Robinson-Schensted algorithms and quantum stochastic double product integrals

by

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Contents

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My research was funded by EPSRC grant number EP/H023364/1.

I wish to dedicate this thesis to champions of human rights and dignity throughout history of China.

Declarations

Chapter 2 is joint work [OP13] with Prof. Neil O'Connell. Chapter 3 is a paper [Pei14]. Chapter 5 is joint work [HP15b] with Prof. Robin Hudson.

Abstract

This thesis is divided into two parts.

In the first part (Chapters 1, 2, 3) various Robinson-Schensted (RS) algorithms are discussed. An introduction to the classical RS algorithm is presented, including the symmetry property, and the result of the algorithm Doob h-transforming the kernel from the Pieri rule of Schur functions h when taking a random word [O'C03a]. This is followed by the extension to a q-weighted version that has a branching structure, which can be alternatively viewed as a randomisation of the classical algorithm. The q-weighted RS algorithm is related to the q-Whittaker functions in the same way as the classical algorithm is to the Schur functions. That is, when taking a random word, the algorithm Doob h-transforms the Hamiltonian of the quantum Toda lattice where h are the q-Whittaker functions. Moreover, it can also be applied to model the q-totally asymmetric simple exclusion process introduced in [SW98]. Furthermore, the q-RS algorithm also enjoys a symmetry property analogous to that of the symmetry property of the classical algorithm. This is proved by extending Fomin's growth diagram technique [Fom79, Fom88, Fom94, Fom95], which covers a family of the so-called branching insertion algorithms, including the row insertion proposed in [BP13].

In the second part (Chapters 4, 5) we work with quantum stochastic analysis. First we introduce the basic elements in quantum stochastic analysis, including the quantum probability space, the momentum and position Brownian motions [CH77], and the relation between rotations and angular momenta via the second quantisation, which is generalised to a family of rotation-like operators [HP15a]. Then we discuss a family of unitary quantum causal stochastic double product integrals E, which are expected to be the second quantisation of the continuous limit W of a discrete double product of aforementioned rotation-like operators. In one special case, the operator E is related to the quantum Lévy stochastic area, while in another case it is related to the quantum 2-d Bessel process. The explicit formula for the kernel of W is obtained by enumerating linear extensions of partial orderings related to a path model, and the combinatorial aspect is closely related to generalisations of the Catalan numbers and the Dyck paths. Furthermore W is shown to be unitary using integrals of the Bessel functions.

Chapter 1

Introduction to Robinson-Schensted algorithms

1.1 Overview

The Robinson-Schensted (RS) algorithm was constructed by Robinson [Rob38] and Schensted [Sch61] for showing explicitly a one-one correspondence between the symmetric group S_n and the set of pairs of standard Young tableaux of size n with the same shape, see e.g. [Ful97] for an exposition of the algorithm and the tableaux, and the Schur functions mentioned below. Such a simple algorithm has over the decades appeared in various contexts.

First of all, naturally they show up in representation theory, not only of S_n, but also of $\mathfrak{gl}_\ell$, and even $U_q(\mathfrak{gl}_\ell)$ when $q \to 0$ [DJM90]. This is because the irreducible representations of $\mathfrak{gl}_\ell$ and $U_q(\mathfrak{gl}_\ell)$ are spanned by the vectors parameterised by the tableaux of the same shape that have entries no greater than ℓ. This connection, together with the branching behaviour of the representations (decomposition of tensor products of natural representations), of the tableaux (the insertion action in the RS algorithm) and of the Schur functions (the Pieri formula), result in an elegant link between the representation theory and the probability theory. In the probabilistic context, a word of independent random letters of categorical distribution in dictionary $\{1, \dots, \ell\}$ can be identified with an ℓ-dimensional simple random walk. When such a word is taken as the input of the RS algorithm, the output random tableaux has a Markovian shape whose transition kernel is the discrete Doob h-transform of that of the random walk, where h are the Schur functions [O'C03a]. This gives the discrete version of the multidimensional Pitman's theorem, whose most familiar face is the 2-d case: Let X be a standard Brownian motion and M

the process defined by $M_t := \sup_{s \le t} B_s$, then $2M - X$ is the 3-d Bessel process.

There are many more applications of the RS algorithm. The algorithm is represented in terms of queues in tandem [O'C03b], as a path transformation named after Pitman which was generalised to the Coxeter groups [BBO05], and as percolation, corner growth model and interacting particles system model [Joh00]. Starting from the paper [BDJ99], the RS algorithm has been tied to the random matrix theory and more specifically is related to the Gaussian Unitary Ensemble(GUE) (and its process version Hermitian Brownian motion) and the Laguerre Ensemble [Bar01, GTW01, OY02, Joh00, Dou03].

As a result of the myriad connections there are many ways to extend the algorithm. One of them is to take a geometric lift, replacing the $(\max, +)$ by $(+, \times)$ in the definition of the Pitman's transform, raising the temperature in the percolation model and turning it into a polymer model [COSZ14, OSZ14, O'C12], where the counterparts of the Schur functions are now the $\mathfrak{gl}_\ell$-Whittaker functions (Whittaker functions). Many of the probabilistic models mentioned up to now belong to the so called Kardar-Parasi-Zhang universality class (see e.g. the survey [Cor14]), where the GUE Tracy-Widom law is a universal limit distribution.

We wanted to find and study a q-weighted RS (q-RS) algorithm, where the q-deformed $\mathfrak{gl}_\ell$-Whittaker functions (q-Whittaker functions) [GLO10] play the same role as the Schur functions and the Whittaker functions in the classical and geometric cases respectively. This was motivated by the fact that on the one hand the q-Whittaker functions, which are Macdonald polynomials [Mac98] when $t = 0$ [GLO11], turn into the Schur functions when $q = 0$, and on the other hand they scale to the Whittaker functions when $q \to 1$ [GLO12]. Indeed, we were able to construct such a q-RS algorithm [OP13](Chapter 2) and prove the corresponding Pitman's theorem. It is very different from the classical and the geometric ones in that it branches even when taking deterministic input. Moreover, the coefficients of the branching satisfies positivity when $q \in [0, 1]$, giving it a probabilistic interpretation as a randomised algorithm. When taking a random walk as the input, it has the first column distributed as the q-totally asymmetric simple exclusion process introduced in [SW98], which was later proved to scale to the Tracy-Widom distribution [FV13].

Later I was able to prove the q-RS algorithm has a symmetry property analogous to the one enjoyed by the classical algorithm [Pei14](Chapter 3). The symmetry property states: if we denote by $\phi_\sigma(P, Q)$ the weight of tableau pair (P, Q) when performing the q-RS algorithm on a permutation σ, then $\phi_{\sigma^{-1}}(Q, P) = \phi_\sigma(P, Q)$ for any σ, P and Q. The proof used the so-called growth graph which is an extension to the growth diagram by Fomin [Fom94, Fom95]. It also generalises to a class of branching insertion algorithms which covers a row insertion version introduced in [BP13].

1.2 The classical Robinson-Schensted algorithms

In this and the rest sections of this chapter, a brief introduction to the ideas in Chapters 2 and 3 is presented. Certain contents will be covered in detail in those chapters.

Fix an arbitrary positive integer ℓ, a *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is an integer vector such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0$. It is also called a *Young diagram* and visualised as an array of left-aligned boxes with λ_i boxes in the ith row. The number of boxes $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$ is called the size of λ and we say λ is a partition of $|\lambda|$, which we denote by $\lambda \vdash |\lambda|$.

A *Young tableau (of rank $\ell - 1$)* is a Young diagram filled with integers from $[\ell] := \{1, 2, \dots, \ell\}$ such that the entries are increasing along the columns and non-decreasing along the rows. Denote by $\mathcal{T}_\ell$ the set of Young tableaux of rank $\ell - 1$. The underlying Young diagram λ of a Young tableau T is called the *shape* of the latter, denoted by $\lambda = \mathrm{sh}T$. A *standard Young tableau* of size n is a Young tableau of size n whose entries are exactly the members of $[n] := \{1, 2, \dots, n\}$. Examples of Young tableaux can be found in Section 2.2. Denote by $\mathcal{S}_n$ and $\mathcal{S}_\lambda$ the set of standard Young tableaux of size n and of shape λ respectively. Let $d_\lambda := |\mathcal{S}_\lambda|$ be the number of standard tableaux of shape λ. This is the dimension of the irreducible representation of S_n corresponding to the conjugacy class λ.

It is well-known (see e.g. [Sag00]) that for a finite group G,

$$|G| = \sum_V (\mathrm{dim}V)^2$$

where V traverses all irreducible representations of G and $\mathrm{dim}V$ is the dimension of V. When $G = S$, this becomes

$$\sum_{\lambda \vdash n} d_\lambda^2 = n!.$$

This identity motivates an explicit bijection between the symmetric group and pairs of standard Young tableaux of the same shape, and the RS algorithm is one such bijection.

There are a few equivalent definitions of the RS algorithm, and a few versions of them. In this section and Section 1.3 I define the column insertion version in two different ways to suit the Chapters 2 and 3.

We call $[\ell]$ an *alphabet*, any $k \in [\ell]$ a *letter*, and a word $w = (w_1, w_2, \dots)$ where $w_i \in [\ell]$ a *word*. For a tableau T we also define λ_j^k to be the number entries less than or equal to k in the jth row of T. The triangular array $(\lambda_j^k)_{1\le j\le k\le \ell}$ is the so called *Gelfand Tsetlin pattern* and satisfies an interlacing relation:

$$\lambda_j^k \le \lambda_{j-1}^{k-1} \le \lambda_{j-1}^k.$$

Clearly given a Gelfand-Tsetlin pattern one can recover the corresponding tableau, thus identifying the two:

$$T \leftrightarrow (\lambda_j^k)_{1\le j\le k\le \ell}.$$

The basic operation of the RS algorithm is the insertion of a letter k into a tableau $P = (\lambda_j^k)_{1\le j\le k\le \ell}$ to obtain $\tilde{P} = I(k, P)$, described in an algorithmic language as follows:

1. Initialise and set k to be j.
2. If $\lambda_{j-1}^{k-1} = \lambda_j^k$ and $j > 1$ then set $j \leftarrow j - 1$; otherwise k displaces the first number s in jth row of the tableau that is greater than k ($s = \infty$ and k is appended at the end of the row if no such number exists) and set $k \leftarrow s$.
3. If $k = \infty$ then we are done; otherwise go to step 2.

The RS algorithm applied to a finite word $w = (w_i)_{1\le i\le n}$ is defined as successively inserting $w_1, w_2, \dots, w_n$ into the empty tableau:

$$P(0) = \emptyset, \quad P(i) = I(w_i, P(i-1)).$$

The output tableau $P = P(n)$ is called the *insertion tableau.* We also define the *recording tableau* Q by a growth of Young diagrams related to P. For $T = (\lambda_j^k)_{1\le j\le k\le \ell}$, let $\lambda^k = (\lambda_j^k)_{1\le j\le k}$ be the Young diagram of the subtableau of T consisting of entries no greater than k, then T can be identified with the growth of Young diagrams

$$T = (\lambda^0 = \emptyset) \prec \lambda^1 \prec \lambda^2 \prec \cdots \prec \lambda^\ell$$

where $\lambda \prec \mu$ means $\mu_{i+1} \leq \lambda_i \leq \mu_i \forall i$. Then Q is defined as

$$Q = P(0) = \emptyset \prec P(1) \prec P(2) \prec \cdots \prec P(n).$$

Since the growth of P is one box per time, Q is a standard tableau. The RS algorithm defines a bijection between the set of words of of length n and the set of pairs $(P, Q) \in \mathcal{T}_\ell \times \mathcal{S}_n$ where $\mathrm{sh}P = \mathrm{sh}Q$. By identifying a permutation $\sigma \in S_n$ with a word $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ the algorithm establishes desired bijection between S_n and $\{(P, Q) \in \mathcal{S}_n \times \mathcal{S}_n : \mathrm{sh}P = \mathrm{sh}Q\}$.

As an example, the Robinson-Schensted algorithm taking the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$ as the input gives the following output of tableau pair:

$$P = \begin{matrix} 1 & 4 \\ 2 & \\ 3 & \end{matrix} \quad , \qquad Q = \begin{matrix} 1 & 3 \\ 2 & \\ 4 & \end{matrix}$$

1.3 The growth diagram

An equivalent definition of the RS algorithm applied to permutations is the growth diagram representation introduced by Fomin [Fom79, Fom88, Fom94, Fom95].

We start with the more general case of an insertion of a word w. For a series of tableau $(P(m))_{m\geq 1} = ((\lambda^i(m)_j)_{1\leq j\leq i\leq \ell})_{m\geq 1}$ we call the superscript i the *level* and the parameter in the bracket m the *time* of a shape $\lambda^i(m)$. When a letter $w_m = k$ is inserted to a tableau $P(m-1) = (\lambda^i_j(m-1))_{1\leq j\leq i\leq \ell}$ to obtain a new tableau $P(m) = (\lambda^i_j(m))_{1\leq j\leq i\leq \ell}$ in the following way. The shapes at level $0, 1, \ldots, k-1$ are unchanged:

$$\lambda^i(m) = \lambda^i(m-1), \qquad 0 \leq i \leq k-1.$$

The shapes at level $i \geq k$ are altered by growing a new box at row j_i, where

$$j_i = \max(\{j \leq j_{i-1} : \lambda^{i-1}_{j-1}(m-1) > \lambda^i_j(m-1)\} \cup \{1\})$$

and the boundary condition $j_{k-1} := k$. This way for $1 \leq i \leq \ell$, $\lambda^i(m)$ only depends on $\lambda^i(m-1), \lambda^{i-1}(m-1)$ and $\lambda^{i-1}(m)$.

When the input is restricted to a permutation of length n, this becomes simpler and more symmetric in level and time, as each number is only inserted once. The insertion process is encoded in a so called growth diagram, a weighted $[0, n] \times$

$[0, n]$ square lattice, such that the vertex (m, i) is labelled with the shape $\lambda^i(m)$. The boxes surrounded by $(m-1, \sigma(m)-1)$, $(m-1, \sigma(m))$, $(m, \sigma(m)-1)$ and $(m, \sigma(m))$ for $1 \le m \le n$ are marked with an X to represent the insertion of the letter $w_m = \sigma(m)$. The shapes $\lambda^0(0), \lambda^0(1), \ldots, \lambda^0(n)$ at level 0 and $\lambda^0(0), \lambda^1(0), \ldots, \lambda^n(0)$ at time 0 are all empty, which are the boundary condition. The growth is a rule of obtaining $\lambda^i(m) = f(\lambda^{i-1}(m-1), \lambda^{i-1}(m), \lambda^i(m-1), \mathrm{hasX})$ where hasX is a boolean variable $\sigma(m) == i$ indicating the existence of an X in the box. The rule f is symmetric:

$$f(\lambda, \mu^1, \mu^2, b) = f(\lambda, \mu^2, \mu^1, b)$$

which proves the symmetry property of the algorithm:

$$(P(\sigma), Q(\sigma)) = (Q(\sigma^{-1}), P(\sigma^{-1})).$$

The growth diagram, and its generalisation to the so-called branching insertion algorithms which includes the q-weighted insertion in Chapter 2, will be discussed in detail in Chapter 3.

1.4 The dynamics of the RS algorithm taking a random word

The Schur function associated with shape λ is defined as a generating function over the tableaux of shape λ:

$$s_\lambda(x) = \sum_{T \in \mathcal{T}_\lambda} a^T = \frac{\det(a_i^{\lambda_j + \ell - j})_{1 \le i,j \le \ell}}{\det(a_i^{\ell - j})_{1 \le i,j \le \ell}}, \tag{1.1}$$

where the weight $a^T := a_1^{\#1\text{'s in } T} a_2^{\#2\text{'s in } T} \ldots a_\ell^{\#\ell\text{'s in } T}$.

When the input word w is random, in the way that there exists probabilities $a_1, a_2, \ldots, a_\ell \in [0, 1]$ such that $a_1 + \cdots + a_\ell = 1$ such that $(w_i)_i$ are i.i.d. random variables with the categorical distribution

$$\mathbb{P}(w_i = j) = a_j,$$

the shape of the output tableaux evolves as a Markov process with the kernel [O'C03a]:

$$p(\lambda, \mu) = \frac{s_\mu(a)}{s_\lambda(a)} \mathbb{I}_{\lambda \nearrow \mu}, \tag{1.2}$$

where $\lambda \nearrow \mu$ means $\lambda \prec \mu$ and $|\lambda| + 1 = |\mu|$, i.e. the Young diagram μ can be obtained by adding a box to λ.

This result is obtained applying the Markov function theorem:

Theorem 1 ([RP81])**.** *Let X be a Markov process with transition kernel P on state space S, and Y a process on T where $Y_t = f(X_t)$ for some f. If there is a kernel $K : T \times S \to [0,1]$ such that $K(y,x) = 0 \forall x \notin f^{-1}(y)$ and $\sum_{x \in S} K(y,x) = 1 \forall y \in T$ and a kernel Q on T satisfying the intertwining relation*

$$KP = QK \tag{1.3}$$

then if $X_0 \sim K(Y_0, \cdot)$ then Y evolves as a Markov chain with kernel Q.

The Pieri rule, which states the Schur function is harmonic with respect to the kernel $\mathbb{I}_{\lambda \nearrow \mu}$:

$$\sum_{\mu : \lambda \nearrow \mu} s_\mu = s_\lambda \tag{1.4}$$

and the structure of s_λ as a generating function contribute to the intertwining relation (1.3) where P is the kernel of the output tableaux, $K(\lambda, T) = \frac{a^T}{s_\lambda} \mathbb{I}_{\mathrm{sh} T = \lambda}$ and Q is the kernel p in (1.2).

In the next Chapter, we exploit equations similar to the generating function structure (1.1) and harmonicity (1.4) to obtain the result in the q-setting.

Chapter 2

A q-weighted Robinson-Schensted algorithm

We introduce a q-weighted version of the Robinson-Schensted (column insertion) algorithm which is closely connected to q-Whittaker functions (or Macdonald polynomials with $t = 0$) and reduces to the usual Robinson-Schensted algorithm when $q = 0$. The q-insertion algorithm is 'randomised', or 'quantum', in the sense that when inserting a positive integer into a tableau, the output is a distribution of weights on a particular set of tableaux which includes the output which would have been obtained via the usual column insertion algorithm. There is also a notion of recording tableau in this setting. We show that the distribution of weights of the pair of tableaux obtained when one applies the q-insertion algorithm to a random word or permutation takes a particularly simple form and is closely related to q-Whittaker functions. In the case $0 \leq q < 1$, the q-insertion algorithm applied to a random word also provides a new framework for solving the q-TASEP interacting particle system introduced (in the language of q-bosons) by Sasamoto and Wadati [SW98] and yields formulas which are equivalent to some of those recently obtained by Borodin and Corwin [BC13] via a stochastic evolution on discrete Gelfand-Tsetlin patterns (or semistandard tableaux) which is coupled to the q-TASEP process. We show that the sequence of P-tableaux obtained when one applies the q-insertion algorithm to a random word defines another, quite different, evolution on semistandard tableaux which is also coupled to the q-TASEP process.

2.1 Introduction

We introduce a q-weighted version of the Robinson-Schensted (column insertion) algorithm which is closely connected to q-Whittaker functions (or Macdonald polynomials with $t = 0$) and reduces to the usual Robinson-Schensted algorithm when $q = 0$. The insertion algorithm is 'randomised', or 'quantum', in the sense that when inserting a positive integer into a tableau, the output is a distribution of weights on a particular set of tableau which includes the output which would have been obtained via the usual column insertion algorithm. As such, it is similar to the quantum insertion algorithm introduced by Date, Jimbo and Miwa [DJM90] (see also [Ber12]) but with different weights. There is also a notion of recording tableau in this setting. We show that the distribution of weights of the pair of tableaux obtained when one applies the insertion algorithm to a random word or permutation takes a particularly simple form and is closely related to q-Whittaker functions. These are functions defined on integer partitions which are eigenfunctions the relativistic Toda chain [Rui90, Rui99, Eti99, GLO10] and simply related to Macdonald polynomials (as a function of the index) with the parameter $t = 0$ [GLO11]. When $q = 0$, they are given by Schur polynomials. Our main result provides a starting point for developing a new combinatorial framework for q-Whittaker functions and related objects, such as Demazure and Kirillov-Reshetikhin crystals. It will be interesting to understand the relation to recent developments in this area, see [HHL05, Len09, RY11, BBL, LL15, ST12, BF14, LS13] and references therein.

In the case $0 \leq q < 1$, the q-insertion algorithm applied to a random word also provides a new framework for solving the q-TASEP interacting particle system introduced (in the language of q-bosons) by Sasamoto and Wadati [SW98] and yields formulas which are equivalent to some of those recently obtained by Borodin and Corwin [BC13] via a stochastic evolution on discrete Gelfand-Tsetlin patterns—or, equivalently, semistandard tableaux—which is coupled to the q-TASEP process. We show that the sequence of P-tableaux obtained when one applies the q-insertion algorithm to a random word defines another, quite different, evolution on semistandard tableaux which is also coupled to the q-TASEP process (after Poissonisation). The q-TASEP process is a particular case of the totally asymmetric zero-range process [BKS12]. See also [BCS14] for related recent work.

When $q \to 1$, the q-Whittaker functions converge with appropriate rescaling to $\mathfrak{gl}_l$-Whittaker functions [GLO12]. The main result of the present chapter can be regarded as a natural (yet non-obvious) discretisation, in time and space, of the main result of the paper [O'C12], which relates a continuous-time version of the geometric Robinson-Schensted-Knuth (RSK) correspondence introduced by A.N. Kirillov [Kir01], with Brownian motion as input, to the open quantum Toda chain with l particles. A discrete time version of that result has been developed in the papers [COSZ14, OSZ14], which is formulated directly in the context of Kirillov's geometric RSK correspondence. The present work differs significantly from [O'C12, COSZ14, OSZ14] in that the analogue of the RSK mapping we consider here is (necessarily) randomised. In the above scaling limit, the q-insertion algorithm we introduce in this chapter should converge in an appropriate sense to the continuous-time version of the geometric RSK mapping considered in [O'C12], which is deterministic, and the main result of this chapter should rescale to the main result of [O'C12]. This can be seen by comparing with the corresponding scaling limits considered in [BC13, GLO12].

The outline of the chapter is as follows. In the next section we give some background on the Robinson-Schensted algorithm. In Section 2.3, we describe the q-weighted version of this algorithm. The main result is presented in Section 2.4. In Section 2.5, we consider the q-insertion algorithm with $0 \leq q < 1$ applied to a random word and explain the connection to the q-TASEP interacting particle system. In Section 2.6 we consider the algorithm applied to a random permutation. The proofs are given in Section 2.7.

2.2 The Robinson-Schensted algorithm

The Robinson-Schensted algorithm is a combinatorial algorithm which plays a fundamental role in the theory of Young tableaux [Rob38, Sch61, Ful97, Sag00, Sta01]. There are two versions, which are in some sense dual to each other, defined via insertion (or 'bumping') algorithms known as *row insertion* and *column insertion*. The column insertion algorithm is also sometimes referred to as the dual RSK algorithm, because it has a natural extension to zero-one matrices which was introduced by Knuth [Knu70]. It is the column insertion version which we consider and generalise in this chapter.

A tableau P is a Young diagram with positive integer entries which are weakly increasing in each row and strictly increasing in each column. The corresponding diagram represents an integer partition which is referred to as the shape of the tableau P and denoted by sh$P$. For example,

$$\begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 & \\ 3 & & & \end{array}$$

is a tableau with shape $(4,3,1)$. To insert a positive integer k into a tableau P, we begin by trying to place that integer at the bottom of the first column of P. If the result is a tableau, we are done. Otherwise, it bumps the smallest entry in that column which is larger than or equal to k. Now proceed by inserting the bumped entry into the second column according to the same rule, and so on, until we have placed a bumped entry at the bottom of column (or on its own in a new column). For example, if we insert the number 2 into the tableau shown above, the outcome is

$$\begin{array}{ccccc} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & & \\ 3 & & & & \end{array}$$

In this example, the 2 in the first column is bumped into the second, the 2 in the second is bumped into the third, the 3 in the third column is bumped into the fourth, and the 3 in the fourth is bumped into a new fifth column on its own. Actually, it will be helpful for later reference to summarise this sequence of events in the following way: in this example, a 2 is inserted into the second row, and a 3 is bumped from the second row and inserted into the first row.

Now, applying this insertion algorithm recursively to a word $w = w_1 \dots w_n \in [l]^n$, starting with an empty tableau and successively inserting the numbers $w_1, w_2, \dots, w_n$, gives rise to a sequence of tableau $P(1), P(2), \dots, P(n) = P$. Note that it is not possible in general to recover the word w from the tableau P. This motivates the notion of a *recording tableau*, which we denote by Q. The tableau Q has size n and is *standard*, that is, it contains each of the numbers $1, 2, \dots, n$ exactly once. If we denote by Q^i the sub-tableau of Q consisting only of those entries which are not greated than i, then Q is defined by the requirement that $\mathrm{sh}Q^i = \mathrm{sh}P(i)$ for $1 \le i \le n$. For example, if $w = 1143232$ then

$$P = \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 2 & 2 & & \\ 3 & & & \end{array} \qquad Q = \begin{array}{cccc} 1 & 2 & 5 & 7 \\ 3 & 4 & & \\ 6 & & & \end{array}$$

The mapping $w \mapsto (P, Q)$ defines a bijection from the set of words $[l]^n$ to the set of pairs $(P, Q) \in \mathcal{T}_l \times \mathcal{S}_n$ such that $\mathrm{sh}P = \mathrm{sh}Q$, where $\mathcal{T}_l$ denotes the set of tableaux with entries from $[l]$ and $\mathcal{S}_n$ denotes the set of standard tableaux of size n. It is the column insertion version of the *Robinson-Schensted correspondence*.

As a warm up for next section, we note that the above column insertion algorithm can also be described in terms of lattice paths, as follows. Suppose we are inserting a number k with $1 \le k \le l$ into a tableau $P \in \mathcal{T}_l$, with resulting tableau $\tilde{P}$. For $1 \le i \le l$, set $\lambda^i = \mathrm{sh}P^i$, and $\tilde{\lambda}^i = \mathrm{sh}\tilde{P}^i$. Let $(e_i,\ 1 \le i \le l)$ denote the standard basis in $\mathbb{Z}^l$. Then $\tilde{\lambda}^i = \lambda^i + e_{j_i}$ where $k = j_{k-1} \ge j_k \ge \cdots \ge j_l \ge 1$ is a weakly decreasing sequence defined by

$$j_i = \max\{\{2 \le m \le j_{i-1} : \lambda^{i-1}_{m-1} - \lambda^i_m > 0\} \cup \{1\}\}, \quad i = k, k+1, \ldots, l.$$

The sequence $k \ge j_k \ge j_{k+1} \ge \cdots j_l \ge 1$ determines a down/right lattice path in $\mathbb{Z}^2$ from (k, k) to $(l+1, j_l)$ by specifying the y-coordinates at which the path moves to the right. From the definition, this path takes a horizontal step to the right $(i, j) \to (i+1, j)$ whenever $\lambda^{i-1}_{j-1} > \lambda^i_j$ or $j = 1$, otherwise it takes a step down $(i, j) \to (i, j-1)$. We will refer to this lattice path as the *insertion path*. The interpretation is as follows. A horizontal portion of the path starting at (i, j) represents inserting an i into the jth row. A vertical portion starting at (i, j) and ending at $(i, j-r)$ indicates that an i is bumped from the jth row to the $(j-r)$th row. For example, the insertion path corresponding to the previous example of inserting a 2 into the tableau

$$\begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 & \\ 3 & & & \end{array}$$

with $l = 3$ is illustrated in Figure 2.1.

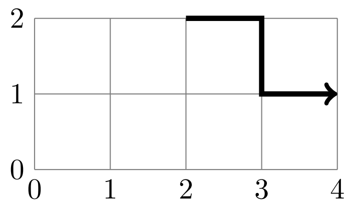


Figure 2.1: An insertion path

2.3 The q-weighted version

In this chapter, we consider the following generalisation of the column insertion algorithm. It is defined by a collection of kernels $I_k(P, \tilde{P})$ which depend on a complex parameter q. We assume throughout that q is not a root of unity. If $0 \le q < 1$, we interpret the quantity $I_k(P, \tilde{P})$ as the probability that, when we insert k into the tableau P, the output is $\tilde{P}$. Recall that the *type* of a tableaux P, which we denote ty$P$, is the composition $\mu = (\mu_1, \mu_2, \ldots)$ where μ_i is the number of i's in P. The set of $\tilde{P}$ for which $I_k(P, \tilde{P}) \neq 0$ has the following properties. The type of $\tilde{P}$ is given by ty$\tilde{P}$ = ty$P + e_k$. The shape of $\tilde{P}$ satisfies sh$\tilde{P}$ = sh$P + e_j$ for some $1 \le j \le k$. Moreover, if we set $\lambda^i = \mathrm{sh}P^i$ and $\tilde{\lambda}^i = \mathrm{sh}\tilde{P}^i$, then there is a weakly decreasing sequence $k = j_{k-1} \ge j_k \ge j_{k+1} \ge \cdots j_l \ge 1$ such that $\tilde{\lambda}^i = \lambda^i$ for $1 \le i < k$ and $\tilde{\lambda}^i = \lambda^i + e_{j_i}$ for $k \le i \le l$. The kernel $I_k(P, \tilde{P})$ is defined to be zero if there is no such sequence; if there is such a sequence, it is given as follows. Define

$$f_0(i,j) = 1 - q^{\lambda^{i-1}_{j-1} - \lambda^i_j}, \qquad f_1(i,j) = \frac{1 - q^{\lambda^{i-1}_{j-1} - \lambda^i_j}}{1 - q^{\lambda^{i-1}_{j-1} - \lambda^{i-1}_j}}, \qquad \text{for } j > 1;$$

$$f_0(i,1) = f_1(i,1) = 1.$$

and set

$$f(i,j) = \begin{cases} f_1(i,j), & \text{if } j = j_{i-1} \text{ and } i \neq k; \\ f_0(i,j), & \text{otherwise.} \end{cases}$$

Then

$$I_k(P, \tilde{P}) = \prod_{i=k}^{l} \left(f(i, j_i) \prod_{j=j_i+1}^{j_{i-1}} (1 - f(i,j)) \right). \tag{2.1}$$

It follows easily from the definition that

$$\sum_{\tilde{P}} I_k(P, \tilde{P}) = 1.$$

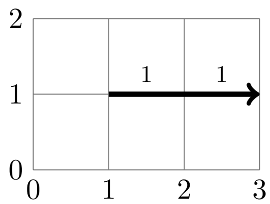
If $0 \le q < 1$, then $I_k(P, \tilde{P}) \ge 0$. In this case, for each k and P, $I_k(P, \cdot)$ defines a probability distribution on $\mathcal{T}_l$ and we interpret $I_k(P, \tilde{P})$ as the probability that, when we insert k into the tableau P, the output is $\tilde{P}$.

The formula (2.1) can be interpreted in terms of insertion paths, as follows. The sequence $k \ge j_k \ge j_{k+1} \ge \cdots j_l \ge 1$ determines a down/right lattice path in $\mathbb{Z}^2$ from (k,k) to the vertical boundary $\{(l+1, j),\ 1 \le j \le k\}$ by specifying the y-

coordinates at which the path moves to the right. The edge weights are $f(i,j)$ on the horizontal edge $(i,j) \to (i+1,j)$ and $1-f(i,j)$ on the vertical edge $(i,j) \to (i,j-1)$, and taking a product of these weights along the path gives the weight $I_k(P,\tilde{P})$ for the corresponding output $\tilde{P}$. We interpret this path as the insertion path associated with q-inserting the number k into P with resulting tableau $\tilde{P}$. As before, a horizontal portion of the path starting at (i,j) represents inserting an i into the jth row. A vertical portion starting at (i,j) and ending at $(i,j-r)$ indicates that an i is bumped from the jth row to the $(j-r)$th row. When $q=0$, there is only one output tableau $\tilde{P}$ with non-zero weight, namely the output of the usual column insertion algorithm. Moreover, if we denote by ω_0 the insertion path corresponding to this tableau and by $S(k,P)$ the set of insertion paths corresponding to the support of $I_k(P,\cdot)$ for nonzero q, then $\omega_0 \in S(k,P)$ and it is the 'highest' path in $S(k,P)$ in the sense that the sequence $k \geq j_k \geq j_{k+1} \geq \cdots j_l \geq 1$ is maximal (in the second example below, it is the path shown on the top left of Figure 2).

Let us compute the kernel $I_k(P,\tilde{P})$ for some concrete examples.

Example 2. Suppose $l=2$. If we are inserting a 1 into $P \in \mathcal{T}_2$ there is only one possible outcome $\tilde{P}$ with $I_1(P,\tilde{P}) \neq 0$, namely the one obtained by the usual column insertion algorithm: the 1 is inserted into the first row, pushing the existing first row over by one. The weighted insertion path in this case is very simple:



For example, if

$$P = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & & & \end{array}$$

then, setting

$$\tilde{P}_1 = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & & & & \end{array}$$

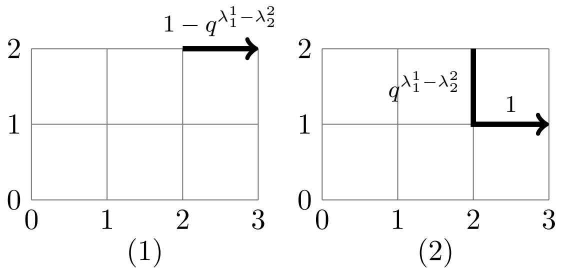
we have

$$I_1(P,\tilde{P}) = \begin{cases} 1 & \text{if } \tilde{P} = \tilde{P}_1 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if we are inserting a 2 there are two possibilities:

1. The 2 is inserted into the second row, pushing the existing second row over by one: this outcome has weight $1 - q^{\lambda_1^1 - \lambda_2^2}$.
2. The 2 is inserted into the first row, pushing the existing 2's over by one: this outcome has weight $q^{\lambda_1^1 - \lambda_2^2}$.

Note that these weights sum to one, as is always the case. The corresponding insertion paths, with edge weights indicated, are:



The quantity $\lambda_1^1 - \lambda_2^2$ is the difference between the number of 1's in the first row and the number of 2's in the second row, see Figure 2.2.

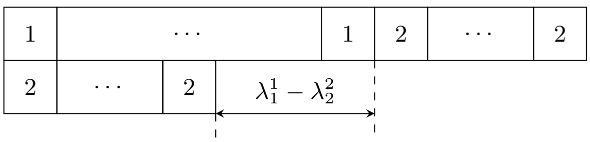


Figure 2.2: The quantity $\lambda_1^1 - \lambda_2^2$ in the exponent in Example 2

For example, inserting a 2 into

$$P = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & & & \end{array}$$

gives

$$I_2(P, \tilde{P}) = \begin{cases} 1 - q & \text{if } \tilde{P} = \tilde{P}_2 \\ q & \text{if } \tilde{P} = \tilde{P}_3 \\ 0 & \text{otherwise.} \end{cases}$$

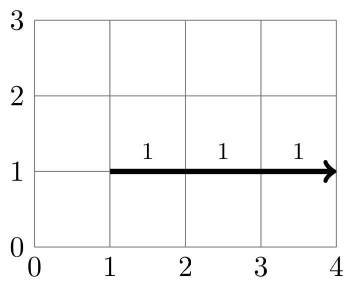
where

$$\tilde{P}_2 = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 2 & & \end{array}$$

and

$$\tilde{P}_3 = \begin{array}{ccccc} 1 & 1 & 2 & 2 & 2 \\ 2 & & & & \end{array}.$$

Example 3. Suppose $l = 3$. If we are inserting a 1 into $P \in \mathcal{T}_3$ there is only one possible outcome $\tilde{P}$ with $I_1(P, \tilde{P}) \neq 0$, namely the one obtained by the usual column insertion algorithm: the 1 is inserted into the first row, pushing the existing first row over by one. The corresponding weighted insertion path is:



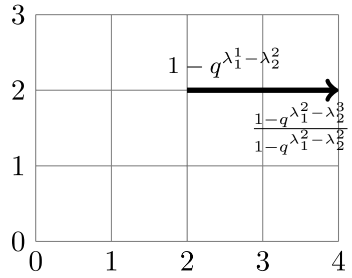
If we are inserting a 2, there are three possible outcomes:

1. The 2 is inserted into the second row, pushing the existing second row over by one: this outcome has weight
$$(1 - q^{\lambda_1^1 - \lambda_2^2}) \frac{1 - q^{\lambda_1^2 - \lambda_2^3}}{1 - q^{\lambda_1^2 - \lambda_2^2}};$$

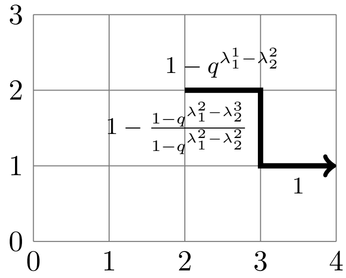
2. The 2 is inserted into the second row, bumping a 3 into the first row: this outcome has weight
$$(1 - q^{\lambda_1^1 - \lambda_2^2}) \left(1 - \frac{1 - q^{\lambda_1^2 - \lambda_2^3}}{1 - q^{\lambda_1^2 - \lambda_2^2}}\right);$$

3. The 2 is inserted into the first row, pushing existing 2's and 3's in first row over by one: this outcome has weight $q^{\lambda_1^1 - \lambda_2^2}$.

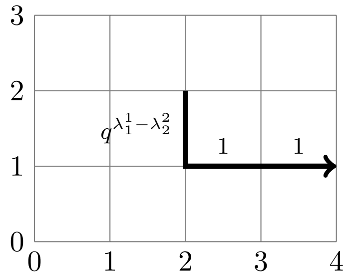
The corresponding insertion paths, with edge weights indicated, are:



(1)

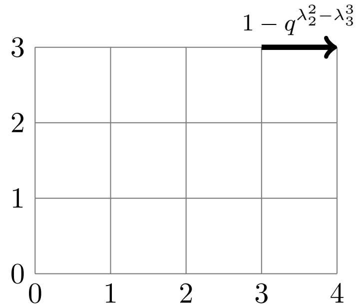


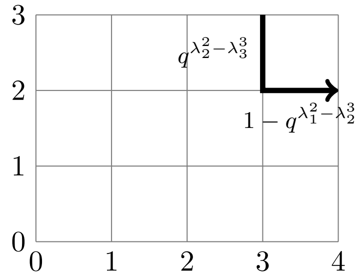
(2)

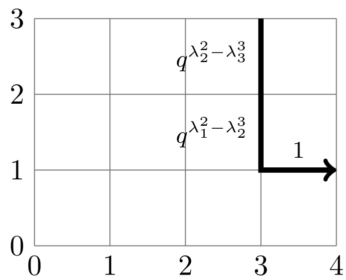


(3)

If we are inserting a 3, there are also three possible outcomes: the 3 is placed in the third, second or first row with respective weights $1-q^{\lambda_2^2-\lambda_3^3}$, $q^{\lambda_2^2-\lambda_3^3}(1-q^{\lambda_1^2-\lambda_2^3})$ and $q^{\lambda_2^2-\lambda_3^3}q^{\lambda_1^2-\lambda_2^3}$. The corresponding insertion paths, with edge weights indicated, are:







The quantities $\lambda_1^1 - \lambda_2^2$, $\lambda_1^2 - \lambda_2^3$, etc. which appear in the above weights are illustrated in Figure 2.3.

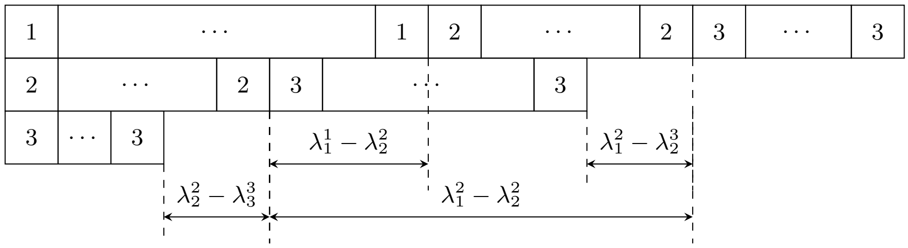


Figure 2.3: The exponent quantities in Example 3.

Example 4. Suppose we are inserting a 3 into

$$P = \begin{array}{llllll} 1 & 2 & 2 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 & & \\ 3 & 4 & & & & \\ 5 & & & & & \end{array} \tag{2.2}$$

The (four) possible output tableaux $\tilde{P}$ and their weights $I_3(P, \tilde{P})$ are shown in Figure 2.4, along with the corresponding weighted insertion paths.

The q-insertion algorithm can be applied to a word $w = w_1 \ldots w_n \in [l]^n$, starting with an empty tableau and successively inserting the numbers $w_1, w_2, \ldots, w_n$, multiplying the weights along each possible sequence of output tableaux $P(1), \ldots, P(n) = P$ to obtain a distribution of weights $\phi_w(P, Q)$ on $\mathcal{T}_l \times \mathcal{S}_n$. More precisely, we define $\phi_w(P, Q)$ recursively as follows. Set

$$\phi_k(P, Q) = \begin{cases} 1 & \text{if } P = k \text{ and } Q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $w \in [l]^n$ and $(\tilde{P}, \tilde{Q}) \in \mathcal{T}_l \times \mathcal{S}_{n+1}$ with $\mathrm{sh}\tilde{P} = \mathrm{sh}\tilde{Q}$, define

$$\phi_{wk}(\tilde{P}, \tilde{Q}) = \sum \phi_w(P, Q) I_k(P, \tilde{P}),$$

where the sum is over $(P, Q) \in \mathcal{T}_l \times \mathcal{S}_n$ with $\mathrm{sh}P = \mathrm{sh}Q$.

We conclude this section by giving a more algorithmic description of the q-insertion algorithm. For this it is convenient to assume $0 \leq q < 1$ and describe it using probabilistic language, although it will be clear how to modify this using

$$\tilde{P} = \begin{array}{llllll} 1 & 2 & 2 & 2 & 3 & 5 \\ 2 & 3 & 3 & 4 & 5 & \\ 3 & 4 & & & & \\ 5 & & & & & \end{array}, \qquad I_3(P, \tilde{P}) = (1-q^2)\frac{1-q^2}{1-q^3}\frac{1-q}{1-q^2};$$

$$\tilde{P} = \begin{array}{lllllll} 1 & 2 & 2 & 2 & 3 & 5 & 5 \\ 2 & 3 & 3 & 4 & & & \\ 3 & 4 & & & & & \\ 5 & & & & & & \end{array}, \qquad I_3(P, \tilde{P}) = (1-q^2)\frac{1-q^2}{1-q^3}\left(1 - \frac{1-q}{1-q^2}\right);$$

$$\tilde{P} = \begin{array}{lllllll} 1 & 2 & 2 & 2 & 3 & 4 & 5 \\ 2 & 3 & 3 & 5 & & & \\ 3 & 4 & & & & & \\ 5 & & & & & & \end{array}, \qquad I_3(P, \tilde{P}) = (1-q^2)\left(1 - \frac{1-q^2}{1-q^3}\right);$$

$$\tilde{P} = \begin{array}{lllllll} 1 & 2 & 2 & 2 & 3 & 3 & 5 \\ 2 & 3 & 4 & 5 & & & \\ 3 & 4 & & & & & \\ 5 & & & & & & \end{array}, \qquad I_3(P, \tilde{P}) = q^2.$$

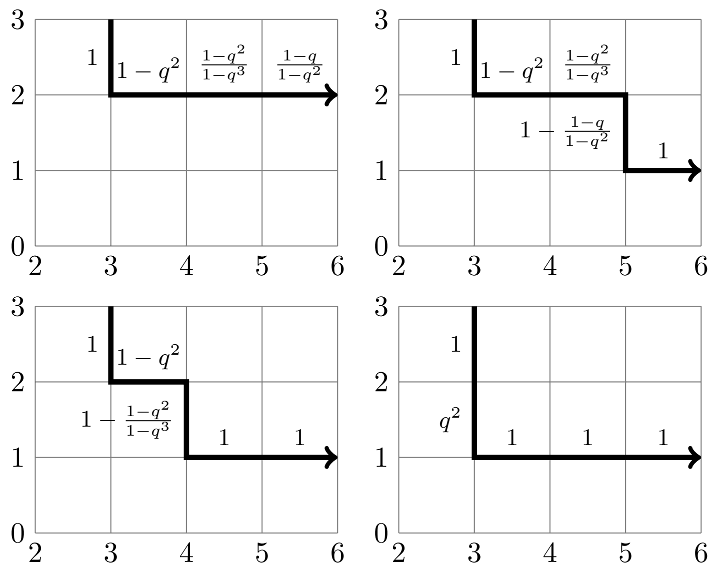


Figure 2.4: The four possible output tableaux $\tilde{P}$, their weights $I_3(P, \tilde{P})$, and the corresponding insertion paths, with edge weights indicated, for $k = 3$ and P given by (2.2).

the language of ‘weights’ in the general case. For reference, we begin with an algorithmic description of the usual column insertion algorithm. Denote the input word by $w \in [l]^n$.

1. Set $i \leftarrow 1$ and $(P, Q) = (\emptyset, \emptyset)$.

2. Set $k \leftarrow w_i$ and $j \leftarrow k$.

3. If $\lambda_{j-1}^{k-1} = \lambda_j^k$ and $j > 1$ then set $j \leftarrow j - 1$; otherwise k displaces the first number s in jth row of the tableau that is larger than k ($s = \infty$ and k is appended at the end of the row if no such number exists) and set $k \leftarrow s$.

4. If $k = \infty$ then append i to Q such that P and Q have the same shape, set $i \leftarrow i + 1$ and go to step (2); otherwise go to step (3).

The q-insertion algorithim is defined as follows. We adopt here the following convention: for $i > 0$, let

$$q^{\lambda_0^{i-1} - \lambda_1^i} = q^{\lambda_0^i - \lambda_1^i} = q^{\lambda_0^i - \lambda_0^{i-1}} = q^{\lambda_i^i - \lambda_{i+1}^i} = q^{\lambda_i^i - \lambda_i^{i-1}} = q^{\lambda_i^{i-1} - \lambda_{i+1}^i} = 0.$$

This convention is used for covering boundary conditions in general arguments. It is only used in the following description of the q-insertion algorithm as well as in Section 2.7.1. Otherwise the undefined λ_j^i for $j > i$ or $j = 0$ are taken to be zero.

1. Set $i \leftarrow 1$ and $(P, Q) = (\emptyset, \emptyset)$.

2. Set $k \leftarrow w_i$, $j \leftarrow k$, $d \leftarrow 0$ and $a_e(m, n) \leftarrow f_e(m, n)$ $\forall e \in \{0, 1\}, 1 \leq n \leq m$.

3. With probability $1 - a_d(k, j)$ set $j \leftarrow j - 1$ and $d \leftarrow 0$; otherwise k displaces the first number s in jth row of the tableau that is larger than k ($s = \infty$ and append k at the end of jth row if no such number exists) and set $k \leftarrow s$ and $d \leftarrow 1$.

4. If $k = \infty$ then append i to Q such that P and Q have the same shape, set $i \leftarrow i + 1$ and go to step (2); otherwise go to step (3).

As is obvious, when $q = 0$ it reduces to the usual column insertion algorithm.

2.4 Main result

The weights $\phi_w(P, Q)$ are quite complicated. The main result of this chapter is that a remarkable simplification occurs when we average over the set of words. Before stating the result, we first introduce two more functions on tableaux and explain their connection to q-Whittaker functions and Macdonald polynomials. Denote the q-Pochhammer symbol by

$$(n)_q := (q; q)_n = (1 - q) \dots (1 - q^n),$$

with the conventions $(n)_0 = (0)_q = 1$, and the q-binomial coefficients by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(n)_q}{(m)_q(n-m)_q}.$$

For $P \in \mathcal{T}_l$ with $\mathrm{sh} P^i = \lambda^i$, $1 \le i \le l$, writing $\lambda = \lambda^l$, define

$$\begin{aligned} \kappa(P) &= \frac{\prod_{j=2}^{l-1} \prod_{i=1}^{j-1} (\lambda_i^j - \lambda_{i+1}^j)_q}{\prod_{j=1}^{l-1} \prod_{i=1}^{j} (\lambda_i^j - \lambda_{i+1}^{j+1})_q (\lambda_i^{j+1} - \lambda_i^j)_q} \\ &= \Delta_l(\lambda)^{-1} \prod_{1 \le j < i \le l} \begin{bmatrix} \lambda_j^i - \lambda_{j+1}^i \\ \lambda_j^i - \lambda_j^{i-1} \end{bmatrix}_q, \end{aligned}$$

where

$$\Delta_l(\lambda) = \prod_{i=1}^{l-1} (\lambda_i - \lambda_{i+1})_q.$$

For $Q \in \mathcal{S}_n$ with $\mathrm{sh} Q^i = \mu^i$, $1 \le i \le n$, define

$$\rho(Q) = \prod_{1 \le i \le j:\ \mu_j^i - \mu_j^{i-1} = 1} (1 - q^{\mu_j^i - \mu_{j+1}^i}).$$

The functions κ and ρ are simply related as follows. Suppose that $l \ge n$ and P has distinct entries $i_1 < i_2 < \cdots < i_n$. Denote by $\hat{P} \in \mathcal{S}_n$ the standard tableau obtained by replacing the entry i_k by k, for each $k = 1, \ldots, n$. Then

$$\kappa(P) = \frac{\rho(\hat{P})}{(1-q)^n \Delta_l(\lambda)}. \tag{2.3}$$

Indeed, using the simple identities,

$$\begin{bmatrix} a \\ 0 \end{bmatrix}_q = 1, \qquad \begin{bmatrix} a \\ 1 \end{bmatrix}_q = \frac{1-q^a}{1-q}, \tag{2.4}$$

we have

$$
\begin{aligned}
\kappa(P) &= \Delta_l(\lambda)^{-1} \prod_{\substack{1\le i<j\le l\\ \lambda_j^i-\lambda_j^{i-1}=1}} \begin{bmatrix} \lambda_j^i - \lambda_{j+1}^i \\ \lambda_j^i - \lambda_j^{i-1} \end{bmatrix}_q \prod_{\substack{1\le i<j\le l\\ \lambda_j^i-\lambda_j^{i-1}=0}} \begin{bmatrix} \lambda_j^i - \lambda_{j+1}^i \\ \lambda_j^i - \lambda_j^{i-1} \end{bmatrix}_q \\
&= \Delta_l(\lambda)^{-1} \prod_{\substack{1\le i<j\le l\\ \lambda_j^i-\lambda_j^{i-1}=1}} \frac{1-q^{\lambda_j^i-\lambda_{j+1}^i}}{1-q} \prod_{\substack{1\le i=j\le l\\ \lambda_j^i-\lambda_j^{i-1}=1}} \frac{1-q}{1-q} \\
&= \frac{\rho(\hat{P})}{(1-q)^n \Delta_l(\lambda)}.
\end{aligned}
$$

The functions κ and ρ are closely related to q-Whittaker functions [Rui90, Eti99, GLO10, GLO12]. Denote by Ω^l the set of partitions with at most l parts. The q-Whittaker function with parameter $a \in \mathbb{C}^l$ is a function on Ω^l defined by

$$\Psi_a(\lambda) = \sum_{P\in\mathcal{T}_l:\ \mathrm{sh}P=\lambda} a^P \kappa(P). \tag{2.5}$$

In [GLO11] it is shown that these functions are given in terms of the Macdonald polynomials $P_\lambda(x;q,t)$ as

$$\Psi_a(\lambda) = \Delta_l(\lambda)^{-1} P_\lambda(a;q,0). \tag{2.6}$$

From this it follows that

$$\Psi_a(\lambda) = \Delta_l(\lambda)^{-1} \sum_\mu k_{\lambda\mu}(q) m_\mu(a) \tag{2.7}$$

where m_μ denote the monomial symmetric functions and

$$k_{\lambda\mu}(q) = \Delta_l(\lambda) \sum_{\mathrm{sh}P=\lambda,\mathrm{ty}P=\mu} \kappa(P) = \sum_\nu K_{\lambda\nu}(q,0) K_{\nu\mu}, \tag{2.8}$$

where $K_{\lambda\mu}(q,t)$ are the two-variable Kostka polynomials [Mac98]. We recall that $K_{\lambda\nu}(q,0) = K_{\lambda'\nu'}(0,q) = K_{\lambda'\nu'}(q)$, where $K_{\lambda\mu}(t) = K_{\lambda\mu}(0,t)$ are the single-variable Kostka polynomials. For an extensive survey of the various properties and interpretations of these polynomials, see [Ana01]. When $q = 0$, $\kappa(P) \equiv 1$ and $k_{\lambda\mu}(0)$ is equal to the Kostka number $K_{\lambda\mu}$, which is the number of tableaux with shape λ

and type μ. In this case, $\Psi_a(\lambda)$ is given by the Schur polynomial

$$\Psi_a(\lambda) = s_\lambda(a) = \sum_\mu K_{\lambda\mu} m_\mu(a).$$

We will also consider the following functions:

$$f^\lambda(q) = \sum_{Q \in \mathcal{S}_n:\ \mathrm{sh} Q = \lambda} \rho(Q).$$

Note that $f^\lambda(0) = f^\lambda$, the number of standard tableaux with shape λ. The relation between $f^\lambda(q)$ and the Whittaker functions Ψ_a is given by the following proposition, which is a straightforward consequence of (2.3). Define

$$\Delta(\lambda) = \prod_{i=1}^{l(\lambda)} (\lambda_i - \lambda_{i+1})_q,$$

where $l(\lambda)$ denotes the number of parts in λ.

Proposition 5. *For each $\lambda \vdash n$,*

$$\lim_{l \to \infty} \Psi_{(1/l)^l}(\lambda) = \frac{f^\lambda(q)}{n!(1-q)^n \Delta(\lambda)}.$$

It follows, using

$$\lim_{l \to \infty} s_\lambda((1/l)^l) = f^\lambda / n!,$$

that $f^\lambda(q)$ is also given, for $\lambda \vdash n$, by

$$f^\lambda(q) = (1-q)^n \sum_\mu K_{\lambda\mu}(q,0) f^\mu.$$

To understand this in terms of specializations, recall that the exponential specialization ex_1 is the homomorphism defined on the ring of symmetric functions by $\mathrm{ex}_1(p_n) = \delta_{n1}$, where p_n are the elementary power sums (see, for example, [Sta01, §7.8]). It follows from the above proposition (or can be seen directly) that

$$f^\lambda(q) = n!(1-q)^n \mathrm{ex}_1\left(P_\lambda(q,0)\right).$$

The q-Whittaker functions Ψ_a are eigenfunctions of Ruijsenaars' relativistic

Toda difference operators [Rui90, Rui99, Eti99, GLO10]. In particular,

$$L\Psi_a = \left(\sum_i a_i\right)\Psi_a, \tag{2.9}$$

where L is the kernel operator defined by

$$L(\lambda,\mu) = \begin{cases} c_i(\lambda) & \text{if } \mu = \lambda + e_i \text{ for some } 1 \le i \le l, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$c_i(\lambda) = \begin{cases} 1 - q^{\lambda_i - \lambda_{i+1} + 1} & \text{for } 1 \le i < l, \\ 1 & \text{for } i = l. \end{cases}$$

Our main result is the following.

Theorem 6. *Let $(P,Q) \in \mathcal{T}_l \times \mathcal{S}_n$ with $shP = shQ = \lambda$. Then*

$$\sum_{w\in[l]^n} \phi_w(P,Q) = (\lambda_l)_q^{-1}\kappa(P)\rho(Q). \tag{2.10}$$

We note the following immediate extension of this identity which is useful for applications. The type of a word w is the composition $\mu = (\mu_1, \mu_2, \ldots)$ where μ_i is the number of i's in w. For $a = (a_1, \ldots, a_l)$ and μ a composition, write $a^\mu = a_1^{\mu_1} \ldots a_l^{\mu_l}$; for $w \in [l]^n$ and $P \in \mathcal{T}_l$, write $a^w = a^{\mathrm{ty}(w)}$ and $a^P = a^{\mathrm{ty}P}$. Now, since $\phi_w(P,Q) = 0$ unless $\mathrm{ty}P = \mathrm{ty}(w)$, we can write

$$\sum_{w\in[l]^n} a^w \phi_w(P,Q) = (\lambda_l)_q^{-1} a^P \kappa(P)\rho(Q). \tag{2.11}$$

Summing (2.11) over P and Q gives

$$\sum_{(P,Q)\in\mathcal{T}_l\times\mathcal{S}_n:\mathrm{sh}P=\mathrm{sh}Q=\lambda}\ \sum_{w\in[l]^n} a^w\phi_w(P,Q) = (\lambda_l)_q^{-1}\Psi_a(\lambda) f^\lambda(q).$$

Note that this implies the Cauchy-Littlewood type identity

$$\sum_{\lambda\vdash n} (\lambda_l)_q^{-1}\Psi_a(\lambda) f^\lambda(q) = \left(\sum_i a_i\right)^n.$$

Theorem 6 also yields some combinatorial formulas.

Corollary 7. *Let $\lambda, \mu \vdash n$ with at most l parts, and let Q be a standard tableau with* *shape λ. Then*

$$P_\lambda(a;q,0) = \sum_{w\in[l]^n} H_Q(w)a^w$$

and

$$k_{\lambda\mu}(q) = \sum_{w\in[l]^n:\ ty(w)=\mu} H_Q(w),$$

where

$$H_Q(w) = \frac{\Delta(\lambda)}{\rho(Q)} \sum_P \phi_w(P,Q).$$

Similarly, for any fixed $P \in \mathcal{T}_l$ with shape $\lambda \vdash n$,

$$f^\lambda(q) = \sum_{w\in[l]^n} G_P(w),$$

where

$$G_P(w) = \frac{(\lambda_l)_q}{\kappa(P)} \sum_Q \phi_w(P,Q).$$

Taking P to be standard with shape $\lambda \vdash n$, this last formula becomes

$$ex_1\left(P_\lambda(q,0)\right) = \frac{1}{n!}\sum_{\sigma\in S_n} H_P(\sigma),$$

where the sum is over permutations and $H_P(\sigma)$ indicates the function H_P evaluated at the word $\sigma^{-1}(1)\ldots\sigma^{-1}(n)$.

When $q = 0$, $H_S(w)$ equals 1 if the Q-tableau obtained by applying the Robinson-Schensted algorithm with column insertion to w is S, and 0 otherwise; similarly, $G_T(w)$ equals 1 if the P-tableau obtained by applying the Robinson-Schensted algorithm with column insertion to w is T, and 0 otherwise. The functions G_T and H_S thus generalise the notions of P-equivalence and Q-equivalence, or Knuth and dual Knuth equivalence, for the Robinson-Schensted algorithm with column insertion (see, for example, [Ful97, Chapter 2 and §A.3]).

The key ingredient in the proof of Theorem 6 in Section 2.7.1 is the following intertwining relation. Define kernel operators K and M by

$$K(\lambda,P) = a^P\kappa(P)\mathbb{I}_{\mathrm{sh}P=\lambda}, \qquad M(P,\tilde{P}) = \sum_{k=1}^{l} a_k I_k(P,\tilde{P}).$$

Proposition 8. *The following intertwining relation holds:*

$$KM = LK \tag{2.12}$$

We remark that (2.12) immediately yields the eigenvalue equation (2.9).

2.5 Stochastic evolutions

If $0 \leq q < 1$ and $a \in \mathbb{R}^l_+$ with $\sum_i a_i = 1$, then

$$\sum_{w \in [l]^n} a^w \phi_w(P,Q) = (\lambda_l)_q^{-1} a^P \kappa(P) \rho(Q) \tag{2.13}$$

defines a probability measure on $\mathcal{T}_l \times \mathcal{S}_n$, which can be interpreted as the distribution of the pair of tableaux obtained when one applies the randomised insertion algotihm to a random word $w_1 \dots w_n$ with each w_i chosen independently at random from $[l]$ according to the probabilities $a_1, \dots, a_l$. If we denote by $\mathcal{L}(m)$ the shape of the tableau obtained after inserting the first m entries $w_1 \dots w_m$ then, given the interpretation of Q as a recording tableau, we conclude by summing (2.13) over P that the sequence of shapes $\mathcal{L}(1), \dots, \mathcal{L}(n)$ is distributed according to

$$\mathbb{P}(\mathcal{L}(1) = \mu^1, \dots, \mathcal{L}(n) = \mu^n) = (\mu^n_\ell)_q^{-1} \Psi_a(\mu^n) \rho(Q),$$

where $Q \in \mathcal{S}_n$ is defined by $\mathrm{sh}Q^i = \mu^i$, $i = 1, \dots, n$. But this can be written as

$$\mathbb{P}(\mathcal{L}(1) = \mu^1, \dots, \mathcal{L}(n) = \mu^n) = \prod_{i=1}^n \frac{\Psi_a(\mu^i)}{\Psi_a(\mu^{i-1})} L(\mu^{i-1}, \mu^i).$$

Since n is arbitrary, we immediately conclude the following. Write $\mu \nearrow \lambda$ if λ is obtained from μ by adding a single box.

Theorem 9. *When applying the randomised insertion algorithm to a random word $w_1 w_2 \dots$ with each w_i chosen independently at random from $[l]$ according to the probabilities $a_1, \dots, a_l$ the sequence of tableaux $\mathcal{P}(n)$, $n \geq 0$ obtained evolves as a Markov chain in $\mathcal{T}_l$ with transition probabilities*

$$M(P, \tilde{P}) = \sum_{k=1}^n a_k I_k(P, \tilde{P}).$$

The sequence of shapes $\mathcal{L}(n) = sh\mathcal{P}(n)$ evolves as a Markov chain in Ω^l with tran-

sition probabilities

$$p(\mu,\lambda)=\frac{\Psi_a(\lambda)}{\Psi_a(\mu)}L(\mu,\lambda)\mathbb{I}_{\mu\nearrow\lambda}.$$

The conditional law of $\mathcal{P}(n)$*, given* $\{\mathcal{L}(1),\dots,\mathcal{L}(n);\ \mathcal{L}(n)=\lambda\}$*, is*

$$\mathbb{P}(\mathcal{P}(n)=P|\ \mathcal{L}(1),\dots,\mathcal{L}(n);\ \mathcal{L}(n)=\lambda)=\frac{K(\lambda,P)}{\Psi_a(\lambda)}.$$

The conditional law of $ty\mathcal{P}(n)$*, given* $\{\mathcal{L}(1),\dots,\mathcal{L}(n);\ \mathcal{L}(n)=\lambda\}$*, is*

$$\mathbb{P}(ty\mathcal{P}(n)=\mu|\ \mathcal{L}(1),\dots,\mathcal{L}(n);\ \mathcal{L}(n)=\lambda)=\frac{a^{\mu}k_{\lambda\mu}(q)}{\Psi_a(\lambda)}.$$

The distribution of $\mathcal{L}(n)$ *is given by*

$$\nu(\lambda):=\mathbb{P}(\mathcal{L}(n)=\lambda)=(\lambda_l)_q^{-1}\Psi_a(\lambda)f^{\lambda}(q).$$

The probability distribution ν is a particular specialisation (and restriction to $\lambda\vdash n$) of the Macdonald measures introduced by Forrester and Rains [PJF02], see also [BC13]. When $q=0$, the above theorem reduces to the fact [O'C03a] that, when applying the usual column insertion algorithm to a random word with probabilities $a_1,\dots,a_l$, the shape of the tableau evolves as a Markov chain with transition probabilities

$$p(\mu,\lambda)=\frac{s_\lambda(a)}{s_\mu(a)}\mathbb{I}_{\mu\nearrow\lambda}.$$

If $a_1>a_2>\dots>a_l$ this Markov chain can be interpreted as a random walk in $\mathbb{N}^l$ with transition probabilities

$$r(\mu,\lambda)=a^{\lambda-\mu}\mathbb{I}_{\mu\nearrow\lambda}$$

conditioned never to exit the Weyl chamber $\{\lambda\in\mathbb{N}^l:\ \lambda_1\ge\dots\ge\lambda_l\}$, which can be identified with Ω^l. This result, which relates to the representation theory of $\mathfrak{gl}_l$, has been generalised to arbitrary complex semisimple Lie algebras in [BBO05, LLP12]. For earlier related work on the asymptotics of longest monotone subsequences in random words, see [TW01]. When $q\to1$ the q-Whittaker functions converge with appropriate rescaling to $\mathfrak{gl}_l$-Whittaker functions [GLO12], and the above theorem should re-scale to the main result of the paper [O'C12], which relates a continuous-time version of the geometric RSK correspondence introduced by A.N. Kirillov [Kir01], with Brownian motion as input, to the open quantum Toda chain with l particles. In this scaling limit, the q-insertion algorithm should converge in an appropriate sense to the continuous-time version of the geometric RSK mapping considered in [O'C12], which is deterministic. The results of [O'C12] have been generalised in [Chh13] (see also [BBO09]) to arbitrary complex semisimple Lie algebras. It is natural to expect the results of the present chapter to admit a similar generalisation.

Example 10. The rank-1 case ($l = 2$) of Theorem 9 is discussed in [O'C14]. Setting $\mathcal{L}^i(n) = \mathrm{sh}\mathcal{P}^i(n)$, the evolution on tableaux in this case is driven by the process $Y(n) = \mathcal{L}^1_1(n) - \mathcal{L}^2_1(n)$, $n \geq 0$, which (setting $p = a_1$) is a birth and death process as illustrated in Figure 2.5.

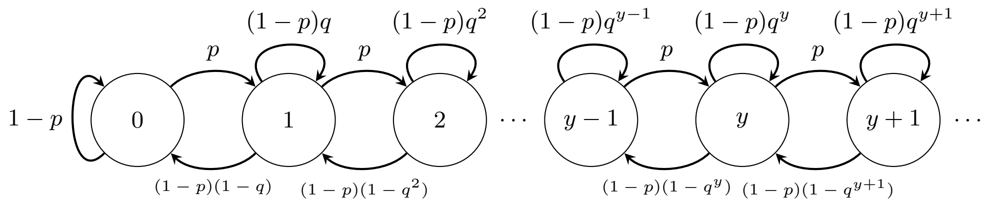


Figure 2.5: The birth-and-death process Y

Example 11. When $l = 3$ the algorithm is more complicated than in the $l = 2$ case because the push-or-bump probability $f_1(3, 2)$ appears. In this case the algorithm with random input is described as follows (cf. Example 2). In the following, w.p. means "with probability".

- w.p. a_1, insert 1 to row 1, pushing 2's and 3's in row 1
- w.p. a_2, insert 2
  - w.p. $1 - q^{\lambda^1_1 - \lambda^2_2}$, the 2 is inserted to row 2 and the displaced 3 is either pushed or bumped
    * w.p. $(1 - q^{\lambda^2_1 - \lambda^3_2})/(1 - q^{\lambda^2_1 - \lambda^2_2})$ the displaced 3 is pushed in row 2
    * w.p. $1 - (1 - q^{\lambda^2_1 - \lambda^3_2})/(1 - q^{\lambda^2_1 - \lambda^2_2})$ the displaced 3 is bumped to row 1
  - w.p. $q^{\lambda^1_1 - \lambda^2_2}$, the 2 is inserted to row 1 and it pushes 3's in row 1
- w.p. a_3, insert 3
  - w.p. $1 - q^{\lambda^2_2 - \lambda^3_3}$, the 3 is inserted to row 3
  - w.p. $q^{\lambda^2_2 - \lambda^3_3}(1 - q^{\lambda^2_1 - \lambda^3_2})$ the 3 is inserted to row 2

– w.p. $q^{\lambda_2^2-\lambda_3^3}q^{\lambda_1^2-\lambda_2^3}$ the 3 is inserted to row 1

The q-insertion algorithm applied to a random word is closely related to the q-TASEP interacting particle system. This is a variation of the totally asymmetric simple exclusion process (TASEP) which was introduced (in the language of q-bosons) and shown to be integrable by Sasamoto and Wadati [SW98], and recently related to q-Whittaker functions by Borodin and Corwin [BC13]. The process is defined as follows. There are l particles on the integer lattice, and we denote their positions by $x_1 > x_2 > \cdots > x_l$. Let $a_1, a_2, \ldots, a_l \in \mathbb{R}_+$. Without loss of generality we can assume $\sum_i a_i = 1$. The particles jump independently to the right by 1 with respective rates

$$r_i = \begin{cases} a_1, & \text{if } i = 1; \\ a_i(1 - q^{x_{i-1}-x_i-1}), & \text{otherwise.} \end{cases}$$

Note that when $x_i + 1 = x_{i-1}$ the rate r_i vanishes, thus enforcing the exclusion rule. Now consider the tableau-valued Markov chain $\mathcal{P}(n)$, $n \geq 0$, defined as above by applying the randomised insertion algorithm applied to a random word with probabilities $a_1, \ldots, a_l$. Setting $\mathcal{L}^i(n) = \mathrm{sh}\mathcal{P}^i(n)$, we see that the process $X_1(n), \ldots, X_l(n),\ n \geq 0$ defined by $X_i(n) = \mathcal{L}^i_i(n) - i + 1$ evolves as a Markov chain with state space $\{x \in \mathbb{Z}^l :\ x_1 > x_2 > \cdots > x_l\}$ and transition probabilities

$$\pi(x, x + e_i) = r_i,\ i = 1, \ldots, l \qquad \pi(x, x) = 1 - \sum_i r_i,$$

where r_i are defined as above. In other words, it is a de-Poissonisation of the q-TASEP process. Denote the q-TASEP process by $\tilde{X}(t),\ t \geq 0$, started with step initial condition $\tilde{X}_i(0) = 1 - i$, $i = 1, \ldots, l$; by Theorem 9, the law of the position of the last particle at time t is given by

$$\begin{aligned} \mathbb{P}(\tilde{X}_l(t) = m - l + 1) &= \sum_{k \geq 0} e^{-t} \frac{t^k}{k!} \sum_{\lambda \vdash k, \lambda^l = m} (\lambda_l)_q^{-1} \Psi_a(\lambda) f^\lambda(q) \\ &= e^{-t} \sum_{\lambda \in \Omega^l, \lambda_l = m} \frac{t^{|\lambda|}}{|\lambda|!} (\lambda_l)_q^{-1} \Psi_a(\lambda) f^\lambda(q). \end{aligned} \tag{2.14}$$

In [BC13], a continuous-time Markov chain on the set of tableaux $\mathcal{T}_l$ (actually discrete Gelfand-Tsetlin patterns, but this is equivalent) was introduced. It has the same fixed time marginals as the Poissonisation of the process $\mathcal{P}(n)$, although the dynamics are quite different. It is also coupled in exactly the same way to the

q-TASEP process and in the paper [BC13] an equivalent expression to (2.14) is obtained via this coupling for the law of $\tilde{X}_l(t)$. See also [BCS14] for related recent work.

2.6 Permutations

If $l = n$ and $P \in \mathcal{S}_n$ with $\mathrm{sh}P = \lambda$, then (2.3) becomes

$$\kappa(P) = \frac{\rho(P)}{(1-q)^n \Delta_n(\lambda)}.$$

Using this, and the fact that $\phi_w(P, Q) = 0$ unless $\mathrm{ty}P = \mathrm{ty}w$, we immediately deduce from Theorem 6 the following corollary.

Corollary 12. *For $P, Q \in \mathcal{S}_n$ with $shP = shQ = \lambda$, we have*

$$\zeta_{P,Q}(q) := \sum_{\sigma \in S_n} \phi_\sigma(P, Q) = \frac{\rho(P)\rho(Q)}{(1-q)^n \Delta(\lambda)}. \tag{2.15}$$

Summing over P and Q gives

$$\theta_\lambda(q) := \sum_{P,Q \in \mathcal{S}_n : \mathrm{sh}P = \mathrm{sh}Q = \lambda} \zeta_{P,Q}(q) = \frac{f^\lambda(q)^2}{(1-q)^n \Delta(\lambda)}.$$

We note that $\sum_{\lambda \vdash n} \theta_\lambda(q) = n!$. When $0 \le q < 1$, the probability measure on integer partitions defined by $\mu_q(\lambda) = \theta_\lambda(q)/n!$ gives the law of the shape of the tableaux obtained when one applied the randomised insertion algorithm to a random permutation. It would be interesting to understand the analogue in this setting of the longest increasing subsequence problem [AD99, BDJ99, Oko01].

For any standard tableau P with entries in $[n]$ and shape λ, its weight $\rho(P)$ is a product of n polynomials of the form of $(1 - q^k)$ and hence $\rho(P)$ is divisible by $(1 - q)^n$. On the other hand, considering the ith and $i + 1$th row in P, each time j a box is added in ith row, a factor $(1 - q^d)$ - where d is the difference between length of the corresponding two rows at time j - appears in $\rho(P)$. For this difference d to reach the value of $\lambda_i - \lambda_{i+1}$ eventually (which it evidently does) all the factors $(1 - q), (1 - q^2), \ldots, (1 - q^{\lambda_i - \lambda_{i+1}})$ must appear at least once. It follows that $\rho(P)$ is also divisible by $\Delta(\lambda)$. Thus, $\zeta_{P,Q}(q) \in \mathbb{Z}[q]$ for each pair (P, Q) and $\theta_\lambda(q) \in \mathbb{Z}[q]$ for each λ.

For any permutation $\sigma \in S_n$, denote by $(P(\sigma), Q(\sigma))$ the pair of tableaux after column inserting σ, and set $F_\sigma(q) = \zeta_{P(\sigma),Q(\sigma)}(q)$. When $n = 2$, the polynomials

$F_\sigma(q)$ and $\theta_\lambda(q)$ are given by

$$F_{12}(q) = 1 - q; \qquad F_{21}(q) = 1 + q.$$

$$\theta_2(q) = 1 + q; \qquad \theta_{1^2}(q) = 1 - q.$$

When $n = 3$, we have

$$F_{123}(q) = (1-q)^2; \quad F_{132}(q) = 1 - q; \quad F_{213}(q) = (1+q)(1-q^2);$$
$$F_{231}(q) = 1 - q^2; \quad F_{312}(q) = 1 - q^2; \quad F_{321}(q) = (1+q)(1+q+q^2).$$

$$\theta_3(q) = (1+q)(1+q+q^2); \quad \theta_{21}(q) = (1-q)(2+q)^2; \quad \theta_{1^3}(q) = (1-q)^2.$$

The polynomials $F_\sigma(q)$ give an alternative interpretation of the probability measure μ_q as the distribution of the shape of the tableaux obtained when one applies the Robinson-Schensted column insertion algorithm to a permutation chosen at random according to the distribution $F_\sigma(q)/n!$.

2.7 Proofs

2.7.1 Proof of Proposition 8

To prove (2.12), we take advantage of the recursive structure of the q-Whittaker functions. Define $\hat{\kappa}$ on $\Omega^l \times \Omega^{l-1}$ by

$$\hat{\kappa}(\lambda^l, \lambda^{l-1}) = \frac{\prod_{i=1}^{l-2} (\lambda_i^{l-1} - \lambda_{i+1}^{l-1})_q}{\prod_{i=1}^{l-1} (\lambda_i^{l-1} - \lambda_{i+1}^{l})_q (\lambda_i^l - \lambda_i^{l-1})_q}$$

and set

$$T = \{(\lambda^l, \lambda^{l-1}) \in \Omega^l \times \Omega^{l-1} : \lambda^{l-1} \prec \lambda^l\},$$

where we write $\lambda \prec \mu$ if $\mu_{i+1} \le \lambda_i \le \mu_i$ for each i.

We begin by verifying the simpler intertwining relation:

$$\hat{K}\hat{M} = L\hat{K}, \tag{2.16}$$

where $\hat{M} : T \times T \to \mathbb{R}_{\geq 0}$ and $\hat{K} : T \to \mathbb{R}_{\geq 0}$ are defined as follows.

$$\hat{M}((\lambda^l, \lambda^{l-1}),(\lambda^l + e_k, \lambda^{l-1})) = a_l(1 - q^{\lambda^{l-1}_{k-1} - \lambda^l_k}) \prod_{i=k}^{l-1} q^{\lambda^{l-1}_i - \lambda^l_{i+1}}, \quad 1 \leq k \leq l;$$

$$\hat{M}((\lambda^l, \lambda^{l-1}), (\lambda^l + e_k, \lambda^{l-1} + e_k)) = \frac{(1 - q^{\lambda^{l-1}_k - \lambda^{l-1}_{k+1}+1})(1 - q^{\lambda^{l-1}_{k-1} - \lambda^l_k})}{1 - q^{\lambda^{l-1}_{k-1} - \lambda^{l-1}_k}}, \quad 1 \leq k \leq l-1;$$

$$\begin{aligned}\hat{M}((\lambda^l,&\lambda^{l-1}), (\lambda^l + e_k, \lambda^{l-1} + e_m)) \\ &= \frac{(1 - q^{\lambda^{l-1}_m - \lambda^{l-1}_{m+1}+1})(1 - q^{\lambda^l_m - \lambda^{l-1}_m})}{1 - q^{\lambda^{l-1}_{m-1} - \lambda^{l-1}_m}}(1 - q^{\lambda^{l-1}_{k-1} - \lambda^l_k}) \prod_{i=k+1}^{m} q^{\lambda^{l-1}_{i-1} - \lambda^l_i}, \\ &\qquad 1 \leq k < m \leq l-1.\end{aligned}$$

$$\hat{K}(\lambda^l, (\tilde{\lambda}^l, \lambda^{l-1})) = a^{\sum_{i=1}^l \lambda^l_i - \sum_{i=1}^{l-1} \lambda^{l-1}_i} \hat{\kappa}(\lambda^l, \lambda^{l-1}) \mathbb{I}_{\lambda^l = \tilde{\lambda}^l}.$$

With a slight abuse of notation we will write $\hat{K}(\lambda^l, \lambda^{l-1})$ as shorthand for $\hat{K}(\lambda^l, (\tilde{\lambda}^l, \lambda^{l-1}))$ since the support of latter is in $\{\lambda^l = \tilde{\lambda}^l\}$. We'll do the same for kernel K.

We will verify the recursive intertwining relation (2.16) directly. The left hand side is given by

$$\begin{aligned}\hat{K}\hat{M}(\lambda^l,&(\lambda^l + e_k, \lambda^{l-1})) = \hat{K}(\lambda^l, \lambda^{l-1})\hat{M}((\lambda^l, \lambda^{l-1}), (\lambda^l + e_k, \lambda^{l-1})) \\ &+ \hat{K}(\lambda^l, \lambda^{l-1} - e_k)\hat{M}((\lambda^l, \lambda^{l-1} - e_k), (\lambda^l + e_k, \lambda^{l-1}))\mathbb{I}_{k \leq l-1} \\ &+ \sum_{m=k+1}^{l-1} \hat{K}(\lambda^l, \lambda^{l-1} - e_m)\hat{M}((\lambda^l, \lambda^{l-1} - e_m), (\lambda^l + e_k, \lambda^{l-1}))\mathbb{I}_{k \leq l-2}.\end{aligned}$$

We calculate each term separately. Set $K' = a_l\hat{K}(\lambda^l, \lambda^{l-1})$.

$$\hat{K}(\lambda^l, \lambda^{l-1})\hat{M}((\lambda^l, \lambda^{l-1}), (\lambda^l + e_k, \lambda^{l-1})) = K'(1 - q^{\lambda^{l-1}_{k-1} - \lambda^l_k}) \prod_{i=k}^{l-1} q^{\lambda^{l-1}_i - \lambda^l_{i+1}}.$$

$$\begin{aligned}
&\hat{K}(\lambda^l,\lambda^{l-1}-e_k)\hat{M}((\lambda^l,\lambda^{l-1}-e_k),(\lambda^l+e_k,\lambda^{l-1}))\\
&\qquad = K'\frac{(1-q^{\lambda^{l-1}_{k-1}-\lambda^{l-1}_k+1})(1-q^{\lambda^{l-1}_k-\lambda^l_{k+1}})}{(1-q^{\lambda^{l-1}_k-\lambda^{l-1}_{k+1}})(1-q^{\lambda^l_k-\lambda^{l-1}_k+1})}\frac{(1-q^{\lambda^{l-1}_k-\lambda^{l-1}_{k+1}})(1-q^{\lambda^{l-1}_{k-1}-\lambda^l_k})}{1-q^{\lambda^{l-1}_{k-1}-\lambda^{l-1}_k+1}}\\
&\qquad = K'(1-q^{\lambda^{l-1}_{k-1}-\lambda^l_k})\frac{1-q^{\lambda^{l-1}_k-\lambda^l_{k+1}}}{1-q^{\lambda^l_k-\lambda^{l-1}_k+1}}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{m=k+1}^{l-1}\hat{K}(\lambda^l,\lambda^{l-1}-e_m)\hat{M}((\lambda^l,\lambda^{l-1}-e_m),(\lambda^l+e_k,\lambda^{l-1}))\\
&\qquad\qquad = K'\sum_{m=k+1}^{l-1}\frac{(1-q^{\lambda^{l-1}_{m-1}-\lambda^{l-1}_m+1})(1-q^{\lambda^{l-1}_m-\lambda^l_{m+1}})}{(1-q^{\lambda^{l-1}_m-\lambda^{l-1}_{m+1}})(1-q^{\lambda^l_m-\lambda^{l-1}_m+1})}\\
&\qquad\qquad\times\frac{(1-q^{\lambda^{l-1}_m-\lambda^{l-1}_{m+1}})(1-q^{\lambda^l_m-\lambda^{l-1}_m+1})}{1-q^{\lambda^{l-1}_{m-1}-\lambda^{l-1}_m+1}}(1-q^{\lambda^{l-1}_{k-1}-\lambda^l_k})\prod_{i=k+1}^{m}q^{\lambda^{l-1}_{i-1}-\lambda^l_i}\\
&\qquad\qquad = K'(1-q^{\lambda^{l-1}_{k-1}-\lambda^l_k})\sum_{m=k+1}^{l-1}(1-q^{\lambda^{l-1}_m-\lambda^l_{m+1}})\prod_{i=k+1}^{m}q^{\lambda^{l-1}_{i-1}-\lambda^l_i}.
\end{aligned}$$

The left hand side of (2.16) is thus given by

$$\begin{aligned}
LHS &= K'(1-q^{\lambda^{l-1}_{k-1}-\lambda^l_k})\left(\prod_{i=k}^{l-1}q^{\lambda^{l-1}_i-\lambda^l_{i+1}}\right.\\
&\quad\left.+\sum_{m=k+1}^{l-1}(1-q^{\lambda^{l-1}_m-\lambda^l_{m+1}})\prod_{i=k+1}^{m}q^{\lambda^{l-1}_{i-1}-\lambda^l_i}\mathbb{I}_{k\le l-2}+\frac{1-q^{\lambda^{l-1}_k-\lambda^l_{k+1}}}{1-q^{\lambda^l_k-\lambda^{l-1}_k+1}}\mathbb{I}_{k\le l-1}\right)\\
&\qquad\qquad = K'(1-q^{\lambda^{l-1}_{k-1}-\lambda^l_k})\frac{1-q^{\lambda^l_k-\lambda^l_{k+1}+1}}{1-q^{\lambda^l_k-\lambda^{l-1}_k+1}}.
\end{aligned}$$

The right hand side is much easier to calculate:

$$\begin{aligned}
L\hat{K}(\lambda^l,(\lambda^l+e_k,\lambda^{l-1})) &= L(\lambda^l,\lambda^l+e_k)\hat{K}(\lambda^l+e_k,\lambda^{l-1})\\
&= K'(1-q^{\lambda^l_k-\lambda^l_{k+1}+1})\frac{1-q^{\lambda^{l-1}_{k-1}-\lambda^l_k}}{1-q^{\lambda^l_k-\lambda^{l-1}_k+1}},
\end{aligned}$$

as required.

We will now prove (2.12) by induction on l. When $l=2$, since $\hat{M}^2$ is the kernel for the whole tableau, the recursive intertwining relation (2.16) is equivalent to the full intertwining relation (2.12). Suppose the statement of the proposition holds for the rank-$(l-2)$ case, that is, for $l-1$. From the definition of K and $\hat{K}$

we have

$$K^l(\lambda^l, \lambda^{1:l-1}) = K^{l-1}(\lambda^{l-1}, \lambda^{1:l-2})\hat{K}^l(\lambda^l, \lambda^{l-1}). \tag{2.17}$$

By the recursive nature of definition of ϕ_w, M^l can be expressed in terms of $\hat{M}^l$, M^{l-1} and L^{l-1}:

$$\begin{aligned} M^l(\lambda^{1:l}, \tilde{\lambda}^{1:l}) = {} & \mathbb{I}_{\lambda^{l-1}=\tilde{\lambda}^{l-1}} \hat{M}^l((\lambda^l, \lambda^{l-1}), (\tilde{\lambda}^l, \tilde{\lambda}^{l-1})) \\ & + \mathbb{I}_{\lambda^{l-1}\nearrow\tilde{\lambda}^{l-1}} \frac{M^{l-1}(\lambda^{1:l-1}, \tilde{\lambda}^{1:l-1})}{L^{l-1}(\lambda^{l-1}, \tilde{\lambda}^{l-1})} \hat{M}^l((\lambda^l, \lambda^{l-1}), (\tilde{\lambda}^l, \tilde{\lambda}^{l-1})). \end{aligned}$$

For partitions λ, μ write $\lambda \rightsquigarrow \mu$ to mean that either $\lambda = \mu$ or $\lambda \nearrow \mu$. Then

$$\begin{aligned} K^l M^l(\lambda^l, (\tilde{\lambda}^l, \lambda^{1:l-1})) = {} & \sum_{\tilde{\lambda}^{1:l-1}:\tilde{\lambda}^{l-1}\rightsquigarrow\lambda^{l-1}} K^l(\lambda^l, \tilde{\lambda}^{1:l-1}) M^l((\lambda^l, \tilde{\lambda}^{1:l-1}), (\tilde{\lambda}^l, \lambda^{1:l-1})) \\ = \sum_{\tilde{\lambda}^{1:l-1}} & \hat{K}^l(\lambda^l, \tilde{\lambda}^{l-1}) K^{l-1}(\tilde{\lambda}^{l-1}, \tilde{\lambda}^{1:l-2}) \Big(\mathbb{I}_{\tilde{\lambda}^{l-1}=\lambda^{l-1}} \hat{M}^l((\lambda^l, \tilde{\lambda}^{l-1}), (\tilde{\lambda}^l, \lambda^{l-1})) \\ + \mathbb{I}_{\tilde{\lambda}^{l-1}\nearrow\lambda^{l-1}} & \frac{M^{l-1}(\tilde{\lambda}^{1:l-1}, \lambda^{1:l-1})}{L^{l-1}(\tilde{\lambda}^{l-1}, \lambda^{l-1})} \hat{M}^l((\lambda^l, \tilde{\lambda}^{l-1}), (\tilde{\lambda}^l, \lambda^{l-1})) \Big) \\ & =: \mathbb{I}_{\tilde{\lambda}^{l-1}=\lambda^{l-1}} \mathrm{I} + \mathbb{I}_{\tilde{\lambda}^{l-1}\nearrow\lambda^{l-1}} \mathrm{II}. \end{aligned}$$

$$\begin{aligned} \mathrm{II} = & \sum_{\tilde{\lambda}^{l-1}\nearrow\lambda^{l-1}} \Bigg(\hat{K}^l(\lambda^l, \tilde{\lambda}^{l-1}) \hat{M}^l((\lambda^l, \tilde{\lambda}^{l-1}), (\tilde{\lambda}^l, \lambda^{l-1})) \\ & \qquad \times \sum_{\tilde{\lambda}^{1:l-2}:\tilde{\lambda}^{l-2}\rightsquigarrow\lambda^{l-2}} K^{l-1}(\tilde{\lambda}^{l-1}, \tilde{\lambda}^{1:l-2}) \frac{M^{l-1}(\tilde{\lambda}^{1:l-1}, \lambda^{1:l-1})}{L^{l-1}(\tilde{\lambda}^{l-1}, \lambda^{l-1})} \Bigg) \\ = & \sum_{\tilde{\lambda}^{l-1}\nearrow\lambda^{l-1}} \Bigg(\hat{K}^l(\lambda^l, \tilde{\lambda}^{l-1}) \hat{M}^l((\lambda^l, \tilde{\lambda}^{l-1}), (\tilde{\lambda}^l, \lambda^{l-1})) \\ & \qquad \times \frac{K^{l-1} M^{l-1}(\tilde{\lambda}^{l-1}, (\lambda^{l-1}, \lambda^{1:l-2}))}{L^{l-1}(\tilde{\lambda}^{l-1}, \lambda^{l-1})} \Bigg) \\ \overset{\text{induction}}{\underset{\text{assumption}}{=\!=\!=}} & \sum_{\tilde{\lambda}^{l-1}\nearrow\lambda^{l-1}} \Bigg(\hat{K}^l(\lambda^l, \tilde{\lambda}^{l-1}) \hat{M}^l((\lambda^l, \tilde{\lambda}^{l-1}), (\tilde{\lambda}^l, \lambda^{l-1})) \\ & \qquad \times \frac{L^{l-1} K^{l-1}(\tilde{\lambda}^{l-1}, (\lambda^{l-1}, \lambda^{1:l-2}))}{L^{l-1}(\tilde{\lambda}^{l-1}, \lambda^{l-1})} \Bigg) \\ = & \sum_{\tilde{\lambda}^{l-1}\nearrow\lambda^{l-1}} \hat{K}^l(\lambda^l, \tilde{\lambda}^{l-1}) \hat{M}^l((\lambda^l, \tilde{\lambda}^{l-1}), (\tilde{\lambda}^l, \lambda^{l-1})) K^{l-1}(\lambda^{l-1}, \lambda^{1:l-2}) \end{aligned}$$

Due to the indicator, when $\tilde{\lambda}^{l-1} = \lambda^{l-1}$,

$$\mathrm{I} = \hat{K}^l(\lambda^l, \tilde{\lambda}^{l-1}) K^{l-1}(\lambda^{l-1}, \lambda^{1:l-2}) \hat{M}^l((\lambda^l, \tilde{\lambda}^{l-1}), (\tilde{\lambda}^l, \lambda^{l-1})).$$

Therefore

$$\begin{aligned}
&K^l M^l(\lambda^l, (\tilde{\lambda}^l, \lambda^{1:l-1})) \\
&\quad = \sum_{\tilde{\lambda}^{l-1} : \tilde{\lambda}^{l-1} \rightsquigarrow \lambda^{l-1}} \hat{K}^l(\lambda^l, \tilde{\lambda}^{l-1}) K^{l-1}(\lambda^{l-1}, \lambda^{1:l-2}) \hat{M}^l((\lambda^l, \tilde{\lambda}^{l-1}), (\tilde{\lambda}^l, \lambda^{l-1})) \\
&\quad = K^{l-1}(\lambda^{l-1}, \lambda^{1:l-2}) \hat{K}^l \hat{M}^l(\lambda^l, (\tilde{\lambda}^l, \lambda^{l-1})) \\
&\quad \overset{(2.16)}{=} K^{l-1}(\lambda^{l-1}, \lambda^{1:l-2}) L^l(\lambda^l, \tilde{\lambda}^l) \hat{K}^l(\tilde{\lambda}^l, \lambda^{l-1}) \\
&\quad \overset{(2.17)}{=} L^l(\lambda^l, \tilde{\lambda}^l) K^l(\tilde{\lambda}^l, \lambda^{1:l-1}) = LK(\lambda^l, (\tilde{\lambda}^l, \lambda^{1:l-1})),
\end{aligned}$$

as required.

2.7.2 Proof of Theorem 6

We will prove the identity (2.11), from which the statement of the theorem follows. From the definition of ϕ_w, for $(P, Q) \in \mathcal{T}_l \times \mathcal{S}_n$ such that $\mathrm{sh}P = \mathrm{sh}Q = \lambda$ and $\mu^i = \mathrm{sh}Q^i$ for $i = 1, \dots, n$, the left hand side of (2.11) can be written as

$$\begin{aligned}
&\sum_{w \in [l]^n} a^w \phi_w(P, Q) \\
&= \sum_{w \in [l]^n} \sum_{(P(i))_{i=1}^{n-1}, \mathrm{sh}P(i) = \mu^i} a^w I_{w_1}(\emptyset, P(1)) \dots I_{w_n}(P(n-1), P) \\
&= \sum_{w \in [l]^n} \sum_{(P(i))_{i=1}^{n-1}, \mathrm{sh}P(i) = \mu^i} (a_{w_1} I_{w_1}(\emptyset, P(1))) \dots (a_{w_n} I_{w_n}(P(n-1), P)) \\
&= \sum_{(P(i))_{i=1}^{n-1}, \mathrm{sh}P(i) = \mu^i} \left(\sum_{w_1 \in [l]} a_{w_1} I_{w_1}(\emptyset, P(1)) \right) \dots \left(\sum_{w_n \in [l]} a_{w_n} I_{w_n}(P(n-1), P) \right) \\
&= \sum_{(P(i))_{i=1}^{n-1}, \mathrm{sh}P(i) = \mu^i} M(\emptyset, P(1)) \dots M(P(n-1), P).
\end{aligned}$$

On the right hand side, from the definition of $\rho(Q)$ and the intertwining relation (2.12),

$$
\begin{aligned}
a^P\kappa(P)\frac{\rho(Q)}{(\lambda_l)_q} &= L(\emptyset,\mu^1)\dots L(\mu^{n-1},\lambda)K(\lambda,P)\\
&= L(\emptyset,\mu^1)\dots L(\mu^{n-2},\mu^{n-1})LK(\mu^{n-1},P)\\
&= L(\emptyset,\mu^1)\dots L(\mu^{n-2},\mu^{n-1})KM(\mu^{n-1},P)\\
&= \sum_{P(n-1):\ \mathrm{sh}P(n-1)=\mu^{n-1}} \Bigg(L(\emptyset,\mu^1)\dots L(\mu^{n-2},\mu^{n-1})\\
&\qquad\qquad \times K(\mu^{n-1},P(n-1))M(P(n-1),P)\Bigg)\\
&= \sum_{\substack{P(n-1),P(n-2):\\ \mathrm{sh}P(n-1)=\mu^{n-1},\mathrm{sh}P(n-2)=\mu^{n-2}}} \Bigg(L(\emptyset,\mu^1)\dots K(\mu^{n-2},P(n-2))\\
&\qquad\qquad \times M(P(n-2),P(n-1))M(P(n-1),P)\Bigg)\\
&= \dots = \sum_{(P(i))_{i=1}^{n-1}:\ \mathrm{sh}P(i)=\mu^i} \Bigg(L(\emptyset,\mu^1)K(\mu^1,P(1))M(P(1),P(2))\times\\
&\qquad\qquad \dots\times M(P(n-1),P)\Bigg).
\end{aligned}
$$

Now, from the definition of L, K and M, for $P(1)\in\mathcal{T}_l$ that has only one entry k and whose shape is $\mu^1=(1)$,

$$L(\emptyset,\mu^1)=1;\quad K(\mu^1,P(1))=M(\emptyset,P(1))=a_k.$$

This completes the proof.

2.7.3 Proof of Proposition 5

Let $\lambda\vdash n$ and note that, for $l>n$, $\Delta_l(\lambda)=\Delta(\lambda)$. We want to show that

$$\lim_{l\to\infty}\Psi_{(1/l)^l}(\lambda)=\frac{f^\lambda(q)}{n!(1-q)^n\Delta(\lambda)}.$$

From the definition of Ψ_a, this is equivalent to

$$\lim_{l\to\infty} l^{-n} \sum_{P\in\mathcal{T}_l\colon\ \mathrm{sh}P=\lambda} \kappa(P) = \frac{f^\lambda(q)}{n!(1-q)^n\Delta(\lambda)}.$$

Write

$$\sum_{P\in\mathcal{T}_l\colon\ \mathrm{sh}P=\lambda} \kappa(P) = A + B$$

where A denotes the sum over tableaux with distinct entries and B denotes the remaining sum. Assume $l > n$. By (2.3), if P has distinct entries, then

$$\kappa(P) = \frac{\rho(\hat{P})}{(1-q)^n\Delta(\lambda)}.$$

Hence

$$l^{-n}A = l^{-n}\binom{l}{n}\sum_{Q\in\mathcal{S}_n}\frac{\rho(Q)}{(1-q)^n\Delta(\lambda)} \to \frac{f^\lambda(q)}{n!(1-q)^n\Delta(\lambda)}$$

as $l\to\infty$. Thus it remains to show that $l^{-n}B \to 0$. We first show that $\kappa(P)$ is bounded for $P\in\mathcal{T}_l$ with $\mathrm{sh}P=\lambda$. To see this, observe that if P has entries from the set $\{i_1,\ldots,i_m\}$ where $i_1<\cdots<i_m$ and $\tilde{P}$ denotes the tableau obtained from P by replacing i_k by k, for each $k=1,\ldots,m$, then $\kappa(\tilde{P})=\kappa(P)$. It follows that

$$\kappa(P) \le \max_{T\in\mathcal{T}_n}\kappa(T) < \infty.$$

Now, by the usual Robinson-Schensted correspondence, the number of $P\in\mathcal{T}_l$ with $\mathrm{sh}P=\lambda$ which don't have distinct entries is at most the number of words $w\in[l]^n$ which don't have distinct entries, and this is given by

$$N(l,n) = l^n - \binom{l}{n}n!.$$

Clearly, $l^{-n}N(l,n)\to 0$ as $l\to\infty$, so we are done.

Chapter 3

A symmetry property for the q-weighted an other branching Robinson-Schensted algorithms

3.1 Introduction

In Chapter 2 a q-weighted version of the Robinson-Schensted algorithm was introduced. In this chapter we show that this algorithm enjoys a symmetry property analoguous to the well-known symmetry property of the Robinson-Schensted algorithm. The proof uses a generalisation of the growth diagram approach introduced by [Fom79, Fom88, Fom94, Fom95].

The insertion algorithm we consider in this chapter is based on column insertion, but the technique applies to any insertion algorithm belonging to a certain class of "branching insertion algorithms", as described in Section 3.7 below. For example, [BP13] have recently introduced a q-weighted version of the row insertion algorithm, which is defined similarly to the column insertion version of [OP13]; they also consider a wider family of such algorithms (and, more generally, dynamics on Gelfand-Tsetlin patterns), some of which fall into the framework considered in the present chapter, and can similarly be shown to have the symmetry property. We discuss such extensions in Section 3.7 below.

The Robinson-Schensted (RS) algorithm is a combinatorial algorithm which was introduced by Robinson [Rob38] and Schensted [Sch61]. It has wide applications in representation theory and probability theory, e.g. last passage percolation, totally asymmetric simple exclusion process / corner growth model, random matrix theory [Joh00], queues in tandem [O'C03b] and more.

There are two versions of RS algorithms, the row insertion and column insertion version. In most of this chapter we deal with column insertion and its q-version. The RS algorithm transforms a word to a tableau pair of the same shape. A word can be treated as a path, hence a random word corresponds to a random walk. When taking such a random walk, the shape of the output tableaux is a Markov chain, whose transition kernel is related to the Schur symmetric functions [O'C03a]. A geometric generalisation transforms a Brownian motion with drift to a Markov process whose generator is related to $\mathsf{GL}(n, \mathbb{R})$-Whittaker functions [O'C12], which are eigenfunctions of the quantum Toda chain [Kos80]. The q-Whittaker functions on the one hand are a generalisation of the Schur symmetric functions and a specialisation of Macdonald (q, t)-symmetric functions when $t = 0$ [Mac98], and on the other hand are eigenfunctions of the q-deformed quantum Toda chain [Rui90, Eti99]. When $q \to 0$ they become the Schur functions and when $q \to 1$ with a proper scaling [GLO12] they converge to Whittaker functions. In the spirit of this connection, a q-weighted Robinson-Schensted algorithm was formulated in [OP13], which transforms the random walk to a Markov chain that is related to q-Whittaker functions.

As is expected the algorithm degenerates to the normal RS algorithm when $q \to 0$. Part of the random P-tableau also has q-TASEP dynamics [OP13], the latter introduced in [SW98], just as the same part of the random P-tableau of normal RS algorithm has TASEP dynamics [O'C03b].

Recently, a q-version of the RS row insertion algorithm was also introduced in [BP13]. It denegerates to the normal RS algorithm with row insertion when $q \to 0$.

In this chapter we show that both algorithms enjoy a symmetry property analogous to the well-known symmetry property of the Robinson-Schensted algorithm. Basically, the symmetry property for the Robinson-Schensted algorithms restricted to permutation inputs is the property that the output tableau pair is interchanged if the permutation is inversed. Knuth [Knu70] generalised the normal row insertion algorithm to one which takes matrix input, which we refer to as Robinson-Schensted-Knuth (RSK) algorithm. For this algorithm the symmetry property is that the output tableau pair is interchanged if the matrix is transposed. Note that the matrix becomes the permutation matrix when the RSK algorithm is restricted to permutation, hence the transposition of the matrix corresponds to inversion of the permutation. Burge gives a similar generalisation of the column insertion algorithm [Bur74], which we refer to as Burge's algorithm.

The symmetry property for the normal RS algorithm is normally discussed in the literature for the row insertion. However, in the permutation case the output tableau pair for row insertion is simply the transposition of the pair for column insertion. Therefore proofs of the symmetry property can be translated to column insertion instantly. In the matrix input case, the Burge and RSK algorithms are also closely related so that proofs can be extended from one to the other naturally.

For a proof of the symmetry property one can see e.g. [Sag00, Sta01, Ful97]. There are a few different approaches to deal with it. One is the Viennot diagram [Vie77] which provides a nice geometric construction of the RS algorithms. In the matrix input case there are two approaches which reduce to the Viennot diagram approach when restricted to permutations. One is the antichain or the inversion digraph construction of the RSK algorithm (which should extend naturally to the Burge algorithm) due to [Knu70]; the other (for both RSK and Burge) is Fomin's matrix-ball construction of [Ful97].

Another method is the growth diagram technique due to Fomin [Fom79, Fom88, Fom94, Fom95]. It can be generalised to the RSK and Burge algorithms. Greene's theorem [Gre74] gives a proof by showing the relation between the lengths of the longest subsequences of the input and the shape of the output. However, the growth diagram approach can be thought of as a fast construction to calculate these lengths.

Among all these techniques, the special structure of growth diagram can be extended to a class of algorithms what we will call branching algorithms, which include the q-weighted column and row insertion algorithms. Therefore it is this approach which we use in this chapter. For a simple description of the technique for normal row insertion see e.g. [Sta01], whose column version will be shown in Section 3.3.

The rest of the chapter is organised in the following way. In Section 3.2 we recall the insertion rule of a letter into a tableau for the normal Robinson-Schensted algorithm (with column insertion). We describe it in a way that suits the growth diagram. In Section 3.3 we describe the insertion rule for a word and state the symmetry property for the RS algorithm applied to permutations. In Section 3.4 and 3.5 we describe the q-weighted insertion algorithm for letters and words in a way which is different, but equivalent to the definition given in [OP13]. In Section 3.6 we state and prove the symmetry property in the q-case. Finally in Section 3.7 we prove the symmetry property for the row insertion algorithm and more generally branching algorithms.

3.2 Classical Robinson-Schensted algorithm

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in W = \{(a_1, a_2, \dots) \in \bigcup_{k=1}^{\infty} \mathbb{N}_{\geq 0}^k, a_1 \geq a_2 \geq \dots\}$ is a vector of weakly decreasing non-negative integer entries. Denote by $l(\lambda)$ the number of positive entries of λ and $|\lambda| = \lambda_1 + \cdots + \lambda_{l(\lambda)}$ the size of the partition. We say λ is a partition of n if $|\lambda| = n$, which we denote by $\lambda \vdash n$. A Young tableau P with shape λ is a left aligned array of $l(\lambda)$ rows of positive integers such that the entries are strictly increasing along each column and weakly increasing along each row, and such that the length of the jth row is λ_j. For example below is a tableau with shape $(4, 3, 2, 2)$.

$$\begin{array}{cccc} 1 & 1 & 3 & 4 \\ 3 & 5 & 8 & \\ 6 & 7 & & \\ 8 & 8 & & \end{array} \tag{3.1}$$

We denote by sh$P$ the shape of tableau P, and $\mathcal{T}_\ell$ the set of all tableaux with entries no greater than ℓ. Denote $[n] = \{1, 2, \cdots, n\}$ for a positive integer n. A standard tableau Q is a tableau with distinct entries from $[|\mathrm{sh}Q|]$. For example below is a standard tableau with shape $(4, 3, 2, 2)$.

$$\begin{array}{cccc} 1 & 3 & 4 & 7 \\ 2 & 5 & 8 & \\ 6 & 10 & & \\ 9 & 11 & & \end{array}$$

We denote by $\mathcal{S}_n$ the set of standard tableaux with shape of size n. For example the above tableau is an element of $\mathcal{S}_{11}$.

For a tableau P, we denote by P^k its subtableau containing all entries no greater than k and call it the kth subtableau of P. For example the 6th subtableau of the tableau shown in (3.1) is

$$\begin{array}{cccc} 1 & 1 & 3 & 4 \\ 3 & 5 & & \\ 6 & & & \end{array}$$

A tableau P can be identified by the shape of its subtableaux, which we usually denote by $\lambda^k = \mathrm{sh}P^k$. Also let $\lambda^0 = \emptyset$ to be the empty partition. Evidently a tableau has only finitely many different λ^i's and there exists an ℓ such that $\lambda^i = \mathrm{sh}P$

for $i \geq \ell$. We call λ^i the ith shape of P. We can identify P with these shapes and write $P = \lambda^0 \prec \lambda^1 \prec \lambda^2 \prec \cdots \prec \lambda^\ell$, where "$\prec$" is an interlacing relation: for $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$, $a \prec b$ means $b_1 \geq a_1 \geq b_2 \geq a_2 \geq \ldots$. For example the tableau P in (3.1) is identified as

$$P = \emptyset \prec 2 \prec 2 \prec 31 \prec 41 \prec 42 \prec 421 \prec 422 \prec 4322.$$

The basic operation of the Robinson-Schensted algorithm is to insert a letter $k \in \mathbb{N}_+ := \mathbb{N}\backslash\{0\}$ into a tableau $P = \emptyset \prec \lambda^1 \prec \lambda^2 \prec \ldots \cdots \prec \lambda^\ell$ and produce a new tableau $\tilde{P} = \emptyset \prec \tilde{\lambda}^1 \prec \tilde{\lambda}^2 \prec \cdots \prec \tilde{\lambda}^\ell$. To do this we first find the lowest row in λ^k such that appending a box at the end of that row would preserve the interlacement between the $k-1$th shape and the kth shape, and append the box to that row. Suppose the row has index j_k, then we find the lowest row in λ^{k+1} that is no lower than row j_k such that appending a box at the end of that row would preserve the interlacement between the kth shape and the $k+1$th shape, and append the box to that row, and so on and so forth. More precisely, define

$$j_{k-1} = k; \quad j_i = \max(\{j \leq j_{i-1} : \lambda^{i-1}_{j-1} > \lambda^i_j\} \cup \{1\}), \qquad i \geq k.$$

Then the new tableau $\tilde{P}$ is defined by

$$\tilde{\lambda}^i = \begin{cases} \lambda^i, & \text{if } i < k; \\ \lambda^i + e_{j_i}, & \text{otherwise.} \end{cases}$$

where e_j is the jth standard basis of $\mathbb{R}^{\mathbb{N}_+}$.

For example, if we insert a 6 into the tableau shown in (3.1), the insertion process is shown as follows:

$$\begin{array}{cccc} \hat{5} & \hat{5} & \hat{5} & \hat{5} \\ \hat{5} & \hat{5} & & \\ \hat{6} & {\color{gray}\hat{6}} & & \end{array} \quad \to \quad \begin{array}{cccc} \hat{6} & \hat{6} & \hat{6} & \hat{6} \\ \hat{6} & \hat{6} & {\color{gray}\hat{7}} & \\ \hat{6} & \hat{7} & & \end{array} \quad \to \quad \begin{array}{cccc} \hat{7} & \hat{7} & \hat{7} & \hat{7} \\ \hat{7} & \hat{7} & \hat{8} & {\color{gray}\hat{8}} \\ \hat{7} & \hat{7} & & \\ \hat{8} & \hat{8} & & \end{array}$$

where each $\hat{i}$ denotes a box in λ^i, and each gray entry denotes an appended box.

The resultant tableau is thus

$$\tilde{P} = \emptyset \prec 2 \prec 2 \prec 31 \prec 41 \prec 42 \prec 422 \prec 432 \prec 4422 = \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 3 & 5 & 7 & 8 \\ 6 & 6 & & \\ 8 & 8 & & \end{array}$$

One can visualise the insertion process by building the shapes up vertically.

$$\begin{tikzcd}
 & \arrow[rr, "\text{Inserting } k"] & & \\
\text{Tableau } P & \lambda^{\ell} \arrow[rr] & \vdots & \tilde{\lambda}^{\ell} & \text{Tableau } \tilde{P} \\
 & \lambda^{3} \arrow[rr] \arrow[u, dotted] & & \tilde{\lambda}^{3} \arrow[u, dotted] & \\
 & \lambda^{2} \arrow[rr] \arrow[u] & & \tilde{\lambda}^{2} \arrow[u] & \\
 & \lambda^{1} \arrow[rr] \arrow[u] & & \tilde{\lambda}^{1} \arrow[u] & \\
 & \emptyset \arrow[rr] \arrow[u] & & \emptyset \arrow[u] &
\end{tikzcd}$$

As we can see, this forms a one-column lattice diagram such that for each $i \leq k$, the vertices labeled with $\lambda^{i-1}, \lambda^{i}, \tilde{\lambda}^{i-1}$ and $\tilde{\lambda}^{i}$ surround a box, which we call the ith box. We can put an X into the kth box to indicate that the number inserted into P is k. The corresponding diagram of the previous example where we insert a 6 into tableau (3.1) is:

$$\begin{tikzcd}
4322 \arrow[r] & 4422 \\
422 \arrow[r] \arrow[u] & 432 \arrow[u] \\
421 \arrow[r] \arrow[u] & 422 \arrow[u] \\
42 \arrow[r] \arrow[u] \arrow[ur, phantom, "X"] & 42 \arrow[u] \\
41 \arrow[r] \arrow[u] & 41 \arrow[u] \\
31 \arrow[r] \arrow[u] & 31 \arrow[u] \\
2 \arrow[r] \arrow[u] & 2 \arrow[u] \\
2 \arrow[r] \arrow[u] & 2 \arrow[u] \\
\emptyset \arrow[r] \arrow[u] & \emptyset \arrow[u]
\end{tikzcd}$$

3.3 Symmetry property for the Robinson-Schensted algorithm

For a word $w = w_1w_2\dots w_n \in [l]^n$, the Robinson-Schensted algorithm starts with the empty tableau $P(0) = \emptyset$. Then w_1 is inserted into $P(0)$ to obtain $P(1)$, then w_2 into $P(1)$ to obtain $P(2)$ and so on. The recording tableau Q is a standard tableau defined by:

$$Q = \mathrm{sh}P(0) \prec \mathrm{sh}P(1) \prec \cdots \prec \mathrm{sh}P(n).$$

For example the following table shows the the process of inserting the word $w = 31342$:

$$\begin{array}{cccccc}
i & 1 & 2 & 3 & 4 & 5 \\
\hline
P(i) & 3 & 1\ 3 & \begin{array}{ll} 1 & 3 \\ 3 & \end{array} & \begin{array}{ll} 1 & 3 \\ 3 & \\ 4 & \end{array} & \begin{array}{lll} 1 & 3 & 3 \\ 2 & & \\ 4 & & \end{array} \\
\hline
Q^i & 1 & 1\ 2 & \begin{array}{ll} 1 & 2 \\ 3 & \end{array} & \begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array} & \begin{array}{lll} 1 & 2 & 5 \\ 3 & & \\ 4 & & \end{array}
\end{array}.$$

The corresponding pair of tableaux, which we denote as $(P(w), Q(w))$ are:

$$(P(w), Q(w)) = \left(\begin{array}{lll} 1 & 3 & 3 \\ 2 & & \\ 4 & & \end{array}, \begin{array}{lll} 1 & 2 & 5 \\ 3 & & \\ 4 & & \end{array} \right)$$

If we denote $(\lambda^k(i))_{1\le k\le \ell}$ as the shape of the subtableaux for $P(i)$, then since $P(i)$ is obtained from $P(i-1)$ by inserting w_i, we can construct a $\{0, 1, \dots, n\} \times \{0, 1, \dots, \ell\} \subset \mathbb{N}^2$ lattice growth diagram by concatenating the one-column lattice diagrams defined in the previous section. This is illustrated in the following picture.

$$
\begin{tikzcd}
\emptyset \arrow[r] & \lambda^{\ell}(1) \arrow[r] & \lambda^{\ell}(2) & \lambda^{\ell}(n-1) \arrow[r] & \lambda^{\ell}(n) \\
\emptyset \arrow[r] \arrow[u] & \lambda^{\ell-1}(1) \arrow[r] \arrow[u] & \lambda^{\ell-1}(2) \arrow[u] \arrow[ur, phantom, "\cdots"] & \lambda^{\ell-1}(n-1) \arrow[r] \arrow[u] & \lambda^{\ell-1}(n) \arrow[u] \\
 & \vdots & & & \vdots \\
\emptyset \arrow[r] & \lambda^{2}(1) \arrow[r] & \lambda^{2}(2) & \lambda^{2}(n-1) \arrow[r] & \lambda^{2}(n) \\
\emptyset \arrow[r] \arrow[u] & \lambda^{1}(1) \arrow[r] \arrow[u] & \lambda^{1}(2) \arrow[u] \arrow[r, phantom, "\cdots"] & \lambda^{1}(n-1) \arrow[r] \arrow[u] & \lambda^{1}(n) \arrow[u] \\
\emptyset \arrow[r] \arrow[u] & \emptyset \arrow[r] \arrow[u] & \emptyset \arrow[u] & \emptyset \arrow[r] \arrow[u] & \emptyset \arrow[u]
\end{tikzcd}
$$

As such, the tableau pair obtained are for P the shapes on the vertices of the rightmost vertical and for Q the shapes on the vertices on the top horizontal line:

$$
\begin{gathered}
P = \lambda^1(n) \prec \lambda^2(n) \prec \cdots \prec \lambda^{\ell}(n); \\
Q = \lambda^{\ell}(1) \nearrow \lambda^{\ell}(2) \nearrow \cdots \nearrow \lambda^{\ell}(n).
\end{gathered}
$$

For example, if we take $\ell = 4$ for word $w = 31342$, the growth diagram is as follows:

$$
\begin{array}{ccccccccccc}
\emptyset & — & 1 & — & 2 & — & 21 & — & 211 & - & 311 \\
| & & | & & | & & | & X & | & & | \\
\emptyset & — & 1 & — & 2 & — & 21 & — & 21 & — & 31 \\
| & X & | & & | & X & | & & | & & | \\
\emptyset & — & \emptyset & — & 1 & — & 1 & — & 1 & — & 11 \\
| & & | & & | & & | & & | & X & | \\
\emptyset & — & \emptyset & — & 1 & — & 1 & — & 1 & — & 1 \\
| & & | & X & | & & | & & | & & | \\
\emptyset & — & \emptyset & — & \emptyset & — & \emptyset & — & \emptyset & — & \emptyset
\end{array}
$$

The Robinson-Schensted algorithm was initially defined to take a permutation as an input and output a pair of *standard* tableaux with the same shape. In this case we take the word identified by $\sigma(1)\sigma(2)\sigma(3)\ldots\sigma(n)$ as the input, which we also denote by σ. A classical result of the algorithm is the symmetry property:

Theorem 13 (See e.g. [Sag00, Sta01, Ful97])**.** *For any permutation σ,*

$$
(P(\sigma^{-1}), Q(\sigma^{-1})) = (Q(\sigma), P(\sigma)).
$$

Sketch proof. We present a growth diagram proof whose row insertion counterpart can be found in [Sta01]. Basically the algorithm is reformulated in a way that is symmetric on the n by n growth diagram. We index a box by (m, k) if its four vertices are $(m-1, k-1)$, $(m-1, k)$, $(m, k-1)$ and (m, k). For any box (m, k) denote by $\lambda, \mu^1, \mu^2, \nu$ the partitions on $(m-1, k-1)$, $(m-1, k)$, $(m, k-1)$ and (m, k) respectively (see the diagram below).

$$\begin{tikzcd}[arrows=-] \mu^1 \arrow[r] \arrow[d] & \nu \arrow[d] \\ \lambda \arrow[r] & \mu^2 \end{tikzcd}$$

The algorithm goes with the following rule:

1. If there's an X in the box and $\lambda = \mu^1 = \mu^2$, then $\nu = \lambda + e_{l(\lambda)+1}$, that is ν is obtained by adding a box that forms a new row itself at the bottom of λ.

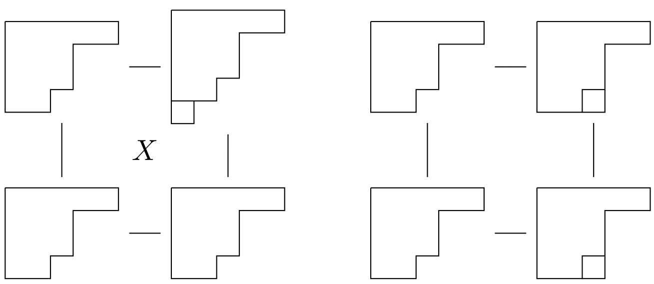
2. If there's no X in the box and $\lambda = \mu^1$, then $\nu = \mu^2$.

3. If there's no X in the box and $\lambda = \mu^2$, then $\nu = \mu^1$.

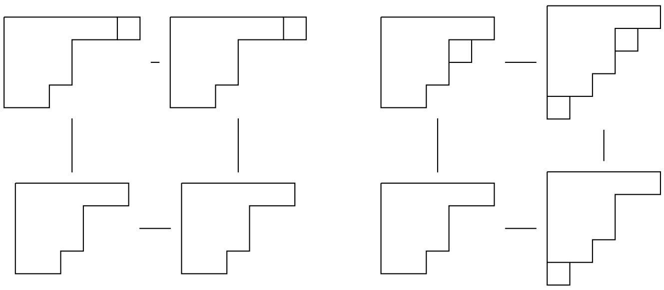
4. If there's no X in the box and $\mu^1 = \lambda + e_i$, $\mu^2 = \lambda + e_j$ with $i \neq j$, then $\nu = \lambda + e_i + e_j = \mu^1 \cup \mu^2$.

5. If there's no X in the box and $\mu^1 = \mu^2 = \lambda + e_i$, then $\nu = \lambda + e_i + e_{i'}$, where $i' = \max(\{j \leq i : \mu^1_{j-1} > \mu^1_j\} \cup \{1\})$.

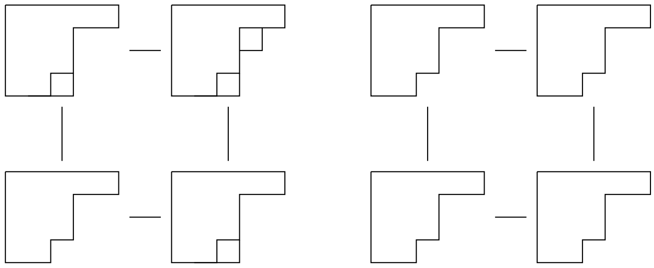
Note that these rules do not apply to the words case, which is why permutation is special. The 5 cases together with a trivial case that belongs to both case 2 and case 3 are illustrated as follows.



Case 1 Case 2



Case 3 Case 4



Case 5 "Boring" intersection of Case 2 and Case 3

This way, all the vertices of the diagram can be labelled recursively given $\emptyset$ as the boundary condition on the leftmost and bottom vertices. One could check that this is indeed equivalent to the definition in the previous section. Moreover, the transposition of the lattice diagram preserves the algorithm. That is: if we put X's in boxes $(\sigma(i), i)_{i \leq n}$ rather than $(i, \sigma(i))_{i \leq n}$, and label each vertex (i, j) with what was labelled on (j, i), we end up with the configuration that is the same as if we apply the rules 1 through 5. This immediately finishes the proof. $\square$

3.4 A q-weighted Robinson-Schensted algorithm

In Chapter 2 a q-weighted Robinson-Schensted algorithm was introduced. In this and the next section we describe the algorithm in a different way from the definition in [OP13]. At the end of next section it is obvious to see that:

Proposition 14. *The algorithm described in this section for inserting a letter to a tableau and in the next section for inserting a word to an empty tableau is an equivalent reformulation of the q-weighted Robinson-Schensted algorithm defined in [OP13].*

When inserting a letter to a tableau it outputs a weigted set of tableaux. To insert a k into P, we start with λ^k, append a box to different possible rows no lower than k, record the index of rows, and for each one of these indices j_k, we obtain a new kth shape $\tilde{\lambda}^k = \lambda^k + e_{j_k}$ with weight $w_0(k, j_k)$; then we proceed to add a box to all possible rows in λ^{k+1} no lower than j_k and obtain a weighted set of new $k+1$th shapes $\{(\tilde{\lambda}^{k+1} = \lambda^{k+1} + e_{j_{k+1}}, w_1(k+1, j_{k+1})) : j_{k+1} \leq j_k\}$; then for each j_{k+1} we obtain the new $k+2$th shapes with weight $w_1(k+2, j_{k+2})$ by adding a box to j_{k+2}th row in λ^{k+2} and so on and so forth. We also prune all the 0-weighted branches.

This way we obtain a forest of trees whose roots are $\tilde{\lambda}^k$'s, leaves are $\tilde{\lambda}^\ell$'s with edges labeled by the weights. If we prepend $\emptyset \prec \lambda^1 \prec \lambda^2 \prec \cdots \prec \lambda^{k-1}$ to this forest we obtain a tree with root $\emptyset$, the first k levels each having one branch with one edge, which we label with weight 1. Then for each leaf $\tilde{\lambda}^\ell$ we obtain a unique tableau by reading its genealogy. We also associate this tableau with a weight which is the product of weights along the edges. By going through all leaves we obtain a set of weighted tableaux, which is the output of q-inserting a k to P.

Now we describe how we calculate the weights when inserting a box to ith shape λ^i with a j_{i-1} (let $j_{k-1} = k$) specified. First let us define functions f_0 and f_1 associated with a pair of nested partitions $\mu \prec \lambda$:

$$f_0(j; \mu, \lambda) = 1 - q^{\mu_{j-1} - \lambda_j}; \qquad f_1(j; \mu, \lambda) = \frac{1 - q^{\mu_{j-1} - \lambda_j}}{1 - q^{\mu_{j-1} - \mu_j}}; \quad 1 < j \leq k$$

$$f_0(1; \mu, \lambda) = f_1(1; \mu, \lambda) = 1.$$

When adding a box to the first affected partition λ^k, the weight $w_0(k, j_k)$ associated with each $j_k \leq k$ is

$$w_0(k, j_k) = f_0(j_k; \lambda^{k-1}, \lambda^k) \prod_{p=j_k+1}^{k} (1 - f_0(p; \lambda^{k-1}, \lambda^k)).$$

For $r > k$, and for a fixed pair of (λ^r, j_{r-1}), when adding a box to row $j_r \leq j_{r-1}$ of λ^r, the corresponding weight $w_1(r, j_r)$ is

$$w_1(r, j_r) \\ = \begin{cases} f_0(j_r; \lambda^{r-1}, \lambda^r) \left(\prod_{p=j_r+1}^{j_{r-1}-1} (1 - f_0(p; \lambda^{r-1}, \lambda^r)) \right) & \\ \times (1 - f_1(j_{r-1}; \lambda^{r-1}, \lambda^r)), & 1 \leq j_r < j_{r-1} \\ f_1(j_r; \lambda^{r-1}, \lambda^r), & j_r = j_{r-1}. \end{cases}$$

For example, below is a path of the tree, i.e. genealogy of a possible output tableau when inserting a 5 into tableau (3.1):

$$\emptyset \xrightarrow{1} \begin{array}{cc} \hat{1} & \hat{1} \end{array} \xrightarrow{1} \begin{array}{cc} \hat{1} & \hat{1} \end{array} \xrightarrow{1} \begin{array}{ccc} \hat{2} & \hat{2} & \hat{3} \\ \hat{3} & & \end{array} \xrightarrow{1} \begin{array}{cccc} \hat{3} & \hat{3} & \hat{3} & \hat{4} \\ \hat{3} & & & \end{array} \xrightarrow{1-q} \begin{array}{cccc} \hat{4} & \hat{4} & \hat{4} & \hat{4} \\ \hat{4} & \hat{5} & & \\ {\color{red}\hat{5}} & & & \end{array}$$

$$\xrightarrow{\left(1-\frac{1-q}{1-q^2}\right)(1-q^2)} \begin{array}{cccc} \hat{5} & \hat{5} & \hat{5} & \hat{5} \\ \hat{5} & \hat{5} & {\color{red}\hat{6}} & \\ \hat{6} & & & \end{array} \xrightarrow{1} \begin{array}{cccc} \hat{6} & \hat{6} & \hat{6} & \hat{6} \\ \hat{6} & \hat{6} & {\color{red}\hat{7}} & \\ \hat{6} & \hat{7} & & \end{array} \xrightarrow{\frac{1-q}{1-q^2}} \begin{array}{cccc} \hat{7} & \hat{7} & \hat{7} & \hat{7} \\ \hat{7} & \hat{7} & \hat{8} & {\color{red}\hat{8}} \\ \hat{7} & \hat{7} & & \\ \hat{8} & \hat{8} & & \end{array}$$

where again each $\hat{r}$ denotes a box in λ^r and the red $\hat{r}$ denotes the new box $\tilde{\lambda}^r/\lambda^r$. We also have $j_5 = 3$, $j_6 = j_7 = j_8 = 2$ in this example. The above output tableau is

$$\tilde{P} = \emptyset \prec 2 \prec 2 \prec 31 \prec 41 \prec 421 \prec 431 \prec 432 \prec 4422 = \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 3 & 5 & 6 & 8 \\ 5 & 7 & & \\ 8 & 8 & & \end{array}$$

with weight $1 \cdot 1 \cdot 1 \cdot 1 \cdot (1-q) \cdot \left(1 - \frac{1-q}{1-q^2}\right)(1-q^2) \cdot 1 \cdot \frac{1-q}{1-q^2} = \frac{q(1-q)^2}{1+q}$. For any two tableaux P and $\tilde{P}$ we denote the weight of obtaining $\tilde{P}$ after inserting k into P by $I_k(P, \tilde{P})$. So in the previous example we have $I_5(P, \tilde{P}) = \frac{q(1-q)^2}{1+q}$.

As we can see, each branching to obtain a weighted set of new ith shapes $\{\tilde{\lambda}^i\}$ is determined by λ^i and j_{i-1}, the latter in turn is identified by the pair $(\lambda^{i-1}, \tilde{\lambda}^{i-1})$. Therefore the branching is determined by the triplet $(\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1})$. While each $\tilde{\lambda}^i$, together with λ^i and λ^{i+1} determines a next branching. Hence the tree has the structure illustrated in Figure 3.1, where we put the weights on the edges. This way we can write the weights:

$$w((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i) = \begin{cases} \mathbb{I}_{\tilde{\lambda}^i = \lambda^i} & i < k \\ w_0(i, j_i) & i = k \\ w_1(i, j_i) & i > k \end{cases}$$

where $\tilde{\lambda}^i = \lambda^i + e_{j_i}$, $j_i \leq j_{i-1}$ for $i \geq k$, and $\mathbb{I}$ is the indicator function.

Therefore, the q-insertion algorithm can be visualised in a "one-column" graph similar to the normal insertion. Each node λ^i (or $\tilde{\lambda}^{j,(p_j)}$) representing the old ith shape (or a new jth shape) is associated with a vertex $(0, i)$ (or $(1, j)$) as the northwest (or northeast) vertex of the ith (or jth) box. Each triplet $(\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1,(p_{i-1})})$

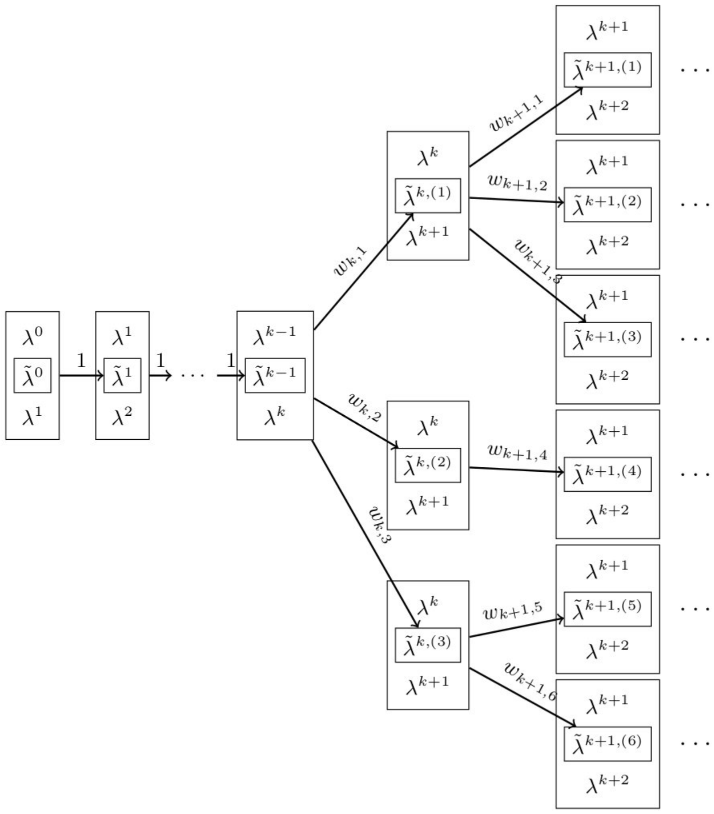


Figure 3.1: The structure of a branching insertion algorithm when inserting a letter k (see Section 3.7, where the q-weighted algorithm is an example).

is represented as a connected triple nodes surrounding the ith box from the south and the west, for which each branch $\tilde{\lambda}^{i,(p_i)}$ is represented as a node that connects to both λ^i and $\tilde{\lambda}^{i-1,(p_{i-1})}$, where we put the weights on both the horizontal and vertical edges.

$$w(\tilde{\lambda}^{i-1,(p_{i-1})} \to \tilde{\lambda}^{i,(p_i)}) = w(\lambda^i \to \tilde{\lambda}^{i,(p_i)}) = w((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1,(p_{i-1})}), \tilde{\lambda}^i), \quad i \geq 1.$$

As is in the normal insertion case, we put an X in the kth box to indicate we are inserting a k. Figure 3.2 is such a visualisation corresponding to the tree in Figure 3.1. All the nodes associated with the right column $((1, i) : 0 \leq i \leq \ell)$ form a tree, such that each $\tilde{P}$ in the output corresponds to a genealogy of a node associated with $(1, \ell)$, whose weight $I_k(P, \tilde{P})$ can be obtained as the product of the weights along the genealogical line of the tree.

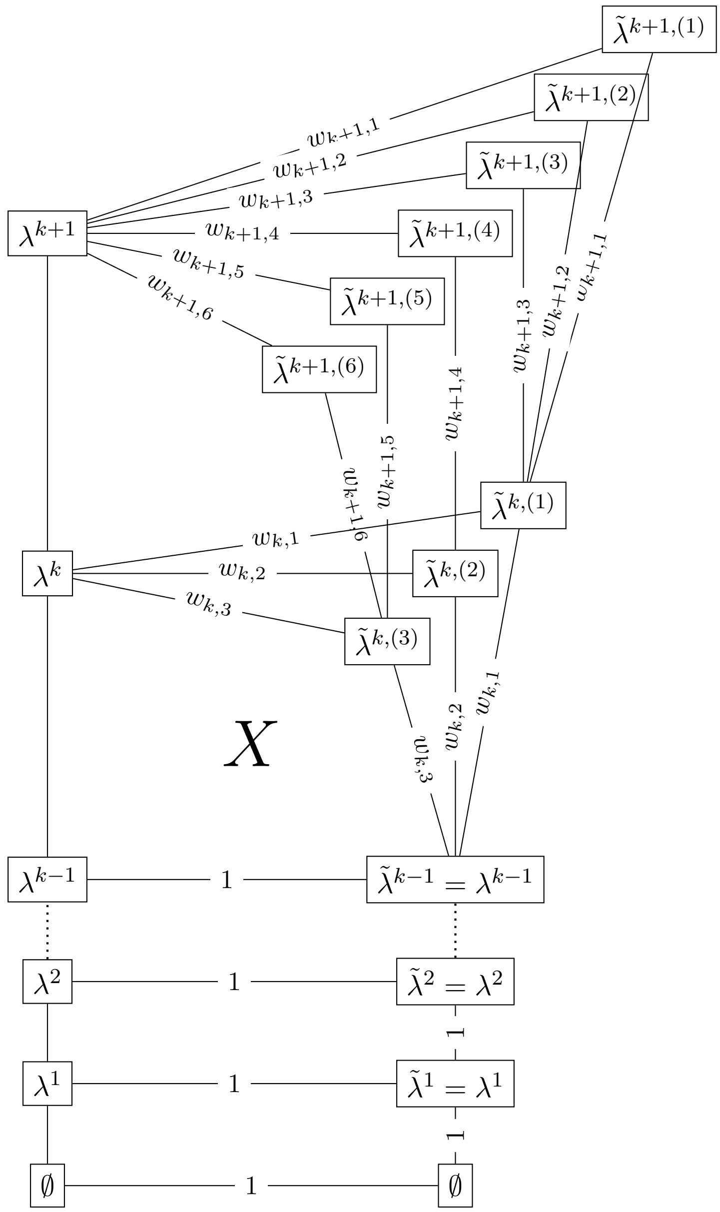


Figure 3.2: The one-column construction of q-inserting a letter to a tableau.

3.5 Word input for the q-weighted Robinson-Schensted algorithm

We can also take a word as input for the q-weighted Robinson-Schensted algorithm and produce a set of weighted pairs of tableaux. We start with the set of only one pair of tableaux $(P(0), Q^0) = (\emptyset, \emptyset)$ with weight 1. Suppose at time m, we have obtained a set of weighted pairs of tableaux

$$\{(P(m)^{(1)}, (Q^m)^{(1)}, w(m)^{(1)}), \cdots, (P(m)^{(p)}, (Q^m)^{(p)}, w(m)^{(p)})\}.$$

We use brackets in superscripts to indicate the different possibilities, in order not to confuse with subtableaux or shapes. We take each triplet $(P(m)^{(i)}, (Q^m)^{(i)}, w(m)^{(i)})$ in the collection, insert w_{m+1} into $P(m)^{(i)}$ to produce a new set of P-tableaux paired with the weight generated during the insertion

$$\{(P(m+1)^{(i,1)}, w^{(i,1)}), (P(m+1)^{(i,2)}, w^{(i,2)}), \cdots, (P(m+1)^{(i,r_i)}, w^{(i,r_i)})\}.$$

Then for each $P(m+1)^{(i,j)}$, $1 \leq j \leq r_i$, we pair it with a $(Q^{m+1})^{(i,j)}$ by adding a box with entry $m+1$ to $(Q^m)^{(i)}$ such that $\mathrm{sh}P(m+1)^{(i,j)} = \mathrm{sh}(Q^{m+1})^{(i,j)}$. We also attach a weight $w(m+1)^{(i,j)} = w(m)w^{(i,j)}$ to this pair. This way, for each triplet $(P(m)^{(i)}, (Q^m)^{(i)}, w(m)^{(i)})$ we have obtained its branching set

$$\begin{aligned}\{&(P(m+1)^{(i,1)}, (Q^{m+1})^{(i,1)}, w(m+1)^{(i,1)}),\\ &\quad (P(m+1)^{(i,2)}, (Q^{m+1})^{(i,2)}, w(m+1)^{(i,2)}),\\ &\qquad\qquad \cdots, (P(m+1)^{(i,r_i)}, (Q^{m+1})^{(i,r_i)}, w(m+1)^{(i,r_i)})\}.\end{aligned}$$

Let i run over $1, 2, \cdots, p$, we obtained a collection of all possible branchings:

$$((P(m+1)^{(i,j)}, (Q^{m+1})^{(i,j)}, w(m+1)^{(i,j)}))_{1\leq i\leq p, 1\leq j\leq r_i}.$$

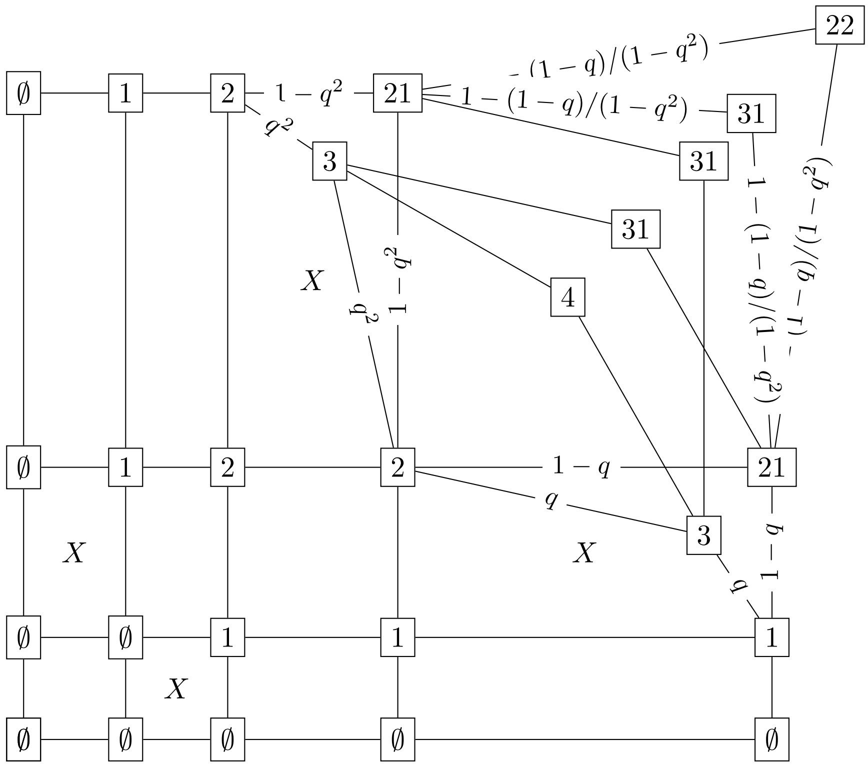
Note by *collection* we mean a vector of objects that allows repeats, as opposed to a *set*. We merge triplets with the same tableau pair by adding up their weights. This way we have obtained the weighted set of pairs of tableaux at time $m+1$.

The output of inserting the word $w_1 w_2 \ldots w_n$ is defined as the weighted set at time n. We denote by $\phi_w(P, Q)$ the weight of pair (P, Q) in this set. This is defined recursively by the following formula (see [OP13]). For P and Q with the same shape of size n and word w of length $n-1$,

$$\phi_{wk}(P, Q) = \sum \phi_w(\hat{P}, Q^{n-1}) I_k(\hat{P}, P),$$

where the sum is over all tableau $\hat{P}$ with the same shape as Q^{n-1}.

This, like in the classical case, can be visualised as a graph whose nodes are associated with vertices on $D_{n,\ell} := \{0, \ldots, n\} \times \{0, \ldots, \ell\}$ lattice diagram by concatenating the graphs of the one-column construction in Figure 3.2 associated with the insertion of each letter. Each possible $\lambda^{k,(p_{m,k})}(m)$ is associated with the vertex (m,k). For example, we obtain the following graph if we apply the algorithm to $2132 \in [3]^4$, where the unlabeled edges have weight 1, which will be the case hereafter.



From the definition of the algorithm, each node $\lambda^{\ell,(p)}(n)$ at (n,ℓ) corresponds to a (P,Q) tableau pair in the output set. Indeed we can trace back the genealogy of $\lambda^{\ell,(p)}(n)$ and find a unique array of indices $(p_{m,k})_{0\le m\le n, 0\le k\le \ell}$ with $p_{n,\ell} = p$ such that nodes $(\lambda^{k,(p_{m,k})}(m))_{m,k}$ are connected and form a diagram that is isomorphic to the lattice diagram $D_{n,\ell}$. Clearly,

$$
\begin{aligned}
P^{(p)} &= \lambda^{0,(p_{n,0})}(n) \prec \lambda^{1,(p_{n,1})}(n) \prec \cdots \prec \lambda^{\ell,(p_{n,\ell})}(n);\\
Q^{(p)} &= \lambda^{\ell,(p_{0,\ell})}(0) \prec \lambda^{\ell,(p_{1,\ell})}(1) \prec \cdots \prec \lambda^{\ell,(p_{n,\ell})}(n),
\end{aligned}
$$

are the corresponding (P,Q) tableau pair. Moreover, we can identify a weight with the node $\lambda^{\ell,(p)}$ by multiplying the weights of all the horizontal (or all the vertical) edges along its genealogical diagram and denote it by $w(\lambda^{\ell,(p)})$:

$$
\begin{aligned}
w(\lambda^{\ell,(p)}(n)) &= \prod_{1\le m\le n, 0\le k\le \ell} w(\lambda^{k,(p_{m-1,k})}(m-1) \to \lambda^{k,(p_{m,k})}(m))\\
&= \prod_{0\le m\le n, 1\le k\le \ell} w(\lambda^{k-1,(p_{m,k-1})}(m) \to \lambda^{k,(p_{m,k-1})}(m)).
\end{aligned}
$$

Then by definition,

$$\phi_w(P,Q) = \sum w(\lambda^{\ell,(p)}(n))\mathbb{I}_{P^{(p)}=P,Q^{(p)}=Q},$$

where the sum is over all nodes $\lambda^{l,(p)}(n)$ at vertex (n,l).

3.6 The symmetry property for the q-weighted RS algorithm with permutation input

When we take a permutation input σ which is identified as a word with distinct letters in the same way as in the normal RS algorithm, we also end up with a symmetry property, which is the main result of this chapter:

Theorem 15. *For any permutation $\sigma \in S_n$ and standard tableau pair (P,Q),*

$$\phi_{\sigma^{-1}}(Q,P) = \phi_\sigma(P,Q).$$

As in the normal RS algorithm case, permutation input are special in that we can restate the insertion rule in a symmetric fashion. For each box we pick a connected triplet on southwest, northwest and southeast corners. That is, suppose the box has index (m,k) we fix arbitrary p_1, p_2 and p_3 such that $(\lambda,\mu^1,\mu^2) := (\lambda^{k-1,(p_1)}(m-1), \lambda^{k,(p_2)}(m-1), \lambda^{k-1,(p_3)}(m))$ is a connected triplet. We denote by N the set of partitions on the vertex (m,k) that are connected to μ^1 and μ^2. And for any $\nu \in N$, we write $w(\mu,\nu)$ instead of $w(\mu^1 \to \nu)$ or $w(\mu^2 \to \nu)$ since they are equal, and for the sake of symmetry. Define an operator $I^k : W \to W$ by

$$I^k\lambda := \lambda + e_{\max\{j\le k:\lambda+e_j\in W\}}.$$

and a set $\Lambda^k(\lambda) \subset W$ by

$$\Lambda^k(\lambda) := \{I^j\lambda : j \le k\},$$

and denote $\Lambda(\lambda) := \Lambda^{l(\lambda)+1}(\lambda)$.

Then N and $(w(\mu,\nu) : \nu \in N)$ belongs to one of the following 5 cases.

1. The box has an X in it, and μ^1, μ^2 are equal to λ. Then N consists of all possible partitions obtained by adding a box to λ:

$$N = \Lambda(\lambda).$$

The weights are

$$w(\mu,\nu) = \begin{cases} q^{\lambda_j} - q^{\lambda_{j-1}}, & \text{if } \nu = \lambda + e_j \text{ for some } j > 1 \\ q^{\lambda_1}, & \text{if } \nu = \lambda + e_1 \end{cases}, \quad \nu \in N.$$

2. The box does not have an X in it and $\mu^1 = \lambda$. Then N is a singleton which is the same as μ^2, and the weight is 1:

$$w(\mu,\nu) = 1, \quad \nu \in N = \{\mu^2\}.$$

3. (The dual case of case 2)The box does not have an X in it and $\mu^2 = \lambda$. Then N is a singleton which is the same as μ^1, and the weight is 1:

$$w(\mu,\nu) = 1, \quad \nu \in N = \{\mu^1\}.$$

4. The box is empty. $\mu^1 = \lambda + e_i$ and $\mu^2 = \lambda + e_j$ for some $i \neq j$. Then N again only contains one element $\mu^1 \cup \mu^2$, with weight 1:

$$w(\mu,\nu) = 1, \quad \nu \in N = \{\mu^1 \cup \mu^2\}.$$

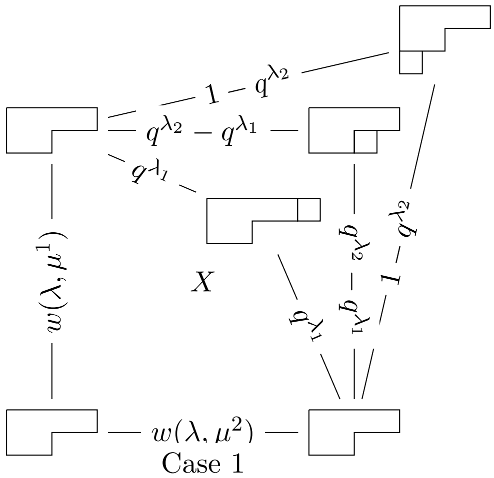
5. The box is empty and $\mu^1 = \mu^2 = \lambda + e_i := \mu$ for some i. Then N consists of all possible partitions that are obtained by adding a box to a row no lower than ith row of μ. That is

$$N = \Lambda^i(\mu).$$

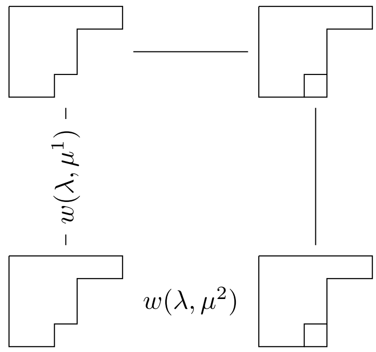
The weights are

$$w(\mu,\nu) = \begin{cases} \frac{1-q^{\lambda_{i-1}-\lambda_i-1}}{1-q^{\lambda_{i-1}-\lambda_i}}, & \nu = \mu + e_i; \\ \frac{1-q}{1-q^{\lambda_{i-1}-\lambda_i}} q^{\lambda_j-\lambda_i-1}(1-q^{\lambda_{j-1}-\lambda_j}), & \nu = \mu + e_j \text{ for some } 2 \le j < i; \\ \frac{1-q}{1-q^{\lambda_{i-1}-\lambda_i}} q^{\lambda_1-\lambda_i-1}, & \nu = \mu + e_1. \end{cases} \quad ,\nu \in N.$$

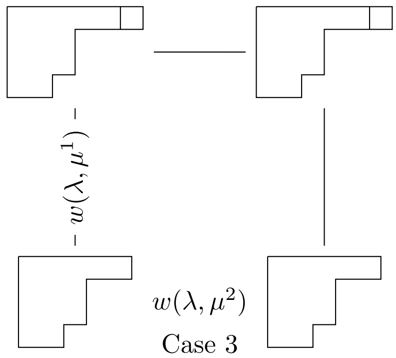
We call this the *growth graph rule* as opposed to the *insertion rule* described in Sections 3.4 and 3.5. Below is a group of illustrations of all 5 cases, where λ is the southwest partition.



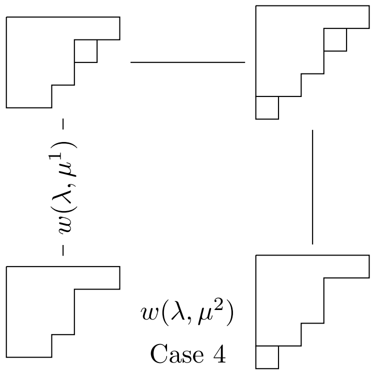
Case 1



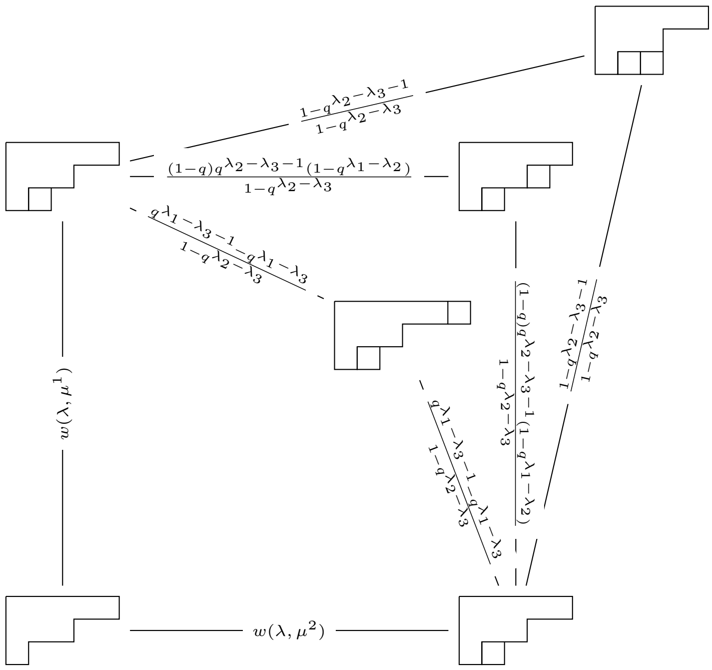
Case 2



Case 3



Case 4



Case 5

Proof of Theorem 15. It now suffices to show that the above constuction agrees with the definition of the algorithm in the preceeding section. That is, for any connected triplets (λ, μ^1, μ^2) surrounding the box (m, k) from the southwest, its branching set with corresponding weights according to the insertion rule agree with N and $w(\mu, \nu)$'s according to the growth graph rule. To distinguish the context of insertion rule and growth graph rule we also write $\lambda = \lambda^{k-1}$, $\mu^1 = \lambda^k$, $\mu^2 = \tilde{\lambda}^{k-1}$ and $\nu = \tilde{\lambda}^k \in N$ in accordance with the definition of the insertion algorithm in Section 3.4. We show this by discussing the location of (m, k) in the permutation matrix of σ, which corresponds to the 5 cases in the growth graph rule.

1. The box (m, k) has an X in it. This means $\sigma_m = k$. So we are inserting a k to the tableau at time m, hence $\lambda^{k-1} = \tilde{\lambda}^{k-1}$, i.e. $\lambda = \mu^2$. Moreover since σ is a permutation, we have $\sigma_i \neq k$ for all $i < m$, hence $\lambda^{k-1} = \lambda^k$, i.e. $\lambda = \mu^1$ (condition of Case 1 satisfied). Any branch of $(\lambda^{k-1}, \lambda^k, \tilde{\lambda}^{k-1}) = (\lambda, \lambda, \lambda)$ is one of the $\lambda + e_j$'s such that the weight $w((\lambda, \lambda, \lambda), \lambda + e_j) = w_0(k, j) \neq 0$.

$$w_0(k, j) = f_0(j; \lambda, \lambda) \prod_{p=j+1}^{k} (1 - f_0(p; \lambda, \lambda))$$
$$= \begin{cases} (1 - q^{\lambda_{j-1} - \lambda_j}) q^{\lambda_j} & \text{if } j > 1 \\ q^{\lambda_1} & \text{if } j = 1. \end{cases}$$

   So the weights agree. All the pruned branches $\lambda + e_j$ with $w_0(k, j) = 0$ are exactly the $\lambda + e_j$'s that are not partitions. Therefore N is exactly the set of all branches of the triplet.

2. There's no X in (i, k) for $1 \leq i \leq m$. This means $\sigma_i \neq k$ for $i \leq m$. Since $\sigma_i \neq k$ for $i < m$, $\lambda^{k-1} = \lambda^k$, i.e. $\lambda = \mu^1$ (condition of Case 2 satisfied). Moreover since $\sigma_m \neq k$, the triplet produces only one branch which is equal to μ^2 with weight 1. This agrees with Case 2.

3. There's no X in (m, i) for $1 \leq i \leq k$. This means $\sigma_m > k$. By the insertion rule, $\lambda^{k-1} = \tilde{\lambda}^{k-1}$, i.e. $\lambda = \mu^2$ (condition of Case 3 satisfied). Again by the same rule the triplet only produces one branch that equals μ^1 with weight 1.

4. There's one X in each of (t, k) and (m, s) for some $t < m$ and $s < k$. Then on the one hand $\sigma_t = k$ so $\lambda^k = \lambda^{k-1} + e_i$ for some i, i.e. $\mu^1 = \lambda + e_i$. On the other hand $\sigma_m < k$ so $\tilde{\lambda}^{k-1} = \lambda^{k-1} + e_j$ for some j and $j_{k-1} = j$, i.e. $\mu^2 = \lambda + e_j$ (Condition of Case 4 or Case 5 satisfied). Therefore the branches of the triplet $(\lambda, \lambda + e_i, \lambda + e_j)$ are in the form of $\tilde{\lambda}^k = \lambda^k + e_{j_k}$ for $j_k \leq j$ with

weights

$$w((\lambda,\lambda+e_i,\lambda+e_j),\lambda+e_i+e_{j_k}) = w_1(k,j_k) = \begin{cases} f_0(j_k;\lambda,\lambda+e_i)\left(\prod_{p=j_k+1}^{j-1}(1-f_0(p;\lambda,\lambda+e_i))\right) & \\ \times(1-f_1(j;\lambda,\lambda+e_i)) & \text{if } j_k<j \\ f_1(j;\lambda,\lambda+e_i) & \text{if } j_k=j \end{cases} \tag{3.2}$$

If $j \neq i$ (condition of Case 4 satisfied), then $f_1(j;\lambda,\lambda+e_i) = (1-q^{\lambda_{j-1}-\lambda_j})/(1-q^{\lambda_{j-1}-\lambda_j}) = 1$. Therefore the branch has only one shape equal to μ^1+e_j with weight 1. This agrees with Case 4.

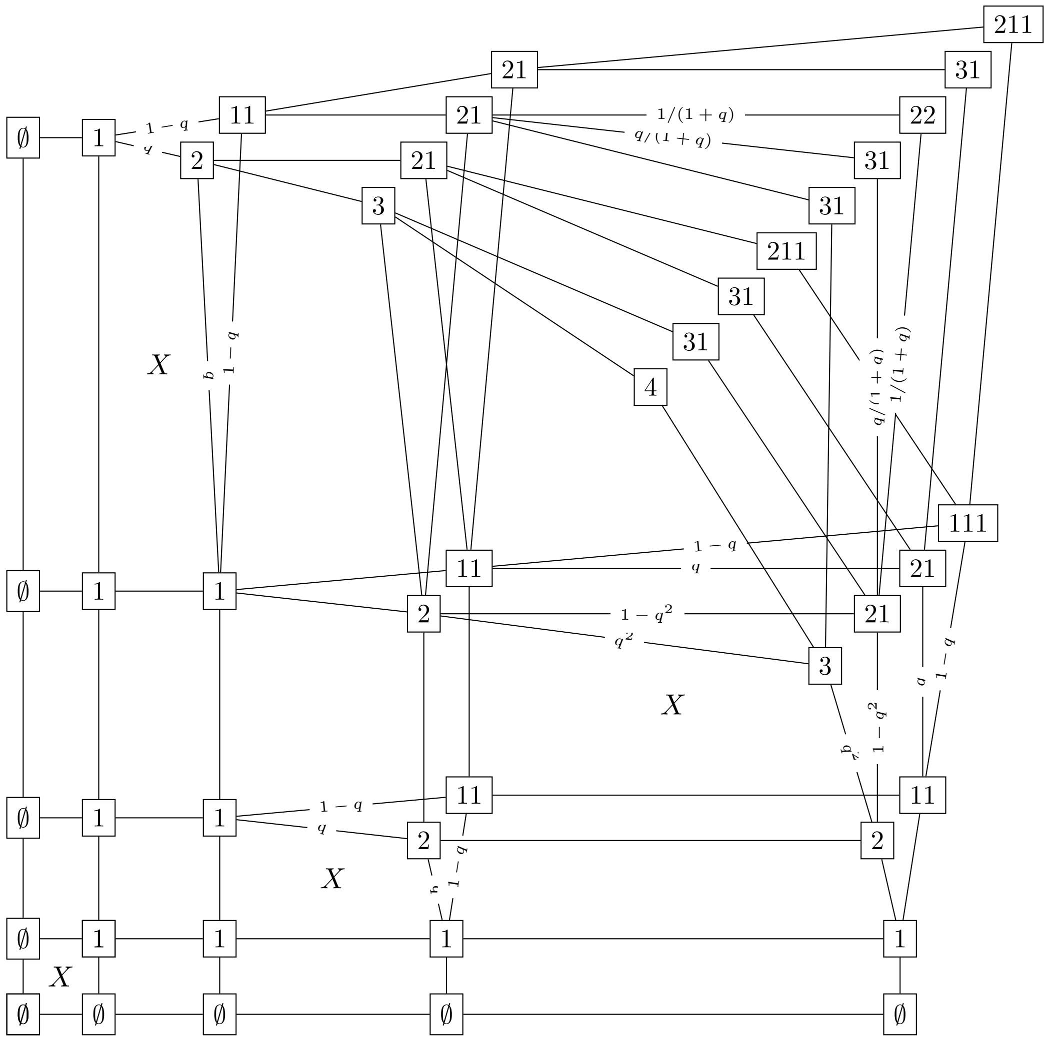
If $j = i$ (condition of Case 5 satisfied), then by (3.2)

$$w_1(k,j_k) = \begin{cases} \frac{1-q^{\lambda_{i-1}-\lambda_i-1}}{1-q^{\lambda_{i-1}-\lambda_i}}; & \text{if } j_k=i; \\ \frac{1-q}{1-q^{\lambda_{i-1}-\lambda_i}}q^{\lambda_j-\lambda_i-1}(1-q^{\lambda_{j-1}-\lambda_j}); & \text{if } 2\le j_k<i; \\ \frac{1-q}{1-q^{\lambda_{i-1}-\lambda_i}}q^{\lambda_1-\lambda_i-1}; & \text{if } j_k=1. \end{cases}$$

So the weights agree with Case 5. Moreover, all the pruned branches are exactly the $\lambda+e_i+e_{j_k}$'s that are not partitions. Therefore N is indeed the set of all branches.

When inverting a permutation σ to σ^{-1}, the X marks are transposed, so is the weighted graph by the symmetry of the rule, thus we have arrived at the conclusion. □

Note that in both rules although we consider an arbitrary box with index (m,k), the rules do not depend on either m or k. This is a key condition for the symmetry property to work and will be generalised in Proposition 18 in the following section. Below is the growth graph of the permutation 1423:



3.7 More insertion algorithms

In [Sta01] the growth diagram technique was used to show the symmetry property for the RS algorithm with row insertion. To row insert a k into a tableau P, we again keep $\lambda^0 \prec \lambda^1 \prec \cdots \prec \lambda^{k-1}$ unchanged. Then we append a box at the end of first row of $\lambda^k, \lambda^{k+1}, \ldots, \lambda^{k_1-1}$, where k_1 is the smallest number in row 1 of P that is larger than k, then we append a box at the end of second row of $\lambda^{k_1}, \lambda^{k_1+1}, \ldots, \lambda^{k_2-1}$, where k_2 is the smallest number in row 2 of P that is larger than k_1, and so on and so forth. More precisely, define:

$$k_0 = k; \quad k_j = \min(\{k' > k_{j-1} : \lambda_j^{k'} > \lambda_j^{k'-1}\} \cup \{\ell + 1\}), \quad j \geq 1.$$

Then the output tableau has:

$$\tilde{\lambda}^i = \begin{cases} \lambda^i, & \text{if } i < k, \\ \lambda^i + e_j, & \text{if } k_{j-1} \leq i < k_j \text{ for some } j. \end{cases}$$

For example if we insert a 3 into the tableau (3.1), the one-column insertion diagram is as follows:

$$\begin{tikzcd}
4322 & 43221 \arrow[l] \\
422 \arrow[u] \arrow[r] & 4221 \arrow[u] \\
421 \arrow[u] \arrow[r] & 4211 \arrow[u] \\
42 \arrow[u] \arrow[r] & 421 \arrow[u] \\
41 \arrow[u] \arrow[r] & 42 \arrow[u] \\
31 \arrow[u] \arrow[r] & 41 \arrow[u] \\
2 \arrow[u] \arrow[r] \arrow[ur, phantom, "X"] & 2 \arrow[u] \\
2 \arrow[u] \arrow[r] & 2 \arrow[u] \\
\emptyset \arrow[u] \arrow[r] & \emptyset \arrow[u]
\end{tikzcd}$$

and the corresponding output tableau is

$$\tilde{P} = \begin{matrix} 1 & 1 & 3 & 3 \\ 3 & 4 & 8 & \\ 5 & 7 & & \\ 6 & 8 & & \\ 8 & & & \end{matrix}$$

The row-insertion of a word is defined in the same way as in column inserting a word. The normal row and column insertions are related in a few ways.

The most elegant one is the duality. Denote by $(P_{\text{col}}(w), Q_{\text{col}}(w))$ and $(P_{\text{row}}(w), Q_{\text{row}}(w))$ the tableau pairs when applying column and row insertions to word w respectively. Also denote by w^r the inverse word of w: $w^r = (w_n, w_{n-1}, \ldots, w_1)$. Moreover, denote by $\text{ev}(Q)$ the evacuation operation on Q, see e.g. [Sag00, Ful97]. Then

Proposition 16 (see e.g. [Ful97]). *For any word w,*

$$P_{col}(w) = P_{row}(w^r).$$

If furthermore w is a permutation, then

$$Q_{col}(w) = (ev(Q_{row}(w^r)))'$$

where T' means the transposition of tableau T.

This duality has a matrix input generalisation where $Q_{\mathrm{row}}(w^r)$ is obtained by a reverse sliding operation, see e.g. [Ful97].

Another simple relation is between the original definitions. Row insertion was initially defined as an algorithm of inserting and bumping based on the ordering of integers. This definition turns into column insertion if one replaces all occurrance of "row" by "column" and replaces the strong order (greater than) with the weak order (greater than or equal to) due to the asymmetry of the ordering in rows and columns in the definition of a tableau. For these definitions see e.g. [Sag00].

The third relation is a bit more complicated. In column insertion, we initialise $j_{k-1} = j$. Then in each step we find the largest row index $j_k \leq j_{k-1}$ for a letter k such that $\lambda^{k-1}_{j_k-1} > \lambda^k_{j_k}$, append a box to $\lambda^k_{j_k}$ and increase k by 1. In row insertion, we initialise $k_0 = k$, and in each step one we find the smallest letter $k_j > k_{j-1}$ for a row index j such that $\lambda^{k_j}_j > \lambda^{k_j-1}_j$, append a box to $\lambda^{k_{j-1}}_j, \dots, \lambda^{k_j-1}_j$ and increase j by 1.

In a recent paper [BP13], a q-weighted Robinson-Schensted algorithm with row insertion was proposed. It is defined in a similar way as the q-column insertion and has the third relation with the column insertion, which we detail below.

In q-column insertion, we initialise $j_{k-1} = k$. Then in each step we run over all row indices $j_k \leq j_{k-1}$ for a letter k, with some weight append a box to $\lambda^k_{j_k}$ and increase k by 1. In q-row insertion, we initialise $k_0 = k$. In each step one we run over all letters $k_j > k_{j-1}$ for a row index j such that, with some weight we append a box to $\lambda^{k_{j-1}}_j, \dots, \lambda^{k_j-1}_j$ and increase j by 1.

Our definition here is a reformulation equivalent to the one in [BP13]. For $\mu \prec \lambda$ define

$$g(1; \mu, \lambda) = 1 - q^{\lambda_1 - \mu_1},$$
$$g(j; \mu, \lambda) = \frac{1 - q^{\lambda_j - \mu_j}}{1 - q^{\mu_{j-1} - \mu_j}} \quad j \geq 2.$$

When q-row inserting a k into a tableau P, we keep the first $k-1$ shapes unchanged: $\tilde{\lambda}^i = \lambda^i, i = 1, \dots, k-1$, with weight 1, $\tilde{\lambda}^k = \lambda + e_1$ and for $i > k$, we have a binary branching for each triplet $(\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1} = \lambda^{i-1} + e_{j_{i-1}})$:

$$u((\lambda^{i-1}, \lambda^i, \lambda^{i-1} + e_{j_{i-1}}), \lambda^i + e_{j_i}) = \begin{cases} g(j_{i-1}; \lambda^{i-1}, \lambda^i), & j_i = j_{i-1} + 1; \\ 1 - g(j_{i-1}; \lambda^{i-1}, \lambda^i), & j_i = j_{i-1}; \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $J_k(P, \tilde{P})$ the weight of obtaining a $\tilde{P}$ after row inserting a k into P.

We define the q-row insertion for word input in the same way as q-column insertion of words: successively q-row inserting letters, multiplying the weights, keeping a tableau Q to record changes for each P, and merging the same tableau pairs by adding up the weights. Denote by $\psi_w(P, Q)$ the weight of obtaining tableau pair (P, Q) after q-row inserting a word w. Then we have the same recursion rule: for a pair of tableaux (P, Q) with the same shape of size n and a word w of length $n-1$,

$$\psi_{wk}(P, Q) = \sum \psi_w(\hat{P}, Q^{n-1}) J_k(\hat{P}, P).$$

where the sum is over all tableau $\hat{P}$ with the same shape as Q^{n-1}. Since the algorithm has a similar triplet branching structure as in Figure 3.1, we can build the one-column insertion construction like in Figure 3.2, and concatenate them into a growth graph. With the same approach we can show the symmetry property for this algorithm.

Theorem 17. *For any permutation $\sigma \in S_n$ and standard tableau pair (P, Q),*

$$\psi_{\sigma^{-1}}(Q, P) = \psi_\sigma(P, Q).$$

A sketch proof. Again denote by λ, μ^1, μ^2 nodes associated with vertices surrounding a box from the south and the west. The set M of all partitions branched from the triplet with $(u(\mu, \nu) : \nu \in M)$ belongs to one of the following 5 cases.

1. The box has an X in it. And μ^1, μ^2 are equal to λ. Then M consists of only one partition $\lambda + e_1$ with weight 1.

2. The box does not have an X in it and $\mu^1 = \lambda$. Then M consists of only one partition μ^2 with weight 1.

3. The box does not have an X in it and $\mu^2 = \lambda$. Then M consists of only one

partition μ^1 with weight 1.

4. The box is empty. $\mu^1 = \lambda + e_i$ and $\mu^2 = \lambda + e_j$ for some $i \neq j$. Then M again only contains one element $\mu^1 \cup \mu^2$ with weight 1.

5. The box is empty and $\mu^1 = \mu^2 = \lambda + e_i := \mu$ for some i. Then $M = \{\lambda + e_i + e_{i+1}, \lambda + 2e_i\}$ with weight

$$u((\lambda, \lambda + e_i, \lambda + e_i), \lambda + e_{i+1}) = \begin{cases} 1 - q, & \text{if } i = 1; \\ \frac{1-q}{1-q^{\lambda_{i-1}-\lambda_i}}, & \text{if } i > 1. \end{cases}$$

$$u((\lambda, \lambda + e_i, \lambda + e_i), \lambda + 2e_i) = \begin{cases} q, & \text{if } i = 1; \\ 1 - \frac{1-q}{1-q^{\lambda_{i-1}-\lambda_i}}, & \text{if } i > 1. \end{cases}$$

The claim is concluded once these symmetric rules are verified to be equivalent to the insertion rule, which is done in the same way as in the proof of Theorem 15. □

More generally, the symmetry property does not require very strong conditions on the insertion algorithms. Here we give a sufficient condition for an insertion algorithm to have this property.

First we define a *branching insertion algorithm* as an algorithm that has the branching structure as in Figure 3.1 when we insert a letter to a tableau. That is, there exists an *initial branching weight function* w_0, a *high level weight function* w_1 and a *low level weight function* w_2 such that when inserting a letter k, the weight of a new ith shape is:

$$w((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i) = \begin{cases} w_0((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i), & \text{if } i = k; \\ w_1((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i), & \text{if } i > k; \\ w_2((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i), & \text{if } i < k. \end{cases}$$

Then we have

Proposition 18. *If a branching insertion algorithm satisfies the following conditions:*

(i) *The insertion of k' into a tableau results in increment of one coordinate by 1 in $\lambda^{k'}, \lambda^{k'+1}, \ldots, \lambda^{\ell}$, while keep $\lambda^0, \lambda^1, \ldots, \lambda^{k'-1}$ unchanged, that is, the support of $w_0((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i)$ and $w_1((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i)$ are in $\{\lambda^i + e_j : j \geq 1\}$ and $w_2((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i) = \mathbb{I}_{\lambda^i = \tilde{\lambda}^i}$,*

(ii) $w_1((\lambda^{m-1}, \lambda^m, \lambda^{m-1} + e_i), \lambda^m + e_j) = \mathbb{I}_{i=j}$ *if* $\lambda^m_i = \lambda^{m-1}_i$;

(iii) $w_0((\lambda,\lambda,\lambda),\lambda+e_i)$ *does not depend on the inserted letter;*

(iv) $w_1((\lambda,\lambda+e_i,\lambda+e_i),\lambda+e_i+e_j)$ *does not depend on the inserted letter,*

then it has the symmetry property.

Sketch proof. We use the same notations λ,μ^1,μ^2 as in descriptions of growth graph rules such that they surround the box (m,k) from the south and the west. The symmetry property is shown once we can construct a symmetric growth graph rule, where the relation of λ, μ^1 and μ^2 and whether there's an X in the box, determines the location of the box in the permutation matrix, and vice versa.

Then the 5 cases of the growth diagram rule are satisfied given these conditions: for Case 1, the equivalence between ($\lambda=\mu^1=\mu^2$ AND X) and ($\sigma_m=k$) is given by (i) and (ii), and the branched shapes and weights are given by (iii); for Case 2 both the equivalence between ($\lambda=\mu^1$ AND NOT X) and ($\sigma_s\neq k\forall s\leq m$) is given by (ii), and the singleton branching with weight 1 is also given by (ii); for Case 3 both the equivalence between ($\lambda=\mu^2$ AND NOT X) and ($\sigma_m>k$) is given by (i), and the singleton shape with weight 1 is also given by (i); for Case 4 and 5 the equivalence between ($\mu^1=\lambda+e_i$ AND $\mu^2=\lambda+e_j$ AND NOT X) and ($\sigma_k^{-1}<m$ AND $\sigma_m<k$) is given by (i)(ii), and the singleton shape with weight 1 in Case 4 is given by (ii) while the branched shapes and weights in Case 5 is given by (iv). □

With this proposition we can test different algorithms for the symmetry property, although not against the symmetry property since it's only a sufficient condition.

For example, one column insertion algorithm proposed by [BP13] satisfies the conditions and has a symmetry property. See Dynamics 3 in Section 6.5.3 and (8.5) in Section 8.2.1 in that paper. Here we restate the definition in the framework of branching insertion algorithms. Denote for $\mu\prec\lambda$:

$$I^j(\lambda;\mu):=\lambda+e_{\max(\{i\leq j:\mu_{i-1}>\lambda_i\}\cup\{1\})}.$$

The algorithm is defined by the following weights

$$
\begin{aligned}
&w_0((\lambda^{k-1}, \lambda^k, \tilde{\lambda}^{k-1}), \tilde{\lambda}^k) = \mathbb{I}_{\tilde{\lambda}^k = I^k(\lambda^k;\lambda^{k-1})},\\
&w_1((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i)\\
&= \begin{cases}
-\frac{q^{\lambda_j^{i-1}-\lambda_{j+1}^i+1}(1-q^{\lambda_j^i-\lambda_j^{i-1}})}{(1-q^{\lambda_j^{i-1}-\lambda_{j+1}^i+1})(1-q^{\lambda_{j-1}^{i-1}-\lambda_j^{i-1}})}, & \\
\qquad\qquad \text{if } \tilde{\lambda}^{i-1} = \lambda^{i-1} + e_j, \tilde{\lambda}^i = \lambda^i + e_{j+1} \text{ for some } j \geq 2; & \\
-\frac{q^{\lambda_1^{i-1}-\lambda_2^i+1}(1-q^{\lambda_1^i-\lambda_1^{i-1}})}{1-q^{\lambda_1^{i-1}-\lambda_2^i+1}}, & \\
\qquad\qquad \text{if } \tilde{\lambda}^{i-1} = \lambda^{i-1} + e_1, \tilde{\lambda}^i = \lambda^i + e_2; & \\
1 + \frac{q^{\lambda_j^{i-1}-\lambda_{j+1}^i+1}(1-q^{\lambda_j^i-\lambda_j^{i-1}})}{(1-q^{\lambda_j^{i-1}-\lambda_{j+1}^i+1})(1-q^{\lambda_{j-1}^{i-1}-\lambda_j^{i-1}})}, & \\
\qquad\qquad \text{if } \tilde{\lambda}^{i-1} = \lambda^{i-1} + e_j, \tilde{\lambda}^i = I^j(\lambda^i; \lambda^{i-1}) \text{ for some } j \geq 2; & \\
1 + \frac{q^{\lambda_1^{i-1}-\lambda_2^i+1}(1-q^{\lambda_1^i-\lambda_1^{i-1}})}{1-q^{\lambda_1^{i-1}-\lambda_2^i+1}}, & \\
\qquad\qquad \text{if } \tilde{\lambda}^{i-1} = \lambda^{i-1} + e_1, \tilde{\lambda}^i = \lambda^i + e_1; & \\
0, & \\
\qquad\qquad \text{otherwise,} &
\end{cases}\\
&w_2((\lambda^{i-1}, \lambda^i, \tilde{\lambda}^{i-1}), \tilde{\lambda}^i) = \mathbb{I}_{\tilde{\lambda}^i = \lambda^i}.
\end{aligned}
$$

We can see from these weight function formulae that this algorithm is different from the q-column insertion algorithm in the previous sections. Qualitatively, it has row-insertion like branching of inserting a bumped letter into a lower row, which never happens in normal column insertion where a bumped letter stays in the same row or is inserted to a higher row. As such, it is more natural to compare the q-column insertion discussed in the previous sections with the q-row insertion, as we have pointed out the similarity of the relation between the q-column insertion and the q-row insertion and the relation between the normal column and row insertions.

In [BP13] another pair of q-RS column and row insertion algorithms called "q-Whittaker-multivariate 'dynamics' with deterministic long-range interactions" were introduced. These are also branching insertion algorithms. We omit the definitions here and point interested readers to Section 8.2.2 of that paper. These algorithms also satisfy the conditions in Proposition 18 and thus enjoy the symmetry property.

Chapter 4

Introduction to quantum stochastic calculus

4.1 Quantum probability

This chapter is a very brief introduction to quantum stochastic calculus. For a comprehensive introduction of this subject see the book by Parthasarathy [Par92]. For basics of C*-algebras including the GNS representation in general see e.g. [Erd03]. For the GNS construction of the symmetric Fock space from the CCR algebra see e.g. [Pet90].

The "quantum" in quantum probability may be interpreted as a close relation to the subject of quantum mechanics, or its nature as a noncommutative version of probability theory where operators are the main objects of interest.

Instead of a sample space, in quantum probability we work on a separable Hilbert space H. Denote by $B(H)$, $O(H)$ and $P(H) \subset O(H) \cap B(H)$ the set of bounded operator, (possibly unbounded) self-adjoint operators and othogonal projections. An *observable*, the noncommutative version of a random variable, is a self-adjoint operator $T \in O(H)$. As such, it is common for observables not to commute with each other, in which case they are called *mutually incompatible* and generally they do not have a joint probability distribution. To describe the probability distribution of an observable T, we need the spectral theorem:

$$T = \int_{\mathbb{R}} \lambda P_T(d\lambda)$$

where $P_T : \mathcal{B}(\mathbb{R}) \to P(H)$ is the spectral measure of T, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. For any bounded Borel function f on $\mathbb{R}$, the operator $f(T)$ can be defined from the spectral measure of T:

$$f(T) := \int_{\mathbb{R}} f(\lambda) P_T(d\lambda).$$

A *state* ρ is a positive linear functional, normalised so that $\rho(I) = 1$ where I is the identity operator. Then the probability measure $\mu_{T,\rho}$ of T in state ρ is defined by

$$\rho(f(T)) = \int_{\mathbb{R}} f(\lambda) \mu_{T,\rho}(d\lambda),$$

for all bounded Borel functions f on $\mathbb{R}$. Most of the time we only work with a *pure state*, that is, a state in the form of $\langle \xi, \cdot \xi \rangle$ for some unital $\xi \in H$, in which case we also refer to ξ as the state.

4.2 The Schrödinger representation

For now we temporarily use the notation H for a Hamiltonian. We start with the harmonic oscillator. It describes the system of a weight attached to the end of a spring and has Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2$$

where $\hat{p}$ and $\hat{q}$ are the momentum and position of the weight, m is the mass and ω the angular velocity.

When quantised, $\hat{p}$ and $\hat{q}$ are replaced by the momentum and position operators p and q, defined by

$$pf(x) := -i\hbar f'(x), \qquad qf(x) = xf(x),$$

where $\hbar$ is the Planck constant. The Hamiltonian is now

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

The time-independent Schrödinger equation is an eigenvalue problem:

$$H\psi = E\psi$$

which, after scaling, i.e. substituting $y = \sqrt{\frac{m\omega}{\hbar}} x$, $\epsilon = \frac{2E}{\hbar\omega}$ and $f(y) = e^{\frac{y^2}{2}} \psi(y)$ turns into the Hermite differential equation:

$$f''(y) - 2yf'(y) + (\epsilon - 1)f(y) = 0,$$

Solving this equation using the series method we obtain evenly-spaced discrete energy level $E = (n + \frac{1}{2})\hbar\omega$ for $n = 0, 1, 2, \dots$ and Hermite polynomials $f(y) = h_n(y)$. The normalisation of $\psi_n = e^{-\frac{x^2}{2}} h_n$ gives the stationary states.

We restore the use of the notation $H = L^2(\mathbb{R})$ as a Hilbert space. It is easier to work with normalised p and q, redefined as follows:

$$p := -\sqrt{2}i\frac{d}{dx}, \qquad q := \sqrt{2}x$$

They are essentially self-adjoint operators on H, meaning that they have unique self-adjoint extensions, and satisfy the canonical commutation relation (CCR)

$$[p, q] = -2i$$

when acting on the Schwarz space

$$\mathcal{S}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : x^n f^{(m)}(x) \to 0 \text{ as } |x| \to \infty, \forall m, n \geq 0\}.$$

The rigorous form of the CCR on H, also known as the Weyl relation, is

$$W(z)W(w) = e^{-i\Im\langle z,w\rangle} W(z + w), \quad z, w \in \mathbb{C} \tag{4.1}$$

where the Weyl operator $W(z) := e^{-ixp+iyq}$ for $z = x + iy \in \mathbb{C}$, and the above CCR can be verified by

$$W(t)f(x) = f(x - \sqrt{2}t), \qquad W(is)f(x) = e^{\sqrt{2}isx} f(x)$$

and the Baker-Campbell-Hausdorff formula. Conversely, if we start from what is called the CCR C*-algebra (defined in the section 4.3) $A(\mathbb{C})$ over $\mathbb{C}$ which is partly characterised by the relation (4.1), then p, q, $W(x)$ and $W(iy)$ as we have defined defined in this chapter are called the *Schrödinger representation* of $A(\mathbb{C})$.

It turns out in the state ψ_0, the observables p and q have standard normal distribution as

$$\langle \psi_0, W(x)\psi_0\rangle = \langle \psi_0, W(ix)\psi_0\rangle = e^{-\frac{x^2}{2}}.$$

Moreover, although p and q do not commute, thus mutually incompatible, they are independent in the state ψ_0 in the sense that

$$\langle \psi_0, W(z)\psi_0 \rangle = e^{-\frac{|z|^2}{2}}.$$

4.3 The Fock representation

There are several different but equivalent definitions of the symmetric Fock space, including writing it as a symmetric tensor algebra, or a Hilbert space generated by the exponential vectors. In this section we use the GNS construction which can be found in [Pet90]. For an introduction to C* algebras and the GNS construction see e.g. [Erd03].

Given a Hilbert space H, the CCR C*-algebra $A(H)$ over H is generated by unitary elements $W(f)$ for $f \in H$, satisfying

1. $W(-f) = W(f)^*$,
2. $W(f)W(g) = \exp(-i\Im\langle f, g\rangle)W(f+g)$.

Let ϕ be a state of $A(H)$ defined by

$$\phi(W(f)) := \exp(-\|f\|^2/2).$$

The GNS representation $F(H)$ of $A(H)$ associated with ϕ is called the (symmetric) *Fock space* over H. We call $W(f) \in A(H)$ the Weyl operators, and denote $[W(f)] \in F(H)$ the induced element in $F(H)$. We call $\Phi := [W(0)] = [I]$ the vacuum state. And define

$$e(f) := \frac{W(f)}{\phi(W(f))}\Phi$$

to be the *exponential vectors*. Then $\{e(f) : f \in H\}$ generates $F(H)$ as a Hilbert space. For any $f \in H$, $W(tu)$ is a strongly continuous one parameter unitary group, hence by Stone's theorem we can define $P(f)$ and $Q(f)$ to be the generator of $(W(-tf))_t$ and $(W(itf))_t$ respectively.

As an example, when $H = \mathbb{C}$, the Fock space $F(H) \approx L^2(\mathbb{R})$, $P(1)$ and $Q(1)$ can be identified with p and q, and the vacuum state Φ corresponds to ψ_0.

When $H = \mathbb{C}^2$, since $F(H_1 \oplus H_2) \approx F(H_1) \otimes F(H_2)$ (with $e(f_1 \oplus f_2) \approx e(f_1) \otimes e(f_2)$), $F(H) \approx L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ and there exist two independent pairs (p, q) and (p', q') such that $W((x+iy, x'+iy')) = \exp(-ixp+iyq-ix'p'+iy'q')$ satisfying

$$[p, q] = [p', q'] = -2i, \quad [p, p'] = [p, q'] = [p', q] = [q, q'] = 0.$$

When $H = L^2(\mathbb{R}_{\geq 0})$, define $P_t = P(\mathbb{I}_{[0,t]})$ and $Q_t = Q(\mathbb{I}_{[0,t]})$. They are Brownian motions in the vacuum state Φ because (where P can be replaced by Q)

$$\langle \Phi, e^{ix_1 P_{t_1}} e^{ix_2 P_{t_2}} \dots e^{ix_n P_{t_n}} \Phi \rangle = \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i x_j (t_i \wedge t_j) \right)$$

Furthermore (P_t) and (Q_t) are also independent in the vacuum state in the sense of the characteristic functions because

$$\begin{aligned} &\mathbb{E}_\Phi \exp\left(i \sum_{j=1}^{m} x_j P_{s_j} + i \sum_{k=1}^{n} y_k Q_{t_k} \right) \\ &\qquad = \mathbb{E}_\Phi \exp\left(i \sum_{j=1}^{m} x_j P_{s_j} \right) \mathbb{E}_\Phi \exp\left(i \sum_{k=1}^{n} y_k Q_{t_k} \right) \end{aligned}$$

We call (P_t) and (Q_t) the *momentum and position Brownian motions.*

4.4 Second quantisation

Given a unitary operator $U \in B(H)$, the second quantisation $\Gamma(U) \in B(F(H))$ is defined by its action on the exponential vectors:

$$\Gamma(U) e(v) = e(Uv).$$

The second quantisation plays an important role in the construction of the quantum Poisson processes (see e.g. [Bou08]). Here we show an explicit formula for the second quantisations of a family of rotation-like operators when $H = \mathbb{C}^2$. Let (p, q) and (p', q') be the two canonical pairs mentioned in Section 4.3. By viewing $pq' - qp'$ as an angular momentum, we establish [HP15b] a relation between it and the rotation via the second quantisation:

$$\Gamma(U_{2x}) = e^{ix(pq' - qp')}$$

where U_x is a rotation on $\mathbb{C}^2$ represented by the matrix

$$U_x = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.$$

This result is extended in [HP15b] to the following:

$$\Gamma(\xi_{\nu,x}) = e^{ix(\lambda(pq'-qp')+\mu(pp'+qq'))},$$

where $\nu = \lambda + i\mu$ and

$$\xi_{\nu,x} = \begin{pmatrix} \cos 2|\nu|x & -\frac{\bar{\nu}}{|\nu|}\sin 2|\nu|x \\ \frac{\nu}{|\nu|}\sin 2|\nu|x & \cos 2|\nu|x \end{pmatrix}$$

This is a key result which will be used in the study of a family of double product integrals in [HP15a](Chapter 5).

Chapter 5

On a family of causal quantum stochastic double product integrals related to Lévy area

We study the family of causal double product integrals

$$\prod_{a<x<y<b}\left(1+i\frac{\lambda}{2}(dP_xdQ_y-dQ_xdP_y)+i\frac{\mu}{2}(dP_xdP_y+dQ_xdQ_y)\right)$$

where P and Q are the mutually noncommuting momentum and position Brownian motions of quantum stochastic calculus. The evaluation is motivated heuristically by approximating the continuous double product by a discrete product in which infinitesimals are replaced by finite increments. The latter is in turn approximated by the second quantisation of a discrete double product of rotation-like operators in different planes due to a result in [HP15b]. The main problem solved in this chapter is the explicit evaluation of the continuum limit W of the latter, and showing that W is a unitary operator. The kernel of W is written in terms of Bessel functions, and the evaluation is achieved by working on a lattice path model and enumerating linear extensions of related partial orderings, where the enumeration turns out to be heavily related to Dyck paths and generalisations of Catalan numbers.

5.1 Introduction

Following Volterra's philosophy of product integrals as continuous limits of discrete products [Sla07], quantum stochastic double product integrals of *rectangular* type have been constructed [HP15b] as limits of discrete approximations obtained by replacing stochastic differentials by discrete increments of the corresponding processes. Such constructions are partially intuitive in character, involving nonrigorous manipulations of unbounded operators. Nevertheless they can be shown to yield explicit rigorously unitary operators which can then be shown in some cases [HJ12] to satisfy the quantum stochastic differential equations (qsde's) whose solutions provide the rigorous definition of the product integral.

In this chapter we initiate the much harder problem of constructing so-called *causal* (or triangular) double product integrals in the same way, first constructing discrete approximations by intuitive methods, which are then shown rigorously to enjoy the property of unitarity, which will allow rigorous verification of the qsde definitions.

The *Fock space* $\mathcal{F}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ is conveniently defined [Par92] as the Hilbert space generated by the *exponential vectors* $e(f), f \in \mathcal{H}$, satisfying

$$\langle e(f), e(g) \rangle = \exp \langle f, g \rangle, \ \ f, g \in \mathcal{H}.$$

Rectangular product integrals live in the tensor product of two Fock spaces. This form of "double" construction was originally motivated by its use to construct explicit solutions of the quantum Yang-Baxter equation with a given classical limit [Hud05], [HP05], of purely algebraic character as formal power series. From the analytic point of view, the alternative causal constructs which are studied in the present chapter which live naturally in a single Fock space are of wider interest.

One example which we consider in some detail is closely related to Lévy's stochastic area [Lév51], and in particular to the Lévy area formula for its characteristic function. In effect we replace the planar Brownian motion by a quantum version in which the two components are the mutually noncommuting momentum and position Brownian motions P and Q of quantum stochastic calculus [CH77], which despite noncommutativity, can be shown to be independent in a certain sense [Hud13]. Other noncommutative analogs of Lévy area are based on free probability [CDM01]; our own less radically noncommutative form is directly related to physical applications [HCHJ13]. It may also offer mathematically significant relations, for example to Riemann zeta values through the links to Euler and Bernoulli numbers [IT10] of the classical Lévy area formula. This is because, while the corresponding probability distribution is the atomic one concentrated at zero, it deforms naturally to the classical distribution at infinite temperature as the Fock "zero temperature" momentum and position processes P and Q are deformed through corresponding finite temperature processes [CH13] to mutually commuting independent Brownian motions.

We denote rectangular and causal product integrals by

$$\prod_{[a,b)\times[c,d)} (1+dr)\,, \prod_{<_{[a,b)}} (1+dr) \tag{5.1}$$

respectively, where $<_{[a,b)}$ is the set $\left\{(x,y)\in\mathbb{R}^2 : a\leq x < y < b\right\}$. They are operators in the Hilbert spaces $\mathcal{F}\left(L^2\left([a,b)\right)\right)\otimes\mathcal{F}\left(L^2\left([c,d)\right)\right)$ and $\mathcal{F}\left(L^2\left([a,b)\right)\right)$ respectively. Both are characterised by the *generator* dr which is a second rank tensor over the complex vector space $\mathcal{I}=\mathbb{C}\left(dP,dQ,dT\right)$ of differentials of the fundamental stochastic processes P, Q and the time process T of the calculus. They have rigorous definitions as solutions of either forward or backward adapted quantum stochastic differential equations [Hud14] in which b or a in 5.1 is the time variable. They are related by the *coboundary relation*

$$\begin{aligned}\prod_{<_{[a,c)}} (1+dr) &= \left(\prod_{<_{[a,b)}} (1+dr)\otimes I\right)\prod_{[a,b)\times[b,c)} (1+dr)\\ &\quad \left(I\otimes\prod_{<_{[b,c)}} (1+dr)\right)\end{aligned}$$

in which the Fock space $\mathcal{F}\left(L^2\left([a,c)\right)\right)$ is canonically split at time $b\in[a,c)$;

$$\begin{aligned}\mathcal{F}\left(L^2\left([a,c)\right)\right) &= \mathcal{F}\left(L^2\left([a,b)\right)\oplus L^2\left([b,c)\right)\right)\\ &= \mathcal{F}\left(L^2\left([a,b)\right)\right)\otimes\mathcal{F}\left(L^2\left([b,c)\right)\right)\end{aligned}$$

allowing it to accommodate the operator $\prod_{[a,b)\times[b,c)} (1+dr)$.

A necessary and sufficient condition that they consist of unitary operators is [Hud14] that

$$dr + dr^\dagger + dr dr^\dagger = 0.$$

Here the space $\mathcal{I}=\mathbb{C}\left\langle dP,dQ,dT\right\rangle$ is equipped with the multiplication given by the quantum Itô product rule

	dP	dQ	dT
dP	dT	$-idT$	0
dQ	idT	dT	0
dT	0	0	0

and $\mathcal{I}\otimes\mathcal{I}$ with the corresponding tensor product multiplication, together with the natural involution † derived from the self-adjointness of P, Q and T.

Two examples of such unitary generators are

$$\begin{aligned} dr_1 &= i\left(dP \otimes dQ - dQ \otimes dP\right), \\ dr_2 &= i\left(dP \otimes dP + dQ \otimes dQ\right). \end{aligned}$$

dr_1 relates to quantum Lévy area in which the independent one-dimensional component Brownian motions of planar Brownian motion are replaced by P and Q. In the same spirit, dr_2 relates to a quantum version of the Bessel process, the radial part of planar Brownian motion. The general form of unitary generator in which the time differential dT does not appear is [HP15b] the real linear combination

$$dr_{\lambda,\mu} = \frac{\lambda}{2} dr_1 + \frac{\mu}{2} dr_2.$$

In this chapter we begin the explicit construction of the unitary causal double product integral

$$E := \prod_{<[a,b)} \left(1 + dr_{\lambda,\mu}\right)$$

as the second quantisation $\Gamma(W)$ of a unitary operator W which differs from the identity operator I by an integral operator on the Hilbert space $L^2([a,b))$ whose kernel will be found explicitly.

5.2 The Lévy stochastic area

Before moving on to construct E, let us take a detour and explain the motivation of this problem.

The stochastic Lévy area introduced in [Lév51] is defined as the signed area formed by connecting the endpoints of a 2-dimensional Brownian path. More specifically, it is defined as

$$L = \frac{1}{2} \int_{0 \le s_1 < s_2 < t} dB^1_{s_1} dB^2_{s_2} - dB^2_{s_1} dB^1_{s_2} ds_1 ds_2,$$

where B^1 and B^2 are two independent Brownian motions. The Lévy area formula shows the characteristic function of L:

$$\mathbb{E} \prod_{0 \le s_1 < s_2 < t} \left(1 + \frac{i\lambda}{2}(dB^1_{s_1} dB^2_{s_2} - dB^2_{s_1} dB^2_{s_2})\right) = \mathbb{E} e^{i\lambda L} = \operatorname{sech} \frac{\lambda t}{2}. \tag{5.2}$$

The Lévy area formula has many interesting connotations. For example there are connections to integrable systems, Bernoulli and Euler polynomials, and hence to the values of the Riemann zeta function [BPY01]. For some recent work and further references see [IT10, IT11]. Also, to within normalisation and rescaling it is equal to its Fourier transform, the density of the corresponding probability distribution, which is a boundary point of the Meixner family [Mei34].

Noncommutative analogues of Lévy area have been previously considered in free probability [CDM01, Ort13, Vic04]. Also in this connection Deya and Schott [DS13] emphasise the primacy of iterated stochastic integrals which accords with our philosophy. But in this chapter we are concerned with a noncommutative analogue of a more conservative kind which arises in quantum stochastic calculus [HP84, Par92], regarded as a noncommutative extension, rather than a radically noncommutative analogue, of Itō calculus. This allows a very natural variant of the area to be constructed using the minimal one-dimensional version of the calculus. It may be regarded as a response to the call [App10] for a study in this quantum context of some of the deeper properties of Brownian motion, as well as a furtherance of the theory of quantum stochastic product integrals [Hud07a, Hud07b, HP81].

By replacing B^1 and B^2 with P and Q, the iterated quantum stochastic integral

$$K(t) = \frac{1}{2}\int_{0\le x<y<t}(dP_x dQ_y - dQ_x dP_y)$$

has some interesting properties [Hud13, CH13]. For example it is evidently invariant under gauge transformations, which replace (P, Q) by (P^θ, Q^θ) where

$$P^\theta = P\cos\theta - Q\sin\theta, \qquad Q^\theta = P\sin\theta + Q\cos\theta;$$

equivalently the corresponding creation and annihilation processes are multiplied by $e^{\pm i\theta}$. In particular, taking $\theta = -\frac{\pi}{2}$ it is invariant under the replacement (P, Q) by $(Q, -P)$. Thus, unlike the separate processes P and Q, it can be canonically "rolled" onto a (one-dimensional) Riemannian manifold, and its multidimensional version [FV10] can similarly be rolled onto a multidimensional manifold, with possible applications to quantum stochastic proofs of index theorems, by identifying the canonical Brownian motion on the manifold generated by the Laplacian as P^θ with arbitrarily chosen θ.

It can also be verified [CH13] that all moments of $K(t)$ vanishes in the vacuum state, so that $K(t)$ vanishes in a probabilistic sense, even though it is not the zero operator.

But it is not K which is the main object of study. Because $\exp(a+b) \neq \exp a \exp b$ when a and b do not commute, the exponential

$$\exp(i\lambda K(t)) = \exp\left(\frac{i\lambda}{2}\int_{0\leq x<y<t}(dP_x dQ_y - dQ_x dP_y)\right)$$

does not reflect in a coherent way the continuous tensor product structure underlying the quantum stochastic calculus. Thus, motivated by the hope of finding quantum extensions of, in particular, the Lévy area formula (5.2), and associated relations with Euler and Bernoulli polynomials [IT11] we investigate the double product integral

$$\prod_{<[a,b)}(1+dr_1) = \prod_{a\leq x<y<b}\left(1+\frac{i\lambda}{2}(dP_x dQ_y - dQ_x dP_y)\right)$$

However, as it turns out, the more general object

$$\begin{aligned} E &= \prod_{<[a,b)}(1+dr_{\lambda,\mu}) \\ &= \prod_{a\leq x<y<b}\left(1+\frac{i\lambda}{2}(dP_x dQ_y - dQ_x dP_y)+\frac{i\mu}{2}(dP_x dP_y + dQ_x dQ_y)\right). \end{aligned}$$

is more fundamental and, surprisingly, simpler to study.

5.3 A discrete double product of unitary matrices

The first stage of the construction of E is similar to that of the rectangular case construction outlined in [HP15b], in that we approximate $\prod_{<[a,b)}(1+dr_{\lambda,\mu})$ by a discrete double product $\Pi_{1\leq j<k\leq N}\left(I+\delta_N^{j,k}r_{\lambda,\mu}\right)$, where $\delta_N^{j,k}r_{\lambda,\mu}$ is obtained from $dr_{\lambda,\mu}$ by replacing each basic differential $dX \in \{dP, dQ\}$ contributing to $dr_{\lambda,\mu} \in \mathcal{I}\otimes\mathcal{I}$ in the first copy of $\mathcal{I}$ by the j-th increment $X_{x_j} - X_{x_{j-1}}$ and in the second copy of $\mathcal{I}$ by the k-th increment $X_{x_k} - X_{x_{k-1}}$ over the equipartition

$$[a,b) = \sqcup_{j=1}^N [x_{j-1}, x_j),\ x_j = a + \frac{j}{N}(b-a) =: a + j\Delta_N.$$

Thus, for example,

$$\begin{aligned}\delta_N^{j,k} r_1 &= \frac{i}{2}\left(\left(P_{x_j} - P_{x_{j-1}}\right) \otimes \left(Q_{x_k} - Q_{x_{k-1}}\right)\right. \\ &\quad \left. - \left(Q_{x_j} - Q_{x_{j-1}}\right) \otimes \left(P_{x_k} - P_{x_{k-1}}\right)\right).\end{aligned}$$

Introducing the standard canonical pairs (p_j, q_j) , $j = 1, 2, ..., N$, given by

$$p_j = \sqrt{\frac{b-a}{N}} \left(P(x_j) - P(x_{j-1})\right),\ q_j = \sqrt{\frac{b-a}{N}} \left(Q(x_j) - Q(x_{j-1})\right),$$

which satisfy the canonical commutation relations

$$[p_j, q_k] = -2i\delta_{j,k}, [p_j, p_k] = [q_j, q_k] = 0, \tag{5.3}$$

we write

$$\delta_N^{j,k} r_1 = i\frac{b-a}{2N}\left(p_j q_k - q_j p_k\right)$$

and more generally

$$\delta_N^{j,k} r_{\lambda,\mu} = i\frac{b-a}{2N}\left(\lambda\left(p_j q_k - q_j p_k\right) + \mu\left(p_j p_k + q_j q_k\right)\right).$$

Our approximation is thus

$$\begin{aligned}\prod_{<_{[a,b)}} (1 + dr_{\lambda,\mu}) &\simeq \Pi_{1\le j<k\le N}\left(I + i\frac{b-a}{2N}\left(\lambda\left(p_j q_k - q_j p_k\right) + \mu\left(p_j p_k + q_j q_k\right)\right)\right) \\ &\simeq \Pi_{1\le j<k\le N} \exp\left(i\frac{b-a}{2N}\left(\lambda\left(p_j q_k - q_j p_k\right) + \mu\left(p_j p_k + q_j q_k\right)\right)\right)\end{aligned} \tag{5.4}$$

for large N.

Temporarily let us fix $j < k$ and write $(p, q) = (p_j, q_j)$, $(p', q') = (p_k, q_k)$ so that

$$[p, q] = -2i,\ \left[p', q'\right] = -2i,\ \left[p, q'\right] = \left[q, p'\right] = \left[p, p'\right] = \left[q, q'\right] = 0. \tag{5.5}$$

We recall [Par92] that, for an arbitrary Hilbert space $\mathcal{H}$ and vector $f \in \mathcal{H}$ the corresponding *Weyl operator* $W(f)$ is the unique unitary operator on $\mathcal{F}(\mathcal{H})$ which acts on each exponential vector $e(g)$, $g \in \mathcal{H}$ as

$$W(f)\, e(g) = e^{-\frac{1}{2}\|f\|^2 - \langle f,g\rangle} e(f+g).$$

The Weyl operators satisfy the Weyl relation

$$W(f)\,W(g) = e^{-2i\mathrm{Im}\langle f,g\rangle}W(f+g)\,. \tag{5.6}$$

A convenient rigorous realisation of two canonical pairs satisfying the commutation relations (5.5) can be constructed in terms of the one-parameter unitary groups of which they are the self-adjoint infinitesimal generators, which are Weyl operators on the Fock space $\mathcal{F}\left(\mathbb{C}^2\right)$ over $\mathbb{C}^2$. Regarding $\mathbb{C}^2$ as a space of column vectors, we take

$$\begin{aligned} e^{ixp} &= W\left((-x,0)^T\right), e^{ixq} = W\left((ix,0)^T\right), \\ e^{ixp'} &= W\left((0,-x)^T\right), e^{ixq'} = W\left((0,ix)^T\right). \end{aligned}$$

for arbitrary $x \in \mathbb{R}$, noting that these four families of Weyl operators are indeed one-parameter unitary groups, and that the commutation relations (5.5) follow by parametric differentiation, for example from the relations

$$\begin{aligned} W\left((-x,0)^T\right)W\left((iy,0)^T\right) &= e^{2ixy}W\left((iy,0)^T\right)W\left((-x,0)^T\right), \\ W\left((0,-x)^T\right)W\left((0,iy)^T\right) &= e^{2ixy}W\left((0,iy)^T\right)W\left((0,-x)^T\right), \end{aligned}$$

all of which are consequences of (5.6).

Theorem 19 below, which is proved in [HP15b], gives a corresponding rigorous explicit form of the self-adjoint operator

$$L(\lambda,\mu) = \lambda\left(pq' - qp'\right) + \mu\left(pp' + qq'\right)$$

in this realisation. Before stating it we recall [Par92] that the *second quantisation* of a unitary operator U on a Hilbert space $\mathcal{H}$ is the unique unitary operator $\Gamma(U)$ on $\mathcal{F}(\mathcal{H})$ which acts on the exponential vectors as

$$\Gamma(U)\,e(f) = e(Uf)\,.$$

It is related to the Weyl operators by

$$\Gamma(U)\,W(f) = W(Uf)\,\Gamma(U) \tag{5.7}$$

for arbitrary $f \in \mathcal{H}$. Second quantisation is multiplicative, in the sense that

$$\Gamma(U_1U_2) = \Gamma(U_1)\,\Gamma(U_2) \tag{5.8}$$

for arbitrary unitary U_1, U_2.

Theorem 19. *$L(\lambda,\mu)$ generates the one-parameter unitary group*

$$e^{ixL(\lambda,\mu)} = \Gamma\left(\begin{bmatrix} \cos(2x|\nu|) & -e^{-i\phi}\sin(2x|\nu|) \\ e^{i\phi}\sin(2x|\nu|) & \cos(2x|\nu|) \end{bmatrix}\right), \ x \in \mathbb{R}$$

where $\nu = \lambda + i\mu = e^{i\phi}|\nu|$ and the matrix operates on column vectors in $\mathbb{C}^2$ by multiplication on the left.

Sketch proof. We follow the proof as in [HP15b]. Denote

$$\xi_{\lambda,\mu}(x) := \begin{bmatrix} \cos 2|\nu|x & -e^{-i\phi}\sin 2|\nu|x \\ e^{i\phi}\sin 2|\nu|x & \cos 2x|\nu| \end{bmatrix}$$

and let K be the generator of $(\Gamma(\xi_{\lambda,\mu}(x)))_x$, that is

$$e^{ixK} = \Gamma(\xi_{\lambda,\mu}(x)).$$

It suffices to show that

$$\begin{aligned} [K,p] &= [L(\lambda,\mu),p], \qquad & [K,q] &= [L(\lambda,\mu),q], \\ [K,p'] &= [L(\lambda,\mu),p'], \qquad & [K,q'] &= [L(\lambda,\mu),q']. \end{aligned}$$

We demonstrate how to obtain the first identity, as the other three can be achieved by following the same procedure. By (5.7),

$$\begin{aligned} &\Gamma(\xi_{\lambda,\mu}(x))e^{iyp}\Gamma(\xi_{\lambda,\mu}(x))^{-1} \\ &\quad = \Gamma(\xi_{\lambda,\mu}(x))W((-y,0)^T)\Gamma(\xi_{\lambda,\mu}(x))^{-1} \\ &\quad = W(\xi_{\lambda,\mu}(x)(y,0)^T) \\ &\quad = W((-y\cos 2|\nu|x, -y\cos\phi\sin 2|\nu|x - iy\sin\phi\sin 2|\nu|x)^T) \\ &\quad = e^{iy(p\cos 2|\nu|x + p'\cos\phi\sin 2|\nu|x - q'\sin\phi\sin 2|\nu|x)}. \end{aligned}$$

Forming $-i\frac{d}{dy}\big|_{y=0}$ we get

$$\Gamma(\xi_{\lambda,\mu}(x))p\Gamma(\xi_{\lambda,\mu}(x))^{-1} = p\cos 2|\nu|x + p'\cos\phi\sin 2|\nu|x - q'\sin\phi\sin 2|\nu|x.$$

Forming $-i\frac{d}{dx}\big|_{x=0}$ we get

$$[K,p] = -2i(\lambda yp' - \mu yq') = [L(\lambda,\mu),p].$$

$\square$

We now use Theorem 19 to construct an explicit second quantisation of the approximation (5.4).

Let us first construct a different realisation of the canonical pairs (p_j, q_j), $j = 1, 2, ..., n$, satisfying (5.3) in the Fock space $\mathcal{F}(\mathbb{C}^n)$ over $\mathbb{C}^n$, by defining

$$e^{ixp_j} = W(-x\varepsilon_j), e^{ixq_j} = W(ix\varepsilon_j)$$

where $(\varepsilon_j)_{j=1}^n$ is the standard orthonormal basis of $\mathbb{C}^n$, $\varepsilon_j = \left(0, ..., \overset{(j)}{1}, 0, ..., 0\right)^\tau$. Correspondingly, in view of Theorem 19, each operator

$$\exp\left(i\frac{b-a}{2N}\left(\lambda\left(p_j q_k - q_j p_k\right) + \mu\left(p_j p_k + q_j q_k\right)\right)\right)$$

is realised as the second quantisation $\Gamma(R^N_{j,k})$ where

$$R^N_{j,k} := \begin{array}{cc} & \begin{array}{ccccccc} & & (j) & & (k) & & \end{array} \\ \begin{array}{c} \\ \\ (j) \\ \\ (k) \\ \\ \\ \end{array} & \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cos\left(\frac{(b-a)}{N}|\nu|\right) & \cdots & -\frac{\bar{\nu}}{|\nu|}\sin\left(\frac{(b-a)}{N}|\nu|\right) & \cdots & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{\bar{\nu}}{|\nu|}\sin\left(\frac{(b-a)}{N}|\nu|\right) & \cdots & \cos\left(\frac{(b-a)}{N}|\nu|\right) & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \end{array}.$$

In view of the multiplicativity property (5.8) the discrete double product (5.4) is correspondingly realised as the second quantisation of the product

$$\prod_{1 \le j < k \le N} R^N_{j,k}. \tag{5.9}$$

We now embed the matrix (5.9) as a unitary operator $\mathcal{W}_N$ on $L^2([a, b[)$ by mapping the standard basis of $\mathbb{C}^N$ to the orthonormal family $(\chi_1, \chi_2, ..., \chi_N)$ of normalized indicator functions

$$\chi_j(x) = \sqrt{\frac{N}{b-a}}\mathbb{I}_{[x_{j-1}, x_j)}.$$

By definition $\mathcal{W}_N$ acts as the identity operator I on $(\chi_1, \chi_2, ..., \chi_N)^\perp$.

Our objective in the remainder of this chapter is to find an explicit form for the (weak) limit

$$W = \lim_{N \to \infty} \mathcal{W}_N$$

and to prove that W is unitarity. The corresponding problems for rectangular unitary product integrals was solved in outline in [HP15b]. The causal case considered here is considerably more difficult, because the method of iterated limits which reduces the rectangular case to a double application of the time-orthogonal unitary dilation of [HIP82], is not applicable. Instead a combinatorial argument based on a lattice path model is used. For a similar alternative approach, avoiding the iterated limit technique, to the rectangular product in the particular case of the generator dr_1 corresponding to the quantum Lévy area, see [HJ12]; however the combinatorics for the rectangular case is much simpler than here and it has no direct relation to Lévy area.

5.4 A lattice path model and linear extensions of partial orderings

So we want to calculate the limit of the triangular double product of $N \times N$ matrices

$$\mathcal{W}_N = \prod_{1 \le j < k \le N} R^N_{j,k}. \tag{5.10}$$

Here, for elements $x_{j,k}$ of an associative algebra having the property that $x_{j,k}$ commutes with $x_{j'k'}$ whenever both $j \neq j'$ and $k \neq k'$ we define the ordered double product $\prod_{1 \le j < k \le N} x_{j,k}$ by any of the equivalent prescriptions

$$\prod_{1 \le j < k \le N} x_{j,k} = \prod_{j=1}^{N-1} \left[\prod_{k=j+1}^{N} x_{j,k} \right] = \prod_{k=2}^{N} \left[\prod_{j=1}^{k-1} x_{j,k} \right] = \prod_{r=1}^{\frac{1}{2}N(N-1)} x_{j_r,k_r}$$

where $\left((j_1, k_1), (j_2, k_2), ..., \left(j_{\frac{1}{2}N(N-1)}, k_{\frac{1}{2}N(N-1)} \right) \right)$ is any ordering of the $\frac{1}{2}N(N-1)$ pairs $(j,k), 1 \le j < k \le N$ which is *allowed,* meaning that it has the property that

$$(j_r, k_r) \text{ precedes } (j_s, k_s) \text{ if both } j_r \le j_s \text{ and } k_r \le k_s. \tag{5.11}$$

In constructing the limit as $N \to \infty$ we use the small angle approximations for sine and cosine, so that

$$\begin{pmatrix} \cos \frac{b-a}{N}|\nu| & -\frac{\bar{\nu}}{|\nu|} \sin \frac{b-a}{N}|\nu| \\ \frac{\nu}{|\nu|} \sin \frac{b-a}{N}|\nu| & \cos \frac{b-a}{N}|\nu| \end{pmatrix} = I + \frac{b-a}{N} \begin{pmatrix} 0 & -\bar{\nu} \\ \nu & 0 \end{pmatrix} + O(N^{-2})$$

hence

$$R^N_{j,k} = I + \frac{(b-a)}{N}(-\bar{\nu}\,|\chi_j\rangle\,\langle\chi_k| + \nu\,|\chi_k\rangle\,\langle\chi_j|) + O(N^{-2}).$$

When there is no ambiguity, for any integers j and k, we use abbreviations $|j\rangle := |\chi_j\rangle$ and $\langle k| := \langle\chi_k|$. Then the product (5.10) becomes

$$\mathcal{W}_N \simeq \prod_{1\le j<k\le N} \left(I + \frac{(b-a)}{N} Z^N(j,k) \right) =: W_N$$

where

$$Z^N(j,k) = -\bar{\nu}\,|j\rangle\,\langle k| + \nu\,|k\rangle\,\langle j|\,.$$

To compute this, we introduce and work on a lattice path model. Consider a lattice $L_s := \{(m,n) : 1 \le m \le s, 0 \le n \le 1\}$. We call $(m,1)_{1\le m\le s}$ the upper vertices, and $(m,0)_{1\le m\le s}$ the lower vertices. Denote by Π_s the set of lattice path $\pi = (m_i, b_i)_{i=1}^s$ satisfying the following two conditions:

1. $m_i = i$ for $i = 1, \ldots, s$
2. there does not exist an i such that $b_i = b_{i+1} = 0$

For convenience, we write $\pi(i) = b_i$ and let $\pi = (\pi(i))_i$. We call any $\pi \in \Pi_s$ a path of length $s-1$. It is straightforward to verify by induction that

$$|\Pi_s| = \mathrm{Fib}_{s+2} = \frac{\Phi^{s+2} - (-\Phi)^{-s-2}}{\sqrt{5}},$$

where Fib_n is the nth Fibonacci number and Φ is the golden ratio $\frac{\sqrt{5}+1}{2}$.

If we assign weight $\theta(v)$ to each vertex v in L_s, then we can define the weight $\theta(\pi)$ of a path $\pi \in \Pi_s$ by the product of the weights of its vertices:

$$\theta(\pi) := \prod_{i=1}^{s} \theta(i, \pi(i)).$$

For any s-array of pairs $\{p_{ij} : 1 \le i \le s, 1 \le j \le 2\}$, define its associated weight $\theta_p(v)$ for any $v = (m, b) \in A$ to be

$$\theta_p(v) = \begin{cases} \nu \left|p_{i2}\right\rangle \left\langle p_{i1}\right|, & \text{if } b = 0; \\ -\bar{\nu} \left|p_{i1}\right\rangle \left\langle p_{i2}\right|, & \text{if } b = 1. \end{cases}$$

Finally, define the weight $\theta_p(\pi)$ of a path in the same way as before.

For example, if we label the vertices by their weights associated to p, then the following is a path of Π_5:

$$\begin{array}{ccccccccc}
-\bar{\nu}\left|p_{11}\right\rangle\left\langle p_{12}\right| & \text{---} & -\bar{\nu}\left|p_{21}\right\rangle\left\langle p_{22}\right| & & -\bar{\nu}\left|p_{31}\right\rangle\left\langle p_{32}\right| & & -\bar{\nu}\left|p_{4,1}\right\rangle\left\langle p_{4,2}\right| & \text{---} & -\bar{\nu}\left|p_{5,1}\right\rangle\left\langle p_{5,2}\right| \\
 & & & \diagdown & & \diagup & & & \\
+\nu\left|p_{12}\right\rangle\left\langle p_{11}\right| & & +\nu\left|p_{22}\right\rangle\left\langle p_{21}\right| & & +\nu\left|p_{32}\right\rangle\left\langle p_{31}\right| & & +\nu\left|p_{4,2}\right\rangle\left\langle p_{4,1}\right| & & +\nu\left|p_{5,2}\right\rangle\left\langle p_{5,1}\right|
\end{array}$$

but not the following because the third edge connects two bottom vertices:

$$\begin{array}{ccccccccc}
-\bar{\nu}\left|p_{11}\right\rangle\left\langle p_{12}\right| & \text{---} & -\bar{\nu}\left|p_{21}\right\rangle\left\langle p_{22}\right| & & -\bar{\nu}\left|p_{31}\right\rangle\left\langle p_{32}\right| & & -\bar{\nu}\left|p_{4,1}\right\rangle\left\langle p_{4,2}\right| & & -\bar{\nu}\left|p_{5,1}\right\rangle\left\langle p_{5,2}\right| \\
 & & & \diagdown & & & & \diagup & \\
+\nu\left|p_{12}\right\rangle\left\langle p_{11}\right| & & +\nu\left|p_{22}\right\rangle\left\langle p_{21}\right| & & +\nu\left|p_{32}\right\rangle\left\langle p_{31}\right| & \text{---} & +\nu\left|p_{4,2}\right\rangle\left\langle p_{4,1}\right| & & +\nu\left|p_{5,2}\right\rangle\left\langle p_{5,1}\right|
\end{array}$$

Any s-array of pairs $p = (p_{ij})_{1 \le i \le s, 1 \le j \le 2}$ satisfying the following condition

$$\mathbb{I}_{p_{i,1}=p_{i+1,1}} + \mathbb{I}_{p_{i,2}=p_{i+1,2}} + \mathbb{I}_{p_{i,2}=p_{i+1,1}} = 1, \qquad p_{i,1} \neq p_{i+1,2} \tag{5.12}$$

can be associated with a path $\pi_p \in \Pi_s$ in the following way:

$$(\pi(i), \pi(i+1)) = \begin{cases} (0,1), & \text{if } p_{i,1} = p_{i+1,1} \\ (1,1), & \text{if } p_{i,2} = p_{i+1,1} \\ (1,0), & \text{if } p_{i,2} = p_{i+1,2} \end{cases}.$$

Note that this is equivalent to

$$\theta_p(\pi_p) = \prod_{i=1}^{s} Z^N(p_{i,1}, p_{i,2}).$$

Lemma 20.

$$W_N = I + \sum_{s=1}^{N(N-1)/2} \tilde{w}_{s,N},$$

where

$$\tilde{w}_{s,N} = \left(\frac{b-a}{N}\right)^s \sum_{(*)} \theta_p(\pi_p)$$

where the domain $(*)$ *of the summation is*

$$\begin{aligned}(1 \leq p_{i,j} \leq N, \forall 1 \leq i \leq s, 1 \leq j \leq 2)\ &AND \\ &(p_{i,1} < p_{i+1,1} < p_{i,2} = p_{i+1,2}\ OR \\ &p_{i,1} < p_{i,2} = p_{i+1,1} < p_{i+1,2}\ OR \\ &p_{i,1} = p_{i+1,1} < p_{i,2} < p_{i+1,2}, \forall 1 \leq i \leq s-1)\end{aligned}$$

Proof. For any rearrangement $(j_i, k_i)_{1 \leq i \leq \frac{N(N-1)}{2}}$ of $\{1 \leq j < k \leq N\}$ satisfying (5.11),

$$\prod_{1 \leq j < k \leq N} (I + Z(j,k)) = \prod_{i=1}^{N(N-1)/2} (I + Z(j_i, k_i)) = I + \sum_{s=1}^{N(N-1)/2} \sum_{(**)} \prod_{r=1}^{s} Z(p_{r1}, p_{r2}),$$

where domain $(**)$ is

$$\begin{aligned}&(p_{11}, p_{12}), (p_{21}, p_{22}), \ldots, (p_{s1}, p_{s2}) \\ &\qquad \text{is a subsequence of } (j_1, k_1), (j_2, k_2), \ldots, (j_{N(N-1)/2}, k_{N(N-1)/2}).\end{aligned}$$

Now for the product $\prod_{r=1}^{s} Z(p_{r1}, p_{r2})$ to be nonzero, the product of $Z(p_{i,1}, p_{i,2}) \times Z(p_{i+1,1}, p_{i+1,2})$ for each i has to be nonzero, that is

$$(-|p_{i,1}\rangle \langle p_{i,2}| + |p_{i,2}\rangle \langle p_{i,1}|)(-|p_{i+1,1}\rangle \langle p_{i+1,2}| + |p_{i+1,2}\rangle \langle p_{i+1,1}|) \neq 0.$$

This in turn is equivalent to

$$(p_{i,2} = p_{i+1,2}) \vee (p_{i,2} = p_{i+1,1}) \vee (p_{i,1} = p_{i+1,1}) \vee (p_{i,1} = p_{i+1,2}).$$

We analyse these four possibilities one by one.

1. If $p_{i,2} = p_{i+1,2}$, then by (5.11), and since $(p_{i,1}, p_{i,2}) \neq (p_{i+1,1}, p_{i+1,2})$, only when $p_{i,1} < p_{i+2,1}$ can the product be nonzero. In this case the coordinates are ordered as $p_{i,1} < p_{i+1,1} < p_{i,2} = p_{i+1,2}$.

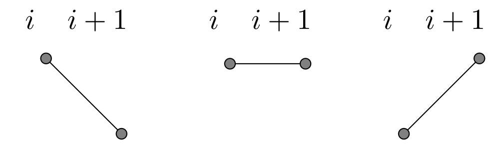
2. If $p_{i,2} = p_{i+1,1}$, then since $p_{i,1} < p_{i,2} = p_{i+1,1}$ and $p_{i,2} = p_{i+1,1} < p_{i+1,2}$, we have that (5.11) is satisfied. Therefore this case is also included / permitted

in the product. The ordering of the coordinates is $p_{i,1} < p_{i,2} = p_{i+1,1} < p_{i+1,2}$.

3. If $p_{i,1} = p_{i+1,1}$, then similar to Case 1, the coordinates have to satisfy $p_{i,1} = p_{i+1,1} < p_{i,2} < p_{i+1,2}$ for the product to be nonzero.

4. If $p_{i,1} = p_{i+1,2}$, then $p_{i,1} = p_{i+1,2} > p_{i+1,1}$ and $p_{i,2} > p_{i,1} = p_{i+1,2}$ violates (5.11), hence this case never happens.

The three feasible cases are illustrated as below.



$$
\begin{array}{ccc}
p_{i1}<p_{i+1,1} & p_{i1}<p_{i+1,1} & p_{i1}=p_{i+1,1} \\
\wedge \qquad \wedge & \wedge \;/\!/\; \wedge & \wedge \qquad \wedge \\
p_{i2}=p_{i+1,2} & p_{i2}<p_{i+1,2} & p_{i2}<p_{i+1,2}
\end{array}
\tag{5.13}
$$

The concatenation of these edges gives a path in Π_s. Case 4 corresponds to a horizontal bottom edge in the path which is not allowed in the definition of Π_s. Therefore we have established a correspondence between the possibilities of orderings in the product and Π_s. $\square$

Denote by A_s^* the set of s-array pairs p satisfying condition $(*)$ in Lemma 20, and $\Omega_\pi := \{p \in A_s^* : \pi_p = \pi\}$. Then

$$\tilde{w}_{s,N} = \left(\frac{b-a}{N}\right)^s \sum_{\pi \in \Pi_s} \sum_{p \in \Omega_\pi} \theta_p(\pi).$$

Given a path $\pi \in \Pi_s$, by the correspondence in (5.13) there exist m_1, m_2, $\ldots$, $m_s \in \{1, 2\}$ such that for any $p \in \Omega_\pi$ and $x \leq s-1$, $p_{x,m_x} = p_{x+1,m'_{x+1}}$, where $m'_x := 3 - m_x$. Therefore, Ω_π is characterised by a partial ordering on the $s+1$ coordinates $p_{1,m'_1}, p_{1,m_1}, p_{2,m_2}, \ldots, p_{s,m_s}$. We call them the *essential coordinates* of p. This also shows we can associate π with $(m_1, m_2, \ldots, m_s)$. In the following we do not differentiate between π and the corresponding partial ordering.

Any ordering $\pi \in \Pi_s$ can be decomposed into (strict) total orderings without any repetition of the essential coordinates and those with repeated essential coordinates. We call any the former orderings B a *linear extension* of π which is denoted by $B \vdash \pi$, and the latter degenerate orderings, which, for reasons that will emerge in the proof of Lemma 21 are ignored. Thus we have

$$\Omega_\pi = \bigsqcup_{B \vdash \pi} B \cup \text{ set of degenerate orderings.}$$

For any $p \in \Omega_\pi$, there exists a $B \vdash \pi$ such that $p \in B$. Denote $(j_p, k_p) = (p_{1,m_1'}, p_{s,m_s})$. In the total ordering imposed by B, let r_B be the number of essential coordinates less than j_p and r_B' the number of those greater than k_p. That is, the essential coordinates are ordered as follows,

$$\begin{aligned} &1 < l_1 < \cdots < l_{r_B} < j_p < m_1 < \cdots < m_{s-1-r_B-r_B'} < k_p < n_1 < \cdots < n_{r_B'} \leq N, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } r_B + r_B' < s \\ &1 < l_1 < \cdots < l_{s-r_B'} < k_p < m_1 < \cdots < m_{r_B+r_B'-s-1} < j_p < n_1 < \cdots < n_{s-r_B} \leq N, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } r_B + r_B' > s \end{aligned}$$

We call (r_B, r_B') the *rank* of B.

Let $\epsilon(\pi)$ be the number of upper vertices of the path π. Since horizontal edges between lower vertices are not allowed, there is at least one upper vertex between two consecutive lower vertices, hence

$$2\epsilon(\pi) \geq s - 1.$$

The location (upper or lower) of the first vertex of π, the number of upper vertices $\epsilon(\pi)$ and the parity of the length of π together determine the number of horizontal edges in π. The cases when $\epsilon(\pi) \approx \frac{s-1}{2}$ are "saturated", meaning there is no horizontal edge in π. This will be later specified and exploited in the proof of Lemma 30.

The weight of π is

$$\theta_p(\pi) = (-\bar{\nu})^{\epsilon(\pi)} \nu^{s-\epsilon(\pi)} \left|j_p\right\rangle \left\langle k_p\right|.$$

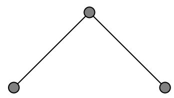
So

$$\begin{aligned} \tilde{w}_{s,N} &= \left(\lambda \frac{b-a}{N}\right)^s \sum_{\pi \in \Pi_s} (-\bar{\nu})^{\epsilon(\pi)} \nu^{s-\epsilon(\pi)} \sum_{p \in \Omega_\pi} \left|j_p\right\rangle \left\langle k_p\right| \\ &\simeq \left(\lambda \frac{b-a}{N}\right)^s \sum_{\pi \in \Pi_s} (-\bar{\nu})^{\epsilon(\pi)} \nu^{s-\epsilon(\pi)} \sum_{B \vdash \Omega_\pi} \sum_{p \in B} \left|j_p\right\rangle \left\langle k_p\right| \\ &= \lambda^s \sum_{\pi \in \Pi_s} (-\bar{\nu})^{\epsilon(\pi)} \nu^{s-\epsilon(\pi)} \sum_{B \vdash \pi} H_s^N(r_B, r_B') + v_{s,N} =: w_{s,N} + v_{s,N}, \end{aligned}$$

where $v_{s,N}$ is the contribution from the degenerate orderings, on which one can carry out the same calculation for $w_{s,N}$ below, and that

$$H_s^N(r,r') = \begin{cases} \left(\frac{b-a}{N}\right)^s \sum_{1<l_1<\cdots<l_r<j<m_1<\cdots<m_{s-1-r-r'}<k<n_1<\cdots<n_{r'}\leq N} |j\rangle\langle k|, & r+r'<s \\ \left(\frac{b-a}{N}\right)^s \sum_{1<l_1<\cdots<l_{s-r'}<k<m_1<\cdots<m_{r+r'-s-1}<j<n_1<\cdots<n_{s-r}\leq N} |j\rangle\langle k|, & r+r'>s \end{cases}.$$

For example, for the following path π of length 2,



$$\begin{array}{ccccc} p_{1,1} & = & p_{2,1} & < & p_{3,1} \\ \wedge & & \wedge & & \wedge \\ p_{1,2} & < & p_{2,2} & = & p_{3,2} \end{array}$$

The ordering of the essential coordinates imposed by π is:

$$(p_{1,1} < p_{1,2} < p_{2,2}) \wedge (p_{2,1} < p_{3,1} < p_{2,2}),$$

and the non-repeated starting and ending coordinates are $j_p = p_{12}$ and $k_p = p_{31}$. The total ordering decomposition of Ω_π is

$$\Omega_\pi = \{p_{11} < p_{12} < p_{31} < p_{22}\} \sqcup \{p_{11} < p_{31} < p_{12} < p_{22}\} \sqcup \{p_{11} < p_{12} = p_{31} < p_{22}\}.$$

The last term is a degenerate case as p_{12} is repeated. There is only one upper vertex, hence this path contributes $-\nu|\nu|^2(H_3^N(2,3) + H_3^N(3,2))$ to $w_{s,N}$.

Define the Volterra-type kernels $>_a^b (x,y) := 1_{a\leq y<x<b}$ and $<_a^b (x,y) :=>_a^b (y,x)$, and $[m,n,p](x,y) := \frac{(x-a)^m}{m!}\frac{(y-x)^n}{n!}\frac{(b-y)^p}{p!}$ and $[m,n,p]^\dagger(x,y) := [m,n,p](y,x)$. The asymptotics of H_s^n can be written down explicitly.

Lemma 21. *$H_s^N(r,r')$ converges weakly to an integral operator $H_s(r,r')$, with the integral kernel*

$$h_s(r,r') = \begin{cases} [r, s-1-r-r', r'] <_a^b, & r+r'<s \\ [s-r', r+r'-s-, s-r]^\dagger >_a^b, & r+r'>s \end{cases}$$

.

Proof. Suppose $r + r' < s$ (the case $r + r' > s$ can be done in the same way). Then

$$\begin{aligned} H_s^N(r, r') &= \left(\frac{b-a}{N}\right)^s \sum_{1 \le j < k \le N} |j\rangle \langle k| \sum_{1 \le l_1 < \cdots < l_r < j < m_1 < \cdots < m_{s-1-r'-r} < k < n_1 < \cdots < n_{r'} \le N} 1 \\ &= \left(\frac{b-a}{N}\right)^s \sum_{1 \le j < k \le N} |j\rangle \langle k| \binom{j-1}{r} \binom{k-j-1}{s-1-r'-r} \binom{N-k}{r'} \end{aligned}$$

We denote $\Delta_N := \frac{b-a}{N}$, then the kernel of $H_s^N(r, r')$ is

$$\begin{aligned} h_s^N(x,y) = &\sum_{1 \le j < k \le N} \mathbb{I}_{A_j}(x) \mathbb{I}_{A_k}(y) \frac{1}{r!(s-1-r-r')!r'!} \\ &\times \prod_{\alpha=0}^{r-1} (x_{j-1} - a - \alpha \Delta_N) \prod_{\beta=0}^{s-2-r-r'} (x_{k-1} - x_j - \beta \Delta_N) \prod_{\gamma=0}^{r'-1} (b - x_k - \gamma \Delta_N). \end{aligned}$$

This, as $N \to \infty$, converges weakly (as an integral kernel) to $[r, s-1-r-r', r'] <_a^b (x, y)$. $\square$

It can also be seen from the proof of this lemma that the degenerate orderings contribute 0 to the total sum. More specifically, the degenerate version of $h_s^N(x, y)$ where there are d repeated essential coordinates is

$$\begin{aligned} q_s^N(x, y) = &\sum_{1 \le j < k \le N} \mathbb{I}_{A_j}(x) \mathbb{I}_{A_k}(y) \frac{1}{r!(s-1-r-r')!r'!} \frac{(b-a)^d}{N^d} \\ &\times \prod_{\alpha=0}^{r-1} (x_{j-1} - a - \alpha \Delta_N) \prod_{\beta=0}^{s-d-2-r-r'} (x_{k-1} - x_j - \beta \Delta_N) \prod_{\gamma=0}^{r'-1} (b - x_k - \gamma \Delta_N) \to 0 \end{aligned}$$

as $N \to \infty$. We will examine carefully the rate of convergence of this lemma and the (in)significance of the degenerate orderings later in the proof of Theorem 25. This lemma immediately gives the following corollary.

Corollary 22. *There exist two integer arrays $(D_{m,n,p;q})_{m,n,p \ge 0, 0 \le q \le m+n+p+1}$ and $(E_{m,n,p;q})_{m,n,p \ge 0, 0 \le q \le m+n+p+1}$ such that $w_{s,N}$ converges weakly as $N \to \infty$ to w_s with kernel*

$$f_s <_a^b + g_s >_a^b$$

where f_s and g_s are defined by

$$f_s = \sum_{m,n,p\geq 0, m+n+p=s-1} \sum_{q=0}^{s} D_{m,n,p;q}(-\bar{\nu})^q \nu^{s-q}[m,n,p],$$
$$g_s = \sum_{m,n,p\geq 0, m+n+p=s-1} \sum_{q=0}^{s} E_{m,n,p;q}(-\bar{\nu})^q \nu^{s-q}[m,n,p]^\dagger.$$

Indeed, $D_{m,n,p;q}$ (resp. $E_{m,n,p;q}$) enumerates the linear extensions of all possible paths of length $m+n+p$ with q upper vertices and rank (m,p) (resp. $(m+n+1, n+p+1)$).

Corollary 23. *The functions f_s and g_s both are symmetric in the following sense:*

$$f_s(x,y) = f_s(a+b-y, a+b-x), \qquad g_s(x,y) = g_s(a+b-y, a+b-x)$$

Proof. This follows from the fact that the path inversion $(i, b_i) \mapsto (i, b_{s+1-i})$ is a weight-preserving bijection between Π_s and itself. □

For example, some calculation yields

$$\begin{aligned}
f_1 &= -\bar{\nu}[0,0,0],\\
f_2 &= -|\nu|^2[0,0,1] - |\nu|^2[1,0,0] + \bar{\nu}^2[0,1,0],\\
f_3 &= (\bar{\nu}|\nu|^2 - \bar{\nu}^3)[0,2,0] + \bar{\nu}|\nu|^2[0,1,1] + \bar{\nu}|\nu|^2[1,1,0] - \nu|\nu|^2[1,0,1],\\
g_1 &= -\bar{\nu}[0,0,0]^\dagger,\\
g_2 &= 0,\\
g_3 &= -\nu|\nu|^2[1,0,1]^\dagger.
\end{aligned} \tag{5.14}$$

The following three theorems are the main results of this chapter:

Theorem 24. *The closed form expression of D and E are:*

$$D_{m,n,p;q} = \begin{cases} \binom{n}{q-1} - \binom{n}{q}, & 2q > m+n+p \\ \binom{n}{q-m} - \binom{n}{q}, & 2q = m+n+p \\ 0, & 2q < m+n+p \end{cases} \tag{5.15}$$

$$E_{m,n,p;q} = \mathbb{I}_{m=p=q,n=0}$$

Proof. See Section 5.5. $\square$

Theorem 25. *The operator W_N converges weakly to*

$$W = I + \sum_{s \geq 1} w_s$$

Proof. See Section 5.6. $\square$

For $j \geq 0$, let B_j be power series in two variables related to the Bessel functions of the first kind J_j.

$$B_j(x, y) := \sum_{n \geq 0} \frac{(-1)^{n+j} x^{n+j} y^n}{(n+j)!n!} = (-1)^j (x/y)^{j/2} J_j(2\sqrt{xy}).$$

Let I be the identity, then the kernel of the operator $W - I$ can be written in terms of B_j.

Theorem 26. *The integral operator $W - I$ has kernel*

$$\begin{aligned} \mathit{ker}(W - I)(x, y) = \Bigg(& \nu B_0((y-a)|\nu|, (b-x)|\nu|) + |\nu| B_1((b-a)|\nu|, (y-x)|\nu|) \\ & - (\nu + \bar{\nu}) \sum_{q \geq 0} B_q((y-x)|\nu|, (b-a)|\nu|) \left(\frac{\bar{\nu}}{|\nu|} \right)^q \Bigg) <_a^b (x, y) \\ & + \nu B_0((y-a)|\nu|, (b-x)|\nu|) >_a^b (x, y). \end{aligned}$$

Moreover, W is unitary.

Proof. See section 5.7. $\square$

For example, when $\mu = 0$ and $\lambda > 0$, the kernel of the operator corresponding to the Lévy stochastic area is

$$\begin{aligned} \ker(W - I)(x, y) = \Bigg(& \lambda B_0((y-a)\lambda, (b-x)\lambda) + \lambda B_1((b-a)\lambda, (y-x)\lambda) \\ - 2\lambda \sum_{q \geq 0} B_q((y-x)\lambda, (b-a)\lambda) \Bigg) & <_a^b (x, y) + \lambda B_0((y-a)\lambda, (b-x)\lambda) >_a^b (x, y). \end{aligned}$$

Moreover by plugging $D_{m,n,p;q}$ and $E_{m,n,p;q}$ into the integral identity (5.18) below, the unitarity of W implies the following combinatorial identity:

$$
\begin{aligned}
D_{\alpha,\beta,\gamma;\xi} - \mathbb{I}_{\alpha=\gamma=\xi-1,\beta=0} - \binom{\alpha+\gamma-1}{\gamma}\mathbb{I}_{\beta+\gamma+1=\alpha=\xi} \\
- \sum_{m=0}^{\alpha}\sum_{p=0}^{\gamma-\alpha+m}\sum_{n=0}^{\alpha+\beta-\gamma-m+p-1} D_{m,n,\alpha+\beta-\gamma-m-n+2p-1;\xi-\gamma+p-1} \\
\times \binom{\alpha}{m}\binom{\gamma-\alpha+m+n-p}{n}\binom{\gamma}{p} \\
+ \sum_{m_1=0}^{\alpha}\sum_{m_2=0}^{\beta}\sum_{n_1=0}^{\gamma-1}\sum_{n_2=0}^{\gamma-1-n_1}\sum_{p_1=0}^{\gamma-1-n_1-n_2}\sum_{t_1=0}^{\xi}(-1)^{\alpha+\gamma-m_1-n_1-p_1+m_2} \\
D_{m_1,n_1+\beta-m_2,p_1;t_1} D_{m_2+\alpha-m_1,n_2,\gamma-1-n_1-n_2-p_1;\alpha+\gamma-\xi-m_1-n_1-p_1+m_2+t_1} \\
\times \binom{\alpha}{m_1}\binom{\beta}{m_2}\binom{n_1+n_2}{n_1}\binom{\gamma-1-n_1-n_2}{p_1} = 0.
\end{aligned}
$$

5.5 Dyck paths and Catalan numbers

For $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$, define the binomial coefficient the usual way

$$\binom{m}{n} := \frac{m!}{n!(m-n)!}\mathbb{I}_{0\leq n\leq m}.$$

For integers m, n, p define a double generalisation of the Catalan numbers and the Catalan's triangle

$$C_{m,n,p} := \binom{m+n}{m} - \binom{m+n}{m+p+1}.$$

For $\alpha, m, n, p \in \mathbb{Z}_{\geq 0}$, denote by $T_{\alpha,m,n,p}$ the set of lattice paths $(\rho_i)_{i=0}^{m+n}$ such that $\rho_0 = \alpha$, $|\rho_i - \rho_{i-1}| = 1$, $\rho_i \geq -p$, $\rho_{m+n} = \alpha + m - n$. That is, $T_{\alpha,m,n,p}$ is the set of Dyck paths starting from α, having m up-steps, n down-steps that never cross the line $y = -p$. By the reflection principle we obtain the following lemma, which shows these numbers have a similar combinatorial interpretation to the Catalan numbers.

Lemma 27. *When $m, n, p \geq 0$ and $m - n \geq -p - 1$, $C_{m,n,p} = |T_{0,m,n,p}|$.*

The doubly generalised Catalan numbers have been discussed in e.g. [Reu14].

When $m, n \geq 0$ and $p = 0$, $C_{m,n,0}$ is reduced to the (m, n)th entry in the Catalan triangle (OEIS:A009766) which we denote by $C_{m,n}$; furthermore when $m = n$, $C_{n,n}$ is the nth Catalan number which we denote by C_n.

The following recurrence relation will be useful:

Lemma 28. *If $n \geq 0$, $m \geq p$ and $m+n+p+1 \geq 0$, then*

$$\sum_{k=0}^{\lfloor \frac{m+p}{2} \rfloor} C_{k+n,k} C_{m-k,p-k} = C_{m+n+1,p}$$

Proof. We first show a basic version of this formula is true: for $n \geq 0$, $m \geq p \geq 0$,

$$\sum_{k=0}^{p} C_{k+n,k} C_{m-k,p-k} = C_{m+n+1,p}.$$

This can be proved using a combinatorial argument similar to one used to prove the recurrence relation of the Catalan numbers which is a special case of the identity above:

$$\sum_{k=0}^{p} C_k C_{p-k} = C_{p+1}.$$

Define a "stopping time" σ on $T_{0,m+n+1,p,0}$ by

$$\sigma(\rho) = \max\{i \geq 0 : \rho_i = n\},$$

then

$$C_{m+n+1,p} = \sum_{k=0}^{p} |\{\rho \in T_{0,m+n+1,p,0} : \sigma(\rho) = 2k+n\}| = \sum_{k=0}^{p} |T_{0,k+n,k,0}||T_{n+1,m-k,p-k,n+1}|$$
$$= \sum_{k=0}^{p} |T_{0,k+n,k,0}||T_{0,m-k,p-k,0}| = \sum_{k=0}^{p} C_{n+k,k} C_{m-k,p-k}.$$

If the condition $n \geq 0, m \geq p$ are retained, but $p < 0$ and $m+n+1+p \geq 0$, then the LHS is zero because the domain of the summation is empty. The RHS is also zero because $\binom{m+n+1+p}{m+n+1} = \binom{m+n+1+p}{m+n+2} = 0$.

Since $m \geq p$, we have $p \leq \lfloor \frac{m+p}{2} \rfloor$. Moreover, for any $k \in (p, \lfloor \frac{m+p}{2} \rfloor]$, $C_{m-k,p-k} = \binom{m+p-2k}{m-k} - \binom{m+p-2k}{m-k+1} = 0$. Therefore we can extend the domain of the summation from $0 \leq k \leq p$ to $0 \leq k \leq \lfloor \frac{m+n}{2} \rfloor$. $\square$

Lemma 29. *For any $B \vdash \pi \in \Pi_s$, if $\pi(0) = 1$ then $r_B = 0$, and if $\pi(0) = 0$ then $r_B > 0$. If $\pi(s) = 1$ then $r'_B = 0$, and if $\pi(s) = 0$ then $r'_B > 0$.*

Proof. We show the claim for $\pi(0)$, as the one for $\pi(s)$ can be deduced from the symmetry property. If $\pi(0) = 1$, then $\pi(1) = 0$ or 1. If $\pi(1) = 0$ then by the correspondence (5.13), for any $p \in \Omega_\pi$, the first four coordinates have the ordering $j_p = p_{1,1} < p_{2,1} < p_{1,2} = p_{2,2}$. Since $p_{1,1} \leq p_{k,1}, k \geq 2$, and $p_{1,1} < p_{1,2} \leq p_{k,2}, k \geq 2$. Thus j_p is the smallest (essential) coordinate and $r_B = 0$. If $\pi(1) = 1$ then $j_p = p_{1,1} < p_{1,2} = p_{2,1} < p_{2,2}$ hence it's also the smallest coordinates and $r_B = 0$.

If $\pi(1) = 0$, then $\pi(1) = 1$ and by (5.13), for any $p \in \Omega_\pi$, the first four coordinates are ordered as $p_{1,1} = p_{2,1} < p_{1,2} = j_p < p_{2,2}$. Hence j_p is greater than at least one other essential coordinate and $r_B > 0$. □

In some extreme cases the coefficient $D_{m,n,p;q}$ can be calculated directly. We denote by $D^\pi_{m,n,p;q}$ the contribution to $D_{m,n,p;q}$ from path π.

Lemma 30.

- *(Case A)* $D_{0,2k,0;k+1} = D^{\vee^k}_{0,2k,0;k+1} = C_k$. *Conversely, if* $m = 0, 2q = n + p + 2$, *then* $D_{m,n,p;q} > 0$ *only if* $p = 0, 2q - 2 = n$.
- *(Case B)* $D_{0,n,2k-n+1;k+1} = D^{\backslash\wedge^k}_{0,n,2k-n+1,k+1} = C_{k,n-k}$ *for* $0 \leq n \leq 2k + 1$.
- *(Case C)* $D_{2k-n+1,n,0;k+1} = D^{\wedge^k/}_{2k-n+1,n,k+1} = C_{k,n-k}$ *for* $0 \leq n \leq 2k + 1$.
- *(Case D)* $D_{r,2k-r-r',r';k} = D^{\wedge^k}_{r,2k-r-r',r';k} = C_{k-r,k-r',r-1} = C_{k-r',k-r,r'-1}$ *for* $0 \leq r + r' \leq 2k$.

  Moreover, $E_{m,n,p;q} = \mathbb{I}_{m=p=q,n=0}$.

Proof. First we show the first identity in each case. In Case D, for there to be k upper vertices and $k + 1$ lower vertices, the path can only be $\wedge^k$.

For Case A, since the rank is $(0, 0)$, by Lemma 29, any path π contributing to $D_{0,2k,0;k+1}$ has to begin and end with upper vertices. Removing these two vertices resulting a path of length $2k - 2$, $k - 1$ upper vertices and k lower vertices, which is the same as Case D. Therefore $\pi = \vee^k$.

With the same arguments the paths for Case B and C are also determined to be $\backslash\wedge^k$ and $\wedge^k/$ respectively.

Now we show the second identity in Case D, as Cases A, B and C are simpler variations of D and can be verified similarly. We achieve this by associating partial orderings with sets of Dyck paths. The path $\pi = \wedge^k$ imposes the following ordering of the essential coordinates:

$$\begin{array}{ccccccccccc} p_{1,1} & < & p_{3,1} & < & p_{5,1} & < & \dots & < & p_{2k-1,1} & < & k_p = p_{2k+1,1} \\ \wedge & & \wedge & & \wedge & & \dots & & \wedge & & \wedge \\ j_p = p_{1,2} & < & p_{2,2} & < & p_{4,2} & < & \dots & < & p_{2k-2,2} & < & p_{2k,2} \end{array}$$

We relabel these coordinates by $t_{1,1} := p_{1,1}, t_{2,1} = p_{3,1}$ and so on, to obtain

$$\begin{array}{ccccccccccc} t_{1,1} & < & t_{2,1} & < & t_{3,1} & < & \dots & < & t_{k,1} & < & t_{k+1,1} \\ \wedge & & \wedge & & \wedge & & \dots & & \wedge & & \wedge \\ t_{1,2} & < & t_{2,2} & < & t_{3,2} & < & \dots & < & t_{k,2} & < & t_{k+1,2} \end{array} \tag{5.16}$$

There is a one-one correspondence between the set of all linear extensions of this partial ordering (namely $\{B : B \vdash \pi\}$) and $T_{0,k+1,k+1,0}$. The Dyck path ρ corresponding to the linear extension $t_{m_1,b_1} < t_{m_2,b_2} < \cdots < t_{m_{2(k+1)},b_{2(k+1)}}$ is defined by

$$\rho(i) = \begin{cases} \rho(i-1)+1, & \text{if } b_i = 1 \\ \rho(i-1)-1, & \text{if } b_i = 2 \end{cases}$$

Clearly, the rank of a linear extension B being (r, r') is equivalent to the corresponding Dyck path starting with r up-steps followed by a down-step and concluding with one down-step with r' up-steps. These cut off the first $r+1$ and the last $r'+1$ steps from the path, making it correspond to $T_{r-1,k-r,k-r',0}$. Therefore

$$D^{\wedge^k}_{r,2k-r-r',r';k} = |T_{r-1,k-r,k-r',0}| = |T_{0,k-r,k-r',r-1}| = C_{k-r,k-r',r-1}.$$

If $r = 0$ or $r' = 0$, then by Lemma 29 $D^{\wedge^k}_{r,2k-r-r',r';q} = 0$, which agrees with $C_{k-r,k-r',r-1}$ as well. On the other hand, since the paths of $T_{0,k,k,0}$ only have k up- and down-steps, the LHS is 0 if $r > k$ or $r' > k$, which agrees with the right hand side.

Finally, the $\wedge^k$ are the only possible paths to contribute to the coefficients E, which record the instances when $k_p < j_p$. The corresponding linear extension $B \vdash \wedge^k$ is $\{t_{1,1} < t_{2,1} < \cdots < t_{k+1,1} < t_{1,2} < t_{2,2} < \cdots < t_{k+1,2}\}$. For any other paths, by Lemma 29, any path starting with \ or ending with / has j_p as the smallest or k_p as the greatest essential coordinate; on the other hand, any horizontal edge will result in $j_p < k_p$. □

From the above proof we can deduce a stronger version of Lemma 29: $D^{\pi}_{0,n,p;q} \neq 0$ only if $\pi(0) = 1$, and for $m \geq 1$, $D^{\pi}_{m,n,p;q} \neq 0$ only if $\pi(0) = 0$ and begins with $\wedge^{m-1}$. We also refer to this stronger version as Lemma 29.

Proof of Theorem 24. Case D covers the $2q = m + n + p$ case; moreover, by the same argument as in the proof of the first identities in each case of Lemma 30, for

$\pi \in \Pi_{m+n+p+s}$,

$$\epsilon(\pi) + 1 \geq m + n + p + 1 - \epsilon(\pi)$$

therefore $D_{m,n,p;q} \neq 0$ only if $2q \geq m + n + p$. Thus it suffices to show

$$D_{m,n,p;q} = C_{q-1,n-q+1}, \quad 2q > m + n + p.$$

We group the paths into ones starting with $\wedge^k/-$ (call the set of such paths $\Pi^{\wedge k}$) and ones starting with $\vee^k-$ (call the set of such paths $\Pi^{\vee k}$). Then by Lemma 29 $D_{m,n,p;q}$ are contributed from $\Pi^{\wedge k}$ if $k - 1 \geq m > 0$, and from $\Pi^{\vee^k}$ if $m = 0$:

$$D_{m,n,p;q} = \begin{cases} \sum_{k \geq m-1} \sum_{\pi \in \Pi^{\wedge k}} D^{\pi}_{m,n,p;q}, & m > 0 \\ \sum_{k \geq 0} \sum_{\pi \in \Pi^{\vee k}} D^{\pi}_{m,n,p;q}, & m = 0 \end{cases}$$

Therefore we divide the proof into two cases, $m = 0$ and $m > 0$. The formula for $D_{m,n,p;q}$ with $m + n + p \leq 2$ can be verified by hand (the reader can check their calculation against (5.14)), so we assume the formula is true for $s \leq m + n + p$, and we want to use induction to verify the formula of $D_{m,n,p;q}$ in general.

5.5.1 $m > 0$

When $m > 0$, for any $\pi \in \Pi^{\wedge k}$,

$$D^{\pi}_{m,n,p;q} = D_{m,2k+1-m,0;k+1} D^{\theta_{2k+2}\pi}_{0,m+n-2k-2,p;q-k-1},$$

where $\theta_r : \Pi_s \to \Pi_{s-r} \forall s \geq r$ is the shifting operator such that $(\theta_r \pi)(j) = \pi(j + r)$.

Summing over k and $\pi \in \Pi^{\wedge k}$ we have

$$\begin{aligned}
D_{m,n,p;q} &= \sum_{k \geq m-1} \sum_{\pi \in \Pi^{\wedge k}} D^{\pi}_{m,n,p;q} \\
&= \sum_{k=m-1}^{\lfloor \frac{m+n-2}{2} \rfloor} \sum_{\pi \in \Pi^{\wedge k}} D_{m,2k+1-m,0;k+1} D^{\theta_{2k+2}\pi}_{0,m+n-2k-2,p;q-k-1} \\
&= \sum_{k=m-1}^{\lfloor \frac{m+n-2}{2} \rfloor} D_{m,2k+1-m,0;k+1} D_{0,m+n-2k-2,p;q-k-1} \\
&= \sum_{k=m-1}^{\lfloor \frac{m+n-2}{2} \rfloor} C_{k,k-m+1} C_{q-k-2,m+n-q-k} \\
&= \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} C_{k+m-1,k} C_{q-m-1-k,n-q+1-k}
\end{aligned}$$

where the last two equalities comes from the Case C in Lemma 30 and the induction assumption.

To apply Lemma 28, we check the three conditions hold: (1) The condition "$n \geq 0$" becomes $m-1 \geq 0$: this is correct as $m > 0$. (2) "$m \geq p$" is $2q \geq m+n+2$: we know that $2q > m+n+p$, so either $2q \geq m+n+2$ or $2q = m+n+1$, in the latter since $m+n+1 \leq m+n+p+1 \leq 2q$, we have $p = 0$ and this is covered by Case C. (3) "$m+n+p+1 \geq 0$" becomes $n \geq 0$, which is evidently true by the definition of $D_{m,n,p;q}$. The upper bound of the summation domain "$\lfloor \frac{m+p}{2} \rfloor$" becomes $\lfloor \frac{n-m}{2} \rfloor$. Therefore we can apply Lemma 28 to the sum above and obtain

$$D_{m,n,p;q} = C_{q-1,n-q+1}.$$

5.5.2 $m = 0$

When $m = 0$, similarly, for any $\pi \in \Pi^{\vee k}$,

$$D^{\pi}_{0,n,p;q} = D_{0,2k,0;k+1} D^{\theta_{2k+1}\pi}_{0,n-2k-1,p;q-k-1}.$$

Again, summing over k and $\pi \in \Pi^{\vee k}$ we have

$$\begin{aligned} D_{0,n,p;q} &= \sum_{k\geq 0} \sum_{\pi \in \Pi^{\vee k}} D^{\pi}_{m,n,p;q} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\pi \in \Pi^{\vee k}} D_{0,2k,0;k+1} D^{\theta_{2k+1}\pi}_{0,n-2k-1,p;q-k-1} \\ &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} D_{0,2k,0;k+1} D_{0,n-2k-1,p;q-k-1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{k,k} C_{q-k-2,n-q-k+1} \end{aligned}$$

where the last equality comes from Case A, the induction assumption and the fact that $C_k = C_{k,k}$. Once again, we want to check the conditions in order to apply Lemma 28. The condition "$n \geq 0$" is obvious. "$m \geq p$" is equivalent to $2q \geq n+3$. Since $2q > n+p$, there are two possibilities apart from "$m \geq p$":

1. $2q = n+p+1$. This is covered by Case B.

2. $2q = n+p+2$ and $p = 0$. This is covered by Case A.

"$m+n+p+1 \geq 0$" is again equivalent to $n \geq 0$, which is evidently true. The upper bound of the summation domain "$\lfloor \frac{m+p}{2} \rfloor$" is $\lfloor \frac{n-1}{2} \rfloor$. Therefore we can apply Lemma 28 to the sum above and obtain:

$$D_{0,n,p;q} = C_{q-1,n-q+1}.$$

□

5.6 Proof of Theorem 25

In this section we often abuse notations and do not differentiate between operators and their kernels. Without loss of generality assume $|\nu| = 1$ (otherwise one can scale (a, x, y, b)). We only consider the generating function of coefficient D, as the case for E can be dealt with similarly. We write

$$\begin{aligned} \phi^N(m,n,p) = &\sum_{1\leq j<k\leq N} \mathbb{I}_{A_j}(x)\mathbb{I}_{A_k}(y) \prod_{\alpha=0}^{m-1}(x_{j-1} - a - \alpha\Delta_N) \prod_{\beta=0}^{n-1}(x_{k-1} - x_j - \beta\Delta_N) \\ &\times \prod_{\gamma=0}^{p-1}(b - x_k - \gamma\Delta_N) \\ \phi(m,n,p) = &(x-a)^m (y-x)^n (b-y)^p \mathbb{I}_{a\leq x<y<b}, \end{aligned}$$

where we recall $\Delta_N = \frac{b-a}{N}$. Then $0 \le \phi^N(m,n,p) \le \phi(m,n,p) \le (b-a)^{m+n+p}$. We also write

$$\mathcal{D}_{m,n,p} = \sum_{q=0}^{s} \frac{D_{m,n,p;q}}{m!n!p!}(-1)^q \nu^{m+n+p+1-2q}.$$

Then

$$W_N = \sum_{s=1}^{N-1} w_{s,N} + \sum_{s=1}^{2N-3} v_{s,N}$$

where $v_{s,N}$ are the degenerate terms, and

$$w_{s,N} = \sum_{m+n+p=s-1} \mathcal{D}_{m,n,p}\phi^N(m,n,p).$$

The reason for the range of the sum for s to be $1 \le s \le N-1$ is because $w_{s,N} = 0$ for $s \ge N$, as $\phi^N(m,n,p) = 0$ for $m+n+p \ge N-1$.

Proof of Theorem 25. We divide the proof into three parts:

1. $\sum_{m+n+p\ge N} \mathcal{D}_{m,n,p}\phi^N(m,n,p) \overset{N\to\infty}{\to} 0$ uniformly on $[a,b)^2$. This shows the limit exists.

2. $\sum_{s=1}^{N-1} w_{s,N}$ converges weakly to W.

3. The degenerate terms vanish uniformly: $\sum_{s=1}^{2N-3} v_{s,N}$ are arbitrarily small as N grows bigger.

5.6.1 Part 1

By the formula of $D_{m,n,p;q}$ a bound can be immediately obtained:

$$0 \le D_{m,n,p;q} \le \binom{n}{\lfloor n/2 \rfloor} \le 2^n.$$

Similarly one can bound the trinomial coefficient $\frac{(m+n+p)!}{m!n!p!} \le 3^{m+n+p}$. Combining these two bounds we obtain

$$|\mathcal{D}_{m,n,p}| = \left| \sum_{q=0}^{m+n+p+1} (-1)^q \nu^{m+n+p+1-2q} \frac{D_{m,n,p}}{m!n!p!} \right| \le (m+n+p+1)\frac{6^{m+n+p}}{(m+n+p)!}$$

Therefore

$$\left|\sum_{m+n+p\geq N}\mathcal{D}_{m,n,p}\phi^N(m,n,p)\right| \leq \sum_{m+n+p\geq N}(m+n+p+1)\frac{(6(b-a))^{m+n+p}}{(m+n+p)!}$$
$$= \sum_{r\geq N}\binom{r+2}{2}(r+1)\frac{(6(b-a))^r}{r!} \to 0$$

as $N \to \infty$.

5.6.2 Part 2

We want to show that for any $\epsilon > 0$ and sufficiently large N, we have $\langle f, \sum_{s\leq N-1}(w_{s,N} - w_s)g\rangle < \epsilon\|f\|_2\|g\|_2$ for testing functions $f, g \in L^2([a,b))$.

Equivalently, we must show that

$$\left|\int_a^b\int_a^b \bar{f}(x,y)\sum_{m+n+p\leq N-2}\mathcal{D}_{m,n,p}(\phi^N(m,n,p)-\phi(m,n,p))g(x,y)dxdy\right| \leq \epsilon\|f\|_2\|g\|_2.$$

We divide it into two further parts.

1. $(x-a)^m(y-x)^n(b-y)^p\sum_{1\leq j<k\leq N} I_{A_j\times A_k}(x,y) \approx (x-a)^m(y-x)^n(b-y)^p\mathbb{I}_{a\leq x<y<b}$,
2. $(x-a)^m(y-x)^n(b-y)^p\sum_{1\leq j<k\leq N}\mathbb{I}_{A_j\times A_k}(x,y) \approx \sum_{1\leq j<k\leq N}\tau^N(j,k;m,n,p)\cdot \mathbb{I}_{A_j}(x)\mathbb{I}_{A_k}(y)$;

where

$$\tau^N(j,k;m,n,p) := \prod_{\alpha=0}^{m-1}(x_{j-1}-a-\alpha\Delta_N)\prod_{\beta=0}^{n-1}(x_{k-1}-x_j-\beta\Delta_N)\prod_{\gamma=0}^{p-1}(b-x_k-\gamma\Delta_N).$$

Part 2.1

We want to show that

$$\left|\int_a^b\int_a^b \bar{f}(x,y)\sum_{m+n+p\leq N-2}\mathcal{D}_{m,n,p}\sum_{1\leq j<k\leq N}(x-a)^m(y-x)^n(b-y)^p\right.$$
$$\left.(\mathbb{I}_{A_j}(x)\mathbb{I}_{A_k}(y)-\mathbb{I}_{a\leq x<y<b})g(x,y)dxdy\right| \leq \epsilon.$$

Denote the left hand side by B_N, then by (1) and that $(x-a)^m(y-x)^n(b-y)^p \leq (b-a)^{m+n+p}$ for $a \leq x \leq y \leq b$, we have that

$$\begin{aligned}
B_N &\leq \sum_{m+n+p \leq N-2} (m+n+p+1)\frac{(6(b-a))^{m+n+p}}{(m+n+p)!} \left| \sum_{j=1}^{N} \int\int_{x_{j-1} \leq x < y < x_j} \bar{f}(x)g(y)dxdy \right| \\
&\leq \sum_{r \leq N-1} (r+1)\binom{r+2}{2}\frac{(6(b-a))^r}{r!} \left| \sum_{j=1}^{N} \int\int_{x_{j-1} \leq x < y < x_j} \bar{f}(x)g(y)dxdy \right|
\end{aligned}$$

The term in the modulus can be bounded by repeated use of Cauchy-Schwartz inequality:

$$\left| \sum_{j=1}^{N} \int\int_{x_{j-1} \leq x < y < x_j} \bar{f}(x)g(y)dxdy \right| \leq \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} |f(x)|dx \int_{x}^{x_j} |g(y)|dy$$

$$\leq \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} |f(x)|\sqrt{x_j - x}\|g\|_2 dx \leq \sum_{j=1}^{N} \frac{(b-a)^2}{2N^2}\|f\|_2\|g\|_2 = \frac{(b-a)^2}{2N}\|f\|_2\|g\|_2.$$

Therefore

$$B_N \leq \sum_{r=0}^{N-1}(r+1)\binom{r+2}{2}\frac{(6(b-a))^r}{r!}\frac{(b-a)^2}{2N}\|f\|_2\|g\|_2 \leq CN^{-1}\|f\|_2\|g\|_2$$

for some constant C, where the second bound comes from the fact that $\sum_{r \geq 0}(r+1)\binom{r+2}{2}\frac{(6(b-a))^r}{r!} < \infty$.

Part 2.2

We establish the following uniform convergence, from which weak convergence will follow:

$$\sum_{m+n+p \leq N-2} \mathcal{D}_{m,n,p} \sum_{1 \leq j < k \leq N} \mathbb{I}_{A_j}(x)\mathbb{I}_{A_k}(y)\left(\tau^N(j,k;m,n,p) - (x-a)^m(y-x)^n(b-y)^p\right) \to 0$$

When $a \le x \le y \le b$, τ^N is non-negative, and for $m+n+p \le N$ we use a telescoping series:

$$\begin{aligned}
&(x-a)^m(y-x)^n(b-y)^p - \tau^N(j,k;m,n,p) \\
&= (x - x_{j-1})\dots \\
&+ (x_{j-1}-a)(x-x_{j-2})\dots \\
&+ (x_{j-1}-a)(x_{j-2}-a)(x-x_{j-3})\dots \\
&+ \dots \\
&+ (x_{j-1}-a)(x_{j-2}-a)(x_{j-3}-a)\dots(x_{k+p-2}-y)(b-y) \\
&+ (x_{j-1}-a)(x_{j-2}-a)(x_{j-3}-a)\dots(b-x_{k+p-2})(x_{k+p-1}-y) \\
&+ (x_{j-1}-a)(x_{j-2}-a)(x_{j-3}-a)\dots(b-x_{k+p-2})(b-x_{k+p-1}) \\
&- (x_{j-1}-a)(x_{j-2}-a)(x_{j-3}-a)\dots(b-x_{k+p-2})(b-x_{k+p-1}) \\
&\le N^{-1}(b-a)^{m+n+p}\left(\sum_{\alpha=1}^{m}\alpha + \sum_{\beta=1}^{n}\beta + \sum_{\gamma=1}^{p}\gamma\right) \\
&\le N^{-1}(b-a)^{m+n+p}\frac{1}{2}(m+n+p)(m+n+p+1).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Bigg| \sum_{m+n+p\le N} \mathcal{D}_{m,n,p} \sum_{1\le j<k\le N} \mathbb{I}_{A_j}(x)\mathbb{I}_{A_k}(y)\big(\tau^N(j,k;m,n,p) \\
&- (x-a)^m(y-x)^n(b-y)^p\big)\Bigg| \le N^{-1}\sum_{r\le N}\binom{k+2}{2}\frac{1}{2}r(r+1)^2\frac{(6(b-a))^r}{r!} \le CN^{-1}.
\end{aligned}$$

5.6.3 Part 3

The degenerate terms are the total orderings of path of length s with some repeated essential coordinates. If such a total ordering J has $s+1-d$ non-repeated coordinates, then we call d the *degree* of degeneration, or we say that there are d degenerations in J. Each degeneration happens on a wedge part of a path, that is, any two essential coordinates $p_{i_1,j_1} = p_{i_2,j_2}$ if and only if they correspond to parts of the same $\wedge^k$ part of a path for some k. On the other hand, degenerations happen in pairs. That is, for an array p, there do not exist three essential coordinates equal to each other, which would violate the partial ordering. Therefore, given a path of length $s-1$, the number of degenerate total orderings with d degenerations is bounded by $\binom{\frac{s-1}{2}}{d}^2$. Moreover, the number of paths of length $s-1$ is the Fibonacci number $\frac{\Phi^{s+2}-(-\Phi)^{-s-2}}{\sqrt{5}}$ where $\Phi = \frac{\sqrt{5}+1}{2}$. Since a path of length $s-1$ can have at most $\frac{s-1}{2}$ wedges, there are at most $\frac{s-1}{2}$ degenerations.

Therefore for each N, we have

$$V_N = \sum_{s=1}^{2N-3} v_{s,N}$$

where $V_{s,N}$ is the counterpart of $W_{s,N}$ that collects all degenerate cases of paths of lengths $s-1$. By applying the calculation in the proof of Lemma 21, we can see the degenerate (of degree d) version of h_s^N is

$$q_s^N(x,y) = \frac{(b-a)^d}{m!n!p!N^d}\phi^N(m,n,p)$$

where $m+n+p+d = s-1$. Thus the sum is bounded uniformly by

$$\frac{(b-a)^{s-1}}{m!n!p!N^d}.$$

Therefore the total sum of degenerate terms is bounded as follows:

$$\begin{aligned}
|V_N| \leq & \left| \sum_{s=1}^{N-1} \sum_{d=1}^{\frac{s-1}{2}} \sum_{m+n+p=s-1-d} C \binom{\frac{s-1}{2}}{d}^2 \Phi^{s+2} \frac{(b-a)^{s-1}}{m!n!p!N^d} \sum_{q=0}^{s} (-1)^q \nu^{s-2q} \right| \\
& \leq \sum_{s=1}^{N-1} \sum_{d=1}^{\frac{s-1}{2}} \sum_{m+n+p=s-1-d} C \binom{\frac{s-1}{2}}{d}^2 \Phi^{s+2} \frac{(3(b-a))^{s-1}}{(s-1)!} sN^{-d} \\
& \leq \sum_{d=1}^{\frac{N-2}{2}} \sum_{s=2d+2}^{N-1} C \frac{(6\Phi(b-a))^{s-1}s}{(s-1)!} \binom{s-d+2}{2} \\
& \leq \sum_{d=1}^{\frac{N-2}{2}} \sum_{s=1}^{N-1} C \frac{(12\Phi(b-a))^{s-1}s}{(s-1)!} \leq C(N^{-1} + \sum_{d=2}^{\frac{N-2}{2}} N^{-2}) \leq CN^{-1}.
\end{aligned}$$

□

5.7 The unitarity of W

Let f and g be the generating functions of D and E:

$$f := \sum_{m,n,p \geq o} \sum_{q=0}^{m+n+p+1} D_{m,n,p;q}[m,n,p](-\bar{\nu})^q \nu^{m+n+p+1-q} = \sum_{s\geq 1} f_s,$$
$$g := \sum_{m,n,p \geq o} \sum_{q=0}^{m+n+p+1} E_{m,n,p;q}[m,n,p]^\dagger(-\bar{\nu})^q \nu^{m+n+p+1-q} = \sum_{s\geq 1} g_s.$$

And the kernel of $W - I$ is

$$f(x,y) <_a^b (x,y) + g(x,y) >_a^b (x,y). \tag{5.17}$$

One can write down the equation that f and g have to satisfy for W to be unitary.

Proposition 31. *For W to be unitary, it suffices to show that for any $a < x < y < b$,*

$$f(x,y) + \overline{g(y,x)} + \int_a^x g(x,z)\overline{g(y,z)}dz + \int_x^y f(x,z)\overline{g(y,z)}dz + \int_y^b f(x,z)\overline{f(y,z)} = 0. \tag{5.18}$$

Proof. For W to be unitary it is necessary and sufficient to show it is both a coisometry and an isometry

$$W^*W = WW^* = I$$

Plugging in (5.17) and using the formulas for kernels of products and adjoints of integral operators we obtain equation (5.18) and three "other" equations:

$$\overline{f(y,x)} + g(x,y) + \int_a^y g(x,z)\overline{g(y,z)}dz + \int_y^x g(x,z)\overline{f(y,z)}dz + \int_x^b f(x,z)\overline{f(y,z)}dz = 0, \qquad a < y < x < b \tag{5.19}$$

$$\overline{g(y,x)} + f(x,y) + \int_a^x f(z,y)\overline{f(z,x)}dz + \int_x^y f(z,y)\overline{g(z,x)}dz + \int_y^b g(z,y)\overline{g(z,x)}dz = 0, \qquad a < x < y < b \tag{5.20}$$

$$\overline{f(y,x)} + g(x,y) + \int_a^y f(z,y)\overline{f(z,x)}dz + \int_y^x g(z,y)\overline{f(z,x)}dz + \int_x^b g(z,y)\overline{g(z,x)}dz = 0, \qquad a < y < x < b \tag{5.21}$$

The equation (5.18) and (5.19) are equivalent: one can interchange x with y and take a conjugate in the former to obtain the latter. So are (5.20) and (5.21). By the symmetry of f and g from Corollary 23, (5.18) and (5.20) are equivalent, hence it suffices to verify (5.18) to show the unitarity of W. □

Proof of Theorem 26. We divide the proof into two parts: first we write down f and g in a more amenable form, then we proceed to proving the integral identity.

5.7.1 The formulas for f and g

As previous, without loss of generality suppose $|\nu| = 1$. Recall that

$$B_j(x,y) = \sum_{n\geq 0} \frac{(-1)^{n+j}x^{n+j}y^n}{(n+j)!n!} = (-1)^j (x/y)^{j/2} J_j(2\sqrt{xy}).$$

Since $E_{m,n,p;q} = \mathbb{I}_{m=p=q,n=0}$,

$$g(x,y) = \nu \sum_{m\geq 0} (-1)^m \frac{(y-a)^m(b-x)^m}{m!m!} = \nu B_0(y-a, b-x). \tag{5.22}$$

Let f_e and f_o be the parts of the sum of f where $m+n+p$ are even and odd respectively, i.e.

$$f_e = \sum_{2|m+n+p} \sum_{q=0}^{m+n+p+1} (x-a)^m (y-x)^n (b-y)^p (-\bar{\nu})^q \nu^{m+n+p+1-q}$$
$$f_o = \sum_{2\nmid m+n+p} \sum_{q=0}^{m+n+p+1} (x-a)^m (y-x)^n (b-y)^p (-\bar{\nu})^q \nu^{m+n+p+1-q}$$

The function f_e can be further divided into the $\binom{n}{q-m}$ part and the rest. Let $u := (x-a)\nu$, $v := (y-x)\nu$, $w := (b-y)\nu$ and $z = -\frac{\bar{\nu}}{\nu} = -\bar{\nu}^2$, then

$$\begin{aligned}
&\nu \sum_{q\geq 0} \sum_{m+n+p=2q} \frac{u^m v^n w^p z^q}{m!n!p!} \binom{n}{q-m} \\
&= \nu \sum_{q\geq 0} \sum_{m+n+p=2q} \frac{u^m v^{2q-m-p} w^p z^q}{m!(2q-m-p)!p!} \binom{2q-m-p}{q-m} \\
&= \nu \sum_{q\geq 0} \sum_{0\leq m,p\leq q} \frac{u^m v^{2q-m-p} w^p z^q}{m!(2q-m-p)!p!} \binom{2q-m-p}{q-m} \\
&= \nu \sum_{q\geq 0} v^{2q} z^q \sum_{m=0}^{q} \frac{(u/v)^m}{m!(q-m)!} \sum_{p=0}^{q} \frac{(w/v)^m}{m!(q-m)!} \\
&= \nu \sum_{q\geq 0} \frac{(u+v)^q (v+w)^q z^q}{q!q!} = \nu B_0(y-a, b-x).
\end{aligned} \tag{5.23}$$

The rest of f_e is slightly more complicated. Let $k = \frac{m+n+p}{2}$. We observe that

$$\begin{aligned}
\sum_q D_{m,n,p;q} z^q - \binom{n}{k-m} z^k = -\binom{n}{k} z^k + \binom{n}{k} z^{k+1} - \binom{n}{k+1} z^{k+1} + \\
\cdots + \binom{n}{n-1} z^n - \binom{n}{n} z^n + \binom{n}{n} z^{n+1} = (z-1) \sum_{q=k}^{n} \binom{n}{q} z^q.
\end{aligned}$$

Therefore the rest of f_e, i.e. the sum excluding the terms corresponding to $\binom{n}{k-m}$ is

$$\begin{aligned}
&\nu(z-1)\sum_{k\geq 0}\sum_{m+n+p=2k}\frac{u^m v^n w^p}{m!n!p!}\sum_{q=k}^{n}\binom{n}{q}z^q \\
&\qquad = \nu(z-1)\sum_{k\geq 0}\sum_{n=k}^{2k}\sum_{q=k}^{n}\binom{n}{q}z^q\frac{(u+w)^{2k-n}}{(2k-n)!}\frac{v^n}{n!} \\
&\qquad = \nu(z-1)\sum_{k\geq 0}\sum_{n=k}^{2k}\sum_{q=k}^{n}\frac{z^q}{q!}\frac{(u+w)^{2k-n}}{(2k-n)!}\frac{v^n}{(n-q)!} \\
&\qquad = \nu(z-1)\sum_{k\geq 0}\sum_{n=0}^{k}\sum_{q=k}^{k+n}\frac{z^q}{q!}\frac{(u+w)^{k-n}}{(k-n)!}\frac{v^{k+n}}{(k+n-q)!} \\
&\qquad = \nu(z-1)\sum_{k\geq 0}\sum_{n=0}^{k}\sum_{q=0}^{n}\frac{z^{k+q}}{(k+q)!}\frac{(u+w)^{k-n}}{(k-n)!}\frac{v^{k+n}}{(n-q)!} \\
&\qquad = \nu(z-1)\sum_{k\geq 0}\sum_{q=0}^{k}\sum_{n=0}^{k-q}\frac{z^{k+q}}{(k+q)!}\frac{(u+w)^{k-q-n}}{(k-q-n)!}\frac{v^{k+q+n}}{n!} \\
&\qquad = \nu(z-1)\sum_{k\geq 0}\sum_{q=0}^{k}\frac{(zv)^{k+q}}{(k+q)!}\frac{(u+v+w)^{k-q}}{(k-q)!} \\
&\qquad = \nu(z-1)\sum_{q\geq 0}\sum_{k\geq 0}\frac{(zv)^{k+2q}}{(k+2q)!}\frac{(u+v+w)^{k}}{k!}
\end{aligned} \tag{5.24}$$

We keep (5.24) to later merge it with a similar term in f_o.

For f_o, let $k=\frac{m+n+p-1}{2}$. We observe that

$$\begin{aligned}
&\sum_q D_{m,n,p;q}z^q-\binom{n}{k-m}z^k \\
&\qquad = \binom{n}{k}z^{k+1}-\binom{n}{k+1}z^{k+1}+\cdots+\binom{n}{n-1}z^n-\binom{n}{n}z^n+\binom{n}{n}z^{n+1} \\
&\qquad = (z-1)\sum_{q=k+1}^{n}\binom{n}{q}z^q+\binom{n}{k}z^{k+1}.
\end{aligned}$$

Following the same procedure which leads (5.24), the sum corresponding to the first term in the RHS is

$$\nu(z-1)\sum_{q\geq 0}\sum_{k\geq 0}\frac{(zv)^{k+2q+1}}{(k+2q+1)!}\frac{(u+v+w)^{k}}{k!} \tag{5.25}$$

whereas the contribution from the second term is computed as follows:

$$\begin{aligned}
&\nu\sum_{k\geq 0}\sum_{m+n+p=2k+1}\frac{u^m v^n w^p}{m!n!p!}\binom{n}{k}z^{k+1}\\
&\qquad=\nu\sum_{k\geq 0}\sum_{n=k}^{2k+1}\frac{v^n}{n!}\frac{(u+w)^{2k+1-n}}{(2k+1-n)!}\binom{n}{k}z^{k+1}\\
&\qquad=\nu\sum_{k\geq 0}\sum_{n=0}^{k+1}\frac{v^{n+k}}{n!k!}\frac{(u+w)^{k+1-n}}{(k+1-n)!}z^{k+1}\\
&\qquad=\nu\sum_{k\geq 0}\frac{v^k}{k!}\frac{z^{k+1}(v+u+w)^{k+1}}{(k+1)!}=B_1(b-a,y-x).
\end{aligned}\tag{5.26}$$

By summing up (5.23), (5.24), (5.25) (5.26) and plugging in u, v, w, z we obtain

$$f(x,y)=\nu B_0(y-a,b-x)+B_1(b-a,y-x)-(\nu+\bar{\nu})\sum_{q\geq 0}B_q(y-x,b-a)\bar{\nu}^q. \tag{5.27}$$

Note that $\sum_{q\geq 0}B_q(y-x,b-a)\bar{\nu}^q=\sum_{q\geq 0}J_q(2\sqrt{(b-a)(y-x)})\left(-\sqrt{\frac{y-x}{b-a}}\bar{\nu}\right)^q$ is a generating function of the Bessel functions.

5.7.2 Verifying the identity (5.18)

We list a few useful properties of B_j (where we let $B_{-1}(x,y):=-B_1(y,x)$):

1. $\partial_x B_j(x,y)=-B_{j-1}(x,y), j\geq 0$,
2. $B_0(x,y)=B_0(y,x)$,
3. $B_j(0,y)=\delta_{j0}, j\geq 0$,
4. $\partial_y B_j(x,y)=B_{j+1}(x,y)$.

And an integral:

$$\begin{aligned}
&\int_y^b B_0(b-x,z-a)B_j(z-y,b-a)dz=-\sum_{k\geq 0}B_k(b-x,z-a)B_{k+j+1}(z-y,b-a)\Big|_y^b\\
&\quad=\begin{cases}-\sum_{k\geq 0}B_k(b-x,b-a)B_{k+j+1}(b-y,b-a), & j\geq 0\\ \sum_{k\geq 0}B_k(b-x,b-a)B_k(b-y,b-a)-B_0(b-x,y-a) & j=-1.\end{cases}
\end{aligned}\tag{5.28}$$

Substituting for f from (5.27) and g from (5.22) into (5.18) gives

$$
\begin{aligned}
& B_1(b-a,y-x)+\int_a^b B_0(z-a,b-x)B_0(z-a,b-y)dz \\
& +\int_y^b B_1(b-a,z-x)B_1(b-a,z-y)dz \\
& +\nu B_0(y-a,b-x)+\bar{\nu}B_0(x-a,b-y)+\bar{\nu}\int_x^b B_1(b-a,z-x)B_0(z-a,b-y)dz \\
& +\nu\int_y^b B_1(b-a,z-y)B_0(z-a,b-x)dz \\
& -(\nu+\bar{\nu})\int_y^b \sum_{q\geq 0} B_1(b-a,z-x)B_q(z-y,b-a)\nu^q dz \\
& -(\nu+\bar{\nu})\int_y^b \sum_{q\geq 0} B_1(b-a,z-y)B_q(z-x,b-a)\bar{\nu}^q dz \\
& -\bar{\nu}(\nu+\bar{\nu})\int_x^b \sum_{q\geq 0} B_0(z-a,b-y)B_q(z-x,b-a)\bar{\nu}^q dz \\
& -\nu(\nu+\bar{\nu})\int_y^b \sum_{q\geq 0} B_0(z-a,b-x)B_q(z-y,b-a)\nu^q dz \\
& +(\nu+\bar{\nu})^2\int_y^b \sum_{q\geq 0} B_q(z-x,b-a)\bar{\nu}^q \sum_{q\geq 0} B_q(z-y,b-a)\nu^q dz \\
& -(\nu+\bar{\nu})\sum_{q\geq 0} B_q(y-x,b-a)\bar{\nu}^q = 0.
\end{aligned}
\tag{5.29}
$$

Let $G_j(x)=\sum_{k\geq 0}\frac{(-1)^k x^k}{k!(k+j)!}$, then for $\alpha>0$ the following two integrals hold:

$$
\begin{aligned}
& \int_0^x G_0(\alpha z)G_0(\beta z)dz=(\alpha-\beta)^{-1}(\alpha x G_1(\alpha x)G_0(\beta x)-\beta x G_1(\beta x)G_0(\alpha x)), \\
& \int_0^z G_1(w)G_1(w+\alpha)dw=\alpha^{-1}(zG_1(z)G_0(z+\alpha)-(z+\alpha)G_1(z+\alpha)G_0(z))+G_1(\alpha).
\end{aligned}
$$

The first of these two integrals is the well-known Lommel's integral, see e.g. Section 11 and 94 of [Bow58]. The second integral written in terms of an indefinite integral of the Bessel functions is

$$
\begin{aligned}
& \int \frac{1}{\sqrt{w^2+\beta^2}} J_1(w)J_1\left(\sqrt{w^2+\beta^2}\right)dw \\
& \quad =\beta^{-2}\left(wJ_1(w)J_0\left(\sqrt{w^2+\beta^2}\right)-\sqrt{w^2+\beta^2}J_1\left(\sqrt{w^2+\beta^2}\right)J_0(w)\right), \beta>0.
\end{aligned}
\tag{5.30}
$$

This is a special case of the so-called Sonine-Gegenbauer type integral (see e.g. page 415 of [Wat95]). However, the authors have not found an explicit formula like the one on the right hand side of (5.30) in the literature.

By these two integrals we have

$$\begin{aligned}
&\int_a^b B_0(z-a,b-x)B_0(z-a,b-y)dz \\
&\quad = (y-x)^{-1}\big((b-y)B_1(b-a,b-y)B_0(b-a,b-x) \\
&\qquad - (b-x)B_1(b-a,b-x)B_0(b-a,b-y)\big) \\
&\int_y^b B_1(b-a,z-x)B_1(b-a,z-y)dz \\
&\quad = -B_1(b-a,y-x) - (y-x)^{-1}\left((b-y)B_1(b-a,b-y)B_0(b-a,b-x)\right. \\
&\qquad \left. -(b-x)B_1(b-a,b-x)B_0(b-a,b-y)\right).
\end{aligned}$$

Therefore the first and the second lines of (5.29) vanishes.

By the integral (5.28) (and interchanging x and y when necessary), the third and the fourth lines are reduced to their real part.

Now if the real part of ν is 0 then we are done. Otherwise by subtracting the first and the second lines and the imaginary part of the third and the fourth lines with is 0 from (5.29), and dividing the remainder by the real part $\frac{\nu+\bar{\nu}}{2}$ of ν, we simplify the integral identity into

$$\begin{aligned}
&B_0(y-a,b-x) + B_0(x-a,b-y) + \int_x^b B_1(b-a,z-x)B_0(z-a,b-y)dz \\
&+ \int_y^b B_1(b-a,z-y)B_0(z-a,b-x)dz - 2\sum_{q\geq 0} B_q(y-x,b-a)\bar{\nu}^q \\
&- 2\int_y^b \sum_{q\geq 0} B_1(b-a,z-x)B_q(z-y,b-a)\nu^q dz \\
&- 2\int_y^b \sum_{q\geq 0} B_1(b-a,z-y)B_q(z-x,b-a)\bar{\nu}^q dz \\
&- 2\bar{\nu}\int_x^b \sum_{q\geq 0} B_0(z-a,b-y)B_q(z-x,b-a)\bar{\nu}^q dz \\
&- 2\nu\int_y^b \sum_{q\geq 0} B_0(z-a,b-x)B_q(z-y,b-a)\nu^q dz \\
&+ 2(\nu+\bar{\nu})\int_y^b \sum_{q\geq 0} B_q(z-x,b-a)\bar{\nu}^q \sum_{q\geq 0} B_q(z-y,b-a)\nu^q dz = 0
\end{aligned}$$

By the integral formulas (5.28), the above identity can be further simplified to

$$
\begin{aligned}
&\sum_{k\geq 0} B_k(b-x,b-a)B_k(b-y,b-a) + \sum_{q>0}\sum_{k\geq 0} B_k(b-x,b-a)B_{k+q}(b-y,b-a)\nu^q \\
&+ \sum_{q>0}\sum_{k\geq 0} B_k(b-y,b-a)B_{k+q}(b-x,b-a)\bar{\nu}^q - \sum_{q\geq 0} B_q(y-x,b-a)\bar{\nu}^q \\
&- \int_y^b \sum_{q\geq 0} B_1(b-a,z-x)B_q(z-y,b-a)\nu^q dz \\
&- \int_y^b \sum_{q\geq 0} B_1(b-a,z-y)B_q(z-x,b-a)\bar{\nu}^q dz \\
&+ (\nu+\bar{\nu})\int_y^b \sum_{q\geq 0} B_q(z-x,b-a)\bar{\nu}^q \sum_{q\geq 0} B_q(z-y,b-a)\nu^q dz = 0
\end{aligned}
$$

It remains to verify the coefficient of ν^q for each $q \in \mathbb{Z}$ in the LHS is 0, which can be done by repeated use of integration by parts and the properties of the B_j functions.

When $q = 0$, the coefficient is

$$
\begin{aligned}
&\sum_{j\geq 0} B_j(b-x,b-a)B_j(b-y,b-a) - B_0(y-x,b-a) \\
&+ \int_y^b -B_1(b-a,z-y)B_0(z-x,b-a) - B_1(b-a,z-x)B_0(z-y,b-a) \\
&+ \sum_{j\geq 0} B_{j+1}(z-x,b-a)B_j(z-y,b-a) + \sum_{j\geq 0} B_{j+1}(z-y,b-a)B_j(z-x,b-a)dz \\
&= \sum_{j\geq 0} B_j(b-x,b-a)B_j(b-y,b-a) - B_0(y-x,b-a) \\
&+ \sum_{j\geq -1} \int_y^b B_j(z-x,b-a)B_{j+1}(z-y,b-a) + B_j(z-y,b-a)B_{j+1}(z-x,b-a)dz \\
&= \sum_{j\geq 0} B_j(b-x,b-a)B_j(b-y,b-a) - B_0(y-x,b-a) \\
&\qquad - \sum_{j\geq 0} B_j(z-x,b-a)B_j(z-y,b-a)\Big|_y^b = 0,
\end{aligned}
$$

where in the sixth line we use integration by parts and the properties of B_j.

When $q > 0$, the coefficient is

$$\sum_{k\geq 0} B_k(b-x,b-a)B_{k+q}(b-y,b-a) + \int_y^b -B_1(b-a,z-x)B_q(z-y,b-a)$$
$$+\sum_{j\geq 0} B_j(z-x,b-a)B_{j+q-1}(z-y,b-a) + \sum_{j\geq 0} B_j(z-x,b-a)B_{j+q+1}(z-y,b-a)dz$$
$$=\sum_{k\geq 0} B_k(b-x,b-a)B_{k+q}(b-y,b-a) + \int_y^b \sum_{j\geq 0} B_j(z-x,b-a)B_{j+q-1}(z-y,b-a)$$
$$+\sum_{j\geq -1} B_j(z-x,b-a)B_{j+q+1}(z-y,b-a)dz$$
$$=\sum_{k\geq 0} B_k(b-x,b-a)B_{k+q}(b-y,b-a) + \int_y^b \sum_{j\geq 0} B_j(z-x,b-a)B_{j+q-1}(z-y,b-a)$$
$$+\sum_{j\geq 0} B_{j-1}(z-x,b-a)B_{j+q}(z-y,b-a)dz$$
$$=\sum_{k\geq 0} B_k(b-x,b-a)B_{k+q}(b-y,b-a) - \sum_{k\geq 0} B_k(z-x,b-a)B_{k+q}(z-y,b-a)\Big|_y^b = 0.$$

When $q < 0$, the coefficient is

$$\sum_{k\geq 0} B_k(b-y,b-a)B_{k+q}(b-x,b-a) - B_q(y-x,b-a)$$
$$+\int_y^b -B_1(b-a,z-y)B_q(z-x,b-a) + \sum_{k\geq 0} B_{k+q-1}(z-x,b-a)B_k(z-y,b-a)$$
$$+\sum_{k\geq 0} B_{k+q+1}(z-x,b-a)B_k(z-y,b-a)dz$$
$$=\sum_{k\geq 0} B_k(b-y,b-a)B_{k+q}(b-x,b-a) - B_q(y-x,b-a)$$
$$+\sum_{k\geq 0}\int_y^b B_{k+q-1}(z-x,b-a)B_k(z-y,b-a) + B_{k+q}(z-x,b-a)B_{k-1}(z-y,b-a)dz$$
$$=\sum_{k\geq 0} B_k(b-y,b-a)B_{k+q}(b-x,b-a) - B_q(y-x,b-a)$$
$$-\sum_{k\geq 0} B_{k+q}(z-x,b-a)B_k(z-y,b-a)\Big|_y^b = 0.$$

$\square$

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