_1, then it is Normal.

Proof By Proposition 11.

9. Complete Normality and T_5.

Definition: CN, T_5

(L, τ) is completely normal if for any fuzzy sets K, U such that $\overline{K} \leq U$, and $K \leq U^\circ$ there exists a set V such that $K \leq V^\circ \leq \overline{V} \leq U$. (L, τ) is T_5 if completely normal $+ T_0 + R_0$.

Proposition 24

Completely Normal $\Rightarrow$ Normal.

10. Perfectly Normal and $T_{5\frac{1}{2}}$

Definition: PN, $T_{5\frac{1}{2}}$

(L, τ) is PN if Normal and every closed set is a countable $\wedge$ of open sets.

(L, τ) is $T_{5\frac{1}{2}}$ if also T_1 .

Proposition 25

Perfect Normality is equivalent to:

"For every closed set K, and open set U such that $K \leq U$, there exist $\{ V_t : t \in [0,1] \}$ such that $(t < s \Rightarrow \overline{V}_t \leq V_s^\circ)$ and $K = V_{0+} \leq V_{1-} = U$."

Or in the pointed case of (L^X, τ), that there exists a continuous function $f\colon X \to [0,1](L)$ such that

$$K = f^{-1}(R_0') \leq f^{-1}(L_1) = U.$$

Proposition 26

Perfectly Normal $\Rightarrow$ Completely Normal.

Proof

Let (L, τ) be PN, and suppose $K \leq U^\circ$ and $\overline{K} \leq U$.

Let $\overline{K} = \bigwedge_{n=1}^{\infty} G_n$ and $U^\circ = \bigvee_{m=1}^{\infty} F_m$ where each G_n is open and each F_m is closed. By normality, we obtain for each m a set A_m such that $F_m \leq A_m^\circ \leq \overline{A}_m \leq U^\circ$, and for each n a set B_n such that $K \leq B_n^\circ \leq \overline{B}_n \leq G_n$.

Let $V_n = A_n^\circ \wedge \left(\bigwedge_{j=1}^{n} B_j^\circ \right)$ and $W_m = \overline{B}_m \vee \left(\bigvee_{j=1}^{m} \overline{A}_j \right)$

Then $V_n \leq W_m$ for all m and n, V_n is open and W_m is closed. Now let $V = \bigvee_{n=1}^{\infty} V_n$ and $W = \bigwedge_{m=1}^{\infty} W_m$.

Then $V \leq W$, V is open and W is closed. Moreover,

$$K \leq U^\circ \wedge \overline{K} = \left(\bigvee_{m=1}^{\infty} F_m \right) \wedge \overline{K} = \bigvee_{m=1}^{\infty} (F_m \wedge \overline{K}) \leq \bigvee_{m=1}^{\infty} V_m = V$$

and

$$W = \bigwedge_{n=1}^{\infty} W_n \leq \bigwedge_{n=1}^{\infty} (G_n \vee U^\circ) = \left(\bigwedge_{n=1}^{\infty} G_n \right) \vee U^\circ = K \vee U^\circ \leq U,$$

as desired.

11. Pseudometrizability and Metrizability

Definition ψM, T_6

(L, τ) has a pseudometric on it if it has a uniformity with base $\{ D_r :$ r is a positive real number $\}$ of symmetric elements satisfying $D_r \circ D_s \leq D_{r+s}$, and if this uniformity generates the topology.

(L, τ) is metrizable (T_6) if also T_0. (We hope to develop these concepts further in a later paper).

Proposition 27

Pseudometrizable $\Rightarrow$ Perfectly Normal.

Metric $\Rightarrow$ $T_{5\frac{1}{2}}$.

Proof

We first prove normality. Suppose $K \leq U$, K is closed and U is open. By appealing to Lemma 1 of [5] we obtain an atomic-like partition $\{K_i\}$ of K, so that $K = \vee K_i$. The complementary version of [5, Lemma 1] yields $U = \wedge U_j$. Since U is open, for each i there is a D_{r_i} such that $D_{r_i}(K_i) \leq U$. Similarly, for each j there is a D_{s_j} such that $D_{s_j}(U_j') \leq K'$, and so since D_{s_j} is symmetric $D_{s_j}(K) \leq U_j$. Thus we have for each i and each j, $D_{r_i}(K_i) \leq U_j$ and $D_{s_j}(K_i) \leq U_j$. In particular, $D_{\frac{1}{2}(r_i+s_j)}(K_i) \leq U_j$ so that $D_{\frac{1}{2}s_j}(D_{\frac{1}{2}r_i}(K_i)) \leq U_j$ for each i. Hence $D_{\frac{1}{2}s_j}(\bigvee_i D_{\frac{1}{2}r_i}(K_i)) \leq U_j$, and therefore $\bigwedge_j D_{\frac{1}{2}s_j}(\bigvee_i D_{\frac{1}{2}r_i}(K_i)) \leq U$. Now $W = \bigvee_i D_{\frac{1}{2}r_i}(K_i)$ is a neighbourhood of K, since $D_{\frac{1}{2}r_i}(K_i)$ is a neighbourhood of K_i.

Moreover, $D_{\frac{1}{2}s_j}(W) \geq \overline{D_{\frac{1}{4}s_j}(W)}$, so that there is a set V such that $K \leq V^\circ \leq \overline{V} \leq U$.

We now show that any closed set K is a G_δ. Again by the complementary version of Lemma 1 of [5], $K = \wedge W_i$. For each i there is a D_r such that $D_r(W_i') \leq K'$, since K' is open. Hence $D_r(K) \leqslant W_i$. Now $s < r$ implies $D_s(K) \leqslant D_r(K)^\circ$, so that $\bigwedge_{r>0} D_r(K)^\circ = K$. In particular, $K = \bigwedge_{n=1}^{\infty} D_{\frac{1}{n}}(K)^\circ$.

Proposition 28

$$L = \bigotimes_{n=1}^{\infty} L_n \quad \text{is pseudometrizable if } L_n \text{ is for each } n.$$

Proof

Given a uniformity $\mathfrak{D}_n$ on each L_n with countable base, we may construct the product uniformity on L, also with a countable base as follows:

Given $D \in \mathfrak{D}_n$ define D^* on L by $D^* = \pi_n^{-1} \circ D \circ \pi_n$. Thus the "pullback" of $\mathfrak{D}_n$ will generate the topology on L produced by "pulling back" τ_n.

Let $\mathfrak{D}$ be the uniformity generated by the "pullback" of $\mathfrak{D}_n$ for all $\mathfrak{D}_n$. The countable bases for each $\mathfrak{D}_n$ will generate a countable sub-base for $\mathfrak{D}$, which in turn will produce a countable base as desired.

12. Summary

(1) $T_i \Rightarrow T_j$ for $i > j$.

(2) $\psi M \Rightarrow PN \Rightarrow CN \Rightarrow N$

$N + R_0 \Rightarrow CR \Rightarrow R \Rightarrow R_2 \Rightarrow R_1$.

(3) (a) CR, R, R_1, and R_0 are productive.

(b) $T_{3\frac{1}{2}}$, T_3, T_2, T_1 and T_0 are productive.

(c) ψM is countably productive.

(d) PN, CN, N are not productive.

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6. UNIFORMITIES ON FUZZY TOPOLOGICAL SPACES, PART II

Uniformities on Fuzzy Topological Spaces, Part II.

1. Introduction

In this paper we tie up some of the loose ends left from our first paper [2]. To achieve this we have had to develop the concepts of products and compactness [3]. Our main achievement in this paper is showing that a compact R_1 fuzzy space has a unique fuzzy uniformity. We also give characterizations of uniformities and pseudometrics. In a future paper we hope to extend the theory of fuzzy uniformities to relate them to compactfications and completions.

Since the writing of our last paper on fuzzy uniformities [2], we have become aware of another paper on fuzzy uniformities by R. Lowen [5]. We note that the definitions are different, and produce different uniformizable spaces. Indeed there is no real relationship with our papers. Moreover, R. Lowen's paper does not fit within the framework of our sequence of papers: that definitions should depend only upon the lattice structure of the lattice of fuzzy sets.

2. Preliminaries

Unfortunately this paper is so dependent upon our previous papers [1], [2], [3], [4], that we cannot give all the notation relevant. The main concepts are as follows:

A fuzzy lattice is a lattice L which is complete, completely distributive and has an order reversing involution $'$.

A topology τ on L is a subset of L closed under arbitrary suprema, and finite infima.

A uniformity on L is a subset $\mathfrak{D}$ of $\mathcal{Q}$, the collection of all sup preserving maps $\varphi\colon L \to L$ such that $a \leq \varphi(a)$, which satisfies:

(Q 1) $\mathfrak{D} \neq \phi$

(Q 2) $D \in \mathfrak{D},\ D \leq E \in \mathfrak{T} \Rightarrow E \in \mathfrak{D}$.

(Q 3) $D, E \in \mathfrak{D} \Rightarrow D \wedge E \in \mathfrak{D}$.

(Q 4) $D \in \mathfrak{D} \Rightarrow D^r \in \mathfrak{D}$.

For more details, see [2], [3], and [4], while noting that some of the definitions and notation given in [2] are later modified in [3] and [4] , in particular the definition of the fuzzy unit interval and naming of the reflection map r.

3. A characterization of Uniformities

In our paper on products of fuzzy topological spaces [3], we obtained a lattice isomorphism between $L \otimes L$ and $\mathfrak{S}$, the sup preserving maps $\varphi\colon L \to L$ (Prop. 6). The correspondence is $\theta\colon A \leftrightarrow \varphi_A$, where $\varphi_A(a) \leq b' \Leftrightarrow (a,b) \in A'$. Thus any structure defined on one of the lattices can be transferred to the other lattice.

Proposition 1

The isomorphism between $L \otimes L$ and $\mathfrak{S}$ induces the following correspondences:

(1) Reflection: (a generalization of "inverse" defined in our first paper on uniformities [2]).

$L \otimes L$: $A^r = \{ (b,a) : (a,b) \in A \}$.

$\mathfrak{S}$: $\varphi^r(b) = \wedge \{a : \varphi(a') \leq b' \}$.

The reflection maps preserves $\vee$, $\wedge$, and $'$.

(2) Complement:

$L \otimes L$: $A' = \{ (x,y) : (\forall (a,b) \in A)(x \leq a' \text{ or } y \leq b') \}$.

$\mathfrak{S}$: $\varphi'(x) = \vee \{ \varphi(a)' : a \nleq x' \}$.

The complement reverses $\vee$ and $\wedge$.

(3) Composition: (From composition of functions in $\mathfrak{S}$).

$L \otimes L$: $A \circ B$ is generated by:

$\{ (a,d) : (\exists b,c)(b \nleq c')((a,b) \in B, (c,d) \in A) \}$.

$\mathfrak{S}$: $(\varphi \circ \psi)(a) = \varphi(\psi(a))$.

Composition is associative, and is distributive over $\vee$.
Also $(A \circ B)^r = B^r \circ A^r$.

(4) Identity Δ:

$L \otimes L$: $\Delta = \{ (x,y) : (\forall a \in L)(x \leq a' \text{ or } y \leq a) \}$.

$\mathfrak{S}$: $\Delta(a) = a$.

Δ is an identity for composition.

Proof

(1). was effectively proved in [2] and has been restated in [3].

(2). That complement reverses $\vee$ and $\wedge$ follows from its properties in $L \otimes L$.

We now show the correspondence is as above. Suppose

$\varphi \in \mathfrak{S}$ corresponds to $A \in L \otimes L$.

$$\begin{aligned} \varphi'(x) \leq y' &\Leftrightarrow (x,y) \in A. \\ &\Leftrightarrow (\forall (a,b) \in A')(a \leq x' \text{ or } b \leq y'). \\ &\Leftrightarrow (a \nleq x' \Rightarrow y \leq \varphi(a)). \\ &\Leftrightarrow y \leq \wedge \{ \varphi(a) : a \nleq x' \}. \end{aligned}$$

Hence $\varphi'(x) = \vee \{ \varphi(a)' : a \nleq x' \}$.

(3). That composition is associative, and is distributive over $\vee$ follows from its definition in $\mathfrak{S}$.

That $(A \circ B)^r = B^r \circ A^r$ is simple to prove, and was initially stated in [2].

Since composition is distributive over $\vee$, we only need to show the formula is true for box sets. Suppose φ corresponds to (a,b), ψ to (c,d) respectively. Then

$$\varphi(x) = \begin{cases} 0 & x \leq a' \\ b & x \nleq a' \end{cases}$$

$$\psi(x) = \begin{cases} 0 & x \leq c' \\ d & x \nleq c' \end{cases}$$

So $$(\varphi \circ \psi)(x) = \begin{cases} 0 & x \leq a' \text{ or } b \leq c' \\ d & \text{else} \end{cases}$$

ie: $\varphi \circ \psi$ corresponds to $\begin{cases} (a,d) & \text{if } b \nleq c' \\ 0 & \text{if } b \leq c' \end{cases}$

This proves the result.

(4). follows by taking the complement of

$$\Delta' = \{ (a,b) : a \leq b' \}.$$

The previous proposition gives us:

Theorem 2

The isomorphism $\theta : L \otimes L \to \mathfrak{S}$, induces the following equivalent definitions of a uniformity:

"A uniformity $\mathfrak{D}$ on a fuzzy lattice L, is a subset $\mathfrak{D}$ of $L \otimes L$ which satisfies:

(PU 1) $\mathfrak{D} \neq \phi$.

(PU 2) $D \in \mathfrak{D} \Rightarrow \Delta \leq D$.

(PU 3) $D \in \mathfrak{D}$, $D \leq E \Rightarrow E \in \mathfrak{D}$.

(PU 4) $D \in \mathfrak{D}$, $E \in \mathfrak{D} \Rightarrow D \wedge E \in \mathfrak{D}$.

(PU 5) $D \in \mathfrak{D} \Rightarrow$ there exists $E \in \mathfrak{D}$ such that $E \circ E \leq D$.

(PU 6) $D \in \mathfrak{D} \Rightarrow D^r \in \mathfrak{D}$. "

The importance of this theorem is that it makes the definition look more conventional (though less intuitive), and it allows us to place a topology on $\mathfrak{D}$ (since we can place a topology on $L \otimes L$).

4. A pointed characterization of pseudometrics.

In the following L shall be a fuzzy lattice, and X an ordinary set. We shall consider L^X as a "pointed" fuzzy lattice.

We have already defined a pseudometric on a lattice as a family $\{ D_s : s > 0 \}$ of symmetric elements in $\mathcal{Q}$ satisfying $D_r \circ D_s \leq D_{r+s}$. Also, we shall add on the condition that $D_r \uparrow 1$ as $r \to \infty$. We did not include this in previous papers, but it is easily rectified by taking a $\{ D_s \}$ and replacing D_s by 1 for $s \geq 1$, say. This is just so that nothing is an "infinite" distance away (which is the case for ordinary (topological) metrics). The topology generated by the pseudometric is the topology with open sets of the form $U = \vee \{ V : D_s(V) \leq U$ for some $s > 0 \}$. We shall identify two psuedometrics $\{ D^1_s \}, \{ D^2_s \}$ if $D^1_{s-} = D^2_{s-}$ and $D^1_{s+} = D^2_{s+}$ for any $s > 0$ (they obviously give the same topology).

Definition (Ref [4, page 16].)

Suppose L is a fuzzy lattice.

We define the fuzzy real line $\mathbb{R}(L)$ as the set of all monotonic increasing maps $\lambda : \mathbb{R} \to L$ for which $\lambda(t) \downarrow 0$ as $t \downarrow -\infty$, and $\lambda(t) \uparrow 1$ as $t \uparrow \infty$. As in the fuzzy unit interval, we identify $\lambda, \mu : \mathbb{R} \to L$ if $\lambda(t+) = \mu(t+)$ and $\lambda(t-) = \mu(t-)$ for any $t \in \mathbb{R}$. We define an order by $\lambda \leq \mu$ if $\mu(t-) \leq \lambda(t-)$, $\mu(t+) \leq \lambda(t+)$, for any $t \in \mathbb{R}$. As before, we may define a topology on the L-fuzzy sets of $\mathbb{R}(L)$ (ie. the maps $\lambda: \mathbb{R}(L) \to L$), by taking $\{ L_t, R_t : t \in \mathbb{R} \}$ as a sub-base, where $L_t(\lambda) = \lambda(t-)$, $R_t(\lambda) = \lambda(t+)'$. We may map $\mathbb{R}$ into $\mathbb{R}(L)$ by defining $\alpha(t) = \begin{cases} 1 & t > \alpha \\ 0 & t < \alpha \end{cases}$

We can then define $[0,1](L) = \{ \lambda \in \mathbb{R}(L) : 0 \le \lambda \le 1 \}$ and $\mathbb{R}^+(L) = \{ \lambda \in \mathbb{R}(L) : \lambda \ge 0 \}$. This gives us the same definition of $[0,1](L)$ as before in [4].

Now consider $\mathbb{R}(L \otimes L)$. We may define reflection $r : \mathbb{R}(L \otimes L) \to \mathbb{R}(L \otimes L)$ by $\lambda^r(t) = \lambda(t)^r$.
We may define composition on $\mathbb{R}(L \otimes L)$ by $(\lambda \circ \mu)(t) = \bigvee_{r+s=t} \lambda(r) \circ \mu(s)$.
Reflection preserves $\vee$ and $\wedge$ (and $\alpha^r = \alpha$ if α is a real number).
Composition is associative and preserves $\vee$ (and $\alpha \circ \beta = \alpha + \beta$ if α and β are real numbers). We may also define $\Delta \in \mathbb{R}(L \otimes L)$ by $\Delta(s) = \begin{cases} 0 & s \le 0 \\ \Delta & s > 0 \end{cases}$.

<u>Theorem 3</u>

Suppose L^X is a pointed fuzzy lattice. Then a pseudometric corresponds to a map

$$d: X \times X \to \mathbb{R}^+(L \otimes L) \quad \text{which satisfies:}$$

(M1) $d(x,x) \ge \Delta$.

(M2) $d(y,x) = d(x,y)^r$.

(M3) $d(x,z) \le d(x,y) \circ d(y,z)$.

Proof

Suppose we are given $\{D_\varepsilon\}$. Identify $L \otimes L$ with $\mathfrak{S}$,

and define $d(x,y) \in \mathbb{R}^+(L \otimes L)$ by

$$d(x,y)(s)(a) = \begin{cases} 0 & s \le 0 \\ D_s(x_a)(y) & s > 0 \end{cases}$$

where x_a is the fuzzy set $x_a(z) = \begin{cases} a & z=x \\ 0 & \text{else} \end{cases}$.

Conversely, given such a map $d: X \times X \to \mathbb{R}^+(L \otimes L)$,

we may define $D_s(U)(y) = \sup_{x \in X} \{ d(x,y)(s)(U(x)) \}$.

Obviously starting with one definition, constructing its corresponding structure, and then constructing the structure corresponding to it, takes us back to the original one. The fact that the new object will satisfy the desired conditions is also obvious.

Note

It is this definition that we alluded to in [2, page 13].

Lemma 4

Suppose L is a fuzzy topological space with a uniformity $\mathfrak{D}$. Then we may define a product uniformity $\tilde{\mathfrak{D}}$ on $L \otimes L$ by making $\{D_i = \pi_i^{-1} \circ D \circ \pi_i : D \in \mathfrak{D},\ i=1,2\}$ a sub-base. This uniformity generates the product topology, and is the smallest uniformity making the projections uniformly continuous. Moreover, the elements $\{\tilde{D} = D_1 \wedge D_2 : D \in \mathfrak{D}\}$ form a base for $\mathfrak{D}$, and satisfy $\tilde{D}(\varphi) = D \circ \varphi \circ D^{-1}$ for $\varphi \in \mathcal{E}(L) \cong L \otimes L$.

Proof

All except the last part are trivial, and have implicitly been used in [4] (by a(quasi) uniformly continuous fuzzy morphism $f: (L, \mathfrak{D}) \to (M, \mathcal{E})$ we mean a morphism such that $f^{-1}(E) = f^{-1} \circ E \circ f \in \mathfrak{D}$ for every $E \in \mathcal{E}$. A more general "pointless" version of Prop. 8 in [2] now says that any (quasi) uniformly continuous morphism is continuous. The proof is the same.). That $\tilde{D}(a,b) = (D(a),D(b))$, is proved by using [2, Lemma 2] and [4, Lemma 6] .

Considering $L \otimes L$ as $\mathcal{E}(L)$, a "box" set (a,b) corresponds to

$$\varphi(c) = \begin{cases} 0 & c \le a' \\ b & \text{else} . \end{cases}$$

Hence

$$D \circ \varphi \circ D^{-1}(c) = \begin{cases} 0 & D^{-1}(c) \le a' \\ D(b) & \text{else} . \end{cases}$$

But $D^{-1}(c) \le a'$ iff $D(a) \le c'$ iff $c \le D(a)'$.

Hence $D \circ \varphi \circ D^{-1} = (D(a), D(b)) = \tilde{D}(\varphi)$ for φ a box set. Distributivity of composition over $\vee$ proves the statement for general φ.

Theorem 5

Suppose L^X has a pseudometric $d: X \times X \to \mathbb{R}^+(L \otimes L)$. Then d is uniformly continuous (and hence continuous).

Proof

The uniformity on $\mathbb{R}^+(L \otimes L)$ is generated by $\{ B_\varepsilon, B^r_\varepsilon : \varepsilon > 0\}$ where $B_\varepsilon(U) = R_{t+\varepsilon}$ where t is the greatest

s such that $U \leq L_s'$

$$= \wedge \{ R_{s-\varepsilon} : U \leq L_s' \} .$$

$$. \quad B_\varepsilon^r(U) = \wedge \{ L_{s+\varepsilon} : U \leq R_s' \} . \quad \text{(see [2, page 19])}.$$

$$\text{Now} \quad d^{-1}(L_t)(x,y)(a) = d(x,y)(t+)(a)$$

$$= D_{t+}(x_a)(y) .$$

That is, $d^{-1}(L_t) = D_{t-}$. Similarly $d^{-1} R_t' = D_{t+}$.

Hence $d^{-1}(B_\varepsilon^r)(U) = D_{(t+\varepsilon)-}$, where t is the smallest

s such that $U \leq D_{s+}$. Hence $\tilde{D}_{\frac{\varepsilon}{3}}(U) \leq D_{\frac{\varepsilon}{3}} \circ D_{t+} \circ D_{\frac{\varepsilon}{3}}$

$$\leq D_{(t + \frac{2}{3}\varepsilon)+}$$

$$\leq D_{(t + \varepsilon)-}$$

$$= d^{-1}(B_\varepsilon)(U) .$$

ie: $\tilde{D}_{\frac{\varepsilon}{3}} \leq d^{-1}(B_\varepsilon)$.

Hence also $\tilde{D}_{\frac{\varepsilon}{3}} \leq d^{-1}(B_\varepsilon)$ (since $\tilde{D}_{\frac{\varepsilon}{3}}$ is symmetric).

This proves that d is uniformly continuous.

5. Uniformities and Compactness

As in [3], we shall say L is compact if every open cover of a closed set has a finite subcover. We showed that this was equivalent to "If $\mathcal{F}$ is a collection of closed sets, closed under finite infima, and if $\wedge \mathscr{F} \leq U$, where U is open, then there exists $F \in \mathscr{F}$ such that $F \leq U$."

Theorem 6

Suppose L is a compact R_1 space (and hence by [4, Prop. 23] completely regular). Then there exists a unique uniformity on L which generates the topology. The uniformity is in fact all neighbourhoods of the diagonal Δ (in the product topology).

Proof

Existence follows by the fact that L is completely regular [2, Theorem 17].

Suppose $\mathfrak{D}$ is a uniformity on L and $D \in \mathfrak{D}$. Then there exists $E \in \mathfrak{D}$, symmetric, such that $E \circ E \leq D$. Now $\tilde{E}$ is an element of the product uniformity and so $\tilde{E}(\Delta) = E \circ E \leq D$ is a neighbourhood of Δ (ref Lemma 4). Also if $E, D \in \mathfrak{D}$ are symmetric, and $E \circ E \leq D$, then $\bar{E} \leq D$ in the product topology. This is because we may find $F \in \mathfrak{D}$, symmetric, such that $F \circ F \leq E$, which implies

$$\begin{aligned} \tilde{F}(E) &= F \circ E \circ F \\ &= F \circ F \circ E \\ &\leq E \circ E \leq D. \end{aligned}$$

By [2, Theorem 15] we have $\bar{E} \leq D$. Thus the closed symmetric neighbourhoods of Δ in $\mathfrak{D}$ form a base for $\mathfrak{D}$.

Now since L is R_1, the smallest set generated by $\tau \cup \tau'$ and containing Δ is closed. Hence $\bar{\Delta} \leq U$ if U is an open neighbourhood of Δ. Thus if $\mathscr{F} = \{ \bar{D} : D \in \mathfrak{D} \}$, then $\mathscr{F}$ is a closed-filter and $\wedge \mathscr{F} = \bar{\Delta} \leq U$. Hence since L is compact, there exists $D \in \mathfrak{D}$ such that $D \leq U$. Hence $\mathfrak{D}$ consists precisely of the neighbourhoods of Δ.

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7. FUTURE DEVELOPMENTS

What are the future developments of fuzzy sets and fuzzy topology? As far as applications go, I cannot say, for I am not familiar with the vast quantities of work that have been done on fuzzy sets. All I can say is that a lot of mathematicians, engineers and people in operations research seem to be very interested and believe that there are many applications in the study of ill defined concepts. As far as I know, my papers are the only papers that have gone this deeply into the general topology related to fuzzy sets - other papers being essentially on a more introductory level. My work definitely seems to be alone in taking my particular "pointless" stance. Hence the question of future developments is very much a matter of what I intend to develop in the future as regards fuzzy topological spaces, since nobody else seems to be working in this area.

The main area I intend to research in is in the formation of Hausdorf compactifications. To do this, I shall need to do a much more careful study of filters and filter pairs. Also I shall need to develop the concepts of convergence of filters, and their relationship with Hausdorfness, so that I can justify the demand that the compactification be Hausdorf and that the initial space by $T_{3\frac{1}{2}}$. The elements of the compactification shall presumably be collections of zero-filters or filter pairs (a zero set being a closed set K_{0+} for which there exist K_t, $t \in [0,1]$, such that $t < s$ implies $\overline{K}_t \leq K_s^{\circ}$).

I then wish to define total boundedness and completions, and relate all of the theory about compactifications, completions and total boundedness to uniformities and proximities. I would also like to discuss questions relating to local properties and paracompactness, which I believe I should be able to define satisfactorily without too much difficulty.

Finally, I would like to drop the complement from the lattice structure, and consider "bitopological" concepts. I believe that I can prove a theorem which says that any reasonable phrased statement in terms of the universal and existential quantifiers, order, suprema and infima, open and closed sets, which is true for all completely distributive lattices with a complement, can be rephrased to give a corresponding true statement for all completely distributive lattices without a complement (subject to replacing an assumption by both the assumption and the dual assumption (formed by interchanging open and closed, reversing order, etc.)). I would also like to see how my definition of compactness relates to (non-fuzzy) bitopological spaces.

APPENDIX

This appendix is meant to clarify any confusion which may be caused by the variations in definitions which occur throughout the five papers in this thesis.

(1) Normality

(2) Uniformities I

(3) Products

(4) Separation Axioms (Version 2)

(5) Uniformities II

(So paper k corresponds to chapter k + 1 for k = 1,2,3,4,5.)

1. In (1) and (2) I use $\cap, \cup, \subseteq$, instead of $\wedge$, $\vee$, $\leq$ in the lattice L^X of fuzzy sets over a set L. This is to imitate the relationship between fuzzy sets and ordinary sets. In (3), (4) and (5) I use the lattice symbols $\wedge, \vee, \leq$.

2. Papers (1) and (2) are written using "pointed" notation, and (3), (4), (5) are written in "pointless" notation. That is, in (1) and (2) the lattice I deal with is assumed to be decomposed into the form $L = M^X$, where M is a fuzzy lattice, and X is a set. In (3), (4) and (5) I make no such assumptions. In (1) and (2) a function (or morphism) is an ordinary set function $f : X \to Y$, which generates a function $f^{-1} : M^Y \to M^X$ which preserves $\wedge, \vee$ and $'$. In (3), (4) and (5) a morphism from L_1 to L_2 is a function from $L_2 \to L_1$ which preserves $\wedge, \vee$, and $'$. Thus a 'pointless' morphism is a generalisation of a 'pointed' function.

Paper (2) could in fact (and should in fact) have been written from the 'pointless' stance. This would stop the redefinition of (quasi)-uniform continuity in (5).

continued

Thus the preliminaries to (3) should be taken as the starting point for all basic "pointless" notation. Any other pointless notation is essentially defined after this (with the exception of uniformities).

Note: In the pointless approach, one can define a relation between L_1 and L_2 as an element of $L_1 \otimes L_2$ or alternatively as a sup preserving map $\phi : L_2 \to L_1$. Thus a morphism is just a special type of relation (one which preserves complement as well). Similarly one can define injective, surjective morphisms, and reflexive, symmetric, transitive relations. An equivalence relation is thus a sup preserving map $\phi: L \to L$ which satisfies (1) $\phi \geq \Delta$, (2) $\phi^2 = \phi$, (3) if $a \in \mathrm{Im}\,\phi$ then $\phi(a') = \phi(a)'$. An equivalence relation corresponds to a "partition" M of L ($M \subseteq L$ is closed under $\vee, \wedge$ and $'$) by making $M = \mathrm{Im}\,\phi$. An element of a quasi- uniformity is of course a reflexive relation on L.

The fuzzy unit interval is first defined in (1), trivially modified in (2), not used in (3) and significantly modified in (4), mainly for its use in (5). The modification in (4) from that of (2) is by replacing $\lambda : \mathbb{R} \to L$ (monotonic decreasing in (2)) by $\lambda' : \mathbb{R} \to L$ (monotonic increasing in (4) where $\lambda'(t) = \lambda(t)'$.

Metrics and pseudometrics are initially defined in (2) and then slightly modified in (5) (not changing the pseudo-metrizability of a space at all). This is just because the initial definition allowed infinite distances, which is unconventional.