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Multiplicative zero-one laws and metric number theory

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Abstract

We develop the classical theory of Diophantine approximation without assuming monotonicity or convexity. A complete 'multiplicative' zero-one law is established akin to the 'simultaneous' zero-one laws of Cassels and Gallagher. As a consequence we are able to establish the analogue of the Duffin-Schaeffer theorem within the multiplicative setup. The key ingredient is the rather simple but nevertheless versatile 'cross fibering principle'. In a nutshell it enables us to 'lift' zero-one laws to higher dimensions.

Keywords: Zero-one laws, metric multiplicative Diophantine approximation, Duffin-Schaeffer theorem

Subject classification: 11J13, 11J83, 11K60

1 Introduction

The theory of multiplicative Diophantine approximation is concerned with the set

$$\mathcal{S}_n^\times(\psi) := \{(x_1,\dots,x_n)\in[0,1]^n : \prod_{i=1}^n \|qx_i\| < \psi(q) \text{ for i.m. } q\in\mathbb{N}\},$$

where $\|qx\| = \min\{|qx-p| : p\in\mathbb{Z}\}$, 'i.m.' means 'infinitely many' and $\psi : \mathbb{N}\to\mathbb{R}_+$ is a a non-negative function. For obvious reasons the function ψ is often referred to as an approximating function. For convenience, we work within the unit cube $[0,1]^n$ rather than $\mathbb{R}^n$; it makes full measure results easier to state and avoids ambiguity. In fact, this is not at all restrictive since the set under consideration is invariant under translation by integer vectors.

Multiplicative Diophantine approximation is currently an active area of research. In particular, the long standing conjecture of Littlewood that states that $\mathcal{S}_2^\times(q\mapsto\varepsilon q^{-1})=\mathbb{R}$ for any $\varepsilon>0$ has attracted much attention – see [1, 16, 18] and references within. In this paper we will address the multiplicative analogue of yet another long standing classical problem; namely, the Duffin-Schaeffer conjecture.

Given $q\in\mathbb{N}$ and $x\in\mathbb{R}$, let

$$\|qx\|' := \min\{|qx-p| : p\in\mathbb{Z},\ (p,q)=1\},$$

and consider the standard simultaneous sets

$$\mathcal{D}_n(\psi) := \{(x_1,\dots,x_n)\in[0,1]^n : \big(\max_{1\leqslant i\leqslant n}\|qx_i\|'\big)^n < \psi(q) \text{ for i.m. } q\in\mathbb{N}\}$$

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and

$$\mathcal{S}_n(\psi) := \{(x_1, \ldots, x_n) \in [0,1]^n : \big(\max_{1 \leqslant i \leqslant n} \|qx_i\|\big)^n < \psi(q) \text{ for i.m. } q \in \mathbb{N}\} .$$

An elegant measure theoretic property of these sets is that they are always of zero or full Lebesgue measure $| \, . \, |$ irrespective of the dimension or the approximating function. Formally, for $n \geqslant 1$ and any non-negative function $\psi : \mathbb{N} \to \mathbb{R}_+$

$$|\mathcal{S}_n(\psi)| \in \{0, 1\} \qquad \text{and} \qquad |\mathcal{D}_n(\psi)| \in \{0, 1\} . \tag{1}$$

The former zero-one law is due to Cassels [7] while the latter is due to Gallagher [10] when $n = 1$ and Vilchinski [19] for n arbitrary. By making use of a refined version of Cassels' zero-one law, Gallagher [12] proved that for $n \geqslant 2$

$$|\mathcal{S}_n(\psi)| = 1 \qquad \text{if} \quad \sum_{q=1}^{\infty} \psi(q) = \infty . \tag{2}$$

Remark 1. Regarding the above statement and indeed the statements and conjectures below, by making use of the Borel-Cantelli Lemma from probability theory, it is straightforward to establish the complementary convergent results; i.e. if the sum in question converges then the set in question is of zero measure.

The case that $n = 1$ is excluded from the statement given by (2) since it is false. Indeed, Duffin & Schaeffer [8] gave a counterexample and formulated an alternative appropriate statement. The Duffin-Schaeffer conjecture[1] states that

$$|\mathcal{D}_n(\psi)| = 1 \qquad \text{if} \quad \sum_{q=1}^{\infty} \left(\frac{\varphi(q)}{q}\right)^n \psi(q) = \infty , \tag{3}$$

where φ is the Euler phi function. The consequence of the zero-one law for $\mathcal{D}_n(\psi)$ is that it reduces the Duffin-Schaeffer conjecture to showing that $|\mathcal{D}_n(\psi)| > 0$. Using this fact the conjecture has been established in the case $n \geqslant 2$ by Pollington & Vaughan [15]. Although various partial results have been obtained in the case $n = 1$, the full conjecture represents a key unsolved problem in number theory. For background and recent developments regarding this fundamental problem see [2, 8, 13, 14]. However, it is worth highlighting the Duffin-Schaeffer theorem which states that (3) holds whenever

$$\limsup_{Q \to \infty} \left(\sum_{q=1}^{Q} \left(\frac{\varphi(q)}{q}\right) \psi(q) \right) \left(\sum_{q=1}^{Q} \psi(q) \right)^{-1} > 0 .$$

Note that this condition implies that the convergence/divergence properties of the sums in (2) and (3) are equivalent.

As already mentioned, the purpose of this paper is to consider the multiplicative setup and in particular, the multiplicative analogue of the Duffin-Schaeffer conjecture. With this in mind, it is natural to define the set

$$\mathcal{D}_n^{\times}(\psi) := \{(x_1, \ldots, x_n) \in [0,1]^n : \prod_{i=1}^{n} \|qx_i\|' < \psi(q) \text{ for i.m. } q \in \mathbb{N}\}.$$

The ultimate goal is to prove the following two statements.

[1] To be precise Duffin and Schaeffer stated their conjecture for $n = 1$. The higher dimensional version is attributed to Sprindžuk – see [17, pg63].

Conjecture 1 *Let $n \geqslant 2$ and $\psi : \mathbb{N} \to [0, \frac{1}{2})$. Then*

$$|\mathcal{S}_n^\times(\psi)| = 1 \qquad \textit{if} \qquad \sum_{q=1}^{\infty} \psi(q) \left(\log \psi(q)^{-1}\right)^{n-1} = \infty\,, \tag{4}$$

Conjecture 2 *Let $n \geqslant 1$ and $\psi : \mathbb{N} \to [0, \frac{1}{2})$. Then*

$$|\mathcal{D}_n^\times(\psi)| = 1 \qquad \textit{if} \qquad \sum_{q=1}^{\infty} \left(\frac{\varphi(q)}{q}\right)^n \psi(q) \left(\log \psi(q)^{-1}\right)^{n-1} = \infty\,. \tag{5}$$

Throughout the paper,

$$\psi(q) \left(\log \psi(q)^{-1}\right)^{n-1} := 0 \quad \text{whenever} \quad \psi(q) = 0\,.$$

In view of the Duffin-Schaeffer counterexample it is necessary to exclude $n = 1$ from the statement of Conjecture 1. Clearly, the Duffin-Schaeffer conjecture and Conjecture 2 coincide when $n = 1$.

Remark 2. For $n \geqslant 2$, the results of Gallagher and Pollington & Vaughan establish the analogues of the above conjectures for the standard simultaneous sets $\mathcal{S}_n(\psi)$ and $\mathcal{D}_n(\psi)$.

1.1 The story so far: convexity versus monotonicity

Throughout this section, assume that $n \geqslant 2$. Geometrically, the multiplicative sets $\mathcal{S}_n^\times(\psi)$ and $\mathcal{D}_n^\times(\psi)$ consist of points in the unit cube that lie within infinitely many 'hyperbolic' domains

$$\mathrm{H} = \mathrm{H}(\psi, \mathbf{p}, q) := \left\{\mathbf{x} \in [0,1]^n : \textstyle\prod_{i=1}^n |x_i - p_i/q| < \psi(q)/q^n\right\}$$

centered around rational points $\mathbf{p}/q$ where $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $q \in \mathbb{N}$. In the case of $\mathcal{D}_n^\times(\psi)$ we impose the additional co-primeness condition $(p_i, q) = 1$ on the rational points. The approximating function ψ governs the size of the domains H. In the case of the standard simultaneous sets $\mathcal{S}_n(\psi)$ and $\mathcal{D}_n(\psi)$ the domains H are replaced by the 'cubical' domains

$$\mathrm{C} = \mathrm{C}(\psi, \mathbf{p}, q) := \left\{\mathbf{x} \in [0,1]^n : \max_{1 \leqslant i \leqslant n} |x_i - p_i/q|^n < \psi(q)/q^n\right\}.$$

The significant difference between the standard and multiplicative situation is that the domains C are convex while the domains H are non-convex. It is this difference that lies behind the fact that Conjectures 1 & 2 are still open whilst their standard simultaneous counterparts have been established – recall we assume that $n \geq 2$. In short, without imposing additional assumptions, convexity is vital in the methods employed by Gallagher and Pollington & Vaughan to establish (2) and (3) respectively. Indeed, their methods can be refined and adapted to deal with lim sup sets arising from more general convex domains but convexity itself seems to be unremovable – see [13, Chp.3] and references within. However, the landscape is completely different if we impose the additional assumption that the approximating function ψ is monotonic. For instance we can then overcome the fact that the domains H associated with the sets $\mathcal{S}_n^\times(\psi)$ and $\mathcal{D}_n^\times(\psi)$ are non-convex and Conjectures 1 & 2 correspond to a well known theorem of Gallagher

[11]. In fact, Gallagher considers lim sup sets arising from more general domains but monotonicity plays a crucial role in his approach and seems to be unremovable. Note that for monotonic ψ the convergence/divergence properties of the sums appearing in (4) and (5) are equivalent and since $\mathcal{S}_n^\times(\psi) \supset \mathcal{D}_n^\times(\psi)$ it follows that Conjecture 2 implies Conjecture 1.

The upshot is that the current body of metrical results for lim sup sets requires that either the approximating domains are convex or that the approximating function is monotonic.

1.2 Statement of results

Our first theorem is the multiplicative analogue of the Cassels-Gallagher zero-one law. It reduces Conjectures 1 & 2 to showing that the corresponding sets are of positive measure. In principal, it is easier to prove positive measure statements than full measure statements. More to the point, there is a well established mechanism in place to obtain lower bounds for the measure of lim sup sets – see §4 below or [3, §8] for a more comprehensive account.

Theorem 1 *Let $n \geqslant 1$ and $\psi : \mathbb{N} \to \mathbb{R}_+$. Then*

$$|\mathcal{S}_n^\times(\psi)| \in \{0,1\} \qquad \textit{and} \qquad |\mathcal{D}_n^\times(\psi)| \in \{0,1\}\,.$$

The proof will rely on the general technique developed in §2 which we refer to as the *cross fibering principle*. Given its simplicity, we suspect that it may well have applications elsewhere in one form or another.

The following theorem represents our 'direct' contribution to Conjectures 1 & 2 and is the complete multiplicative analogue of the Duffin-Schaeffer theorem.

Theorem 2 *Let $n \geqslant 1$, $\psi : \mathbb{N} \to [0, \frac{1}{2})$. Then*

$$|\mathcal{S}_n^\times(\psi)| = 1 = |\mathcal{D}_n^\times(\psi)| \qquad \textit{if} \qquad \sum_{q=1}^{\infty} \psi(q)\left(\log \psi(q)^{-1}\right)^{n-1} = \infty$$

and

$$\limsup_{Q\to\infty} \sum_{q=1}^{Q} \left(\frac{\varphi(q)}{q}\right)^n \psi(q)\left(\log \psi(q)^{-1}\right)^{n-1} \left(\sum_{q=1}^{Q} \psi(q)\left(\log \psi(q)^{-1}\right)^{n-1}\right)^{-1} > 0\,. \tag{6}$$

In turn,

$$|\mathcal{S}_n^\times(\psi)| = 0 = |\mathcal{D}_n^\times(\psi)| \qquad \textit{if} \qquad \sum_{q=1}^{\infty} \psi(q)\left(\log \psi(q)^{-1}\right)^{n-1} < \infty\,. \tag{7}$$

Note that the 'additional' assumption (6) implies that the convergence/divergence properties of the sums within Conjectures 1 & 2 are equivalent.

Remark 3. Theorem 2 enables us to establish the complete analogue of Gallagher's multiplicative theorem [11] within the framework of the 'p-adic Littlewood Conjecture' – see §4.1. It is also worth pointing out that the same arguments that show that the Duffin-Schaeffer theorem is valid for example when $\psi(q)$ is monotonic with q restricted to a lacunary sequence, or when $\psi(q)$ is arbitrary with q restricted to the sequence of primes, are equally applicable within the context of Theorem 2.

Remark 4. In the case when $\psi(q) \leqslant q^{-\delta}$ for all sufficiently large $q \in \mathbb{N}$ and some fixed $\delta > 0$ the term $\log \psi(q)^{-1}$ can replaced with $\log q$ throughout the statement of Theorem 2. However, in general this 'modified' version (in which $\log \psi(q)^{-1}$ is replaced with $\log q$) of the divergence part of Theorem 2 is false[2]. For instance, assume that $n \geq 2$ and let P be an infinite collection of primes such that $\sum_{p\in P}(\log\log p)^{n-1}(\log p)^{-1} < \infty$ and $\psi(q) = (\log q)^{-1}$ if $q \in P$ and 0 otherwise. Then

$$\sum_{q=1}^{\infty} \psi(q)\left(\log \psi(q)^{-1}\right)^{n-1} < \sum_{p\in P}^{\infty} \frac{(\log\log q)^{n-1}}{\log p} < \infty$$

and so $|\mathcal{S}_n^\times(\psi)| = 0 = |\mathcal{D}_n^\times(\psi)|$ by the convergence part of Theorem 2. However,

$$\sum_{q=1}^{\infty} \psi(q)(\log q)^{n-1} \geqslant \sum_{q=1}^{\infty} \frac{(\log q)^{n-1}}{\log q} \geqslant \sum_{q=1}^{\infty} 1 = \infty$$

since $n \geq 2$, and

$$\sum_{q=1}^{Q} \left(\frac{\varphi(q)}{q}\right)^n \psi(q)(\log q)^{n-1} \asymp \sum_{q=1}^{Q} \psi(q)(\log q)^{n-1}$$

since ψ is supported on primes and thus $\varphi(q) = q - 1$ whenever $\psi(q) \neq 0$. Thus, the 'modified' version of the divergence part of Theorem 2 would be false for this ψ. Here and elsewhere $a \asymp b$ means that a and b are comparable, that is $a \ll b$ and $a \gg b$, where $\ll$ and $\gg$ are the Vinogradov symbols that indicate an inequality with an unspecified positive multiplicative constant.

2 Cross Fibering Principle

Let X and Y be two non-empty sets. Let $S \subset X \times Y$. Given $x \in X$, the set

$$S_x := \{y : (x, y) \in S\} \subset Y$$

will be called a *fiber of S through x*. Similarly, given $y \in Y$, the set

$$S^y := \{x : (x, y) \in S\} \subset X$$

will be called a *fiber of S through y*. Given a measure μ over X, we will say that $A \subset X$ is *μ-trivial* if A is either null or full with respect to μ; that is

$$\mu(A) = 0 \qquad \text{or} \qquad \mu(X \setminus A) = 0\,.$$

It is an immediate consequence of Fubini's theorem (see below) that

$$S \text{ is } \mu \times \nu\text{-trivial} \quad \Longrightarrow \quad \mu\text{-almost every fiber } S_x \text{ is } \nu\text{-trivial}, \tag{8}$$

[2]The authors are grateful to the anonymous reviewer of the paper who has pointed a mistake in the earlier version of the paper, where we miss out the fact that such a simplification is not always possible.

and likewise

$$S \text{ is } \mu\times\nu\text{-trivial} \quad \Longrightarrow \quad \nu\text{-almost every fiber } S^y \text{ is } \mu\text{-trivial.} \tag{9}$$

Neither of these implications can be reversed in their own right. However, if the right hand side statements are combined together then we actually have a criterion which we will refer to as the *cross fibering principle.*

Theorem 3 *Let μ be a σ-finite measure over X, ν be a σ-finite measure over Y and $S \subset X \times Y$ be a $\mu\times\nu$-measurable set. Then*

$$S \textit{ is } \mu\times\nu\textit{-trivial} \quad \Longleftrightarrow \quad \begin{array}{c} \mu\textit{-almost every fiber } S_x \textit{ is } \nu\textit{-trivial} \\ \& \\ \nu\textit{-almost every fiber } S^y \textit{ is } \mu\textit{-trivial.} \end{array} \tag{10}$$

The proof of this theorem will make use of the following form of Fubini's theorem which can be found in [5, pg.233] and [9, §2.6.2].

Fubini's Theorem *Let μ be a σ-finite measure over X and ν be a σ-finite measure over Y. Then $\mu\times\nu$ is a regular measure over $X\times Y$ such that*

(i) *If A is a μ-measurable set and B is a ν-measurable set then $A\times B$ is a $\mu\times\nu$-measurable set and*

$$(\mu\times\nu)(A\times B) = \mu(A)\cdot\nu(B)\,.$$

(ii) *If S is a $\mu\times\nu$-measurable set, then*

$$S^y \quad \textit{is } \mu\textit{-measurable for } \nu\textit{-almost all } y,$$

$$S_x \quad \textit{is } \nu\textit{-measurable for } \mu\textit{-almost all } x,$$

the functions

$$X\to\overline{\mathbb{R}}\,:\,x\mapsto\nu(S_x) \qquad \textit{and} \qquad Y\to\overline{\mathbb{R}}\,:\,y\mapsto\mu(S^y) \tag{11}$$

are integrable and

$$(\mu\times\nu)(S) = \int \mu(S^y)d\nu = \int \nu(S_x)d\mu. \tag{12}$$

2.1 Proof of Theorem 3

The measures μ and ν are σ-finite. Thus, without loss of generality we can assume that the measures are finite and indeed that they are probability measures; that is

$$\mu(X) = 1 = \nu(Y)\,.$$

Necessity ($\Longrightarrow$). Without loss of generality, we can assume that $(\mu\times\nu)(S) = 0$ since otherwise we can replace S by its complement $X \setminus S$. Therefore, both the integrals appearing in (12) vanish. Note that the integrals themselves are obtained by integrating the non-negative functions (11). The upshot is that these functions vanish almost everywhere with respect to the appropriate measures which in turn implies the right hand side of (10).

Sufficiency ($\Longleftarrow$). Let $\tilde{X}$ be the set of $x \in X$ such that S_x is ν-measurable and trivial. Similarly, let $\tilde{Y}$ be the set of $y \in Y$ such that S^y is μ-measurable and trivial. In view of part (ii) of Fubini's theorem and the right hand side of (10) we have that both $\tilde{X}$ and $\tilde{Y}$ are sets of full measure; that is $\mu(X \setminus \tilde{X}) = 0$ and $\nu(Y \setminus \tilde{Y}) = 0$. In particular, $\tilde{X}$ is μ-measurable and $\tilde{Y}$ is ν-measurable. Now partition $\tilde{X}$ and $\tilde{Y}$ into two disjoint subsets as follows:

$$X_0 := \{x \in \tilde{X} : \nu(S_x) = 0\}, \qquad Y_0 := \{y \in \tilde{Y} : \mu(S^y) = 0\},$$

$$X_1 := \tilde{X} \setminus X_0 = \{x \in \tilde{X} : \nu(S_x) = 1\}, \qquad Y_1 := \tilde{Y} \setminus Y_0 = \{y \in \tilde{Y} : \nu(S^y) = 1\}.$$

Let $\mathcal{X}_A$ denote the characteristic function of a set A. By definition and part (ii) of Fubini's theorem, the functions (11) almost everywhere coincide with the functions $\mathcal{X}_{X_1}$ and $\mathcal{X}_{Y_1}$. Since the functions (11) are integrable, the functions $\mathcal{X}_{X_1}$ and $\mathcal{X}_{Y_1}$ are also integrable and so it follows that the sets X_1 and Y_1 are respectively μ and ν-measurable. This together with the fact that $\tilde{X}$ and $\tilde{Y}$ are respectively μ and ν-measurable, implies that $X_0 = \tilde{X} \setminus X_1$ is μ-measurable and $Y_0 = \tilde{Y} \setminus Y_1$ is ν-measurable.

Observe that $\mu(X_0) + \mu(X_1) = \mu(\tilde{X}) = 1$ and $\nu(Y_0) + \nu(Y_1) = \nu(\tilde{Y}) = 1$. Let us assume that the sets X_i and Y_i are non-trivial. In other words,

$$0 < \mu(X_i) < 1 \quad \text{and} \quad 0 < \nu(Y_i) < 1 \qquad \text{for} \quad i = 0, 1\,. \tag{13}$$

By part (i) of Fubini's theorem, the set $M := X_0 \times Y_1$ is $\mu \times \nu$-measurable. Now consider the set $S \cap M$ and observe that $M^y = X_0$ if $y \in Y_1$ and $M^y = \emptyset$ otherwise. Therefore, on using the first equality of (12) we obtain that

$$(\mu \times \nu)(S \cap M) = \int \mu(S^y \cap M^y) d\nu = \int \mu(S^y \cap X_0)\ \mathcal{X}_{Y_1}(y) d\nu. \tag{14}$$

By definition, for $y \in Y_1$ the set S^y is full in X and thus is full in X_0. As a consequence, we have that $\mu(S^y \cap X_0) = \mu(X_0)$ for $y \in Y_1$. Therefore, (13) and (14) imply that

$$(\mu \times \nu)(S \cap M) = \int \mu(X_0) \mathcal{X}_{Y_1}(y) d\nu = \mu(X_0)\nu(Y_1) > 0. \tag{15}$$

On the other hand, observe that $M_x = Y_1$ if $x \in X_0$ and $M_x = \emptyset$ otherwise. Then, on using the second equality of (12) we obtain that

$$(\mu \times \nu)(S \cap M) = \int \nu(S_x \cap M_x) d\mu = \int \nu(S_x \cap Y_1) \mathcal{X}_{X_0}(x) d\mu. \tag{16}$$

By definition, for $x \in X_0$ the set S_x is null and so $\nu(S_x \cap Y_1) = 0$ for $x \in X_0$. Therefore, (16) implies that

$$(\mu \times \nu)(S \cap M) = \int 0\, d\mu = 0.$$

This contradicts (15). Therefore at least one of the sets X_i and Y_i must be trivial. This together with (12) implies that S is trivial and thereby completes the proof.

Remark 5. Using induction Theorem 3 can be easily extended to the product of any finite number of measure spaces.

3 Proof of Theorem 1

The proof is by induction. Consider the set $\mathcal{S}_n^\times(\psi)$. When $n=1$, we have that $\mathcal{S}_1^\times(\psi)=\mathcal{S}_1(\psi)$ and Cassels' zero-one law implies that $\mathcal{S}_1^\times(\psi)$ is μ-trivial where μ is one-dimensional Lebesgue measure on $X:=[0,1]$.

Now assume that $n>1$ and that Theorem 1 is true for all dimensions $k<n$. Given a k-tuple $(x_1,\dots,x_k)\in[0,1]^k$, consider the function

$$\psi_{(x_1,\dots,x_k)}(q) \;:=\; \frac{\psi(q)}{\|qx_1\|\dots\|qx_k\|}\,.$$

Here we adopt the convention that $\alpha/0:=+\infty$ if $\alpha>0$ and that $\alpha/0:=0$ if $\alpha=0$. With reference to §2, let $Y:=[0,1]^{n-1}$ and let ν be $(n-1)$-dimensional Lebesgue measure on Y. Furthermore, let $S:=\mathcal{S}_n^\times(\psi)$. Then it is readily verified that for any $x_1\in X$ the fiber S_{x_1} is equal to the set $\mathcal{S}_1^\times(\psi_{(x_1)})$ and similarly for any $(x_2,\dots,x_n)\in Y$ the fiber $S^{(x_2,\dots,x_n)}$ is equal to the set $\mathcal{S}_{n-1}^\times(\psi_{(x_2,\dots,x_n)})$. In view of the induction hypothesis, we have that S_{x_1} is μ-trivial and $S^{(x_2,\dots,x_n)}$ is ν-trivial. Therefore, by Theorem 3 it follows that S is $\mu\times\nu$-trivial. In other words, the n-dimensional Lebesgue measure of $\mathcal{S}_n^\times(\psi)$ is either zero or one. This establishes Theorem 1 for the set $\mathcal{S}_n^\times(\psi)$.

Apart from obvious notational changes, the proof for the set $\mathcal{D}_n^\times(\psi)$ is exactly the same as above except for that fact that when $n=1$ we appeal to Gallagher's zero-one law rather than Cassels' zero-one law.

3.1 A multiplicative zero-one law for linear forms

In what follows $m\geq1$ and $n\geq1$ are integers. Given a 'multi-variable' approximating function $\Psi:\mathbb{Z}^n\to\mathbb{R}_+$, let $\mathcal{S}_{n,m}^\times(\Psi)$ denote the set of $\mathbf{X}\in[0,1]^{mn}$ such that

$$\Pi(\mathbf{qX}+\mathbf{p})<\Psi(\mathbf{q}) \tag{17}$$

holds for infinitely many $(\mathbf{p},\mathbf{q})\in\mathbb{Z}^n\times\mathbb{Z}^m\setminus\{\mathbf{0}\}$. Here $\Pi(\mathbf{y}):=\prod_{i=1}^n|y_i|$ for a vector $\mathbf{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$, $\mathbf{X}$ is regarded as an $m\times n$ matrix and $\mathbf{q}$ is regarded as a row vector. Thus, $\mathbf{qX}\in\mathbb{R}^n$ represents a system of n real linear forms in m variables. Naturally, let $\mathcal{D}_{m,n}^\times(\Psi)$ denote the subset of $\mathcal{S}_{m,n}^\times(\Psi)$ corresponding to $\mathbf{X}\in[0,1]^{mn}$ for which (17) holds infinitely often with the additional co-primeness condition $(p_i,\mathbf{q})=1$ for all $1\leqslant i\leqslant n$. Clearly, when $m=1$ and $\Psi(q)=\psi(|q|)$ the sets $\mathcal{S}_{m,n}^\times(\Psi)$ and $\mathcal{S}_n^\times(\psi)$ coincide as do the sets $\mathcal{D}_{m,n}^\times(\Psi)$ and $\mathcal{D}_n^\times(\psi)$.

The following statement is the natural generalisation of Theorem 1 to the linear forms framework. It also gives a positive answer to Question 4 raised in [4].

Theorem 4 *Let $m,n\geq1$ and $\Psi:\mathbb{Z}^n\to\mathbb{R}_+$ be a non-negative function. Then*

$$|\mathcal{S}_{m,n}^\times(\Psi)|\in\{0,1\}\qquad\textit{and}\qquad|\mathcal{D}_{m,n}^\times(\Psi)|\in\{0,1\}\,.$$

In view of the linear forms version of the Cassels-Gallagher zero-one law established in [4], the proof of Theorem 4 is pretty much the same as the proof of Theorem 1 with obvious modification. More specifically, all that is required from [4] is Theorem 1 with $n=1$.

4 Proof of Theorem 2

To begin with, we recall that $\mathcal{S}_n^\times(\psi) \supset \mathcal{D}_n^\times(\psi)$ and therefore is suffices to prove the divergence part for $\mathcal{D}_n^\times(\psi)$ and the convergence part for $\mathcal{S}^\times(\psi)$ only. Regarding the divergence case, by Theorem 1, we are done if we can show that $|\mathcal{D}_n^\times(\psi)| > 0$. Given $q \in \mathbb{N}$, let

$$A_q := \big\{(x_1, \dots, x_n) \in [0,1)^n : x_1 \cdots x_n \leqslant \psi(q)\big\}$$

and

$$B_q := \left\{ \mathbf{x} \in [0,1)^n \ : \begin{array}{l} q\mathbf{x} - \mathbf{p} \in A_q \text{ for some } \mathbf{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n \\ \text{with } (p_i, q) = 1 \text{ for all } i \end{array} \right\}.$$

Note that $A_q = B_q = \emptyset$ whenever $\psi(q) = 0$ and that

$$B := \limsup_{q\to\infty} B_q \ \subset \ \mathcal{D}_n^\times(\psi)\,.$$

Thus it suffices to prove that $|B| > 0$. For this purpose we will use the following generalisation of the divergent part of the standard Borel-Cantelli lemma, see for example [17, Lemma 5].

Lemma 1 *Let (Ω, A, μ) be a probability space and $\{E_q\} \subseteq A$ be a sequence of sets such that $\sum_{q=1}^\infty \mu(E_q) = \infty$. Then*

$$\mu(\limsup_{q\to\infty} E_q) \ \geq \ \limsup_{Q\to\infty} \frac{\Big(\sum_{s=1}^Q \mu(E_s)\Big)^2}{\sum_{s,t=1}^Q \mu(E_s \cap E_t)} \quad .$$

Naturally we shall use this lemma with $E_q = B_q$. The following estimates for the measure of $|B_q|$ can be found in [10, §§1,2] – they make use of the assumption that $0 \leqslant \psi(q) \leqslant 1/2$. For $q \in \mathbb{N}$

$$|A_q| \asymp \psi(q) \left(\log \psi(q)^{-1}\right)^{n-1}$$

and

$$|B_q| = (\varphi(q)/q)^n |A_q| \asymp (\varphi(q)/q)^n \psi(q) \left(\log \psi(q)^{-1}\right)^{n-1} \quad .$$

Then, by (6), we have that for infinitely many Q

$$\sum_{q=1}^Q |B_q| \asymp \sum_{q=1}^Q \psi(q) \left(\log \psi(q)^{-1}\right)^{n-1} \asymp \sum_{q=1}^Q |A_q|\,. \tag{18}$$

Together with the divergent sum hypothesis this implies that

$$\sum_{q=1}^\infty |B_q| = \infty\,. \tag{19}$$

Regarding the measures of overlaps, Lemma 2 in [11] implies that

$$|B_q \cap B_{q'}| \leqslant |A_q|\,|A_{q'}| \qquad \text{for } q \neq q' \tag{20}$$

with $\psi(q) \neq 0$ and $\psi(q') \neq 0$. Note that (20) is valid if $\psi(q) = 0$ or $\psi(q') = 0$ since we have zero on both sides of the inequality. Since (18) diverges, $\sum_{q=1}^{Q} |A_q| \geqslant 1$ for Q sufficiently large and so

$$\sum_{q,q'=1}^{Q} |B_q \cap B_{q'}| \leqslant \Big(\sum_{q=1}^{Q} |A_q|\Big)^2 + \sum_{q=1}^{Q} |A_q| \leqslant 2\Big(\sum_{q=1}^{Q} |A_q|\Big)^2.$$

This together with (18) and (19), gives via Lemma 1 that $|\limsup_{q\to\infty} B_q| > 0$ and thereby proves the divergence case of Theorem 2.

The convergence case is a consequence of, for example, Theorem 13 from [17, §5]. Before saying how to derive it note that the role of m and n is reversed in [17, §5] compared to the present paper. Thus, with reference to Theorem 13 in [17, §5] one has to take $n = 1$, $S = \mathbb{N}$ and $A(\overline{a})$ to consist of $(x_1, \ldots, x_m) \in [0,1)^m$ such that $\|x_1\| \cdots \|x_m\| < \psi(\overline{a})$ for $\overline{a} \in S$. Note that any point in $A(\overline{a})$ is obtained from a point of A_q (introduced above) by applying relevant symmetries $x_i \mapsto 1 - x_i$. This gives that $|A(\overline{a})| \ll |A_q| \ll \psi(q)\left(\log \psi(q)^{-1}\right)^{n-1}$ and hence the condition $\sum_{\overline{a}} |A(\overline{a})| < \infty$ which is required to derive our Theorem 2 for convergence from Theorem 13 in [17, §5].

4.1 An application to p-adic approximation

Theorems 1 & 2 settle the conjecture and problem stated in [6, §4.5] regarding the multiplicative set $\mathcal{S}_n^{\times}(\psi)$. In particular, as a consequence of Theorem 2 we are able to prove the following generalisation of the main result appearing in [6]. In short the statement corresponds to the complete analogue of Gallagher's multiplicative theorem [11] within the framework of the 'p-adic Littlewood Conjecture' – for further details see [1, 6] and references within. Given a prime p, we denote by $|q|_p$ the p-adic norm of $q \in \mathbb{Z}$.

Theorem 5 *Let $k \in \mathbb{N}$, $p_1, \ldots, p_k$ be distinct prime numbers and $F : \mathbb{N} \to \mathbb{R}_+$ be a positive function such that*

$$F(q) = F(q') \ \textit{whenever}\ |q|_{p_i} = |q'|_{p_i} \ \textit{for all}\ i. \tag{21}$$

Let $\Psi : \mathbb{N} \to \mathbb{R}_+$ will be a positive decreasing function. If

$$\sum_{q=1}^{\infty} \frac{\Psi(q)}{F(q)} \left(\log_+ \frac{F(q)}{\Psi(q)}\right)^{n-1} \tag{22}$$

converges, where $\log_+ x := \log\max\{2, x\}$, then for almost every $(x_1, \ldots, x_n) \in \mathbb{R}^n$ the inequality

$$F(q)\|qx_1\| \cdots \|qx_n\| < \Psi(q) \tag{23}$$

has only finitely many solutions $q \in \mathbb{N}$. On the other hand, if (22) diverges then for almost every $(x_1, \ldots, x_n) \in \mathbb{R}^n$ inequality (23) has infinitely many solutions $q \in \mathbb{N}$.

Proof of Theorem 5. We will use Theorem 2 with $\psi(q) = \Psi(q)/F(q)$. Indeed, the case of convergence is a straightforward application of Theorem 2 as in this case the convergence of (22) implies the convergence condition within (7).

In what follows we consider the divergence case. First of all observe that $\|qx_1\|\cdots\|qx_n\| \leqslant 2^{-n}$ for all $q \in \mathbb{N}$. Therefore, without loss of generality we can assume that $\Psi(q)/F(q) < 2^{-n}$ for all $q \in \mathbb{N}$. Furthermore, by replacing $\Psi(q)$ with $2^{-n}\Psi(q)$ if necessary, we can assume that $\Psi(q)/F(q) < e^{-n}$ for all q. In particular, this means that $\log_+ \frac{F(q)}{\Psi(q)} = \log \frac{F(q)}{\Psi(q)}$.
Throughout, $\mathbb{Z}_+$ will denote non-negative integers, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k$ and $|\alpha| = \max_i \alpha_i$. Each $q \in \mathbb{N}$ can be uniquely written as $q = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \ell$ for some $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\ell \in \mathbb{N}$ with $(\ell, p_1 \cdots p_k) = 1$. Then, by (21), the monotonicity of Ψ and the assumption that $\Psi(q)/F(q) < e^{-n}$, the function

$$f_\alpha(\ell) := \frac{\Psi(q)}{F(q)} \left(\log \frac{F(q)}{\Psi(q)}\right)^{n-1} \quad \text{where } q = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \ell \quad \text{and} \quad (\ell, p_1 \cdots p_k) = 1$$

is monotonically decreasing in ℓ for each fixed α. Label the numbers ℓ_i $(i \in \mathbb{N})$ with $(\ell_i, p_1 \cdots p_k) = 1$ in increasing order; i.e. $\ell_1 < \ell_2 < \ell_3 < \ldots$. By (22),

$$\sum_{\alpha \in \mathbb{Z}_+^k} \sum_{i=1}^{\infty} f_\alpha(\ell_i) = \infty. \tag{24}$$

Thus, by Theorem 2 with $\psi(q) = \Psi(q)/F(q)$, to complete the proof of Theorem 5 it suffices to show that for sufficiently large Q

$$\sum_{|\alpha| \leqslant Q} \sum_{i \leqslant Q} \left(\frac{\varphi(p_1^{\alpha_1} \ldots p_k^{\alpha_k} \ell_i)}{p_1^{\alpha_1} \ldots p_k^{\alpha_k} \ell_i}\right)^n f_\alpha(\ell_i) \gg \sum_{|\alpha| \leqslant Q} \sum_{i \leqslant Q} f_\alpha(\ell_i)\,. \tag{25}$$

Since $\varphi(p_1^{\alpha_1} \cdots p_k^{\alpha_k} \ell_i) = \prod_{i=1}^k (1 - p_i^{-1}) p_1^{\alpha_1} \cdots p_k^{\alpha_k} \varphi(\ell_i)$, inequality (25) would follow on showing that for each fixed $\alpha \in \mathbb{Z}_+^k$

$$\sum_{i \leqslant Q} \left(\frac{\varphi(\ell_i)}{\ell_i}\right)^n f_\alpha(\ell_i) \gg \sum_{i \leqslant Q} f_\alpha(\ell_i) \tag{26}$$

with the implied constant being independent of α. Lemma 2 from [6] gives that $\sum_{j \leqslant i} \varphi(\ell_j)/\ell_j \gg i$. This together with Jensen's inequality implies that $\sum_{j \leqslant i} \big(\varphi(\ell_j)/\ell_j\big)^n \gg i$. Then, by partial summation and the monotonicity of f_α, for each fixed α and $Q > 1$ we have that

$$\begin{aligned}
\sum_{i \leqslant Q} \left(\frac{\varphi(\ell_i)}{\ell_i}\right)^n f_\alpha(\ell_i) &= \sum_{i \leqslant Q} (f_\alpha(\ell_i) - f_\alpha(\ell_{i+1})) \sum_{j=1}^{i} \left(\frac{\varphi(\ell_j)}{\ell_j}\right)^n + \\
&\qquad + f_\alpha(\ell_{Q+1}) \sum_{j=1}^{Q} \left(\frac{\varphi(\ell_j)}{\ell_j}\right)^n \\
&\gg \sum_{i \leqslant Q} i\big(f_\alpha(\ell_i) - f_\alpha(\ell_{i+1})\big) + Q f_\alpha(\ell_{Q+1}) = \sum_{i \leqslant Q} f_\alpha(\ell_i).
\end{aligned}$$

This establishes (26) and thus completes the proof.

Remark 6. It is impossible to replace $\log \frac{F(q)}{\Psi(q)}$ with $\log q$ within (22). To see that this is so, let $k = n = 1$, $p_1 = p$, $\Psi(q) = q^{-2}$ and $F(q) = |q|_p^2 (1 - \log |q|_p)^2$. Write each $q \in \mathbb{N}$ as $p^\alpha \ell$ with $\alpha \in \mathbb{Z}_+$, $\ell \in \mathbb{N}$ and $(p, \ell) = 1$. Then,

$$\frac{\Psi(q)}{F(q)} = \frac{(p^\alpha \ell)^{-2}}{p^{-2\alpha}(1 + \alpha \log p)^2} \asymp \frac{1}{\ell^2 (1 + \alpha)^2}.$$

Consequently (22) is comparable to

$$\sum_{\alpha\geqslant 0}\ \sum_{(\ell,p)=1}\ \frac{1}{\ell^2(1+\alpha)^2}\ \log(\ell^2(1+\alpha)^2) \ll \sum_{\alpha\geqslant 0}\ \sum_{\ell\geqslant 1}\ \frac{(\log \ell)(\log(1+\alpha))}{\ell^2(1+\alpha)^2} < \infty\,.$$

On the other hand,

$$\sum_{q=1}^{\infty}\frac{\Psi(q)}{F(q)}\ (\log q)^{n-1} \asymp \sum_{\alpha\geqslant 0}\ \sum_{(\ell,p)=1}\ \frac{1}{\ell^2(1+\alpha)^2}\ \log(p^{\alpha}\ell) \gg \sum_{\alpha\geqslant 0}\ \frac{1}{1+\alpha} = \infty\,.$$

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