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PROJECTIVE FIBRED SCHEMES

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Projective Fibred Schemes.

Abstract.

This thesis takes the construction (due to Grothendieck) of the projective fibred scheme of a coherent sheaf and investigates certain aspects of its geometry, (and, in Chapter IV, topology). In Chapter I, after quoting the basic definitions, of the projective fibred scheme of $\mathcal{E}$ with its projection $\pi : \mathbb{P}\mathcal{E} \longrightarrow X$ and fundamental invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}\mathcal{E}$, and giving some simple illustrative examples, we turn (§§ 2,3) to some particular features of the geometry, notably the Fitting subschemes and the dominating component (and interactions between the two). Here and throughout most of the work we keep close to geometrical intuition by considering only coherent modules on locally-noetherian, reduced schemes. In an appendix we introduce some sheaves that are in a sense universal for coherent sheaves on projective varieties. (These do not play any essential role in the rest of the thesis).

Chapter II is concerned with the canonical homomorphism α, $\mathcal{E} \longrightarrow \pi_* \mathcal{O}(1)$ which is known to be an isomorphism when $\mathcal{E}$ is locally-free. We extend this result to a larger class of sheaves and show, for example, that α is an isomorphism if $\mathbb{P}\mathcal{E}$ is normal. In view of this result, and for general reasons, it is of interest to look for examples of smooth projective fibred schemes. This we do in Chapter III and show that a "generic" Module $\mathcal{E}$ of the type that locally has a resolution $0 \longrightarrow \mathcal{O}_U^p \xrightarrow{\varphi} \mathcal{O}_U^q \longrightarrow \mathcal{E}_U \longrightarrow 0$, (U,smoot where $\varphi(x)=0$, has smooth $\mathbb{P}\mathcal{E}$ in a neighbourhood of $\mathbb{P}\mathcal{E}_{(x)}$.

Chapter IV considers the (singular) cohomology ring of $\mathbb{P}\mathcal{E}$ when $\mathcal{E}$ is a sheaf on a complex variety (with the classical topology). We include a discussion of the effect on cohomology of blowing-up a smooth variety with centre a smooth subvariety.

Preface.

The projective fibred scheme ("fibré projectif") construction of Grothendieck (EGA II § 4) associates to each quasi-coherent $\mathcal{O}_X$-Module $\mathcal{E}$ on a scheme X, an X-scheme $\pi : \mathbb{P}\mathcal{E} \longrightarrow X$, whose geometric fibres are projective spaces. Two extreme examples are well-known. First, if $\mathcal{E}$ is locally-free of rank r the fibres of $\mathbb{P}\mathcal{E}$ have constant dimension r-1 and $\mathbb{P}\mathcal{E}$ is essentially an algebraic projective bundle. Second, if $\mathcal{E}$ is the sheaf of ideals defining a regular subscheme Y of a regular noetherian scheme X then $\mathbb{P}\mathcal{E}$ is obtained by blowing up X with centre Y, a construction which (at least for the case: Y, a point) dates from the classical era of the subject. However, the projective fibred scheme of a general coherent sheaf $\mathcal{E}$, which appears to combine many of the features of these two extreme examples (cf. Chapter I § 3(iii)), has not received much attention. (We should mention (15) which shows how certain classical problems on projective embeddings of varieties can be expressed in terms of projective fibred schemes.) We envisage that the construction may play a central role in a future theory seeking to distinguish and classify Modules by geometric properties of their associated fibred schemes.

Our theme then is the geometric theory of modules; or rather, since such a theory is not yet realised, we take some tentative steps towards it. With the exception of Chapter I § 1 and Chapter II § 1, the work presented here is thought to be original. When it leans on work of others (especially Chapter I § 2 and Appendix, and Chapter IV § 1,) due acknowledgement is made.

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CONTENTS

Chapter I Elements of the Geometry of Projective Fibred Schemes.

Introduction.

The chapter is divided into 3 sections and an appendix. §1 contains the basic definitions, summarised from EGA II, and adds some examples to motivate the rest of the chapter. An important characteristic of a projective fibred scheme $\mathbb{P}\mathcal{E}$ when $\mathcal{E}$ is coherent is the way the dimensions of the fibres vary. This is expressed by the Fitting subschemes of $\mathcal{E}$ in X which are global analogues of the Fitting invariants of a module in commutative algebra. We give the definitions in § 2. Intuitively the Fitting subscheme $F_n(\mathcal{E})$ is the subscheme of X over which the fibres of $\mathbb{P}\mathcal{E}$ have dimension $\geqslant$ n-1.

§ 3 treats of some further geometrical features of $\mathbb{P}\mathcal{E}$ where $\mathcal{E}$ is a coherent Module over an integral, noetherian scheme X. We are concerned especially with the dominating component of $\mathbb{P}\mathcal{E}$, $Q(\mathcal{E})$, which is the unique irreducible component of $\mathbb{P}\mathcal{E}$ with surjective projection $\pi: Q(\mathcal{E}) \longrightarrow X$. If $\mathcal{E}$ is a torsion-free sheaf this is reflected in some properties of $Q(\mathcal{E})$, wth applications in Chapters II and IV.

In the Appendix we define some sheaves $\mathcal{U}$, on projective spaces over an algebraically closed field k, which are universal for coherent sheaves on projective varieties over k in the sense that $\mathbb{P}\mathcal{E}$ is induced by embedding the variety X in some projective space and restricting $\mathbb{P}\mathcal{U}$ to X.

§1 Basic Definitions and Examples.

If V is a vector space over a field k we write $\mathbb{P}V$ for the projective space of 1-dimensional linear subspaces of the dual space $V^\vee = \mathrm{Hom}(V,k)$. This is dual to the classical construction of the "projective space associated to a vector space": the difference is forced because we wish to globalise the construction as follows. To each quasi-coherent sheaf over a scheme X there is associated a X-scheme, $\pi : \mathbb{P}\mathcal{E} \longrightarrow X$, (called the "projective fibred scheme" of $\mathcal{E}$), such that the fibre $\pi^{-1}(x)$ over a point x in X is canonically isomorphic to $\mathbb{P}(\mathcal{E}\otimes k(x))$. The construction of $\mathbb{P}\mathcal{E}$ is due to Grothendieck (EGA II§4), and is a case of a more general construction, that of the homogeneous spectrum of a sheaf of graded algebras (EGA II§§2,3). In this section we reproduce the details of this construction, thereby fixing our notation (which is in the main consistent with that of EGA).

We first define the homogeneous spectrum of a ring. Let S be a commutative graded ring: that is, as abelian group $S = \bigoplus_{n\geq 0} S_n$ and the multiplication in S is such that $S_n S_m \subset S_{n+m}$ $(m,n \in \mathbb{Z})$. An element $x \in S_r$ is called homogeneous of degree r.

Given $x \in S_r$ $(r > 0)$, let $T = \{x^n \mid n \in \mathbb{Z}^+\}$, (by convention, $x^0 = 1$). T is a multiplicatively closed subset of S and the associated ring of fractions, $S_x = T^{-1}S$, is graded by putting $(S_x)_d = \{f/x^n \mid f \in S_{rn+d}, n \in \mathbb{Z}^+\}$. Let $S_{(x)} = (S_x)_0 = \{f/x^n \mid f \in S_{rn}, n \in \mathbb{Z}^+\}$ (EGA II 2.2).

Let $S_+ = \bigoplus_{n>0} S_n$ and assume that S_+ is generated (as a ring or as an ideal in S, it makes no difference) by S_1. An ideal $\underline{p}$ of S is said to be graded if $\underline{p} = \bigoplus_{n \geq 0} (\underline{p} \cap S_n)$. With these preliminaries the homogeneous spectrum of S (called "Proj S") can be defined as follows:

(i) Its underlying set is the set of graded prime ideals $\underline{p}$ of S such that $S_+ \not\subset \underline{p}$.

(ii) The subsets $D_+(x) = \{ \underline{p} \in \mathrm{Proj}\ S \mid x \notin \underline{p} \}$, where $x \in S_n$ for some $n > 0$, form a basis for a topology on Proj S.

(iii) Proj S has the structure of a ringed space determined by $\mathcal{O}(D_+(x)) = S_{(x)}$.

With this structure Proj S is a separated scheme (EGA II Prop. 2.4.2). Each $D_+(x)$ with its induced ringed space structure is isomorphic to the affine scheme Spec $S_{(x)}$.

If A is a ring and S a graded A-algebra with X = Proj S then $\mathcal{O}_X$ is an A-Algebra: in other words X is a seperated scheme over Spec A (EGA II Prop. 2.4.6).

If M is a graded module (graded by both positive and negative indices, $M = \bigoplus_{n \in \mathbb{Z}} M_n$) over a graded ring or A-algebra, S, then M determines a sheaf of modules on X. For $x \in S_r$, let $M_{(x)} = (M_x)_0 = \{ m/x^n \mid m \in M_{rn}, n \geq 0 \}$. Then $M_{(x)}$ is a module over $S_{(x)}$. Let d, e be integers (>0), and let $f \in S_d$, $g \in S_e$. There is a canonical isomorphism of rings $S_{(fg)} \xrightarrow{\cong} (S_{(f)})_{g^d/f^e}$, and if we identify these two rings there is a canonical isomorphism of modules $M_{(fg)} \xrightarrow{\cong} (M_{(f)})_{g^d/f^e}$. Hence (EGA II 2.2.3) there are canonical homomorphisms $S_{(f)} \longrightarrow S_{(fg)}$ and $M_{(f)} \longrightarrow M_{(fg)}$, which together define the sheaf of modules on Proj S:

EGA II Proposition 2.5.2.

On $X = \operatorname{Proj} S$ there is a unique quasi-coherent $\mathcal{O}_X$-Module $\widetilde{M}$ such that for $f \in S_r$ we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$. The restriction homomorphism $\Gamma(D_+(f), \widetilde{M}) \longrightarrow (D_+(fg), \widetilde{M})$, for f, g homogeneous in S_+, is the canonical homomorphism $M_{(f)} \longrightarrow M_{(fg)}$. $\square$

As an important example of this construction there are the sheaves $\mathcal{O}_X(n)$ $(n \in \mathbb{Z})$ defined below. For $n \in \mathbb{Z}$, $M(n)$ is the graded module over S defined by $(M(n))_k = M_{n+k}$. In particular $S(n)$ is a graded S-module (where $S_n = 0$ for $n < 0$), and we put $\mathcal{O}_X(n) = \widetilde{S(n)}$. Assuming that S_+ is generated by S_1 we have (EGA 11 2.5.9) $\mathcal{O}_X(n)$ is an invertible $\mathcal{O}_X$-Module for all $n \in \mathbb{Z}$. (2.5.14) For $m, n \in \mathbb{Z}$, $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n)$ and $\mathcal{O}_X(n) = (\mathcal{O}_X(1))^{\otimes n}$, both up to canonical isomorphism. (2.5.15) For any graded S-Module M, $\widetilde{M(n)} = \widetilde{M} \otimes \mathcal{O}_X(n)$, up to canonical isomorphism.

The above constructions have the following functorial property, (EGA II 2.8.9) . Let A, A' be two rings and $\psi : A' \longrightarrow A$ a ring homomorphism defining a morphism Ψ : Spec $A \longrightarrow$ Spec A'. Let S' be a graded A'-algebra and let $S = S' \otimes_{A'} A$, which is an A-algebra graded by the $S'_n \otimes_{A'} A$. Let M' be a graded S'-module and put $M = M' \otimes_{A'} A = M' \otimes_{S'} S$. Let $Y = \operatorname{Spec} A$, $Y' = \operatorname{Spec} A'$, $X = \operatorname{Proj} S$ $X' = \operatorname{Proj} S'$. Then there is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d] & X' \arrow[d, "\pi"] \\ Y \arrow[r, "\Psi"'] & Y' \end{tikzcd}$$

which (EGA II Proposition 2.8.10) identifies the scheme X with the product $X' \times_{Y'} Y$. (Φ is the morphism defined by $\underline{p} \longmapsto \varphi^{-1}(\underline{p})$ where $\varphi : S' \longrightarrow S$, $\varphi(s') = s' \otimes 1$). Further there is a canonical isomorphism $\Phi^*(\widetilde{M'}) \xrightarrow{\cong} \widetilde{M}$.

In particular, if $A = T^{-1}A'$ for some multiplicatively closed subset T of A and $\psi : A' \longrightarrow A$ is the canonical homomorphism, then Spec A $\hookrightarrow$ Spec A' is an open embedding and $X = \mathrm{Proj}(S' \otimes_{A'} A)$ is canonically identified with $\pi^{-1}(\mathrm{Spec}\ A)$. Consequently (EGA Prop 3.1.2) if Y is a scheme and $\mathcal{S}$ a sheaf of graded algebras over Y, then Proj $\mathcal{S}$ can be constructed as a scheme over Y by gluing ("recollement").

Further, if $\mathcal{M}$ is a sheaf of graded $\mathcal{O}_Y$-modules, the sheaves $\Gamma(U, \mathcal{M})^{\sim}$ define, by the gluing process, a sheaf of modules $\widetilde{\mathcal{M}}$ on Proj $\mathcal{S}$. In particular, if $\mathcal{S}$ is generated by $\mathcal{S}_1$, then the $\mathcal{O}(n)$ define invertible sheaves on Proj $\mathcal{S}$: $\mathcal{O}(1)$ is called the <u>fundamental sheaf</u>.

By (EGA II 3.3.2), for each $n \geqslant 0$ there is a canonical homomorphism of $\mathcal{O}_X$-Modules $\rho_n: \mathcal{M}_n \longrightarrow \pi_*\widetilde{\mathcal{M}}(n)$, where π is the projection Proj $\mathcal{S} \longrightarrow Y$. ρ_n is defined over each open $U \subset X$ by mapping $m \in \Gamma(U, \mathcal{M}_n)$ to the element $\tilde{m}$ of $\Gamma(U, \pi_*\widetilde{\mathcal{M}}(n)) = \Gamma(\pi^{-1}U, \widetilde{\mathcal{M}}(n))$ which restricts to $m/1$ in each $\Gamma(D_+(f), \widetilde{\mathcal{M}}(n))$, $f \in \Gamma(U, \mathcal{S}_1)$. In particular we note for later reference the morphisms

$$\mathcal{S}_n \longrightarrow \pi_*\mathcal{O}(n).$$

We complete this summary of the construction of homogeneous spectra with the result (EGA II 3.6) which shows the construction is functorial in a restricted sense. Suppose $\mathcal{S}$, $\mathcal{S}'$ are two graded Algebras, each generated by its component of degree 1, and suppose $\varphi : \mathcal{S} \longrightarrow \mathcal{S}'$ is a graded homomorphism which is surjective. Let $X = \mathrm{Proj}\ \mathcal{S}$, $X' = \mathrm{Proj}\ \mathcal{S}'$. Then if p is a point in X', i.e a graded prime ideal of $\mathcal{S}'(U)$ such that $\mathcal{S}_1(U) \not\subset p$, $\varphi^{-1}(p)$ is a graded prime ideal of $\mathcal{S}(U)$ and $\mathcal{S}_1(U) \not\subset \varphi^{-1}(p)$. Hence $p \longmapsto \varphi^{-1}(p)$ defines a function $\Phi : X' \longrightarrow X$. It can be verified that Φ is the set-theoretic map underlying a morphism of schemes (which we shall also denote by Φ) and that Φ is in fact a closed embedding ("immersion fermé"). Further (3.6.3) $\mathcal{O}_{X'}(n)$ is canonically isomorphic to $\Phi^*\mathcal{O}_X(n)$. For these facts, generalisations and further properties, we refer the reader to EGA II § 3.

The two examples of the Proj construction with which we shall be principally concerned are (i) the projective fibred scheme of a quasi-coherent $\mathcal{O}_X$-Module (EGA II § 4) and (ii) the blow-up ("éclatement") of a quasi-coherent sheaf of ideals in $\mathcal{O}_X$ (that is, of a subscheme of X), (EGA II § 8).

(i) Projective fibred schemes.

Definition. (EGA II 4.1.1) If $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$-Module, $\mathbb{P}\mathcal{E}$, the projective fibred scheme of $\mathcal{E}$, is the X-scheme, Proj $\mathcal{S}\mathcal{E}$ where $\mathcal{S}\mathcal{E}$ is the symmetric Algebra of $\mathcal{E}$.

Let us summarise the construction of $\mathcal{S}\mathcal{E}$: If E is a module over a ring A, the symmetric algebra of E is the A-algebra $S_A E$ with a canonical A-linear monomorphism $i: E \longrightarrow S_A E$ satisfying the universal property :-

if $i': E \longrightarrow S'$ is an A-module homomorphism where S' is an algebra over A, there is a unique A-algebra homomorphism $\alpha: S_A E \longrightarrow S'$ such that $\alpha \circ i = i'$.

This property defines $S_A E$ uniquely up to isomorphism. We shall sometimes abbreviate notation and write SE for $S_A E$ when no confusion can result. SE is a graded algebra over A such that $(SE)_0 \cong A$, $(SE)_1 \cong E$ and i is a monomorphism onto $(SE)_1$.

Lemma (1.1)

If B is an algebra over A, $A \longrightarrow B$, there is a natural isomorphism $S_A E \otimes_A B \cong S_B(E \otimes_A B)$.

Proof.

It suffices to show $i \otimes 1: E \otimes B \longrightarrow S_A E \otimes B$ has the universal property:

if S' is a B-algebra and we have a B-linear homomorphism $i': E \otimes_A B \longrightarrow S'$, there is a unique B-algebra homomorphism

$\alpha : S_A E \otimes B \longrightarrow S'$ such that $\alpha_\circ(i \otimes 1) = i'$.

The universal property of $i: E \longrightarrow S_A E$ gives a unique A-algebra homomorphism $\alpha' : S_A E \longrightarrow S'$ such that the diagram below commutes:

$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[d] & S_A E \arrow[d, "\alpha'"] \\
E \otimes B \arrow[r, "i'"] & S'
\end{tikzcd}$$

α' induces $\alpha = \alpha' \otimes 1_B : S_A E \otimes B \longrightarrow S' \otimes B = S'$. Then the following diagram commutes and so α has the required property.

$$\begin{tikzcd}
E \arrow[rr] \arrow[d] & & S_A E \arrow[dr] \arrow[dd, "\alpha'"'] & \\
E \otimes B \arrow[rrr, "i \otimes 1"] \arrow[dr, "i'"'] & & & S_A E \otimes B \arrow[dl, "\alpha"] \\
& S' \arrow[r, equal] & S' \otimes B &
\end{tikzcd}$$

Uniqueness of α follows from the fact that $\mathrm{im}(S_A E)$ generates $S_A E \otimes B$ as a B-module. $\square$

By applying the lemma in the case $B = AT^{-1}$ a ring of fractions of A, it can be seen that if X is a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_X$-Module then the symmetric algebras of the modules $\mathcal{E}(U)$, (U open affine in X) form a quasi-coherent $\mathcal{O}_X$-Algebra $\mathcal{S}\mathcal{E}$.

We list below some basic properties of $\mathbb{P}\mathcal{E} = \mathrm{Proj}\,\mathcal{S}\mathcal{E}$.

<u>Proposition</u> (1.2)

(i) If $f: Y \longrightarrow X$ is a morphism of schemes and $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$-Module, there is a canonical isomorphism $\Phi : \mathbb{P}(f^*\mathcal{E}) \longrightarrow Y \times_X \mathbb{P}\mathcal{E}$, and if $\mathcal{O}(1)'$ is the fundamental invertible sheaf on $\mathbb{P}(f^*\mathcal{E})$ there is an isomorphism $\mathcal{O}(1)' \cong (p.\Phi)^* \mathcal{O}(1)$ where $p: Y \times_X \mathbb{P}\mathcal{E} \longrightarrow \mathbb{P}\mathcal{E}$ is the projection.

(ii) If $\mathcal{E} \longrightarrow \mathcal{E}'$ is an epimorphism of $\mathcal{O}_X$-Modules, there is induced a closed embedding of X-schemes: $\mathbb{P}\mathcal{E}' \xrightarrow{j} \mathbb{P}\mathcal{E}$, such that $j^* \mathcal{O}(1)_{\mathcal{E}} \cong \mathcal{O}(1)_{\mathcal{E}'}$.

(iii) Since $(\mathcal{S}\mathcal{E})_1 = \mathcal{E}$ (gives a canonical morphism $\alpha: \mathcal{E} \longrightarrow \pi_* \mathcal{O}(1)$. The homomorphism $\pi^* \mathcal{E} \longrightarrow \mathcal{O}(1)$ (EGA II 4.1.5.1) defined as the composite of α^* with the canonical homomorphism $\pi^* \pi_* \mathcal{O}(1) \longrightarrow \mathcal{O}(1)$, is surjective.

Remarks.

(i) is the global version of Lemma (1.1) above. If we apply it to the case $Y = \mathrm{Spec}\ k$ (k, a field), so $f: \mathrm{Spec}\ k \longrightarrow X$ is a geometrical (k-valued) point of X, we see that the fibre $Y \times_X \mathbb{P}\mathcal{E}$ is isomorphic to $\mathbb{P}(\mathcal{E} \otimes k)$. Here $\mathcal{E} \otimes k$ is a vector space over k, so $S(\mathcal{E} \otimes k)$ is a polynomial algebra and $\mathrm{Proj}(S(\mathcal{E} \otimes k)) = \mathbb{P}(\mathcal{E} \otimes k)$ is the associated projective space. Thus the geometric fibres of $\pi: \mathbb{P}\mathcal{E} \longrightarrow X$ are projective spaces, which explains the terminology "projective fibred schemes".

(ii) follows from (p.5) above since an epimorphism $\mathcal{E} \longrightarrow \mathcal{E}'$ induces an epimorphism $\mathcal{S}\mathcal{E} \longrightarrow \mathcal{S}\mathcal{E}'$. $\square$

If $\mathcal{E}$ is coherent and locally free $\mathbb{P}\mathcal{E}$ may be considered to be a projective bundle of locally constant rank. For general $\mathcal{E}$ the dimensions of the geometrical fibres of $\mathbb{P}\mathcal{E}$ vary (in fact, upper semicontinuously over X) and it is this which makes the geometry of $\mathbb{P}\mathcal{E}$ potentially interesting. It is to be hoped that study of the geometry of projective fibred schemes will yield insights into the algebraic structure of modules and Modules. Experiment shows, however, that the structure of $\mathbb{P}\mathcal{E}$ can be far from transparent (for example, unpleasant singularities can, and often do, occur), and a preliminary step, before any such program can be instigated, must be to isolate for study a class of sheaves large enough to be interesting, whose projective fibred schemes have good geometric properties. This is the philosophy behind the investigations of Chapters II and III.

(ii) Blowing-up a quasi-coherent sheaf of ideals.

Suppose X is a scheme and $\mathcal{J}$ is a quasi-coherent sheaf of ideals in $\mathcal{O}_X$. Let $gr(\mathcal{J})$ be the graded $\mathcal{O}_X$-Algebra $\bigoplus_{n=0}^{\infty} \mathcal{J}^n$

Definition (EGA II 8.1.3)

The X-scheme $\rho : \mathrm{Proj}\,(gr(\mathcal{J})) \longrightarrow X$ is obtained by blowing-up the ideal $\mathcal{J}$. Alternatively we say $\mathrm{Proj}(gr(\mathcal{J}))$ is the blow-up of X with centre $\mathcal{J}$ (or centre Y, if Y is the subscheme of X defined by $\mathcal{J}$). We write $\tilde{X} = \mathrm{Proj}(gr(\mathcal{J}))$.

Since $gr(\mathcal{J})_+$ is generated by $gr(\mathcal{J})_1 = \mathcal{J}$, we have the canonical invertible $\mathcal{O}_{\tilde{X}}$-Module, $\mathcal{O}(1)$.

If U is an open set in X such that $U \cap Y = \phi$ then $\Gamma(U, \mathcal{J}) = 0$ and $\Gamma(U, gr(\mathcal{J})) = \Gamma(U, \mathcal{O}_X)$. Therefore ρ restricts to an isomorphism $\rho^{-1}(U) \longrightarrow U$.

The inclusion $\mathcal{J} = (gr(\mathcal{J})_1) \hookrightarrow gr(\mathcal{J})$ induces an $\mathcal{O}_X$-Algebra epimorphism $S\mathcal{J} \longrightarrow gr(\mathcal{J})$, and hence a closed embedding $i : \tilde{X} \hookrightarrow \mathbb{P}\mathcal{J}$. (If X is irreducible then so is $\tilde{X}$ and, anticipating §3(i), we may say $\tilde{X}$ is the dominating component of $\mathbb{P}\mathcal{J}$.)

An important case occurs when Y is a subscheme regularly embedded in a noetherian scheme X. Then i is an isomorphism. By localising at the ideal defining Y, this follows from the following characterisation of regular noetherian local rings.

Proposition (1.3)

Suppose A is a noetherian local ring with maximal ideal m. A is regular if and only if the graded A-algebra homomorphism:

$$\varphi : S_A(m) \longrightarrow \sum_{n=0}^{\infty} m^n$$

is an isomorphism.

Proof.

We start from another characterisation of regular noetherian local rings: A is regular if and only if (see e.g. (16))

$$\varphi' : S_{A/m}(m/m^2) \longrightarrow \sum_{n=0}^{\infty} (m^n/m^{n+1}) \quad \text{is an isomorphism.}$$

Note that $S_{A/m}(m/m^2) \cong S_A(m)/mS_A(m)$. Therefore if φ is an isomorphism so is φ' and A is regular.

Conversely if A is regular, φ' is an isomorphism and, by Nakayama's lemma, φ is surjective. To prove φ is injective we have a commutative diagram:

$$\begin{tikzcd}
(S_A(m))_{n+1} \arrow[r, "\psi_n"] \arrow[d, "\varphi_{n+1}"] & (S_A(m))_n \arrow[r] \arrow[d, "\varphi_n"] & S_A(m)_n/m(S_A(m))_n \arrow[r] \arrow[d] & 0 \\
m^{n+1} \arrow[r, hook] & m^n \arrow[r] & m^n/m^{n+1} \arrow[r] & 0
\end{tikzcd}$$

where ψ_n is determined by $\psi_n(x_1.x_2.\cdots.x_{n+1}) = ((x_1x_2).x_3.\cdots.x_{n+1})$.

If $\alpha \in \ker \varphi_n$, then $\alpha = \psi_n(\alpha')$ for some $\alpha' \in \ker \varphi_{n+1}$

Consequently $\ker \varphi_n \subset \bigcap_{i=1}^{\infty} m^i(S_A(m))_n$

$= 0$ since $(S_A(m))_n$ is a finitely-generated A-module and A is noetherian.

Therefore φ is an isomorphism. $\square$

Let us give a few low-dimensional examples of projective fibred schemes to illustrate some of the features that are considered in general in §§2,3.

1. Blowing-up the point 0 in $\mathbb{C}^2$.

$\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x,y]$. The blow-up in question is $\mathbb{P}(\tilde{M})$ where M is the ideal $\langle x,y \rangle$ in $A = \mathbb{C}[x,y]$, considered as an A-module: $M \cong Ae \oplus Af / \langle ye-xf \rangle$. Then $\mathbb{P}(\tilde{M})$ is the subvariety of $\operatorname{Proj}(A[e,f]) = \mathbb{P}^1 \times \mathbb{C}^2$, defined by the graded ideal generated by $ye-xf$. The exceptional fibre of $\pi : \mathbb{P}(\tilde{M}) \longrightarrow \mathbb{C}^2$ is $\pi^{-1}(0) = \mathbb{P}^1 \times \{0\}$ with dimension 1. The other geometric fibres are points.

2. To illustrate the effect of torsion in the sheaf $\mathcal{E}$, consider the module, $M' \cong Ae \oplus Af / \langle x(ye-xf)\rangle$, (A, as above) Then $ye-xf$ is a torsion element in M. $\mathbb{P}(\tilde{M}')$ is the union of irreducible components $\mathbb{P}(\tilde{M})$ and $\mathbb{P}^1 \times Y$ where Y is the "y-axis" in $\mathbb{C}^2$. (Since M is obtained from M' by factoring out the torsion, $\mathbb{P}(\tilde{M})$ embeds in $\mathbb{P}(\tilde{M}')$ by Proposition (1.2)(ii).)

3. Even when $\mathcal{E}$ is torsion-free $\mathbb{P}\mathcal{E}$ may still be reducible. For example $\mathbb{P}(\tilde{M} \oplus \tilde{M})$ is reducible as may be seen by dimensional considerations. The fibre $\pi^{-1}(0)$ has dimension 3 which is equal to the dimension of $\pi^{-1}(\mathbb{C}^2-0)$. Hence $\pi^{-1}(0)$ is an irreducible component of $\mathbb{P}(\tilde{M} \oplus \tilde{M})$ and the other component is the closure of $\pi^{-1}(\mathbb{C}^2-0)$, $Q(\tilde{M} \oplus \tilde{M})$ say.

$$M \oplus M \cong A(e_1, e_2, f_1, f_2) / \langle ye_1 - xf_1,\ ye_2 - xf_2 \rangle$$

Therefore in $S_A(M \oplus M)$, $x(e_1f_2-e_2f_1) = y(e_1f_2-e_2f_1) = 0$

Hence $Q(\tilde{M} \oplus \tilde{M})$ is defined by the ideal generated by

ye_1-xf_1, ye_2-xf_2, and $e_1f_2-e_2f_1$. $Q(\tilde{M} \oplus \tilde{M})$ is not itself a projective fibred scheme. The fibre over 0 of $Q(\tilde{M} \oplus \tilde{M})$ is the surface $e_1f_2-e_2f_1 = 0$; the other fibres are projective lines.

Here $Q(\tilde{M} \oplus \tilde{M})$ is an example of what we call the dominating component, which we define in §3. First in §2 we define the Fitting subschemes which describe the way the dimensions of the geometric fibres of a $\mathbb{P}\mathcal{E}$ vary over the points of the base scheme.

§2 The Fitting subschemes of a coherent sheaf.

The Fitting invariants of a module of finite presentation over a ring A are certain ideals in A, defined, for instance, in (12) (Appendix 4-3(b) p.145), and well-known to students of Commutative Algebra. Their construction has the property of "commuting with localisation" necessary to capture the interest of geometers, which allows the definition to be globalised so that to a coherent $\mathcal{O}_X$-Module over a scheme X we associate some subschemes of X: these we propose to call the Fitting subschemes of the Module. Our treatment is essentially that of (12) but with more geometric emphasis.

We begin with a well-known corollary to Nakayama's lemma.

Lemma (2.1)

Suppose A is a local ring with maximal ideal $\underline{m}$ and residue field $k = A/\underline{m}$, and M is a finitely generated module over A. If $a_1, \ldots, a_t$ are elements of M such that their residues generate the vector space $M/\underline{m}M$ then $a_1, \ldots, a_t$ generate M.

Proof.

Let N be the submodule of M generated by $a_1, \ldots, a_t$. Then $M = \underline{m}M + N$; therefore $M/N = \underline{m}(M/N)$ and Nakayama's lemma implies $M/N = 0$. $\square$

Suppose $\mathcal{E}$ is a coherent $\mathcal{O}_X$-Module on a scheme X: i.e. each point p of X has an affine neighbourhood $U = \operatorname{Spec} A$ and a finitely presentable module M over A such that if $V' = \operatorname{Spec} A_{S'}$, $V = \operatorname{Spec} A_S$, are affine open subspaces of U with $V' \subset V$ then $\mathcal{E}(V) = M \otimes A_S$, and the restriction homomorphism

$$\begin{array}{ccc} \mathcal{E}(V) & \longrightarrow & \mathcal{E}(V') \\ \| & & \| \\ M\otimes A_S & \longrightarrow & M\otimes A_{S'} \end{array}$$

is induced by the canonical homomorphism $A_S \longrightarrow A_{S'}$. We denote this by $\mathcal{E}_U = \tilde{M}$. If $q = \dim(k(p)\otimes\mathcal{E})$ where $k(p) = A_p/pA_p$, the residue field at p, then Lemma (2.1) shows that U may be chosen so that $M = \mathcal{E}(U)$ is generated by q elements. (First choose U' such that $\mathcal{E}_{U'} = \tilde{M}'$ and M' contains elements $a_1,\dots,a_q$ whose germs at p generate $\mathcal{E}_p = \tilde{M}'_p$. Let N be the submodule of M' generated by $a_1,\dots,a_q$; then $(M'/N)_p = M'_p/N_p = 0$ and so there is a neighbourhood U of p such that $(M'/N)^{\sim}_U = 0$. Hence $\mathcal{E}_U = \tilde{M}'_U = \tilde{N}_U$ and U has the required property.) Then the restriction of $\mathcal{E}$ to U has a resolution:

$$\mathcal{O}_U{}^p \longrightarrow \mathcal{O}_U{}^q \longrightarrow \mathcal{E}_U \longrightarrow 0 \quad .$$

Let e be the function defined on the point-set of X by

$$e(x) = \dim\,(k(x)\otimes\mathcal{E}).$$

If $x\in U$ then $e(x) \leqslant q = e(p)$; i.e. e is upper semi- continuous at p. Hence for $n\in\mathbb{Z}$, $F_n = \{x\in X \mid e(x)\geqslant n\}$ is closed in X. In fact, each F_n can be canonically endowed with a structure sheaf so that it becomes a closed subscheme of X.

First consider the affine case, $X = \operatorname{Spec} A$ and M a finitely presentable A-module with a free resolution

$$A^p \xrightarrow{\psi} A^q \longrightarrow M \longrightarrow 0 \quad .$$

ψ can be represented by a $(p\times q)$-matrix $\Psi = (\psi_{ij})_{\substack{i=1,\dots,p\\ j=1,\dots,q}}$ with entries in A, (i.e. $\psi(a_1,\dots,a_p) = (b_1,\dots,b_q)$ where $b_j = \sum_{i=1}^{p}\psi_{ij}a_i$). For $1\leqslant n\leqslant q$ let $I_n(M,\psi)$ be the ideal in A generated by the subdeterminants of Ψ of size $(q-n+1)\times(q-n+1)$. (If there are no such subdeterminants, i.e. if $q-n\geqslant p$, then $I_n(M,\psi) = 0$.) When $n>q$, let $I_n(M,\psi) = A$. It is shown below that each $I_n(M,\psi)$ depends solely on M and is independent of the resolution used to define it. So we may use the notation $I_n(M)$. $I_n(M)$ is the <u>n^{th} Fitting invariant of M</u>. (cf. (12) p.145)

Lemma (2.2)

Given two free resolutions of M:

$$A^p \xrightarrow{\psi} A^q \xrightarrow{\varphi} M \longrightarrow 0$$
$$A^r \xrightarrow{\psi'} A^s \xrightarrow{\varphi'} M \longrightarrow 0$$

defining, for each n, ideals $I_n = I_n(M,\psi)$ and $I'_n = I_n(M,\psi')$ respectively, then $I_n = I'_n$.

Proof.

We may suppose A is a local ring: for if A_p is a localisation of A at a prime $p \in \operatorname{Spec} A$, tensoring a resolution of M by A_p gives a resolution of M_p,

$$A_p{}^p \xrightarrow{\overline{\psi}} A_p^q \longrightarrow M_p \longrightarrow 0$$

which shows that $I_n(M_p, \overline{\psi}) = I_n(M,\psi)_p \lhd A_p$. Then the lemma in case A is local implies $I_n(M)_p = I'_n(M)_p$ for all $p \in \operatorname{Spec} A$ and it follows that $I_n(M) = I'_n(M)$ for general A. So we assume that A is local.

For this proof we regard $I_n(\alpha)$ as a function of a matrix α. Then if ψ is a $(p \times q)$-matrix, α a $(p' \times p)$-matrix and β a $(p \times q)$-matrix, we have $I_n(\psi\alpha) \leqslant I_n(\psi)$ and $I_n(\beta\psi) \leqslant I_n(\psi)$. Hence if α, β are invertible, $I_n(\beta\psi\alpha) = I_n(\psi)$.

Note that $I_n(M,\psi)$ is unaltered if we modify the resolution by introducing a direct summand

$$A^p \oplus A^t \xrightarrow{\psi \oplus I^t} A^q \oplus A^t \longrightarrow M \longrightarrow 0,$$

the effect on ψ being to augment it with a $(t \times t)$ identity matrix thus:

$$\left(\begin{array}{c|c} \psi & 0 \\ \hline 0 & I^t \end{array}\right)$$

In particular, the cases $q - n \leqslant p$ and $n > q$ can be seen to present no anomalies. In short, we have $I_n(\psi) = I_n(\tilde{\psi})$ where $\tilde{\psi} = \psi \oplus I^t$.

Consider the two given resolutions of M. Since A^q is free, there exists a homomorphism $\alpha' : A^q \longrightarrow A^s$ such that $\varphi'\alpha' = \varphi$.

If $\alpha'': A^t \longrightarrow \operatorname{im}\psi'$ is an epimorphism, the direct sum $\alpha = \alpha' \oplus \alpha'': A^q \oplus A^t \longrightarrow A^s$ is also an epimorphism. For if $a \in A^s$, there exists $a' \in A^q$ such that $\varphi(a') = \varphi'(a)$; then $a - \alpha'(a') \in \ker \varphi' = \operatorname{im} \psi' = \operatorname{im} \alpha''$.

Since $\operatorname{im}\psi' \leq \operatorname{im} \alpha\tilde{\psi}$ there exists a homomorphism $\beta: A^r \longrightarrow A^p \oplus A^t$ such that $\psi' = \alpha\tilde{\psi}\beta$

$$\begin{tikzcd}
A^p \oplus A^t \arrow[r, "\tilde{\psi}"'] & A^q \oplus A^t \arrow[r, "\varphi"'] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, equal] & 0 \\
A^r \arrow[u, "\beta"] \arrow[r, "\psi'"] & A^s \arrow[r, "\varphi'"] & M \arrow[r] & 0
\end{tikzcd}$$

Then $I_n(\psi') = I_n(\alpha\tilde{\psi}\beta) \leq I_n(\alpha\tilde{\psi})$. We claim that $I_n(\alpha\tilde{\psi}) \leq I_n(\tilde{\psi})$. Since α is an epimorphism, $A^q \oplus A^t \cong A^s \oplus \ker\alpha$. $\ker\alpha$ is a projective module and therefore free (since A is a local ring) of rank $u = q+t-s$. There is a commutative triangle:

$$\begin{tikzcd}
A^q \oplus A^t \arrow[r, "\cong"] \arrow[dr, "\alpha"'] & A^s \oplus A^u \arrow[d] \\
 & A^s
\end{tikzcd}$$

Since $\ker\alpha \leq \operatorname{im}\tilde{\psi}$, $\operatorname{im}\tilde{\psi} = (\operatorname{im}\tilde{\psi} \cap A^s) \oplus \ker\alpha = \operatorname{im}\alpha\tilde{\psi} \oplus A^u$.
Therefore there exists $\beta': (A^p \oplus A^t) \oplus A^u \longrightarrow A^p \oplus A^t$ such that the following triangle commutes:

$$\begin{tikzcd}
(A^p \oplus A^t) \oplus A^u \arrow[d, "\beta'"'] \arrow[drr, "\alpha\tilde{\psi} \oplus I^u"] & & \\
A^p \oplus A^t \arrow[r, "\tilde{\psi}"] & A^q \oplus A^t \arrow[r, phantom, "\cong"] & A^s \oplus A^u
\end{tikzcd}$$

Hence $I_n(\alpha\tilde{\psi}) = I_n(\alpha\tilde{\psi} \oplus I^u) = I_n(\tilde{\psi}\beta') \leq I_n(\tilde{\psi})$

Therefore $I_n(\psi') \leq I_n(\tilde{\psi}) = I_n(\psi)$. Similarly $I_n(\psi) \leq I_n(\psi')$ which proves the lemma and establishes that the ideals $I_n(\psi)$ are invariants of M. $\square$

Lemma (2.3)

The Fitting invariants of a module commute with change of rings: i.e. if B is an A-algebra, $\omega : A \longrightarrow B$, M a finitely presentable A-module, and M_B is the B-module $M \otimes_A B$, then

$$I_n(M_B) = (I_n(M))B.$$

Proof.

If $A^p \xrightarrow{\psi} A^q \longrightarrow M \longrightarrow 0$ is a free resolution of M then its tensor product with B is a free resolution of M_B,

$$B^p \xrightarrow{\psi'} B^q \longrightarrow M \longrightarrow 0,$$

where ψ' is represented by the matrix $(\omega(\psi_{ij}))$: the lemma follows immediately.

Corollary (2.4)

Let $p \in \operatorname{Spec} A$ and $k = A_p/pA_p$, the residue field at p. Then $\dim(M\otimes k) \geqslant n$ if and only if $I_n(M) \leqslant p$.

Proof.

By definition $I_n(M\otimes k) = \begin{cases} 0 & \text{if } n \leqslant \dim(M\otimes k) \\ k & \text{if } n > \dim(M\otimes k) \end{cases}$, while by Lemma (2.3) $I_n(M\otimes k)$ is the residue at p of $I_n(M)$. Therefore $I_n(M\otimes k) = 0$ if and only if $I_n(M) \leqslant p$.

Corollary (2.5)

Fitting invariants commute with localisation (putting $B = A_S$ in Lemma (2.3)); consequently, if $\mathcal{E}$ is a coherent $\mathcal{O}_X$-Module, the n^{th} Fitting invariants of the modules $\mathcal{E}(U)$ (for U affine open in X) form a quasi-coherent sheaf $\mathcal{J}_n$ of ideals in $\mathcal{O}_X$, of finite type, such that $\mathcal{O}_X/\mathcal{J}_n$ is supported on F_n. (This last assertion follows from Corollary (2.4)).

Definition (2.6)

The n^{th} Fitting subscheme of $\mathcal{E}$ is the closed subscheme F_n with structure sheaf $\mathcal{O}_X/\mathcal{J}_n$.

§3

(i) Suppose X is a noetherian scheme which is integral (i.e. reduced and irreducible), and $\mathcal{E}$ is a coherent $\mathcal{O}_X$-Module. This section is concerned with some aspects of the geometry of $\mathbb{P}\mathcal{E}$ which involve the Fitting subschemes of $\mathcal{E}$.

The generic rank of $\mathcal{E}$ is defined to be the rank (dimension) of the generic fibre, $\dim_K(\mathcal{E} \otimes K)$, where K is the function field of X. Let N be the generic rank of $\mathcal{E}$.

If $V = X - Y$ where Y is the $(N+1)^{\text{th}}$ Fitting subscheme of $\mathcal{E}$, then $\mathcal{E}_V$ is a locally-free $\mathcal{O}_V$-Module of rank N.

$\pi^{-1}(V) = \mathbb{P}(\mathcal{E}_V)$ is reduced and irreducible: we write $\mathcal{Q}(\mathcal{E})$ for the (scheme-theoretic) closure of $\mathbb{P}(\mathcal{E}_V)$ in $\mathbb{P}(\mathcal{E})$, and call $\mathcal{Q}(\mathcal{E})$ the dominating component of $\mathbb{P}\mathcal{E}$. It is the unique irreducible component of $\mathbb{P}\mathcal{E}$ with surjective projection onto X. (If $\mathcal{E}$ is a torsion sheaf, $N = 0$ and $\mathcal{Q}(\mathcal{E})$ is empty).

To give an explicit algebraic description of $\mathcal{Q}(\mathcal{E})$, suppose M is a finitely-presentable module over a reduced noetherian ring A. We are, of course, interested in the case $U = \operatorname{Spec} A$ is affine open in X such that $\mathcal{E}_U = \tilde{M}$. Since X is irreducible it follows that for each generic point (minimal prime ideal) $p \in \operatorname{Spec} A$, $A_p = K$ and $\dim_K(M \otimes A_p) = N$. In particular, the $(N+1)^{\text{th}}$ Fitting ideal, I_{N+1} is not included in p.

If $s \in I_{N+1}$, $(s \neq 0)$ then M_s is a locally-free (that is, projective) module over A_s. The kernel of the ring homomorphism, $S_A M \longrightarrow S_{A_s} M_s$ is the graded ideal

$$\mathcal{X}_{(s)} = \{\sigma \in S_A M \mid s^n \sigma = 0 \text{ for some } n \in \mathbb{Z}\}.$$

It follows that, over $U = \operatorname{Spec} A$, $\mathcal{Q}(\mathcal{E})$ is defined by the graded ideal $\mathcal{X} = \bigcap_{s \in I_{N+1}} \mathcal{X}_{(s)} = \left\{ \sigma \in S_A M \mid I_{N+1} \leqslant \sqrt{\operatorname{ann}(\sigma)} \right\}$.

We collect together some basic properties of χ in the the following proposition.

Proposition (3.1)

(i) $\chi \cap (S_A M)_0 = 0$

(ii) Let $\underline{D}(M)$ be the set of prime ideals q in $S_A M$ such that $q \cap (S_A M)_0$ is a minimal prime in $(S_A M)_0 \cong A$. Then

$\chi = \bigcap \{q \mid q \in \underline{D}(M)\}$.

(iii) If A is a domain, χ is prime.

Consequently if A is a domain, χ is the unique smallest prime ideal in $S_A M$ such that $\chi \cap (S_A M)_0 = 0$.

(iv) If T is a multiplicatively-closed subset of A (and the ring of fractions $T^{-1}(S_A M)$ is identified with $S_{T^{-1}A}(T^{-1}M)$) then $T^{-1}\chi(M) = \chi(T^{-1}M)$. In particular (letting $T = A - p$ where p is a prime ideal in A) the construction of χ commutes with localisation in A. Thus χ defines a coherent sheaf of graded ideals in $\mathfrak{S}\mathcal{E}$ and hence a subscheme of $\mathbb{P}\mathcal{E}$.

Proof.

(i) If $a \in \chi \cap (S_A M)_0$ and $s \in I_{N+1}$, then $s^n a = 0$ for some n. Hence $(sa)^n = 0$ and, since A is reduced, $sa = 0$. Since for each minimal prime ideal p of A there exists $s \in I_{N+1}$ such that $s \notin p$ it follows that $a \in p$. Thus $a \in \bigcap p = 0$.

(ii) Suppose $q \in \underline{D}(M)$. If $x \in \chi$ and $x \notin q$, then for $s \in I_{N+1}$, $s^n x = 0$ implies $s \in q$ and so $I_{N+1} \leqslant q \cap (S_A M)_0$, which is not the case. Therefore $\chi \subset q$, and $\chi \subset \bigcap \{q \mid q \in \underline{D}(M)\}$.

If $s \in A$, there is a one-one correspondence between those prime ideals of $S_A M$ that do not contain s and the prime ideals of $(S_A M)_s = S_{A_s} M_s$, given by $q \longmapsto s^{-1}q$.
Further, if $s \notin q$, then $q \in \underline{D}(M)$ if and only if $s^{-1}q \in \underline{D}(M_s)$.

If $s \in I_{N+1}$ then M_s is locally-free. Therefore if p is a prime ideal in A_s then $pS_{A_s} M_s$ is prime in $S_{A_s} M_s$. It follows that

$\bigcap\{q \mid q \in \underline{D}(M_s)\} = \bigcap\{pS_{A_s}M_s \mid p \lhd A,\ p \text{ minimal}\}$

$= 0$ [To prove this note that we may reduce to the case, M_s free: for localisation commutes with formation of finite intersections and since A_s is noetherian it has only finitely many minimal prime ideals. But if M_s is free $S_{A_s}M_s$ is a polynomial ring and the result follows because A_s is reduced.]

Let $x \in \bigcap\{q \mid q \in \underline{D}(M)\}$ and let $\bar{x}$ be the image of x in A_s. Then $\bar{x} \in \bigcap\{q \mid q \in \underline{D}(M_s)\} = 0$. Therefore $s^n x = 0$ for some n, and $x \in \chi_{(s)}$. It follows that $\bigcap\{q \mid q \in \underline{D}(M)\} \subset \bigcap \chi_{(s)} = \chi$.

(iii) Suppose A is a domain and $\sigma_1, \sigma_2 \notin \chi$. Then there exist $s_1, s_2 \in I_{N+1}$ such that $s_1^n \sigma_1 \neq 0$ and $s_2^n \sigma_2 \neq 0$ for all n. Hence $\sigma_1/1 \neq 0 \in (SM)_{s_1}$ and $\sigma_2/1 \neq 0 \in (SM)_{s_2}$. $(SM)_{s_i}$ (i=1,2) and $(SM)_{s_1 s_2}$ are domains, so $\sigma_1\sigma_2/1 \neq 0 \in (SM)_{s_1 s_2}$; i.e. for all n, $(s_1 s_2)^n \sigma_1 \sigma_2 \neq 0$. Therefore $\sigma_1\sigma_2 \notin \chi$.

(iv) If $\bar{\sigma} \in T^{-1}\chi(M)$, $\bar{\sigma} = \sigma/t$ $(\sigma \in \chi(M), t \in T)$ and $\bar{s} \in I_{N+1}(T^{-1}M) = T^{-1}I_{N+1}(M)$, $\bar{s} = s/t'$ then $s^n \sigma = 0$ for some n; therefore $\bar{s}^n \bar{\sigma} = 0$ and $\bar{\sigma} \in \chi(T^{-1}M)$.

Conversely suppose $\bar{\sigma} \in \chi(T^{-1}M)$, $\bar{\sigma} = \sigma/t$ $(\sigma \in A,\ t \in T)$. If $s \in I_{N+1}(M)$ then $s/1 \in I_{N+1}(T^{-1}M)$ and $(s/1)^n \bar{\sigma} = 0$ for some n. Therefore $t' s^n \sigma = 0$ for some $t' \in T$; $\bar{\sigma} = (\sigma t')/(t' t) \in \chi_{(s)}.T^{-1}$. If $\{s_1, \dots, s_r\}$ is a finite set of generators for I_{N+1}, we have $\chi(T^{-1}M) \subset \bigcap_{i=1}^{r}\left(\chi_{(s_i)}.T^{-1}\right) = \left(\bigcap_{i=1}^{r}\chi_{(s_i)}\right).T^{-1} = \chi.T^{-1}$. $\square$

Remark: (ii) implies that $Q(\mathcal{E})$ is reduced, whether $\mathbb{P}\mathcal{E}$ is reduced or not. (Indeed the scheme-theoretic closure of a reduced scheme is always reduced (EGA* I 6.10.6)).

(ii) Torsion and torsion-free sheaves.

An integral noetherian scheme, X, has a quasi-coherent sheaf $\mathcal{R}(X)$, the "sheaf of rational functions", such that if U = SpecA is an affine open subscheme of X then $\Gamma(U,\mathcal{R}(X))$ is the total ring of fractions of A; i.e. the ring of fractions a/b, where $a,b \in A$ and b is not a zero-divisor in A. (EGA* I 8.3)

If $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$-Module, the canonical monomorphism $\mathcal{O}_X \longrightarrow \mathcal{R}(X)$ defines, by tensoring, a homomorphism of $\mathcal{O}_X$-Modules $t : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{R}(X)$.

The torsion sub-Module of $\mathcal{E}$ is by definition (EGA* I 8.4), the kernel $\tau\mathcal{E}$ of t :

$$0 \longrightarrow \tau\mathcal{E} \longrightarrow \mathcal{E} \xrightarrow{t} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{R}(X).$$

$\mathcal{E}$ is said to be torsion-free if $\tau\mathcal{E} = 0$, and $\mathcal{E}$ is a torsion sheaf if $\mathcal{E} = \tau\mathcal{E}$.

For any $\mathcal{E}$, $\mathcal{E}/\tau\mathcal{E}$ is torsion-free and $\tau\mathcal{E}$ is a torsion sheaf.

Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-Module and Spec A = U an affine open subscheme of X such that $\mathcal{E}_U = \tilde{M}$. Suppose K is the total ring of fractions of A. Then $\Gamma(U,\tau\mathcal{E})$ is the kernel:

$$0 \longrightarrow \Gamma(U,\tau\mathcal{E}) \longrightarrow M \longrightarrow M \otimes_A K$$
$$m \longmapsto m \otimes 1$$

Hence $\Gamma(U,\tau\mathcal{E}) = \{ m \in M \mid sm = 0 \text{ for some } s \in A,\ s \text{ not a zero-divisor} \}$

Definition: $m \in M$ is a torsion element if $sm = 0$ for some $s \in A$ where s is not a zero-divisor. The set of torsion elements of M forms a submodule τM, the torsion submodule of M.

Thus $\mathcal{E}$ is a torsion-free $\mathcal{O}_X$-Module if and only if for all affine open $U \subset X$, $\mathcal{E}(U)$ has trivial torsion submodule.

Let us see what implications this has for the dominating component of $\mathcal{E}$. Suppose $0 \neq s \in I_{N+1}$, where s is not a zero-divisor. (Such an s exists since the set of zero-divisors in A is the union of the minimal prime ideals of A, $\bigcup_i p_i$; and $I_{N+1} \subset \bigcup_i p_i$ only if $I_{N+1} \subset p_i$ for some i, which is not the case). Then $(\sigma \in \chi)$ $s^n \sigma = 0$ for some n and so σ is a torsion element. Therefore $\chi \cap M \subset \tau M$. In particular, if $\mathcal{E}$ is torsion-free, $\chi \cap M = 0$.

In order to interpret this geometrically we make a definition.

Definition (3.2)

If $S = \bigoplus_{i=0}^{\infty} S_i$ is a graded ring and q a graded ideal of S defining a closed subscheme $\mathcal{Q}$ in Proj S, then the <u>linear hull</u> of $\mathcal{Q}$ is the closed subscheme of Proj S defined by the graded ideal generated by $q \cap S_1 = q_1$.

This definition extends in the obvious fashion to apply to a closed subscheme $\mathcal{Q}$ of Proj $\mathcal{S}$, where $\mathcal{S}$ is a graded $\mathcal{O}_X$-Algebra.

It is easy to see that forming the linear hull commutes with change of base. Let $S_0 \longrightarrow S_0'$ be an S_0-algebra, and $S' = S \otimes_S S_0'$ inducing the commutative diagram

$$\begin{tikzcd}
\text{Proj } S' \arrow[r, "\alpha"] \arrow[d] & \text{Proj } S \arrow[d] \\
\text{Spec } S_0' \arrow[r] & \text{Spec } S_0
\end{tikzcd} .$$

Let $q' = qS'$, the ideal defining the closed subscheme $Q' = \alpha^{-1}Q$ in Proj S'. Then if $\overline{Q}$ (respectively $\overline{Q'}$) is the linear hull of Q (resp. Q') in Proj S (resp. Proj S'), we have $\overline{Q'} = \alpha^{-1}\overline{Q}$.

Note that if k is a field and S is the polynomial ring $k[t_0, \ldots, t_n]$ then Proj $S = \mathbb{P}_n(k)$ (n-dimensional projective space over k), and $\overline{Q}$ is the smallest (reduced) linear subspace which contains Q as a subscheme. Q is said to be <u>normally embedded</u> in $\mathbb{P}_n(k)$ if $\overline{Q} = \mathbb{P}_n(k)$.

Consider now from this point of view the subscheme $Q(\mathcal{E})$ in $\mathbb{P}\mathcal{E}$. We have the following proposition.

Proposition (3.3)

If $\mathcal{E}$ is a coherent $\mathcal{O}_X$-Module, the linear hull of $Q(\mathcal{E})$ in $\mathbb{P}\mathcal{E}$ is the subscheme $\mathbb{P}(\mathcal{E}/\tau\mathcal{E})$. Consequently $\mathcal{E}$ is torsion-free if and only if the linear hull of $Q(\mathcal{E})$ is $\mathbb{P}\mathcal{E}$ itself.

Proof.

We have shown above that if $\mathcal{E}$ is torsion-free then $\chi \cap M = 0$, where $M = \mathcal{E}(U)$, U affine open in X and χ is the graded ideal defining $Q(\mathcal{E})_U$ in $\mathbb{P}(\mathcal{E}_U) = \mathrm{Proj}(SM)$. Hence the linear hull of $Q(\mathcal{E})$ is defined by the trivial ideal and is therefore $\mathbb{P}\mathcal{E}$ itself.

$\mathbb{P}(\mathcal{E}/\tau\mathcal{E})$ is a subscheme of $\mathbb{P}\mathcal{E}$ by Proposition(1.2)(ii),and since $\mathcal{E}/\tau\mathcal{E}$ is torsion-free, we have $\bar{Q}(\mathcal{E}/\tau\mathcal{E}) = \mathbb{P}(\mathcal{E}/\tau\mathcal{E})$. Here $\bar{Q}(\mathcal{E}/\tau\mathcal{E})$ is the linear hull of $Q(\mathcal{E}/\tau\mathcal{E})$ in $\mathbb{P}(\mathcal{E}/\tau\mathcal{E})$, but this is clearly the same as the linear-hull in $\mathbb{P}\mathcal{E}$. Further the definition of $Q(\mathcal{E})$ as a scheme-theoretic closure implies that $Q(\mathcal{E}) = Q(\mathcal{E}/\tau\mathcal{E})$, and the proposition follows. $\square$

Such results as these are perhaps more geometrically suggestive if expressed as far as possible in terms of the geometric fibres. Accordingly, and with a glance ahead towards Proposition (3.5), we state

Corollary (3.4)

If the coherent Module $\mathcal{E}$ is torsion-free then each geometric fibre of the dominating component $Q(\mathcal{E})$ is normally embedded in its geometric fibre of $\mathbb{P}\mathcal{E}$. The converse implication holds provided $\mathbb{P}\mathcal{E}$ is a reduced scheme.

Proof.

This is a consequence of the second assertion of Proposition (3.3) and the preceding remarks on linear hulls and change of base. The assumption that $\mathbb{P}\mathcal{E}$ be reduced is necessary for the inference:

if L is a subscheme of $\mathbb{P}\mathcal{E}$ such that for each geometric point of X, Spec $k \longrightarrow X$, we have $L \times_X \operatorname{Spec} k = \mathbb{P}\mathcal{E} \times_X \operatorname{Spec} k$, then $L = \mathbb{P}\mathcal{E}$. We set L to be the linear hull of $Q(\mathcal{E})$. $\square$

Let us give a specific example to illustrate these considerations. Consider the $\mathcal{O}_X$-Module $\mathcal{E}$ where $X = \mathbb{P}_3(k) = \operatorname{Spec} k[x,y,z]$ (k, a field), and $\mathcal{E} = \tilde{M}$ where M is the ideal generated by x^2, xyz, y^2z in $k[x,y,z]$, considered as a module over $k[x,y,z]$. Then M is a torsion-free module and, writing $u=x^2$, $v=xyz$, $w=y^2z$, M is generated by u,v,w, subject to relations $yzu-xv = yv-xw = 0$. (For if $\alpha x^2 + \beta xyz + \gamma y^2 z = 0$, $\alpha,\beta,\gamma \in k[x,y,z]$ then

yz divides α, $\alpha = yz\alpha'$ say, and

x divides γ, $\gamma = x\gamma'$ say; so

$xyz(x\alpha' + \beta + y\gamma') = 0$, $\beta = -x\alpha' - y\gamma'$, and $\alpha u + \beta v + \gamma w =$ $\alpha'(yzu - xv) - \gamma'(yv - xw)$.) Therefore $\mathbb{P}\mathcal{E}$ is the subscheme of $X \times \mathbb{P}_2(k)$ defined by equations

$$(*) \qquad \begin{cases} yzu - xv = 0 \\ yv - xw = 0, \end{cases}$$

where $(u:v:w)$ are homogeneous coordinates in $\mathbb{P}_2(k)$. Hence the dimensions of the geometric fibres of $\mathbb{P}\mathcal{E}$ over k-valued points of X are as follows:

point	$x=y=0$	$x=z=0,\ y\neq 0$	otherwise
dimension	2	1	0

$(*)$ imply $y(v^2 - zuw) = 0$ and it may be seen that $Q(\mathcal{E})$ is defined by the 3 equations $yzu - xv = 0$, $yv - xw = 0$, $v^2 - zuw = 0$. When $x=y=0$ the fibre $Q(\mathcal{E})_{(0,0,z)}$ is the conic in $\mathbb{P}\mathcal{E}_{(0,0,z)}\ (\cong \mathbb{P}_2)$ defined by $v^2 - zuw = 0$, which degenerates into 2 coincident lines when $z=0$. In either case $Q(\mathcal{E})_{(0,0,z)}$ is normally embedded in $\mathbb{P}\mathcal{E}_{(0,0,z)}$. $Q(\mathcal{E})_{(x,y,z)} = \mathbb{P}\mathcal{E}_{(x,y,z)}$ over all other points (x,y,z).

As a consequence of Proposition (3.3) we have the following result which finds application as a key step in the proof of the main result of Chapter II(Theorem 2.7, p.51).

Proposition (3.5)

Suppose X is an integral noetherian scheme, and $\mathcal{E}$ is a torsion-free coherent $\mathcal{O}_X$-Module with projective fibred scheme $\pi : \mathbb{P}\mathcal{E} \longrightarrow X$ and assume $\mathbb{P}\mathcal{E}$ is reduced: then the coherent $\mathcal{O}_X$-Module $\pi_* \mathcal{O}(1)$ is torsion-free.

Proof.

Let $U = \operatorname{Spec} A$ be an affine open subscheme of X such that $\mathcal{E}_U = \tilde{M}$ and consider $\sigma \in \pi_* \mathcal{O}(1)(U) = \Gamma(\pi^{-1}U, \mathcal{O}(1))$. Considering σ as a section of the canonical line bundle its zeros form a closed subscheme Z_σ of codimension ≤ 1 in $\mathbb{P}\mathcal{E}$. (To be precise, suppose $V \subset \pi^{-1}U$ is an open subscheme such that $\Gamma(V, \mathcal{O}(1))$ is a free $\mathcal{O}(V)$-module of rank 1 generated by e, say. Then $\sigma|_V = se$ for some $s \in \mathcal{O}(V)$ and $Z_\sigma \cap V$ is the subscheme defined by the principal ideal (s) in $\mathcal{O}(V)$.)

Now suppose $a \in A$ where a is not a zero-divisor and $a\sigma = 0$. Then if U' is a Zariski-dense open subset of U such that $a|_{U'}$ is invertible, σ vanishes on $\pi^{-1}U'$. Consequently, by the definition of $Q(\mathcal{E})$ as a scheme-theoretic closure, we have $Q(\mathcal{E}) \cap \pi^{-1}U$ is a subscheme of Z_σ. If we consider the restriction σ_0 of σ to a geometric fibre $\mathbb{P}_0$ of $\mathbb{P}\mathcal{E} \longrightarrow X$, σ_0 is a section of the hyperplane bundle: hence Z_{σ_0} is either a hyperplane in $\mathbb{P}_0$ or the whole of $\mathbb{P}_0$. But Z_{σ_0} contains the geometric fibre of $Q(\mathcal{E})$, whence Corollary (3.4) implies $Z_{\sigma_0} = \mathbb{P}_0$. Therefore, since $\mathbb{P}\mathcal{E}$ is reduced, $Z_\sigma = \mathbb{P}\mathcal{E}$, i.e. $\sigma = 0$, and hence $\pi_* \mathcal{O}(1)$ is torsion-free. $\square$

(iii) We conclude §3 with a construction which sheds some light on the geometry of $Q(\mathcal{E})$ and how it is related to the Fitting subschemes of $\mathcal{E}$. The key point is contained in the following lemma.

Lemma (3.6)

Suppose $\mathcal{E}$ is a torsion-free $\mathcal{O}_X$-Module which is not locally-free. Let $q = \max(e(x) \mid x \in X)$ (where $e(\ )$ is the fibre-dimension function defined p. 13), and F_q the q^{th} Fitting subscheme of $\mathcal{E}$ in X. Suppose $\rho : X' \longrightarrow X$ is the blow-up of X with centre F_q, and let $\mathcal{E}'$ be the torsion-free $\mathcal{O}_{X'}$-Module $\mathcal{E}' = \rho^*\mathcal{E}/\tau(\rho^*\mathcal{E})$. Then $\dim(k(x) \otimes \mathcal{E}') < q$ for all $x \in X'$.

Proof.

Let $x \in X$ be a point such that $e(x) = q$. Then, as shown (p.), there is an affine open neighbourhood, $U = \operatorname{Spec} A$, of x and an A-module $M = \mathcal{E}(U)$, where M has a presentation

$$A^p \xrightarrow{\ \Psi\ } A^q \xrightarrow{\ \varphi\ } M \longrightarrow 0,$$

i.e. M is generated by q elements $a_1, \ldots, a_q$.

Ψ is represented by a matrix (Ψ_{ij}) where $\Psi_{ij} \in \underline{x}$, $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant p$. (Here $\underline{x}$ is the point x as ideal in A). Since $\mathcal{E}$ is not locally-free, $\Psi_{ij} = 0$ for some ij. Then $F_q \cap U$ is defined by the ideal $I = I_q$ generated by $\{\Psi_{ij} \mid 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant p\}$.

The open subset of X', $\rho'(U) = \operatorname{Proj}(\sum_{n=0}^{\infty} I^n)$ is covered by the affine open sets $U_{ij} = \operatorname{Spec} A_{ij}$, where $\Psi_{ij} \neq 0$ and

$$A_{ij} = \left(\left(\sum_{n=0}^{\infty} I^n\right)_{\Psi_{ij}}\right)_0 .$$

Here, and below, it is necessary to distinguish between an element $\sigma \in I$ considered as a member of $(\sum_{n=0}^{\infty} I^n)_0 = A$ and considered as a member of $(\sum_{n=0}^{\infty} I^n)_1 = I$. We hope to avoid notational confusion by writing $\bar{\sigma}$ when the latter is meant. Thus, $\bar{\Psi}_{ij} \in (\sum_{n=0}^{\infty} I^n)_1$.

Assume for example that $\Psi_{11} \neq 0$ and consider $M_{11} = A_{11} \otimes_A M$ as a module over A_{11}. M_{11} has a presentation:

$$A_{11}^p \longrightarrow A_{11}^q \xrightarrow{1\otimes\varphi} M_{11} \longrightarrow 0.$$

Then $b = \sum_{i=1}^{q} (\bar{\Psi}_{i1}/\bar{\Psi}_{11}) \otimes a_i$ is a torsion element in M_{11}: for $\Psi_{11}b = \sum \Psi_{i1} \otimes a_i = 1 \otimes \sum \Psi_{i1}a_i = 0 \in M_{11}$. Note that Ψ_{11} is not a zero-divisor in A_{11} (although it may be in A); for there is a non-(zero-divisor) $\sigma \in I$ and then $\sigma = \Psi_{11}(\bar{\sigma}/\bar{\Psi}_{11})$ in A_{11} implies Ψ_{11} is not a zero-divisor in A_{11}.

Now consider $M_{11}/A_{11}b$: this has a presentation with matrix

$$\left(\begin{array}{c|c} 1 & \\ \bar{\Psi}_{21}/\bar{\Psi}_{11} & \\ \vdots & \Psi_{ij} \\ \bar{\Psi}_{q1}/\bar{\Psi}_{11} & \end{array}\right)$$

Hence the q^{th} Fitting ideal $I_q(M_{11}/A_{11}b) = A_{11}$. Since $M_{11}/\tau M_{11}$ is a quotient of $M_{11}/A_{11}b$, it follows that $I_q(M_{11}/\tau M_{11}) = A_{11}$.

Applying this argument to each U_{ij} yields $F_q(\mathcal{E}') = \phi$, which is the assertion of the lemma. $\square$

We have produced a birational surjective morphism, $\rho : X' \longrightarrow X$ and a torsion-free $\mathcal{O}_{X'}$-Module $\mathcal{E}' = \rho^*\mathcal{E}/\tau(\rho^*\mathcal{E})$ such that $\mathcal{E}, \mathcal{E}'$ have the same generic rank and $\max_{x\in X} \dim \mathcal{E}_{(x)} > \max_{x\in X'} \dim \mathcal{E}'_{(x)}$, provided $\mathcal{E}$ is not locally-free. Thus we may define recursively, morphisms $\rho^{(n-1)}: X^{(n)} = (X^{(n-1)})' \longrightarrow X^{(n-1)}$ and $\mathcal{O}_{X^{(n)}}$-Modules $\mathcal{E}^{(n)} = (\mathcal{E}^{(n-1)})' = \rho^{(n-1)*}\mathcal{E}^{(n-1)}/\tau(\rho^{(n-1)*}\mathcal{E}^{(n-1)})$ until when $n = M$ say, $\max_{x\in X^{(n)}} \dim \mathcal{E}^{(n)}_{(x)} =$ generic rank of $\mathcal{E} = N$, and so $\mathcal{E}^{(M)}$ is locally free.

Let ρ be the composite

$$\tilde{X} = X^{(M)} \longrightarrow X^{(M-1)} \longrightarrow \cdots \longrightarrow X' \longrightarrow X$$

and $\mathcal{F} = \mathcal{E}^{(M)}$, the locally-free $\mathcal{O}_{\tilde{X}}$-Module of rank N, the generic rank of $\mathcal{E}$. There is a canonical epimorphism $\rho^*\mathcal{E} \longrightarrow \mathcal{F}$. By Proposition (1.2), this induces a closed embedding $\mathbb{P}\mathcal{F} \hookrightarrow \mathbb{P}(\rho^*\mathcal{E}) = \mathbb{P}\mathcal{E} \times \tilde{X}$ and hence a commutative diagram

$$
(3.7) \qquad
\begin{tikzcd}
 & \mathbb{P}\mathcal{E}\times\tilde{X} \arrow[dr] & \\
\mathbb{P}\mathcal{F} \arrow[ur, hook] \arrow[rr, "\tilde{\rho}"'] \arrow[d] & & \mathbb{P}\mathcal{E} \arrow[d] \\
\tilde{X} \arrow[rr, "\rho"] & & X
\end{tikzcd}
$$

$\mathbb{P}\mathcal{F}$ is an integral scheme and $\tilde{\rho}$ is a proper morphism: therefore $\tilde{\rho}(\mathbb{P}\mathcal{F})$ is closed and must be $Q(\mathcal{E})$. Thus $\tilde{\rho}$ induces a proper surjective birational morphism $\mathbb{P}\mathcal{F} \longrightarrow Q(\mathcal{E})$. We may say, using the suggestive traditional language, that $Q(\mathcal{E})$ is covered by fibres (projective spaces of dimension N-1)lying over points of X and infinitely-near points of the Fitting subschemes of $\mathcal{E}$ in X.

Appendix: Universal Sheaves for Projective Geometry.

Throughout this appendix we work over an algebraically closed field k. We propose to define some coherent sheaves, (over projective spaces), which we dignify with the title "universal", and their associated Fitting subschemes, the "universal Fitting subschemes", in view of the following property. If X is a projective variety over k (in the sense of Serre(FAC) i.e. a reduced scheme of finite type over k which embeds as a closed subscheme of some projective space $\mathbb{P}(k^n)$), and $\mathcal{E}$ a coherent $\mathcal{O}_X$-Module, there are embeddings $X \hookrightarrow \mathbb{P}_N$ (for some N depending on $\mathcal{E}$) such that the Fitting subschemes of $\mathcal{E}$ in X are the restrictions of the universal Fitting subschemes in $\mathbb{P}_N$.

Suppose V,W are finite-dimensional vector spaces over k of dimensions p,q respectively. The space of k-linear maps, Hom(V,W), is naturally isomorphic to $\check{V}\otimes W$ Let P be the projective space $\mathbb{P}((\check{V}\otimes W)^{\vee})$ with projection $\pi : P \longrightarrow \operatorname{Spec} k$. We have the canonical epimorphism (Prop.(1.2)(iii)):

$$\pi^*((\check{V}\otimes W)^{\vee}) \longrightarrow \mathcal{O}_P(1) \longrightarrow 0$$

Dualising and tensoring with $\mathcal{O}(1)$ yields:

$$0 \longrightarrow \mathcal{O}_P \longrightarrow (\pi^*(\check{V}\otimes W))(1).$$

There are natural isomorphisms:

$$(\pi^*(\check{V}\otimes W))(1) \cong (\pi^*V)^{\vee}\otimes(\pi^*W)(1) \cong \mathrm{Hom}_{\mathcal{O}_P}(\pi^*V, (\pi^*W)(1))$$

Therefore there is induced a map

$$k = \Gamma\mathcal{O}_P \longrightarrow \Gamma(\mathrm{Hom}_{\mathcal{O}_P}(\pi^*V,(\pi^*W)(1))),$$

and we let the image of 1 be ξ in $\Gamma(\mathrm{Hom}_{\mathcal{O}_P}(\pi^*V,(\pi^*W)(1)))$. Thus ξ is a homomorphism of $\mathcal{O}_P$-Modules, $\pi^*V \xrightarrow{\xi} (\pi^*W)(1)$.

Definition:

The universal $\mathcal{O}_P$-Module $\mathcal{U}(V,W)$ is the cokernel of ξ :

$$\pi^*V \xrightarrow{\xi} (\pi^*W)(1) \longrightarrow \mathcal{U}(V,W) \longrightarrow 0$$

We have made this definition independent of any choice of bases in the vector spaces V,W. However, it may be that the construction is more perspicuous when described in terms of such bases. Let $\{v_1,\dots,v_p\}$, $\{w_1,\dots,w_q\}$ be bases of V,W respectively, $\{\check{v}_1,\dots,\check{v}_p\}$ the dual basis of V, and $L = \check{V} \otimes W$. Then $P = \mathrm{Proj}(S_k\check{L}) = \mathrm{Proj}\ k[\check{\varphi}_{ij}]$ where φ_{ij} $i=1,\dots,q; j=1,\dots,p$ is the basis of L given by $\varphi_{ij} = \check{v}_j \otimes w_i$. The canonical epimorphism $\pi^*\check{L} \longrightarrow \mathcal{O}_P(1)$ is défined over $D_+(\check{\varphi}_{ij})$ by $\check{\varphi}_{\alpha\beta} \longmapsto (\check{\varphi}_{\alpha\beta}/\check{\varphi}_{ij})u_{ij}$ where u_{ij} is the canonical generator of $\mathcal{O}(1)(D_+(\check{\varphi}_{ij}))$. Hence, taking the dual homomorphism as explained above, ξ is defined over $D_+(\check{\varphi}_{ij})$ by

$$1 \longmapsto \left[\left(\begin{matrix} \cdot & \cdot \\ \check{\varphi}_{\alpha\beta}/\check{\varphi}_{ij} \\ \cdot & \cdot \end{matrix}\right)\right] \otimes u_{ij} \quad \in \quad L \otimes_k \mathcal{O}_P(1)(D_+(\check{\varphi}_{ij}))$$

Here a section $\sum_{ij} \varphi_{ij} \otimes \psi_{ij}$ in $L \otimes_k \mathcal{O}_P(1)(D_+(\check{\varphi}_{ij}))$ is written in matrix form:

$$\begin{pmatrix} \psi_{11} & \cdot & \cdot & \psi_{1p} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \psi_{q1} & \cdot & \cdot & \psi_{qp} \end{pmatrix}$$

by virtue of the isomorphism $\pi^*L(1) = \mathrm{Hom}_P(\pi^*V, (\pi^*W)(1))$.

A closed point t in $D_+(\check{\varphi}_{ij})$ is a one-dimensional subspace of $\check{\check{L}} = L$, and if t is generated by $\sum t_{\alpha\beta}\varphi_{\alpha\beta}$, we may write the homogeneous coordinates $(t_{\alpha\beta})$ of t in matrix form

$$\begin{pmatrix} t_{11} & \cdot & \cdot & t_{1p} \\ \vdots & \cdot & \cdot & \vdots \\ \cdot & \cdot & \cdot & \cdot \\ t_{q1} & \cdot & \cdot & t_{qp} \end{pmatrix} \quad \text{where} \quad t_{ij} \neq 0.$$

Thus if $\bar{t}_{\alpha\beta} = t_{\alpha\beta}/t_{ij}$ (so $(\bar{t}_{\alpha\beta})$ are inhomogeneous coordinates for t) the geometric residue of ξ over t can be identified with the linear morphism $V \longrightarrow W$ whose matrix with respect to the chosen bases is:

$$\begin{pmatrix} \bar{t}_{11} & \cdot & \cdot & \bar{t}_{1p} \\ \vdots & & & \vdots \\ \bar{t}_{q1} & \cdot & \cdot & \bar{t}_{qp} \end{pmatrix}$$

In this formulation it becomes clear that the Fitting subschemes of $\mathcal{U}(V,W)$ in P are (projective) determinantal varieties: F_n is defined by the graded ideal in $k[\tilde{\varphi}_{11}, \cdots, \tilde{\varphi}_{qp}]$ generated by the $(q-n+1) \times (q-n+1)$ subdeterminants of the matrix

$$\begin{pmatrix} \tilde{\varphi}_{11} & \cdots & \tilde{\varphi}_{1p} \\ \vdots & & \vdots \\ \tilde{\varphi}_{q1} & \cdots & \tilde{\varphi}_{qp} \end{pmatrix}.$$

Such varieties have been widely studied. For example, the results of (8) imply that the singularities of F_n are normal. In fact F_{n+1} is the singular locus of F_n.

The rest of this appendix is devoted to the constructions that demonstrate the universality of the sheaves $\mathcal{U}$.

Let X be a projective variety over k, with $\mathcal{O}(1)$ a very ample invertible $\mathcal{O}_X$-Module, and $\mathcal{E}$ a coherent sheaf on X. For $m \gg 0$, $\mathcal{E}(m) = \mathcal{E} \otimes \mathcal{O}(1)^m$ is generated by its global sections: i.e. writing $\mathcal{E}' = \mathcal{E}(m)$, if $(\Gamma\mathcal{E}')_X$ is the $\mathcal{O}_X$-Module $\pi^*\Gamma\mathcal{E}'$ where $\pi : X \longrightarrow \operatorname{Spec} k$ is the structure morphism and $\Gamma\mathcal{E}' = \pi_*\mathcal{E}'$ (a vector space over k), then the canonical morphism $(\Gamma\mathcal{E}')_X \longrightarrow \mathcal{E}'$ is an epimorphism. (4) Let $\mathcal{K}$ be the kernel, so we have the short exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow (\Gamma\mathcal{E}')_X \longrightarrow \mathcal{E}' \longrightarrow 0.$$

For $n \gg 0$, $\mathcal{K}(n)$ is generated by global sections and we have the exact sequence;

$$\begin{tikzcd}
\Gamma(\mathcal{K}(n))_X \arrow[rr, "\tilde{\varphi}"] \arrow[dr] & & (\Gamma\mathcal{E}')_X(n) \arrow[r] & \mathcal{E}'(n) \arrow[r] & 0 \\
& (n) \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$\tilde{\varphi}$ defines a global section φ of the locally free $\mathcal{O}_X$-Module $\operatorname{Hom}_{\mathcal{O}_X}(\Gamma(\mathcal{K}(n))_X, (\Gamma\mathcal{E}')_X(n)) \cong (\Gamma(\mathcal{K}(n))_X^\vee \otimes_{\mathcal{O}_X} \Gamma(\mathcal{E}')_X(n))$. Note that if m is large enough then $\mathcal{K}_x \neq 0$ and $\varphi(x) \neq 0$ for all closed points $x \in X$.

In general if $\mathcal{F}$ is a locally free $\mathcal{O}_X$-Module, a never-zero section defines an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{G}$ is locally-free. Hence the dual sequence

$$0 \longrightarrow \mathcal{G}^\vee \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{O}_X \longrightarrow 0$$

is exact and induces an embedding $X \hookrightarrow \mathbb{P}(\mathcal{F}^\vee)$.

In this way φ determines $X \longrightarrow \mathbb{P}((\Gamma(\mathcal{K}(n))_X^\vee \otimes (\Gamma\mathcal{E}')_X(n))^\vee)$. We have isomorphisms of schemes

$$\mathbb{P}((\Gamma(\mathcal{K}(n))_X^\vee \otimes_{\mathcal{O}_X} (\Gamma\mathcal{E}')_X(n))^\vee) \cong \mathbb{P}((\Gamma(\mathcal{K}(n))_X^\vee \otimes_{\mathcal{O}_X} (\Gamma\mathcal{E}')_X)) \cong$$

$X \times \mathbb{P}_N$ where $\mathbb{P}_N$ is the projective space $\mathbb{P}((\Gamma(\mathcal{K}(n))^\vee \otimes_k (\Gamma\mathcal{E}'))^\vee)$. By projecting onto $\mathbb{P}_N$ we have defined a morphism $j_n : X \longrightarrow \mathbb{P}_N$. Concerning this the main result is the following theorem. The structure of the proof is modelled on Hartshorne's treatment of the projective embedding induced by an ample line bundle ((4) Chapter I § 3).

Theorem (A.1)

There is an integer η (depending on $\mathcal{E}$) such that if $n > \eta$, j_n is a closed embedding.

Proof.

Let $x \in X$ be a closed point. We first show there is a positive integer n_x, depending on x, such that if $n \geqslant n_x$, then j_n is an isomorphism at x: i.e. j_n separates x from other points of X and from its infinitely near points. This means (i) if $y \in X$, $y \neq x$, then $j_n(y) \neq j_n(x)$, and (ii) if $\mathfrak{J}_{n,x} : \mathcal{O}_{\mathbb{P},j(x)} \longrightarrow \mathcal{O}_{X,x}$ is the induced local homomorphism of local rings, $\mathfrak{J}_{n,x}$ is surjective. By the usual Nakayama's lemma argument (ii) is equivalent to:

(ii)' the induced morphism of cotangent spaces,

$$\underline{m}_{\mathbb{P},j(x)} \Big/ \underline{m}^2_{\mathbb{P},j(x)} \longrightarrow \underline{m}_x \Big/ \underline{m}^2_x \qquad \text{is surjective.}$$

Let I_x be the sheaf of ideals in $\mathcal{O}_X$ defined by the closed subscheme $\{x\}$. To establish (i), let n be a positive integer such that $I_x(n')$ is generated by its global sections when $n' > n_o$. Then we have the following commutative diagram with exact top row and exact columns:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathcal{K}(n)) \otimes I_x(n') \arrow[r] \arrow[d] & \Gamma(\mathcal{K}(n))_X(n') \arrow[d] \\
 & \mathcal{K}(n) \otimes I_x(n') \arrow[r] \arrow[d] & (\mathcal{K}(n))(n') = \mathcal{K}(n+n') \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

If $y \in X$, $y \neq x$, there exists $s \in \Gamma(\mathcal{K}(n)) \otimes \Gamma(I_x(n'))$ such that $s(y) \neq 0$. If $\bar{s}$ is the image in $\Gamma\mathcal{K}(n+n')$ of s, then $\bar{s}(y) \neq 0$, while $\bar{s}(x) = 0$. It follows that $j_{n+n'}(x) \neq j_{n+n'}(y)$, as required.

We now consider (ii)'. Let $\sigma \in (\mathrm{Hom}(\Gamma(\mathcal{K}(n)), \Gamma\mathcal{E}'))$ be such that $\sigma(\varphi(x)) \neq 0$, and let $S_{\sigma,n} = S_k(\mathrm{Hom}(\Gamma(\mathcal{K}(n)),\Gamma\mathcal{E}')^{\vee})_{(\sigma)}$. If $e_1, \cdots, e_t$ is a basis for $\mathrm{Hom}(\Gamma(\mathcal{K}(n)),\Gamma\mathcal{E}')^{\vee}$ then a typical element of $S_{\sigma,n}$ can be written $p(e_1, \cdots, e_t)/\sigma^r$ where p is a homogeneous polynomial of degree r in t variables. Spec $S_{\sigma,n}$ is an open affine neighbourhood of $j_n(x)$ in $\mathbb{P}_N$. There is an affine neighbourhood, Spec A, of x in X such that $j_n(\mathrm{Spec}\ A) \subset \mathrm{Spec}\ S_{\sigma,n}$, and $j_n|\mathrm{Spec}\ A$ is induced by the k-algebra morphism $S_{\sigma,n} \longrightarrow A$, "evaluation at φ". Localising at x, suppose now that $A = \mathcal{O}_{X,x}$: we want to show that $S_{\sigma,n} \longrightarrow \mathcal{O}_{X,x}/\underline{m}_x$ is surjective.

We have a short exact sequence

$$0 \longrightarrow I_x \longrightarrow \mathcal{O}_X \longrightarrow k_x \longrightarrow 0$$

where k_x is the sheaf supported at x with stalk k. Let $\mathcal{K}_0 = \mathcal{K} \otimes k_x$: $\mathcal{K}_0$ is a non-trivial linear subspace of $\Gamma\mathcal{E}'$. Let H be a hyperplane of $\Gamma\mathcal{E}'$ such that $\Gamma\mathcal{E}' = \mathcal{K}_0 + H$. Then the composite

$$\mathcal{K} \longrightarrow (\Gamma\mathcal{E}')_X \longrightarrow (\Gamma\mathcal{E}'/H)_X$$

induces a surjection, $\mathcal{K}_0 \longrightarrow \Gamma\mathcal{E}'/H$, on the geometric stalks at x. By Nakayama's lemma, $\mathcal{K} \otimes \mathcal{O}_x/I_x^2 \longrightarrow (\Gamma\mathcal{E}'/H) \otimes \mathcal{O}_x/I_x^2$ is surjective.

To justify the next step of the proof we insert the following

<u>Lemma</u> (A.2)

If $\mathcal{F}$ is a coherent sheaf on X and is generated by its global sections, then for a closed point $x \in X$ the k-linear map

$$(*) \qquad \Gamma(\mathcal{F}(1)) \longrightarrow \mathcal{F}(1)_x/\underline{m}_x^2$$

is surjective.

Proof.

Note that if (*) is surjective, and $\mathcal{G}$ is a coherent quotient of $\mathcal{F}$ then (*) remains surjective when $\mathcal{F}$ is replaced by $\mathcal{G}$. $\mathcal{F}$ is generated by its global sections and is therefore a quotient of $\bigoplus_n \mathcal{O}_X$ for some n. So it suffices to prove the lemma for the case $\mathcal{F} = \mathcal{O}_X$; but this case is clear since $\mathcal{O}_X(1)$ is very ample. $\square$

We apply the lemma to the sheaf $\mathcal{K}(n)$ which is generated by its global sections: so $\Gamma(\mathcal{K}(n+1)) \longrightarrow \mathcal{K}(n+1)_x/\underline{m}_x^2$ is surjective.

Now we have a commutative diagram

$$\begin{tikzcd}
\Gamma(\mathcal{K}(n+1)) \arrow[r] \arrow[d] & \mathcal{K}(n+1)_x/\underline{m}_x^2 \arrow[r] & (\Gamma\mathcal{E}'/H)\otimes\left(\mathcal{O}_X(n+1)_x/\underline{m}_x^2\right) \\
\Gamma(\mathcal{K}(n+1))\otimes\mathcal{O}_{X,x} \arrow[rr, "\varphi_x"] & & (\Gamma\mathcal{E}')\otimes\mathcal{O}_X(n+1)_x \arrow[u]
\end{tikzcd}$$

where the two maps in the top row are surjective. (The second is obtained from $\mathcal{K}\otimes\mathcal{O}_X/I_x^2 \longrightarrow (\Gamma\mathcal{E}'/H)\otimes\mathcal{O}_X/I_x^2$ by taking tensor product with $\mathcal{O}(n+1)_x$.) Suppose we fix isomorphisms $\Gamma\mathcal{E}'/H \cong k$ (k-linear) and $(\Gamma\mathcal{E}'/H)\otimes\mathcal{O}_X(n+1)_x/\underline{m}_x^2 \cong \mathcal{O}_{X,x}/\underline{m}_x^2$ (an isomorphism of modules over $\mathcal{O}_{X,x}$). The above diagram becomes:

$$\begin{tikzcd}
\Gamma(\mathcal{K}(n+1)) \arrow[r, "w"] \arrow[d] & \mathcal{O}_{X,x}/\underline{m}_x^2 \\
\Gamma(\mathcal{K}(n+1))\otimes\mathcal{O}_{X,x} \arrow[r] & (\Gamma\mathcal{E}')\otimes\mathcal{O}_X(n+1)_x \cong (\Gamma\mathcal{E}')\otimes\mathcal{O}_{X,x} \arrow[u]
\end{tikzcd}$$

Let $a \in \mathcal{O}_{X,x}/\underline{m}_x^2$ and let $\bar{a} \in \Gamma(\mathcal{K}(n+1))$ be chosen such that $w(\bar{a}) = a$. We define an element $\bar{\bar{a}} \in \mathrm{Hom}(\Gamma(\mathcal{K}(n+1)),\Gamma\mathcal{E}')^\vee$ by $\bar{\bar{a}}(f) = pf(\bar{a})$ where $f \in \mathrm{Hom}(\Gamma(\mathcal{K}(n+1)),\Gamma\mathcal{E}')$ and p is the map $\Gamma\mathcal{E}' \longrightarrow \Gamma\mathcal{E}'/H \xrightarrow{\cong} k$. Thus $\bar{\bar{a}} \in S_{\sigma,n+1}$ and the above commutative square expresses the fact that $\bar{\bar{a}}$ maps onto a under the morphism $S_{\sigma,n+1} \longrightarrow \mathcal{O}_{X,x}/\underline{m}_x^2$. This morphism is therefore surjective and it follows that the induced morphism

$$\underline{m}_{\mathbb{P},j}(x)\big/\underline{m}^2_{\mathbb{P},j}(x) \longrightarrow \underline{m}_x\big/\underline{m}_x^2 \quad \text{is surjective.}$$

Thus we have property (ii)'.

To complete the proof of the theorem it suffices to note that for each n the set $U_n = \{x \in X \mid j_n$ is an isomorphism at $x\}$ is Zariski open in X. (This follows from EGA I 6.5.4). We have shown that $X = \bigcup_n U_n$ and $U_n \subset U_{n+1} \subset \ldots$. Therefore $X = \bigcup_{n=1}^{\eta} U_n$ for some η. Hence j_n is an isomorphism at every point x of X if $n > \eta$. Therefore j_n is a closed embedding if $n > \eta$. $\square$

If $V = \Gamma(\mathcal{K}(n))$ and $W = \Gamma\mathcal{E}'$, $j = j_n$ embeds X in $P = \mathbb{P}((\check{V}\otimes W)^{\vee})$. Let $\mathcal{U} = \mathcal{U}(V,W)$ be the universal sheaf on P. In order to justify our use of the term "universal" it remains to check that $j^*\mathcal{U} \cong \mathcal{E}'(n) \ \ (= \mathcal{E}(m+n))$.

As stated (p. 31) the exact sequence

$$V_X \xrightarrow{\tilde{\varphi}} W_X(n) \longrightarrow \mathcal{E}'(n) \longrightarrow 0$$

induces an embedding $s : X \longrightarrow \mathbb{P}(\check{L}\otimes \mathcal{O}_X(-n))$ $(L = \check{V}\otimes W)$ (this by Proposition (1.2)(ii) where $\mathcal{E}' = \mathcal{O}_X$). Hence $s^*\mathcal{O}_{\mathbb{P}(\check{L}_X(-n))}(1) = \mathcal{O}_X$.

Let π_1, π_2 be the projections:

$$\begin{array}{ccc} & \mathbb{P}(\check{L}\otimes\mathcal{O}_X(-n)) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ X & & P = \mathbb{P}(L) \end{array}$$

Then $\mathcal{O}_{\mathbb{P}(\check{L}_X(-n))}(1) = \pi_1^*\mathcal{O}_X(-n)\otimes\pi_2^*\mathcal{O}_P(1)$, where $\mathcal{O}_P(1)$ is the fundamental invertible sheaf on P. Therefore $j^*\mathcal{O}_P(1) = s^*\pi_2^*\mathcal{O}_P(1) = s^*(\mathcal{O}_{\mathbb{P}(\check{L}_X(-n))}(1)\otimes\pi_1^*\mathcal{O}_X(n)) = s^*\pi_1^*\mathcal{O}_X(n) = \mathcal{O}_X(n)$.

Therefore applying j^* to the sequence which defines $\mathcal{U}$:

$$V_P \xrightarrow{\xi} W_P(1) \longrightarrow \mathcal{U} \longrightarrow 0$$

gives

$$V_X \longrightarrow W_X(n) \longrightarrow j^*\mathcal{U} \longrightarrow 0,$$

and it may be seen that $j^*\xi = \tilde{\varphi}$; therefore $j^*\mathcal{U} \cong \mathcal{E}'(n) = \mathcal{E}(m+n)$.

Thus any coherent $_X$-Module on a projective variety is, up to tensoring with an invertible sheaf (a"Serre twist"), induced from a universal Module. Since tensoring with an invertible sheaf does not alter the projective fibred scheme and the Fitting subschemes, (i.e. $\mathbb{P}(\mathcal{E}(m+n)) \cong \mathbb{P}\mathcal{E}$ and $F_n(\mathcal{E}(m+n)) = F_n(\mathcal{E})$) all projective fibred schemes of coherent sheaves on X and the Fitting subschemes are induced from the universal examples by restriction.

Chapter II The homomorphism $\alpha : \mathcal{E} \longrightarrow \pi_* \mathcal{O}(1)$.

Introduction.

It is natural to ask how much is lost in passing from a Module to its projective fibred scheme. We shall show that, given a projective fibred scheme with projection $\pi : \mathbb{P}\mathcal{E} \longrightarrow X$ and its fundamental invertible sheaf $\mathcal{O}(1)$, then in certain circumstances the Module $\mathcal{E}$ can be reconstructed. It is well known that if $\mathcal{E}$ is locally-free the canonical homomorphism $\alpha : \mathcal{E} \longrightarrow \pi_* \mathcal{O}(1)$ (Chapter I, Proposition (1.2)(iii)) is an isomorphism, and it is this result that we seek to extend to more general coherent sheaves. Our main results are Theorems (2.7) and (3.4).

The proofs make essential use of the vector fibred scheme construction. §1 is mainly devoted to the definition and basic properties of the construction, summarised from EGA* I 9.4, sufficient for the applications in the following sections. The main theme of the chapter, the discussion of α, begins with §2, and Theorem (2.7) is proved, under the assumption that the coherent Module $\mathcal{E}$ satisfies a condition (Definition (2.4)) of a rather technical nature. §3 discusses further this condition and gives examples of situations in which it is satisfied.

§1 Vector fibred schemes.

The vector fibred scheme associated with a quasi-coherent $\mathcal{O}_X$-Module $\mathcal{E}$ is the X-scheme, affine over X, $\pi : \mathbb{V}\mathcal{E} \longrightarrow X$ where $\mathbb{V}\mathcal{E}$ = Spec $\mathfrak{S}\mathcal{E}$. That is, if U = SpecA is an affine open in X such that $\mathcal{E}|U = \widetilde{E}$ for an A-module E then $\pi^{-1}U \cong$ Spec S where S is the symmetric algebra of E).

The principal properties of this construction are developed in EGA* I.9.4, where the results of this section may be found, with the exception of Proposition (1.3) (which pushes to a categorical conclusion ideas implicit in EGA* 1.9.4.14) and Proposition (1.4) (which we require to apply in § 2).

There is a functor (EGA* 1.9.4.8), $F_{\mathcal{E}} : \underline{Sch}^{o}_{X} \longrightarrow \underline{Ens}$, defined as follows: if $f: T \longrightarrow X$ is an X-scheme, $F_{\mathcal{E}}(T) = \mathrm{Hom}_{\mathcal{O}_T}(f^*\mathcal{E}, \mathcal{O}_T)$, and if $g : T' \longrightarrow T$ is an X-morphism of X-schemes then $F_{\mathcal{E}}(g) : F_{\mathcal{E}}(T) \longrightarrow F_{\mathcal{E}}(T')$ maps $(u : f^*\mathcal{E} \longrightarrow \mathcal{O}_T)$ to the composite $f'^*\mathcal{E} = g^*f^*\mathcal{E} \xrightarrow{g^*u} g^*\mathcal{O}_T \longrightarrow \mathcal{O}_{T'}$.

<u>Lemma (1.1)</u> (EGA* 1 Proposition 9.4.9).

For any quasi-coherent $\mathcal{O}_X$-Module $\mathcal{E}$ the functor $F_{\mathcal{E}}$ is representable by the X-scheme $\mathbb{V}\mathcal{E}$ and the $\mathcal{O}_{\mathbb{V}\mathcal{E}}$-homomorphism $\pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{V}\mathcal{E}}$ induced by the canonical homomorphism $\mathcal{E} \longrightarrow \mathfrak{S}\mathcal{E}$

Proof.

This follows from the sequence of canonical isomorphisms functorial in T (in the category of X-schemes):

$$\mathrm{Hom}_X(T, \mathbb{V}\mathcal{E}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X\text{-Alg}}(\mathfrak{S}\mathcal{E}, f_*\mathcal{O}_T)$$ (where f is the structure morphism $T \longrightarrow X$)

$\xrightarrow{\approx} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, f_*\mathcal{O}_T)$ (by the universal property of the symmetric algebra)

$\xrightarrow{\approx} \mathrm{Hom}_{\mathcal{O}_T}(f^*\mathcal{E}, \mathcal{O}_T)$ (since the functor f^* is coadjoint to f_*, (EGA 0.4.4.3.1))

$= F_{\mathcal{E}}(T)$.

The following facts about $\mathbb{V}\mathcal{E}$, quoted without proof, follow easily from the definition or from the above universal property.

Lemma (1.2) (EGA* I Proposition 9.4.11)

(i) $\mathbb{V}\mathcal{E}$ is a contravariant functor in $\mathcal{E}$ from the category of quasi-coherent $\mathcal{O}_X$-Modules to the category of affine X-schemes.

(ii) If $\mathcal{E}$ is an $\mathcal{O}_X$-Module of finite type (respectively of finite presentation), $\mathbb{V}\mathcal{E}$ is of finite type (respectively, of finite presentation) over X.

(iii) If $\mathcal{E}, \mathcal{F}$ are two quasi-coherent $\mathcal{O}_X$-Modules, $\mathbb{V}(\mathcal{E} \oplus \mathcal{F})$ is canonically isomorphic to $\mathbb{V}\mathcal{E} \times_X \mathbb{V}\mathcal{F}$.

(iv) Let $g : X' \longrightarrow X$ be a morphism: for each quasi-coherent $\mathcal{O}_X$-Module $\mathcal{E}$, $\mathbb{V}(g^*\mathcal{E})$ is canonically isomorphic to $(\mathbb{V}\mathcal{E})_{(X')} = \mathbb{V}\mathcal{E} \times_X X'$.

(v) The X-morphism $\mathbb{V}\mathcal{F} \longrightarrow \mathbb{V}\mathcal{E}$ induced by an epimorphism of quasi-coherent $\mathcal{O}_X$-Modules, $\mathcal{E} \longrightarrow \mathcal{F}$, is a closed immersion. $\square$

Remarks: (a) As a special case of (iv) let $X' = \mathrm{Spec}\, K$, for a field K: K is an extension of $k(x)$ where $x = g(X')$ and $\mathbb{V}\mathcal{E} \times_X \mathrm{Spec}\, K$ is canonically isomorphic to $\mathbb{V}(g^*\mathcal{E})$ $= \mathbb{V}(\mathcal{E}^x \otimes_{k(x)} K)$ (where $\mathcal{E}^x$ is the vector space over $k(x)$,

$\mathcal{E}_x/\underline{m}_x\mathcal{E}_x$). The K-valued points of $\mathbb{V}(\mathcal{E}^x \otimes_{k(x)} K)$ form the space $\mathrm{Hom}_{\mathrm{Spec}\,k}(\mathrm{Spec}\,K, \mathbb{V}(\mathcal{E}^x \otimes_{k(x)} K)) \cong \mathrm{Hom}_k(\mathcal{E}^x \otimes_{k(x)} K, K) = (\mathcal{E}^x \otimes_{k(x)} K)^\vee$. Thus the geometric fibre of $\mathbb{V}\mathcal{E} \dashrightarrow X$ over the K-valued point g is the K-vector space dual to $\mathcal{E}^x \otimes_{k(x)} K$, which justifies the term "vector fibred scheme".

(b) In 1.2(iv) "canonically" should be interpreted as follows: a morphism of $\mathcal{O}_X$-Modules $\varphi : \mathcal{E} \dashrightarrow \mathcal{F}$ induces the X-morphism $\mathbb{V}(\varphi) : \mathbb{V}\mathcal{F} \dashrightarrow \mathbb{V}\mathcal{E}$ and then, by change of base, an X'-morphism $\mathbb{V}\mathcal{F} \times X' \dashrightarrow \mathbb{V}\mathcal{E} \times X'$. φ also induces a morphism of $\mathcal{O}_{X'}$-Modules $g^*\varphi : g^*\mathcal{E} \dashrightarrow g^*\mathcal{F}$ and hence an X'-morphism $\mathbb{V}(g^*\mathcal{F}) \dashrightarrow \mathbb{V}(g^*\mathcal{E})$. These two X'-morphisms are compatible with the isomorphisms of 1.2(iv): i.e. we have a commutative diagram

$$\begin{array}{ccc} \mathbb{V}(g^*\mathcal{F}) & \longrightarrow & \mathbb{V}(g^*\mathcal{E}) \\ \downarrow \wr\!\!\! \parallel & & \downarrow \wr\!\!\! \parallel \\ \mathbb{V}(\mathcal{F}) \times X' & \longrightarrow & \mathbb{V}(\mathcal{E}) \times X' \end{array}$$

Similar remarks apply to 1.2(iii).

We proceed now to some categorical algebraic properties of the functor $\mathbb{V}$. Let $X[T]$ be the X-scheme $\mathbb{V}(\mathcal{O}_X)$ (EGA*1 9.4.12). If $f : Y \longrightarrow X$ is an X-scheme, $\mathcal{A}(Y)$ is the $\mathcal{O}_X$-Algebra $f_*\mathcal{O}_Y$. Then we have isomorphisms (EGA*1 9.4.13),

$$\mathrm{Hom}_X(Y, X[T]) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, f_*\mathcal{O}_Y) \cong \Gamma(X, \mathcal{A}(Y)),$$

and $\Gamma(X, \mathcal{A}(Y))$ has a ring structure, functorial in Y, which displays $X[T]$ as a ring X-scheme (i.e. a ring object in the category of X-schemes). If $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$-Module then $\mathrm{Hom}_X(Y, \mathbb{V}\mathcal{E}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}_{(Y)}, \mathcal{O}_Y) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{A}(Y))$

and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{A}(Y))$ is a module over the ring $\Gamma(X, \mathcal{A}(Y))$. Thus $\pi : \mathbb{W}\mathcal{E} \dashrightarrow X$ is a module X-scheme over the ring X-scheme $X[T]$.

Suppose $\mathcal{E}, \mathcal{F}$ are two quasi-coherent $\mathcal{O}_X$-Modules. Let $\varphi : \mathrm{Hom}_X(\mathbb{W}\mathcal{E}, \mathbb{W}\mathcal{F}) \dashrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{S}\mathcal{E})$ be the isomorphism defined by $\mathrm{Hom}_X(\mathbb{W}\mathcal{E}, \mathbb{W}\mathcal{F}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_X\text{-Alg.}}(\mathcal{S}\mathcal{F}, \mathcal{S}\mathcal{E}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{S}\mathcal{E})$. The canonical monomorphism $\mathcal{E} \dashrightarrow \mathcal{S}\mathcal{E}$ induces a monomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}) \dashrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{S}\mathcal{E})$ and we thereby identify $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})$ with a subgroup of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{S}\mathcal{E})$. Let $L = \varphi^{-1}(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}))$. L is characterised as a subset of $\mathrm{Hom}_X(\mathbb{W}\mathcal{E}, \mathbb{W}\mathcal{F})$ by the following result.

Proposition (1.3).

Suppose $\alpha \in \mathrm{Hom}_X(\mathbb{W}\mathcal{E}, \mathbb{W}\mathcal{F})$: then $\alpha \in L$ if and only if α is a morphism of module X-schemes.

Consequently, $\mathbb{W}(\ )$ is a fully faithful contravariant functor from the category of quasi-coherent $\mathcal{O}_X$-Modules to the category of module X-schemes over the ring X-scheme $X[T]$.

Proof.

Suppose $\alpha \in L$ and $\varphi(\alpha) = \alpha' \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})$. Then for each X-scheme $f : Y \dashrightarrow X$ the following diagram commutes by virtue of the universal property (Lemma (1.1)):

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{A}Y) \arrow[r, "(-\circ\alpha')"'] \arrow[d, phantom, "\wr\|"] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{A}Y) \arrow[d, phantom, "\wr\|"] \\
\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}_{(Y)}, \mathcal{O}_Y) \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}_{(Y)}, \mathcal{O}_Y) \arrow[d, phantom, "\wr\|"] \\
\mathrm{Hom}_X(Y, \mathbb{W}\mathcal{E}) \arrow[r, "(\alpha\circ-)"] & \mathrm{Hom}_X(Y, \mathbb{W}\mathcal{F})
\end{tikzcd}$$

The top row is a $\Gamma(X, \mathcal{A}Y)$-module homomorphism: therefore α is a morphism of module X-schemes.

Conversely let $\alpha \in \mathrm{Hom}_X(\mathbb{V}\mathcal{E}, \mathbb{V}\mathcal{F})$ with α a morphism of module X-schemes. Let $\varphi(\alpha) = \alpha' \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathfrak{S}\mathcal{E})$: then for any X-scheme $f: Y \longrightarrow X$, α' induces a module homomorphism

$$\mathrm{Hom}_{\mathcal{O}_X\text{-Alg}}(\mathfrak{S}\mathcal{E}, \mathcal{A}Y) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{A}Y).$$

The module structure of $\mathrm{Hom}_{\mathcal{O}_X\text{-Alg}}(\mathfrak{S}\mathcal{E}, \mathcal{A}Y)$, defined by the isomorphism $\mathrm{Hom}_{\mathcal{O}_X\text{-Alg}}(\mathfrak{S}\mathcal{E}, \mathcal{A}Y) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{A}Y)$, is given explicitly as follows (cf. EGA*I 9.4.14): if $h, h' \in \mathrm{Hom}_{\mathcal{O}_X\text{-Alg}}(\mathfrak{S}\mathcal{E}, \mathcal{A}Y)$ and $s_1, \ldots, s_n$ are sections of $\mathcal{E}$ over an open set U in X and $t \in \Gamma(U, \mathcal{A}X)$, then

$$(h + h')(s_1 s_2 \cdots s_n) = \prod_{i=1}^{n}(h(s_i) + h'(s_i))$$

and

$$(t.h)(s_1 s_2 \cdots s_n) = t^n . \prod_{i=1}^{n} h(s_i).$$

Now let Y be the X-scheme $\mathbb{V}(i^*\mathcal{E})[T]$ where $i : U \hookrightarrow X$ is the inclusion morphism and T is an indeterminate. Then $\mathcal{A}Y = (\mathfrak{S}\mathcal{E})[T] \big| U$. If $h \in \mathrm{Hom}_{\mathcal{O}_X\text{-Alg}}(\mathfrak{S}\mathcal{E}, \mathcal{A}Y)$ we have $T.(h \circ \alpha') = (T.h) \circ \alpha'$. If $x \in \Gamma(U, \mathcal{F})$ and $\alpha'(x) = z_0 + \cdots + z_n$ where $z_i \in \Gamma(U, \mathfrak{S}\mathcal{E})_i$ then $((T.h) \circ \alpha')(x) = (T.h)(z_0 + \cdots + z_n) = \sum_{i=0}^{n} T^i h(z_i)$, and if we choose h such that h(U) is the canonical inclusion $\mathfrak{S}\mathcal{E}(U) \longrightarrow \mathfrak{S}\mathcal{E}(U)[T]$, then

$$\sum_{i=0}^{n} T^i z_i = \sum_{i=0}^{n} T z_i .$$

Therefore $z_i = 0$ if $i \neq 1$ and $\alpha' \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})$. $\square$

If K is a field and $i : \mathrm{Spec}\, K \longrightarrow X$ is a K-valued point of X with $i(\mathrm{Spec}\, K) = x \in X$, then $\mathrm{Hom}_X(\mathrm{Spec}\, K, \mathbb{V}\mathcal{E})$ is the space of K-valued points of the fibre $\pi^{-1}(x) = \mathbb{V}\mathcal{E}^x$ (i.e. the K-rational geometric fibre over x, in the terminology of EGA*I 9.4.10, and of remark (a) p.39 above): $\mathrm{Hom}_X(\mathrm{Spec}\, K, X[T]) = \Gamma(X, i_* K) = K$. Hence Proposition (1.3) confirms (and generalises) the fact that, if $\alpha \in L \subset \mathrm{Hom}_X(\mathbb{V}\mathcal{E}, \mathbb{V}\mathcal{F})$ then α induces a vector space homomorphism on each geometric fibre, $\mathrm{Hom}_X(\mathrm{Spec}\, K, \mathbb{V}\mathcal{E}) \longrightarrow \mathrm{Hom}_X(\mathrm{Spec}\, K, \mathbb{V}\mathcal{F})$.

We express this by saying, α is linear on geometric fibres. The following Proposition is a result in the converse direction, proved under the assumption that $\mathbb{V}\mathcal{E}$ is a reduced scheme; it may be considered a strengthening of Proposition (1.3) in that case.

Proposition (1.4)

If $\mathcal{E}, \mathcal{F}$ are quasi-coherent $\mathcal{O}_X$-Modules such that $\mathbb{V}\mathcal{E}$ is reduced and if $\alpha : \mathbb{V}\mathcal{E} \longrightarrow \mathbb{V}\mathcal{F}$ is a morphism over X, then α is induced by an $\mathcal{O}_X$-Module morphism $\mathcal{F} \longrightarrow \mathcal{E}$ if and only if α is linear on all geometric fibres.

Proof.

It remains to show that if $\alpha \in \mathrm{Hom}_X(\mathbb{V}\mathcal{E}, \mathbb{V}\mathcal{F})$ is linear on geometric fibres then $\varphi(\alpha) \in \mathrm{Hom}(\mathcal{F}, \mathcal{E})$, i.e. if $\varphi(\alpha)_1$ is the component of $\varphi(\alpha)$ in $\mathrm{Hom}(\mathcal{F}, \mathcal{E})$ and $\beta = \varphi(\alpha) - \varphi(\alpha)_1 \in \mathrm{Hom}(\mathcal{F}, S\mathcal{E})$ then $\beta = 0$. Assume $\beta \neq 0$ and restrict to an affine open $U = \mathrm{Spec}\, A \subset X$ such that $\mathcal{F}_U = \tilde{F}$, $\mathcal{E}_U = \tilde{E}$ and $\beta | U = b \in \mathrm{Hom}_A(F, SE)$, $b \neq 0$. Since SE is reduced there exists a prime ideal p in SE such that $b(F) \not\subset p$. Then $p \cap (SE)_0 = q$ is a prime ideal in A such that $b(F) \not\subset qSE$: therefore if $K = A_q/qA_q$, the residue field at q, b induces a non-zero homomorphism

$b_K = b \otimes 1 : F \otimes_A K \longrightarrow SE \otimes_A K = S_K(E \otimes K)$. Thus α fails to be linear on the K-rational geometric fibre over $q \in X$.

A similar argument proves the following, in a "classical" algebro-geometric context:

Proposition (1.5)

If X is a reduced scheme of finite type over an algebraically closed field K, and $\mathcal{E}, \mathcal{F}$ are coherent sheaves on X such that $\mathbb{V}\mathcal{E}$ is reduced and $\alpha : \mathbb{V}\mathcal{E} \longrightarrow \mathbb{V}\mathcal{F}$ is an X-morphism, then α is induced by an $\mathcal{O}_X$-Module homomorphism $\mathcal{F} \longrightarrow \mathcal{E}$ if and only if α is linear on each geometric (K-rational) fibre over a closed point of X.

Proof.

With notation as in the proof of (1.4), E is of finite type over A and SE is of finite type over K: it follows that the intersection of the maximal ideals of SE is 0, and so the ideal p can be assumed to be maximal. Then $q = p \cap (SE)_0$ is maximal in A, (for if $q \underset{\neq}{<} q' \underset{\neq}{\triangleleft} A$, the ideal of SE generated by q' and p is proper and strictly larger than p). Thus q is a closed (K-rational) point of X, and the result follows as in (1.4). $\square$

<u>Remark</u> on the relations between locally-free coherent sheaves, their associated fibred schemes, and algebraic vector bundles:

If X is a scheme defined over a field k, and $F \longrightarrow X$ is an algebraic vector bundle with fibre k^r, then the sheaf of germs of algebraic sections of F, $\mathcal{F}$, is a locally-free $\mathcal{O}_X$-Module of rank r. The k-valued points of the vector fibred scheme $\mathbb{V}\mathcal{F}$ form a vector bundle <u>dual</u> to F.

On $\mathbb{P}\mathcal{F}$, $\mathcal{O}(1)$ is the sheaf of germs of sections of the "hyperplane bundle", the line bundle (dual to the Hopf bundle) whose global sections are linear forms on each fibre of $\mathbb{P}\mathcal{F}$. Hence $\mathbb{V}(\mathcal{O}(1))$ should be regarded as the (algebraic analogue of) the Hopf bundle. If $\pi : \mathbb{P}\mathcal{F} \longrightarrow X$ is the projection then $\pi^* \mathcal{F} \twoheadrightarrow \mathcal{O}(1)$ induces an embedding $\mathbb{V}(\mathcal{O}(1)) \hookrightarrow \pi^* \mathbb{V}\mathcal{F}$ and projecting onto $\mathbb{V}\mathcal{F}$ gives a morphism $\mathbb{V}(\mathcal{O}(1)) \longrightarrow \mathbb{V}\mathcal{F}$, which can be identified with blowing-up the zero section in $\mathbb{V}\mathcal{F}$.

§2.

In this section and the next we show that, under certain conditions, the canonical $\mathcal{O}_X$-Module morphism $\alpha : \mathcal{E} \longrightarrow \pi_*\mathcal{O}(1)$ is an isomorphism. In order to construct an inverse morphism $\beta : \pi_*\mathcal{O}(1) \longrightarrow \mathcal{E}$, when it exists, we use the vector fibred scheme construction described in §1. That is, we shall first construct a morphism of schemes $v : \mathbb{V}\mathcal{E} \longrightarrow \mathbb{V}(\pi_*\mathcal{O}(1))$ and show that $v = \mathbb{V}(\beta)$ for some β. Sufficient conditions for the existence of β are made the hypothesis of Theorem(2.7) below, and further elucidated in the following §3.

Our results are proved under appropriate "finiteness" assumptions on schemes and sheaves. Thus, in the sequel X will be a reduced noetherian scheme and all $\mathcal{O}_X$-Modules considered will be coherent.

Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-Module such that the projection $\pi : \mathbb{P}\mathcal{E} \longrightarrow X$ is surjective; i.e. $\mathcal{E}$ is not a torsion sheaf. The canonical epimorphism $\pi^*\mathcal{E} \longrightarrow \mathcal{O}(1)$ induces a morphism of vector fibred schemes over $\mathbb{P}\mathcal{E}$, $h' : \mathbb{V}(\mathcal{O}(1)) \longrightarrow \mathbb{V}(\pi^*\mathcal{E})$. This is a closed immersion by Lemma (1.2)(v). $\mathbb{V}(\pi^*\mathcal{E})$ is canonically isomorphic with $\mathbb{P}\mathcal{E} \times_X \mathbb{V}\mathcal{E}$ ((1.2)(iv)): composing h' with the projection onto $\mathbb{V}\mathcal{E}$ defines a morphism of schemes h.

$$\begin{tikzcd} \mathbb{V}(\mathcal{O}(1)) \arrow[r] \arrow[drr, "h"'] & \mathbb{V}(\pi^*\mathcal{E}) \cong & \mathbb{P}\mathcal{E} \times_X \mathbb{V}\mathcal{E} \arrow[d] \\ & & \mathbb{V}\mathcal{E} \end{tikzcd}$$

Similarly the epimorphism $\pi^*\pi_*\mathcal{O}(1) \longrightarrow \mathcal{O}(1)$ induces a morphism $u : \mathbb{V}(\mathcal{O}(1)) \longrightarrow \mathbb{V}(\pi_*\mathcal{O}(1))$.

<u>Lemma (2.1)</u>

(i) The following diagram is commutative:

$$\begin{tikzcd} \mathbb{P}\mathcal{E} \arrow[r, "j"'] \arrow[d, "\pi"'] & \mathbb{V}(\mathcal{O}(1)) \arrow[r] \arrow[d, "h"] & \mathbb{P}\mathcal{E} \arrow[d, "\pi"] \\ X \arrow[r, "i"] & \mathbb{V}\mathcal{E} \arrow[r] & X \end{tikzcd} \qquad (2.1.1)$$

(here i,j are the immersions onto the respective zero sections).

Given a k-valued point of X, $* = \operatorname{Spec} k \longrightarrow X$ (k, a field) the diagram induced from (2.1.1) by restriction to fibres over $*$ is canonically isomorphic with

$$\begin{tikzcd} \mathbb{P}(V) \arrow[r] \arrow[d] & \check{H} \arrow[r] \arrow[d, "h_*"] & \mathbb{P}(V) \arrow[d] \\ * \arrow[r] & \check{V} \arrow[r] & * \end{tikzcd}$$

where V is the vector space over k, $\mathcal{E}_{(*)}$, H is the fundamental line bundle (= "hyperplane bundle") on $\mathbb{P}(V)$ and $h_* : \check{H} \longrightarrow \check{V}$ collapses the zero section of H.

(ii) The restriction of $h : \mathbb{V}(\mathcal{O}(1)) - j(\mathbb{P}\mathcal{E}) \longrightarrow \mathbb{V}\mathcal{E} - i(X)$ is an isomorphism.

Proof.

(i) The commutativity of (2.1.1) is clear from the definition of h. Further, construction of (2.1.1) commutes with change of base X; for, taking account of remark (b) after Lemma (1.2), if $f : X' \longrightarrow X$ is an X-scheme then the morphism (defined similarly to h) $h' : \mathbb{V}(\mathcal{O}(1)') \longrightarrow \mathbb{V}(\mathcal{E}')$, where $\mathcal{E}' = f^*\mathcal{E}$ and $\mathcal{O}(1)'$ is the fundamental invertible sheaf on $\mathbb{P}(\mathcal{E}')$, is canonically identifiable with the pull-back of h by the projection $\mathbb{V}(\mathcal{E}') \longrightarrow \mathbb{V}(\mathcal{E})$. In particular when $X' = \operatorname{Spec} k$ the second assertion of (i) follows from remark (a) after Lemma (1.2) and the identification of h in the case $\mathcal{E} = V$, a vector space over k:

$$\begin{tikzcd} \check{H} \arrow[r] \arrow[dr, "h"'] & \check{V} \times \mathbb{P}(V) \arrow[d] \\ & \check{V} \end{tikzcd}$$

(ii) Let $U = \operatorname{Spec} A$ be an affine open subset of X such that $\mathcal{E}|U = \tilde{E}$ for an A-module E. Then $\mathbb{V}(\mathcal{E}|U) - i(U)$ is covered by affine open sets $\operatorname{Spec}((S_A E)_\sigma)$ where $\sigma \in E$ $(\sigma \neq 0)$. (As in Chap.I §1 the subscript σ means "localisation away from σ" : $(S_A E)_\sigma$ is a graded ring with component of degree 0 denoted by $(S_A E)_{(\sigma)}$.)

$$h^{-1}(\operatorname{Spec}(S_A E)_\sigma) = \operatorname{Spec} R_\sigma \quad \text{where}$$

$$R_\sigma = ((S_A E)_{(\sigma)}[t])_t = (S_A E)_{(\sigma)}[t\,,\, t^{-1}].$$

The restriction of h to $\operatorname{Spec} R_\sigma$ corresponds to the A-algebra homomorphism $(S_A E)_\sigma \longrightarrow (S_A E)_{(\sigma)}[t\,,\, t^{-1}]$ uniquely determined by $x \longmapsto (x/\sigma)t$ when $x \in E$. (In particular $\sigma \longmapsto t$, and so $1/\sigma \longmapsto t^{-1}$.) This homomorphism is evidently surjective, and if $p/\sigma^d \longrightarrow 0$ then $\sigma^r p = 0$ for some r, so $p/\sigma^d = 0$ and the homomorphism is injective, therefore an isomorphism. $\square$

Lemma (2.2)

u is constant on each fibre $h_0^{-1}(x)$, $x \in \mathbb{V}(\pi_* \mathcal{O}(1))$.

Proof.

Either $x \in \mathbb{V}\mathcal{E} - i(X)$ and $h_0^{-1}(x)$ has a single point or $x \in i(X)$ and $h_0^{-1}(x)$ is a fibre of $\pi : \mathbb{P}\mathcal{E} \dashrightarrow X$ mapped by u_0 to the point $\pi(h^{-1}(x))$. $\square$

It follows that there is a unique (set-theoretic) map $v_0 : \mathbb{V}\mathcal{E} \dashrightarrow \mathbb{V}(\pi_* \mathcal{O}(1))$ such that $(v_0)(h_0) = u_0$. We want to assert that in fact there is a morphism of X-schemes $v : \mathbb{V}\mathcal{E} \dashrightarrow \mathbb{V}(\pi_* \mathcal{O}(1))$ with $vh = u$.

$$\begin{tikzcd}
\mathbb{V}(\mathcal{O}(1)) \arrow[d, "h"'] \arrow[dr, "u"] & \\
\mathbb{V}\mathcal{E} \arrow[r, dashed, "v"'] & \mathbb{V}(\pi_* \mathcal{O}(1))
\end{tikzcd}$$

For this purpose there is the following lemma (EGA **II** 8.11.1):

Lemma (2.3)

Let U, V be two schemes, $h = (h_o, \lambda) : U \longrightarrow V$ a surjective morphism. We suppose that

1) $\lambda : \mathcal{O}_V \longrightarrow h_* \mathcal{O}_U = (h_o)_* \mathcal{O}_U$ is an isomorphism;

2) the underlying space of V is the quotient space of the underlying space of U by the relation $R : h_o(x) = h_o(y)$ (which condition is always satisfied when the morphism h is open or closed or, a fortiori, proper).

Then, for any scheme W, the map $\mathrm{Hom}(V,W) \longrightarrow \mathrm{Hom}(U,W)$, which takes a morphism $v : V \longrightarrow W$ to the morphism $u = vh$ $(u : U \longrightarrow W)$, is a bijection from Hom(V,W) to the set of morphisms $u = (u_o, \mu) : U \longrightarrow W$ such that u_o be constant on each fibre $h_o^{-1}(x)$. $\square$

Note that our morphism $h : \mathbb{V}(\mathcal{O}(1)) \longrightarrow \mathbb{V}\mathcal{E}$ is surjective (by virtue of Lemma (2.1) and our assumption that $\pi : \mathbb{P}\mathcal{E} \longrightarrow X$ is surjective), and satisfies condition 2): in fact h is a projective morphism in the sense of EGA II Definition(5.5.2):

A morphism of schemes $h : X \longrightarrow Y$ is <u>projective</u> if there is a quasi-coherent $\mathcal{O}_Y$-Module $\mathcal{E}$ of finite type such that X is Y-isomorphic to a closed subscheme of $\mathbb{P}\mathcal{E}$.

EGA II Theorem (5.5.3): any projective morphism is proper.

We delay until §3 of this chapter the discussion of when h satisfies condition 1), so we make the following convenient definition:

Definition (2.4)

A coherent $\mathcal{O}_X$-Module $\mathcal{E}$ satisfies <u>condition (A)</u> if the morphism $\lambda : \mathcal{O}_{\mathbb{V}\mathcal{E}} \longrightarrow h_* \mathcal{O}_{\mathbb{V}\mathcal{O}(1)}$ is an isomorphism.

Throughout the remainder of this section $\mathcal{E}$ will be a coherent $\mathcal{O}_X$-Module which satisfies condition (A).

Proposition (2.5)

(i) There is a unique morphism $v : \mathbb{V}\mathcal{E} \longrightarrow \mathbb{V}(\pi_* \mathcal{O}(1))$ such that $vh = u$.

(ii) v is an X-morphism and is linear on geometric fibres.

Proof.

(i) This now follows immediately from Lemmas (2.2) and (2.3).

(ii) We have the commutative diagram:

$$\begin{tikzcd}
\mathbb{V}\mathcal{O}(1) \arrow[rr, "u'"] \arrow[dr] \arrow[ddd, "h"] & & \mathbb{V}(\pi^*\pi_*\mathcal{O}(1)) \arrow[dl] \arrow[ddd] \\
& \mathbb{P}\mathcal{E} \arrow[d, "\pi"] & \\
& X & \\
\mathbb{V}\mathcal{E} \arrow[ur] & & \mathbb{V}(\pi_*\mathcal{O}(1)) \arrow[ul]
\end{tikzcd}$$

and a morphism $v : \mathbb{V}\mathcal{E} \longrightarrow \mathbb{V}(\pi_* \mathcal{O}(1))$ completing the commutative square

$$\begin{tikzcd}
\mathbb{V}\mathcal{O}(1) \arrow[r] \arrow[d, "h"] & \mathbb{V}(\pi^*\pi_*\mathcal{O}(1)) \arrow[d] \\
\mathbb{V}\mathcal{E} \arrow[r, "v"'] & \mathbb{V}(\pi_*\mathcal{O}(1))
\end{tikzcd}$$

The commutativity of the triangle

$$\begin{tikzcd}
& X & \\
\mathbb{V}\mathcal{E} \arrow[ur] \arrow[rr, "v"'] & & \mathbb{V}(\pi_*\mathcal{O}(1)) \arrow[ul]
\end{tikzcd}$$

follows by elementary diagram chasing, since h is an epimorphism. Thus v is an X-morphism.

To show that v is linear on geometric fibres we make use of the following construction. Suppose given a commutative square of schemes

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "h"'] & S \arrow[d, "g"] \\
Y \arrow[r, "i"'] & X
\end{tikzcd}$$

and an $\mathcal{O}_S$-Module $\mathcal{K}$. Then there is an $\mathcal{O}_Y$-Module homomorphism

$\omega : i^*g_*\mathcal{K} \longrightarrow h_*f^*\mathcal{K}$ defined as follows:

(h^*, h_*) is a pair of adjoint functors (EGA 0 4.3) and there is the natural isomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{O}_Y}(i^*g_*\mathcal{K},\ h_*f^*\mathcal{K}) \xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathcal{O}_Z}(h^*i^*g_*\mathcal{K},\ f^*\mathcal{K})$$

[Recall that if $u : i^*g_*\mathcal{K} \longrightarrow h_*f^*\mathcal{K}$ is an $\mathcal{O}_Y$-Module homomorphism then $\varphi(u) : h^*i^*g_*\mathcal{K} \longrightarrow f^*\mathcal{K}$ is the composite of $h^*(u) : h^*i^*g_*\mathcal{K} \longrightarrow h^*h_*f^*\mathcal{K}$ and the canonical homomorphism (4.4.3.3) $h^*h_*f^*\mathcal{K} \longrightarrow f^*\mathcal{K}$.] To define ω it suffices to specify $\varphi(\omega) : h^*i^*g_*\mathcal{K} \longrightarrow f^*\mathcal{K}$; but $h^*i^*g_*\mathcal{K} = f^*g^*g_*\mathcal{K}$ and we put $\varphi(\omega) = f^*(e)$ where e is the canonical homomorphism $g^*g_*\mathcal{K} \longrightarrow \mathcal{K}$.

Returning to the proof, suppose $i : \mathrm{Spec}\ k \longrightarrow X$ is a k-valued point of X and $i^{\bullet}\mathbb{P}\mathcal{E} = \mathbb{P}\mathcal{E} \times \mathrm{Spec}\ k$ is the corresponding geometric fibre of $\mathbb{P}\mathcal{E}$; so we have the diagram

$$\begin{tikzcd}
i^{\bullet}\mathbb{P}\mathcal{E} \arrow[r, "i'"] \arrow[d, "\pi'"] & \mathbb{P}\mathcal{E} \arrow[d, "\pi"] \\
\mathrm{Spec}\ k \arrow[r, "i"] & X
\end{tikzcd}$$

There is the morphism $\omega : i^*\pi_*\mathcal{O}(1) \longrightarrow \pi'_*i'^*\mathcal{O}(1)$ (as defined above) and $\pi'_*i'^*\mathcal{O}(1)$ is canonically isomorphic to $\mathcal{E}\otimes k$. Thus ω induces $\mathbb{V}(\omega) : \mathbb{V}(\mathcal{E}\otimes k) \longrightarrow \mathbb{V}(i^*\pi_*\mathcal{O}(1))$, and we claim that $\mathbb{V}(\omega)$ is precisely the morphism induced by $v : \mathbb{V}\mathcal{E} \longrightarrow (\pi_*\mathcal{O}(1))$ when restricted to the geometric fibre. To check this it suffices, (since $\mathbb{V}(i'^*\mathcal{O}(1)) \longrightarrow \mathbb{V}(i^*\mathcal{E}) \cong \mathbb{V}(\pi'_*i'^*\mathcal{O}(1))$ is a birational map) to verify that the following triangle commutes:

$$\begin{tikzcd}
i^{\bullet}\mathbb{V}(\mathcal{O}(1)) = \mathbb{V}(i'^*\mathcal{O}(1)) \arrow[d] \arrow[dr, "i^{\bullet}(u)"] & \\
\mathbb{V}(i^*\mathcal{E}) \cong \mathbb{V}(\pi'_*i'^*\mathcal{O}(1)) \arrow[r] & \mathbb{V}(i^*\pi_*\mathcal{O}(1)) = i^{\bullet}\mathbb{V}(\pi_*\mathcal{O}(1))
\end{tikzcd}$$

This diagram can be augmented as follows:

$$\begin{tikzcd}
\mathbb{V}(i'^*\mathcal{O}(1)) \arrow[rr, "i^\bullet(\tilde{u})"] \arrow[dr] \arrow[ddr] \arrow[rr, phantom, shift right=6ex, "(*)"] & & \mathbb{V}(i'^*\pi^*\pi_*\mathcal{O}(1)) \arrow[d, equal] \\
& \mathbb{V}(\pi'^*\pi'_*i'^*\mathcal{O}(1)) \arrow[r] \arrow[d] & \mathbb{V}(\pi'^*i^*\pi_*\mathcal{O}(1)) \arrow[d] \\
& \mathbb{V}(\pi'_*i'^*\mathcal{O}(1)) \arrow[r] & \mathbb{V}(i^*\pi_*\mathcal{O}(1))
\end{tikzcd}$$

and it remains to show that the upper triangle (*) is commutative. (*) is induced by the following diagram of Module homomorphisms:

$$\begin{tikzcd}
i'^*\mathcal{O}(1) & i'^*\pi^*\pi_*\mathcal{O}(1) \arrow[l] \arrow[d, equal] \\
\pi'^*\pi'_*i'^*\mathcal{O}(1) \arrow[u] & \pi'^*i^*\pi_*\mathcal{O}(1) \arrow[l, "\pi'^*(\omega)"']
\end{tikzcd}$$

which commutes by definition of ω. Thus our claim is justified. In particular it follows that $i^\bullet(v) : \mathbb{V}(\mathcal{E}\otimes k) \longrightarrow \mathbb{V}(i^*\pi_*\mathcal{O}(1))$ is k-linear. $\square$

<u>Corollary (2.6)</u>

If $\mathbb{V}\mathcal{E}$ is reduced, there exists a unique $\mathcal{O}_X$-Module homomorphism $\beta : \pi_*\mathcal{O}(1) \longrightarrow \mathcal{E}$ such that $v = \mathbb{V}(\beta)$.

Proof.

This follows from Proposition (1.4). $\square$

With the following theorem we reach our objective in this section.

<u>Theorem (2.7)</u>

Suppose X is a reduced noetherian scheme and $\mathcal{E}$ is a coherent, torsion-free $\mathcal{O}_X$-Module [satisfying condition (A)] such that $\mathbb{V}\mathcal{E}$ is reduced. Then $\beta: \pi_*\mathcal{O}(1) \longrightarrow \mathcal{E}$ is inverse to the canonical homomorphism $\alpha : \mathcal{E} \longrightarrow \pi_*\mathcal{O}(1)$.

Proof.

First note that if $\mathbb{V}\mathcal{E}$ is reduced then so is $\mathbb{P}\mathcal{E}$. In fact, if S is a reduced graded ring and $x \in S_1$ then $S_{(x)}$ is reduced, (using the notation of Chapter I §1): for if $\frac{t}{x^n}$ is a nilpotent element of $S_{(x)}$, $\left(\frac{t}{x^n}\right)^d = 0$ for some $d > 0$; hence $x^r t^d = 0$ for some r, $(xt)^{r+d} = 0$ and so $xt = 0 \in S$ and $\frac{t}{x^n} = 0$.

We have the following diagram:

$$\begin{tikzcd}
 & \mathbb{V}(\mathcal{O}(1)) \arrow[dl, "h"'] \arrow[d, "u"] \arrow[dr, "h"] & \\
\mathbb{V}\mathcal{E} \arrow[r, "v = \mathbb{V}(\beta)"'] & \mathbb{V}(\pi_* \mathcal{O}(1)) \arrow[r, "\mathbb{V}(\alpha)"'] & \mathbb{V}\mathcal{E}
\end{tikzcd} \tag{2.7.1}$$

[The right-hand triangle commutes since $\pi^*\mathcal{E} \longrightarrow \mathcal{O}(1)$ factors $\pi^*\mathcal{E} \longrightarrow \pi^*\pi_*\mathcal{O}(1) \longrightarrow \mathcal{O}(1)$ inducing a commutative digram:

$$\begin{tikzcd}
\mathbb{V}(\mathcal{O}(1)) \arrow[r, "u'"] \arrow[d, "u"'] \arrow[drr, "h"] & \mathbb{V}(\pi^*\pi_*\mathcal{O}(1)) \arrow[r] \arrow[dl] & \mathbb{V}(\pi^*\mathcal{E}) \arrow[d] \\
\mathbb{V}(\pi_*\mathcal{O}(1)) \arrow[rr, "\mathbb{V}(\alpha)"'] & & \mathbb{V}\mathcal{E}
\end{tikzcd}$$
.]

Applying Lemma (2.3) and using again the fact that $\mathcal{E}$ satisfies condition (A), (2.7.1) implies $\mathbb{V}\alpha \circ \mathbb{V}\beta = \mathrm{id}_{\mathbb{V}\mathcal{E}}$; i.e. $\mathbb{V}(\beta \circ \alpha) = \mathrm{id}_{\mathbb{V}\mathcal{E}}$. Hence, by Proposition (1.4), $\beta \circ \alpha = \mathrm{id}_{\mathcal{E}}$. It follows that α is a monomorphism onto a direct summand of $\pi_*\mathcal{O}(1)$. Let $\mathcal{K}$ be a direct complement of $\alpha(\mathcal{E})$, so that $\pi_*\mathcal{O}(1) \cong \mathcal{E} \oplus \mathcal{K}$.

In the case $\mathcal{E}$ is locally-free, $\alpha : \mathcal{E} \longrightarrow \pi_*\mathcal{O}(1)$ is an isomorphism. This well-known fact (cf. EGA III Proposition 2.1.15) is easily verified as follows: suppose $\mathcal{E}_U = \tilde{E}$ where $U = \operatorname{Spec} A$ and E is a free A-module with basis $\{x_1, \cdots, x_n\}$. Then $S_A E$ is the polynomial algebra $A[x_1, \cdots, x_n]$, and each $\tilde{\sigma} \in \Gamma(U, \pi_*\mathcal{O}(1)) = \Gamma(\pi^{-1}U, \mathcal{O}(1))$ is a family $\sigma_i \in (S_A E)_{(x_i)}$ $(i = 1, \cdots, n)$ such that $x_i \sigma_i = x_j \sigma_j \in (S_A E)_{(x_i x_j)}$. We may write $\sigma_i = p_i / x_i^m$ where each p_i is a polynomial of degree m and homogeneous.

Then $x_i x_j^m p_i = x_j x_i^m p_j$ implies x_i^{m-1} divides p_i. Consequently we may put $m = 1$ and $\sigma_i = t/x_i$ where $t \in A[x_1, \cdots, x_n]_1 \cong E$, and t is independent of i. The map $\tilde{\sigma} \longmapsto t$, $\Gamma(U, \pi_* \mathcal{O}(1)) \longrightarrow E$, is inverse to $\alpha_U : E \longrightarrow \Gamma(U, \pi_* \mathcal{O}(1))$ so α_U is an isomorphism.

Returning to the general case, there is an open dense set U in X such that $\mathcal{E}_U$ is locally-free, and therefore $\mathcal{K}_U = 0$. Consequently $\mathcal{K}$ is a torsion sheaf. But if $\mathcal{E}$ is torsion-free, Proposition (3.5) of Chapter I implies that $\pi_* \mathcal{O}(1)$ is torsion-free, and therefore $\mathcal{K} = 0$. $\square$

§3

We take up the question left open in §2 and discuss some conditions sufficient for the coherent $\mathcal{O}_X$-Module $\mathcal{E}$ to satisfy condition(A). In this connection the notion of a normal scheme is relevant. We recall the definitions:

An integral domain A is <u>normal</u> if A is integrally closed in its field of fractions.

A scheme is <u>normal</u> if it is reduced and irreducible, and has a covering by affine open sets, Spec A_i, where each A_i is a normal integral domain.

The following result is standard and well-known,(e.g EGAIII 4.3.12).

<u>Lemma (3.1)</u>

If $f: P \longrightarrow Q$ is a surjective proper birational morphism where P,Q are integral locally noetherian schemes and Q is normal, then $\mathcal{O}_Q \longrightarrow f_* \mathcal{O}_P$ is an isomorphism.

Proof.

Since f is proper $f_* \mathcal{O}_P$ is a coherent, in particular a finite $\mathcal{O}_Q$-Module. Since f is birational $f_* \mathcal{O}_P$ is a sub-Algebra of $\mathcal{R}(Q)$, the sheaf of rational functions on Q. Hence the normality of Q implies $\mathcal{O}_Q = f_* \mathcal{O}_P$. $\square$

If $\mathcal{E}$ is a coherent $\mathcal{O}_X$-Module such that $\mathbb{V}\mathcal{E}$ is a normal, locally noetherian scheme, then $\mathbb{P}\mathcal{E}$, and hence $\mathbb{V}\mathcal{O}(1)$, is integral and $\mathbb{V}\mathcal{O}(1) \longrightarrow \mathbb{V}\mathcal{E}$ is a surjective proper birational morphism: consequently Lemma (3.1) implies condition (A) is satisfied in this case.

In order to state a conclusion which depends on properties of $\mathbb{P}\mathcal{E}$ rather than $\mathbb{V}\mathcal{E}$ we show that the normality of $\mathbb{P}\mathcal{E}$ is equivalent to that of $\mathbb{V}\mathcal{E}$.

Theorem (3.2)

Assume $\mathcal{E}$ is a coherent $\mathcal{O}_X$-Module where X is a locally noetherian scheme. Then $\mathbb{P}\mathcal{E}$ is a normal scheme if and only if $\mathbb{V}\mathcal{E}$ is a normal scheme.

Proof.

Suppose $\mathbb{V}\mathcal{E}$ is normal: then since $\mathbb{V}(\mathcal{O}(1)) - j(\mathbb{P}\mathcal{E}) = \mathbb{V}\mathcal{E} - i(X)$ (Lemma (2.1)(ii)), $\mathbb{V}(\mathcal{O}(1)) - j(\mathbb{P}\mathcal{E})$ is normal. Clearly this implies $\mathbb{V}(\mathcal{O}(1))$ is normal, and to deduce that $\mathbb{P}\mathcal{E}$ is normal we apply the following well-known result:

A ring B is normal if and only if the polynomial ring $B[x]$ is normal.

Conversely suppose $\mathbb{P}\mathcal{E}$ is normal. Using our usual notation (as in the proof of Lemma (2.1)(ii)) and writing $S = S_A E$, we have $S_{(\sigma)}$ is a normal ring for each $\sigma \in S_1 = E$. Let K_σ be the field of fractions of $S_{(\sigma)}$ Then K, the field of fractions of S, is a simple transcendental extension of K_σ: $K_\sigma \longrightarrow K_\sigma(t) = K$. We regard S as a subring of $S_{(\sigma)}[t]$ via the monomorphism defined by $x \longmapsto (x/\ n)t^n$ for $x \in S_n$. Thus we have the sequence of subalgebras of K:

$$S \subset S_{(\sigma)}[t] \subset S_{(\sigma)}[t,t^{-1}] \cong S_\sigma \subset K.$$

Since $S_{(\sigma)}$ is integrally-closed in K_σ, $S_{(\sigma)}[t]$ is integrally-closed in $K_\sigma(t) \cong K$. If $y \in K$ and y is intgral over S then, a fortiori, y is integral over $S_{(\sigma)}[t]$. Therefore $y \in S_{(\sigma)}[t] \subset S_\sigma$. Hence $y \in S_\sigma$ for all $\sigma \in S_1 \cong E$, consequently $y \in S$ and S is integrally-closed in K. That is, S is normal and thus $\mathbb{V}\mathcal{E}$ is normal. □

Corollary (3.3)

If $\mathbb{P}\mathcal{E}$ is normal then $\mathcal{E}$ satisfies condition (A). □

We may now state as a consequence of Theorem (2.7),

Theorem (3.4)

Suppose X is a reduced noetherian scheme and $\mathcal{E}$ is a coherent torsion-free $\mathcal{O}_X$-Module such that $\mathbb{P}\mathcal{E}$ is a normal scheme. Then the canonical homomorphism $\alpha : \mathcal{E} \longrightarrow \pi_* \mathcal{O}(1)$ is an isomorphism. □

As an application of Theorem (3.4) we prove the following:

Theorem (3.5)

Suppose X is an integral noetherian scheme and $\mathcal{E}$ is a torsion-free coherent $\mathcal{O}_X$-Module of generic rank N, such that $\mathbb{P}\mathcal{E}$ is a normal scheme. Then there exists a proper surjective birational morphism $\rho : \tilde{X} \longrightarrow X$ and a locally-free $\mathcal{O}_{\tilde{X}}$-Module of rank N, $\mathcal{F}$, such that $\mathcal{E} \cong \rho_* \mathcal{F}$.

Proof.

The construction described in Chapter I§3(iii) gives a commutative diagram

$$\begin{tikzcd} \mathbb{P}\mathcal{F} \arrow[r, "\tilde{\rho}"'] \arrow[d, "\pi'"] & \mathbb{P}\mathcal{E} \arrow[d, "\pi"] \\ \tilde{X} \arrow[r, "\rho"'] & X \end{tikzcd}$$

where ρ, $\tilde{\rho}$ are proper birational surjective morphisms and $\mathcal{F}$ is locally-free of rank N. Note that since $\mathbb{P}\mathcal{E}$ is normal, in particular integral, $\mathbb{P}\mathcal{E} = Q(\mathcal{E})$. Let $\mathcal{O}_{\mathcal{E}}(1)$, $\mathcal{O}_{\mathcal{F}}(1)$ be the fundamental invertible sheaves on $\mathbb{P}\mathcal{E}$, $\mathbb{P}\mathcal{F}$ respectively. Then $\mathcal{O}_{\mathcal{F}}(1) \cong \tilde{\rho}^* \mathcal{O}_{\mathcal{E}}(1)$ (by Chap. I, Proposition (1.2)(i) and (ii)). Since $\mathcal{F}$ is locally-free $\pi'_* \mathcal{O}_{\mathcal{F}}(1) \cong \mathcal{F}$. Since $\mathbb{P}\mathcal{E}$ is normal, $\tilde{\rho}_* \mathcal{O}_{\mathbb{P}\mathcal{F}} \cong \mathcal{O}_{\mathbb{P}\mathcal{E}}$ (Lemma (3.1)), and $\tilde{\rho}_* \mathcal{O}_{\mathbb{P}\mathcal{F}}(1) \cong \tilde{\rho}_* \tilde{\rho}^* \mathcal{O}_{\mathcal{E}}(1) \cong \mathcal{O}_{\mathcal{E}}(1) \otimes \tilde{\rho}_* \mathcal{O}_{\mathbb{P}\mathcal{F}} \cong \mathcal{O}_{\mathcal{E}}(1)$.

Therefore $\rho_* \mathcal{F} \cong \rho_* \pi'_* \mathcal{O}_{\mathcal{F}}(1) \cong \pi_* \tilde{\rho}_* \mathcal{O}_{\mathcal{F}}(1) = \pi_* \mathcal{O}_{\mathcal{E}}(1)$

$= \mathcal{E}$, by theorem (3.4). $\square$

The interest of such theorems depends upon the prevalence or otherwise of sheaves with normal projective fibred schemes. Here are some examples:

(1) Sheaves $\mathcal{E}$ with smooth $\mathbb{P}\mathcal{E}$. In the following chapter we shall investigate such sheaves (on a smooth variety defined over a field) and show that there do indeed exist non-trivial examples.

(2) The result of blowing-up a normal scheme with centre a regularly embedded subscheme is a normal scheme.

(3) If $\mathcal{E}$ is a coherent Module with $\mathbb{P}\mathcal{E}$ normal and $\mathcal{F}$ is a locally-free coherent Module then $\mathbb{P}(\mathcal{E} \oplus \mathcal{F})$ is normal. [For if $\mathbb{P}\mathcal{E}$ is normal, $\mathbb{V}\mathcal{E}$ is normal, hence $\mathbb{V}(\mathcal{E} \oplus \mathcal{F})$ is normal and therefore $\mathbb{P}(\mathcal{E} \oplus \mathcal{F})$ is normal.] In contrast if $\mathbb{P}\mathcal{E}$ is smooth and $\mathcal{E}$ is not locally-free then $\mathbb{P}(\mathcal{E} \oplus \mathcal{F})$ is not smooth.

We close the chapter with a single example to show how α may fail to be an isomorphism if normality conditions are not satisfied. Consider the following example of a non-normal isolated singularity in a complex surface ((13) p.54). $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^4$ is the morphism $\Phi(u,v) = (w,x,y,z)$ where $w=v$, $x=uv$, $y=u^3v$, $z=u^4v$, so the image of Φ is the surface $V = \operatorname{Spec} A$ where

$$A = \mathbb{C}[w,x,y,z] \,/\, \langle wz-xy,\ x^3-yw^2 \rangle$$

Then V is not normal at 0, since, in the language of the theory of complex analytic spaces, u^2v is a function weakly holomorphic but not holomorphic at 0 on V. In fact

$(*)$ $u^2v = x^2/w = yw/x = y^2/z = zx/y$ on V wherever the terms are defined. But if I is the maximal ideal $\langle w,x,y,z\rangle$ in A of the point 0 in V, then $(*)$ shows that u^2v defines a section of $\mathcal{O}(1)$ over $\mathbb{P}(\tilde{I})$ which is not in the image of α.

Chapter III Smooth Projective Fibred Schemes.

Throughout the following chapter X will be a scheme defined over a field k, locally of finite type and smooth over k.

§1. A smoothness criterion for $\mathbb{P}\mathcal{E}$.

We consider the question of smoothness for $\mathbb{P}\mathcal{E}$ where $\mathcal{E}$ is a coherent $\mathcal{O}_X$-Module.

In the first place we may assume, without essential loss of generality, that k is algebraically closed: for if $\bar{k}$ is the algebraic closure of k, $\tilde{X} = X \times \operatorname{Spec} \bar{k}$, $\bar{\mathcal{E}} = \mathcal{E} \otimes \mathcal{O}_{\tilde{X}}$, then $\mathbb{P}\bar{\mathcal{E}} \cong \mathbb{P}\mathcal{E} \times \tilde{X} \cong \mathbb{P}\mathcal{E} \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ and $\mathbb{P}\bar{\mathcal{E}}$ is smooth over $\bar{k}$ if and only if $\mathbb{P}\mathcal{E}$ is smooth over k, (since $k \longrightarrow \bar{k}$ is faithfully flat).

Assume then k is algebraically closed: we proceed to formulate the smoothness question in terms of local generators and relations for $\mathcal{E}$ in a neighbourhood of a closed (i.e. k-rational) point of X. Each such point $x \in X$ has an affine open neighbourhood $U = \operatorname{Spec} A$ where A is a regular noetherian k-algebra such that the restriction of $\mathcal{E}$ to U can be resolved:

$$\mathcal{O}_U^p \longrightarrow \mathcal{O}_U^q \longrightarrow \mathcal{E} \longrightarrow 0 \qquad \text{for some } p,q.$$

This situation is expressible in a number of equivalent ways as follows. Let $\mathcal{E}(U) = E$, so E is an A-module with resolution

$$A^p \xrightarrow{f} A^q \longrightarrow E \longrightarrow 0 .$$

If f is represented by a matrix (f_{ij}) $(1 \leq i \leq q, 1 \leq j \leq p)$, with entries in A, (i.e. $f(a_1, \cdots, a_p) = (b_1, \cdots, b_q)$ where $b_i = \sum_{j=1}^{p} f_{ij} a_j$), we have a k-algebra homomorphism $k[x_{11}, \cdots x_{ij}, \cdots x_{qp}] \longrightarrow A$ defined by $x_{ij} \longmapsto f_{ij}$. Equivalently, we have a morphism of affine schemes, $\varphi : U \longrightarrow \mathbb{A}^{pq} = \operatorname{Spec} k[x_{11}, \cdots x_{ij}, \cdots x_{qp}]$. Clearly φ determines $\mathcal{E}$ over U, and is determined by $\mathcal{E}$ once a choice of resolution has been made.

The vector space over k of closed points of $\mathbb{A}^{pq}$ is here identified with $L(k^p, k^q)$, the space of k-linear maps $k^p \longrightarrow k^q$, or with $M_{p,q}(k)$, the space of p×q matrices over k.

Let χ be the matrix of indeterminates $\begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{q1} & \cdots & x_{qp} \end{pmatrix}$

Let J_r be the ideal in $k[x_{11}, \ldots, x_{qp}]$ generated by the $(r+1) \times (r+1)$ subdeterminants of χ. Let Γ_r be the closed subscheme of $\mathbb{A}^{pq}$ defined by the ideal J_r. Note that $\varphi^{-1}\Gamma_r$ is the (q-r)th Fitting subscheme of $\mathcal{E}_U$. By an elementary lemma of linear algebra, a matrix (with entries in a field) has rank not greater than r if and only if every $(r+1) \times (r+1)$ subdeterminant vanishes. Consequently the closed points of Γ_r are just those matrices over k of rank $\leqslant$ r. We have $0 = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_r \subset \Gamma_{r+1} \cdots \subset \mathbb{A}^{pq} = \Gamma_{\min(p,q)}$ where Γ_r is a closed subscheme of Γ_{r+1}. In fact Γ_{r-1} is the singular locus of Γ_r and $\Gamma_r - \Gamma_{r-1}$ is smooth of dimension $r(p+q-r)$. For these facts and other properties of the determinantal varieties Γ_r, see e.g. (9) and (8).

Returning to the principal object of study, we have defined the morphism $\varphi : U \longrightarrow \mathbb{A}^{pq}$. If the generic rank of $\mathcal{E}$, $\dim_K(E \otimes_A K)$ (where K is the field of fractions of A), is q-r, then $\varphi(U) \subset \Gamma_r$. We seek conditions (of transversality type) on φ necessary and sufficient for $\mathbb{P}\mathcal{E}$ to be smooth in a neighbourhood of a point $y \in \pi^{-1}(x)$. We shall apply the Jacobian criterion for smoothness in the following version.

Lemma (1.1) (Jacobian criterion)

Suppose X is a locally noetherian scheme over k, Y is a closed subscheme with J its sheaf of ideals, x is a point of Y, such that X is smooth at x and $\dim_x X = n$. Then the following are equivalent:

(i) Y is smooth at x and $\dim_x Y = n-p$;

(ii) $\dim_x Y \geqslant n-p$ and there exist elements $g_1, \cdots, g_p$ in J_x such that $dg_1, \ldots, dg_p$ are linearly independent in $\Omega^1_X(x)$.

Further, if Y is smooth at x and $\dim_x Y = n-p$, then $g_1, \cdots, g_m \in J_x$ generate J_x if and only if $\operatorname{rank}(dg_1, \cdots, dg_m) = p$.

(A useful reference for the Jacobian condition and related matters is (1) Chapter VII §5; the above is contained in Theorem 5.8 and Corollary 5.9).

$\mathbb{P}\mathcal{E}_U$ is defined as a subscheme of $U \times \mathbb{P}_{q-1} = \operatorname{Proj} A[z_1, \cdots, z_q]$ by the graded ideal of $A[z_1, \cdots, z_q]$ generated by $F_1, \ldots, F_p$ where $F_j = \sum_{i=1}^{q} f_{ij} z_i$. By suitable choice of bases we can assume $\varphi(x)$ is the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & I_{r'} \end{pmatrix}$$

where $I_{r'}$ is the $r' \times r'$ identity matrix, $r' = \operatorname{rank} f(x) = q - \dim_{k(x)}(\mathcal{E} \otimes k(x))$. That is, $f_{ij}(x) = \begin{cases} 1 & \text{if } i > q-r' \text{ and } j = i+p-q \\ 0 & \text{otherwise} \end{cases}$.

$\mathbb{P}(\mathcal{E})_{(x)} \subset \mathbb{P}_{q-1}$ is the subspace of dimension $q-1-r'$ defined by $z_q = z_{q-1} = \cdots = z_{q-r'+1} = 0$. Thus $D_+(z_j) \cap \mathbb{P}(\mathcal{E})_{(x)} = \emptyset$ if $j > q-r'$.

We now consider $D_+(z_j) \cap \mathbb{P}(\mathcal{E})_{(x)}$ for $j \leqslant q-r'$. For convenience take $j=1$. In $D_+(z_1)$ we replace homogeneous coordinates $(z_1 : \ldots : z_q)$ by inhomogeneous coordinates $(\tilde{z}_2, \ldots, \tilde{z}_q)$ $(\tilde{z}_i = z_i/z_1, i=2, \ldots, q)$ and write $\tilde{F}_j = \sum_{i=1}^{q} f_{ij} \tilde{z}_i$, $j=1, \ldots, p$.

At this stage it is convenient to introduce parameters on X near x.. X is smooth at x and so there is a Zariski neighbourhood of x (which we suppose to be U itself, shrinking U if necessary), with an étale morphism $U \longrightarrow \mathbb{A}^n(k)$ $(n = \dim_x X)$. Equivalently there is a system of parameters $x_1, \cdots, x_n \in \underline{x}$ (where $\underline{x}$ is the maximal ideal in A of the point x), such that if $k[t_1, \cdots, t_n] \longrightarrow A$ is the k-algebra homomorphism determined by $t_i \longmapsto x_i$, then A is étale over $k[t_1, \cdots, t_n]$.

The cotangent space to X at x is the vector space $\Omega^1_X(x) = \underline{x}/\underline{x}^2$ (over the field $A/\underline{x} = k$), which has a basis $\{dx_1, \cdots, dx_n\}$ where dx_i is the residue class of x_i, mod $\underline{x}^2$. If $F \in A$, then $dF(x) \in \Omega^1_X(x)$ and we write $dF(x) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x) dx_i$, where $\frac{\partial F}{\partial x_i}(x)$ is in $A/\underline{x}$.

If $(x,z) \in U \times D_+(z_1) \subset U \times \mathbb{P}_{q-1}$, then

$$\Omega^1_{U \times \mathbb{P}_{q-1}}(x,z) \cong \Omega^1_U(x) \oplus \Omega^1_{\mathbb{P}_{q-1}}(z), \text{ with basis}$$

$\{dx_1, \cdots, dx_n, d\tilde{z}_2, \cdots, d\tilde{z}_q\}$. In terms of this basis $d\tilde{F}_j(x,z) =$

$$d\Big(\sum_{i=1}^{q} f_{ij}\tilde{z}_i\Big)(x,z) = \sum_{i=1}^{q}\left(\sum_{k=1}^{n} \frac{\partial f_{ij}}{\partial x_k}(x)\, dx_k\right) \tilde{z}_i + \sum_{i=2}^{q} f_{ij}(x) d\tilde{z}_i.$$

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{q} \frac{\partial f_{ij}}{\partial x_k}(x)\tilde{z}_i\right) dx_k + \sum_{i=2}^{q} f_{ij}(x) d\tilde{z}_i.$$

Therefore, the Jacobian matrix, J, of $(d\tilde{F}_j)_{j=1,\cdots,p}$ at (x,z) has the form:

$$J = \begin{array}{c} \begin{array}{cc} \overleftrightarrow{\;\;n\;\;} & \overleftrightarrow{\;\;q-1\;\;} \end{array} \\ \left(\begin{array}{c|c|c} & 0 & 0 \\ J' & & \\ \cline{2-3} & 0 & I_{r'} \end{array}\right) \end{array} \updownarrow p$$

where the submatrix J' has $\sum_{i=1}^{q} \frac{\partial f_{ij}}{\partial x_k}(x)\, \tilde{z}_i$ as entry in the (j,k) position.

J has rank r if and only if the matrix J'', formed of the first $p-r'$ rows of J', has rank $r-r'$. J'' may be interpreted as the matrix of the linear map, the composite of:

$$T_x \xrightarrow{D\varphi} L(k^p, k^q) \longrightarrow L(\ker f(x), k^q/\ker z).$$

Here T_x is the tangent space $(\underline{x}/\underline{x}^2)^\vee$ at x, $D\varphi$ is the derivative of φ at x, ker z is a hyperplane in k^q (strictly, $z \in \mathbb{P}_{q-1}$

$= \mathbb{P}(k^q)$ so z is a one-dimensional subspace of $(k^q)^\vee$ and $\ker z$ is the kernel of a generator of z).

Let $A_z = \{\alpha \in L(k^p, k^q) \mid \alpha(\ker f(x)) \subset \ker z\}$. Then rank $J'' = r-r'$ if and only if the linear space $D\varphi(T_x) \cap A_z$ has codimension $r-r'$ in $D\varphi(T_x)$. Thus we are led to the following result.

Lemma (1.2)

If y is a closed point in $Q(\mathcal{E})$ with $\pi(y) = x$ and $y = (x,z) \in \{x\} \times \mathbb{P}_{q-1}$, then $\mathbb{P}\mathcal{E}$ is smooth at y if and only if

$$\operatorname{codim}(D\varphi(T_x) \cap A_z,\ D\varphi(T_x)) = r-r'.$$

Note that $r-r'$ is the "excess dimension" of the fibre $\mathbb{P}\mathcal{E}_{(x)}$; i.e. $r-r' = \dim \mathbb{P}\mathcal{E}_{(x)} -$ (generic rank of $\mathcal{E}$)

Proof.

If $\operatorname{codim}(D\varphi(T_x) \cap A_z, D\varphi(T_x)) = r-r'$ then rank $J = r$. Since $\dim_y \mathbb{P}\mathcal{E} \geqslant \dim_y Q(\mathcal{E}) = (n+q-1)-r$, the Jacobian Criterion (Lemma (1.2)) implies $\mathbb{P}\mathcal{E}$ is smooth at y.

Conversely if $\mathbb{P}\mathcal{E}$ is smooth at y then $\dim_y \mathbb{P}\mathcal{E} = \dim_y Q(\mathcal{E}) = (n+q-1)-r$ and the final assertion of Lemma (1.2) implies rank $J = r$. Therefore $\operatorname{codim}(D\varphi(T_x) \cap A_z, D\varphi(T_x)) = r-r'$. $\square$

To interpret this lemma as a transversality statement we suppose that $f(x) = 0$; this is permissible since (cf. p.13) E is generated in a neighbourhood of x by $q = \dim_{k(x)}(E \otimes k(x))$ elements. Then $r' = 0$ and $A_z = \{\alpha \in L(k^p, k^q) \mid \operatorname{im}\alpha \subset \ker z\}$. Hence

$$\dim \Gamma_r - \dim(\Gamma_r \cap A_z) = r(p+q-r) - r(p+q-1-r) = r.$$

The fact that $\varphi(U) \subset \Gamma_r$, U is smooth, $\varphi(x) = 0$ and Γ_r is a cone with vertex 0, together imply that $D\varphi(T_x) \subset \Gamma_r$. Let $w \in \Gamma_r \cap A_z \cap D\varphi(T_x)$ where w is a regular point of Γ_r; then $D\varphi(T_x)$ is transverse to $\Gamma_r \cap A_z$ in Γ_r at w if and only if $\operatorname{codim}(D\varphi(T_x) \cap A_z, D\varphi(T_x)) = r$. This shows that if transversality holds at one such point w it holds at all such points, and we say simply that $D\varphi(T_x)$ is transverse to $\Gamma_r \cap A_z$ in Γ_r. Thus we have:

Corollary (1.3)

(i) If $y = (x,z) \in Q(\mathcal{E})$, as in Lemma (1.2), then $\mathbb{P}\mathcal{E}$ is smooth at y if and only if $D\varphi(T_x)$ is transverse to $\Gamma_r \cap A_z$ in Γ_r.

(ii) $\mathbb{P}\mathcal{E}$ is smooth at all points of $\mathbb{P}\mathcal{E}_{(x)} = \pi^{-1}(x)$ (and therefore smooth in a neighbourhood of $\mathbb{P}\mathcal{E}_{(x)}$) if and only if $D\varphi(T_x)$ is transverse to $\Gamma_r \cap A_z$ in Γ_r for all $z \in \mathbb{P}_{q-1}$.

Proof.

(i) follows from Lemma (1.2) and the remarks above.

(ii) : If $\mathbb{P}\mathcal{E}$ is smooth at $y \in Q(\mathcal{E})_{(x)}$ then $\mathbb{P}\mathcal{E}$ is locally irreducible at y and therefore there is an open neighbourhood V of y in $\mathbb{P}\mathcal{E}$ such that $\mathbb{P}\mathcal{E} \cap V = Q(\mathcal{E}) \cap V$. Hence $Q(\mathcal{E})_{(x)}$ contains an open subset of $\mathbb{P}\mathcal{E}_{(x)}$ and it follows that $Q(\mathcal{E})_{(x)} = \mathbb{P}\mathcal{E}_{(x)}$. Now (ii) follows from (i).

§2 Examples of Smooth Projective Fibred Schemes.

There is one case where the criterion of Corollary (1.3)(ii) can be used to demonstrate the existence of smooth projective fibred schemes. Let $r = p < q$: that is, we consider sheaves $\mathcal{E}$ with resolution $0 \longrightarrow \mathcal{O}_U^p \xrightarrow{\Psi} \mathcal{O}_U^q \longrightarrow \mathcal{E}_U \longrightarrow 0$ where Ψ is a monomorphism with $\Psi(x) = 0$.

For this case $\Gamma_r = L(k^p,k^q)$. We write $T = D\varphi(T_x)$ and say T is good if T is transverse to A_z for all $z \in \mathbb{P}_{q-1}$. We consider the question of the existence of good T of dimension s. Let $G_{s,pq-s}$ be the Grassmann variety of vector subspaces of dimension s in $L(k^p,k^q)$. Let $\Omega_z = \{ T \in G_{s,pq-s} \mid \operatorname{codim}(T \cap A_z, T) \leqslant r-1 \}$; i.e. Ω_z is the set of T not transverse to A_z. Ω_z is a Schubert subvariety of $G_{s,pq-s}$. (For facts on Grassmann varieties and Schubert subvarieties we refer to (9)). Then $\dim G_{s,pq-s} = s(pq-s)$ and $\dim \Omega_z = (s-r+1)(pq-p-s+r-1) + (pq-s)(r-1)$

$$= s(pq-s) - (s-p+1) \quad \text{(since } r=p\text{)}.$$

Let $\mathcal{B} = \{\langle z,T\rangle \in \mathbb{P}_{q-1} \times G_{s,pq-s} \mid \operatorname{codim}(T \wedge A_z, T) \leqslant r-1\}$. Then $\mathcal{B}$ is a closed subvariety of $\mathbb{P}_{q-1} \times G_{s,pq-s}$ of dimension $s(pq-s)-(s-p-q+2)$. If π is the projection $\mathcal{B} \longrightarrow G_{s,pq-s}$, $\langle z,T\rangle \longmapsto T$, then $\pi\mathcal{B}$ is a closed subvariety of $G_{s,pq-s}$, which is strictly contained in $G_{s,pq-s}$ if $s > p+q-2$. The set of good T is the complement of $\pi\mathcal{B}$ in $G_{s,pq-s}$ and therefore this is a non-empty Zariski-open subset provided $s > p+q-2$.

From these considerations we may say that, provided X is smooth of dimension $s > p+q-2$, a "generic" $\mathcal{O}_X$-Module of the type considered (i.e. with a resolution in a neighbourhood U of x

$$0 \longrightarrow \mathcal{O}_U^p \longrightarrow \mathcal{O}_U^q \longrightarrow \mathcal{E}_U \longrightarrow 0,$$

where $q = \dim \mathcal{E}_{(x)}$) has a projective fibred scheme which is smooth in a neighbourhood of $\mathbb{P}\mathcal{E}_{(x)}$. The significance of the condition $\dim X > p+q-2$ is perhaps more evident when it is expressed in the form $q \leqslant n + \frac{1}{2}(s-n)$ where $s = \dim X$, and $n = q-p = q-r$ is the generic rank of $\mathcal{E}$.

In the above U is assumed to be a Zariski neighbourhood of x: however we may admit a larger class of sheaves by supposing only that U is an étale neighbourhood of x, i.e. that there exists a Zariski neighbourhood U' of x and a (surjective) étale morphism $U \longrightarrow U'$ such that $\mathcal{E} \otimes \mathcal{O}_U$ has a resolution

$$0 \longrightarrow \mathcal{O}_U^p \longrightarrow \mathcal{O}_U^q \longrightarrow \mathcal{E} \otimes \mathcal{O}_U \longrightarrow 0.$$

(This follows either from the method of proof used in §1, or by noting that $\mathbb{P}(\mathcal{E} \otimes \mathcal{O}_U) = \mathbb{P}\mathcal{E}_{U'} \times U$ is smooth if and only if $\mathbb{P}\mathcal{E}_{U'}$ is smooth).

Chapter IV The Cohomology of Complex Projective Fibred Schemes.

We consider algebraic varieties over the complex field $\mathbb{C}$ (in the sense of Serre (FAC); that is, varieties are ~~reduced~~ separated schemes of finite type over $\mathbb{C}$). In addition to the Zariski topology each such variety has an associated structure as a complex analytic space with the "classical" topology. When we write of the cohomology groups $H^*(V, \mathbb{C})$ or $H^*(V, \mathbb{Z})$ of a variety V we mean singular cohomology (with appropriate coefficients) of V with this topology.

Our task is to calculate as far as possible the cohomology ring of the projective fibred scheme $\mathbb{P}\mathcal{E}$ of a coherent sheaf $\mathcal{E}$ on a complete, smooth variety V, in terms of the cohomology of V and discrete invariants (such as characteristic classes) of $\mathcal{E}$. This is most feasible when $\mathbb{P}\mathcal{E}$ is itself smooth but even for this case our results are not complete, and we end with a conjecture.

Our approach is via the construction described in Chapter I, § 3(iii) which involves successive blowings-up of subvarieties of V. We therefore begin by discussing in some detail the effect on cohomology of blowing up a regular subvariety of a smooth variety. These results may well be known although it seems no clear statement of them appears in the literature; (but see (14) and remarks below).

It is to be expected that the results of this chapter have analogues which hold for varieties over any algebraically closed field of characteristic 0 and suitable algebraically defined cohomology theory, such as the Algebraic de Rham cohomology, introduced by Grothendieck in (11) and systematically treated by Hartshorne (5). However in the present state of algebro-geometric knowledge the finite characteristic case cannot be treated because (i) there is as yet no satisfactory cohomology theory in characteristic p, and (ii) "resolution of singularities" is not available.

§1 Cohomology and blowing-up.

Suppose Y is a smooth closed subvariety of a smooth complete variety X. Let X' be the blow-up of X with centre Y. We propose to describe the cohomology ring $H^*(X',Z)$ in terms of the cohomology of X and Y and invariants of the embedding $Y \hookrightarrow X$. Our result and many of the formal details of the proof are modelled on Manin's account (14) of the K-theory of X' (i.e. the Grothendieck group of locally-free sheaves on X'). Differences (and simplifications) in the proof for cohomology result from the richer structure of cohomology compared with K-theory (in particular, the relative cohomology groups and the exact sequences).

The method turns upon the following easy, formal, and known result.

Lemma (1.0)

Suppose given the following commutative "prism" of abelian groups such that the two triangles are exact and γ is an isomorphism:

$$\begin{tikzcd}
 & C' \arrow[dl, "k'"'] & \\
A' \arrow[rr, "i'"] & & B' \arrow[ul, "j'"'] \\
 & C \arrow[uu, "\gamma"', "\cong"] \arrow[dl, "k"'] & \\
A \arrow[uu, "\alpha"] \arrow[rr, "i"'] & & B \arrow[uu, "\beta"'] \arrow[ul, "j"]
\end{tikzcd}$$

Then the following triangle is exact:

$$\begin{tikzcd}
 & A' \oplus B \arrow[dr, "i'+\beta"] & \\
A \arrow[ur, "-\alpha \oplus i"] & & B' \arrow[ll, "k \circ \gamma^{-1} \circ j'"]
\end{tikzcd}$$

Proof: Standard diagram chasing. $\square$

Suppose $f: X' \longrightarrow X$ is the blow-up of X with centre Y, let $Y' = f^{-1}Y$ and let $g: Y' \longrightarrow Y$ be the restriction of f to Y'. Let $m = \operatorname{codim}_{\mathbb{C}}(Y,X)$ and $r = \dim_{\mathbb{C}}(X)$.

If N is the (algebraic) normal bundle of the embedding $i : Y \longrightarrow X$, the sheaf of (germs of) sections of N, $\mathcal{n}$, is a locally free $\mathcal{O}_Y$-Module of rank m, isomorphic to $(\mathcal{J}/\mathcal{J}^2)^\vee$, where $\mathcal{J}$ is the sheaf of ideals defining Y in X. Thus $Y' \cong \mathbb{P}(\mathcal{J}/\mathcal{J}^2)$.

Let N' be the normal bundle (rank 1) of the embedding $j: Y' \longrightarrow X'$. The sheaf of sections of N' is dual to $\mathcal{O}(1)$, the fundamental invertible sheaf on $(\mathcal{J}/\mathcal{J}^2)$. Hence N' can be identified with the bundle of closed points of $\mathbb{V}(\mathcal{O}(1))$.

We refer to (10) for the theory of Chern classes of coherent locally-free sheaves in algebraic geometry and to (e.g.)(2) for the topological theory. If F is a complex vector bundle of rank n on a complex variety Y the Chern class $c_i(F)$ is a cohomology class in $H^{2i}(Y,\mathbb{Z})$, $i=1,\cdots,n$. The conventions of algebraists and topologists agree in that if $\mathcal{F}$ is the sheaf of sections of the algebraic vector bundle F then $c_i(\mathcal{F}) = c_i(F)$. The Chern classes are characterised by the following well-known result.

<u>Proposition (1.1)</u>(Cohomology of projective bundles).

Let F be a rank n complex algebraic vector bundle on a complex variety Y, $\mathcal{F}$ the coherent locally-free sheaf of (germs of) algebraic sections of F, and $\pi : \mathbb{P}\mathcal{F} \longrightarrow Y$ the associated projective fibred variety. Then π induces a monomorphism $\pi^* : H^*(Y) \longrightarrow H^*(\mathbb{P}\mathcal{F})$ on cohomology (with integer coefficients), and $H^*(\mathbb{P}\mathcal{F})$, considered as an algebra over $H^*(Y)$ by virtue of π^*, is generated by $h = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}\mathcal{F})$, subject to the relation

$$\sum_{i=0}^{n} (-1)^i c_i(F) h^{n-i} = 0 \qquad (c_0 = 1) \qquad \square$$

Thus $H^*(Y')$ is determined by $H^*(Y)$ and the Chern classes of $\check{N}$.

We shall take for granted the theory of Poincaré duality in homology and cohomology. Consider the homology exact sequence of the pair (X,Y) (coefficients in $\mathbb{Z}$):

$$(1.2)\qquad \cdots \longrightarrow H_i(Y) \longrightarrow H_i(X) \longrightarrow H_i(X,Y) \longrightarrow H_{i-1}(Y) \longrightarrow \cdots$$

We have isomorophisms, given by duality:

$$H_i(X) \cong H^{2r-i}(X)$$
$$H_i(Y) \cong H^{2r-2m-i}(Y) .$$

Substituting in (1.2) yields:

$$(1.3)\qquad \cdots \longrightarrow H^{2r-2m-i}(Y) \longrightarrow H^{2r-i}(X) \longrightarrow H_i(X,Y) \longrightarrow \cdots .$$

We write $H^*(\ ) = \bigoplus_{i=0}^{\infty} H^i(\ )$, and $H_*(\ ) = \bigoplus_{i=0}^{\infty} H_i(\ )$, thus suppressing the grading, and express (1.3) as an exact triangle:

$$\begin{tikzcd}
 & H_*(X,Y) \arrow[dl] & \\
H^*(Y) \arrow[rr, "i_*"'] & & H^*(X) \arrow[ul]
\end{tikzcd}$$

Here i_* is the so-called Gysin (or umkehrungs) homomorphism induced by the inclusion $i\colon Y \longrightarrow X$.

The pair (X',Y') gives rise to a similar triangle and we can construct the following diagram:

$$(1.4)\qquad \begin{tikzcd}
 & H_*(X,Y) \arrow[dl] \arrow[dd, "f_*"] & \\
H^*(Y') \arrow[rr, "j_*"'] & & H^*(X') \arrow[ul] \\
 & H_*(X,Y) \arrow[dl] & \\
H^*(Y) \arrow[uu, "\alpha"] \arrow[rr, "i_*"'] & & H^*(X) \arrow[uu, "f^*"'] \arrow[ul]
\end{tikzcd}$$

The homomorphism $\alpha : H^*(Y) \longrightarrow H^*(Y')$ is defined by $\alpha(y) = g^*(y).c_{m-1}(F)$, where F is the rank $m-1$ vector bundle on Y' defined by the short exact sequence:

$$0 \longrightarrow N' \longrightarrow g^*N \longrightarrow F \longrightarrow 0.$$

Since f induces a homeomorphism $X' - Y' \xrightarrow{\approx} X - Y$ it follows that $f_*: H_*(X',Y') \longrightarrow H_*(X,Y)$ is an isomorphism. The commutativity of (1.4) follows from a number of formulae which we collect together as

Lemma (1.5)

(i) if $y \in H^*(Y)$, $i^* i_* y = y.c_m(N)$;

(ii) $g_* c_{m-1}(F) = 1$;

(iii) if $y \in H^*(Y)$, $f^* i_* y = j_*(g^* y.c_{m-1}(F))$;

(iv) $f_* f^* = \mathrm{id.} : H^*(X) \longrightarrow H^*(X)$;

(v) if $y \in \ker j_* \subset H^*(Y')$ then $y = g^* g_* y.c_{m-1}(F)$.

Proof.

(i): This is a standard consequence of the theory of Chern classes, (see, for example, (7)).

(ii) If we write $c = 1 + c_1 + c_2 + \ldots$ for the total Chern class, then formally

$$c(F) = g^* c(N)/c(N') = g^* c(N)/(1 - h) \quad \text{where } h = -c_1(N').$$

Hence $c_{m-1}(F) = \sum_{i=0}^{m-1} g^* c_{m-1-i}(N) h^i$ and

$$g_* c_{m-1}(F) = \sum_{i=0}^{m-1} c_{m-1-i}(N) g_*(h^i) = 1, \quad \text{since}$$

$g_*(h^i) = 0$ for i $m-1$, and $g_*(h^{m-1}) = 1$.

(iv): By virtue of the identity $f_*(f^* x.y) = x.f_* y$ it suffices to prove $f_* 1 = 1$. This follows from the fact that if $\eta_X \in H_{2r}(X)$, $\eta_{X'} \in H_{2r}(X')$ are respectively the fundamental homology classes of X, X', then $f_* \eta_{X'} = \eta_X$. (cf. (3)).

(iii): $f_*(f^* i_* y - j_*(g^* y.c_{m-1}(F))) = f_* f^* i_* y - i_* g_*(g^* y.c_{m-1}(F))$

$= i_* y - i_*(y.g_* c_{m-1}(F)) = 0$. (by (ii) and (iv)).

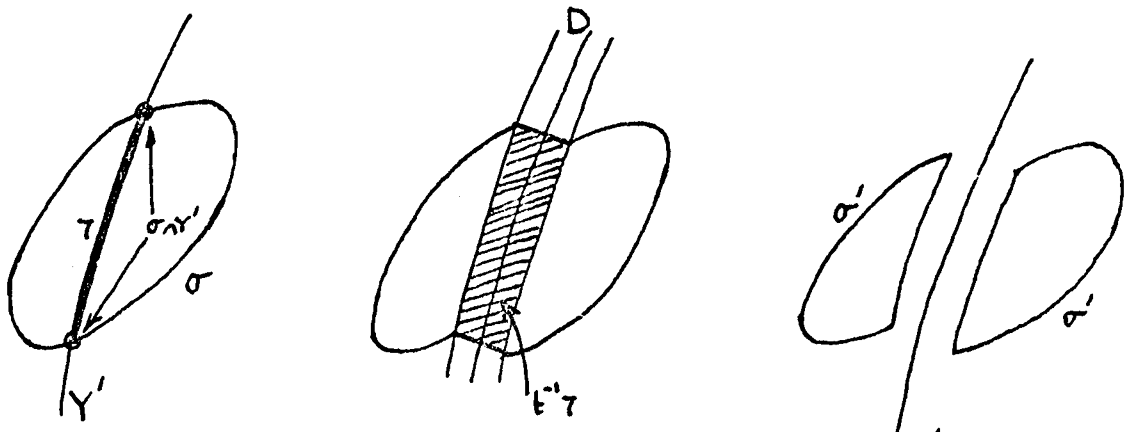
$$j^*(f^* i_* y - j_*(g^* y.c_{m-1}(F)) = g^* i^* i_* y - j^* j_*(g^* y.c_{m-1}(F))$$

$= g^*(y.c_m(N)) - (g^* y.c_{m-1}(F)) c_1(N')$ (by (i) and the analogous result $j^* j_* y = y.c_1(N')$)

$= 0$ since $g^* c_m(N) = c_{m-1}(F).c_1(N')$.

Therefore it suffices to prove that $\ker(f_*) \cap \ker(j^*) = 0$.

We sketch a geometrical proof (from a simplicial point of view —— as proved by S. Lojasiewicz, complex analytic spaces are triangulable). Suppose σ is a simplicial n-cycle whose homology class in $H_n(X')$ is the Poincaré dual of a cohomology class $s \in \ker(f_*) \cap \ker(j^*)$. σ may be assumed transverse to Y' (by subdividing and making each simplex transverse to Y'). Since $j^* s = 0$, $\sigma \cap Y'$ is an $(n-2)$-cycle homologous to 0 in Y'. Then there exists an n-cycle σ', homologous to σ, such that the simplices of σ' lie in $X' - Y'$. For suppose $\partial\tau = \sigma \cap Y'$, where τ is a $(n-1)$-chain in Y'. Y' can be "thickened" to a (triangulated) disc-bundle neighbourhood D of Y' in X' with projection $t: D \longrightarrow Y'$. [There is a deformation retract along the fibres of t, $X' - Y' \longrightarrow X' - D$]. Then $t^{-1}\tau$ is an $(n+1)$-chain and $\sigma \sim (\sigma - \partial(t^{-1}\tau)) \sim \sigma'$ where σ' is as required. (See diagrams). Now $f(\sigma') \sim 0$ in X and if $f(\sigma') = \partial\tau'$, τ' may be made transverse to Y in X. Then τ'', the "proper transform" of τ', (i.e. the closure in X' of $f^{-1}(\tau' \cap (X-Y))$, is a chain in X' such that $\partial\tau'' = \sigma'$, i.e. $\sigma' \sim 0$.



1. 2. 3.

(v): A typical element $y \in H^*(Y')$ can be written uniquely in the form $y = \sum_{i=0}^{m-1} a_i h^i$ where $a_i \in H^*(Y)$. If $j_* y = 0$ then $0 = j^* j_* y = -yh. = -\sum_{i=0}^{m-1} a_i h^{i+1}$. Therefore (by Proposition (1.1)) $a_{m-1} g^* c_{m-i}(N) = a_{i-1}$ $(i=1,\cdots,m)$. Hence $y = \sum_{i=0}^{m-1} a_{m-1} g^* c_{m-i-1}(N) h^i$ $= a_{m-1} c_{m-1}(F)$. If $a_{m-1} = g^* a$, then $g_* y = a . g_* c_{m-1}(F) = a$ (by (ii)) and $g^* g_* y . c_{m-1}(F) = g^* a . c_{m-1}(F) = y$. $\square$

Applying Lemma (1.0) to (1.4) gives an exact triangle

$$(1.6)\qquad \begin{tikzcd} H^*(Y) \arrow[rr, "-\alpha\oplus i_*"] & & H^*(Y')\oplus H^*(X) \arrow[dl, "j_*+f^*"] \\ & H^*(X') \arrow[ul] & \end{tikzcd}$$

Further α is a monomorphism (for if $0 = \alpha y = g^* y.c_{m-1}(F)$, then $0 = g_*(g^* y.c_{m-1}(F)) = y.\ g_* c_{m-1}(F) = y$); therefore $-\alpha\oplus i_*$ is a monomorphism and (1.6) yields a short exact sequence

$$(1.7)\qquad 0 \longrightarrow H^*(Y) \xrightarrow{-\alpha\oplus i_*} H^*(Y')\oplus H^*(X) \xrightarrow{j_*+f^*} H^*(X') \longrightarrow 0$$

The homomorphisms α, i_*, j_*, f^* respect the grading and raise degrees by $2(m-1)$, $2m$, 2, 0, respectively. Therefore, restoring the grading in (1.7) we have $H^s(X') \cong \dfrac{H^{s-2}(Y')\ \oplus\ H^s(X)}{(-\alpha+i_*)\ H^{s-2m}(Y)}$

(In particular if $s < 2m$, $H^s(X') \cong H^{s-2}(Y') \oplus H^s(X)$.)

Thus (1.7) determines the additive structure of $H^*(X')$. We turn now to consider the multiplication.

Let $H^*(Y)_N$ be the ring with additive group $H^*(Y)$ and multiplication $\circ$ defined by $y_1\circ y_2 = y_1 y_2 c_m(N)$. Thus $H^*(Y)_N$ is an associative ring (without unit), commutative in the sense that $y_1\circ y_2 = (-1)^{\deg y_1.\deg y_2}\ y_2\circ y_1$. This definition is motivated by the property : $i_*: H^*(Y)_N \longrightarrow H^*(X)$ is a ring homomorphism, (for $i_*(y_1\circ y_2) = i_*(y_1 y_2 c_m(N)) = i_*(y_1(i^* i_* y_2))$
$= (i_* y_1)(i_* y_2)$).

Similarly we define the ring $H^*(Y')_{N'}$ with multiplication $y_1\circ y_2 = y_1 y_2 c_1(N')$.

Finally we define multiplication in $H^*(Y')_{N'} \oplus H^*(X)$ by

$$(y_1, x_1)\circ(y_2, x_2) = ((g^* i^* x_1) y_2 + y_1(g^* i^* x_2) + y_1\circ y_2\ ,\ x_1 x_2).$$

It is straightforward to verify that with this definition $H^*(Y')_{N'} \oplus H^*(X)$ becomes an associative ring, which is commutative, again in the evident "graded" sense.

The following lemma shows that the multiplicative structure of $H^*(X')$ is also determined by (1.7).

Lemma (1.8)

The additive homomorphisms

$$-\alpha \oplus i_* : H^*(Y)_N \longrightarrow H^*(Y')_{N'} \oplus H^*(X)$$
$$j_* + f^* : H^*(Y')_{N'} \oplus H^*(X) \longrightarrow H^*(X')$$

appearing in (1.7), are ring homomorphisms.

Proof.

If $y_1, y_2 \in H^*(Y)$ then $(-\alpha \oplus i_*)y_1 \, (-\alpha \oplus i_*)y_2 =$

$$\begin{aligned}
&(-(g^*y_1)c_{m-1}(F),\ i_*y_1)\circ(-(g^*y_2)c_{m-1}(F),\ i_*y_2) = \\
&(-(g^*i^*i_*y_1)(g^*y_2)c_{m-1}(F) -(g^*y_1)c_{m-1}(F)(g^*i^*i_*y_2) + \\
&\qquad (g^*y_1)c_{m-1}(F)(g^*y_2)c_{m-1}(F)c_1(N'),\ i_*y_1 \cdot i_*y_2) = \\
&(-g^*(y_1y_2c_m(N))c_{m-1}(F),\ i_*(y_1\circ y_2)) \qquad (\text{since } c_m(N) = c_{m-1}(F)c_1(N') \\
&= (-\alpha \oplus i_*)(y_1 \circ y_2). \qquad \text{and } i^*i_*y = yc_m(N) \ )
\end{aligned}$$

If $(y_1,x_1),(y_2,x_2) \in H^*(Y')_{N'} \oplus H^*(X)$ then

$$\begin{aligned}
&(j_* + f^*)((y_1,x_1)\circ(y_2,x_2)) = \\
&\quad (j_* + f^*)((g^*i^*x_1)y_2+y_1(g^*i^*x_2)+ y_1y_2c_1(N'),\ x_1x_2) = \\
&\quad j_*((g^*i^*x_1)y_2 + y_1(g^*i^*x_2) + y_1y_2c_1(N')) + f^*(x_1x_2) = \\
&\quad j_*((j^*f^*x_1)y_2 + y_1(j^*f^*x_2) + y_1y_2c_1(N')) + f^*(x_1x_2) = \\
&\quad (f^*x_1)(j_*y_2) + (j_*y_1)(f^*x_2) + (j_*y_1)(j_*y_2) + (f^*x_1)(f^*x_2) = \\
&\quad (j_*y_1 + f^*x_1)(j_*y_2 + f^*x_2). \qquad \square
\end{aligned}$$

§2 Cohomology of $\mathbb{P}\mathcal{E}$.

If $\mathcal{F}$ is a locally-free coherent sheaf on a complete variety X then Proposition (1.1) shows that $H^*(\mathbb{P}\mathcal{F})$ isdetermined by $H^*(X)$ and theChern classes of $\mathcal{F}$. It would be over-optimistic to expect any such neat result for $H^*(\mathbb{P}\mathcal{E})$ when $\mathcal{E}$ is a general torsion-free coherent sheaf, but we can prove some partial results in this direction. The construction described in Chapter I, 3(iii) defines a locally-free sheaf $\mathcal{F}$ closely related to the given sheaf $\mathcal{E}$ and we can use this to approach the cohomology of $\mathbb{P}\mathcal{E}$.

Chapter I (3.7) gives a commutative square:

$$\begin{tikzcd} \mathbb{P}\mathcal{F} \arrow[r, "\tilde{\rho}"'] \arrow[d] & Q(\mathcal{E}) \arrow[d] \\ \tilde{X} \arrow[r, "\rho"'] & X \end{tikzcd}$$

where $\rho, \tilde{\rho}$ are proper birational surjective morphisms and $\mathcal{F}$ is a locally-free sheaf on $\tilde{X}$, a quotient sheaf of $\rho^*\mathcal{E}$, of rank equal to the generic rank of $\mathcal{E}$. ρ is the composite of a succession of blowings-up centred at Fitting subschemes.

Results of Hironaka on resolution of singularities (6) show that the sequence of blowings-up can be replaced by another sequence whose centres are smooth subschemes. Hironaka proves ((6), consequence 1 of Corollary 1, p.144) that if $\tilde{X}$, X are birational varieties there exists $X'' \longrightarrow X$, obtained by a finite succession of blowings-up with non-singular centres, such that X'' dominates $\tilde{X}$. Thus in our case there exists a birational morphism $X'' \xrightarrow{\rho'} \tilde{X}$ such that $\rho \circ \rho'$ can be realised by a succession of blowings-up with non-singular centres. Then $\mathcal{F}' = \rho'^*\mathcal{F}$ is locally-free on X'' and, combining the above commutative square with the commutative square:

$$\begin{tikzcd} \mathbb{P}\mathcal{F}' \arrow[r] \arrow[d] & \mathbb{P}\mathcal{F} \arrow[d] \\ X'' \arrow[r] & \tilde{X} \end{tikzcd}$$

we have

(2.1)
$$\begin{tikzcd} \mathbb{P}\mathcal{F}' \arrow[r, "\tilde{\rho}''"] \arrow[d, "\pi'"'] & Q(\mathcal{E}) \arrow[d, "\pi"] \\ X'' \arrow[r, "\rho''"] & X \end{tikzcd}$$

As a first consequence we may state

<u>Proposition (2.2)</u>

The homomorphism $\pi^*: H^*(X) \longrightarrow H^*(Q(\mathcal{E}))$ induced by the projection π is a monomorphism.

Proof.

$\rho''^* : H^*(X) \longrightarrow H^*(X'')$ is a monomorphism (Lemma (1.5)(iv)) and $\pi'^* : H^*(X'') \longrightarrow H^*(\mathbb{P}\mathcal{F}')$ is a monomorphism (Proposition (1.1)). Therefore $\pi'^*\rho''^* = \tilde{\rho}''^*\pi^*$ is a monomorphism and so π^* is a monomorphism. $\square$

Since $\tilde{\rho}''$ is birational we have (as in Lemma (1.5)(iv))
$$\tilde{\rho}''_*\tilde{\rho}''^* = \text{id.} : H^*(Q(\mathcal{E})) \longrightarrow H^*(Q(\mathcal{E})).$$
Consequently $H^*(Q(\mathcal{E}))$ can be identified, via $\tilde{\rho}''^*$, with a subring of $H^*(\mathbb{P}\mathcal{F}')$ and $H^*(Q(\mathcal{E}))$ is additively a direct summand of $H^*(\mathbb{P}\mathcal{F}')$. As groups $H^*(Q(\mathcal{E})) \cong H^*(\mathbb{P}\mathcal{F}')/\ker \tilde{\rho}''_*$.

If X is smooth then, given sufficient information on the centres of the blowings-up of X and the Chern classes of $\mathcal{F}'$, the cohomology of $\mathbb{P}\mathcal{F}'$ can be determined in principle by the results of §1.

Since $H^*(Q(\mathcal{E})) \cong H^*(\mathbb{P}\mathcal{F}')/\ker \tilde{\rho}''_*$ we are interested in characterising $\ker \tilde{\rho}''_*$. If $y \in \ker \tilde{\rho}''_*$ then, writing $h = c_1(\mathcal{O}(1))$ in $H^*(Q(\mathcal{E})) \subset H^*(\mathbb{P}\mathcal{F})$,
$$\rho''_*\pi'_*(yh^i) = \pi_*\tilde{\rho}''_*(yh^i) = \pi_*(\tilde{\rho}''_*y)h^i = 0;\ \text{i.e.}\ \pi'_*(yh^i)$$
is in $\ker \rho''_*$. Thus if $K = \{y \in H^*(\mathbb{P}\mathcal{F}) \mid \pi'_*(yh^i) \in \ker \rho''_* \text{ for all } i\}$ there is a group epimorphism $H^*(Q(\mathcal{E})) \longrightarrow H^*(\mathbb{P}\mathcal{F})/K$. This is as much as we are able to prove but it seems reasonable to conjecture that if $\mathbb{P}\mathcal{E}$ is smooth (so $\mathbb{P}\mathcal{E} = Q(\mathcal{E})$), then $H^*(\mathbb{P}\mathcal{E}) \cong H^*(\mathbb{P}\mathcal{F})/K$.

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