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Differentiability of solutions of stationary Fokker–Planck–Kolmogorov equations with respect to a parameter

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We obtain sufficient conditions for the differentiability of solutions to stationary Fokker–Planck–Kolmogorov equations with respect to a parameter. In particular, this gives conditions for the differentiability of stationary distributions of diffusion processes with respect to a parameter.

Keywords Stationary Fokker–Planck–Kolmogorov equation; Differentiability with respect to a parameter.

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1. Introduction and main results

The goal of this paper is to give broad sufficient conditions for the differentiability of solutions to stationary Fokker–Planck–Kolmogorov equations

$$\partial_{x_i}\partial_{x_j}(a^{ij}_\alpha \mu_\alpha) - \partial_{x_i}(b^i_\alpha \mu_\alpha) = 0$$

with respect to a parameter. In particular, we obtain sufficient conditions for the differentiability of invariant measures of diffusion processes with respect to a parameter. Our conditions are expressed in terms of Lyapunov functions and apply to unbounded coefficients. The results of [18] and [23], where the problem was first studied, are generalized and reinforced in the case of one-fold differentiability: substantially broader assumptions about the coefficients are considered, the main novelty is that rapidly growing coefficients are allowed. Dependence of solutions on parameters, in particular, differentiability and continuity with respect to parameters, obviously belongs to questions of general interest, which are important both for the theory and diverse applications such as control theory (see, e.g., [1] and [17]). However, the case of equations on the whole space has not been studied in sufficient generality so far (except for the already cited pioneering papers [18] and [23], where the case of bounded coefficients was examined). The results of this paper are new even in the one-dimensional case. Our conditions become especially simple in the case where a^{ij}_α and $\partial_\alpha a^{ij}_\alpha$ are uniformly bounded and $\partial_{x_i} a^{ij}_\alpha$, $\partial_\alpha \partial_{x_i} a^{ij}_\alpha$, b^i_α, $\partial_\alpha b^i_\alpha$ have at most polynomial growth: just the relation $\lim\limits_{|x|\to\infty} \sup_\alpha \langle b_\alpha(x), x\rangle = -\infty$ for the drift coefficient b_α is needed. Some auxiliary results obtained below on solvability of non-homogeneous Fokker–Planck–Kolmogorov equations and related a priori estimates can be useful in other problems such as discrete approximations.

Let us explain our framework. Suppose first that we are given a single second order elliptic operator

$$L\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi,$$

where the usual summation with respect to repeated indices is meant, a^{ij} and b^i are real Borel functions on $\mathbb{R}^d$, and the matrix $A(x) = (a^{ij}(x))_{i,j\le d}$ is positive-definite for each x. We say that a bounded Borel measure μ satisfies the stationary Fokker–Planck–Kolmogorov equation

$$\partial_{x_i}\partial_{x_j}(a^{ij}\mu) - \partial_{x_i}(b^i\mu) = 0,$$

or, in a shorter form,

$$L^*\mu = 0 \tag{1.1}$$

on a domain Ω in $\mathbb{R}^d$ (in our main results $\Omega = \mathbb{R}^d$) if the coefficients a^{ij} and b^i are locally integrable in Ω with respect to the measure $|\mu|$ (which holds automatically for locally bounded coefficients) and we have the integral identity

$$\int L\varphi\,d\mu = 0 \quad \forall\,\varphi \in C_0^\infty(\Omega).$$

For example, this equation holds for stationary probabilities of the diffusion process governed by the stochastic equation

$$d\xi_t = \sqrt{2A(\xi_t)}dw_t + b(\xi_t)dt.$$

Suppose now that for every $\alpha \in [0,1]$ we are given a second order elliptic operator

$$L_\alpha\varphi = a_\alpha^{ij}\partial_{x_i}\partial_{x_j}\varphi + b_\alpha^i\partial_{x_i}\varphi$$

with coefficients satisfying certain conditions specified below. Suppose also that for each α there is a unique probability measure μ_α satisfying the stationary Fokker–Planck–Kolmogorov equation

$$L_\alpha^*\mu_\alpha = 0 \tag{1.2}$$

in the sense explained above. The goal of this paper is to provide broad sufficient conditions for the continuity and differentiability of μ_α and its density ϱ_α with respect to the parameter α. In particular, if there is a diffusion $\xi_{\alpha,t}$ with generator L_α and a stationary distribution μ_α, our results provide broad conditions for the continuity and differentiability of the density of μ_α with respect to the parameter α.

Recall that the Sobolev class $W^{p,1}(U)$ on a domain U in $\mathbb{R}^d$ consists of all functions $f \in L^p(U)$ having generalized derivatives $\partial_{x_i}f \in L^p(U)$ and is equipped with the Sobolev norm

$$\|f\|_{p,1} = \|f\|_p + \|\partial_{x_1}f\|_p + \cdots + \|\partial_{x_d}f\|_p,$$

where $\|\cdot\|_p$ denotes the L^p-norm. The class $C_b^k(\Omega)$ consists of functions on Ω with k bounded continuous derivatives and $C_b^\infty(\Omega)$ is the intersection of these classes.

It is known (see [4], [6]) that if for every ball U in Ω there exists a number $p = p(U) > d$ such that $a^{ij}|_U \in W^{p,1}(U)$, $b^i|_U \in L^p(U)$ and $\inf_U \det A > 0$, then any solution μ to equation (1.1) has a continuous density ϱ whose restriction to every ball U belongs to the Sobolev class $W^{p,1}(U)$ with the corresponding $p = p(U) > d$. Moreover, if $\mu \ge 0$ is not identically zero and Ω is connected, then $\varrho > 0$.

In this case the equation $L^*\mu = 0$ can be written as the equation

$$\partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho) = 0$$

for ϱ (understood in the sense of distributions) and further transformed into the divergence form equation

$$\operatorname{div}(A\nabla\varrho - (b - \operatorname{div}A)\varrho) = 0, \quad \operatorname{div}A = (\partial_{x_j}a^{1j},\dots,\partial_{x_j}a^{dj}).$$

There is a vast literature devoted to the theory of such equations, see, e.g., [16], [21], [22], and references in [6].

A sufficient condition for the existence of a probability solution to (1.1) on the whole space under the local assumptions mentioned above is the existence of a Lyapunov function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ and $LV(x) \leq -\kappa < 0$ outside of a compact set, see [10] or a somewhat weaker result in [7].

A sufficient condition for the uniqueness of a probability solution to (1.1) under the same local assumptions is the existence of a Lyapunov function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ and $LV(x) \leq qV(x)$ for some number $q \geq 0$, see [6], [9], and [12]. In particular, the existence condition above ensures also the uniqueness.

In case of coefficients depending on a parameter $\alpha \in [0,1]$, we need uniformity in α of the above conditions. Namely, we assume throughout that we deal with real coefficients a^{ij}_α and b^i_α on $\mathbb{R}^d$, Borel measurable in (x,α) and satisfying the following conditions:

the matrices $A_\alpha(x) = (a^{ij}_\alpha(x))_{i,j\leq d}$ are symmetric and for every ball $U \subset \mathbb{R}^d$ we have

$$\sup_\alpha \|a^{ij}_\alpha\|_{W^{p,1}(U)} \leq M_1(U) < \infty, \quad \sup_\alpha \|b^i_\alpha\|_{L^p(U)} \leq M_2(U) < \infty, \tag{1.3}$$

where $p = p(U) > d$, and for all x we have

$$A_\alpha(x) \geq c_0 \mathrm{I}, \quad c_0 > 0, \tag{1.4}$$

where I is the unit operator and c_0 is a constant (independent of U).

Unlike the case of a boundary value problem on a bounded domain with a nice boundary, where the differentiability of solutions with respect to a parameter under our basic assumptions follows relatively easily from suitable a priori estimates and compactness of embeddings (see, e.g., [14, Chapter X, Section 5, Theorem 15, Chapter III, Section 6]), the case of the whole space is more subtle and much less studied. Already in the one-dimensional case with $A_\alpha = 1$ (where a probability solution is unique) and smooth $b_\alpha(x)$ the continuity of the density in α can fail (see Example 1.8).

The lack of compactness will be compensated by suitable Lyapunov functions. The concept of uniform tightness of families of measures will be useful.

Recall that a family $\mathcal{M}$ of probability measures is uniformly tight if, for each $r > 0$, there is a compact set K such that $\mu(\mathbb{R}^d \backslash K) \leq r$ for all $\mu \in \mathcal{M}$. A necessary and sufficient condition for the uniform tightness is the existence of a locally bounded Borel function $W \geq 0$ such that $\lim\limits_{|x|\to\infty} W(x) = +\infty$ and

$$\sup_{\mu\in\mathcal{M}} \int_{\mathbb{R}^d} W\, d\mu < \infty.$$

The case of continuity is much easier and here we have the following result (in which (1.4) is replaced by a local bound).

Proposition 1.1. *Suppose that* (1.3) *holds,* $\inf_{\alpha, x\in U} \det A_\alpha(x) > 0$ *for every ball* U *and that the family of measures* μ_α (*that are unique probability solutions to the corresponding equations* (1.2)) *is uniformly tight. Assume also that, for every ball* U*, the restrictions of* a^{ij}_α *and* b^i_α *to* U *are continuous in* α *in the space* $L^1(U)$*. Then, one can choose densities* ϱ_α *of* μ_α *such that the function* $\varrho_\alpha(x)$ *will be jointly continuous. In* *addition, the mapping $\alpha \mapsto \varrho_\alpha$ with values in $L^1(\mathbb{R}^d)$ is continuous, i.e., the mapping $\alpha \mapsto \mu_\alpha$ is continuous in the variation norm.*

A sufficient condition for the uniform tightness of the measures μ_α is the existence of a single Lyapunov function V such that $V(x) \to +\infty$ and $\sup_\alpha L_\alpha V(x) = -\infty$ as $|x| \to \infty$. Certainly, this condition ensures also the existence and uniqueness of solutions. In order to have the local continuity in L^1 it is enough to have the usual continuity of the coefficients in α along with their uniform integrability on balls.

The case of differentiability is much harder and requires some auxiliary results presented in the next section.

Recall that a mapping $\alpha \mapsto f_\alpha$ from $(0,1)$ to $L^p(U)$ is differentiable if there is a mapping $\alpha \mapsto g_\alpha$ from $(0,1)$ to $L^p(U)$ such that $(f_{\alpha+s} - f_\alpha)/s \to g_\alpha$ in $L^p(U)$ as $s \to 0$ for each fixed $\alpha \in (0,1)$. If $\alpha \mapsto g_\alpha$ is continuous, then f_α is said to be continuously differentiable in L^p.

Suppose that for every ball U there is a number $p_0 = p_0(U) > d$ such that the mappings

$$\alpha \mapsto a_\alpha^{ij}|_U \text{ and } \alpha \mapsto b_\alpha^i|_U \quad \text{are continuously differentiable in } L^{p_0}(U). \tag{1.5}$$

Note that this condition is fulfilled if, in addition to (1.3), the functions a_α^{ij}, $\partial_{x_k} a_\alpha^{ij}$ and b_α^i are differentiable in α and their derivatives in α are continuous in α and locally bounded in both variables.

Set

$$A_\alpha^i := (a_\alpha^{i1}, \ldots, a_\alpha^{id}), \quad \operatorname{div} A_\alpha := (\operatorname{div} A_\alpha^1, \ldots, \operatorname{div} A_\alpha^d),$$
$$B_\alpha := \partial_\alpha b_\alpha = (\partial_\alpha b_\alpha^1, \ldots, \partial_\alpha b_\alpha^d),$$
$$S_\alpha := \partial_\alpha A_\alpha = (\partial_\alpha a_\alpha^{ij})_{i,j \le d}, \quad R_\alpha^i = \partial_\alpha \partial_{x_j} a_\alpha^{ij}, \quad R_\alpha := (R_\alpha^1, \ldots, R_\alpha^d) = \operatorname{div} S_\alpha.$$

We assume that

$$\sup_{\alpha, x} \|S_\alpha(x)\| \le \lambda_0 < \infty. \tag{1.6}$$

This condition is obviously fulfilled if $A_\alpha(x)$ does not depend on α or is uniformly Lipschitzian in α.

Condition (1.6) implies that in (1.5) actually a stronger condition on the diffusion coefficient is fulfilled: the functions $\alpha \mapsto a_\alpha^{ij}|_U$ are continuously differentiable in every $L^{p_1}(U)$ with $p_1 < \infty$. In particular, we can take $p_1 = p_1(U) > \dfrac{dp}{p-d}$, where $p = p(U)$ is the number from (1.3).

Our main theorem is this.

Theorem 1.2. *Let* (1.3), (1.4), (1.5), *and* (1.6) *hold. Suppose that $V \in C^2(\mathbb{R}^d)$ and W is a locally integrable function such that*

$$\lim_{|x|\to\infty} V(x) = +\infty, \quad \lim_{|x|\to\infty} W(x) = +\infty, \quad \sup_\alpha L_\alpha V(x) \le -W(x) \text{ if } |x| \ge R \tag{1.7}$$

for some $R > 0$. Assume also that for some numbers $C_V > 0, m \ge 1$ we have

$$\sup_\alpha \Bigl(|A_\alpha^{-1/2}(b_\alpha - \operatorname{div} A_\alpha)|^2 + |A_\alpha^{-1/2}(\partial_\alpha b_\alpha - \partial_\alpha \operatorname{div} A_\alpha)|^2 + |L_\alpha V| \Bigr) \le C_V + C_V V^m W, \tag{1.8}$$

Finally, assume that for some $\varepsilon < 1/(4m+1)$ there is a ball outside of which

$$\sup_\alpha \langle A_\alpha \nabla V, \nabla V \rangle \le \varepsilon V W. \tag{1.9}$$

Then $\partial_\alpha \varrho_\alpha$ *exists and for each* $\alpha \in (0,1)$ *satisfies the equation*

$$L_\alpha^* \partial_\alpha \varrho_\alpha = \operatorname{div}(B_\alpha \varrho_\alpha - R_\alpha \varrho_\alpha - S_\alpha \nabla \varrho_\alpha). \tag{1.10}$$

In addition, the mapping $\alpha \mapsto \varrho_\alpha$ *with values in* $L^1(\mathbb{R}^d)$ *is differentiable.*

Finally, if the diffusion matrix A *does not depend on* α*, then* (1.8) *can be replaced by the simpler condition* $\sup_\alpha(|A^{-1/2}\partial_\alpha b_\alpha|^2 + |L_\alpha V|) \le C_V + C_V V^m W$.

Thus, the theorem employs seven conditions (1.3)–(1.9) (or four global conditions (1.6)–(1.9) once we fix our local assumptions), but if $A_\alpha = \mathrm{I}$, $|b_\alpha|$ and $|\partial_\alpha b_\alpha|$ have polynomial bounds, then, by taking $V(x) = |x|^2$, it suffices to have only one condition that $\lim\limits_{|x|\to\infty} \sup_\alpha \langle b_\alpha(x), x\rangle = -\infty$.

Let us briefly comment on the hypotheses of this theorem.

Remark 1.3. (i) As explained above, condition (1.7) ensures the existence and uniqueness of probability solutions to (1.2) for each α. It also ensures the uniform boundedness of the integrals of W with respect to the measures μ_α; moreover, in Lemma 2.2 we shall see that for each $k < 4m+1$ the integrals of $V^k W$ against μ_α are uniformly bounded. It is worth noting that, as shown in [11], the existence of a certain Lyapunov function of class $W^{d,2}_{loc}(\mathbb{R}^d)$ is necessary for the existence of a probability solution μ to (1.1) such that $|a^{ij}(x)|/(1+|x|^2), |b^i(x)|/(1+|x|)$ are μ-integrable.

(ii) Note also that if A is constant (independent of α) and nondegenerate, then (1.4) and (1.6) are fulfilled (along with the first condition in (1.3)) and $R_\alpha = S_\alpha = 0$.

(iii) If A_α is Lipschitzian in α, then (1.8) implicitly yields that b_α is locally bounded outside of some ball, since, on every bounded set where $\sup_\alpha L_\alpha V \le -W$, the right-hand side of (1.8) is dominated by $C + C\sup_\alpha |b_\alpha|$, while the left-hand side dominates a multiple of $\sup_\alpha |b_\alpha|^2$. However, the last assertion of the theorem allows locally unbounded drifts in the case of the diffusion matrix independent of α.

(iv) It follows from (1.4) that (1.8) is ensured by the estimate

$$\sup_\alpha \Bigl(|b_\alpha - \operatorname{div} A_\alpha|^2 + |\partial_\alpha b_\alpha - \partial_\alpha \operatorname{div} A_\alpha|^2 + |L_\alpha V|\Bigr) \le C_V + C_V V^m W.$$

However, for growing diffusion coefficients the operators $A_\alpha^{-1/2}$ in (1.8) can help. Certainly, for uniformly bounded A_α both estimates are equivalent.

Let us briefly explain the idea of our proof. Given a sequence $h_k \to 0$, we consider the differences $\delta_k \varrho = (\varrho_\alpha - \varrho_{\alpha-h_k})/h_k$ and observe that they satisfy non-homogeneous equations

$$L_\alpha^* \delta_k \varrho = \operatorname{div} F_k$$

with certain vector fields F_k. It would be nice to obtain some uniform bounds on these solutions and their appropriate convergence. It turns out that our rather general assumptions about the coefficients do not allow to justify this procedure directly (at least, we have not managed to do this), which leads to an additional technical step at which the above plan is realized for less general coefficients. However, an appropriate approximation brings our proof to the end. This plan requires a preliminary study of the above non-homogeneous equation, which has already been investigated in [5], however, here we obtain new existence results for this equation along with certain a priori estimates that can be useful for other purposes.

Immediate examples are cases with uniformly elliptic diffusion matrices and polynomial or exponential bounds on the drift coefficients possessing a sufficient dissipativity. In these examples, rather technical conditions (1.8) and (1.9) are easily verified.

Corollary 1.4. *Suppose that A_α, A_α^{-1} and $\partial_\alpha A_\alpha$ are uniformly bounded,* (1.5) *holds and that for all i, j, l we have*

$$|\partial_{x_l} a_\alpha^{ij}(x)| + |\partial_\alpha \partial_{x_l} a_\alpha^{ij}(x)| + |b_\alpha^i(x)| + |\partial_\alpha b_\alpha^i(x)| \le C + C|x|^k \quad \forall\, x, \alpha$$

for some constants C and k. Assume also that

$$\lim_{|x|\to\infty} \sup_\alpha \langle b_\alpha(x), x\rangle = -\infty.$$

Then $\varrho_\alpha(x)$ is differentiable in α and $\partial_\alpha \varrho_\alpha(x)$ satisfies the equation indicated in the theorem.

Proof. Let us take $V(x) = |x|^2$ and $W(x) = -\sup_\alpha \langle x, b_\alpha(x)\rangle$. Then

$$L_\alpha V(x) = 2\mathrm{trace}\, A_\alpha(x) + 2\langle x, b_\alpha(x)\rangle \le -W(x)$$

outside of some ball. Clearly, for each $\varepsilon > 0$ outside of some ball we also have

$$\langle A_\alpha(x)\nabla V(x), \nabla V(x)\rangle = 4\langle A_\alpha(x)x, x\rangle \le \varepsilon |x|^2 W(x).$$

In addition, there is a number C_1 such that

$$|L_\alpha V(x)| \le C_1 + C_1|x|^{2k+1}.$$

Therefore, all hypotheses of the theorem are satisfied (with $m = k + 1/2$). $\square$

Corollary 1.5. *Suppose that the operator norms of A_α, A_α^{-1} and $\partial_\alpha A_\alpha$ are uniformly bounded, condition* (1.5) *holds and that for all i, j, l we have*

$$|\partial_{x_l} a_\alpha^{ij}(x)| + |\partial_\alpha \partial_{x_l} a_\alpha^{ij}(x)| + |b_\alpha^i(x)| + |\partial_\alpha b_\alpha^i(x)| \le C \exp(q|x|^\beta) \quad \forall\, x, \alpha$$

for some positive numbers C, q, and β. Assume also that there is a number

$$\gamma > (9 \sup_{\alpha,x} \|A_\alpha(x)\| + 1/4) q\beta$$

such that outside of some ball we have

$$\sup_\alpha \langle b_\alpha(x), x\rangle \le -\gamma |x|^\beta.$$

Then $\varrho_\alpha(x)$ is differentiable in α and $\partial_\alpha \varrho_\alpha(x)$ satisfies the equation indicated in the theorem.

Proof. Let us take $V(x) = \exp(q|x|^{2s})$, $s = \beta/2$, $c = \sup_{\alpha,x} \|A_\alpha(x)\|$. We have $V(x) = f(V_0(x))$, where $V_0(x) = |x|^2$, $f(u) = \exp(qu^s)$. Hence

$$f'(u) = qsu^{s-1} f(u), \quad f''(u) = qs(s-1)u^{s-2} f(u) + q^2 s^2 u^{2s-2} f(u),$$

which gives the equality

$$\begin{aligned} L_\alpha V(x) &= qs\langle x, x\rangle^{s-1} V(x) L_\alpha V_0(x) \\ &\quad + 4\Big(qs(s-1)\langle x, x\rangle^{s-2} V(x) + q^2 s^2 \langle x, x\rangle^{2s-2} V(x)\Big)\langle A_\alpha(x)x, x\rangle \\ &= qs\langle x, x\rangle^{s-1} V(x)\Big(L_\alpha V_0(x) + 4(s-1)\langle A_\alpha(x)x/|x|, x/|x|\rangle + qs\langle x, x\rangle^s\Big). \end{aligned}$$

Therefore, once $qs < 2\gamma$, the right-hand side is dominated outside of some ball by the function

$$W(x) := -\kappa\langle x, x\rangle^{2s-1}V(x) = -\kappa\langle x, x\rangle^{\beta-1}V(x), \quad \text{where} \quad \kappa = qs(2\gamma - qs).$$

On the other hand, for each $\delta > 0$ there is $C_\delta > 0$ such that

$$|L_\alpha V(x)| \le C_\delta + C_\delta V^{2+\delta}(x),$$

since $|L_\alpha V_0(x)| \le 2\operatorname{trace} A_\alpha(x) + 2C|x|V(x)$. Finally,

$$|A_\alpha^{1/2}\nabla V(x)|^2 \le 4q^2s^2|x|^{2\beta-2}V^2(x) \le \varepsilon V(x)W(x)$$

outside of a sufficiently large ball depending on a given $\varepsilon < (4m+1)^{-1}$, where $m = 2+\delta$ and $\delta > 0$ is small enough so that $4cqs < \varepsilon(2\gamma - qs)$; such a choice is possible, since $4cqs < (2\gamma - qs)/9$ due to the estimate $\gamma > (9c + 1/4)q\beta$. □

Example 1.6. Let $A = \mathrm{I}$, $b(x) = -x + h_\alpha(x)$, where $\sup_{\alpha,x} |h_\alpha(x)| < \infty$, $h_\alpha(x)$ is continuously differentiable in α, and $|\nabla h_\alpha(x)| \le C\exp(q|x|^2)$, $q < 1/20$. Then probability solutions μ_α to the corresponding equations (1.2) exist, are unique and have densities ϱ_α differentiable in α.

Example 1.7. (The case considered in [18] and [23].) Let the coefficients $a_\alpha^{ij}(x)$ and $b_\alpha^i(x)$ be of class C_b^1 in both variables, let $A_\alpha^{-1}(x)$ be uniformly bounded, and let $\sup_\alpha\langle b_\alpha(x), x\rangle \to -\infty$ as $|x| \to \infty$. Then $\varrho_\alpha(x)$ is continuously differentiable in both variables.

Note that in applications of these results to stationary distributions of diffusions governed by stochastic equations $d\xi_{\alpha,t} = \sigma_\alpha(\xi_{\alpha,t})dw_t + b_\alpha(\xi_{\alpha,t})dt$ the hypotheses must be checked for the matrices $A_\alpha = \sigma_\alpha\sigma_\alpha^*/2$.

Let us consider examples showing that certain additional assumptions, besides smoothness of the coefficients, are needed to guarantee even the continuity of densities with respect to the parameter.

Example 1.8. One can find a bounded function $b_\alpha(x)$, $(x, \alpha) \in \mathbb{R}\times\mathbb{R}$, of class C^∞ in both variables such that the integral

$$J_\alpha = \int_{-\infty}^{+\infty} \exp\int_0^x b_\alpha(y)\,dy\,dx$$

exists, but is not continuous at $\alpha = 0$. It is not difficult to give explicit examples of such functions; it suffices to take a positive integrable smooth function g such that g'/g is bounded (say, $(1 + x^2)^{-1}$) and set $g(\alpha, x) = g(x) + \alpha g(\alpha x)$; in this case the integral in x is not continuous in α at the origin. Then the probability density

$$\varrho_\alpha(x) = J_\alpha^{-1}\exp\int_0^x b_\alpha(y)\,dy, \quad b_\alpha(x) = \partial_x g(\alpha, x)/g(\alpha, x),$$

satisfies the equation $\varrho_\alpha'' - (b_\alpha\varrho_\alpha)' = 0$, but $\varrho_\alpha(x)$ is discontinuous in α at $\alpha = 0$ for all x. A bit more involved example (see the next example) provides bounded $b_\alpha(x)$ that is Lipschitzian in α (in the example above $\partial_\alpha b_\alpha(x)$ is not uniformly bounded). It is also worth noting that if we consider our equation with a parameter as an equation with an extra variable (or pass to a system of equations), then we obtain a degenerate equation.

Example 1.9. Let us give an explicit example (suggested by I.S. Yaroslavtsev) of a uniformly bounded Lipschitzian function $b_\alpha(x)$ (in particular, with bounded $\partial_\alpha b_\alpha(x)$) such that the probability solution $\varrho_\alpha(x)$ to the corresponding equation $L_\alpha^*(\varrho_\alpha dx) = 0$ is not continuous in α.

Set $b_\alpha(x) = -(x+1)/(|x|+1) + \alpha$ if $x < 0$, $\alpha \in [0,1)$ and define $b_\alpha(x)$ for $x \geq 0$, $\alpha \in [0,1)$ as follows. Let $b_0(x) = -(x+1)^{-1/2}$. Set $\varphi_1(x) = 2(x+1)^{-1/2}$ and $\varphi_2(x) = 2\varphi_1(x)$.

On the domain $\alpha \leq \varphi_1(x)$, that is, $0 \leq x \leq 4\alpha^{-2} - 1$, we set

$$b_\alpha(x) = -(x+1)^{-1/2} + \alpha.$$

On the curve $(x, \varphi_1(x))$ our function equals $(x+1)^{-1/2}$. Let us observe that

$$\begin{aligned}\int_0^{4\alpha^{-2}-1} \exp \int_0^x b_\alpha(y)\,dy\,dx &= \int_0^{4\alpha^{-2}-1} \exp\bigl(\alpha x - 2(x+1)^{1/2}\bigr)\,dx \\ &\geq e^{-4/\alpha} \int_0^{4\alpha^{-2}-1} e^{\alpha x}\,dx = \frac{1}{\alpha}(e^{-\alpha} - e^{-4/\alpha}),\end{aligned}$$

which tends to $+\infty$ as $\alpha \to 0$. This yields that, independently of how we define b on the remaining domain, the integral J_α introduced in the previous example tends to $+\infty$ as $\alpha \to 0$. Therefore, the density ϱ_α defined by the expression in that example tends to zero as $\alpha \to 0$, which ensures the desirable discontinuity at $\alpha = 0$.

Finally, on the domain $\varphi_1(x) < \alpha < \varphi_2(x)$ we set

$$b_\alpha(x) = -(x+1)^{-1/2} - (\alpha - \varphi_2(x))$$

and on the domain $\alpha \geq \varphi_2(x)$ we set $b_\alpha(x) = -(x+1)^{-1/2}$. It is clear that $|b_\alpha(x)| \leq 2$, b is Lipschitzian separately in x and in α, hence is Lipschitzian in both variables, and $|\partial_\alpha b_\alpha(x)| \leq 1$, more precisely, in the interiors of the domains bounded by the two curves defined above $\partial_\alpha b_\alpha(x)$ is 1, -1 and 0, respectively (and is 1 for $x < 0$). For $\alpha < 0$ we set $b_\alpha = b_{|\alpha|}$. The corresponding solution $\varrho_\alpha(x)$ is discontinuous at $\alpha = 0$, as explained above. This property can be retained by smoothing b and making it differentiable in α everywhere with uniformly bounded partial derivatives $\partial_\alpha b_\alpha(x)$ and $\partial_x b_\alpha(x)$.

It is instructive to see which conditions of the theorem cannot be ensured in this example. Here Corollary 1.4 almost applies with $V(x) = x^2$ and for any fixed α we have $b_\alpha(x)x < -|x|^{1/2}/2$ outside of some interval, depending on α, but there is no uniformity in α.

2. AUXILIARY RESULTS

A useful fact employed below is that in the case where $LV(x) \leq -1$ outside of a ball and μ is a probability solution to the equation $L^*\mu = 0$, we have $|LV| \in L^1(\mu)$. Actually, the following is true (see [6]): if

$$LV \leq \Psi - \Phi,$$

where Ψ and Φ are Borel functions such that $\Psi \in L^1(\mu)$ and $\Phi \geq 0$, then

$$\int_{\mathbb{R}^d} \Phi\,d\mu \leq \int_{\mathbb{R}^d} \Psi\,d\mu. \tag{2.1}$$

It will be important below that if a function u on a domain Ω satisfies the equation

$$\partial_{x_i}(a^{ij}\partial_{x_j}u)+\partial_{x_i}(b^i u)=\operatorname{div}G,$$

where $G=(G^i)$ is a measurable vector field and

$$\|a^{ij}\|_{W^{p,1}(\Omega)}+\|b^i\|_{L^p(\Omega)}+\sup_{\Omega}|\det(a^{ij})|^{-1}\le K$$

with some $p>d$, then for every ball U with compact closure in Ω there is a constant $C(p,K,U,\Omega)$ that depends only on p, K, U and the distance from U to the boundary of Ω such that

$$\|u\|_{W^{p,1}(U)}\le C(p,K,U,\Omega)\Bigl(\|u\|_{L^1(\Omega)}+\|G\|_{L^p(\Omega)}\Bigr). \tag{2.2}$$

The Sobolev embedding theorem yields also a bound

$$\sup_U|u|\le C'(p,K,U,\Omega)\Bigl(\|u\|_{L^1(\Omega)}+\|G\|_{L^p(\Omega)}\Bigr), \tag{2.3}$$

where C' depends on the same objects as C. In particular, having a family of solutions to different equations with a common bound K, we obtain the uniform boundedness in the Sobolev norm on any inner ball, provided we have their uniform boundedness in L^1 on a slightly larger ball along with a common bound for the L^p-norms of the right-hand sides on that larger ball. A detailed proof can be found, e.g., in [20].

If $G=0$ and $u\ge 0$ in Ω, then, according to Harnack's inequality,

$$\sup_{x\in U}u(x)\le H(K,U,\Omega)\inf_{x\in U}u(x), \tag{2.4}$$

where the number $H(K,U,\Omega)$ depends only on p, K, U and the distance from U to the boundary of Ω.

Lemma 2.1. *Suppose that* (1.3) *holds, the family* $\{\mu_\alpha\}$ *is uniformly tight, and, for each closed ball* U*, we have* $\inf_{\alpha,x\in U}\det A_\alpha(x)>0$ *and the mappings* $\alpha\mapsto a^{ij}_\alpha|_U$ *and* $\alpha\mapsto b^i_\alpha|_U$ *with values in* $L^1(U)$ *are continuous. Then, for every ball* U*, the continuous versions of the densities* ϱ_α *satisfy the estimate*

$$\inf_\alpha\min_{x\in U}\varrho_\alpha(x)\ge m(U)>0,$$

where $m(U)$ *does not depend on* α.

Proof. Suppose that there is a sequence $\alpha_n\to\alpha$ in $[0,1]$ for which $\min_{x\in U}\varrho_{\alpha_n}(x)\to 0$. It follows by (2.4) and (2.2) that passing to a subsequence we can assume that the functions ϱ_{α_n} converge locally uniformly to some function ϱ. By the uniform tightness, we have also convergence in $L^1(\mathbb{R}^d)$ and ϱ is a probability density. It is readily seen that $L^*_\alpha\varrho=0$, since $L_{\alpha_n}\varphi\to L_\alpha\varphi$ in $L^1(\mathbb{R}^d)$ for each smooth φ with compact support. Hence ϱ is positive by Harnack's inequality, which leads to a contradiction. □

We need also the following a priori estimate for a probability solution μ of the equation $L^*\mu=0$.

Lemma 2.2. *Let* $k\ge 1$. *Suppose that*

$$LV\le -W \quad \textit{and} \quad \langle A\nabla V,\nabla V\rangle\le\varepsilon VW$$

outside of some compact set S_0*, where* $0\le\varepsilon<k^{-1}$. *Then*

$$\int_{\mathbb{R}^d\backslash S_0}V^kW\,d\mu\le (k+1)^{-1}(1-k\varepsilon)^{-1}\int_{S_0}|LV^{k+1}|\,d\mu. \tag{2.5}$$

Proof. Let us consider the function $V_0 = V^{k+1}$. We have

$$LV_0 = (k+1)V^k LV + k(k+1)V^{k-1}\langle A\nabla V, \nabla V\rangle \le -(k+1)(1-k\varepsilon)V^k W$$

outside of S_0. Hence we can apply estimate (2.1) with functions $\Psi = |LV^{k+1}|I_{S_0}$ and $\Phi = (k+1)(1-k\varepsilon)V^k W I_{\mathbb{R}^d\setminus S_0}$. $\square$

Once a probability solution μ to the equation $L^*\mu = 0$ exists, it satisfies (under our local assumptions about A and b, see (1.3) and (1.4)) the following estimate (see [5]):

$$\int_{\mathbb{R}^d}\Bigl|\frac{A^{1/2}\nabla\varrho}{\varrho}\Bigr|^2\,d\mu \le \int_{\mathbb{R}^d}|A^{-1/2}(b - \operatorname{div} A)|^2\,d\mu, \quad \operatorname{div} A := (\partial_{x_j}a^{1j}, \dots, \partial_{x_j}a^{1j}), \tag{2.6}$$

provided the right-hand side is finite and

$$\liminf_{r\to\infty}\int_{r\le|x|\le 2r}\Bigl[r^{-2}|a^{ij}| + r^{-1}|\partial_{x_k}a^{ij}|\Bigr]\,d\mu = 0. \tag{2.7}$$

The last assumption is fulfilled, e.g., if the mapping A is Lipschitzian or, more generally, if the functions $|a^{ij}(x)|/(1+|x|^2)$ and $|\partial_{x_k}a^{ij}(x)|/(1+|x|)$ are μ-integrable. In particular, this condition is satisfied if a^{ij} and $\partial_{x_k}a^{ij}$ are μ-integrable on the whole space. However, it is not known whether (2.6) is satisfied for all solutions without the extra assumption (2.7). For this reason, we show in the next lemma that in the presence of a suitable Lyapunov function even without (2.7) there is a unique probability solution satisfying (2.6).

Lemma 2.3. *Suppose that the coefficients a^{ij} and b^i satisfy our local assumptions (see (1.3) and (1.4)) and there is a function $V \in C^2(\mathbb{R}^d)$ such that $\lim\limits_{|x|\to\infty} V(x) = +\infty$ and outside of some ball*

$$LV(x) \le -1, \quad \psi(|x|)|A^{-1/2}(b - \operatorname{div} A)(x)|^2 \le |LV(x)|,$$

where ψ is a locally bounded Borel function on $[0,+\infty)$ with $\lim\limits_{t\to+\infty}\psi(t) = +\infty$. Then there is a unique probability solution μ to the equation $L^*\mu = 0$ such that (2.6) holds provided the right-hand side is finite.

Proof. It is known that for almost every $t \in \mathbb{R}$ the compact set $U_t = \{V \le t\}$ has boundary of finite perimeter (see [15, Section 5.5] and [24, Chapter 5]; certainly, if we had $V \in C^d(\mathbb{R}^d)$, then by Sard's theorem $V^{-1}(t)$ would be a C^1-surface for almost each t, but we do not assume such a regularity of V). Hence there is an increasing sequence $t_n \to +\infty$ of points with this property. Set $U_n = U_{t_n}$. Let

$$f^i = b^i - \partial_{x_j}a^{ij}, \quad h = \partial_{x_i}f^i.$$

According to [21] (see also [22]), for each n, there is a solution $w_n \in W_0^{2,1}(U_n)$ to the Dirichlet problem

$$\partial_{x_i}(a^{ij}\partial_{x_j}w_n) - \partial_{x_i}(f^i w_n) = h,$$

where $W_0^{2,1}(U_n)$ is the closure of $C_0^\infty(U_n)$ in $W^{2,1}(U_n)$. Therefore, the function

$$\varrho_n := w_n + 1$$

satisfies the homogeneous equation $L^*\varrho_n = 0$ in U_n and the boundary condition $\varrho_n|_{\partial U_n} = 1$ in the sense that $\varrho_n - 1 \in W_0^{2,1}(U_n)$. Let us observe that it follows from [22] (Theorem 2 applies with $\gamma = 0$) that $\varrho_n \ge 0$ and consequently by Harnack's inequality $\varrho_n > 0$ in U_n. Indeed, the hypotheses of [22] are satisfied due to our choice of

t_n which makes possible to use the Gauss–Green formula for U_n (see [24, Section 5.8]). Let us normalize our solutions in such a way that ϱ_n becomes a probability density on U_n for all n. Then, due to the existence of a Lyapunov function, by a standard procedure (see, e.g., [6] and [7]), one can select a subsequence in $\{\varrho_n\}$ that locally uniformly converges to a probability solution ϱ of the equation $L^*\varrho = 0$. It is also known that in this situation there is a number M such that

$$\int_{U_n} |LV|\varrho_n\,dx \le M \quad \forall\, n. \tag{2.8}$$

This can be derived from (2.1) applied to $\Psi = |LV|I_{U_{n_1}}$ and $\Phi = |LV|I_{\mathbb{R}^d\setminus U_{n_1}}$ for a suitable number n_1.

Finally, we verify (2.6) for this particular solution. To this end, we multiply the equation for ϱ_n by $\log\varrho_n - c_n$, where c_n is the constant boundary value of ϱ_n (obtained after normalization, so that it need not be 1 anymore), and integrate by parts (which is possible due to the above choice of U_n) obtaining the equality

$$\begin{aligned}\int_{U_n} \langle A\nabla\varrho_n, \varrho_n^{-1}\nabla\varrho_n\rangle\,dx &= \int_{U_n} \langle b - \operatorname{div} A, \nabla\varrho_n\rangle\,dx\\ &= \int_{U_n} \langle A^{-1/2}(b - \operatorname{div} A), \varrho_n^{-1}A^{1/2}\nabla\varrho_n\rangle\varrho_n\,dx,\end{aligned}$$

where we used that $\nabla(\log\varrho_n - c_n) = \nabla\varrho_n/\varrho_n$ and canceled ϱ_n where possible. Applying the Cauchy inequality to the right-hand side, we arrive at the uniform estimate

$$\int_{U_n} \Bigl|\frac{A^{1/2}\nabla\varrho_n}{\varrho_n}\Bigr|^2 \varrho_n dx \le \int_{U_n} |A^{-1/2}(b - \operatorname{div} A)|^2\,\varrho_n dx. \tag{2.9}$$

Let us show that

$$\lim_{n\to\infty}\int_{U_n} |A^{-1/2}(b - \operatorname{div} A)|^2\,\varrho_n dx = \int_{\mathbb{R}^d} |A^{-1/2}(b - \operatorname{div} A)|^2\,\varrho dx. \tag{2.10}$$

Let $\varepsilon > 0$. Take $R > 0$ such that $|A^{-1/2}(b - \operatorname{div} A)(x)|^2 \le \varepsilon|LV(x)|$ whenever $|x| > R$. Then

$$\int_{|x|>R} |A^{-1/2}(b - \operatorname{div} A)(x)|^2\varrho_n(x)\,dx \le \varepsilon M.$$

By Fatou's theorem the same is true for ϱ in place of ϱ_n. Since $\varrho_n \to \varrho$ locally uniformly, we obtain equality (2.10). For every smooth compactly supported vector field v we have

$$\int_{\mathbb{R}^d} \Bigl\langle \frac{\nabla\varrho}{\varrho}, v\Bigr\rangle \varrho\,dx = \lim_{n\to\infty}\int_{U_n} \Bigl\langle \frac{\nabla\varrho_n}{\varrho_n}, v\Bigr\rangle \varrho_n\,dx,$$

since the left-hand side is the integral of $-\varrho\operatorname{div} v$, which is the limit of the integrals of $-\varrho_n\operatorname{div} v$. Combined with (2.9) and (2.10) this yields (2.6). Finally, as noted above, the uniqueness of a probability solution follows from the estimate $LV \le -1$ outside of a ball. □

Having an operator L satisfying the same local assumptions as L_α (see (1.3) and (1.4)), let us consider the equation

$$L^*w = \operatorname{div}(\varrho F), \tag{2.11}$$

where ϱ is a probability solution of the equation $L^*\varrho = 0$ and F is a Borel vector field such that $|A^{-1/2}F|^2\varrho \in L^1(\mathbb{R}^d)$. We arrive at this equation by formally differentiating (1.2) in α.

Writing $w = v\varrho$, we obtain the following equation on v:

$$\operatorname{div}(\varrho A\nabla v) + \operatorname{div}(vb_0) = \operatorname{div}(\varrho F), \quad b_0^i = \partial_{x_j}(a^{ij}\varrho) - b^i\varrho. \tag{2.12}$$

Let us observe that

$$\operatorname{div}(vb_0) = \langle \nabla v, b_0\rangle.$$

Indeed,

$$\operatorname{div} b_0 = 0$$

due to the equality $\partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho) = 0$. Therefore, (2.12) can be rewritten as

$$\operatorname{div}(\varrho A\nabla v) + \langle \nabla v, b_0\rangle = \operatorname{div}(\varrho F). \tag{2.13}$$

In the next section we use the results of this section on equation (2.11) in the situation where $\varrho = \varrho_\alpha$ and

$$F = B_\alpha - R_\alpha - S_\alpha \frac{\nabla\varrho_\alpha}{\varrho_\alpha},$$

$$B_\alpha = (\partial_\alpha b_\alpha^1, \dots, \partial_\alpha b_\alpha^d), \quad R_\alpha = (R_\alpha^1, \dots, R_\alpha^d), \quad R_\alpha^i = \partial_\alpha \partial_{x_j} a_\alpha^{ij}, \quad S_\alpha = (\partial_\alpha a_\alpha^{ij})_{i,j\le d}.$$

Note that vector fields F of such a form appear in the equations satisfied by the derivatives $\partial_\alpha\varrho_\alpha$.

Proposition 2.4. *Suppose that v is a solution to* (2.13) *on the domain $\Omega = \{V < R\}$, where $V \in C^2(\Omega)$ is a nonnegative function such that there exist a measurable function $W \ge 1$, a measurable function $\Psi \ge 0$ and a number $R_0 \in (0, R)$ such that*

$$LV(x) \le \Psi(x) - W(x) \quad \textit{if } V(x) \ge R_0.$$

Then

$$\begin{aligned}\int_{R_0<V<R} v^2 W\varrho\,dx \le{}& 2\int_{V<R_0} v^2|LV|\varrho\,dx + 6R\int_{V<R}|A^{1/2}\nabla v|^2\varrho\,dx \\ &+ 2\int_{R_0<V<R} v^2\Psi\varrho\,dx + 2R\int_{V<R}|A^{-1/2}F|^2\varrho\,dx \\ &+ 4\int_{V<R}|A^{-1/2}F|^2|A^{1/2}\nabla V|^2 W^{-1}\varrho\,dx.\end{aligned} \tag{2.14}$$

If $|A^{1/2}\nabla V|^2 \le C_V V W + C_V$ with some number $C_V \ge 1$, then

$$\begin{aligned}\int_{R_0<V<R} v^2 W\varrho\,dx \le{}& 2\int_{V<R_0} v^2|LV|\varrho\,dx + 6R\int_{V<R}|A^{1/2}\nabla v|^2\varrho\,dx \\ &+ 2\int_{R_0<V<R} v^2\Psi\varrho\,dx + C_V(6R+1)\int_{V<R}|A^{-1/2}F|^2\varrho\,dx.\end{aligned} \tag{2.15}$$

Proof. We multiply equation (2.13) by $v\psi$, where $\psi \in C_0^\infty(\Omega)$, integrate by parts (which is possible due to our assumptions about the coefficients yielding the Sobolev regularity of all solutions) and obtain the equality

$$\int_\Omega |A^{1/2}\nabla v|^2\psi\varrho\,dx = \frac{1}{2}\int_\Omega v^2 L\psi\,\varrho\,dx + \int_\Omega \langle F, \nabla\psi\rangle v\varrho\,dx + \int_\Omega \langle F, \nabla v\rangle\psi\varrho\,dx, \tag{2.16}$$

in which we used the intermediate equalities

$$
\begin{aligned}
&\int_\Omega \operatorname{div}(\varrho A\nabla v)v\psi\,dx = -\int_\Omega \langle A\nabla v,\nabla v\rangle\psi\varrho\,dx - \int_\Omega v\langle A\nabla v,\nabla\psi\rangle\varrho\,dx\\
&= -\int_\Omega \langle A\nabla v,\nabla v\rangle\psi\varrho\,dx - \int_\Omega \langle\nabla(v^2/2), A\nabla\psi\rangle\varrho\,dx\\
&= -\int_\Omega \langle A\nabla v,\nabla v\rangle\psi\varrho\,dx + \int_\Omega \frac{v^2}{2}\operatorname{div}(A\nabla\psi)\,\varrho\,dx + \int_\Omega \frac{v^2}{2}\langle A\nabla\psi,\nabla\varrho\rangle\,dx\\
&= -\int_\Omega \langle A\nabla v,\nabla v\rangle\psi\varrho\,dx + \int_\Omega \frac{v^2}{2}\operatorname{div}(A\nabla\psi)\,\varrho\,dx + \int_\Omega \frac{v^2}{2}\langle\nabla\psi, A\nabla\varrho\rangle\,dx,
\end{aligned}
$$

$$
\int_\Omega \langle\nabla v, b_0\rangle v\psi\,dx = \int_\Omega \langle\nabla(v^2/2), b_0\rangle\psi\,dx = -\int_\Omega \frac{v^2}{2}\langle b_0,\nabla\psi\rangle\,dx,
$$

where in the latter identity we used the condition that $\operatorname{div} b_0 = 0$. Finally,

$$
\operatorname{div}(A\nabla\psi) - \langle b_0,\nabla\psi\rangle = L\psi.
$$

Let $R_0 < N < R_1 < R$. Let us take $\psi = \zeta_N(V) - R_1$, where $\zeta_N \in C^2(\mathbb{R})$, $\zeta_N(t) = t$ if $t \le N$, $\zeta_N(t) = (R_1+N)/2$ if $t \ge R_1$, and $0 \le \zeta_N' \le 1$, $\zeta_N'' \le 0$. Note that the function ψ belongs to the class $C^2(\mathbb{R}^d)$ and vanishes if $V \ge R_1$. Taking into account that $\psi \le 0$, $\nabla\psi = \zeta_N'(V)\nabla V$, $L\psi = LV$ on Ω_0 and that outside of

$$
\Omega_0 = \{V < R_0\}
$$

we have

$$
L\psi = \zeta_N'(V)LV + \zeta_N''(V)\langle A\nabla V,\nabla V\rangle \le \zeta_N'(V)LV \le \zeta_N'(V)\Psi - \zeta_N'(V)W,
$$

we conclude that (2.16) yields the estimate

$$
\begin{aligned}
\int_{\Omega\setminus\Omega_0} v^2\zeta_N'(V)W\varrho\,dx &\le \int_{\Omega_0} v^2|LV|\varrho\,dx + 2\int_\Omega |A^{1/2}\nabla v|^2|\psi|\varrho\,dx\\
&\quad + \int_{\Omega\setminus\Omega_0} v^2\Psi\varrho\,dx + 2\int_\Omega \langle F,\nabla\psi\rangle v\varrho\,dx + 2\int_\Omega \langle F,\nabla v\rangle\psi\varrho\,dx.
\end{aligned}
$$

Since

$$
2\int_\Omega \langle F,\nabla\psi\rangle v\varrho\,dx \le \int_\Omega \zeta_N'(V)\Big[\frac{1}{2}v^2W + 2W^{-1}\langle F,\nabla V\rangle^2\Big]\varrho\,dx,
$$

$$
2\int_\Omega \langle F,\nabla v\rangle\psi\varrho\,dx \le \int_\Omega \Big[|A^{1/2}\nabla v|^2|\psi| + |A^{-1/2}F|^2|\psi|\Big]\varrho\,dx,
$$

we arrive at the estimate

$$
\begin{aligned}
\int_{\Omega\setminus\Omega_0} v^2\zeta_N'(V)W\varrho\,dx &\le 2\int_{\Omega_0} v^2|LV|\varrho\,dx + 6\int_\Omega |A^{1/2}\nabla v|^2|\psi|\varrho\,dx\\
&\quad + 2\int_{\Omega\setminus\Omega_0} v^2\Psi\varrho\,dx + 4\int_\Omega |A^{-1/2}F|^2|A^{1/2}\nabla V|^2W^{-1}\varrho\,dx\\
&\quad + 2\int_\Omega |A^{-1/2}F|^2|\psi|\varrho\,dx,
\end{aligned}
$$

which completes the proof by letting $N \to R$, since $|\psi| \le R$. $\square$

Corollary 2.5. *Suppose that* v *is a solution to* (2.13) *on the domain* $\Omega=\{V<R\}$, $R\ge 1$, *where* $V\in C^2(\Omega)$ *is a nonnegative function and there exist a measurable function* $W\ge 1$ *and a number* $R_0\in(0,R)$ *such that*

$$LV(x)\le -W(x)\quad \textit{if } V(x)\ge R_0.$$

If $|A^{1/2}\nabla V|^2\le C_V+C_V VW$ *with some number* $C_V\ge 1$, *then for any* $k\ge 0$ *we have*

$$\begin{aligned}\int_{R_0<V<R} & v^2V^kW\varrho\,dx\\ &\le M(R_0)C_VR^{k+1}\biggl(\sup_{\{V<R_0\}}(\varrho v^2)+\int_{V<R}\Bigl[|A^{1/2}\nabla v|^2+|A^{-1/2}F|^2\Bigr]\varrho\,dx\biggr), \qquad (2.17)\end{aligned}$$

where the number $M(R_0)$ *is independent of* v *and depends only on* R_0 *and the bounds on the coefficients on* $\{V<R_0\}$.

In the formulation of the next proposition two numbers $P(\Omega_0)$ and $H_0=H_0(\Omega_0)$ are employed. The first one depends only on the domain $\Omega_0=\{V<R_0\}$, where $R_0>0$ will be picked later. This is the number in the Poincaré inequality

$$\int_{\Omega_0}\varphi^2\,dx\le P(\Omega_0)\int_{\Omega_0}|\nabla\varphi|^2\,dx$$

valid for every function $\varphi\in W^{2,1}(\Omega_0)$ with zero integral over Ω_0. There is also a refined version of this inequality: if S_0 is a fixed ball containing the closure of Ω_0 (for the later use we assume also that $\operatorname{dist}(\Omega_0,\partial S_0)=1$), then

$$\int_{S_0}\varphi^2\,dx\le P(S_0,\Omega_0)\int_{S_0}|\nabla\varphi|^2\,dx$$

for every function $\varphi\in W^{2,1}(S_0)$ with zero integral over Ω_0 (see [24, Theorem 4.4.2]).

The second number $H_0=H_0(\Omega_0,S_0)$ is Harnack's constant for the operator L on the same ball S_0. With this constant one has (2.4) for every positive solution u of the equation $L^*u=0$ on S_0, namely,

$$\sup_{x\in U}u(x)\le H_0\inf_{x\in U}u(x) \qquad (2.18)$$

for each ball $U\subset\Omega_0$. This number depends only on S_0, Ω_0, and the coefficients of L through the $W^{p,1}(S_0)$-norms of a^{ij}, the $L^p(S_0)$-norms of b^i, and $\inf_{S_0}\det A$.

Proposition 2.6. *Suppose that there exist a function* $V\in C^2(\mathbb{R}^d)$, *a locally integrable function* $W\ge 1$ *and a number* $C_V\ge 1$ *such that*

$$\lim_{|x|\to+\infty}V(x)=+\infty,\quad \langle A\nabla V,\nabla V\rangle\le C_V+C_VVW$$

and for some $R_0>0$ *we have*

$$LV(x)\le -W(x)\quad \textit{if}\ \ x\notin\Omega_0:=\{V<R_0\}.$$

Let ϱ *be the unique probability solution of the equation* $L^*\varrho=0$. *Assume also that*

$$\int_{\mathbb{R}^d}|A^{-1/2}F|^2\varrho\,dx\le M_F<\infty \qquad (2.19)$$

and for some numbers $m\ge 1$ *and* $t>1$

$$\int_{\mathbb{R}^d}V^{2m+1+t}W\varrho\,dx\le M_0<\infty. \qquad (2.20)$$

Then, there exists a solution w of equation (2.11) *with the following property:*

$$\int_{\mathbb{R}^d} WV^m|w|\,dx \le (M_*M_F)^{1/2}\frac{1}{t-1} + (M_*M_F)^{1/2}2^{2m+1+t}M_0, \tag{2.21}$$

where M_* is a number that depends only on the constants in (1.3) *and* (1.4) *for a fixed ball S_0 containing the closure of Ω_0, say, a fixed ball S_0 such that* $\operatorname{dist}(S_0,\Omega_0)=1$, *and also on the integral of $|W|$ over $\{V\le 1\}$.*

Proof. We seek for a solution w of the form

$$w = v\varrho,$$

where v satisfies equation (2.13). Let $U_n=\{V<n\}$, $n>R_0$. Let v_n be the solution to the Dirichlet problem

$$\operatorname{div}(\varrho A\nabla v_n) + \langle b_0, \nabla v_n\rangle = \operatorname{div}(F\varrho), \quad v_n|_{\partial U_n}=0.$$

This solution exists due to our assumptions about the coefficients, see [21]. Multiplying the equation by v_n, integrating over U_n and using the integration by parts formula we obtain the equality

$$-\int_{U_n}|A^{1/2}\nabla v_n|^2\varrho\,dx + \int_{U_n} v_n\langle b_0,\nabla v_n\rangle\,dx = -\int_{U_n}\langle v_n, F\rangle\,\varrho\,dx,$$

where the second term on the left vanishes, since $\operatorname{div} b_0 = 0$ and $v_n\nabla v_n = \nabla(v_n^2)/2$. The integrand on the right is estimated by $|A^{1/2}\nabla v_n|^2/2 + |A^{-1/2}F|^2/2$, which yields the estimate

$$\int_{U_n}|A^{1/2}\nabla v_n|^2\varrho\,dx \le \int_{U_n}|A^{-1/2}F|^2\varrho\,dx.$$

Therefore,

$$\int_{U_n}|A^{1/2}\nabla v_n|^2\varrho\,dx \le M_F. \tag{2.22}$$

We now change the function v_n (keeping the same notation) by subtracting its integral over the domain $\Omega_0=\{V<R_0\}$, which yields a function satisfying the same equation (but not the boundary condition, of course) and having the zero integral over Ω_0. Obviously, these new functions v_n satisfy (2.22). The Poincaré inequality (see above) yields the bound

$$\begin{aligned}\int_{\Omega_0} v_n^2\varrho\,dx &\le \sup_{\Omega_0}\varrho\int_{\Omega_0}v_n^2\,dx \le P(\Omega_0)\sup_{\Omega_0}\varrho\int_{\Omega_0}|\nabla v_n|^2\,dx\\ &\le c(\Omega_0)P(\Omega_0)\sup_{\Omega_0}\varrho\,(\inf_{\Omega_0}\ \varrho)^{-1}\int_{\Omega_0}|A^{1/2}\nabla v_n|^2\varrho\,dx\\ &\le c(\Omega_0)P(\Omega_0)H_0M_F. \end{aligned}\tag{2.23}$$

However, we need more: we need a bound on the integral of $|v_n|^2(|LV|+1)$ over Ω_0. Since $|LV|+1$ is integrable on Ω_0, it suffices to have a uniform bound on $\sup_{\Omega_0}|v_n|$. The desired bound is ensured by (2.3), where we take $U=S_0$ (a ball whose interior contains the closure of Ω_0) and $\Omega=S_1$ is the ball with the same center and the radius increased by 1. Again by the Poincaré inequality we obtain

$$\|v_n\|^2_{L^1(S_1)} \le \|v_n\|^2_{L^2(S_1)}|S_1| \le c(S_1)P(S_1,\Omega_0)H_0M_F.$$

So we have

$$\sup_{\Omega_0}|v_n|^2 \le M_1M_F, \quad M_1 = C'(p,K,S_0,S_1)^2c(S_1)P(S_1,\Omega_0)^2|S_1|,$$

where the number K is determined by (1.3) and (1.4) according to (2.3).

By the previous proposition (see (2.17)) we arrive at the following estimate for all $k\ge n$:

$$\begin{aligned}\int_{U_n\backslash\Omega_0} v_k^2W\,\varrho\,dx &\le 2\int_{\Omega_0} v_n^2|LV|\varrho\,dx + 6nM_F + C_V(6n+1)M_F\\ &\le 2M_1M_F\int_{\Omega_0}|LV|\varrho\,dx + 6nM_F + C_V(6n+1)M_F \le M_2M_Fn,\end{aligned}$$

where M_2 is a number determined by the regarded norms of the coefficients on the ball S_1, $\sup_{S_1}\varrho$, $\inf_{S_1}\det A$, and also some universal constants (entering through the Poincaré, Sobolev, and Harnack inequalities). Increasing M_2 we can assume that

$$\int_{U_n} v_k^2W\,\varrho\,dx \le M_3n \quad \forall\, n,\, k\ge n. \tag{2.24}$$

It follows by (2.22), (2.23) and the Poincaré inequality that on every fixed ball U the sequence of functions v_n with $n\ge n(U)$ is bounded in the Sobolev norm of $W^{2,1}(U)$. Since these functions satisfy the elliptic equation whose coefficients satisfy the above mentioned conditions, we conclude by (2.2) that this sequence is bounded also in the Sobolev space $W^{p,1}(U)$, where $p=p(U)>d$, hence is uniformly bounded and contains a subsequence convergent uniformly on U to some function v. Using the diagonal procedure we pick a subsequence convergent locally uniformly to a common function v such that $v\in W^{p,1}(U)$ for every ball U with the respective $p=p(U)>d$. It is also possible to ensure that on each ball the restrictions of v_n converge to the restriction of v weakly in the respective $W^{p,1}(U)$. Obviously, v satisfies the desired equation on the whole space. By Fatou's theorem and (2.24) we have

$$\int_{U_n} v^2W\,\varrho\,dx \le M_3n \quad \forall\, n. \tag{2.25}$$

We now show that $WV^mv\varrho$ is integrable on the whole space. For any $n>1$, by the Cauchy inequality and (2.25) we have

$$\begin{aligned}\int_{n-1\le V\le n} WV^m|v|\varrho\,dx &\le n^m\int_{n-1\le V\le n} W|v|\varrho\,dx\\ &\le M_3^{1/2}n^{m+1/2}\biggl(\int_{n-1\le V\le n} W\varrho\,dx\biggr)^{1/2}\\ &\le M_3^{1/2}n^{-t} + M_3^{1/2}n^{2m+1+t}\int_{n-1\le V\le n} W\varrho\,dx\\ &\le M_3^{1/2}n^{-t} + M_3^{1/2}2^{2m+1+t}\int_{n-1\le V\le n} V^{2m+1+t}W\varrho\,dx.\end{aligned}$$

The integral of $WV^m|v|\varrho$ over $\{V\le 1\}$ is dominated by the square root of M_3 multiplied by the integral of $W\varrho$ over $\{V\le 1\}$. Therefore, increasing M_3, we arrive at the estimate

$$\int_{\mathbb{R}^d} WV^m|v|\varrho\,dx \le M_3^{1/2}\frac{1}{t-1} + M_3^{1/2}2^{2m+1+t}\int_{\mathbb{R}^d} V^{2m+1+t}W\varrho\,dx,$$

which is the desired bound. $\square$

Proposition 2.7. *If $F = 0$, then any solution w of equation* (2.11) *satisfying the condition*

$$\int_{\mathbb{R}^d} (|LV| + \langle A\nabla V, \nabla V\rangle)\, |w|\, dx < \infty$$

has the form $w = \lambda\varrho$, where λ is a constant. Therefore, a solution to (2.11) *in the class of functions satisfying the above condition is unique up to adding functions of the form $\lambda\varrho$.*

Proof. It suffices to show that any solution w of the homogeneous equation is proportional to ϱ, because ϱ is a solution. Let $v = w/\varrho$. Then v satisfies equation (2.13). Let f be a smooth function on $[0, +\infty)$ and let $\psi \in C_0^2(\mathbb{R}^d)$. We have (see [3, Lemma 1])

$$\int_{\mathbb{R}^d} |\sqrt{A}\nabla v^+|^2 f''(v^+)\psi\varrho\, dx = \int_{\mathbb{R}^d} f(v^+) L\psi\, \varrho\, dx - f'(0)\int_{\mathbb{R}^d} v^- L\psi\, \varrho\, dx,$$

where $v^+ = \max\{v, 0\}$ and $v^- = -\min\{v, 0\}$. Set $f(t) = (1+t)^{-1}$ and $\psi_N = \varphi(V/N)$, where $\varphi \in C_0^\infty(\mathbb{R}^d)$, $0 \le \varphi \le 1$, $\varphi(x) = 1$ if $|x| \le 1$, $|\varphi'| \le 1$, $|\varphi''| \le 1$. Then

$$L\psi_N = N^{-1}\varphi'(V/N)LV + N^{-2}\varphi''(V/N)\langle A\nabla V, \nabla V\rangle.$$

Hence

$$\begin{aligned} 2\int_{\mathbb{R}^d} & |\sqrt{A}\nabla v^+|^2 (1+v^+)^{-3}\psi_N \varrho\, dx \\ & \le N^{-1}\int_{\mathbb{R}^d} (|LV| + \langle A\nabla V, \nabla V\rangle)\varrho\, dx + N^{-1}\int_{\mathbb{R}^d} (|LV| + \langle A\nabla V, \nabla V\rangle)|w|\, dx. \end{aligned}$$

The right-hand side tends to zero as $N \to \infty$. In addition, $\psi_N \to 1$. Hence $\nabla v^+ = 0$ a.e., so $v^+ = \text{const}$. Replacing w by $-w$, we conclude that $v^- = \text{const}$. Thus, $v = \text{const}$. $\square$

Note that the integrability condition required in this proposition is fulfilled if we have the estimate $|LV| + \langle A\nabla V, \nabla V\rangle \le C_V + C_V V^m W$ assumed in the main theorem and $V^m W$ is integrable.

It should be also observed that the uniform bound (1.4) was never used in this section in its full strength: it would suffice to require this lower bound on each ball U with a constant $c(U)$ depending on U.

3. PROOFS

We first prove the continuity result, which is very simple.

Proof of Proposition 1.1. Let $\alpha_n \to \alpha$ in $[0, 1]$. As explained in the previous section, it follows from our assumptions that, for every ball U, the restrictions of the densities ϱ_α to U are uniformly bounded in the Sobolev norm of $W^{p,1}(U)$, hence are uniformly bounded and uniformly Hölder continuous. Therefore, there is a subsequence $\{\alpha_{n_j}\}$ such that the functions ϱ_α with the respective indices converge uniformly on balls to some continuous function ϱ. Since the measures μ_α are uniformly tight by assumption, we conclude that $\nu = \varrho\, dx$ is a probability measure. By convergence of densities we obtain convergence in variation, i.e. convergence of densities in $L^1(\mathbb{R}^d)$. It is clear that ν satisfies the equation $L_\alpha^* \nu = 0$ (here the local L^1-continuity of the coefficients in α is used to take limits under the integral sign), whence by the assumed uniqueness we have $\nu = \mu_\alpha$. Since this is true for any subsequence in the original sequence, our assertion is proven. $\square$

Remark 3.1. A similar (and even simpler) proof yields an alternative continuity result under incomparable conditions: if the functions $(x,\alpha) \mapsto a^{ij}_\alpha(x)$ and $(x,\alpha) \mapsto b^i_\alpha(x)$ are continuous, the matrix $A(x)$ is nonnegative-definite (possibly degenerate), the family $\{\mu_\alpha\}$ is uniformly tight, and for each α the measure μ_α is a unique probability solution to the equation $L^*_\alpha \mu = 0$, then the mapping $\alpha \mapsto \mu_\alpha$ is continuous with values in the space of measures with the weak topology.

Remark 3.2. In the situation of the theorem one can deal with parameters α belonging to a compact interval $[\tau, \tau_1]$ in $(0,1)$. Since by our assumption the mapping $\alpha \mapsto b_\alpha|_U$ is L^p-differentiable for every ball U, it is possible to choose versions of the functions b_α that are absolutely continuous in α, namely, one can use the version given by

$$b_\alpha(x) = b_\tau(x) + \int_\tau^\alpha \partial_s b_s(x)\, ds.$$

For this version we have

$$|b_\alpha(x)| \le |b_\tau(x)| + \int_\tau^{\tau_1} |\partial_s b_s(x)|\, ds,$$

hence $\sup_\alpha |b_\alpha(x)|$ is locally integrable (this function is Lebesgue measurable, since the coefficients are jointly Borel measurable, see [2, Corollary 2.12.8]), moreover, it is locally in L^p. Then the function $\sup_\alpha |L_\alpha V(x)|$ is locally integrable, because the terms with second derivatives of V are locally uniformly bounded in x and α. Hence in (1.7) without loss of generality we can assume that $\sup_\alpha L_\alpha V(x) \le -W(x)$ on the whole space. However, it will be more convenient to redefine W by 1 on a suitable ball.

Proof of Theorem 1.2. We can assume that $V \ge 1$. By assumption,

$$L_\alpha V(x) \le -W \le -1$$

outside of some ball. Since V is continuous and $\lim\limits_{|x|\to\infty} V(x) = +\infty$, we can assume that this holds outside of the set $\Omega_0 = \{V \le R_0\}$ for some $R_0 \ge 1$. Let us set $W(x) = 1$ if $x \in \Omega_0$. Then $W \ge 1$.

Let us prove the differentiability with respect to α in the special case where $A_\alpha(x)$ and $b_\alpha(x)$ do not depend on α for all x outside of some common ball U, i.e.,

$$A_{\alpha+h}(x) = A_\alpha(x), \quad b_{\alpha+h}(x) = b_\alpha(x) \quad \text{for all } h \text{ and all } x \notin U.$$

Suppose that α is fixed and a sequence of nonzero numbers h_k tends to zero. Set

$$\delta_k \varrho = h_k^{-1}(\varrho_\alpha - \varrho_{\alpha-h_k}), \quad \delta_k b = h_k^{-1}(b_\alpha - b_{\alpha-h_k}),$$
$$\delta_k a^{ij} = h_k^{-1}(a^{ij}_\alpha - a^{ij}_{\alpha-h_k}), \quad \delta_k A = (\delta_k a^{ij})_{i,j\le d}$$

Of course, these functions depend also on α, which is suppressed in our notation, since α is a fixed point where the differentiability is verified. Observe that $\delta_k a^{ij} = 0$ and $\delta_k b = 0$ outside of U for all k. Each function $\delta_k \varrho$ satisfies the equation

$$L^*_\alpha \delta_k \varrho = \operatorname{div}(F_k \varrho_\alpha), \tag{3.1}$$

where

$$F_k \varrho_\alpha = -\delta_k A \nabla \varrho_{\alpha-h_k} - \delta_k \partial_{x_i} A^i \varrho_{\alpha-h_k} + \delta_k b \varrho_{\alpha-h_k},$$

$$\delta_k \partial_{x_i} A^i = h_k^{-1}(\partial_{x_i} a^{i1}_\alpha - \partial_{x_i} a^{i1}_{\alpha-h_k}, \ldots, \partial_{x_i} a^{id}_\alpha - \partial_{x_i} a^{id}_{\alpha-h_k}).$$

The vector field F_k vanishes outside of U and its $L^r(U)$-norm is bounded by a number $C(U)$ independent of k for some $r = r(U) > d$. Indeed, by (2.2) the functions $\varrho_{\alpha-h_k}$ are uniformly bounded on U. By Proposition 1.1 they converge to ϱ_α in $L^1(U)$ and pointwise, hence also in $L^q(U)$ for each $q < \infty$. According to (1.5), for some $p_0 = p_0(U) > d$, the mappings $\delta_k \partial_{x_i} A^i$ and $\delta_k b$ converge in $L^{p_0}(U)$ to $\partial_\alpha \partial_{x_i} A^i_\alpha$ and $\partial_\alpha b$, respectively. Therefore,

$$-\delta_k \partial_{x_i} A^i \varrho_{\alpha-h_k} + \delta_k b \varrho_{\alpha-h_k} \to -\partial_\alpha \partial_{x_i} A^i_\alpha \varrho_\alpha + \partial_\alpha b \varrho_\alpha$$

in $L^{p_0}(U)$. In addition, again by (2.2) the mappings $\nabla\varrho_{\alpha-h_k}$ are uniformly bounded in $L^p(U)$, where $p = p(U) > d$, and by convergence of $\varrho_{\alpha-h_k}$ to ϱ_α they weakly converge in $L^p(U)$ to $\nabla\varrho_\alpha$. Next, according to (1.5) and the comment made below (1.6), the mappings $\delta_k A$ converge to $\partial_\alpha A_\alpha$ in $L^{p_1}(U)$ for some $p_1 = p_1(U) > dp/(p-d)$. By Hölder's inequality the mappings $\delta_k A \nabla\varrho_{\alpha-h_k}$ are uniformly bounded in $L^s(U)$ with $s = p_1 p/(p_1 + p)$. Note that $s > d$. Recall also that by Lemma 2.1 the functions ϱ_α are locally uniformly separated from zero. Therefore, we have the following weak convergence in $L^s(U)$ with $s > d$:

$$F_k \to F := -\partial_\alpha A_\alpha \nabla\varrho_\alpha/\varrho_\alpha - \partial_\alpha \partial_{x_i} A^i_\alpha + \partial_\alpha b.$$

Note also that $\delta_k \varrho$ satisfies the following conditions:

$$\int_{\mathbb{R}^d} \delta_k \varrho \, dx = 0, \quad \int_{\mathbb{R}^d} (|L_\alpha V| + \langle A_\alpha \nabla V, \nabla V\rangle)|\delta_k \varrho| \, dx < \infty.$$

Indeed, by Lemma 2.2 for every α the function $V^m W \varrho_\alpha$ is integrable, where m is the number from the hypotheses of the theorem, moreover,

$$\sup_\alpha \int_{\mathbb{R}^d} V^m W \varrho_\alpha \, dx \le M_1 < \infty. \tag{3.2}$$

According to the results of the previous section (see Proposition 2.6 and Proposition 2.7), for each k, there are a solution u_k to the equation $L^*_\alpha u_k = \operatorname{div}(F_k \varrho_\alpha)$ and a constant λ_k such that

$$\delta_k \varrho = u_k - \lambda_k \varrho_\alpha$$

and

$$\int_{\mathbb{R}^d} V^m W |u_k| \, dx \le M,$$

where M does not depend on k. The latter follows by (2.21), since we have (2.20) and (2.19) holds for $F = F_k$ with a constant independent of k due to the fact that the fields F_k have supports in U and are uniformly bounded in $L^p(U)$. It is clear that

$$\lambda_k = \int_{\mathbb{R}^d} u_k \, dx.$$

Hence $|\lambda_k| \le \|u_k\|_{L^1(\mathbb{R}^d)} \le M$. Therefore, by (3.2) we have

$$\int_{\mathbb{R}^d} V^m W |\delta_k \varrho| \, dx \le M + M\|V^m W\|_{L^1(\mu_\alpha)} \le M + MM_1.$$

Passing to a subsequence and using our local estimates (2.2), we can assume that the functions $\delta_k \varrho$ converge locally uniformly to some function w_α. By Fatou's theorem

$$\int_{\mathbb{R}^d} V^m W |w_\alpha| \, dx \le M + M\|V^m W\|_{L^1(\mu_\alpha)} \le M + MM_1.$$

In addition, the functions $\delta_k \varrho$ converge to w_α in $L^1(\mathbb{R}^d)$ (since $\lim\limits_{|x|\to\infty} W(x) = +\infty$) and

$$\int_{\mathbb{R}^d} w_\alpha \, dx = 0.$$

Moreover, the function w_α satisfies the equation

$$L_\alpha^* w_\alpha = \mathrm{div}(\varrho_\alpha F).$$

It remains to observe that w_α satisfies the hypotheses of Proposition 2.7 about uniqueness. Thus, for each sequence $h_k \to 0$ the continuous functions $\delta_k \varrho$ converge locally uniformly to one and the same limit w_α. Therefore, we have $w_\alpha = \partial_\alpha \varrho_\alpha$.

We now proceed to the general case. We can assume that our parameters take values in an interval of length less than $c_0(2\lambda_0+1)^{-1}$ (and less than 1), where λ_0 and c_0 are constants from (1.4) and (1.6), which by (1.6) yields the estimate

$$\|A_\alpha(x) - A_{\alpha_0}(x)\| \le \frac{c_0}{2}$$

for all α, α_0 and x. It follows that for any number $\theta \in [0,1]$ we have

$$\|(\theta A_\alpha(x) + (1-\theta)A_{\alpha_0}(x))^{-1/2} A_\alpha(x)^{1/2}\| \le 2. \tag{3.3}$$

Indeed, let us observe that for any nonnegative operator T and symmetric operator D such that $T \ge c_0\mathrm{I}$ and $\|D\| \le \varepsilon$, where $\varepsilon < c_0/2$, we have $T - \varepsilon\mathrm{I} \le T + D \le T + \varepsilon\mathrm{I}$ in the sense of quadratic forms, hence (see [19, Chapter VIII, Problem 50])

$$(T-\varepsilon\mathrm{I})^{1/2} - T^{1/2} \le (T+D)^{1/2} - T^{1/2} \le (T+\varepsilon\mathrm{I})^{1/2} - T^{1/2},$$

so that

$$-\frac{\varepsilon}{2c_0^{1/2}}\mathrm{I} \le (T+D)^{1/2} - T^{1/2} \le \frac{\varepsilon}{(2c_0)^{1/2}}\mathrm{I},$$

which yields that $\|(T+D)^{1/2} - T^{1/2}\| \le \varepsilon(2c_0)^{-1/2}$. Therefore,

$$\begin{aligned}
\|(T+D)^{-1/2}T^{1/2}\| &= \|(T+D)^{-1/2}((T+D)^{1/2} + T^{1/2} - (T+D)^{1/2})\| \\
&\le 1 + \|(T+D)^{-1/2}\| \, \|T^{1/2} - (T+D)^{1/2}\| \\
&\le 1 + \|(T+D)^{-1/2}\| \frac{\varepsilon}{(2c_0)^{1/2}} \le 1 + c_0^{1/2}\|(T+D)^{-1/2}\|.
\end{aligned}$$

This yields (3.3) if we take $T = A_\alpha(x)$ and $D = (1-\theta)(A_{\alpha_0}(x) - A_\alpha(x))$, that is, $T + D = \theta A_\alpha(x) + (1-\theta)A_{\alpha_0}(x)$, because $\theta A_\alpha(x) + (1-\theta)A_{\alpha_0}(x) \ge c_0\mathrm{I}$, hence $\|(\theta A_\alpha(x) + (1-\theta)A_{\alpha_0}(x))^{-1/2}\| \le c_0^{-1/2}$.

Let $\psi_N(x) = \psi(|x| - N + 1)$, where $\psi \in C^\infty(\mathbb{R})$, $0 \le \psi \le 1$, $-2 \le \psi' \le 0$, $\psi(s) = 1$ if $s \le 1$ and $\psi(s) = 0$ if $s \ge 2$. In addition, we take ψ such that $|\psi'(s)|^2 \le C_0\psi(s)$ with some $C_0 > 0$. Then $0 \le \psi_N \le 1$, $\psi_N(x) = 1$ if $|x| < N$ and $\psi_N(x) = 0$ if $|x| > N+1$, $|\nabla\psi_N| \le 2$, $|\nabla\psi_N|^2 \le C_0\psi_N$. Fix some α_0 (say, the middle of the interval) and set

$$L_{\alpha,N} = \psi_N L_\alpha + (1-\psi_N)L_{\alpha_0},$$

$$A_{\alpha,N} = \psi_N A_\alpha + (1-\psi_N)A_{\alpha_0}, \quad b_{\alpha,N} = \psi_N b_\alpha + (1-\psi_N)b_{\alpha_0}.$$

We observe that the corresponding coefficients $a^{ij}_{\alpha,N}(x)$ and $b^i_{\alpha,N}(x)$ do not depend on α if $|x| > N+1$, once N is fixed. Moreover,

$$L_{\alpha,N}V = \psi_N L_\alpha V + (1-\psi_N)L_{\alpha_0}V \le -W$$

and

$$\langle A_{\alpha,N}\nabla V, \nabla V\rangle \le \varepsilon V W$$

outside of the same ball as in the case of L_α. We also have

$$|L_{\alpha,N}V| + |A_{\alpha,N}^{-1/2}(b_{\alpha,N} - \operatorname{div} A_{\alpha,N})|^2 + |A_{\alpha,N}^{-1/2}(\partial_\alpha b_{\alpha,N} - \partial_\alpha \operatorname{div} A_{\alpha,N})|^2 \\ \le \widetilde{C}_V + \widetilde{C}_V V^m \quad (3.4)$$

with $\widetilde{C}_V = 18C_V + 16\lambda_0^2 c_0^{-1}$, since

$$\operatorname{div} A_{\alpha,N} = \psi_N \operatorname{div} A_\alpha + (1-\psi_N)\operatorname{div} A_{\alpha_0} + (A_\alpha - A_{\alpha_0})\nabla\psi_N,$$

$$\partial_\alpha \operatorname{div} A_{\alpha,N} = \psi_N \operatorname{div} \partial_\alpha A_\alpha + \partial_\alpha A_\alpha \nabla\psi_N, \quad \partial_\alpha b_{\alpha,N} = \psi_N \partial_\alpha b_\alpha,$$

where $\|A_\alpha(x) - A_{\alpha_0}(x)\| \le \lambda_0|\alpha-\alpha_0| \le \lambda_0$ by (1.6). Indeed,

$$A_{\alpha,N}^{-1/2}(b_{\alpha,N} - \operatorname{div} A_{\alpha,N}) = \psi_N A_{\alpha,N}^{-1/2} A_\alpha^{1/2} A_\alpha^{-1/2}(b_\alpha - \operatorname{div} A_\alpha) \\ + (1-\psi_N) A_{\alpha,N}^{-1/2} A_{\alpha_0}^{1/2} A_{\alpha_0}^{-1/2}(b_{\alpha_0} - \operatorname{div} A_{\alpha_0}) + A_{\alpha,N}^{-1/2}(A_\alpha - A_{\alpha_0})\nabla\psi_N,$$

$$A_{\alpha,N}^{-1/2}(\partial_\alpha b_{\alpha,N} - \partial_\alpha \operatorname{div} A_{\alpha,N}) = \psi_N A_{\alpha,N}^{-1/2} A_\alpha^{1/2} A_\alpha^{-1/2}(\partial_\alpha b_\alpha - \partial_\alpha \operatorname{div} A_\alpha) \\ - A_{\alpha,N}^{-1/2}\partial_\alpha A_\alpha \nabla\psi_N.$$

Hence, by (3.3), the norm of the first of these two vectors is dominated by the sum of the norms of $A_\alpha^{-1/2}(b_\alpha - \operatorname{div} A_\alpha)$ and $A_{\alpha_0}^{-1/2}(b_{\alpha_0} - \operatorname{div} A_{\alpha_0})$ and $c_0^{-1/2}\lambda_0$, and similarly for the second vector.

In addition, for every ball U we have

$$\sup_\alpha \|a_{\alpha,N}^{ij}\|_{W^{p,1}(U)} \le 2\sup_\alpha \|a_\alpha^{ij}\|_{W^{p,1}(U)}, \quad \sup_\alpha \|b_{\alpha,N}^i\|_{L^p(U)} \le \sup_\alpha \|b_\alpha^i\|_{L^p(U)}$$

for the corresponding $p = p(U) > d$, and also

$$\psi_N(x)A_\alpha(x) + (1-\psi_N(x))A_{\alpha_0}(x) \ge c_0 \cdot \mathrm{I} \quad \forall x.$$

Defining $S_{\alpha,N}$ for the mapping $A_{\alpha,N}$ by the same formula as S_α for A_α, due to (1.6), we have

$$\sup_{\alpha,N} \|S_{\alpha,N}\| \le \lambda_0 < \infty. \quad (3.5)$$

For each N there exist probability solutions $\varrho_{\alpha,N}$ of the equations

$$L_\alpha^* \varrho_{\alpha,N} = 0.$$

As shown above, there exist the derivatives $w_{\alpha,N} = \partial_\alpha \varrho_{\alpha,N}$ satisfying the equations

$$L_{\alpha,N}^* w_{\alpha,N} = \operatorname{div}(B_{\alpha,N}\varrho_{\alpha,N} - R_{\alpha,N}\varrho_{\alpha,N} - S_{\alpha,N}\nabla\varrho_{\alpha,N}),$$

where $B_{\alpha,N} = \partial_\alpha b_{\alpha,N}$, $R_{\alpha,N} = \operatorname{div} S_{\alpha,N}$, as in the case of the original operators L_α. Moreover, we have

$$\int_{\mathbb{R}^d} V^m W |w_{\alpha,N}|\, dx \le M, \quad (3.6)$$

where M does not depend on N and α. Indeed, as above, we can construct a solution $\widetilde{w}_{\alpha,N}$ for which this estimate holds. To this end we observe that by Lemma 2.3 and (3.4) we have

$$\int_{\mathbb{R}^d} \Bigl|\frac{A_\alpha^{1/2}\nabla\varrho_{\alpha,N}}{\varrho_{\alpha,N}}\Bigr|^2 \varrho_{\alpha,N}\,dx \le \int_{\mathbb{R}^d} |A_{\alpha,N}^{-1/2}(b_{\alpha,N} - \operatorname{div} A_{\alpha,N})|^2\,\varrho_{\alpha,N}\,dx$$

$$\le (1+c_0)\widetilde{C}_V + (1+c_0)\widetilde{C}_V \int_{\mathbb{R}^d} V^m W\,\varrho_{\alpha,N}\,dx,$$

which is uniformly bounded in α and N, since the integrals of $V^m W\varrho_{\alpha,N}$ are uniformly bounded by Lemma 2.2. Therefore, by the equality

$$A_{\alpha,N}^{-1/2} S_{\alpha,N}\nabla\varrho_{\alpha,N} = A_{\alpha,N}^{-1/2} S_{\alpha,N} A_{\alpha,N}^{-1/2} A_{\alpha,N}^{1/2}\nabla\varrho_{\alpha,N}$$

combined with (1.4) and (3.5) we have

$$\sup_{\alpha,N}\int_{\mathbb{R}^d} \Bigl|\frac{A_\alpha^{-1/2} S_{\alpha,N}\nabla\varrho_{\alpha,N}}{\varrho_{\alpha,N}}\Bigr|^2 \varrho_{\alpha,N}\,dx \le c_0^{-1}\lambda_0^2 \sup_{\alpha,N}\int_{\mathbb{R}^d} \Bigl|\frac{A_\alpha^{1/2}\nabla\varrho_{\alpha,N}}{\varrho_{\alpha,N}}\Bigr|^2 \varrho_{\alpha,N}\,dx < \infty.$$

We have also a uniform bound for the integrals of $|A_{\alpha,N}^{-1/2}(B_{\alpha,N}-R_{\alpha,N})|^2\varrho_{\alpha,N}$, because

$$B_{\alpha,N} - R_{\alpha,N} = \psi_N(B_\alpha - R_\alpha) - S_\alpha\nabla\psi_N,$$

so that

$$|A_{\alpha,N}^{-1/2}(B_{\alpha,N} - R_{\alpha,N})| \le C_2 + C_2 V^m W$$

with some constant C_2. Hence we ensure condition (2.19), so that estimate (2.21) in Proposition 2.6 yields (3.6) for $\widetilde{w}_{\alpha,N}$. By the uniqueness result (Proposition 2.7) we have the equality

$$w_{\alpha,N}(x) = \widetilde{w}_{\alpha,N}(x) - \varrho_{\alpha,N}(x)\int_{\mathbb{R}^d} \widetilde{w}_{\alpha,N}\,dx,$$

which yields the desired estimate (3.6) for $w_{\alpha,N}$, because the integrals of $V^m W\varrho_{\alpha,N}$ are uniformly bounded by Lemma 2.2.

Passing to a subsequence and using (2.2), we conclude that for each α the sequence of functions $\varrho_{\alpha,N}$ converges uniformly in x to the unique solution ϱ_α of the equation $L_\alpha^*\mu = 0$ and the sequence of functions $w_{\alpha,N}$ converges to a solution w_α of the equation

$$L_\alpha^* w = \operatorname{div}(B_\alpha\varrho_\alpha - R_\alpha\varrho_\alpha - S_\alpha\nabla\varrho_\alpha) \tag{3.7}$$

that satisfies the same bound as in (3.6). It follows that

$$\int_{\mathbb{R}^d} w_\alpha\,dx = 0,$$

hence w_α is a unique solution to (3.7) with zero integral such that $V^m W w_\alpha$ is integrable.

We now observe that the solutions w_α (as well as $w_{\alpha,N}$) satisfying the conditions

$$\int_{\mathbb{R}^d} w_\alpha\,dx = 0, \quad \sup_\alpha \int_{\mathbb{R}^d} V^m W|w_\alpha|\,dx < \infty \tag{3.8}$$

are continuous in α locally uniformly in x. Indeed, if $\alpha_k \to \alpha$, then the sequence $\{w_{\alpha_k}\}$ contains a subsequence convergent locally uniformly in x to some function w (this follows by (2.2)). Due to our assumption that the mappings $\alpha \mapsto \partial_\alpha a_\alpha^{ij}$, $\alpha \mapsto \partial_\alpha b_\alpha^i$,

$\alpha \mapsto \partial_\alpha \partial_{x_k} a_\alpha^{ij}$ are continuous with values in $L^1(U)$ for each ball U and $\varrho_\alpha(x)$ is jointly continuous by Proposition 1.1, we see that w is a solution to the equation

$$L_\alpha^* w = \operatorname{div}(B_\alpha \varrho_\alpha - R_\alpha \varrho_\alpha - S_\alpha \nabla \varrho_\alpha)$$

and this solution satisfies the above estimate by Fatou's theorem. Hence w coincides with our unique solution w_α. Since this is true for each sequence $\{\alpha_k\}$, we obtain that $\{w_{\alpha_k}\}$ converges to w_α. The continuity is proven.

It remains to show that $w_\alpha = \partial_\alpha \varrho_\alpha$. Let us fix x. By the Newton–Leibniz formula

$$\varrho_{\alpha,N}(x) = \varrho_{\alpha_0,N}(x) + \int_{\alpha_0}^{\alpha} w_{s,N}(x)\, ds.$$

All limiting functions $\varrho_\alpha(x)$ and $w_\alpha(x)$ are continuous in α, as shown above. In addition, letting $U(x,r)$ be the ball of radius r centered at x, we have by (2.2)

$$\begin{aligned}|w_{\alpha,N}(x)| &\le \|w_{\alpha,N}\|_{W^{p,1}(U(x,1))}\\ &\le C\bigl(\|w_{\alpha,N}\|_{L^1(U(x,2))} + \|B_{\alpha,N}\varrho_{\alpha,N} - R_{\alpha,N}\varrho_{\alpha,N} - S_{\alpha,N}\nabla\varrho_{\alpha,N}\|_{L^p(U(x,2))}\bigr).\end{aligned}$$

The right-hand side is uniformly bounded in α and N (once x is fixed). Therefore,

$$\sup_{\alpha,N} |w_{\alpha,N}(x)| < \infty.$$

Passing to the limit as $N \to \infty$, we obtain the equality

$$\varrho_\alpha(x) = \varrho_{\alpha_0}(x) + \int_{\alpha_0}^{\alpha} w_s(x)\, ds,$$

whence we conclude that $\partial_\alpha \varrho_\alpha(x) = w_\alpha(x)$. The assertion about the L^1-differentiability of $\alpha \mapsto \varrho_\alpha$ follows from (3.8), which allows to show the L^1-convergence to zero of the ratio

$$\frac{\varrho_{\alpha+h} - \varrho_\alpha}{h} - w_\alpha = h^{-1}\int_\alpha^{\alpha+h} (w_s - w_\alpha)\, ds,$$

reducing it to the L^1-convergence on balls. The general case of the theorem is proven.

In the special case where the diffusion matrix does not depend on α we have $R_\alpha = 0$ and $S_\alpha = 0$, so in the right-hand side of (3.7) we have only one term $\operatorname{div}(B_\alpha \varrho_\alpha)$ with $B_\alpha = \partial_\alpha b_\alpha$. Hence we can obtain (2.19) immediately from the given bound on B_α. $\square$

Remark 3.3. (i) It follows from the proof (or from the L^1-differentiability) that

$$\int_{\mathbb{R}^d} \partial_\alpha \varrho_\alpha(x)\, dx = 0.$$

(ii) The main theorem can be combined with the results of [5] on Sobolev regularity of non-homogeneous equations in order to ensure the differentiability of $\alpha \mapsto \varrho_\alpha$ with values in $W^{r,1}(\mathbb{R}^d)$. For example, suppose that in the main theorem A_α, A_α^{-1}, ∇a^{ij}, and $\partial_\alpha A_\alpha$ are uniformly bounded and

$$|b_\alpha|^p + |\partial_\alpha b_\alpha|^p + |\partial_\alpha \operatorname{div} A_\alpha|^p \le C_V + C_V V^m W$$

with some $p > d$. Then we have the differentiability in $W^{r,1}(\mathbb{R}^d)$ for any $r < p$.

(iii) Condition (1.6) has been essential in estimating the integral of the expression $|A_\alpha^{-1/2} S_\alpha \nabla \varrho_\alpha|^2/\varrho_\alpha$ and a similar integral for $\varrho_{\alpha,N}$, since we have had an a priori bound just for the integral of $|\nabla \varrho_\alpha|^2/\varrho_\alpha$, so that a growing $\|S_\alpha\|$ could destroy this estimate. However, under assumptions similar to those used in Corollary 1.5 it is proved in [8] that there is a bound for the integral of $|\nabla \varrho_\alpha|^p/\varrho_\alpha$ with a sufficiently large $p > d$. This enables us to replace (1.6) by an exponential bound and make our condition on $\partial_\alpha A_\alpha$ closer to that of the condition on $\partial_\alpha b_\alpha$.

(iv) The main theorem and its corollaries extend to the case where the parameter α takes values in $\mathbb{R}^n$; this case can be also deduced from the scalar case.

(v) Analogous results can be obtained by the same method for equations on manifolds; some ingredients of the proofs are already developed in [13].

(vi) Finally, let us observe that a similar method enables one to obtain higher differentiability of ϱ_α with respect to α (considered in [18] for coefficients of class C_b^k), which will be the subject of another paper (in order not to overload this paper with additional technicalities).

Remark 3.4. It would be tempting to prove the theorem along the following lines: it is known that under our assumptions the solutions ϱ_α can be obtained as limits of the normalized positive solutions to the equations $L_\alpha^*\varrho_{\alpha,n} = 0$ on increasing domains $U_n = \{V < n\}$; e.g., one can use solutions to boundary value problems with constant boundary conditions. Such solutions are differentiable with respect to α and the derivative in α satisfies the required equation in U_n. Then the problem is to obtain convergence of these derivatives. Proposition 2.4 seems to be a suitable tool, moreover, we apply it in a similar situation. However, in that situation we deal with zero boundary condition, which is very essential.

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