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AN EXTENSION OF A THEOREM OF HARTSHORNE

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Abstract. We extend a classical theorem of Hartshorne concerning the connectedness of the punctured spectrum of a local ring by analyzing the homology groups of a simplicial complex associated with the minimal primes of a local ring.

1. introduction

R. Hartshorne proved in [Har62, Proposition 2.1] that, if $(R, \mathfrak{m})$ is a noetherian local ring whose depth is at least 2, then the punctured spectrum (*i.e.* $\mathrm{Spec}(R)\backslash\{\mathfrak{m}\}$) is connected. In this note we partially extend this result to the case depth ≥ 3. We express our extension in terms of a simplicial complex (which already appeared in [Lyu07, Theorem 1.1]) defined as follows.

Definition 1.1. Let R be a noetherian commutative ring and let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the minimal primes of R. Assume that R is either local with maximal ideal $\mathfrak{m}$ or that R is graded with $\mathfrak{m}$ the ideal of elements of positive degrees. A simplicial complex $\Delta(R)$ (or Δ whenever R are clear from the context) is defined as follows: $\Delta(R)$ is the simplicial complex on the vertices $1, \ldots, n$ such that a simplex $\{i_0, \ldots, i_s\}$ is included in $\Delta(R)$ if and only if $\sqrt{\mathfrak{p}_{i_0} + \cdots + \mathfrak{p}_{i_s}} \neq \mathfrak{m}$.

Note that the punctured spectrum of R (*i.e.* $\mathrm{Spec}(R)\backslash\{\mathfrak{m}\}$) is connected (as a topological space under the Zariski topology) if and only if $\widetilde{H}_0(\Delta(R), G) = 0$ for some (equivalently every) non-zero abelian group, where $\widetilde{H}_0(\Delta(R), G)$ is the 0-th reduced singular homology of the simplicial complex $\Delta(R)$ with coefficients in G. Thus Hartshorne's theorem says that if $\mathrm{depth}R \geq 2$, then $\widetilde{H}_0(\Delta(R), G) = 0$. Our main result is the following extension of Hartshorne's result.

Theorem 1.2. *Let $(R, \mathfrak{m}, k)$ be a noetherian commutative complete local ring of characteristic p. Assume that the residue field k is separably closed and that* $\mathrm{depth}(R) \geq 3$. *Then*
$$\widetilde{H}_0(\Delta(R); k) = \widetilde{H}_1(\Delta(R); k) = 0.$$

We also have a graded analog of Theorem 1.2 over any separably closed field (not just in characteristic p).

Theorem 1.3. *Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be a standard $\mathbb{N}$-graded ring over a separably closed field k (i.e. $R_0 = k$ and R is finitely generated by R_1 as a k-algebra). If*

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depth$(R) \geq 3$, *then*

$$\widetilde{H}_0(\Delta(R);k) = \widetilde{H}_1(\Delta(R);k) = 0.$$

Hartshorne's theorem provides a sufficient condition for depth$(R/I) \leq 1$. Namely, if $\widetilde{H}_0(\Delta(R);k) \neq 0$, then depth$(R/I) \leq 1$. Our Theorem 1.3 provides a sufficient condition for depth$(R/I) \leq 2$. Namely, if $\widetilde{H}_1(\Delta(R);k) \neq 0$, then depth$(R/I) \leq 2$.

We also show by an example (Example 3.4) that Theorem 1.3 (and hence also Theorem 1.2) does not necessarily hold if one replaces k in $\widetilde{H}_*(\Delta(R);k)$ with $\mathbb{Z}$.

We do not know whether the assumptions that R is complete and the residue field is separably closed in Theorem 1.2 or the assumption that k is separably closed in Theorem 1.3 can be removed. Neither do we know whether the analogue of Theorem 1.2 in characteristic 0 holds.

Finally, it would be very interesting to know whether there exists a similar connection between higher values of the depth of R and the vanishing of higher homology groups of the complex $\Delta(R)$. When R is a Stanley-Reisner ring we obtain a natural extension of Theorem 1.3 (Corollary 3.5) and show that if depth $R \geq d$ then $\widetilde{H}_j(\Delta(R),k) = 0$ for all $0 \leq j \leq d-2$.

2. Proof of Theorem 1.2

In this section, we will prove our main technical result that implies Theorems 1.2 and 1.3. To this end, we recall the definition of the cohomological dimension.

Definition 2.1. Let I be an ideal of a noetherian commutative ring R. Then the *cohomological dimension* of the pair (R,I), denoted by cd(R,I), is defined by

$$\mathrm{cd}(R,I) := \max\{j | H^j_I(R) \neq 0\},$$

where $H^j_I(R)$ is the jth local cohomology module of R supported at I.

We will also consider the following spectral sequence ([ÀMGLZA03, p. 39])

$$E_1^{-a,b} = \bigoplus_{i_0<\cdots<i_a} H^b_{\mathfrak{p}_{i_0}+\cdots+\mathfrak{p}_{i_a}}(R) \Rightarrow H^{b-a}_{\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n}(R) = H^{b-a}_I(R). \tag{2.0.1}$$

where R is a noetherian commutative ring, I is an ideal of R, and $\{\mathfrak{p}_1,\dots,\mathfrak{p}_s\}$ is the set of minimal primes of I.

To prove Theorem 1.2, we will need the following results which are completely characteristic-free; they hold even in mixed-characteristic.

Lemma 2.2. *Let I be an ideal of a d-dimensional noetherian complete local domain $(R,\mathfrak{m},k)$ and let $\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$ be the set of minimal primes of I. Then*

$$E_2^{-t,d} \cong H_t(\Delta_n, \Delta(R/I); H^d_{\mathfrak{m}}(R)),$$

where $E_2^{-t,d}$ is the E_2-term of the spectral sequence (2.0.1), Δ_n denotes the full n-simplex, and $H_t(\Delta_n,\Delta(R/I);H^d_{\mathfrak{m}}(R))$ denotes the singular homology of the pair $(\Delta_n,\Delta(R/I))$ with coefficients in $H^d_{\mathfrak{m}}(R)$.

Proof. The spectral sequence (2.0.1) with $b=d$ gives us a complex

$$\cdots \to E_1^{-3,d} \to E_1^{-2,d} \to E_1^{-1,d} \to \cdots \tag{2.0.2}$$

whose homology is $E_2^{*,d}$. By the Hartshorne-Lichtenbaum vanishing theorem ([Har68, Theorem 3.1]), for each ideal $\mathfrak{a}$ of R, one has $H^d_{\mathfrak{a}}(R) = 0$ if and only if $\sqrt{\mathfrak{a}} \neq \mathfrak{m}$. Consequently, the complex (2.0.2) becomes

$$\cdots \to \bigoplus_{\substack{j_0<\cdots<j_t;\\ \sqrt{\mathfrak{p}_{j_0}+\cdots+\mathfrak{p}_{j_t}}=\mathfrak{m}}} H^d_{\mathfrak{m}}(R) \to \bigoplus_{\substack{i_0<\cdots<i_{t-1};\\ \sqrt{\mathfrak{p}_{i_0}+\cdots+\mathfrak{p}_{i_{t-1}}}=\mathfrak{m}}} H^d_{\mathfrak{m}}(R) \to \cdots \tag{2.0.3}$$

Let $C^*(\Delta_n)$ and $C^*(\Delta(R/I))$ denote the complexes of singular chains of Δ_n and $\Delta(R/I)$ with coefficients in $H^d_{\mathfrak{m}}(R)$ respectively and let $\widetilde{C}^*$ denote the complex of (2.0.3). One can see that the condition imposed on $\sqrt{\mathfrak{p}_1+\cdots+\mathfrak{p}_t}$ appearing in (2.0.3) is the opposite to the one imposed in the definition of $\Delta(R/I)$; consequently, one has a short exact sequence

$$0 \to C^*(\Delta(R/I)) \to C^*(\Delta_n) \to \widetilde{C}^* \to 0,$$

i.e. $\widetilde{C}^*$ is the complex of singular chains of the pair $(C^*(\Delta_n), C^*(\Delta(R/I)))$ with coefficients in $H^d_{\mathfrak{m}}(R)$. Hence the homology of the complex (2.0.3) is the relative homology $H_*(\Delta_n, \Delta(R/I); H^d_{\mathfrak{m}}(R))$, *i.e.*

$$E_2^{-t,d} \cong H_t(\Delta_n, \Delta(R/I); H^d_{\mathfrak{m}}(R)),$$

for each integer t. □

Theorem 2.3. *Let I be an ideal of a d-dimensional noetherian complete local domain $(R, \mathfrak{m}, k)$. Assume that $\operatorname{cd}(R, I) \leq d-3$ and $\operatorname{cd}(R, \mathfrak{p}_j) \leq d-2$ for each minimal prime $\mathfrak{p}_j$ of I. Then*

$$\widetilde{H}_0(\Delta(R/I); H^d_{\mathfrak{m}}(R)) = \widetilde{H}_1(\Delta(R/I); H^d_{\mathfrak{m}}(R)) = 0$$

where $\widetilde{H}_*(\Delta(R/I); H^d_{\mathfrak{m}}(R))$ denotes the reduced singular homology of $\Delta(R/I)$ with coefficients in $H^d_{\mathfrak{m}}(R)$.

Proof. Assume that I has n minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, and we will consider the spectral sequence (2.0.1).

Grothedieck's Vanishing Theorem ([BS98, 6.1.2]) asserts that $H^j_I(M) = 0$ for any ideal I of a d-dimensional noetherian commutative ring A, any A-module M, and any integer $j > d$. Therefore, for each $r \geq 2$, one has

$$E_1^{-2-r,d+r-1} = \bigoplus_{i_0<\cdots<i_{2+r}} H^{d+r-1}_{\mathfrak{p}_{i_0}+\cdots+\mathfrak{p}_{i_{2+r}}}(R) = 0;$$

hence the (incoming) differential $E_r^{-2-r,d+r-1} \to E_r^{-2,d}$ is 0. Since $E_1^{0,d-1} = \bigoplus_j H^{d-1}_{\mathfrak{p}_j}(R)$ and $\operatorname{cd}(R, \mathfrak{p}_j) \leq d-2$, we have $E_1^{0,d-1} = 0$ and hence the (outgoing) differential $E_2^{-2,d-1} \to E_2^{0,d-1}$ is 0. If $r \geq 3$, then $-2+r > 0$ and clearly $E_r^{-2+r,d-r+1} = 0$. This shows that, for $r \geq 2$, all incoming and outgoing differentials from $E_r^{-2,d}$ are 0; consequently $E_2^{-2,d} = E_\infty^{-2,d}$. On the other hand, since $\operatorname{cd}(R, I) \leq d-3$, we have $H^{d-2}_I(R) = 0$. Therefore $E_2^{-2,d} = 0$. Similarly, one can also show that $E_2^{-1,d} = 0$. On the other hand, it follows from the Hartshorne-Lichtenbaum vanishing theorem ([Har68, Theorem 3.1]) that $E_1^{0,d} = 0$. Thus, $E_2^{0,d} = 0$.

Therefore, it follows from Lemma 2.2 that

$$H_2(\Delta_n, \Delta(R/I); H^d_{\mathfrak{m}}(R)) \cong E_2^{-2,d} = 0 \text{ and } H_1(\Delta_n, \Delta(R/I); H^d_{\mathfrak{m}}(R)) \cong E_2^{-1,d} = 0.$$

The long exact sequence of homology

$$\cdots \to H_j(\Delta_n; H^d_{\mathfrak{m}}(R)) \to H_j(\Delta_n, \Delta(R/I); H^d_{\mathfrak{m}}(R)) \to H_{j-1}(\Delta(R/I); H^d_{\mathfrak{m}}(R))$$
$$\to H_{j-1}(\Delta_n; H^d_{\mathfrak{m}}(R)) \to \cdots$$

implies that

$$H_1(\Delta(R/I); H^d_{\mathfrak{m}}(R)) \cong H_1(\Delta_n; H^d_{\mathfrak{m}}(R)) = 0$$
$$H_0(\Delta(R/I); H^d_{\mathfrak{m}}(R)) \cong H_0(\Delta_n; H^d_{\mathfrak{m}}(R)) = H^d_{\mathfrak{m}}(R)$$

where $H_1(\Delta_n; H^d_{\mathfrak{m}}(R)) = 0$ since Δ_n is contractible. Therefore, we have

$$\widetilde{H}_1(\Delta(R/I); H^d_{\mathfrak{m}}(R)) = 0$$
$$\widetilde{H}_0(\Delta(R/I); H^d_{\mathfrak{m}}(R)) = 0.$$

This finishes the proof of our theorem. □

Remark 2.4. By the Universal Coefficient Theorem ([Hat02, Theorem 3A.3]), for each topological space X and an abelian group G, there is a short exact sequence

$$0 \to H_i(X; \mathbb{Z}) \otimes_{\mathbb{Z}} G \to H_i(X; G) \to \mathrm{Tor}_1(H_{i-1}(X; \mathbb{Z}), G) \to 0.$$

Since $H_0(X; \mathbb{Z})$ is always a free $\mathbb{Z}$-module, we have $\mathrm{Tor}_1(H_0(X; \mathbb{Z}), G) = 0$ and hence $H_1(X; G) \cong H_1(X; \mathbb{Z}) \otimes_{\mathbb{Z}} G$. In particular, with notation as in Theorem 2.3, we have

$$H_1(\Delta(R/I); H^d_{\mathfrak{m}}(R)) \cong H_1(\Delta(R/I); \mathbb{Z}) \otimes_{\mathbb{Z}} H^d_{\mathfrak{m}}(R)$$
$$H_1(\Delta(R/I); k) \cong H_1(\Delta(R/I); \mathbb{Z}) \otimes_{\mathbb{Z}} k$$

Under the assumptions of Theorem 2.3, for each nonzero element $r \in R$, the short exact sequence $0 \to R \xrightarrow{r} R \to R/(r) \to 0$ induces an exact sequence

$$\cdots \to H^d_{\mathfrak{m}}(R) \xrightarrow{r} H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R/(r)) = 0,$$

where $H^d_{\mathfrak{m}}(R/(r)) = 0$ since $\dim(R/(r)) < d$. This shows that $rH^d_{\mathfrak{m}}(R) = H^d_{\mathfrak{m}}(R)$ for each nonzero element $r \in R$. Therefore,

(1) when R doesn't have any integer torsion (*i.e.* when R contains $\mathbb{Q}$ or doesn't contain a field), $H_1(\Delta(R/I); \mathbb{Z}) \otimes_{\mathbb{Z}} H^d_{\mathfrak{m}}(R) \neq 0$ if and only if $H_1(\Delta(R/I); \mathbb{Z})$ contains a copy of $\mathbb{Z}$;
(2) when the characteristic of R is p, $H_1(\Delta(R/I); \mathbb{Z}) \otimes_{\mathbb{Z}} H^d_{\mathfrak{m}}(R) \neq 0$ if and only if $H_1(\Delta(R/I); \mathbb{Z})$ contains either a copy of $\mathbb{Z}$ or a copy of $\mathbb{Z}/p\mathbb{Z}$.

Consequently, when R contains a field, it is clear that $H_1(\Delta(R/I); \mathbb{Z}) \otimes_{\mathbb{Z}} H^d_{\mathfrak{m}}(R) = 0$ if and only if $H_1(\Delta(R/I); \mathbb{Z}) \otimes_{\mathbb{Z}} k = 0$, *i.e.*

$$\widetilde{H}_1(\Delta(R/I); H^d_{\mathfrak{m}}(R)) = 0 \Leftrightarrow H_1(\Delta(R/I); k) = 0.$$

On the other hand, it should be clear that

$$\widetilde{H}_0(\Delta(R/I); H^d_{\mathfrak{m}}(R)) = 0 \Leftrightarrow H_0(\Delta(R/I); k) = 0$$

always holds.

To prove Theorem 1.2, we also need the following result due to Peskine-Szpiro.

Theorem 2.5 (Remarque on p. 386 in [PS73]). *Let R be a d-dimensional regular ring of characteristic p and let I be an ideal of R. Then*

$$\mathrm{cd}(R, I) \leq d - \mathrm{depth}(R/I).$$

Proof of Theorem 1.2. By the Cohen Structure Theorem, there is a complete regular local ring $(S, \mathfrak{n}, k)$ that maps onto R. Let I denote the kernel of the surjection $S \to R$. Then it is straightforward to check that, if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is the set of minimal primes of R, then the preimages of $\mathfrak{p}_i$, $\{\widetilde{\mathfrak{p}}_1, \dots, \widetilde{\mathfrak{p}}_n\}$, are the minimal primes of I in S. It is also clear that

$$\sqrt{\mathfrak{p}_{i_0} + \cdots + \mathfrak{p}_{i_s}} \neq \mathfrak{m} \Leftrightarrow \sqrt{\widetilde{\mathfrak{p}}_{i_0} + \cdots + \widetilde{\mathfrak{p}}_{i_s}} \neq \mathfrak{n}$$

and consequently $\Delta(R) = \Delta(S/I)$.

Since each $\widetilde{\mathfrak{p}}_j$ is a prime ideal in S, the punctured spectrum of $S/\widetilde{\mathfrak{p}}_j$ is connected. Hence $\mathrm{cd}(S, \widetilde{\mathfrak{p}}_j) \leq \dim(S) - 2$ by [PS73, Corollaire 5.5]. Furthermore, since $\mathrm{depth}(S/I) \geq 3$, we know that $\mathrm{cd}(S, I) \leq \dim(S) - 3$ according to Theorem 2.5. Therefore, our conclusion follows directly from Theorem 2.3 and Remark 2.4. □

To prove Theorem 1.3, the following result of Varbaro is needed.

Theorem 2.6 (Theorem 3.5 in [Var13])**.** *Let I be an homogeneous ideal of $R = k[x_1, \dots, x_d]$ where k is a field of characteristic 0. If* $\mathrm{depth}(R/I) \geq 3$*, then* $\mathrm{cd}(R, I) \leq d - 3$*.*

Proof of Theorem 1.3. Since R is standard graded, one can write $R = k[x_1, \dots, x_n]/I$ for a homogeneous ideal I of $S = k[x_1, \dots, x_n]$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the minimal primes of I in S. Then $\mathrm{cd}(S, \mathfrak{p}_j) \leq \dim(S) - 2$ by [PS73, Corollaire 5.5], [Ogu73, Corollary 2.11], and [HL90, Theorem 2.9]. And one has $\mathrm{cd}(S, I) \leq \dim(S) - 3$ according to Theorem 2.5 (characteristic p) and Theorem 2.6 (characteristic 0). Therefore, our conclusion follows directly from Theorem 2.3 and Remark 2.4. □

We conclude this section with the following corollary.

Corollary 2.7. *Let $(R, \mathfrak{m}, k)$ be an equicharacteristic complete regular local ring whose residue field is separably closed. Assume that an ideal I of R can be generated by $d - 3$ elements, where $d = \dim(R)$. Then*

$$\widetilde{H}_0(\Delta(R/I); k) = \widetilde{H}_1(\Delta(R/I); k) = 0.$$

Proof. Since I can be generated by $d - 3$ elements, $\mathrm{cd}(R, I) \leq d - 3$. Same as in the proof of Theorem 1.2, we have $\mathrm{cd}(R, \mathfrak{p}) \leq d - 2$ for each minimal prime of I. Therefore, our corollary follows directly from Theorem 2.3. □

Remark 2.8. In [Fal80, Theorem 6], it is proved that, *if an ideal I of a d-dimensional complete local domain $(R, \mathfrak{m})$ can be generated by $d-2$ elements, then* $\mathrm{Spec}(R) \backslash \{\mathfrak{m}\}$ *is connected*. One may consider Corollary 2.7 as an extension of Faltings' connectedness theorem. Of course it would be very interesting to know whether the requirement that R be regular can be removed or at least weakened.

3. An application to Stanley-Reisner rings

Let S be a polynomial ring $k[x_1, \dots, x_n]$ where k is any field, and let K be a simplicial complex with vertices $\{x_1, \dots, x_n\}$. In this section we explore the properties of $\Delta(R)$ when R is the Stanley-Reisner ring $R = k[\Delta] = S/I(K)$. We then apply this to construct Example 3.4 that shows that Theorem 1.2 does not hold if the reduced homology groups $\widetilde{H}_*(\Delta(R); -)$ are computed with coefficients in $\mathbb{Z}$.

We first recall a definition and a well known result.

Definition 3.1. Let K be a simplicial complex and let $K_1, \ldots, K_t$ be subcomplexes of K. The *nerve of* $K_1, \ldots, K_t$, denoted $\mathcal{N}(K_1, \ldots, K_t)$, is the simplicial complex with vertices $K_1, \ldots, K_t$ and faces consisting of sets $\{K_{i_1}, \ldots, K_{i_\ell}\}$ such that $K_{i_1} \cap \cdots \cap K_{i_\ell} \neq \emptyset$.

The usefulness of this construction derives from the following theorem (cf. [Bjö95, Theorem 10.6].)

Theorem 3.2. *With the notation above, assume further that $K_1 \cup \cdots \cup K_t = K$. If every non-empty intersection $K_{i_1} \cap \cdots \cap K_{i_\ell}$ is contractible, K and $\mathcal{N}(K_1, \ldots, K_t)$ are homotopy equivalent (and hence have same homology groups.)*

We can now prove the main result of this section.

Theorem 3.3. *Let $S = k[x_1, \ldots, x_n]$, let K be a simplicial complex with vertex set $V = \{x_1, \ldots, x_n\}$, and let $I = I(K)$ be the Stanley-Reisner ideal corresponding to K. Then $\Delta(S/I)$ is homotopy equivalent to K.*

Proof. Let $K_1, \ldots, K_t$ be the facets (i.e., maximal faces) of K and note that since every intersection of facets is a face, Theorem 3.2 implies that K and $\mathcal{N}(K_1, \ldots, K_t)$ are homotopy equivalent.

The minimal primes of I are generated by complements of facets of K (cf. [BH93, Theorem 5.1.4]) thus $\Delta(S/I)$ is the simplicial complex whose vertices are the complements of facets of K and whose faces consist of

$$\{\{V \setminus K_{i_1}, \ldots, V \setminus K_{i_\ell}\} \,|\, (V \setminus K_{i_1}) \cup \cdots \cup (V \setminus K_{i_\ell}) \neq V\}.$$

We conclude the proof by exhibiting the isomorphism of simplicial complexes $\Phi : \mathcal{N}(K_1, \ldots, K_t) \to K$ given by $\Phi(K_i) = V \setminus K_i$. $\square$

Example 3.4. Let K be Reisner's six-point triangulation of the projective plane (cf. [Rei76, Remark 3]) and let $R = k(K)$ be the Stanley-Reisner ring associated to K. [Rei76, Theorem 1] shows that R is Cohen-Macaulay if and only if the characteristic of k is not 2 thus its depth of R is 3 when the characteristic of k is not 2. On the other hand the previous theorem implies that $\widetilde{H}_1(\Delta(R); \mathbb{Z}) = \widetilde{H}_1(K; \mathbb{Z}) \neq 0$. We conclude that the analogue of Theorem 1.3 with homology computed with integer coefficients does not hold.

We can now extend Theorem 1.3 for Stanley-Reisner rings as follows.

Corollary 3.5. *Let $S = k[x_1, \ldots, x_n]$, let K be a simplicial complex with vertex set $V = \{x_1, \ldots, x_n\}$, and let $I = I(K)$ be the Stanley-Reisner ideal corresponding to K. If $\operatorname{depth} S/I \geq d$ then $\widetilde{H}_j\left(\Delta(S/I), k\right) = 0$ for all $0 \leq j \leq d-2$.*

Proof. The Auslander-Buchsbaum theorem implies that the condition $\operatorname{depth} S/I \geq d$ is equivalent to $\operatorname{pdim}_S S/I \leq n - d$. This translates, using Hochster's formula (cf. [Hoc77], [MS05, Corollary 5.12]), to the vanishing of $\widetilde{H}_{\#W-n+j-1}\left(K_W, k\right)$ for all $0 \leq j \leq d-1$ and all $W \subseteq V$, where K_W denotes the restriction of K to the subset of its vertices W. In particular, this holds for $W = V$, giving $\widetilde{H}_{j-1}\left(K, k\right) = 0$ for all $0 \leq j \leq d-1$. Theorem 3.3 now implies $\widetilde{H}_{j-1}\left(\Delta(S/I), k\right) = 0$ for all $0 \leq j \leq d-1$. $\square$

We can also reinterpret Corollary 2.7 in the case of Stanley-Reisner ideals as follows. Let K be a simplicial complex with vertex set $V = \{x_1, \ldots, x_n\}$, let

$R = k[x_1, \ldots, x_n]$ and let $I = I(K)$ be the Stanley-Reisner ideal corresponding to K. The ideal I is generated by $n - t$ elements if and only if the Alexander dual K^* has at most $n - t$ facets. Also $\widetilde{H}_j(\Delta(R/I), k) = \widetilde{H}_j(K, k) = \widetilde{H}_{n-3-j}(K^*, k)$ where the first equality follows from Theorem 3.3 and the second from Alexander Duality (cf. [MS05, Theorem 5.6]). Now Corollary 2.7 implies the following: if a simplicial complex Δ with n vertices has at most $n - 3$ facets, then $\widetilde{H}_{n-3}(\Delta, k) = \widetilde{H}_{n-4}(\Delta, k) = 0$. This also leads to the following natural generalization of Corollary 2.7.

Proposition 3.6. *For a simplicial complex Δ with n vertices and at most μ facets, $\widetilde{H}_i(\Delta, k) = 0$ for all $i \geq \mu - 1$.*

Proof. Let $K_1 \cup \cdots \cup K_s$ be the distinct facets of K and let $\mathcal{N} = \mathcal{N}(K_1, \ldots, K_s)$. Note that $\dim \mathcal{N} \leq s - 1$ and that $\mathcal{N}$ is $(s-1)$-dimensional precisely when it is a simplex. An application of Theorem 3.2 gives $\widetilde{H}_i(\Delta, k) = \widetilde{H}_i(\mathcal{N}, k)$. and this vanishes for all $i \geq s - 1$, and since $s \leq \mu$, $\widetilde{H}_i(\Delta, k) = 0$ for all $i \geq \mu - 1$. □

Corollary 3.7. *Let K be a simplicial complex with vertex set $V = \{x_1, \ldots, x_n\}$, let $R = k[x_1, \ldots, x_n]$ and let $I = I(K)$ be the Stanley-Reisner ideal corresponding to K. If I is generated by $n - t$ elements then $\widetilde{H}_i(\Delta(R/I); k) = 0$ for all $0 \leq i \leq t - 2$.*

Proof. Using the previous discussion we reduce the problem to showing that $\widetilde{H}_{n-3-j}(K^*, k)$ vanishes for $0 \leq j \leq t - 2$ and this follows from the application of Proposition 3.6 to K^*. □

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References

[ÀMGLZA03] J. Àlvarez Montaner, R. García López, and S. Zarzuela Armengou: *Local cohomology, arrangements of subspaces and monomial ideals*, Adv. Math. **174** (2003), no. 1, 35–56. 1959890 (2004a:13012)

[Bjö95] A. Björner: *Topological methods*, Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 1819–1872.

[BS98] M. P. Brodmann and R. Y. Sharp: *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, vol. 60, Cambridge University Press, Cambridge, 1998. MR1613627 (99h:13020)

[BH93] W. Bruns and J. Herzog: *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956 (95h:13020)

[Fal80] G. Faltings: *Some theorems about formal functions*, Publ. Res. Inst. Math. Sci. **16** (1980), no. 3, 721–737. 602466 (82h:14004)

[Har62] R. Hartshorne: *Complete intersections and connectedness*, Amer. J. Math. **84** (1962), 497–508. 0142547 (26 #116)

[Har68] R. Hartshorne: *Cohomological dimension of algebraic varieties*, Ann. of Math. (2) **88** (1968), 403–450. 0232780 (38 #1103)

[Hat02] A. Hatcher: *Algebraic topology*, Cambridge University Press, Cambridge, 2002. 1867354 (2002k:55001)

[Hoc77] M. Hochster: *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975), Dekker, New York, 1977, pp. 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26. 0441987 (56 #376)

[HL90] C. Huneke and G. Lyubeznik: *On the vanishing of local cohomology modules*, Invent. Math. **102** (1990), no. 1, 73–93. 1069240 (91i:13020)

[Lyu07] G. Lyubeznik: *On some local cohomology modules*, Adv. Math. **213** (2007), no. 2, 621–643. 2332604 (2008e:13021)

[MS05] E. Miller and B. Sturmfels: *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. 2110098 (2006d:13001)

[Ogu73] A. Ogus: *Local cohomological dimension of algebraic varieties*, Ann. of Math. (2) **98** (1973), 327–365. 0506248 (58 #22059)

[PS73] C. Peskine and L. Szpiro: *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 47–119. MR0374130 (51 #10330)

[Rei76] G. A. Reisner: *Cohen-Macaulay quotients of polynomial rings*, Advances in Math. **21** (1976), no. 1, 30–49. 0407036 (53 #10819)

[Var13] M. Varbaro: *Cohomological and projective dimensions*, Compos. Math. **149** (2013), no. 7, 1203–1210. 3078644

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