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BLIND IMAGE DECONVOLUTION USING THE SYLVESTER RESULTANT MATRIX

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ABSTRACT

This paper uses techniques from computational algebraic geometry to perform blind image deconvolution, such that prior knowledge of the point spread function (PSF) is not required to compute a deblurred form of a given blurred image. In particular, it is shown that the Sylvester resultant matrix enables the PSF to be calculated by two approximate greatest common divisor computations. These computations, and not greatest common divisor computations, are required because of the noise that is present in the exact image and PSF. The computed PSF is then deconvolved from the blurred image in order to calculate the deblurred image. The experimental results show consistently good results for the deblurred image and PSF, and they are compared with the results from other methods for blind image deconvolution.

Index Terms— Image restoration, blind image deconvolution, Sylvester matrix, greatest common divisor

1. INTRODUCTION

Blind image deconvolution (BID) is the process of obtaining a true image from a distorted version of it, possibly using prior knowledge of the true image or the function, called the point spread function (PSF), that causes the blur [1]. There are many sources of blur, including motion of the camera and/or object, imperfections in the lens, and variations in the air, for example, turbulence, and they result in the intensity of a given image not being recorded exactly, such that the intensity of a given pixel in the recorded image is influenced by neighbouring pixels in the image [2]. Moreover, the recorded image is frequently corrupted by random noise that is generated from the optical device and leads to measurement errors [3]. The blurred image is formed by the convolution of the exact image and the PSF, and this explains the term blind image deconvolution because the reconstructed image is computed by deconvolving the PSF, which may only be known partially, from the blurred image [4, 6]. This blind deconvolution operation is required in several applications, including video-conferencing, and astronomical and medical imaging, but it may be difficult or impossible to calculate the PSF a priori, which makes BID a challenging problem.

If G is the blurred image, F is the exact image, P is the PSF, E is the measurement error, N is the noise and $\otimes$ denotes convolution, then the blurring model is

$$\mathsf{G} = \mathsf{F} \otimes (\mathsf{P} + \mathsf{E}) + \mathsf{N}. \tag{1}$$

This is the most general model for the formation of a blurred image because it contains two sources of uncertainty that contribute to the degradation of the exact image F.

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The objective of this paper is to solve the BID problem, such that prior knowledge of the exact image and the PSF are not required. It is shown that approximate greatest common divisor (AGCD) computations allow the PSF to be computed. These calculations are carried out using the Sylvester resultant matrix [7], such that two AGCD computations are performed on the blurred image, and the deblurred image is then computed by deconvolving the PSF from the blurred image. Although AGCD computations have been used previously for image deblurring [6, 8, 9], the work described in this paper differs from the work in [6, 8, 9] and other works that use AGCD computations for image deblurring because (a) it is not assumed that the level of noise is known, and (b) it is not assumed that the PSF is known.

This paper describes initial results from the application of a non-linear structure-preserving matrix method [18] to the Sylvester matrix, and the computation of the PSF from the decomposition of this matrix, for the solution of the BID problem. The signal-to-noise ratio (SNR) of the blurred images used in the examples in this paper is much lower than in other work, and since, as noted above, the noise level and PSF need not be known, the method proposed in this paper has practical advantages with respect to methods that assume one or both of these quantities are known. The results in Section 5 are obtained with images of size 128×128 pixels, and they are typical of the results obtained with larger, rectangular images. These results are encouraging and work is currently focussed on the replacement of the singular value decomposition of the Sylvester matrix by its QR decomposition because this will reduce the complexity of the algorithm since use will be made of its update formula for the calculation of the horizontal and vertical extents of the PSF.

2. RELATED WORK

There are several algorithms for image restoration, and they can be classified according to whether the PSF is, or is not, known. If the PSF is known, image restoration can be performed using regularised filtering [2, 19] image denoising [10], Wiener filtering [11, 19], or the Lucy-Richardson algorithm [12, 19]. If, however, the PSF is not known, then Bayesian theory for iterative blind image deconvolution [5, 13], Bussgang deconvolution [14], and methods of maximum likelihood [15], constant modulus [16] and the greatest common divisor of two polynomials [6, 8, 9], can be used.

3. PROBLEM FORMULATION

An image of size $M \times N$ can be represented as a matrix F,

$$\mathsf{F} = \begin{bmatrix} f(0,0) & f(0,1) & \cdots & f(0,N-1) \\ f(1,0) & f(1,1) & \cdots & f(1,N-1) \\ \vdots & \vdots & \vdots & \vdots \\ f(M-1,0) & f(M-1,1) & \cdots & f(M-1,N-1) \end{bmatrix},$$

whose entries $f(i,j)$ are the pixel values and coefficients of a bivariate polynomial $F(x,y)$,

$$F(x,y) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} f(i,j) x^i y^j.$$

Similarly, the matrix $P \in \mathbb{R}^{C \times D}$ is the PSF array whose entries $p(k,l)$ are the coefficients of a bivariate polynomial $P(x,y)$,

$$P(x,y) = \sum_{k=0}^{C-1} \sum_{l=0}^{D-1} p(k,l) x^k y^l, \tag{2}$$

and it is not assumed that the supports of the PSF in the x and y directions are equal. The blurred image is represented as a bivariate polynomial $G(x,y) = F(x,y)P(x,y)$, which is equal to

$$\sum_{i,j,s,t} f(i,j) p(s-i, t-j) x^s y^t, \tag{3}$$

where $s = i + k$ and $t = j + l$. This equation shows that the pixel values of the blurred image G are equal to the two-dimensional convolution of the pixel values of the exact image F and PSF. The coefficient of $x^s y^t$ in the bivariate polynomial $G(x,y)$ of G is given by

$$\sum_{i,j} f(i,j) p(s-i, t-j), \tag{4}$$

and this is the value of the pixel at (s,t) in G.

The product (3) shows that the blurred image G is represented by a bivariate polynomial of degrees $(M+C-2)$ and $(N+D-2)$ in x and y respectively, and thus its coefficients are stored in a matrix of order $(M+C-1) \times (N+D-1)$. This matrix is larger than the matrix representation of the exact image F because of the extra rows at the top and bottom, and the extra columns on the left and right, of G. These extra rows and columns in G define the boundary conditions, and the blurred nature of G is most easily seen by its representation by a bivariate polynomial of higher degrees than the degrees of the bivariate polynomial representation of F.

It is assumed the PSF is separable,

$$p(s-i, t-j) = p_c(s-i) p_r(t-j),$$

where the subscripts c and r denote column and row respectively. The two-dimensional convolution (4) is therefore equal to the product of two one-dimensional convolutions,

$$\sum_{i=0}^{M-1} p_c(s-i) \sum_{j=0}^{N-1} f(i,j) p_r(t-j), \tag{5}$$

which is equal to

$$\sum_{j=0}^{N-1} p_r(t-j) \sum_{i=0}^{M-1} f(i,j) p_c(s-i). \tag{6}$$

Since the expressions (5) and (6) are equal, it is adequate to consider one of them because the analysis for the other form follows identically. Consider the form in (5), which shows that the contribution of the row component $p_r(t-j)$ of the PSF to the blurred image G is

$$\sum_{j=0}^{N-1} f(i,j) p_r(t-j), \quad i = 0, \ldots, M-1. \tag{7}$$

This equation shows that the contribution of the row component of the PSF to each row of G can be calculated by the multiplication of the polynomial forms of the exact image F and this component of the PSF, that is, $G_{r,i}(y) = F_{r,i}(y) P_r(y)$, where $G_{r,i}(y)$ is the polynomial form of the ith row of the blurred image due to the row component $P_r(y)$ of the PSF, and $F_{r,i}(y)$ is the polynomial form of the ith row of the exact image. It therefore follows from (7) that if rows i and j are considered, then $G_{r,i}(y) = F_{r,i}(y) P_r(y)$ and $G_{r,j}(y) = F_{r,j}(y) P_r(y)$, and thus if the polynomials $F_{r,i}(y)$ and $F_{r,j}(y)$ are coprime, then the row component of the PSF is equal to the greatest common divisor (GCD) of the polynomial forms of the ith and jth rows of the given blurred image, $P_r(y) = \text{GCD}(G_{r,i}, G_{r,j})$. It is clear that this operation can be repeated for columns k and l of the blurred image, that is, $P_c(x) = \text{GCD}(G_{c,k}, G_{c,l})$, where the polynomial $P(x,y)$, which is defined in (2), is given by $P(x,y) = P_c(x) P_r(y)$.

The discussion above assumes that noise is absent from the blurred image G, but this condition is not satisfied in practical examples. In particular, the presence of uncertainty E in the PSF and noise N, as shown in (1), implies that the GCD of two polynomials cannot be considered, and that it is necessary to consider an AGCD because the blurred image is defined by inexact (noisy) polynomials. The equations that define $P_c(x)$ and $P_r(y)$ are therefore given by

$$P_c(x) = \text{AGCD}(G_{c,k}, G_{c,l}),$$

and

$$P_r(y) = \text{AGCD}(G_{r,i}, G_{r,j}),$$

and thus a separable PSF can be computed by performing two AGCD computations. The exact image can then be recovered by deconvolving the PSF from the blurred image G after computations have been performed on G. The next section considers the application of the Sylvester resultant matrix to AGCD computations.

4. THE SYLVESTER MATRIX AND IMAGE DEBLURRING

The GCD $\hat{d}(y)$ of two exact polynomials $\hat{p}(y)$ and $\hat{q}(y)$, of degrees m and n respectively, can be computed from their Sylvester resultant matrix $S(\hat{p}, \hat{q})$, which is a square matrix of order $m+n$ [20]. In particular, the degree $\hat{t}$ of $\hat{d}(y)$ is equal to the rank loss of $S(\hat{p}, \hat{q})$, and the coefficients of $\hat{d}(y)$ are contained in the last non-zero row of the upper triangular matrices R and U of, respectively, the QR and LU decompositions of $S(\hat{p}, \hat{q})$. If inexact forms $p(y)$ and $q(y)$ of, respectively, $\hat{p}(y)$ and $\hat{q}(y)$ are given, then it can be assumed, without loss of generality, that $p(y)$ and $q(y)$ are coprime and $S(p,q)$ is therefore of full rank [17]. This is the situation that corresponds to a blurred image, and it may therefore be necessary to specify a threshold, which is a function of the SNR, that can be applied to the singular values of $S(p,q)$ [8].

The specification of a threshold necessarily implies that the SNR is known, but the satisfaction of this condition cannot be guaranteed in practical problems because the SNR may not be known, or it may only be known approximately. In these circumstances, it is difficult or impossible to specify a threshold. It has, however, been shown that if the given inexact polynomials $p(y)$ and $q(y)$ are processed before $S(p,q)$ is constructed, then t, the degree of an AGCD of $p(y)$ and $q(y)$, can be computed without knowledge of the SNR [17]. These preprocessing operations are:

1. The normalisation of each polynomial by the geometric mean of its coefficients.

2. The replacement of $g(y)$ by $\alpha g(y)$, where α is a non-zero constant.
3. The substitution of the independent variable y by the independent variable w,

$$y = \theta w, \tag{8}$$

where θ is a parameter.

The first preprocessing operation is necessary because $S(p, q)$ has a partitioned structure, that is,

$$S(p, q) = \begin{bmatrix} C(p) & D(q) \end{bmatrix}, \tag{9}$$

where the entries of $C(p) \in R^{(m+n)\times n}$ are the coefficients of $p(y)$, the entries of $D(q) \in R^{(m+n)\times m}$ are the coefficients of $q(y)$, and $C(p)$ and $D(q)$ are Tœplitz matrices. It follows that if the coefficients of $p(y)$ are much smaller or larger (in magnitude) than the coefficients of $q(y)$, then incorrect results are obtained because $S(p, q)$ is unbalanced. It is therefore desirable to balance the matrix, and this is achieved by normalising the coefficients of $p(y)$ and $q(y)$. It is shown in [17] that normalisation of $p(y)$ and $q(y)$ by the geometric means of their coefficients has advantages with respect to normalisation by the 2-norms of their coefficients.

The second preprocessing operation follows from the scale invariance property of the GCD of $\hat{p}(y)$ and $\hat{q}(y)$. Specifically, the GCD of two polynomials is defined to within an arbitrary non-zero scale factor α, that is,

$$\text{GCD}(\hat{p}, \hat{q}) \sim \text{GCD}(\hat{p}, \alpha\hat{q}), \tag{10}$$

where $\sim$ denotes equivalence to within the scale factor α. It follows, however, from (9) that $S(\hat{p}, \alpha\hat{q}) \neq \alpha S(\hat{p}, \hat{q})$, and thus scaling $\hat{q}(y)$ by α does not scale $S(\hat{p}, \hat{q})$ by α. If exact polynomials are considered and all computations are performed symbolically, then α can set equal to one, but if inexact polynomials $p(y)$ and $q(y)$ are considered, then the examples in [17] show that the computed value of t is a function of α. This is an unsatisfactory result because an AGCD of $p(y)$ and $q(y)$ also satisfies (10), that is, it is a function of the roots of $p(y)$ and $q(y)$, and it is independent of α.

The third preprocessing operation follows because computations on polynomials whose coefficients vary widely in magnitude may be unstable, and it is therefore desirable to minimise this ratio [21]. The substitution (8) introduces the parameter θ, and it is shown in [7, 17] that the optimal values of α and θ minimise the ratio of the maximum entry (in magnitude) to the minimum entry (in magnitude) of $S(\tilde{p}, \alpha\tilde{q})$, where $\tilde{p}(w) = p(\theta w)$ and $\tilde{q}(w) = q(\theta w)$, and that this minimisation leads to a linear programming problem.

After these preprocessing operations have been implemented, the value of t, that is, the degree of the polynomial representation of the PSF, can be computed by using properties of the subresultant matrices of $S(\tilde{p}, \alpha\tilde{q})$. Two methods for the computation of t that yield the desired result $t = \hat{t}$ are discussed in [17], and this computation enables the coefficients of an AGCD of $\tilde{p}(w)$ and $\alpha\tilde{q}(w)$, that is, the coefficients of the polynomial form of the PSF, to be determined. In particular, it is shown in [7] that the method of non-linear structured total least norm (SNLTN) [18] enables an AGCD of $\tilde{p}(w)$ and $\alpha\tilde{q}(w)$ to be computed to high accuracy, such that the error between the GCD of the exact polynomials $\hat{p}(\theta w)$ and $\hat{q}(\theta w)$, and an AGCD of $p(\theta w)$ and $\alpha q(\theta w)$, is small. It is noted that the method of SNLTN yields a much better estimate of an AGCD of two inexact polynomials $p(y)$ and $q(y)$ than do the QR and LU decompositions of $S(p, q)$.

The method described above for BID is shown in Algorithm 1.

Algorithm 1 BID using the Sylvester matrix
BEGIN
1- Read in a distorted image G.
2- Choose two rows (r_1, r_2) and two columns (c_1, c_2) of G, and calculate their AGCDs $P_c(x)$ and $P_r(y)$.
3- Calculate the PSF $P(x, y) = P_c(x)P_r(y)$.
4- Deconvolve $P(x, y)$ from the blurred image.
END

The method of SNLTN allows an improved (reduced noise) form of each row and each column of the given blurred image to be computed, such that when the computed PSF is deconvolved from the improved form of the blurred image, the computation is equivalent to polynomial division in which the denominator polynomial (the PSF) is an exact divisor of the numerator polynomial (the improved form of the deblurred image). It therefore follows that the error between the exact image and the deblurred image is small.

5. EXPERIMENTAL RESULTS

This section presents the results from the method described in this paper. The first experiment is performed on two images, each of which is 128×128 pixels. Each exact image is convolved with a separable PSF of size 7×7 pixels. The relative errors due to the PSF and additive noise are, respectively, 10^{-3} and 10^{-7} for the first image (Map), and 10^{-4} and 10^{-7}, respectively, for the second image (Grass). The SNR is defined as

$$\text{SNR} = 10 \log_{10} \frac{P_{signal}}{P_{noise}} \text{ dB},$$

where P_{signal}, the signal power of an image F of size $M \times N$, is given by

$$P_{signal} = \|F - F^*\|_F, \quad F^* = \frac{\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i, j)}{MN},$$

and P_{noise}, the noise power of a blurred image G of the exact image F, is given by

$$P_{noise} = \|(G - F) - (G - F)^*\|_F,$$

where

$$(G - F)^* = \frac{\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} (g(i, j) - f(i, j))}{MN}.$$

It was noted in Section 3 that G is larger than F because of the convolution operation. The expression for P_{noise} assumes, therefore, that the border pixels of G that are introduced as a consequence of the convolution operation are removed, such that F and G are the same size.

Exact, blurred and restored images are shown in Figures 1 and 2, where the methods M1, M2, M3, M4 and M5 are defined in Table 1. It is seen that the method presented in this paper yields significantly improved deblurred images with respect to the deblurred images obtained from the Lucy-Richardson algorithm, regularised filtering and the method of maximum likelihood. The work in this paper does not require prior knowledge of the PSF and noise level, and it therefore has advantages with respect to the other methods used to obtain the images in Figures 1 and 2 because these methods require thresholds for the computation of the horizontal and vertical extents of the PSF, and the noise level.

Table 1 shows two error measures for the images in Figures 1 and 2, and it is seen that they are smaller for the method discussed in this paper than for the other methods. The root mean square error (RMSE) and the normalised absolute error (NAE) between F, whose pixel values are $f(i,j)$, and G, whose pixel values are $g(i,j)$, are

$$\mathrm{RMSE} = \sqrt{\frac{\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} e^2(i,j)}{MN}},$$

and

$$\mathrm{NAE} = \frac{\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} |e(i,j)|}{\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} |f(i,j)|},$$

where $e(i,j) = g(i,j) - f(i,j)$.

Table 1: Comparison of 5 deblurring methods. M1: The method described in this paper, M2: Lucy-Richardson, M3: Regularised filter, M4: Wiener filter, M5: Maximum likelihood.

Image	Methods	SNR	RMSE	NAE
Map	M1	40.22	1.92913e − 05	3.45823e − 03
	M2	3.19	8.45883e − 04	1.28538e − 01
	M3	9.42	5.11916e − 04	8.60910e − 02
	M4	38.10	2.57312e − 05	4.65718e − 03
	M5	3.21	8.43858e − 04	1.28191e − 01
Grass	M1	56.21	2.75215e − 06	4.54241e − 04
	M2	0.48	1.05311e − 03	1.56465e − 01
	M3	5.80	5.66495e − 04	8.85092e − 02
	M4	47.36	4.81265e − 06	7.63011e − 04
	M5	0.48	1.05503e − 03	1.56816e − 01

The results of the second experiment are shown in Figure 3, which contains an exact image, a blurred form of this image and a restored (deblurred) form of the blurred image, obtained using the method described in this paper. The SNRs of the blurred and deblurred images are 2.21 dB and 47.66 dB respectively, and this significant improvement is readily apparent from visual inspection of the images.

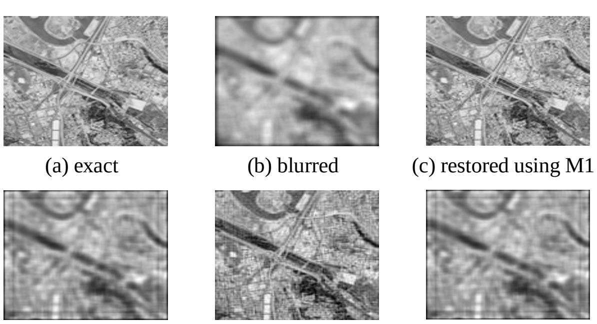
(a) exact (b) blurred (c) restored using M1
(d) restored using M2 (e) restored using M3 (f) restored using M5

Fig. 1: Results for the restoration of an aerial map.

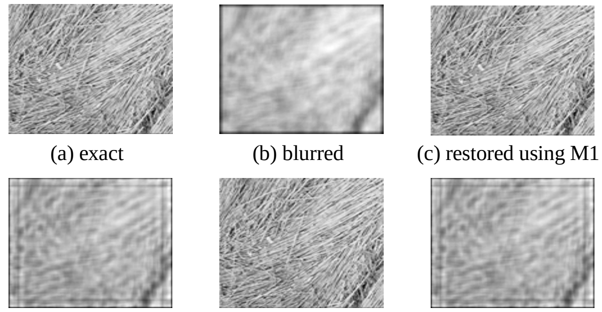
(a) exact (b) blurred (c) restored using M1
(d) restored using M2 (e) restored using M4 (f) restored using M5

Fig. 2: Results for the restoration of an image of grass.


(a) exact (b) blurred image, SNR = 2.21 dB (c) restored image, SNR = 47.66 dB

Fig. 3: The exact, blurred and restored images for the second example.

6. CONCLUSION

This paper has presented a method to perform BID using robust AGCD computations, such that neither the noise level nor the PSF need be known. It has been shown that excellent results are obtained when the rows and columns of the blurred image are preprocessed, and the PSF is computed using a structure-preserving matrix method. These computations enable the PSF to be computed, such that the deconvolution of the PSF from the blurred image corresponds to polynomial division with a very small error, that is, the error between the exact and restored images is also small.

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