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On finite-dimensional global attractors of homeomorphisms

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Abstract

Let E be a normed linear space and suppose that $\mathbb{A}$ is the global attractor of either (i) a homeomorphism $F\colon E \to E$ or (ii) a semigroup $S(\cdot)$ on E that is injective on $\mathbb{A}$. In both cases $\mathbb{A}$ has trivial shape, and the dynamics on $\mathbb{A}$ can be described by a homeomorphism $F\colon \mathbb{A} \to \mathbb{A}$ (in the second case we set $F = S(t)$ for some $t > 0$). If the topological dimension of $\mathbb{A}$ is finite we show that for any $\epsilon > 0$ there is an embedding $e\colon \mathbb{A} \to \mathbb{R}^k$, with $k \sim \dim(\mathbb{A})$, and a (dynamical) homeomorphism $f\colon \mathbb{R}^k \to \mathbb{R}^k$ such that F is conjugate to f on $\mathbb{A}$ (i.e. $F|_{\mathbb{A}} = e^{-1} \circ f \circ e$) and f has a global attractor A_f with $e(\mathbb{A}) \subseteq A_f \subseteq N(e(\mathbb{A}), \epsilon)$. In other words, we show that the dynamics on $\mathbb{A}$ are essentially finite dimensional.

We characterise subsets of $\mathbb{R}^n$ that can be the global attractors of homeomorphisms as cellular sets, give elementary proofs of various topological results connected to Borsuk's theory of shape and cellularity in Euclidean spaces, and prove a controlled homeomorphism extension theorem. We also show that we could achieve $e(\mathbb{A}) = A_f$ under the assumption of a stronger controlled homeomorphism extension theorem.

1. Introduction

One can recast many of the important equations of mathematical physics within the framework of infinite-dimensional dynamical systems, i.e. dynamical systems evolving in an infinite-dimensional phase space. The theory of such systems has been systematically developed over the last three decades, and is well covered in the monographs by Babin & Vishik (1992), Chepyzhov & Vishik (2002), Chueshov (2002), Hale (1988), Ladyzhenskaya (1991), Robinson (2001), and Temam (1988). One of the most striking results in this theory is that in many interesting examples the long-time dynamics can be captured by a finite-dimensional subset of the ambient (infinite-dimensional) phase space, the 'global attractor'.

However, this statement says nothing a priori about the dynamics restricted to the attractor, and it is natural to ask in what sense (if any) these dynamics are themselves finite dimensional. This question was first posed in this generality by Eden et al. (1994), and subsequently discussed by Robinson (1999) and Romanov (2000). Ideally one would construct a finite-dimensional ordinary differential equation whose dynamics reproduce those on the attractor. This is certainly possible if the original system possesses an inertial manifold (Foias et al., 1988), but the existence of such an object requires restrictive conditions (a 'spectral gap condition' on the linear part of the equation) that prevent the theory being applicable to many important examples, such as the two-dimensional Navier–Stokes equations.

However, the construction of such an ODE seems very difficult, essentially because it would require a bi-Lipschitz embedding of the attractor into a Euclidean space (this can be slightly weakened to allow logarithmic corrections, see Pinto de Moura et al., 2011). An intrinsic characterisation of sets that admit such an embedding is a major open problem in the theory of metric spaces (see Heinonen, 2003, for example), and there are examples due to Eden et al.

(2013) that show that such an embedding (even with a logarithmic correction) is in general not possible for the attractors of infinite-dimensional dynamical systems.

Thus in this paper we aim to reproduce not the continuous dynamics on the attractor, but the discrete dynamics that come from considering the time T map of the flow, for some fixed $T > 0$. In fact we will consider the discrete problem from the outset, in which case the attractor arises from the iteration of some given homeomorphism.

At the heart of our construction is a classical theorem due to Menger (1926) and Nöbeling (1931) (see also Hurewicz & Wallman, 1941; Robinson, 2011), which guarantees that any finite-dimensional compact metric space can be embedded into a finite-dimensional Euclidean space of comparable dimension. The following theorem states this more precisely.

THEOREM 1.1 Menger–Nöbeling.   Let (X, d) be a compact metric space of dimension $\leq d$. Then the set of all homeomorphisms of X onto a subset of $\mathbb{R}^{2d+1}$ is dense in $C^0(X, \mathbb{R}^{2d+1})$. [In fact the homeomorphisms form a dense G_δ in $C^0(X, \mathbb{R}^{2d+1})$.]

We use this theorem to make a homeomorphic copy X of our original finite-dimensional attractor, along with its dynamics, into some $\mathbb{R}^k$. The main task is to find a way to extend the embedded dynamics from X onto the whole of $\mathbb{R}^k$, and to make the set X an attractor (or as nearly as possible) for the resulting dynamics.

In Section 2 we show that cellularity characterises the global attractors of homeomorphisms in Euclidean spaces, based in part on previous work of Garay (1991). In Section 3 we show that global attractors of homeomorphisms, and of continuous time semigroups, have trivial shape, a property that is topologically invariant. We recall in Section 4 the cellularity criterion of McMillan (1964), which allows us to boost a set with trivial shape to one that is cellular by adding an extra dimension to the ambient space. In Section 5 we prove a theorem that provides a controlled extension of a homeomorphism from a compact subset of $\mathbb{R}^k$ to a map on $\mathbb{R}^{2k}$, based on a trick of Klee (1955). Finally we combine these techniques in Section 6 to show that discrete dynamics on finite-dimensional global attractors are no more complicated than the dynamics on global attractors of homeomorphisms in finite-dimensional spaces.

However, the main result of the paper (Theorem 6.1) does not produce a homeomorphism that has the desired set X as its global attractor, but only as a subset of the global attractor. In the final section we show that the set X itself could be made the global attractor given a certain controlled homeomorphism extension theorem that (in some sense) 'respects the cellularity of X'. Details are given in Proposition 7.1.

2.   Global attractors and cellularity

2.1.   Global attractors in normed spaces

Suppose that E is a normed space, and that $F \colon E \to E$ is a continuous map. Then $\mathbb{A} \subseteq E$ is a *global attractor* for F if

(1) $\mathbb{A}$ is compact;
(2) $\mathbb{A}$ is invariant, i.e. $F(\mathbb{A}) = \mathbb{A}$; and
(3) $\mathbb{A}$ attracts bounded sets, i.e. for every bounded subset B of E,

$$\mathrm{dist}(F^n(B), \mathbb{A}) \to 0 \qquad \text{as} \qquad n \to \infty,$$

where $\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|$.

Note that if the global attractor exists it is unique and is the minimal closed set that attracts bounded sets (both follow from the fact that if Z attracts bounded sets then $\mathrm{dist}(\mathbb{A}, Z) = \mathrm{dist}(F^n(\mathbb{A}), Z) \to 0$ as $n \to \infty$).

Condition (3) is equivalent to requiring that for any bounded set B of E and every $\epsilon > 0$ there exists n_0 such that $F^n(B) \subseteq N(\mathbb{A}, \epsilon)$ whenever $n \geq n_0$. Here $N(\mathbb{A}, \epsilon)$ denotes the set $\{x \in E : \ \mathrm{dist}(x, \mathbb{A}) < \epsilon\}$. In particular, this implies that for any bounded set B containing $\mathbb{A}$, the equality $\mathbb{A} = \bigcap_n F^n(B)$ holds.

The existence of a global attractor is equivalent to the existence of a compact attracting set K, i.e. a compact set K satisfying (3) above in place of $\mathbb{A}$.

THEOREM 2.1.   *The map F has a global attractor $\mathbb{A}$ if and only if it has a compact attracting set K, and in this case*

$$\mathbb{A} = \bigcap_{j=0}^{\infty} F^j(K). \tag{2.1}$$

Proof. It is relatively straightforward to show that given the existence of a compact attracting set K, for any bounded set B the set

$$\omega(B) := \bigcap_{k \geq 0} \overline{\bigcup_{n \geq k} F^n(B)} \tag{2.2}$$
$$= \{x \in E : \ x = \lim_{j \to \infty} F^{n_j}(b_j), \ n_j \to \infty, \ b_j \in B\}$$

is a subset of K that is compact, invariant, and attracts B (see Hale, 1988, or Theorem 11.3 in Robinson, 2011, for example). We now show that $\omega(K)$ is the global attractor. Consider

$$\mathbb{A} = \overline{\bigcup_{B \text{ bounded}} \omega(B)}. \tag{2.3}$$

Since $\omega(B) \subseteq K$ for every bounded B, this is a closed subset of K, and so compact. It is clearly invariant since every $\omega(B)$ is invariant, and it attracts every bounded set, so it must be the global attractor since this is unique. It is immediate from (2.3) that $\omega(K) \subseteq \mathbb{A}$. Since $\mathbb{A}$ is the minimal closed set that attracts bounded sets, $\mathbb{A} \subseteq K$ and hence $\mathbb{A} = \omega(\mathbb{A}) \subseteq \omega(K)$.

Finally, to show that (2.1) holds, note that $\omega(K) \subseteq K$ and then, since $\omega(K)$ is invariant,

$$\omega(K) \subseteq \bigcap_{k \geq 0} F^k(K),$$

and it follows from (2.2) that

$$F^k(K) \subseteq \overline{\bigcup_{n \geq k} F^n(K)} \qquad \Rightarrow \qquad \bigcap_{k=0}^{\infty} F^k(K) \subseteq \omega(K),$$

which yields (2.1).   □

In a normed space we write B_R to denote the open ball and $\overline{B}_R$ the closed ball of radius R and with centre 0. A closed subset C of a normed space is a *cell* if $(C, \mathrm{int}(C))$ is homeomorphic to $(\overline{B}_1, B_1)$ (evidently the choice of radius 1 in this definition is irrelevant). Note that, as a consequence of the theorem on invariance of domain, if E is finite dimensional then any subset C of E that is homeomorphic to $\overline{B}_1$ is a cell in this sense.

We now show that any global attractor of a *homeomorphism* must be strongly cellular. A set X is *cellular* in E if

$$X = \bigcap_{j=1}^{\infty} C_j$$

for some sequence (C_j) of cells that is strictly decreasing in the sense that $C_{j+1} \subseteq \text{int}(C_j)$ for each j. The sequence (C_j) is then called a cellular sequence for X. If, in addition, for any open set U containing X there is a j such that $C_j \subseteq U$ then X is *strongly cellular* (correspondingly, (C_j) is then said to be a strongly cellular sequence for X). Cellularity and strong cellularity are equivalent in finite-dimensional spaces because of their local compactness (see Garay, 1991, and McCoy, 1973).

LEMMA 2.2.   *If $\mathbb{A}$ is the global attractor of a homeomorphism $F\colon E \to E$, where E is a normed space, then $\mathbb{A}$ is strongly cellular.*

Proof.   Choose R sufficiently large that $\mathbb{A} \subseteq B_R$. Since $\mathbb{A}$ is the global attractor, there exists an n such that

$$F^n(\overline{B}_R) \subseteq B_R.$$

For each $j = 1, 2, \ldots$ let $C_j = F^{nj}(\overline{B}_R)$. By definition F^{nj} restricts to a homeomorphism from $\overline{B}_R$ onto C_j. Moreover, since F^{nj} is defined on the whole space E, it maps the interior of $\overline{B}_R$ onto the interior of C_j. Therefore it provides a homeomorphism from $(\overline{B}_R, B_R)$ onto $(C_j, \text{int}(C_j))$, proving that the C_j are all cells. The choice of n guarantees that $C_{j+1} \subseteq F^{nj}(B_R) = \text{int}(C_j)$ for each j, so the C_j form a strictly decreasing sequence.

Clearly

$$\mathbb{A} \subseteq \bigcap_{j=1}^{\infty} C_j$$

because $\mathbb{A}$ is invariant. Since $\mathbb{A}$ is the global attractor of F it has to attract the bounded set $\overline{B}_R$, so given any open $U \supset \mathbb{A}$ there exists j such that $C_j \subseteq U$. This readily implies that

$$\mathbb{A} = \bigcap_{j=1}^{\infty} C_j,$$

so $\mathbb{A}$ is strongly cellular. $\square$

2.2.   *Global attractors in Euclidean spaces*

We now want to show that any cellular subset of $\mathbb{R}^n$ can be the global attractor of some homeomorphism on $\mathbb{R}^n$. We start by showing that any cellular set X in $\mathbb{R}^n$ is 'pointlike', i.e. $\mathbb{R}^n \setminus X \simeq \mathbb{R}^n \setminus \{0\}$.

LEMMA 2.3 (After Brown, 1960).   *If $X \subseteq \mathbb{R}^n$ is cellular then there exists a continuous map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $g^{-1}(\{0\}) = X$ and $g|_{\mathbb{R}^n \setminus X}\colon \mathbb{R}^n \setminus X \to \mathbb{R}^n \setminus \{0\}$ is a homeomorphism. Moreover, if $X \subseteq B_R$, then g can be chosen to be the identity outside B_R.*

Proof.   Let

$$X = \bigcap_{j=1}^{\infty} Q_j$$

where each Q_j is a cell with $Q_{j+1} \subseteq \operatorname{int}(Q_j)$. Perhaps discarding the first few Q_j we can assume that Q_1 is a subset of the interior of the ball $Q := \overline{B}_R$.

Let $g_1 \colon Q \to Q$ be a homeomorphism such that $g_1|\partial Q = \operatorname{id}$ and $g_1(Q_1) \subseteq B_{R/2}$. To see that such a homeomorphism exists, first note that there exists $0 < r < R$ such that Q_1 is contained in B_r. Consider a strictly increasing continuous map $a \colon [0, R] \to [0, R]$ such that $a(R) = R$, a is the identity near 0 and $a(r) = R/2$. Then the map $g_1 \colon Q \to Q$ given by

$$g_1(x) = a(|x|)\frac{x}{|x|}$$

is a homeomorphism (the fact that a is the identity near 0 guarantees that h is continuous) such that $g_1(B_r)$, and hence also $g_1(Q_1)$, is contained in $B_{R/2}$.

Now, given g_{j-1}, let $g_j \colon Q \to Q$ be a homeomorphism such that

$$g_j(x) = g_{j-1}(x) \quad \text{for } x \in Q \setminus Q_{j-1} \qquad \text{and} \qquad g_j(Q_j) \subseteq B_{R/(j+1)}.$$

(That such a homeomorphism exists follows from a similar argument to that given above.)

By the construction, $g_k|(Q \setminus Q_j) = g_j|(Q \setminus Q_j)$ and $g_k(Q_j) = g_j(Q_j)$ if $k \geq j \geq 1$. Now define a map $g \colon Q \to Q$ by setting $g|(Q \setminus Q_j) = g_j|(Q \setminus Q_j)$ if $j \geq 1$ and $g(X) = \{0\}$. Then $g|\partial Q = \operatorname{id}$, $g|(Q \setminus X)$ is an embedding whose image is $g(Q \setminus X) = Q \setminus \bigcap_{j=1}^{\infty} g_j(Q_j) = Q \setminus \{0\}$, and g is continuous at the points of X as $g(Q_j) \subseteq B(0, r/(j+1))$ if $j \geq 1$. Finally extend g to $\mathbb{R}^n$ by letting g be the identity on $\mathbb{R}^n \setminus Q$. □

Using this we can show that any cellular $X \subseteq \mathbb{R}^n$ is the global attractor of some homeomorphism. In fact one can use almost exactly the same proof to define an abstract flow on $\mathbb{R}^n$ (even a smooth one generated by an ordinary differential equation) that has X as the global attractor; for details on these refinements see Günther & Segal (1993), Günther (1995) and Pinto de Moura et al. (2011).

THEOREM 2.4 (After Theorem 2.7 in Garay, 1991). *If X is a cellular subset of $\mathbb{R}^n$ then there exists a homeomorphism $h \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $h(x) = x$ for all $x \in X$ and X is the global attractor for the dynamical system generated by h. Moreover, if $X \subseteq B_R$, there exists a constant $\rho > 0$ such that $h(B_r) \subseteq B_{r-\rho}$ for every $r \geq R$.*

Proof. After rescaling, we may assume without loss of generality that $R > 1$. Choose $R > R' > 1$ such that $X \subseteq B_{R'}$ and let $g \colon \mathbb{R}^n \to \mathbb{R}^n$ be the map given by Lemma 2.3, with the property that g is the identity outside the ball $B_{R'}$. Consider the annuli

$$R_k = \{x \in \mathbb{R}^n : \ 2^{-(k+1)} \leq |x| \leq 2^{-k}\}$$

for $k = 0, 1, 2, \ldots$. Each R_k is a compact subset of $\mathbb{R}^n \setminus \{0\}$, and so $g^{-1} \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus X$ is uniformly continuous on each set $R_k \cup R_{k+1}$; in particular there exists a b_k such that if $x, y \in R_k \cup R_{k+1}$ with $x = r\xi$ and $y = s\xi$, for some ξ with $|\xi| = 1$, then

$$|g^{-1}(r\xi) - g^{-1}(s\xi)| \leq 2^{-k} \tag{2.4}$$

provided that $|r - s| \leq b_k$. Redefine b_k (if necessary) to ensure that $R - b_0 > R'$ and

$$b_k < \min(b_{k-1}/2, 2^{-(k+3)}),$$

and let $\beta \colon [0, \infty) \to [0, \infty)$ be the function with $\beta(0) = 0$, $\beta(2^{-k}) = b_k$ for each $k \in \mathbb{N}$, $\beta|_{[2^{-(k+1)}, 2^{-k}]}$ affine for each $k \in \mathbb{N}$, and $\beta(r) = b_0$ for $r \geq 1$. Now set $\alpha(r) = r - \beta(r)$ and note that (i) $\alpha(0) = 0$, $\alpha(r) \to \infty$ as $r \to \infty$, and $\alpha \colon [0, \infty) \to [0, \infty)$ is strictly increasing, so α is a homeomorphism; (ii) $\alpha^k(r) \to 0$ as $k \to \infty$ for any $r > 0$; and (iii) $|r - \alpha(r)| = \beta(r) \to 0$

as $r \to 0$. For $x \notin X$ let

$$h(x) = g^{-1}\left[\alpha(|g(x)|)\frac{g(x)}{|g(x)|}\right],$$

and for $x \in X$ set $h(x) = x$. Clearly X is the global attractor of this homeomorphism, since

$$g[h^k(x)] = \alpha^k(|g(x)|)\frac{g(x)}{|g(x)|},$$

$\alpha^k(r) \to 0$ as $k \to \infty$, and $\mathrm{dist}(y, X) \to 0$ as $g(y) \to 0$.

This mapping satisfies the requirements of the theorem; the only possible issue is continuity at each $x \in X$. First observe the following: if $y \in g^{-1}(R_k)$ then $y = g^{-1}(r\xi)$ for some r, ξ with $2^{-(k+1)} \leq r \leq 2^{-k}$ and $|\xi| = 1$, and so by (2.4)

$$|y - h(y)| = |g^{-1}(r\xi) - g^{-1}(\alpha(r)\xi)| \leq 2^{-k},$$

since $|r - \alpha(r)| = \beta(r) \leq b_k$, and thus also $\alpha(r) \geq 2^{-(k+2)}$ implying that $\alpha(r)\xi \in R_k \cup R_{k+1}$. Now fix $x \in X$ and $\epsilon > 0$. Choose $N > 0$ such that $2^{-N} < \epsilon/2$ and $0 < \delta < \epsilon/2$ so small that $|y - x| < \delta$ implies $y \in X$ or $y \in g^{-1}(R_k)$ for some $k \geq N$. Then if $|y - x| < \delta$

$$|h(x) - h(y)| \leq |x - y| + |y - h(y)| \leq \epsilon/2 + 2^{-N} < \epsilon,$$

which shows that h is continuous.

Finally, pick $x \in \mathbb{R}^n$ with $|x| = r \geq R$. Then $g(x) = x$ because g is the identity outside B_R, and $\beta(|x|) = b_0$ because R was assumed to be bigger than 1. Thus

$$\alpha(|g(x)|)\frac{g(x)}{|g(x)|} = (|x| - b_0)\frac{x}{|x|} = (r - b_0)\frac{x}{|x|}$$

has modulus $r - b_0 \geq R - b_0 > R'$, so g^{-1} leaves it fixed. Consequently h transforms the sphere of radius r onto the sphere of radius $r - b_0$, and therefore $h(B_r) \subseteq B_{r-b_0}$. □

Note that the results of Lemma 2.2 and Theorem 2.4 show that cellularity is the characteristic feature of global attractors of homeomorphisms in Euclidean spaces.

In comparison with the results in the paper by Garay (1991), we note that his definition of a global attractor X (a 'globally asymptotically stable compact invariant set' in his terminology) requires X to be asymptotically stable and to attract points of E. If E is finite dimensional this notion of attractor coincides with the one used in this paper, but this is no longer true when E is infinite dimensional.

3. Global attractors have trivial shape

3.1. Global attractors of homeomorphisms

Given a finite-dimensional global attractor $\mathbb{A}$ of a homeomorphism on some normed space E, our aim is to construct a homeomorphism on some $\mathbb{R}^n$ that has a homeomorphic copy A of $\mathbb{A}$ as a global attractor. We have shown that to be a global attractor in $\mathbb{R}^n$ the set A must be cellular, but cellularity is not a topological property, i.e. a priori there is no way to guarantee that A is cellular, even though $\mathbb{A}$ is.

In order to circumvent this problem we introduce some more refined topological ideas from the theory of shape due to Borsuk (1975). Here we follow Garay (1991) and relate trivial shape to contractibility properties (the equivalence of Borsuk's original definition with that given here follows from Borsuk (1970) and Hyman (1969)).

DEFINITION 3.1.   Let A be a subset of B. A *contraction of A in B* is a continuous map

$$F\colon A \times [0,1] \to B$$

such that $F_0 = \mathrm{id}_A$ and $F_1 \equiv$ constant map, where F_t means the partial map $F_t\colon A \to B$ given by $F_t(p) := F(p,t)$. If such a contraction exists we say that A is *contractible in B*. If a set A is contractible in itself we simply say that A is *contractible*.

An easy but important remark is the following.

REMARK 3.2.   If A is contractible in B and $A_0 \subseteq A$, then A_0 is also contractible in B. A suitable contraction may be obtained by restricting a contraction of A in B.

As a simple but useful example, we note that any ball B in a normed space is contractible, since there is an obvious contraction onto its centre. Namely, if $B = x_0 + B_r$, then

$$F(x,t) := x_0 + (1-t)(x-x_0)$$

provides a contraction of B in itself.

The following is an extremely trivial proposition, but nevertheless we include it here for comparison purposes with Proposition 3.7 below.

PROPOSITION 3.3.   *If $h\colon A \to A'$ is a homeomorphism and A is contractible, then so is A'. That is, "being contractible" is a topological property.*

Proof.   Let $F\colon A \times [0,1] \to A$ be a contraction. Then

$$h \circ F \circ (h^{-1} \times \mathrm{id}_{[0,1]})\colon A' \times [0,1] \to A'$$

is a contraction of A'. $\square$

Now we introduce the property we are interested in, which is weaker than being contractible.

DEFINITION 3.4.   Let X be a compact subset of a normed space E. We say that X has *trivial shape* if for every neighbourhood U of X in E, X is contractible in U.

Observe that we do not require the existence of a contraction of X in itself, but that there exist contractions of X that take place in arbitrarily small neighbourhoods of X in E. When E is a finite-dimensional Euclidean space, a set X having trivial shape is also said to be *cell-like*.
We now show that strongly cellular sets have trivial shape.

LEMMA 3.5.   *If E is a normed space and $X \subseteq E$ is strongly cellular, then X has trivial shape.*

Proof.   Since X is strongly cellular, given any neighbourhood U of X we can find a cell C such that $X \subseteq C \subseteq U$. Any ball is contractible, therefore (Proposition 3.3) any cell is contractible. Therefore (Remark 3.2) any subset of a cell is contractible within that cell. Thus X is contractible within C, and so within U. It follows that X has trivial shape. $\square$

The following corollary, an immediate consequence of this result and Lemma 2.2, gives one indication why this definition is potentially interesting.

COROLLARY 3.6.   *If E is a normed space and $\mathbb{A}$ is the global attractor of a homeomorphism $F\colon E \to E$, then $\mathbb{A}$ has trivial shape.*

However, unlike cellularity, having trivial shape is a topological property.

PROPOSITION 3.7.   *Let $h\colon X \to X'$ be a homeomorphism between two compact sets X and X' contained in normed spaces E and E'. Then X has trivial shape if, and only if, X' has trivial shape.*

Proof.   We assume that X has trivial shape. Let $\widehat{h}\colon E \to E'$ be a continuous extension of h; this exists by Dugundji's extension of Tietze's theorem (Dugundji, 1951).

Let U' be a neighbourhood of X' in E' and consider $U := \widehat{h}^{-1}U'$, which is a neighbourhood of X in E. Since X has trivial shape, it is contractible in U; let $F\colon X \times [0,1] \to U$ be a contraction. Then

$$\widehat{h} \circ F \circ (h^{-1} \times \mathrm{id}_{[0,1]})\colon X' \times [0,1] \to U'$$

is a contraction of X' in U'. $\qquad\square$

One may wonder whether global attractors are actually contractible. The answer is, in general, negative. A quick way to prove this is to observe that a contractible set must be path connected, and then construct examples where global attractors exist that are not path connected. This can be done even in the plane, a good example being the 'topologist's sine curve'

$$\{(0,y)\colon\ -1 \leq y \leq 1\}\ \cup\ \{(x, \sin(1/x))\colon\ x \in (0,1]\} \subseteq \mathbb{R}^2.$$

This set is cellular, as can be easily checked from a drawing. Theorem 2.4 guarantees that it is the global attractor of some homeomorphism, but it is not path connected.

3.2.   *Global attractors of semiflows*

The following result shows that attractors with trivial shape also arise in more general situations. A semiflow $S(\cdot)\colon E \to E$ is a family of maps $\{S(t)\colon\ t \geq 0\}$ such that $S(0)$ is the identity map, $S(t + s) = S(t)S(s)$ for all $t, s \geq 0$, and $(t, p) \mapsto S(t)(p)$ is continuous.

The definition of a global attractor of a semiflow is an obvious analogue of that of a continuous map.

PROPOSITION 3.8.   *If $\mathbb{A}$ is the global attractor of a semiflow $S(\cdot)$ on a normed space E then $\mathbb{A}$ has trivial shape.*

The result appears in Garay (1991), but our proof is much simpler. Note that this result is not a simple consequence of applying Corollary 3.6 to the map $S(T)$ for some fixed $T > 0$, since there is no reason why $S(T)$ should be a homeomorphism.

Proof.   Let U be a neighbourhood of $\mathbb{A}$. We need to show that $\mathbb{A}$ is contractible in U; that is, there exist a continuous map $F\colon \mathbb{A} \times [0,1] \to U$ and a point $* \in U$ such that $F(p, 0) = p$ and $F(p, 1) = *$ for every $p \in \mathbb{A}$.

Choose any $q \in \mathbb{A}$ and let $G \colon \mathbb{A} \times [0,1] \to E$ be defined as $G(p,t) := q + (1-t)(p-q)$. Clearly G is a continuous map such that $G(p,0) = p$ and $G(p,1) = q$ for every $p \in \mathbb{A}$. Since $\mathbb{A} \times [0,1]$ is compact and G is continuous, its image $C := G(\mathbb{A} \times [0,1])$ is compact. Thus there exists $T > 0$ such that $S(t)(C) \subseteq U$ for every $t \geq T$, because $\mathbb{A}$ attracts compact subsets of E.

Let $H \colon \mathbb{A} \times [0,1] \to U$ be the composition $H := S(T) \circ G$ (notice that the range of H is now U). Denote $* := S(T)(q)$. Then, since $S(t)\mathbb{A} = \mathbb{A}$ for each $t \geq 0$, we have $* \in \mathbb{A} \subseteq U$. Clearly H is continuous; it also satisfies $H(p,0) = S(T)(p)$ and $H(p,1) = *$ for every $p \in \mathbb{A}$. Thus it is almost a contraction of $\mathbb{A}$ in U, the only issue being the fact that $H(p,0) = S(T)(p)$ rather than $H(p,0) = p$. However, this is easy to fix, as follows. Let $F \colon \mathbb{A} \times [0,1] \to U$ be defined as

$$F(p,t) := \begin{cases} S(2Tt)(p) & \text{if } 0 \leq t \leq 1/2, \\ H(p, 2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

It is straightforward to check that F is continuous and satisfies the required properties $F(p,0) = p$ and $F(p,1) = *$ for every $p \in \mathbb{A}$. $\square$

We note that in many interesting examples one can show that, at least on the global attractor, the semigroup is injective. In this case, it follows that for any time $t > 0$, the time-t map $S(t)$ is a homeomorphism. What follows is therefore applicable to both the global attractors of homeomorphisms, and to the time t map on any global attractor of a semigroup.

4. Trivial shape and cellularity in Euclidean spaces

Now suppose that we begin with a set $\mathbb{A}$ that is the attractor of a homeomorphism (or a semiflow) on a normed space. We therefore know from Corollary 3.6 (or Proposition 3.8) that $\mathbb{A}$ has trivial shape. If $\mathbb{A}$ is finite dimensional, then we can use Theorem 1.1 to find an embedding $e \colon \mathbb{A} \to \mathbb{R}^n$ for some n. Since trivial shape is a topological property (Proposition 3.7), it follows that $e(\mathbb{A}) \subseteq \mathbb{R}^n$ has trivial shape.

However, in order to make $e(\mathbb{A})$ the global attractor of a homeomorphism on $\mathbb{R}^n$ it must be cellular. We can obtain a cellular set by appealing to the following result due essentially to McMillan (1964), but given in precisely the form we require in Daverman (1986, III.18, Corollary 5A). This gives cellularity of $X_0 \times \{0\}$ in $\mathbb{R}^{n+1}$ whenever X_0 has trivial shape.

THEOREM 4.1 McMillan–Daverman.   *If X_0 is a compact subset of $\mathbb{R}^n$ that has trivial shape then $X_0 \times \{0\} \subseteq \mathbb{R}^{n+1}$ is cellular.*

Daverman includes the hypothesis $n \geq 3$ in his statement of the theorem, but the result actually holds for $n \geq 1$, since in dimensions one and two being cellular and having trivial shape are equivalent. For $n = 1$ this follows from the fact that a set having trivial shape is connected, so the set X_0 of Theorem 4.1 must be a compact interval, and then clearly $X_0 \times \{0\} \subseteq \mathbb{R}^2$ is cellular. The case $n = 2$ is proved as Corollary 4C in Section III.15 of Daverman (1986).

COROLLARY 4.2.   *Let $\mathbb{A}$ be the global attractor of a homeomorphism or a semiflow on a normed space E, and assume $\dim(\mathbb{A}) \leq k$. Then there exists a homeomorphism $e \colon \mathbb{A} \to X \subseteq \mathbb{R}^{2k+2}$ such that X is cellular in $\mathbb{R}^{2k+2}$.*

Proof.   By Theorem 1.1 there exists a homeomorphism $e \colon \mathbb{A} \to X_0 \subseteq \mathbb{R}^{2k+1}$. By identifying $\mathbb{R}^{2k+1}$ with $\mathbb{R}^{2k+1} \times \{0\} \subseteq \mathbb{R}^{2k+2}$ we may think of e as a homeomorphism onto the subset $X := X_0 \times \{0\} \subseteq \mathbb{R}^{2k+2}$. Corollary 3.6 or Proposition 3.8 guarantees that the set $\mathbb{A}$ has trivial

shape, and the same is true of X_0 by Proposition 3.7. Then Theorem 4.1 implies that X is cellular in $\mathbb{R}^{2k+2}$. $\qquad\square$

5. Extension of homeomorphisms from compact subsets of $\mathbb{R}^n$

With the above results we can guarantee that the global attractor $\mathbb{A} \subseteq E$ has trivial shape and find a homeomorphism $e \colon \mathbb{A} \to X \subseteq \mathbb{R}^{2k+2}$ such that X is cellular. Note that the dynamics on $\mathbb{A}$ are reproduced on X by means of the homeomorphism $f = e \circ F \circ e^{-1}$ (the dynamics on $\mathbb{A}$ and X are conjugated by e).

We now need to extend the homeomorphism $f \colon X \to X$ to a homeomorphism on the whole of $\mathbb{R}^{2k+2}$. Since we have the freedom to increase the dimension of the ambient space (we have done this once already), we can use the elegant trick to due to Klee (1955, statement (3.3)), as outlined in Proposition 5.3 below.

We make use of the following elementary results on extension of continuous functions.

LEMMA 5.1.   *Let $X \subseteq B_R \subseteq \mathbb{R}^n$ be a compact set and $f \colon X \to X$ be a continuous function. Then there exists a continuous extension $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ of f such that $\varphi(\overline{B}_R) \subseteq \overline{B}_R$ and φ is the identity outside B_R.*

Proof.   Let D denote the boundary of B_R and extend f to $X \cup D$ by letting it be the identity on D (this is still a continuous map). Considering this as a map from $X \cup D$ into $\overline{B}_R$, the Tietze extension theorem can be used to obtain a continuous $\varphi \colon \overline{B}_R \to \overline{B}_R$ such that $\varphi|_X = f$ and $\varphi|_D = \mathrm{id}$. It only remains to set $\varphi(x) := x$ for $x \notin B_R$. $\qquad\square$

The second simple lemma will be crucial in obtaining a controlled extension in the subsequent theorem.

LEMMA 5.2.   *Let $X \subseteq B_R \subseteq \mathbb{R}^n$ be a compact set. Then there exists a homeomorphism $c \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $c|_{X \cup \{0\}} = \mathrm{id}_{X \cup \{0\}}$ and $c(B_r) \subseteq B_{r/2}$ for every $r \geq 2R$.*

Proof.   Choose $R^* < R$ be so close to R that $X \subseteq B_{R^*}$. Let $\theta \colon [0, \infty) \to [0, \infty)$ be the (unique) continuous map that is affine on each of the intervals $[0, R^*]$, $[R^*, 2R]$ and $[2R, \infty)$ and such that (i) $\theta(r) = r$ for $0 \leq r \leq R^*$, (ii) $\theta(r) = r/2$ for $r \geq 2R$. Notice that θ is strictly increasing and surjective, hence a homeomorphism.

Define $c \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$c(p) := \begin{cases} 0 & \text{for } p = 0, \\ \theta(|p|)p/|p| & \text{for } p \neq 0. \end{cases} \tag{5.1}$$

Clearly c is continuous except possibly at $p = 0$. However for $p \in B_{R^*}$ we have $c(p) = p$, so c is continuous at $p = 0$ too. This also shows that $c|_X = \mathrm{id}_X$. Replacing θ in (5.1) by θ^{-1} yields an expression for c^{-1}, which is thus also continuous, and hence c is a homeomorphism.

It only remains to show that $c(B_r) \subseteq B_{r/2}$ for every $r \geq 2R$. Thus, let $r \geq 2R$ and pick $p \in B_r$. If $|p| \leq 2R$ then $\theta(|p|) \leq R$ and so $c(p) \in B_R \subseteq B_{r/2}$. If $|p| > 2R$ then $\theta(|p|) = |p|/2 < r/2$, so $c(p) \in B_{r/2}$. $\qquad\square$

We now prove our controlled extension theorem.

PROPOSITION 5.3.   *Let $X \subseteq B_R \subseteq \mathbb{R}^n$ be a compact set and $f \colon X \to X$ a homeomorphism. Then there exists a homeomorphism $\hat{f} \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that*

$$\hat{f}(x, 0_n) = (f(x), 0_n) \qquad \text{for all } x \in X,$$

where 0_n is the origin in $\mathbb{R}^n$, and such that

$$(x, y) \in B_r \times B_r \quad \Rightarrow \quad \hat{f}(x, y) \in B_r \times B_r \tag{5.2}$$

for any $r \geq R$.

Proof. We use Lemma 5.1 to extend the continuous map $f \colon X \to \mathbb{R}^n$ to a continuous map $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ which is the identity outside B_R and for which $\varphi(\overline{B}_R) \subseteq \overline{B}_R$, and again to extend the continuous map $f^{-1} \colon X \to \mathbb{R}^n$ to another continuous map $\psi \colon \mathbb{R}^n \to \mathbb{R}^n$ which is the identity outside B_R and for which $\psi(\overline{B}_R) \subseteq \overline{B}_R$.

Let $f_1, f_2 \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the homeomorphisms defined by

$$f_1(x, y) = (x, y + \varphi(x)) \qquad \text{and} \qquad f_2(x, y) = (2y + \psi(x), x)$$

for $x, y \in \mathbb{R}^n$. One can check that these are homeomorphisms since their inverses are given explicitly by

$$f_1^{-1}(x, y) = (x, y - \varphi(x)) \qquad \text{and} \qquad f_2^{-1}(x, y) = (y, (x - \psi(y))/2).$$

Define $g = f_2^{-1} \circ f_1$; this is a homeomorphism of $\mathbb{R}^{2n}$ since it is the composition of homeomorphisms. Notice that for every $x \in X$

$$f_1(x, 0) = (x, \varphi(x)) = (x, f(x))$$

and

$$f_2(f(x), 0) = (\psi(f(x)), f(x)) = (x, f(x)),$$

so

$$g(x, 0) = (f_2^{-1} \circ f_1)(x, 0) = f_2^{-1}(x, f(x)) = (f(x), 0).$$

The argument to this point provides a homeomorphism that extends f. We now combine this with the homeomorphism of Lemma 5.2 to obtain the controlled extension we require.

Let $c \colon \mathbb{R}^n \to \mathbb{R}^n$ be the homeomorphism constructed in Lemma 5.2, which gives rise to another homeomorphism $\hat{c} \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ setting $\hat{c}(x, y) := (c(x), c(y))$. Finally, define $\hat{f} := \hat{c} \circ g$. We claim that $\hat{f}$ satisfies the required properties.

Since $\hat{c}|_{X \times \{0\}} = \mathrm{id}_{X \times \{0\}}$, clearly $\hat{f}(x, 0) = (f(x), 0)$ for $x \in X$. Now let $r \geq R$ and pick $(x, y) \in B_r \times B_r$. Then $|\varphi(x)| \leq r$, so

$$(x', y') = f_1(x, y) = (x, y + \varphi(x)) \in B_r \times B_{2r},$$

and similarly $|\psi(y')| \leq 2r$, so

$$g(x, y) = f_2^{-1}(x', y') = (y', (x' - \psi(y'))/2) \in B_{2r} \times B_{2r}.$$

Now $2r \geq 2R$ so $c(B_{2r}) \subseteq B_r$, and hence $\hat{f}(x, y) = \hat{c}(g(x, y)) \in B_r \times B_r$. □

REMARK 5.4.   In the applications of Proposition 5.3 below it would suffice for the condition $\hat{f}(B_r \times B_r) \subseteq B_r \times B_r$ of (5.2) to hold only for a restricted set of radii, e.g. those of the form $r = 2^j R$ with $j \geq 0$.

It is natural to ask whether the doubling of the dimension used in Proposition 5.3 is really required. Certainly one cannot, in general, extend a homeomorphism $f \colon X \to X$, where X

is a compact subset of $\mathbb{R}^n$, to a homeomorphism on the whole of $\mathbb{R}^n$. For instance, let $X \subseteq \mathbb{R}^2$ be the union of two concentric circumferences and let $f\colon X \to X$ be a homeomorphism that interchanges the two connected components of X. Then f cannot be extended to a homeomorphism of all of $\mathbb{R}^2$ because such an extension would have to send one of the bounded components of $\mathbb{R}^2 \setminus X$ onto the unbounded one, which is impossible. This same example can be turned into a connected set by joining the two circumferences by a radial segment lying between them.

In the application of Proposition 5.3 to our dynamical problem in the next section, X will be cellular. Moreover, if the original F that generates the dynamics on E is the time-one map of a flow, then $F|_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}$ is isotopic to the identity and the same is true of $f\colon X \to X$. In this case a (hard) theorem of Oversteegen & Tymchatyn (2010) shows that f can indeed be extended to a homeomorphism of all of $\mathbb{R}^2$.

However, in dimensions 3 and higher, even in the favourable case just mentioned that X is cellular and f is isotopic to the identity, extending f to a homeomorphism of all $\mathbb{R}^n$ may not be possible. To illustrate this point we shall consider the arc A shown in Figure 1 (a detailed description can be found in Moise, 1977). It is the union of two subarcs A_1 and A_2, the first consisting of infinitely many trefoil knots that accumulate at one of its endpoints p and the second being just a straight line segment pq.
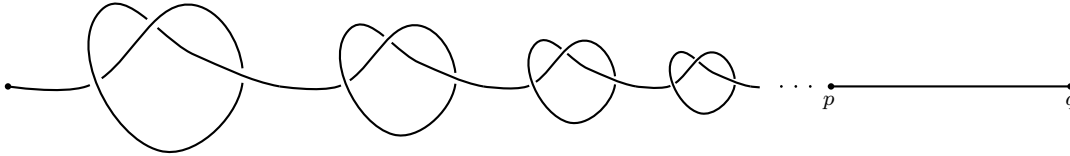

FIGURE 1. *The arc* A

It is easy to see from the drawing that: (i) A is cellular and (ii) each point of the arc other than p has a neighbourhood U in $\mathbb{R}^3$ such that there exists a homeomorphism of $\mathbb{R}^3$ that sends $U \cap A$ into a straight line segment (A is said to be 'locally polyhedral' at these points). Although more difficult to prove (we refer the reader again to Moise, 1977), it can be shown that (iii) p does not have this property. Also notice that, by definition, (iv) if $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism such that $h(A) = A$ and the arc is locally polyhedral at a point p', then it is also locally polyhedral at $h(p')$ (that is, being locally polyhedral at a point is preserved under ambient homeomorphisms leaving A invariant).

Now let X be the arc A and $f\colon X \to X$ a homeomorphism such that $f(p) \neq p$. For instance, if we think of the arc as being parametrised by some homeomorphism $\alpha\colon [0,1] \longrightarrow A$, we could choose f to send $\alpha(t)$ onto $\alpha(t^2)$. This f is isotopic to the identity and its only fixed points are the endpoints of the arc. If f could be extended to a homeomorphism $\hat{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$, by (ii), (iii), and (iv) above necessarily $\hat{f}(p) = p$, contradicting the condition that $f(p) \neq p$. Thus we have exhibited a cellular compactum $X \subseteq \mathbb{R}^3$ and a homeomorphism f of X (isotopic to the identity) that has no extension to a homeomorphism of $\mathbb{R}^3$.

6. *Finite-dimensional dynamics*

Finally we can combine these ingredients to show that the dynamics on $\mathbb{A}$ occur within the global attractor of a finite-dimensional system.

THEOREM 6.1. *Let $\mathbb{A}$ be the global attractor of a homeomorphism $F\colon E \to E$ with $\dim(\mathbb{A}) \leq k$. For any $\epsilon > 0$ there exist homeomorphisms*

$$e\colon \mathbb{A} \to A \subseteq \mathbb{R}^{4k+4} \qquad and \qquad f\colon \mathbb{R}^{4k+4} \to \mathbb{R}^{4k+4}$$

such that A is invariant under f and the dynamics on $\mathbb{A}$ induced by F and the dynamics on A induced by f are conjugate under e, i.e.

$$F|_{\mathbb{A}} = e^{-1} \circ f \circ e,$$

and such that f has a global attractor A_f with

$$A \subseteq A_f \subseteq N(A, \epsilon),$$

where

$$N(A, \epsilon) = \{ y \in \mathbb{R}^{4k+4} : \ \mathrm{dist}(y, A) < \epsilon \}.$$

In other words, $A \simeq \mathbb{A}$ is 'almost' the global attractor of f; and the dynamics of $F|_{\mathbb{A}}$ are no more complicated than those of f.

Proof.   Corollary 4.2 provides a homeomorphism $e\colon \mathbb{A} \to X_1 \subseteq \mathbb{R}^{2k+2}$ where X_1 is cellular in $\mathbb{R}^{2k+2}$. Using Theorem 2.4 we may find a homeomorphism $h\colon \mathbb{R}^{2k+2} \to \mathbb{R}^{2k+2}$ such that X_1 is the global attractor of h and $h|_{X_1} = \mathrm{id}$. Choose R such that $X_1 \subseteq B_R$. Then we can assume there is a constant $\rho > 0$ such that $h(B_r) \subseteq B_{r-\rho}$ for $r \geq R$ and, in particular, $h^m(B_r) \subseteq B_{\max(r - m\rho, R)}$ for all $m \geq 1$ and $r > 0$.

The map $f_1\colon X_1 \to X_1$ defined by $f_1(x) = e \circ F \circ e^{-1}(x)$ is a homeomorphism. We use Proposition 5.3 to produce a homeomorphism $\hat{f}_1\colon \mathbb{R}^{4k+4} \to \mathbb{R}^{4k+4}$ such that

$$\hat{f}_1(x, 0) = (f_1(x), 0) \qquad \text{for all } x \in X_1,$$

i.e. $\hat{f}_1$ extends f_1 from the set $A := X_1 \times \{0_{2k+2}\}$ to all of $\mathbb{R}^{4k+4}$. We can choose $\hat{f}_1$ so that

$$(x, y) \in B_r \times B_r \quad \Rightarrow \quad \hat{f}_1(x, y) \in B_r \times B_r$$

for any $r \geq R$.

Consider the homeomorphism $\hat{h}\colon \mathbb{R}^{4k+4} \to \mathbb{R}^{4k+4}$ given by $\hat{h}(x, y) := (h(x), y/2)$ for $x, y \in \mathbb{R}^{2k+2}$. Clearly A is a global attractor for $\hat{h}$ and $\hat{h}|_A = \mathrm{id}$. Let $K = \overline{B}_R \times \overline{B}_R$, and choose so large an $m \geq 1$ that $\hat{h}^m(K) \subseteq N(A, \epsilon)$. Set $f := \hat{h}^m \circ \hat{f}_1$. We claim that f satisfies the required properties.

It is clear that $f|_A = f_1|_A$, since $\hat{h}$ is the identity on A. Thus A is a compact invariant set under f, and $e^{-1} \circ f \circ e = F|_{\mathbb{A}}$. We show that K is an attracting set for f. Then f has a global attractor A_f by Theorem 2.1, and $A \subseteq A_f \subseteq K$. Moreover,

$$A_f = f(A_f) \subseteq f(K) = \hat{h}^m \hat{f}_1(K) \subseteq \hat{h}^m(K) \subseteq N(A, \epsilon).$$

To show that K is an attracting set, consider the mapping $\phi(r) := \max(r - m\rho, r/2^m, R)$. We claim that

$$f(B_r \times B_r) \subseteq B_{\phi(r)} \times B_{\phi(r)} \qquad \text{for } r > 0.$$

This is easily seen to be true for $r \leq R$ because

$$f(B_R \times B_R) \subseteq B_R \times B_R,$$

while for $r > R$ we have

$$\hat{f}_1(B_r \times B_r) \subseteq B_r \times B_r$$

and then

$$\hat{h}^m \hat{f}_1(B_r \times B_r) \subseteq B_{\max(r - m\rho, R)} \times B_{r/2^m} \subseteq B_{\phi(r)} \times B_{\phi(r)}.$$

Now let C be a bounded subset of $\mathbb{R}^{4k+4}$ and choose r such that $C \subseteq B_r \times B_r$. For n sufficiently large $\phi^n(r) = R$, in which case

$$f^n(C) \subseteq f^n(B_r \times B_r) \subseteq B_{\phi^n(r)} \times B_{\phi^n(r)} = B_R \times B_R,$$

more than is required. □

7. Conclusion and open problems

Theorem 6.1 shows that the dynamics on a finite-dimensional global attractor of a homeomorphism are no more complicated than the dynamics that can arise in a finite-dimensional system.

However, it is natural to wonder whether it would in fact be possible to construct a homeomorphism $f \colon \mathbb{R}^{4k+4} \to \mathbb{R}^{4k+4}$ such that $A_f = A$ in Theorem 6.1, i.e. finite-dimensional global attractors of homeomorphisms can always be realised, along with their dynamics, as global attractors in finite-dimensional spaces. Even if one allows the finite-dimensional map to be continuous rather than a homeomorphism, to our knowledge this problem is still open.

We can reformulate this problem in a more topological way. Suppose that $X \subseteq B_R$ is a cellular subset of $\mathbb{R}^n$ written as the intersection of the cellular sequence (C_j),

$$X = \bigcap_{j=1}^{\infty} C_j.$$

Also, let a homeomorphism $f \colon X \to X$ be given. Proposition 5.3 is a controlled extension result: it provides a homeomorphism $\hat{f} \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that extends f (in the sense that $\hat{f}(x, 0_n) = (f(x), 0_n)$ for every $x \in X$) and such that

$$\hat{f}(B_r \times B_r) \subseteq B_r \times B_r \tag{7.1}$$

for every $r \geq R$. We are now going to show that if the extension $\hat{f}$ can also be controlled near X (and not only for $r \geq R$), then it is possible to achieve $A_f = A$ in Theorem 6.1.

PROPOSITION 7.1. *Suppose that Proposition 5.3 could be strengthened so that, in addition to (7.1) (possibly weakened as in Remark 5.4), the relation*

$$\hat{f}(C_j \times \overline{B}_{R/2^{j-1}}) \subseteq C_j \times \overline{B}_{R/2^{j-1}} \tag{7.2}$$

held for every $j \geq 1$. Then in Theorem 6.1 we could achieve $A_f = A$. That is, if $\mathbb{A}$ is the global attractor of a homeomorphism $F \colon E \to E$ with $\dim(\mathbb{A}) \leq k$, there exist homeomorphisms

$$e \colon \mathbb{A} \to A \subseteq \mathbb{R}^{4k+4} \qquad and \qquad f \colon \mathbb{R}^{4k+4} \to \mathbb{R}^{4k+4}$$

such that the dynamics on $\mathbb{A}$ and A are conjugate under e, and A is the global attractor of f.

Proof. Let e, X_1, B_R, h, f_1, $\hat{h}$ and K be as in the proof of Theorem 6.1. We have $h(\overline{B}_R) \subseteq \overline{B}_{R-\rho}$ for some $\rho > 0$ so $h(\overline{B}_R) \subseteq B_R$. For each $j = 1, 2, \dots$ let

$$C_j = h^{j-1}(\overline{B}_R);$$

clearly (C_j) is a cellular sequence for X_1. Now pick an extension $\hat{f}_1 \colon \mathbb{R}^{4k+4} \to \mathbb{R}^{4k+4}$ that satisfies both (7.1) and (7.2) in place of $\hat{f}$ (and with $n = 2k + 2$). Define $f := \hat{h} \circ \hat{f}_1$. The same argument as that given in Theorem 6.1 proves that for any bounded set C we have $f^n(C) \subseteq K$ for all sufficiently large n, so f has a global attractor $A_f \subseteq K$. Clearly $A \subseteq A_f$. Moreover, observe that

$$f(C_j \times \overline{B}_{R/2^{j-1}}) = \hat{h}(\hat{f}_1(C_j \times \overline{B}_{R/2^{j-1}})) \subseteq \hat{h}(C_j \times \overline{B}_{R/2^{j-1}}) = C_{j+1} \times \overline{B}_{R/2^j}$$

for every $j \geq 1$, so

$$f^n(C_j \times \overline{B}_{R/2^{j-1}}) \subseteq C_{j+n} \times \overline{B}_{R/2^{j+n-1}}$$

for all $j \geq 1$ and $n \geq 1$. Now, for every $n \geq 1$ the case $j = 1$ yields

$$f^n(K) = f^n(C_1 \times \overline{B}_R) \subseteq C_{n+1} \times \overline{B}_{R/2^n}.$$

By Theorem 2.1 we get

$$A_f = \bigcap_{n=1}^{\infty} f^n(K) \subseteq \bigcap_{n=1}^{\infty} (C_{n+1} \times \overline{B}_{R/2^n}) = X_1 \times \{0\} = A,$$

as required. $\qquad \square$

In view of the usefulness of having a controlled extension result that satisfies both (7.1) and (7.2) we restate it more formally as an open question.

QUESTION 7.2. Let $X \subseteq B_R \subseteq \mathbb{R}^n$ be a compact set, $f \colon X \to X$ a homeomorphism, and (C_j) a cellular sequence for X. Does there exist a homeomorphism $\hat{f} \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$\hat{f}(x, 0_n) = (f(x), 0_n) \qquad \text{for all } x \in X,$$

where 0_n is the origin in $\mathbb{R}^n$;

$$\hat{f}(B_{2^j R} \times B_{2^j R}) \subseteq B_{2^j R} \times B_{2^j R}$$

for every $j \geq 0$; and

$$\hat{f}(C_j \times \overline{B}_{R/2^{j-1}}) \subseteq C_j \times \overline{B}_{R/2^{j-1}}$$

for every $j \geq 1$?

As remarked in the introduction, in the case of semiflows the problem of embedding the attractor along with its dynamics seems much more difficult and is still entirely open.

8. Acknowledgments

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