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ACCESSIBILITY AND SINGULAR FOLIATIONS

by

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ABSTRACT

In Part One we study the partition of a finite-dimensional manifold M into the accessible sets of an arbitrary system A of isotopy families of local diffeomorphisms of M and, in particular, into the accessible sets of an arbitrary system of differentiable vectorfields on M.

In Part Two we generalize the methods of Part One to study the integrability of singular distributions on infinite-dimensional manifolds.

In Part Three we return to finite-dimensional manifolds and use the results of Part One to study in detail the contrasting properties of integrability and irreducibility of systems of vectorfields on M.

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Theorem 3 in Part One answers a question put by Professor S.A. Robertson in connection with his work on mobility and co-mobility. I am grateful to him for asking it and for taking an interest in my work.

I want to thank David Chillingworth for taking the trouble to read and answer my long letters and for many hours of stimulating discussions and much constructive criticism. In particular, his original proof [14] convinced me that the methods of Part One can be modified to fit the infinite dimensional case and inspired the work which is presented in Part Two.

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PREFACE

1. RESULTS. The contents of this thesis is divided into three parts. Part One is (except in the Introduction) identical to a paper which will appear in the Proceedings of the London Mathematical Society. Part Two generalizes the results of Part One to infinite-dimensional (Banach) manifolds and Part Three consists of two shorter papers which could be considered as applications of the results of Part One.

The main result of Part One is to show that the accessible sets of an arbitrary collection A of 'arrows' (= isotopy families of local diffeomorphisms) on a manifold M possess a differentiable structure which makes them into connected immersed submanifolds of M. We also show that this differentiable structure is <u>unique</u> and that, more generally, every differentiable function $N \to M$ which factors set-theoretically through an accessible set L of A factors <u>differentiably</u> through L. Moreover, a similar result holds if we replace N by a locally connected topological space and substitute 'continuous' for 'differentiable'. In short, the accessible sets of A are almost as well-behaved as embedded submanifolds.

As for the 'differentiability transversally to the accessible sets' we show that the latter fit together to form a <u>foliation with singularities</u> (a regular foliation if they happen to be all of the same dimension). For example, the partition of the plane into the graphs of the functions $y^3 = (x-c)^3$ does not represent the collection of accessible sets of any set A of arrows on R^2.

There is a simple description of the tangent spaces of the accessible sets of A which allows us to compute their dimension and to give a necessary and sufficient condition (homogeneity) that A be <u>integrable</u> (that is, that it <u>span</u> the tangent spaces of its accessible sets). This condition is later shown to imply the 'classical' integrability conditions such as the Frobenius theorem and Nagano's results on integrability of (possibly singular) real-analytic distributions [7].

In Theorem 2 we show that, if $\sim$ is an arbitrary equivalence relation on M, then there exists the greatest foliation with singularities whose leaves are inscribed in the equivalence classes of $\sim$. If ϕ is a local diffeomorphism of M such that $x \sim y$ implies $\phi(x) \sim \phi(y)$, then ϕ is a local diffeomorphism for the differentiable structure defined by this foliation. The collection of $\sim$-preserving vectorfields on M is closed under formation of the Lie bracket (§5).

In Theorem 3 we show that every subgroup G of Diff M defines a foliation with singularities whose leaves are the orbits of the isotopy component G_o of id_M in G. The orbits of G are unions of G_o-orbits of constant dimension. In Theorems 4 and 5 we state a similar result for groupoids of germs of local diffeomorphisms and for arbitrary collections of differentiable vectorfields.

In Theorem 6 we re-formulate our integrability condition in terms of Lie brackets. We also show how it implies various other integrability conditions and give some examples illustrating their relationship.

In §6 of Part One we introduce the concept of 'multiarrows'. This is a convenient gadget for replacing Lie brackets (infinitesimal commutators) by 'finite' commutators. It is used here to give a direct proof of the so-called Chow's theorem.

In Part Two we define the differentiability of (possibly singular) distributions on infinite-dimensional (Banach) manifolds and show that it can be described in terms of vector-valued one-forms. The main result (Theorem 1) states that a weakly differentiable, possibly singular, distribution is integrable if and only if it is homogeneous. We give some other necessary and sufficient conditions of integrability and show that an integrable distribution B defines a unique differentiable structure σ on M such that (M,σ) is an integral submnifold of B. Further, σ is a foliation with singularities and the connected components of σ are the accessible sets of B.

In §8 of Part Two we use Lie derivatives to give some necessary and sufficient conditions that a vectorfield X respects a distribution B and hence deduce the corresponding conditions of homogeneity. We also show how these conditions imply the standard Frobenius theorem and prove that a real-analytic (possibly singular) distribution is integrable if and only if it is involutive and locally everywhere defined.

In the last section of Part Two we introduce the concepts of a neat leaf of a distribution and a neat submanifold of M and discuss a related unsolved problem.

In Part Three, §1, we return to the integrability of a system of vectorfields on a finite dimensional manifold and answer in full some of the problems which were left open in Part One. In particular, we show that, contrary to the claims in [6], [10] and [11], the condition that a set S of vectorfields be 'locally of finite type' is not sufficient for its integrability. We give some related necessary and sufficient conditions of integrability for the C^∞ case and show that they are not sufficient in the real-analytic case.

Finally, in §2 of Part Three we prove that the set of irreducible pairs of C^k-vectorfields on M is C^k-generic for every $k \geq 1$. (A pair $S = \{X,Y\}$ of vectorfields on M is irreducible if the accessible sets of S coincide with the connected components of M; the result has been known for $k \geq 2n$ [19]).

2. CONTEXT. Although the motif of this thesis goes back to Caratheodory's work on Thermodynamics (cf. Math. Annalen 67, 1909) the main reference is undoubtedly the 1939 paper of Wei-Liang Chow [2]. Chow's results can be summarized in our notation as follows. Let S be a system of C^1 vectorfields on a manifold M and assume that

(1) $$\dim \bar{S}(y) = \text{const} \quad \text{for} \quad y \in \Omega,$$

where Ω is a neighbourhood of a point x in M(cf. Part One, §4.3). Then the accessible sets of S define a regular foliation on a (possibly smaller) neighbourhood Ω' of x, which is tangent to the distribution $(\bar{S}(y) : y \in \Omega')$. The regularity condition (1) is of course a major restriction on the class of the admissible systems of vectorfields, whereas the results of Part One are valid for an arbitrary system S (and, more generally, for an arbitrary collection of 'arrows' on M).

As far as I know, the first proof of a 'Frobenius theorem' for singular (but real-analytic) distributions was published by Tadashi Nagano in 1966 [7]. I have also benefited a great deal from reading the papers [5] and [21] of Robert Hermann and from Claude Lobry's 1970 paper [6].

A preliminary version of Part One appeared as [11]. It has since transpired that some of the results (notably much of §5 and, to a lesser extent, the assertion of Theorem 5) partially overlap with the recent work of Héctor J. Sussmann [9], [10].

CONTENTS

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*) §1 represents an abbreviated version of the pre-print [12].

PART ONE

Accessible Sets, Orbits, and Foliations with Singularities

INTRODUCTION

In §§1-3 we prove a general theorem on accessible sets of collections of 'arrows' on the C^q-manifold M, where $1 \leq q \leq \omega$. In §4 we apply this result to the following situations:

(1) the partition of M into G-orbits, where G is an arbitrary subgroup of Diff (M);

(2) a similar situation, in which G is replaced by an arbitrary groupoid Γ of germs of local diffeomorphisms of M; and

(3) the partition of M into the accessible sets of an arbitrary collection of vectorfields on M.

In §5 we study the tangent spaces of accessible sets and obtain various generalizations of the Frobenius theorem. Finally in §6 we introduce the concept of a multiarrow and give a direct proof of the so-called Chow's theorem.

§1. ACCESSIBLE SETS FORM A FOLIATION WITH SINGULARITIES

Let M be a finite-dimensional paracompact C^q-manifold, $1 \leq q \leq \omega$. The word 'differentiable' always refers to this fixed class C^q. A subset L of M is said to be a <u>k-leaf</u> of M if there exists a differentiable structure σ on L such that (i) (L,σ) is a connected k-dimensional immersed submanifold of M and (ii) if N is an arbitrary locally connected topological space and $f : N \to M$ is a continuous function such that $f(N) \subset L$, then $f : N \to (L,\sigma)$ is continuous.

It follows from the properties of immersions that if $f : N \to M$ is a differentiable mapping of manifolds such that $f(N) \subset L$, then $f : N \to (L,\sigma)$ is also differentiable. In particular, σ is the <u>unique</u> differentiable structure on L which makes L into an immersed k-dimensional submanifold of M. Since M is paracompact, every connected immersed submanifold of M is separable, and so L does not admit a differentiable structure of a connected immersed submanifold of M of a dimension other than k.

Every embedded connected submanifold of M is a leaf, and so is the flow-line of the irrational flow if M is the two-torus.

We say that $\underset{\sim}{F}$ is a <u>C^q-foliation of M with singularities</u> if $\underset{\sim}{F}$ is a partition of M into C^q-leaves of M, such that, for every $x \in M$, there exists a local C^q-chart ψ of M with the following properties:

(a) The domain of ψ is of the form $U \times W$, where U is an open neighbourhood of 0 in R^k, W is an open neighbourhood of 0 in R^{n-k} and k is the dimension of the leaf through x.

(b) $\psi(0,0) = x$.

(c) If L is a leaf of $\underset{\sim}{F}$, then $L \cap \psi(U \times W) = \psi(U \times \ell)$, where $\ell = \{w \in W : \psi(0,w) \in L\}$.

Let $d(x,\underset{\sim}{F})$ denote the dimension of the leaf of $\underset{\sim}{F}$ which contains the point x. It follows immediately from the above definition that the function $x \to d(x,\underset{\sim}{F})$ is semi-continuous below. We write $(M,\underset{\sim}{F})$ for the C^q-manifold with the same underlying set as M and with the C^q-structure of the disjoint sum of the leaves of $\underset{\sim}{F}$; in general, this is a disconnected manifold whose components are not necessarily all of the same dimension.

By a <u>local diffeomorphism</u> of M we mean a diffeomorphism of one open subset of M onto another. We say that a differentiable function $a : R \times M \to M$ is an <u>arrow</u> if its domain is an open subset of $R \times M$ and if it satisfies the following two conditions: (i) for every $t \in R$, $a^t = a(t,-)$ is a local diffeomorphism of M (possibly with the empty domain) and (ii) if (t,x) belongs to the domain of a, then so does (s,x) for every s between 0 and t and $a(0,x) = x$. We write $\dot{a}(t,x)$ for the tangent vector at t of the curve $a(-,x)$, and $(a^t)^*(x)$ for the differential at x of the function $a^t : M \to M$. If $y = a(t,x)$, then $\dot{a}(t,x) \in T_yM$ and $(a^t)^*(x)$ is a linear mapping $T_xM \to T_yM$.

An example of an arrow is the <u>flow</u> of a differentiable vectorfield (§4.3). In general, an arrow a does not necessarily satisfy the condition $a^{t+s} = a^t \circ a^s$.

Let A be a collection of arrows on M. We write θA for the set of all local diffeomorphisms ϕ of M such that $\phi = a^t$ for some $a \in A$ and $t \in R$, and let ΨA denote the set which consists of the identity mapping of M and all the local diffeomorphisms of the form $\phi_1 \circ \phi_2 \circ \ldots \circ \phi_p$, where p is an arbitrary positive integer and ϕ_i or ϕ_i^{-1} belongs to θA for $1 \le i \le p$. We write $y = x \bmod A$ if $y = \phi(x)$ for some $\phi \in \Psi A$. This is clearly an equivalence relation on M; its equivalence classes are termed the <u>accessible sets</u> of A.

Given $x \in M$, we consider two vector subspaces, $A(x)$ and $\bar{A}(x)$, of the tangent space $T_x M$, spanned respectively by the sets $\{\dot{a}(t,y) : a \in A,\ a(t,y) = x\}$ and $\{\phi^*(y).w : \phi \in \Psi A,\ \phi(y) = x,\ w \in A(y)\}$. If $\phi \in \Psi A$ and $\phi(x) = y$, then clearly $\phi^*(x).\bar{A}(x) = \bar{A}(y)$. Thus, if $x = y \bmod A$, then $\dim \bar{A}(x) = \dim \bar{A}(y)$.

THEOREM 1. Let $\underset{\sim}{F}$ be the partition of M into the accessible sets of A. Then $\underset{\sim}{F}$ is a foliation with singularities and, for every $x \in M$, $T_x(M,\underset{\sim}{F}) = \bar{A}(x)$.

In particular, every accessible set of A is a leaf of M and thus admits a unique differentiable structure of a connected immersed submanifold of M.

Let $\sim$ be an equivalence relation on M. We say that a local diffeomorphism ϕ <u>preserves</u> $\sim$ if $\phi(x) \sim x$ whenever x belongs to the domain of ϕ. We say that ϕ <u>respects</u> $\sim$ if $\phi(x) \sim \phi(y)$ whenever $x \sim y$ and both x and y belong to the domain of ϕ. An arrow a preserves (or respects) $\sim$ if so does the local diffeomorphism a^t for every $t \in R$.

THEOREM 2. Let $\sim$ be an equivalence relation on M and let A be the collection of all the arrows on M which preserve $\sim$. If $\phi \in$ Loc Diff M and ϕ respects $\sim$, then $\phi \in$ Loc Diff $(M, \underset{\sim}{F})$, where $\underset{\sim}{F} = \underset{\sim}{F}(A)$ is the partition of M into the accessible sets of A.

§2 HOMOGENEOUS AND SYMMETRIC ENVELOPES

A collection A of arrows on M is said to be homogeneous if $A(x) = \bar{A}(x)$ for every $x \in M$, that is if $\phi^*(x).A(x) \subset A(y)$ whenever ϕ or ϕ^{-1} belongs to θA and $\phi(x) = y$; A is symmetric if $\phi \in \theta A$ implies that ϕ^{-1} is a composition of finitely many members of θA.

We write $A_o(x)$ for the vector subspace of $T_x M$ spanned by the set $\{\dot{a}(0,x) : a \in A,\ (0,x) \in \text{domain}(a)\}$.

PROPOSITION 2.1. If A is a collection of arrows on M, then there exists a symmetric homogeneous collection of arrows B such that the accessible sets of A and B are the same and $\bar{A}(x) = B(x) = B_o(x)$ for every $x \in M$

LEMMA 2.2. Let $A^{\#}$ be the collection of all arrows which can be written as $a^{t+s} \circ (a^s)^{-1}$ or $a^{s-t} \circ (a^s)^{-1}$ for some $a \in A$ and $s \in R$. Then

(a) $\Psi A^{\#} = \Psi A$;

(b) $A^{\#}$ is symmetric;

(c) $A_o^{\#}(x) = A^{\#}(x) = A(x)$ for every $x \in M$;

(d) the accessible sets of $A^{\#}$ and A are the same; and

(e) if A is homogeneous, then so is $A^{\#}$.

PROOF. Note that the assertion (d) follows at once from (a) and that (e) follows from (a) and (c). Let $b^t = a^{t+s} \circ (a^s)^{-1}$ and $c^t = a^{s-t} \circ (a^s)^{-1}$. If x lies in the domain of b^t, then it lies in the domain of b^τ for every τ between 0 and t, and so b (and similarly c) is indeed an arrow. Taking s = 0 (t = s) we get $b^t = a^t$ ($c^s = (a^s)^{-1}$) and so $\theta A \subset \theta A^{\#} \subset \Psi A$ and $(\theta A)^{-1} \subset \theta A^{\#}$, proving the assertions (a) and (b). Finally the equations

$$\dot{b}(t,x) = \dot{a}(t+s, (a^s)^{-1}.x),$$
$$\dot{c}(t,x) = -\dot{a}(s-t, (a^s)^{-1}.x)$$

and

$$\dot{b}(0,a^s.x) = \dot{a}(s,x)$$

show that $A^{\#}(x) \subset A(x) \subset A_0^{\#}(x)$, which proves the assertion (c).

LEMMA 2.3. Let A^* be the collection of all the arrows a on M such that:

(i) the domain of a is of the form $J \times V$, where J is an open interval in R and V is an open subset of M.

(ii) there exists $b \in A^{\#}$ and $\phi \in \Psi A$ such that, for every t in J and x in V,

$$a(t,x) = \phi \circ b^t \circ \phi^{-1} .x .$$

Then

(a) every $\phi \in \Psi A^*$ is a restriction of some local diffeomorphism in ΨA;

(b) the accessible sets of A^* and A are the same;

(c) $A^*(x) = \bar{A}(x)$ for every $x \in M$; and

(d) A^* is a homogeneous set of arrows.

PROOF. Note that the assertion (d) follows at once from (a) and (c). Since (a), (b) and the inclusion $A^*(x) \subset \bar{A}(x)$ are obvious, it remains to show that $\bar{A}(x) \subset A^*(x)$. Let $w \in A(y)$, $\phi \in \Psi A$ and $\phi(y) = x$. By Lemma 2.2c $w = \dot{b}(0,y)$ for some $b \in A^{\#}$. There exists $\delta > 0$ such that $a^t = \phi b^t \phi^{-1}$ is defined for $|t| < \delta$ on a sufficiently small neighbourhood of x. Hence $\phi^*(y).w = \dot{a}(0,x) \in A^*(x)$, Q.E.D.

Proposition 2.1 is now proved by taking $B = (A^*)^{\#}$.

§3. THE PROOF OF THEOREMS 1 AND 2

LEMMA 3.1 Let L be a subset of M. For every $x \in L$, let $L(x)$ be a vector subspace of $T_x M$. Assume that $\dim L(x) = k$ for every $x \in L$ and that, for every $x \in L$, there exists a local chart ψ of M such that

(a) the domain of ψ is $U \times W$, where U and W are open neighbourhoods of the origin in R^k and R^{n-k} respectively;

(b) $\psi(0,0) = x$;

(c) $L \cap \psi(U \times W) = \psi(U \times \ell)$, where $\ell = \{s \in W : \psi(0,s) \in L\}$; and

(d) $D_i\psi(t,s) \in L(\psi(t,s))$ for $1 \leq i \leq k$ and all $(t,s) \in \psi^{-1}(L)$.

Then there exists a differentiable structure σ on L with the following properties:

(i) (L,σ) is an immersed submanifold of M and $T_x(L,\sigma) = L(x)$ for every $x \in L$;

(ii) If $f : N \to M$ is a differentiable mapping of manifolds such that $f(N) \subset L$ and $f^*(\xi).T_\xi N \subset L(f(\xi))$ for every $\xi \in N$, then $f : N \to (L,\sigma)$ is differentiable.

(iii) Every connected component of (L,σ) is a leaf of M.

REMARKS. (1) It follows from (ii) that σ is the <u>unique</u> differentiable structure on L which satisfies the condition (i). (2) Apart from (iii), the assertions of this lemma do <u>not</u> depend on the paracompactness of M.

The proof of Lemma 3.1 consists in piecing together some *) of the arguments in [1] and we give it here merely for the sake of completeness.

PROOF. A function $\psi : U \times W \to M$ is said to be a <u>privileged chart</u> of M if it is a local chart of M and if it satisfies the conditions (a), (c) and (d) above.

(A) Let $\psi : U \times W \to M$ be a privileged chart of M and let $f : N \to M$ be a differentiable mapping of a connected manifold N into M such that $f(N) \subset L \cap \psi(U \times W)$ and, for every $\xi \in N$, $f^*(\xi).T_\xi N \subset L(f(\xi))$. Then there exists a constant $w \in W$ such that $f(N) \subset \psi(U \times \{w\})$.

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*) See pp. 91-95; it is assumed there that $L = M$, and the assertion (ii) is not formulated.

To prove this, let $p : R^k \times R^{n-k} \to R^{n-k}$ be the projection on the second coordinate and let $g = p \circ \psi^{-1}$. As $g \circ \psi.(t,s) = s$, we have $D_i(g \circ \psi)(t,s) = g^*(\psi(t,s)).D_i\psi(t,s) = 0$ for $1 \leq i \leq k$. If $\psi(t,s) \in L$, then $D_1\psi(t,s)$, $D_2\psi(t,s), \ldots, D_k\psi(t,s)$ span $L(\psi(t,s))$, which proves that $g^*(x).L(x) = 0$ for every $x \in L \cap \psi(U \times W)$. Hence $(g \circ f)^*(\xi) = 0$ for every $\xi \in N$. As N is connected, $g \circ f$ is constant on N, Q.E.D.

(B) Let Ψ be the collection of all the functions of the form $\psi_w = \psi(-,w) : U_\psi \to L$, where ψ is some privileged chart of M with the domain $U_\psi \times W_\psi$, $w \in W_\psi$, and $\psi(0,w) \in L$. Let $f : N \to M$ be a differentiable mapping of a manifold N into M such that $f(N) \subset L$ and, for every $\xi \in N$, $f^*(\xi).T_\xi N \subset L(f(\xi))$. If $\psi_w \in \Psi$, then $G = f^{-1}(\psi_w(U_\psi))$ is an open subset of N, and $(\psi_w)^{-1} \circ f : G \to R^k$ is a differentiable function.

Indeed, it follows immediately from (A) that G is the union of some of the connected components of the open set $f^{-1}(\psi(U_\psi \times W_\psi))$, and it is obvious that $(\psi_w)^{-1} \circ f = \psi^{-1} \circ f|_G$.

(C) Ψ is an atlas of a differentiable structure σ on L which satisfies the assertions (i) and (ii).

For it follows immediately from (B) that the charts of Ψ are mutually compatible; as their ranges cover the whole of L, Ψ is an atlas of a differentiable structure σ, which obviously satisfies the assertion (i). The assertion (ii) follows from (B).

(D) By paracompactness of M, every connected component L_o of (L,σ) is separable in the topology $\tau(\sigma)$ of the differentiable structure σ. Let $\psi : U \times W \to M$ be a privileged chart of M and let $\ell_o = \{s \in W : \psi(0,s) \in L_o\}$. Since $\{\psi(U,s) : s \in \ell_o\}$ is a collection of mutually disjoint $\tau(\sigma)$-open subsets of L_o, $\psi(\{0\} \times \ell_o)$ is an isolated subset of L_o. Hence ℓ_o is countable and, therefore, completely disconnected subset of W. It follows that $U \times \{0\}$ is a connected component of $U \times \ell_o$ in the product topology of $U \times W$, and so $\psi(U \times \{0\})$ is a connected component of $\psi(U \times \ell_o) = \psi(U \times W) \cap L_o$ in the induced topology $\tau(L_o,M)$. Since U can be taken arbitrarily small, we have proved that <u>every point of L_o has a fundamental system of $\tau(\sigma)$-neighbourhoods that are $\tau(L_o,M)$-connected components of $\tau(L_o,M)$-neighbourhoods</u>.

It follows trivially from the definitions and the above assertion that L_o is a leaf of M, which concludes the proof of the lemma.

PROOF OF THEOREM 1. By Proposition 2.1, we may assume that A is homogeneous and that $A(x) = A_o(x)$ for every $x \in M$.

Let L be an accessible set of A and let $x \in L$. Choose $a_i \in A$, $1 \le i \le k$, such that $\dot{a}_i(0,x)$ form a basis of $A(x)$ and let $\Phi(t_1, t_2,\dots,t_k,y) = a_1^{t_1} \circ a_2^{t_2} \circ \dots \circ a_k^{t_k}(y)$. Then Φ is a differentiable function $R^k \times M \to M$ and we may assume that the domain of Φ is of the form $U \times V$, where U is an open neighbourhood of the origin in R^k and V is a neighbourhood of x in M. It is obvious that

(i) for every $t \in U$ and $y \in V$, $\Phi(t,y) = y \pmod A$ and $\Phi(0,y) = y$;

(ii) $D_i\Phi(0,x) = \dot{a}_i(0,x)$ for $1 \le i \le k$. We claim that

(iii) for every $t \in U$, $y \in V$ and i between 1 and k, $D_i\Phi(t,y) \in A(\Phi(t,y))$.

To prove (iii) note that, for example, $D_2\Phi(t,y) = (a_1^{t_1})^*(z).\dot{a}_2(t_2,w)$ where $a_2(t_2,w) = z$ and $a_1(t_1,z) = \Phi(t,y)$. Since $\dot{a}_2(t_2,w) \in A(z)$, the result follows from the homogeneity of A.

Let now $n = \dim M$ and let Q be an $(n-k)$-dimensional submanifold of M such that $x \in Q$ and $T_xM = T_xQ + A(x)$. Let $f : W \to Q$ be a local chart for Q such that $f(0) = x$ and $f(W) \subset V$ and let $\psi : U \times W \to M$ be defined by $\psi(t,s) = \Phi(t,f(s))$. Since the rank of ψ at $(0,0)$ is n, ψ is a local chart of M for sufficiently small U and W. It is easy to check that ψ satisfies the conditions of Lemma 3.1 with $L(y) = A(y)$ and that the condition (c) remains valid if L is replaced by an <u>arbitrary</u> accessible set of A.

Let σ be the differentiable structure on L whose existence is asserted in Lemma 3.1 It remins to show that (L,σ) is a <u>connected</u> immersed submanifold of M. Let $u(t) = a(t,x)$, where $a \in A$ and $x \in L$. Then $u : R \to M$ is differentiable, the range of u is contained in L and $\dot{u}(t) \in A(u(t))$ for every t in the domain of u. By Lemma 3.1 (ii), the 'arrow-path' $u : R \to (L,\sigma)$ is differentiable and it remains to note that any two points of L can be joined by a succession of such arrow-paths (traced forwards or backwards).

LEMMA 3.2. Let ~ be an arbitrary equivalence relation on M and let A be the set of all arrows which preserve ~. Then A is symmetric and homogeneous and $A(x) = A_0(x)$ for every x in M.

PROOF. It is clear that $A^{\#} \subset A$ and $A^* \subset A$. (See Lemmas 2.2 and 2.3.)

PROOF OF THEOREM 2. Consider $\phi \in \mathrm{Loc\ Diff}\ (M)$ such that ϕ respects ~. Let $x \in$ domain (ϕ). Let L be the accessible set of A through x and let L' be the accessible set of A through $\phi(x)$. Let $k = \dim L$ and let $a_1, a_2, \ldots, a_k$ be members of A such that $\dot{a}_i(0,x)$ form a basis of $A(x)$. If $b_i^t = \phi a_i^t \phi^{-1}$, then $b_i \in A$, $1 \le i \le k$. If $y = f(t_1, t_2, \ldots, t_k) = a_1^{t_1} \circ a_2^{t_2} \circ \ldots \circ a_k^{t_k}(x)$ and $|t_i|$ are sufficiently small, then $\phi(y) = b_1^{t_1} \circ b_2^{t_2} \circ \ldots \circ b_k^{t_k}(\phi(x))$ and so $\phi(y) \in L'$. Since the rank of $f : R^k \to L$ is k, we have proved that there exists a neighbourhood U of x in L such that $\phi(U) \subset L'$. Since $\phi : U \to M$ is differentiable and L' is a leaf of M, it follows that $\phi : U \to L'$ is differentiable, Q.E.D.

§4 COROLLARIES OF THEOREMS 1 AND 2

4.1 Let G be a subgroup of Diff(M). Two elements g and h of G are said to be G-isotopic if there exists a differentiable mapping $a : R \times M \to M$ such that $a^t \in G$ for every $t \in R$, $a^t = g$ for $t \le 0$ and $a^t = h$ for $t \ge 1$. This is an equivalence relation and the component G_o of the identity is a normal subgroup of G.

THEOREM 3. (a) Let $\underset{\sim}{F} = \underset{\sim}{F}(G_o)$ be the partition of M into G_o-orbits. Then $\underset{\sim}{F}$ is a foliation with singularities.

(b) $G \subset \mathrm{Diff}(M,\underset{\sim}{F})$ and every G-orbit consists of G_o-orbits of constant dimension.

(c) If G/G_o is countable, then every G-orbit admits a unique structure of a separable immersed submanifold of M.

PROOF. Let A be the set of all differentiable mappings $R \times M \to M$ such that $a^t \in G$ for every $t \in R$ and $a^t = id_M$ for $t \le 0$. Then A is a symmetric set of arrows and the accessible sets of A are the orbits of G_o. The assertion (b) follows at once from Theorem 2 if we take $\sim$ to be the equivalence relation defined by the action of G_o and use the fact that G_o is normal. The assertion (c) follows easily from (b) and the fact that every G_o-orbit is a leaf of M.

4.2 If $\phi \in$ Loc Diff (M) and x belongs to the domain of ϕ, let $\gamma(x,\phi)$ denote the germ of ϕ at x. Let $e(x) = \gamma(x, id_M)$ and let $\Delta = \Delta(M)$ be the groupoid of all germs of local diffeomorphisms of M. Let $\alpha : \Delta \to M$ and $\omega : \Delta \to M$ be the projections onto the initial and final points respectively, so that $\alpha(\gamma(x,\phi)) = x$ and

Let Γ be a subgroupoid of Δ. We say that $g \in \Gamma$ and $h \in \Gamma$ are Γ-isotopic if $\alpha(g) = \alpha(h)$ and if there exists an open neighbourhood U of $\alpha(g)$ and a differentiable mapping $a : R \times U \to M$ such that (i) $\gamma(\alpha(g),a^t) \in \Gamma$ for every $t \in R$, (ii) $\gamma(\alpha(g),a^t) = g$ for $t \leq 0$ and (iii) $\gamma(\alpha(g),a^t) = h$ for $t \geq 1$. Let $\Gamma_o = \{g \in \Gamma : g$ is Γ-isotopic to $e(\alpha(g))\}$. Then Γ_o is a subgroupoid of Γ and g and h are Γ-isotopic if and only if $\Gamma_o g = \Gamma_o h$.

THEOREM 4. The assertions of Theorem 3 remain valid if we replace G by Γ, G_o by Γ_o and $G \subset \mathrm{Diff}(M,\underset{\sim}{F})$ by $\tilde{\Gamma} \subset \mathrm{Loc\ Diff}\ (M,\underset{\sim}{F})$, where $\phi \in \tilde{\Gamma}$ if and only if $\gamma(x,\phi) \in \Gamma$ for every x in the domain of ϕ.

4.3 If X is a differentiable vectorfield on M, $\exp X$ denotes the <u>flow</u> of X, so that $t \to \exp X.(t,x)$ is the integral curve of X passing through x at $t = 0$. If S is a set of vectorfields on M we put $\exp S = \{\exp X : X \in S\}$. It is clear that $\exp S$ is a symmetric set of arrows; the <u>accessible sets</u> of S are, by definition, the accessible sets of $\exp S$.

We write θS, ΨS, $S(x)$ and $\bar{S}(x)$ instead of $\theta \exp S$, $\Psi \exp S$, $(\exp S)(x)$ and $\overline{(\exp S)}(x)$, so that $S(x)$ is the vector subspace of $T_x M$ spanned by $\{X(x) : X \in S\}$, and $\bar{S}(x)$ is spanned by all the vectors of the form $\phi^*(y).X(y)$, where $\phi \in \Psi S$, $\phi(y) = x$ and $X \in S$.

THEOREM 5. Let $\underset{\sim}{F} = \underset{\sim}{F}(S)$ be a partition of M into the accessible sets of S. Then $\underset{\sim}{F}$ is a foliation with singularities*) and $T_x(M,\underset{\sim}{F}) = \bar{S}(x)$ for every x in M.

COROLLARY. $T_x(M,\underset{\sim}{F}) = S(x)$ for every $x \in M$ if and only if $\exp S$ is a homogeneous set of arrows, that is if and only if $(\exp X^t)*(x).S(x) \subset S(y)$ whenever $X \in S$ and $\exp X(t,x) = y$.

§5. LIE BRACKETS AND SUFFICIENT CONDITIONS OF HOMOGENEITY

LEMMA 5.1. Assume that $\underset{\sim}{F}$ is a partition of M into immersed submanifolds and let S be the collection of all vectorfields on M that leave $\underset{\sim}{F}$ invariant. Then

(a) $X \in S$ if and only if $X(x) \in T_x(M,\underset{\sim}{F})$ for every $x \in M$;

(b) If X and Y belong to S, then $[X,Y](x) \in T_x(M,\underset{\sim}{F})$ for every $x \in M$; and so

(c) if $q = \infty$ or ω, then S is closed under formation of the Lie bracket.

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*) See also [10], where it is proved that the accessible sets of S are immersed submanifolds of M (but not that they fit together to form a foliation with singularities). The 'D-invariance' in [10] is equivalent to our 'homogeneity'.

PROOF. The assertion (a) depends on the existence and uniqueness theorem for ordinary differential equations (see [8], Lemma 2.4) and (b) follows from the fact that $(M,\underset{\sim}{F})$ is an immersed submanifold of M (see [4], (17.14.3.5)).

COROLLARY 5.1. Let $\sim$ be an arbitrary equivalence relation on M and let S be the set of all vectorfields on M that leave the equivalence classes of $\sim$ invariant. If $q = \infty$ or ω, then S is closed under formation of the Lie bracket.

PROOF. If A is the collection of all the arrows on M that preserve $\sim$ and $\underset{\sim}{F} = \underset{\sim}{F}(A)$, then clearly $\exp S \subset A$ and $X \in S$ if and only if X leaves $\underset{\sim}{F}$ invariant.

THEOREM 6. Let $q \geq 2$ and let S be a set of C^1-vectorfields on M. The following assertions are equivalent:

(i) $\exp S$ is homogeneous.

(ii) Given $X \in S$ and $x \in M$, there exists $\varepsilon > 0$, a finite set $\{X_1, X_2, \ldots, X_p\} \subset S$ and continuous*) functions $\lambda_{ij} :]-\varepsilon, \varepsilon[\to R$ $(1 \leq i, j \leq p)$ such that (a) the vectors $X_1(x), X_2(x), \ldots, X_p(x)$ span $S(x)$, (b) for every $t \in]-\varepsilon, \varepsilon[$ and j between 1 and p

$$[X, X_j](u(t)) = \sum_{i=1}^{p} \lambda_{ij}(t) X_i(u(t)),$$

where $u(t) = \exp X.(t,x)$ and (c) $X_i(u(t))$ span $S(u(t))$.

PROOF. The implication (i) $\Rightarrow$ (ii) follows easily from Theorem 5 and Lemma 5.1. Assume (ii), and let $\Phi(t) = (\exp X^t)^*(x)$. If $y = u(t)$, we must show that $\Phi(t).S(x) = S(y)$. By a compactness argument, it is sufficient to prove this for $|t| < \varepsilon$. Put $V_j(t) = \Phi(t)^{-1}.X_j(u(t))$. Then $V_j(t) \in T_x M$ and (using, for example, the formula (17.14.3.2) of [4])

$$\dot{V}_j(t) = \Phi(t)^{-1}.[X, X_j](u(t)) = \Phi(t)^{-1} \sum_{i=1}^{p} \lambda_{ij}(t) X_i(u(t)) = \sum_{i=1}^{p} \lambda_{ij}(t) V_i(t).$$

Let $H : T_x M \to R$ be a linear functional and let $h_i(t) = \langle H, V_i(t) \rangle$ and $h(t) = (h_1(t), h_2(t), \ldots, h_p(t)) \in R^p$. Then $\dot{h}(t) = \Lambda(t) h(t)$, where $\Lambda(t)$ is the $p \times p$ matrix with entries λ_{ij}. Thus $h(t) = 0$ if and only if $h(0) = 0$, and it follows that the vectors $V_1(t), V_2(t), \ldots, V_p(t)$ span the same subspace of $T_x M$ as the vectors $V_1(0), V_2(0), \ldots, V_p(0)$. Since $V_i(0) = X_i(x)$ span $S(x)$ and $V_i(t) = \Phi(t)^{-1}.V_i(u(t))$ span

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*) It is sufficient to assume that λ_{ij} are Lebesgue integrable.

$\Phi(t)^{-1}.S(y)$, we have $S(x) = \Phi(t)^{-1}.S(y)$, Q.E.D.

Let R^x denote the ring of germs at x of real-valued C^q-functions and let S^x denote the module over R^x generated by the vectorfields in S. Following Lobry [6], we say that S is locally of finite type if, for every $x \in M$, there exists a finite set $F \subset S$ such that (i) $F(x) = S(x)$ and (ii) $[S,F]^x \subset F^x$. Consider the following conditions on S (where we assume for simplicity that $q = \infty$ or ω):

(H) $\exp S$ is a homogeneous set of arrows;

(1) S is locally of finite type;

(2) S is closed under formation of the Lie bracket and, for every $x \in M$, S^x is a finitely generated module over R^x;

(3) S is closed under formation of the Lie bracket and $\dim S(x)$ is locally constant on M;

(4) S is closed under formation of the Lie bracket, $q = \omega$, and the vectorfields in S are defined everywhere on M.

(5) $\dim S(x) \leq 1$ for every $x \in M$.

(6) S is closed under formation of the Lie bracket.

(7) $\tilde{S}$ is locally of finite type and $q \neq \omega$. Here $\tilde{S}$ is the set of all C^∞ vectorfields X such that $X(y) \in S(y)$ for ever $y \in$ domain X.

PROPOSITION 5.2. The conditions (3), (4), (5) and (7) imply (H).

The assertions (5)$\Rightarrow$(H) and (3)$\Rightarrow$(H) are easily deduced from Theorem 6 and are left to the reader. The proof of (4)$\Rightarrow$(H) is given in Nagano's paper [7]; a simpler proof (of a slightly stronger result) using Theorem 6 is given in [12]. The only proof of (7)$\Rightarrow$(H) known to the author is given in [12].

It is claimed in [6] (and also in [10] and [11]) that (1) $\Rightarrow$ (H), but this assertion is false [12].

REMARKS. (a) The examples below show that (6) $\not\Rightarrow$ (H), (H) $\not\Rightarrow$ (1) and ((5) and (6)) $\not\Rightarrow$ (2).

(b) Note that (3) $\Longrightarrow$ (H) in combination with Theorem 5 gives the classical Frobenius theorem and (4) $\Longrightarrow$ (H) together with Theorem 5 give Nagano's theorem on integrability of real-analytic distributions with singularities [7]. Thus Theorems 5 and 6 and Proposition 5.2 taken together can be regarded as a generalization of the Frobenius theorem.

COROLLARY 5.2. Let $q = \infty$ and let $[S]$ denote the smallest set of vectorfields on M which contains S and is closed under formation of the Lie bracket. Let L be an accessible set of S. Then $[S](x) \subset T_xL$ for every $x \in L$. If $\dim[S](x)$ is constant on L, then $[S](x) = T_xL$ for every $x \in L$.

PROOF. The first part follows at once from Lemma 5.1. Let $\dim[S](x)$ be constant on L. Without loss of generality, we may assume that $L = M$. Since $S \subset [S]$, L is an accessible set of $[S]$. By Proposition 5.2, $[S]|_L$ is homogeneous, and so the assertion follows from the Corollary of Theorem 5.

EXAMPLE 5.3. Let $\phi : R \to R$ be defined by $\phi(x) = 0$ for $x \leq 0$ and $\phi(x) = e^{-1/x}$ for $x > 0$. Let $M = R^3$ and let $S = \{X,Y\}$, where $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial x} + \phi(x-1)\frac{\partial}{\partial y} + \phi(-1-x)\frac{\partial}{\partial z}$. It is easy to check that $L = R^3$ and that $\dim [S](x,y,z) = \dim S(x,y,z) = 1$ if $-1 \leq x \leq 1$ and 2 otherwise. In particular, $\exp[S]$ is not homogeneous.

EXAMPLE 5.4. Let $M = R^2$ and let S consist of the vectorfield $\frac{\partial}{\partial x}$ and all the vectorfields of the form $\phi(y)e^x\frac{\partial}{\partial y}$, where $\phi : R \to R$ is a differentiable function such that $\phi(x) = P(1/x).\exp(-1/x^2)$ for some real polynomial P and all $x \neq 0$. If $\underset{\sim}{F}$ is the partition of R^2 into the accessible sets of S, then clearly $\underset{\sim}{F}$ = {upper half-plane, x-axis, lower half-plane} and $S(x) = T_x(R^2,\underset{\sim}{F})$ for every $x \in R^2$. By the Corollary of Theorem 5, $\exp S$ is homogeneous. We claim that S is not locally of finite type.

Indeed, assume that $F = \{X_1,X_2,\ldots,X_p\}$ satisfies the assumptions (i) and (ii) of Lobry's condition at the origin of R^2. It follows from (i) that $\frac{\partial}{\partial x} \in \{X_1,X_2,\ldots,X_p\}$, say $\frac{\partial}{\partial x} = X_1$. Let $X_i(x,y) = \phi_i(y)e^x\frac{\partial}{\partial y}$, $2 \leq i \leq p$, and let $X(x,y) = \phi(y)e^x\frac{\partial}{\partial y} \in S$. By (ii),

$$\phi(y)e^x\frac{\partial}{\partial y} = [\frac{\partial}{\partial x}, \phi(y)e^x\frac{\partial}{\partial y}] = [X_1,X] = \lambda_1(x,y)\frac{\partial}{\partial x} + \sum_{i=2}^{p}\lambda_i(x,y)\phi_i(y)e^x\frac{\partial}{\partial y}$$

for x and y sufficiently near the origin. Comparing the coefficients at $\frac{\partial}{\partial y}$ and setting $x = 0$, we see that $\phi(y) = \sum_{i=2}^{p}\alpha_i(y)\phi_i(y)$. There exists an integer $k > 0$ such that, for $2 \leq i \leq p$, $\phi_i(y).y^k.\exp(1/y^2)$ is continuous at $y = 0$, and so $\phi(y).y^k.\exp(1/y^2)$ is also continuous. Setting $\phi(y) = (1/y)^{k+1}.\exp(-1/y^2)$ for $y \neq 0$, we arrive at a contradiction.

EXAMPLE 5.5. Let $M = R$ and let S be the set of all vectorfields of the form $\phi(x)\frac{\partial}{\partial x}$, where ϕ is as in Example 5.4. Then S satisfies the conditions (5) and (6) of Proposition 5.2, but the condition (2) breaks down at the origin.

§6. MULTIARROWS. A DIRECT PROOF OF CHOW'S THEOREM

6.1. Throughout this section, we assume that $q = \infty$ or ω and let $[S]$ denote the smallest set of vectorfields on M which contains S and is closed under formation of the Lie bracket. The following theorem follows from the results of Chow [2] and is usually referred to under his name ([5], [6], [8]).

THEOREM (Chow). Let L be an accessible set of S and let $x \in L$. If $\dim[S](x) = \dim M$, then L is an open subset of M.

PROOF. This follows immediately from Theorem 5 and Lemma 5.1.

In this section we given an alternative proof of Chow's theorem, which is based on the concept of a <u>multiarrow</u> (see 6.3).

6.2. Let $f : R^k \to R^n$ be a smooth function defined in a neighbourhood of the origin of R^k. Let $x \in R^n$ and assume that $f(t) = x$ whenever at least one component of $t = (t_1,t_2,\ldots,t_k)$ is zero. Clearly, $D_1^{p_1} D_2^{p_2} \ldots D_k^{p_k} f(0,0,\ldots,0) = 0$ whenever some $p_i = 0$. In particular,

(a) $$f(t) = x + t_1 t_2 \dots t_k \bar{D}f(0) + \omega(t).t^{(k+1)}$$

where $\bar{D} = D_1 D_2 \dots D_k$, $\omega(t)$ is a symmetric $(k+1)$-linear mapping $(R^k)^{k+1} \to R^n$ which depends differentiably on t, and $t^{(k+1)} =$ $= (t,t,\dots,t) \in (R^k)^{k+1}$ ([3], (8.14. 3), (8.12.7)). If $\phi : R^n \to R^m$ is a smooth function defined in a neighbourhood of x, it is easily checked that

(b) $$\bar{D}(\phi \circ f)(0) = \phi^*(x).\bar{D}f(0).$$

In particular, $\bar{D}f(0)$ is a well-defined vector in $T_x M$ if R^n above is replaced by a smooth manifold M.

Note also that $(t_2,t_3,\dots,t_k) \to D_1 f(0,t_2,t_3,\dots,t_k)$ is a smooth function from R^{k-1} into the vector space $T_x M$ and that $\bar{D}f(0)$ is its $(k-1)^{st}$ mixed partial derivative. As $D_1 f(0,t_2,t_3,\dots,t_k) = 0$ whenever one of the components of $(t_2,t_3,\dots t_k)$ is zero, the above arguments show that

$$D_1 f(0,t_2,t_3,\dots,t_k) = t_2 t_3 \dots t_k \bar{D}f(0) + \tilde{\omega}(\bar{t}).\bar{t}^{(k)},$$

where $\bar{t} = (t_2,t_3,\dots,t_k)$. Hence (at least if $t_2 t_3 \dots t_k \neq 0$),

$$D_1 f(0,\bar{t}) = t_2 t_3 \dots t_k a(\bar{t}),$$

where $a(\bar{t}) \to \bar{D}f(0)$ as $\bar{t} \to 0$.

6.3 A smooth function $a : R^k \times M \to M$ is a <u>multiarrow of order k</u> if

(a) a is defined on an open neighbourhood of $0 \times M$;

(b) for every t in R^k, $a^t = a(t,-)$ is a diffeomorphism of an open subset of M onto an open subset of M;

(c) $a(t,x) = x$ whenever (t,x) belongs to the domain of a and at least one component of $t = (t_1,t_2,\dots,t_k)$ is zero.

If $a(t,x) = y$, $1 \le i \le k$, we write $D_i a(t,x) \in T_y M$ for the partial derivative of the function $a(-,x) : R^k \to M$ and $a^*(t,x) : T_x M \to T_y M$ for the differential of $a^t = a(t,-) : M \to M$. It follows from (6.2) that $\bar{D}a(0,x) = D_1 D_2 \ldots D_k a(0,0,\ldots,0,x)$ is a well-defined element of $T_x M$ and that, for $\bar{t} = (t_2,t_3,\ldots,t_k) \in R^{k-1}$, $t_2 t_3 \ldots t_k \neq 0$,

(d) $$D_1 a(0,\bar{t},x) = t_2 t_3 \ldots t_k X(\bar{t},x), \quad \lim_{\bar{t}\to 0} X(\bar{t},x) = \bar{D}a(0,x).$$

We write A_k for the set of all k-multiarrows on M and put $A = \bigcup_{k=1}^{\infty} A_k$. If $a \in A$, $\bar{D}a$ denotes the smooth vectorfield $M \to TM : x \to \bar{D}a(0,x)$. The <u>bracket</u> of $a \in A_k$ and $b \in A_\ell$ is the $k+\ell$-multiarrow $[a,b]$ defined by

(e) $$[a,b](t,s,x) = (b^s)^{-1} \circ (a^t)^{-1} \circ (b^s) \circ (a^t).(x)$$

LEMMA 6 .3.

$$\bar{D}[a,b] = [\bar{D}a,\bar{D}b].$$

PROOF. Let $f(t,x) = (a^t)^{-1}(x)$ and $g(s,x) = (b^s)^{-1}(x)$. Fixing $x \in M$ and the local coordinates, we assume that $M = R^n$. Let $\phi(t,s) = [a,b](t,s,x) = g(s,f(t,b(s,a(t,x))))$, so that

$$\bar{D}[a,b](x) = \bar{D}_t \bar{D}_s \phi(0,0).$$

By (6.2.(b)),

$$\bar{D}_s\phi(t,0) = \bar{D}_s g(0,f(t,a(t,x)) +$$
$$+ g^*(0,f(t,a(t,x)).f^*(t,a(t,x)).\bar{D}_s b(0,a(t,x)) =$$
$$= \bar{D}_s g(0,x) + f^*(t,a(t,x)).\bar{D}b(a(t,x)).$$

If $\psi(t) = f^*(t,a(t,x))$ and $\bar{t} = (t_2,\ldots,t_k)$, then

$$D_1\psi(0,\bar{t}) = D_1 f^*(0,\bar{t},x) + f^{**}(0,\bar{t},x).D_1 a(0,\bar{t},x)$$
$$= D_1 f^*(0,\bar{t},x)$$

because $f^*(0,\bar{t},x) \equiv \mathrm{id}$ and so $f^{**}(0,\bar{t},x) = 0$. Hence $\bar{D}\psi(0) = \bar{D}f^*(0,x)$ and

$$\bar{D}_t\bar{D}_s\phi(0,0) = (\bar{D}f)^*(0,x).\bar{D}b(x) + f^*(0,x)(\bar{D}b)^*(x).\bar{D}a(x) =$$
$$= (\bar{D}b)^*(x).\bar{D}a(x) + (\bar{D}f)^*(0,x).\bar{D}b(x).$$

Differentiating the equation $f(t,a(t,x)) = x$, we immediately see that $\bar{D}f(0,x) = -\bar{D}a(x)$, and so (cf. [4], 17.14.3.2)

$$\bar{D}_t\bar{D}_s\phi(0,0) = (\bar{D}b)^*(x).\bar{D}a(x) - (\bar{D}a)^*(x)\bar{D}b(x)$$
$$= [\bar{D}a,\bar{D}b](x),$$

Q.E.D.

6.4 Let $\sim$ be an equivalence relation on M. We say that a multiarrow a preserves $\sim$ if $a(t,x) \sim x$ for all (t,x) in the domain of a. Let $\tilde{A}$ denote the set of all the multiarrows on M which preserve $\sim$.

LEMMA 6.4.1. Let $\tilde{V} = \overline{D}\tilde{A}$ be the collection of all the vectorfields of the form $\bar{D}a$, $a \in \tilde{A}$. Then $\tilde{V}$ is closed under formation of the Lie bracket.

PROOF. This follows at once from Lemma 6.3

LEMMA 6.4.2. If $\dim\tilde{V}(x) = d$, then there exists an immersion $\psi : R^d \to M$ such that

(a) ψ is defined on a neighbourhood of $0 \in R^d$ and $\psi(0) = x$.

(b) $\psi(t) \sim x$ for every t in the domain ψ.

PROOF. Choose $a_i \in \tilde{A}$ so that the vectors $\bar{D}a_1(x), \bar{D}a_2(x),\ldots,\bar{D}a_d(x)$ are linearly independent. It follows from (6.3.(d)) that there exist $\lambda_i \in R^{k_i-1}$ such that the vectors $D_1a_i(0,\lambda_i,x)$ are also linearly independent. Consider the arrows b_i defined by $b_i^t = a_i(t,\lambda_i,-)$

and let $\psi(t_1, t_2, \ldots, t_d) = b_1^{t_1} \circ b_2^{t_2} \circ \ldots \circ b_d^{t_d}.(x)$. It is clear that ψ satisfies the conditions (a) and (b) of the lemma and that $D_i\psi(0) = D_1 a_i(0, \lambda_i, x)$, so that ψ restricts to an immersion on a sufficiently small neighbourhood of the origin.

PROOF OF CHOW'S THEOREM. Let $\sim$ be the relation $x = y (\mathrm{mod} \exp S)$. It is clear that $S \subset \tilde{\mathcal{V}}$ and so, by Lemma 6.4.1, $[S] \subset \tilde{\mathcal{V}}$. Hence $\dim \tilde{\mathcal{V}}(x) = \dim M$ and Lemma 6.4.2 implies that there is an open set U in M such that $x \in U \subset L$. If $y \in L$, then $y = \phi(x)$ for some $\phi \in \Psi S$. Let W be a neighbourhood of x contained in $U \cap (\mathrm{domain}(\phi))$. Then $y \in \phi(W) \subset L$, which proves the assertion of the Theorem.

PART TWO

Integrability of singular distributions on infinite-dimensional manifolds

INTRODUCTION

Let M be a C^q Banach manifold, where $2 \leq q \leq \omega$. To simplify the notation, we assume that M is modelled on a single Banach space E. The word <u>differentiable</u> refers to a fixed class C^r, where $1 \leq r \leq q-1$ and we take $\infty = \infty - 1$ and $\omega = \omega - 1$.

A distribution B on M is a family $(B_x : x \in M)$, where each B_x is a topological direct summand of the tangent space $T_x M$. B is <u>regular</u> if it defines a differentiable subbundle of the tangent bundle TM, otherwise B is <u>singular</u>. If B is singular, then B_x need not be isomorphic to B_y even if x and y lie in the same connected component of M. (0.1)

By an <u>immersion</u> we always mean a <u>split</u> immersion, so that a differentiable function $f : N \to M$ is an immersion if and only if, for every x in N, the differential $f^*(x)$ is an isomorphism of $T_x N$ onto a topological direct summand of $T_{f(x)}M$.

An <u>immersed submanifold</u> of M is a subset L of M together with a differentiable structure σ on L such that the inclusion mapping of L into M is an immersion. We identify the tangent space $T_x(L,\sigma)$ at $x \in L$ with the corresponding subspace of $T_x M$. L is an <u>integral manifold</u> of the distribution B if $T_x(L,\sigma) = B_x$ for every $x \in L$. (0.2)

We say that B is an <u>integrable distribution</u> if there exists a differentiable structure σ on M such that (M,σ) is an integral manifold of B. (0.3)

In §§4 and 5 we define the differentiability of (possibly singular) distributions and show that it can be described in terms of vector-valued one-forms on M.

Our main results are collected in §7, where we show that a differentiable (possibly singular) distribution is integrable if and only if it is <u>homogeneous</u> (0.6) and give some other necessary and sufficient conditions of integrability. We also prove that the differentiable structure σ which makes M into an integral manifold of B is unique, that every integral manifold of B is an open submanifold of (M,σ) and that σ is a foliation with singularities as defined in §2.

A differentiable vectorfield X, defined on an open subset of M, is said to <u>lie in B</u> of $X(x) \in B_x$ for every x in the domain of X. We say that a vectorfield X (which does <u>not</u> necessarily lie in B) <u>respects</u> B if (0.4) (0.5)

$$(X^t)^*(x).B_x = B_y$$

whenever $X^t.x = y$. B is said to be <u>homogeneous</u> if every vector-field in B respects B. (0.6)

In §8 we give some necessary and sufficient conditions that X respect B, formulated in terms of Lie brackets, and deduce the corresponding conditions for the homogeneity (and so integrability) of B, which generalise Theorem 6 of Part One. In particular, we recover the <u>standard Frobenius theorem</u> (SFT) on the integrability of <u>regular</u> distributions, as stated, for example, in [13] or [16].

We also prove that a real analytic (possibly singular) distribution is integrable if and only if it is involutive and locally everywhere defined.

Finally, in §9, we introduce the concept of a neat leaf and discuss a related unsolved problem.

Just as is the case with SFT, the proofs of our results are fairly simple and have a 'coordinate free', rather than a true 'functional-analytic' flavour. I hope that they will pave the way for some future 'hard' theorems.

§1. TOPOLOGICAL DIRECT SUMMANDS

1.1 Given two topological vector spaces E and F (over $\mathbb{R}$), L(E,F) denotes the vector space of all continuous linear mappings $E \to F$. If E and F are normed (or normable), we shall always think of L(E,F) as a normed (normable) space with the usual 'sup over the unit ball of E' norm (or the corresponding topology). LIS(E,F) denotes the subspace of L(E,F) consisting of the toplinear isomorphisms, and we write End(E) and GL(E) instead of L(E,E) and LIS(E,E).

Recall that a vector subspace F of a Hausdorff topological vector space E is a <u>direct summand</u> of E if it satisfies one of the following equivalent conditions:

(1) there exists a topological vector space G and a toplinear isomorphism $\alpha : E \to F \times G$ such that

$$id_F = p \circ \alpha \circ i,$$

where $i : F \to E$ is the inclusion and $p : F \times G \to F$ is the coordinate projection;

(2) there exists a subspace G of E such that the mapping $F \times G \to E$: $(x,y) \to x + y$ is a toplinear isomorphism;

(3) there exists a continuous linear projection $P \in End(E)$ such that $F = Ker\,P$;

(4) there exists a continuous linear projection $Q \in End(E)$ such that $F = Im\,Q$.

It is easy to check that a closed, finite codimensional subspace of E is always a direct summand.

If E is locally convex, then, by the Hahn-Banach theorem, every finite-dimensional subspace of E is a direct summand.

If E is a Fréchet space, then, by the closed graph theorem, F is a direct summand of E if and only if

(5) F is closed and there exists a closed subspace G of E such that $F \cap G = 0$ and $F + G = E$.

1.2 Let $GL(E|F)$ denote the subset of $GL(E)$ consisting of those toplinear automorphisms of E which map F into itself, and let $GL_o(E|F)$ consist of those members of $GL(E|F)$ that restrict to an automorphism of F.

PROPOSITION 1.2. Let E be a Banach space and let F be a direct summand of E. Then $GL_o(E|F)$ is open and closed in $GL(E|F)$. If F is finite dimensional or finite codimensional, then $GL_o(E|F)$ coincides with $GL(E|F)$.

PROOF. Let $F_1 = F$ and let F_2 be some topological complement of F. Let i_k be the inclusion mapping $F_k \to E$ and let p_1 denote the projection of E onto F_1 along F_2, and p_2 the complementary projection of E onto F_2. An operator a in End(E) is represented by a matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{k\ell} = p_k a i_\ell$ belongs to $L(F_\ell, F_k)$. It is clear that $GL_o(E|F)$ is <u>open</u> in $GL(E|F)$, being an inverse image of the open subset $GL(F)$ of $End(F)$ under the continuous mapping $a \to a_{11}$. We prove that $GL_o(E|F)$ is <u>closed</u> in $GL(E|F)$ by showing that it is the inverse image of zero under the mapping

$$GL(E|F) \to L(F_1,F_2) \; : \; a \to p_2 \circ (a)^{-1} \circ i_1 \; .$$

To check this, let

$$a = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in GL(E|F)$$

and

$$a^{-1} = b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} .$$

From $ab = ba = id_E$ we obtain

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} id_{F_1} & 0 \\ 0 & id_{F_2} \end{pmatrix} .$$

If $b_{21} = 0$, then $a_{11}b_{11} = b_{11}a_{11} = id_{F_1}$, and a belongs to $GL_o(E|F)$. If $a \in GL_o(E|F)$, then a_{11} is surjective, and so $b_{21}a_{11} = 0$ implies $b_{21} = 0$.

If F_1 is finite co-dimensional, then F_2 is finite-dimensional and $a_{22}b_{22} = id_{F_2}$ implies that a_{22} is injective. Since $a_{22}b_{21} = 0$, we again obtain $b_{21} = 0$ and so $a \in GL_o(E|F)$.

§2. BOXES, SLICES AND FOLIATIONS

If E is a normed space, we put $E^{\varepsilon} = \{x \in E : \|x\| < \varepsilon\}$, so that E^1 is the open unit ball of E. A C^r box of the manifold M is a triple (ψ,U,W), where U and W are Banach spaces and ψ is a C^r diffeomorphism of $U^1 \times W^1$ onto an open subset of M. By a slice of (ψ,U,W) we mean any of the mappings

$$\psi(-,w) : U^1 \to M, \quad w \in W^1.$$

A differentiable structure σ on the underlying set of M is a C^r foliation of M if, given $x \in M$, there exists a C^r box (ψ,U,W) of M such that

(1) $\psi(0,0) = x$;

(2) $\psi(-,0) : U^1 \to M$ is a chart of σ; and

(3) every slice of (ψ,U,W) is a differentiable function of U^1 into (M,σ).

(Here and below, (M,σ) denotes the underlying set of M equipped with the differentiable structure σ.) A foliation σ is <u>regular</u> if, given x in M, there exists a C^r box (ψ,U,W) of M such that $\psi(0,0) = x$ and

(4) every slice of (ψ,U,W) is a chart of σ.

A foliation σ which fails to be regular is said to be <u>singular</u>.

More generally, let (ψ,U,W) be a differentiable box of M and let N be an arbitrary immersed submanifold of M. We say that (ψ,U,W) <u>cuts N in open slices</u> if

(a) $N_w = \psi(-,w)^{-1}(N)$ is an open subset of U for every $w \in W$; and

(b) $\psi(-,w) : N_w \to N$ is differentiable for every $w \in W^1$.

We say that the box (ψ,U,W) is <u>parallel</u> to N if the tangent spaces to the slices are <u>contained</u> in the tangent spaces of N at every point of intersection, that is if

$$D_1\psi(u,w).U \subset T_{\psi(u,w)}N$$

whenever $\psi(u,w) \in N$.

PROPOSITION 2.1. A box of M cuts an immersed submanifold N of M in open slices if and only if it is parallel to it.

PROOF. Assume that (ψ,U,W) is parallel to N and let H be a neighbourhood of $\bar{h} = \psi(\bar{u},\bar{w})$ in N. It is sufficient to show that $\psi(\bar{u}+u,\bar{w}) \in H$ for u in some neighbourhood U^{ε} of the origin in U, for this implies that $N_{\bar{w}}$ is an open subset of U and that

$\psi(-,\bar{w}) : N_{\bar{w}} \to N$ is continuous. The differentiability of this function then follows from the fact that $\psi(-,\bar{w}) : U^1 \to M$ is differentiable and N is an immersed submanifold of M.

Taking a suitable local chart, we may identify a neighbourhood of $\bar{h}$ in M with an open subset Ω of E and assume that $\bar{h}$ is the origin of E and that H is the open unit ball F^1 of some direct summand F of E.

We may also assume that $(\bar{u},\bar{w})$ is the origin of $U \times W$ and that $\psi(u,w) \in \Omega$ for all (u,w) in $U^1 \times W^1$. Let

$$a = D\psi(0,0) \in LIS(U \times W, E).$$

Since (ψ,U,W) is parallel to N, we have

(1) $D_1\psi(u,w).U \subset F$ whenever $\psi(u,w) \in H$, and in particular

(2) $$a = aj_W p_W$$

where j_W is the inclusion $W \to U \times W : w \to (0,w)$ and p_W is the coordinate projection.

Let now G be a complement of F in E, $p_G \in End(E)$ the projection onto G along F and $p_F \in End(E)$ the complementary projection onto F. Put

$$\pi_G = p_W a^{-1} p_G a j_W \in End(W)$$

and

$$\pi_F = p_W a^{-1} p_F a j_W \in End(W).$$

Using (2), it is easily checked that

$$\pi_G^2 = \pi_G, \quad \pi_F^2 = \pi_F \quad \text{and} \quad \pi_G + \pi_F = id_W.$$

Hence π_G and π_F are complementary projections and W is the direct sum of

$$V = \pi_F W \quad \text{and} \quad Z = \pi_G W.$$

Moreover, it is easy to show that

(3) $$b = p_G a j_Z \in LIS(Z,G),$$

where j_Z is the inclusion $Z \to U \times W$: $z \to (0,z)$ and p_G is now considered as a map $E \to G$.

Let now

$$\theta : U \times V \times Z \to G$$

be defined by

$$\theta(u,v,z) = p_G \psi(u,v+z).$$

Then

(4) $\theta(u,v,z) = 0$ if and only if $\psi(u,v+z) \in H$, and so, by (1),

(5) $\theta(u,v,z) = 0$ implies $D_1\theta(u,v,z) = 0$.

Since

(6) $D_3\theta(0,0,0) = b \in LIS(Z,G)$, if follows from the implicit function theorem ([3], 10.2.1) that there exists $\varepsilon > 0$, $\delta > 0$, and a differentiable function

$$\phi : U^\varepsilon \times V^\varepsilon \to Z^\delta$$

such that, for every (u,v,z) in $U^\varepsilon \times V^\varepsilon \times Z^\delta$,

(7) $D_3\theta(u,v,z) \in LIS(Z,G)$

and

(8) $\theta(u,v,z) = 0$ if and only if $z = \phi(u,v)$.

Thus $\theta(u,v,\phi(u,v)) = 0$ for all (u,v) in $U^\varepsilon \times V^\varepsilon$. Differentiating by u, we obtain

$$D_1\theta(u,v,\phi(u,v)) + D_3\theta(u,v,\phi(u,v)).D_1\phi(u,v) = 0$$

and so, by (5), (7) and (8),

$$D_1\phi(u,v) = 0$$

for every (u,v) in $U^\varepsilon \times V^\varepsilon$. Hence

$$\phi(u,v) = \phi(0,v),$$

and it follows from (8) and (4) that

$$\psi(u,v+\phi(0,v)) \in H$$

for every (u,v) in $U^\varepsilon \times V^\varepsilon$. Since $\phi(0,0) = 0$, we have

$$\psi(u,0) \in H$$

for every u in V^ε, Q.E.D.

REMARKS. 1) A shorter proof, based on the existence theorem for ordinary differential equations, can be given if ψ and N are at least of the order C^2, or if N is finite-dimensional. If ψ or N are of order C^1, then the right-hand sides of the corresponding differential equations are generally only continuous, and if N is infinite-dimensional, then the existence theorem no longer applies.

2) A cautionary example against relaxing the assumptions of Proposition 2.1 is given by $M = R^2$, $N = R \times 0$, and $\psi(u,w) = (u,(u-w)^3)$ (see Fig. 1).

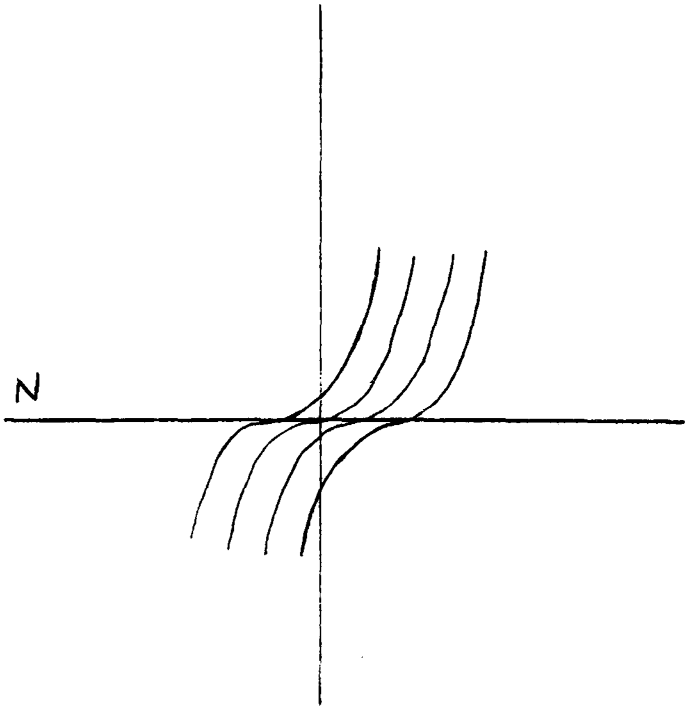

Figure 1

§3. C^1 SUBMANIFOLDS AND FLOWS

PROPOSITION 3.1. Let N be a C^1 immersed submanifold of M and let X be a C^1 vectorfield on M such that $X(x) \in T_xN$ for every x in N. Let

$$\Delta = \{(t,x) \in \mathbb{R} \times N : X^t.x \in N \text{ and } X^s.x \in N \text{ for all s between } 0 \text{ and } t\} .$$

Then Δ is open in $R \times N$ and the function

$$\Delta \to N : (t,x) \to X^t.x$$

is differentiable.

REMARK. We cannot use the usual existence, uniqueness and 'dependence on initial conditions' theorems because the 'restriction' of X to N is generally only a C^0 vectorfield. The proof is similar to the proof of Proposition 2.1 and is given here only for the sake of completeness.

LEMMA 3.2. Assume that a C^1 function $\phi : R \times M \to M$ satisfies the following properties:

(i) the domain of ϕ is an open neighbourhood of $0 \times M$ and $\phi(0,x) = x$ for every $x \in M$;

(ii) $D_1\phi(t,x) \in T_yN$ whenever $\phi(t,x) = y \in N$.

If $x \in N$, then there exists a neighbourhood H of x in N and $\varepsilon > 0$ such that

(a) $\phi(t,y) \in N$ for every $(t,y) \in R^\varepsilon \times H$ and

(b) $\phi : R^\varepsilon \times H \to N$ is differentiable.

PROOF. Using a suitable chart to identify a neighbourhood of x in M with an open subset Ω of E, we may assume that x is the origin of E and that there exist two complementary closed subspaces F and G of E such that $\Omega = F^1 + G^1$ and F^1 is a neighbourhood of x in N.

Let p_F be the projection of E onto F along G and let p_G be the complementary projection of E onto G. Since $\phi(t,y) \in F^1$ implies $\phi(t,y) = p_F\phi(t,y)$ and $p_F \circ \phi$ is a differentiable function into F, it is sufficien to find $\varepsilon > 0$ such that $\phi(t,y) \in F^1$ for every $(t,y) \in R^\varepsilon \times F^\varepsilon$.

If $h : R \times F \times G$ is defined by $h(t,\xi,\eta) = p_G\phi(t,\xi+\eta)$, then, by (ii),

(1) $D_1h(t,\xi,\eta) = 0$ whenever $h(t,\xi,\eta) = 0$.

Since

(2) $D_3h(0,\xi,\eta) = p_G \circ D_2\phi(0,\xi+\eta) \circ i_G = p_G \circ i_G = id_G$, (where i_G is the inclusion $G \to E : \eta \to (0,\eta)$) and since $h(0,0,0) = 0$, the inverse function theorem implies the existence of $\varepsilon > 0$, $\delta > 0$ and a C^1 function $\theta : R^\varepsilon \times F^\varepsilon \to G^\delta$ such that, for all $(t,\xi,\eta) \in R^\varepsilon \times F^\varepsilon \times G^\delta$,

(3) $D_3h(t,\xi,\eta) \in GL(G)$ and

(4) $h(t,\xi,\eta) = 0$ is equivalent to $\eta = \theta(t,\xi)$.

Differentiating by t the equation $h(t,\xi,\theta(t,\xi)) = 0$, and using (1) and (3), we deduce that

$$D_1\theta(t,\xi) = 0$$

and so, for $(t,\xi) \in R^\varepsilon \times F^\varepsilon$,

$$\theta(t,\xi) = \theta(0,\xi).$$

Hence $\theta(t,\xi) = p_G\phi(0,\xi+\theta(t,\xi)) = h(0,\xi,\theta(t,\xi)) = h(0,\xi,\theta(0,\xi)) = 0$, and therefore

$$h(t,\xi,0) = h(t,\xi,\theta(t,\xi)) = 0,$$

which proves that $\phi(t,\xi) \in F$ for every $(t,\xi) \in R^\varepsilon \times F^\varepsilon$, Q.E.D.

PROOF OF PROPOSITION 3.1. Let $(t,x) \in \Delta$ and assume, for example, that $t \geq 0$. Assume that $\tau \in [0,t]$ has the following property:

(5) there exists a neighbourhood H of x in N and $\varepsilon > 0$ such that $\phi([\tau-\varepsilon,\tau+\varepsilon] \times H) \subset N$ and $\phi : [\tau-\varepsilon,\tau+\varepsilon] \times H \to N$ is differentiable.

Using Lemma 3.2 and the equation $\phi(\tau+s,y) = \phi(\delta,\phi(\tau+s-\delta,y))$, it is easy to show that the set I_o of those $\tau \in [0,t]$ which satisfy (5) is open and closed in $I = [0,t]$ and contains 0. Hence $I_o = I$, Q.E.D.

§4. DIFFERENTIABLE DISTRIBUTIONS

We say that the distribution B is <u>differentiable</u> at a point $x \in M$, if there exists a <u>differentiable</u> section f of the bundle $LIS(T_xM, TM)$, defined on a neighbourhood Ω of x, such that $f(x)B_x = B_x$ and $f(y)B_x \subset B_y$ for every y in Ω. We call such a section a <u>stem</u> of B at x and we usually assume that $f(x) = id_{T_xM}$.

A stem f can also be described as a vector-bundle isomorphism

$$\Omega \times T_xM \to TM_\Omega$$

which is the identity on $\{x\} \times T_xM$ and maps $\Omega \times B_x$ into $B|_\Omega$. The stem f is said to be <u>regular</u> if $f(y)B_x = B_y$ for every $y \in \Omega$.

Locally*), f is represented by a differentiable function

$$\Omega \to GL(E).$$

There is a weaker definition of differentiability, where we only ask for the function

$$\Omega \times T_xM \to TM \;:\; (y,v) \to f(y).v$$

to be differentiable (implying that f is C^{r-1}). Such f is called a <u>weak stem</u> of B and B is then said to be <u>weakly differentiable at x</u>.

(4.1) The distribution B is <u>differentiable</u> (or <u>weakly differentiable</u>) it is differentiable (weakly differentiable) at every x in M. B is <u>regular (0.1)</u> if and only if it has a regular <u>differentiable</u> stem at every $x \in M$.

---

*) This word always signals that we are using some local chart to identify a neighbourhood of a point in M with an open subset of E, tangent spaces with E, vectorfields with their principal parts,...

If B_x is <u>finite dimensional</u>, then B is differentiable at x if and only if there exist differentiable vectorfields $X_1, X_2, \ldots, X_k$ in B whose values at x form a basis of B_x.

To see this, assume that $TM|_\Omega = \Omega \times E$, and put,

$$g(y).v = \ell_1(v)X_1(y) + \ell_2(v)X_2(y) + \ldots + \ell_k(v)X_k(y),$$

where $\ell_i : E \to R$ are continuous linear functionals such that $\ell_i(X_j(x)) = \delta_{ij}$. It is then sufficient to put

$$f(y) = \mathrm{id}_E + g(y) - g(x)$$

and to note that $f(y) \in GL(E)$ for y sufficiently near to x.

If $\operatorname{codim} B_x < \infty$, then, as we show in the next section, the differentiability of B at x can be described in terms of finitely many real-valued differentiable one-forms.

§5. DIFFERENTIABILITY AND VECTOR-VALUED ONE-FORMS

We recall that, given a Banach space F, an F-valued one-form on M is a differentiable section of the vector bundle $L(TM,F) = \bigcup_{x \in M} L(T_xM,F)$. Locally, such a form is represented by a C^r-mapping $\Omega \to L(E,F)$ (cf. [13], 8.3.1).

PROPOSITION 5.1. A distribution B is differentiable at x if and only if there exists a Banach space F and F-valued differentiable one-form ω, defined on a neighbourhood Ω of x, such that

(a) $\omega(x) : T_xM \to F$ is surjective and $\mathrm{Ker}\,\omega(x) = B_x$; and

(b) for every y in Ω, $\mathrm{Ker}\,\omega(y) \subset B_y$.

PROOF. We may assume that $TM|_\Omega = \Omega \times E$. Let G be a complement of B_x in E, and let $j : G \to E$ be the inclusion mapping. By the closed graph theorem, $\omega(x) \circ j \in \mathrm{LIS}(G,F)$, and, taking Ω sufficiently small, we may assume that $\omega(y) \circ j \in \mathrm{LIS}(G,F)$ for every $y \in \Omega$. Let

$$\gamma(y) = j \circ (\omega(y) \circ j)^{-1} : F \to E$$

and put

$$p(y) = \gamma(y)\,\omega(y) : E \to E.$$

Since $\omega(y)\gamma(y) = id_F$, we have $\omega(y)p(y) = \omega(y)$ and $p^2(y) = p(y)$. Put

$$f(y) = id_E + p(x) - p(y).$$

Taking Ω smaller if necessary, we may assume that $f(y) \in GL(E)$ for every y in Ω. If $g \in G$, then $p(x)g = g$ and

$$\omega(y)f(y)g = 2\omega(y)g - \omega(y)p(y)g = \omega(y)g.$$

If $b \in B_x$, then $p(x)b = 0$ and

$$\omega(y)f(y)b = \omega(y)b - \omega(y)p(y)b = 0.$$

Hence

$$\omega(y)f(y)(b+g) = \omega(y)g = 0 \text{ if and only if } g = 0,$$

and therefore

$$f(y)B_x = \mathrm{Ker}\,\omega(y) \subset B_y,$$

so that f is a differentiable stem*) of B.

---

*) Note that f is <u>regular</u> if $\mathrm{Ker}\,\omega(y) = B_y$ for every y in Ω.

Conversly, given a stem f of B at x, it is sufficient to put

$$F = p(T_x M) \text{ and } \omega(y) = p \circ (f(y))^{-1},$$

where $p \in \mathrm{End}(T_x M)$ is a projection and $B_x = \mathrm{Ker}\, p$.

COROLLARY 5.1. Suppose that $\mathrm{codim}\, B_x = k < \infty$. Then B is differentiable at x if and only if there exist differentiable real-valued one-forms $\omega_1, \omega_2, \dots, \omega_k$ on a neighbourhood Ω of x such that

(a) $\omega_1(x), \omega_2(x), \dots, {}_k(x)$ is a basis of the annihilator B_x^o of B_x in T_x^*M; and,

(b) for every $y \in \Omega$, B_y^o is contained in the span of $\omega_1(y), \omega_2(y), \dots, \omega_k(y)$.

PROOF. Take $F = R^k$ and $\omega = (\omega_1, \omega_2, \dots, \omega_k)$.

§6. HOMOGENEOUS STEMS AND LOCAL INTEGRABILITY

Throughout this section we assume that f is a <u>weak stem</u> (§4) of the distribution B at a point $x \in M$. We say that f is <u>homogeneous</u> if the vectorfields

$$y \to f(y).v, \quad v \in B_x$$

<u>respect</u> the distribution B (see Introduction, (0.5)).

A differentiable box (ψ, U, W) is said to be <u>parallel</u> to B if

$$D_1\psi(u,w).U \subset B_{\psi(u,w)}$$

for every $(u,w) \in U^1 \times W^1$. Recall (§2) that a <u>slice</u> of (ψ, U, W) is any of the functions

$$\psi(-,w) : U^1 \to M , w \in W^1.$$

By abuse of language, the set $\psi(U^1,w)$, together with its differentiable structure of an embedded submanifold of M, is also called a slice of ψ.

PROPOSITION 6.1. Let f be a homogeneous weak stem of B at the point x. There exists a differentiable box (ψ,U,W) of M such that

(1) $\psi(0,0) = x$;

(2) (ψ,U,W) is parallel to B;

(3) the slice $\psi(U^1,0)$ of ψ is an integral manifold of B; and

(4) every point of $\psi(U^1,0)$ can be reached from x along an integral curve of a vectorfield in B.

COROLLARY 6.1. Let N be an integral manifold of B. Then

(5) (ψ,U,W) cuts N in open slices and

(6) if $Z_x = \psi(U^1,0)$, then $Z_x \cap N$ is an open subset of both Z_x and N and inherits the same differentiable structure from Z_x and N.

PROOF OF THE COROLLARY. The assertion (5) follows from (2) above and from Proposition 2.1. The assertion (6) follows at once from (5), (3), and the inverse function theorem.

PROOF OF PROPOSITION 6.1. We may assume that the domain Ω is an open subset of E and that

$$f : \Omega \to GL(E),$$

where the function

$$\Omega \times E \to E : (y,v) \to f(y).v$$

is differentiable. Let

$$\xi(t,y,u)$$

be the value at t of the maximal solution of the equations

$$\dot{\xi} = f(\xi)u, \quad \xi(0) = y. \tag{6.1.1}$$

Then

(A) $\mathrm{dom}\xi$ is an open neighbourhood of $(0,x,0)$ in $R \times \Omega \times E$;

(B) $\xi : \mathrm{dom}\xi \to \Omega$ is differentiable;

(C) if $\xi(t,y,u) \in \mathrm{dom}\xi$ and $s \neq 0$ is a real number, then $(t/s,y,su) \in \mathrm{dom}\xi$ and $\xi(t/s,y,su) = \xi(t,y,u)$.

Here (A) and (B) are deduced easily from, say [3], (10.7.4) and (C) follows from the uniqueness theorem because $\eta(t) = \xi(t/s,y,su)$ is a solution of (6.1.1).

(D) $D_3\xi(t,y,u)B_x \subset B_z$, $z = \xi(t,y,u)$.

To prove this, fix $u \in B_x$ and $y \in \Omega$ and put $y_t = \xi(t,y,u)$, $\alpha_t = D_3\xi(t,y,u)$ and $\gamma_t = D_2\xi(t,y,u)$. Let the function $g : \Omega \to E$ be defined by

$$g(y) = f(y).u.$$

Then $\alpha_t \in \mathrm{End}(E)$, $\gamma_t \in GL(E)$ and (cf. [3], (10.7.3.1), (10.8.4.1))

$$\dot{y}_t = g(y), \quad y_o = y \; ;$$

(6.1.D.1)
$$\dot{\gamma}_t = Dg(y_t)\gamma_t, \quad \gamma_o = id_E \; ;$$
$$\dot{\alpha}_t = Dg(y_t)\alpha_t + f(y_t), \quad \alpha_o = 0.$$

Hence

(6.1.D.2)
$$\alpha_t = \gamma_t \int_0^t (\gamma_s)^{-1} f(y_s)ds \; .$$

By homogeneity of f, the vectorfield $f(-).u$ respects B and we have

$$(\gamma_s)^{-1} B_{y_s} = B_y$$

(cf. Introduction, (0.5)) and so

$$(\gamma_s)^{-1} f(y_s) B_x \subset B_y .$$

Hence

$$\int_0^t (\gamma_s)^{-1} f(y_s)ds$$

maps B_x into B_y, and α_t maps B_x into $\gamma_t\ B_y$. Using the homogeneity of f once more, we have

$$\alpha_t B_x \subset \gamma_t B_y = B_{y_t} = B_z ,$$

which proves (D).

Note also that $u = 0$ implies $y_t = y_o = y$, $\gamma_t = \gamma_o = id_E$, and therefore

$$\alpha_t = \int_0^t f(y)ds = tf(y),$$

or

(E)
$$D_3\xi(t,y,0) = tf(y)$$

It follows from (A) and (C) that there exists $\varepsilon > 0$ such that $\xi(1,y,u)$ is defined for $\|x-y\| < \varepsilon$ and $\|u\| < \varepsilon$. Since

$$D_3\xi(1,x,0) = f(x) = id_E ,$$

we may assume that

$$D_3\xi(1,x,u) \in GL(E) \quad \text{for} \quad \|u\| < \varepsilon.$$

We claim that, for $u \in B_x$ and $\|u\| < \varepsilon$,

(F) $$D_3\xi(1,x,u).B_x = B_z, \text{ where } z = \xi(1,x,u).$$

To prove (F), put $z_u = \xi(1,x,u)$, $\beta_u = D_3\xi(1,x,u)$ and $\delta_u = D_2\xi(1,x,u)$. The homogeneity of f implies that

$$\delta_u B_x = B_{z_u} \quad \text{for} \quad u \in B_x.$$

Now, by (D), $u \in B_x$ implies that

$$\beta_u B_x \subset B_{z_u}$$

and so

$$(\delta_u)^{-1}\beta_u B_x \subset B_x,$$

or (cf. §1.2)

$$\theta(u) = (\delta_u)^{-1}\beta_u \in GL(E|B_x)$$

Since $B_x^\varepsilon = \{u \in B_x : \|u\| < \varepsilon\}$ is <u>connected</u>, $\theta : B_x^\varepsilon \to GL(E|B_x)$ is continuous, and $\theta(0) = (\delta_o)^{-1}\beta_o = id_E \in GL_o(E|B_x)$, it follows from Proposition 1.2 that

$$\theta(u) \in GL_o(E|B_x)$$

for every $u \in B_x^\varepsilon$. Hence $\theta(u)B_x = B_x$ and so

$$\beta_u B_x = \delta_u \theta(u) B_x = \delta_u B_x = B_{z_u}$$

for $u \in B_x^\varepsilon$, which proves the assertion (F).

(G) Taking s = t in (C) we see that, for $u \in B_x$,

$$\xi(1,x,tu) = \xi(t,x,u)$$

is an integral curve of a differentiable vectorfield in B.

Let now Q be a submanifold of Ω such that $x \in Q$ and T_xQ is a complement of B_x in E, and let

$$\phi : W^1 \to Q, \quad \phi(0) = x$$

be a diffeomorphism of the unit ball W^1 of a Banach space W onto a neighbourhood of x in Q. Let $U = B_x$ and let ψ from $U \times W$ into M be defined by

$$\psi(u,w) = \xi(1,\phi(w),u).$$

Since $D_2\xi(1,x,0) = id_E = D_3\xi(1,x,0)$ (see (F)), it follows from the closed graph theorem that

$$D\psi(0,0) \in LIS(U \times W,E)$$

and we may assume that ψ is a diffeomorphism of $U^\varepsilon \times W^\varepsilon$ onto an open subset of Ω. Multiplying the norms of U and W by $1/\varepsilon$, we turn U^ε and W^ε into U^1 and W^1 and (ψ,U,W) into a differentiable box of M. The assertions (2), (3) and (4) of Proposition 6.1 follow now from (D), (F) and (G).

§7. INTEGRABLE DISTRIBUTIONS

Recall that a distribution B on M is homogeneous if every differentiable vectorfield in B respects B (see Introduction, (0.6) and (0.7)) and that B is said to be integrable if there exists a differentiable structure σ on M such that (M,σ) is an integral manifold of B.

THEOREM 1. Let B be a weakly differentiable distribution on M.

(a) The following conditions are equivalent:

(1) B is integrable.

(2) For every x in M, there exists an integral manifold of B containing x.

(3) B is homogeneous.

(4) For every x in M, there exists a homogeneous weak stem of B at x.

(b) If B is integrable, then there exists a unique differentiable structure σ on M such that (M,σ) is an integral manifold of B. Furthermore,

(5) Every integral manifold of B is an open submanifold of (M,σ).

(6) σ is a foliation of M in the sense of §2.

(7) Two points of M belong to the same connected component of (M,σ) if and only if they can be joined by finitely many integral curves of differentiable vectorfields in B.

PROOF OF THEOREM 1. To show that $(2) \Rightarrow (3)$, let X be a differentiable vectorfield in B, let $x \in$ domain of X and let Z be an integral manifold of B through the point x. By Proposition 3.1, there exists a neighbourhood H_1 of x in Z and $\varepsilon_1 > 0$ such that $X^t.y \in Z$ for $|t| < \varepsilon$ and $y \in H^1$ and $(t,y) \to X^t.y : R \times Z \to Z$

is differentiable. Choose $\varepsilon > 0$ and a neighbourhood H of x such that $X^t.y \in H_1$ for $|t| < \varepsilon$ and $y \in H$, $H \subset H_1$ and $\varepsilon \leq \varepsilon_1$. Differentiating the equations $X^{-t}(X^t.y) = y = X^t(X^{-t}.y)$ with respect to $y \in Z$ we prove that, for $|t| < \varepsilon$, $(X^t)*(x)$ is an isomorphism of $T_xZ = B_x$ onto $T_yZ = B_y$. The result now follows from Lemma 1.1, Part Three.

Let us now assume that B satisfies (4). For every $x \in M$, let Z_x be the integral manifold of B described in Corollary 6.1. By (6.1.6), the differentiable structures of Z_x and Z_y coincide on $Z_x \cap Z_y$ and hence define a differentiable structure σ on M such that each Z_x is an open submanifold of (M,σ) ([13], 5.2.4). It is clear that (M,σ) is an integral manifold of B and so $(4) \Rightarrow (1)$. Furthermore, it follows from (6.1.6) that every integral manifold of B is an open submanifold of σ. Hence $(4) \Rightarrow (5)$. In particular, σ is the <u>unique</u> differentiable structure which makes M into an integral manifold of B.

The assertion (6) now follows at once from (6.1.2), (6.1.3) and Proposition 2.1, where we take $N = (M,\sigma)$.

Finally let ζ be the equivalence relation 'x and y can be joined by finitely many integral curves of differentiable vectorfields in B'. By Proposition 3.1, the integral curves of vectorfields in B are continuous as functions $R \to (M,\sigma)$, and so the equivalence classes of ζ are <u>connected</u>. On the other hand, (6.1.4) shows that each equivalence class of ζ is <u>open</u> in (M,σ), which proves the assertion (7).

§8. THE USE OF LIE BRACKETS

8.1. NOTATION. If X and Y are differentiable vectorfields on M and $x_t = X^t.x$, then locally,

$$[X,Y](x_t) = DY(x_t)X(x_t) - DX(x_t)Y(x_t) = \frac{d}{dt}Y(x_t) - DX(x_t)Y(x_t).$$

This leads to the following definitions.

(8.1.1) A vectorfield over a curve $\sigma : I \to M$ (where I is an interval in R) is a curve $Y : I \to TM$ such that $Y(t) \in T_{\sigma(t)}M$ for all $t \in I$.

(8.1.2) If $\sigma(t) = x_t = X^t.x$ for some differentiable vectorfield X on an open subset of M, and if Y is a differentiable vectorfield over σ, we define the vectorfield $[X,Y]$ over σ by the local coordinate formula

$$[X,Y](t) = \frac{d}{dt}Y(t) - DX(x_t)Y(t).$$

(To show that this formula behaves well under C^2 changes of coordinates, let ϕ be a C^2 local diffeomorphism of E and let

$$\tilde{Y}(t) = D\phi(x_t).Y(t), \quad \tilde{X}(x) = D\phi(\phi^{-1}(x)).X(\phi^{-1}(x)).$$

Then

$$D\tilde{X}(x).v = D^2\phi(\phi^{-1}(x)).(D\phi^{-1}(x).v).X(\phi^{-1}(x)) + $$
$$+ D\phi(\phi^{-1}(x)).DX(\phi^{-1}(x)).D\phi^{-1}(x).v,$$

and it is easily checked that

$$\frac{d}{dt}\tilde{Y}(t) - D\tilde{X}(\phi(x_t)).\tilde{Y}(t) = D\phi(x_t).[X,Y](t).)$$

(8.1.3) Let F be a Banach space and let

$$f : I \to L(F,TM)$$

be a C^1 curve in the vector-bundle $L(F,TM)$ such that $f(t) \in L(F,T_{\sigma(t)}M)$ where $\sigma(t) = x_t = X^t.x$ as above. If $v \in F$, we write

(8.1.3.1) $\bar{v} : I \to TM : t \to f(t).v$ for the corresponding vectorfield over σ. The Lie derivative of f with respect to X is the curve

$$L_X f : I \to L(F,TM)$$

(which again covers σ), defined by the formula

(8.1.3.2) $L_X f(t) : F \to T_{\sigma(t)}M : v \to [X,\bar{v}](t)$, or, locally,

(8.1.3.3) $L_X f(t) = \dot{f}(t) - DX(x_t) \circ f(t)$.

8.2 A CONDITION OF RESPECTABILITY. The next theorem is a straight generalization of Theorem 6 in Part One.

THEOREM 2. Let X be a C^1 vectorfield on an open subset of M, $x \in$ domain X, $x_t = X^t.x$, and $\gamma_t = (X^t)^*(x)$. Let B be an arbitrary (not necessarily differentiable) distribution on M. Assume that there exist $\varepsilon > 0$, a Banach space F, a differentiable function

$$f : R^\varepsilon \to L(F,TM)$$

and a continuous function

$$\Lambda : R^\varepsilon \to \mathrm{End}(F),$$

such that

(8.2.1) f covers the integral curve $t \to x_t$; amd, for all $t \in R^\varepsilon$,

(8.2.2) $f(t)F = B_{x_t}$ and

(8.2.3) $L_X f(t) = f(t)\Lambda(t)$.

Then, for all $t \in R^{\varepsilon}$,

(8.2.4) $$\gamma_t B_x = B_{x_t}.$$

COROLLARY 2. If the assumptions of Theorem 2 are satisfied at every $x \in$ domain X, then X respects the distribution B (Lemma 1.1, Part Three).

In particular, a weakly differentiable distribution B on M is integrable if (and, as we shall see in the next section, only if) the assumptions of Theorem 2 are satisfied for every differentiable vectorfield X in B and every $x \in$ domain X.

REMARK. It will be seen from the proof of Theorem 2 that its assertion is valid even if the vector spaces in the distribution B are <u>not</u> direct summands of the tangent spaces. We have to assume, however, that they are closed.

PROOF OF THEOREM 2. Let
$$\alpha_t = (\gamma_t)^{-1} f(t) : F \to T_x M.$$
Locally,
$$\dot{\alpha}_t = -\gamma_t^{-1}\dot{\gamma}_t\gamma_t^{-1} f(t) + \gamma_t^{-1}\dot{f}(t) = (\gamma_t)^{-1}(\dot{f}(t) - DX(x_t)f(t))$$
and so (8.1.3.3)
$$\dot{\alpha}_t = (\gamma_t)^{-1} L_X f(t) = (\gamma_t)^{-1} f(t)\Lambda(t) = \alpha_t \circ \Lambda(t).$$
Let now $h \in (T_x M)^*$ and let
$$h_t = h\alpha_t = (\alpha_t)^* h$$
Then
$$\dot{h}_t = h_t \Lambda(t) = \Lambda^*(t) h_t,$$

and, by the uniqueness theorem, $h_t = 0$ if and only if $h_o = 0$. This means that h vanishes on $B_x = \alpha_o F$ if and only if it vanishes on $\alpha_t F = (\gamma_t)^{-1} f(t) F = (\gamma_t)^{-1} B_{x_t}$, and so, by the Hahn-Banach theorem,

$$B_x = (\gamma_t)^{-1} B_{x_t},$$

Q.E.D.

8.3. COVARIANT STEMS. Let X be a C^1 vectorfield on an open subset of M and let $x \in$ domain X and $x_t = X^t.x$. If B is a distribution on M, we define an <u>X-covariant B-stem at x</u> as a C^1 function

$$f : R^\varepsilon \to \mathrm{LIS}\,(T_x M, TM)$$

which covers the integral curve $t \to x_t$ and satisfies the following conditions:

(8.3.1) $f(0) = id_{T_x M}$, and, for all $t \in R^\varepsilon$,

(8.3.2) $f(t)B_x = B_{x_t}$ and

(8.3.3) $L_X f(t) B_x \subset B_{x_t}$,

or equivalently,

(8.3.3.a) $[X,\bar{v}](t) \in B_{x_t}$ for all $v \in B_x$,

where the vectorfield $\bar{v}$ over t is defined by (8.1.3.1)

Let now

$$i : B_x \to T_x M \quad \text{and} \quad p : T_x M \to B_x$$

be the inclusion mapping and the projection along some complement of B_x in $T_x M$. Let

$$\Lambda(t) = f(t)^{-1} L_X f(t) \in \mathrm{End}(T_x M)$$

and let

$$\bar{f} : R^{\varepsilon} \to L(B_x, TM)$$

and

$$\bar{\Lambda} : R \to \mathrm{End}(B_x)$$

be defined by $\bar{f}(t) = f(t) \circ i$ and $\bar{\Lambda}(t) = p \circ \Lambda(t) \circ i$. As these functions obviously satisfy the conditions of Theorem 2, we have proved the following lemma.

LEMMA 8.3. Let $\gamma_t = (X^t)^*(x)$ and assume that there exists and X-covariant B-stem at x defined for $|t| < \varepsilon$. Then, for $|t| < \varepsilon$,

$$\gamma_t B_x = B_{x_t}.$$

THEOREM 3. Let X be a C^1 vectorfield defined on an open subset of M and let B be an arbitrary (not necessarily differentiable) distribution on M. Then X respects B if and only if, for every x in the domain of X, there exists an X-covariant B-stem at x.

PROOF. If X-covariant B-stems exist, then the assertion follows at once from Lemma 8.3 and from Lemma 1.1 in Part Three. Conversly, if X respects B and x belongs to the domain of X, we put

$$f(t) = \gamma_t = (X^t)^*(x) \in \mathrm{LIS}(T_x M, T_{x_t} M).$$

Locally,

$$L_X f(t) = \dot{\gamma}_t - DX(x_t)\gamma_t = 0$$

and so f obviously satisfies the conditions (8.3.1) - (8.3.3), Q.E.D.

COROLLARY 3. A weakly differentiable distribution B is integrable if and only if, for every differentiable vectorfield X in B, and for every $x \in$ domain X, there exists an X-covariant B-stem at x.

This follows at once from Theorems 1 and 2.

8.4. THE STANDARD FROBENIUS THEOREM. Recall that a distribution B is said to be involutive if the Lie bracket of any two differentiable vectorfields in B lies in B. The following proposition follows at once from ([4], 17.14.3.5).

PROPOSITION 8.4. Every integrable distribution is involutive.

We can now state the 'standard' Frobenius theorem.

THEOREM 4 (Frobenius). A regular distribution B is integrable if and only if it is involutive.

PROOF. Let X be a vectorfield in B, x a point in the domain of X and $x_t = X^t.x$. Let Ω be a neighbourhood of x in M and let $f : \Omega \to \mathrm{LIS}(T_xM,TM)$ be a regular C^r stem of B at the point x. By regularity , $f(y)B_x = B_y$ for every y in U and so, if B is involutive, $t \to f(x_t)$ is an X-covariant B-stem at x, and the integrability of B follows at once from Corollary 3.

REMARKS. 1) Note that the stem f in the proof above has to be differentiable (rather than weakly differentiable, cf. the definition of the regular distribution in (4.1)).

2) There exist involutive non-integrable C^∞ distributions on R^2 (cf. Example 8.5.2). Hence, the regularity of B in Theorem 4 is, in general, essential. We shall see below that the situation is simpler in the real-analytic case.

8.5. REAL ANALYTIC DISTRIBUTIONS. The basic facts about real analytic functions on Banach spaces and real analytic manifolds are collected in [13], §§3 and 5. We need the result that the integral curves of real analytic vectorfields are real analytic functions of time ([13], 9.1.8).

If $B = (B_x : x \in M)$ is a distribution on M and T is a subset of M, we say that $B|T$ is <u>spanned</u> by a set S of vectorfields on M if $B_y = \text{span}\{X(y) : X \in S\}$ for every $y \in T$.

We write $C^\omega(B,T)$ for the set of real analytic vectorfields in B whose domain includes T, and we say that B is <u>locally everywhere defined</u> if, for every vectorfield X in B, and every $x \in$ domain X, there exists $\varepsilon > 0$ such that

$$B|\{x_t : |t| < \varepsilon\} \text{ is spanned by } C^\omega(B,\{x_t : |t| < \varepsilon\}),$$

where $x_t = X^t.x$. Note that the word 'locally' refers here to a small portion of an integral curve, rather than to a neighbourhood of x in M (cf. Example 8.5.1).

The next theorem generalizes a result of Nagano [7].

THEOREM 5. A real analytic distribution is integrable if and only if it is involutive and locally everywhere defined.

PROOF. Let B be an involutive, locally everywhere defined real analytic distribution on M, X a real analytic vectorfield in B, $x \in$ domain X, $x_t = X^t.x$ and $\gamma_t = (X^t)^*(x)$.

Let Ω be a neighbourhood of x and let

$$f : \Omega \to L(T_xM, TM)$$

be a real analytic stem of B at x. If B is the distribution on Ω given by

$$\tilde{B}_y = f(y)B_x,$$

then, clearly, $t \to f(x_t)$ is an X-covariant $\tilde{B}$-stem at x, and so, by Lemma 8.3, there exists $\varepsilon > 0$ s.t.

$$(8.5.0) \qquad \gamma_t B_x = \tilde{B}_{x_t} \subset B_{x_t} \qquad \text{for} \quad |t| < \varepsilon.$$

Let now $v \in B_{x_t}$. Then $v = Y(x_t)$ for some real analytic vectorfield Y in B. Since B is locally everywhere defined, we may assume that $x_s \in$ domain Y for all s, $|s| < \varepsilon$. Writing $Y(s)$ instead of $Y(x_s)$, we put

$$v(s) = (\gamma_s)^{-1}Y(s) \in T_xM.$$

A simple computation of the usual kind shows that

$$\dot{v}(s) = (\gamma_s)^{-1}[X,Y](s) = (\gamma_s)^{-1}(\mathrm{ad}X.Y)(s).$$

Hence, by induction,

$$\frac{d^n}{ds^n} v(s) = (\gamma_s)^{-1}((\mathrm{ad}X)^n.Y)(s),$$

and in particular, as B is involutive,

$$\frac{d^n}{ds^n} v(0) = ((\mathrm{ad}X)^n.Y)(x) \in B_x$$

Let now h be an arbitrary linear functional in $(T_xM)^*$ and let

$$H(s) = h.v(s).$$

Then H is a real analytic function of s; if h vanishes at B_x, then all the derivatives of H vanish at $s = 0$ and so $H(s) \equiv 0$. Hence, by the Hahn-Banach theorem,

$$v(t) = (\gamma_t)^{-1}Y(x_t) \in B_x,$$

and so $(\gamma_t)^{-1}B_{x_t} \subset B_x$, or $B_{x_t} \subset \gamma_t B_x$. Combining this with (8.5.0), we see that

$$\gamma_t B_x = B_{x_t} \quad \text{for} \quad |t| < \varepsilon$$

and the result now follows from Theorem 1 and from Lemma 1.1 in Part Three.

EXAMPLE 8.5.1. Let M = R and let B be the real analytic distribution on R spanned by the vectorfields X_1 and X_2, where $X_1 = 0$ and X_2 is defined on R^+ by $X_2(x) = (1/x).\partial/\partial x$. Then B is clearly <u>integrable</u>, and the origin of R has <u>no</u> neighbourhood Ω such that

$$B|\Omega \text{ is spanned by } C^\omega(B,\Omega).$$

EXAMPLE 8.5.2. Let $M = R^2$ and let B be the real analytic distribution on R^2 spanned by the vectorfields X_1 and X_2, where $X_1 = \partial/\partial\xi$ and X_2 is defined for $\xi > 0$ by $X_2(\xi,\eta) = (1/\xi)\partial/\partial\eta$. (See Fig. 2)

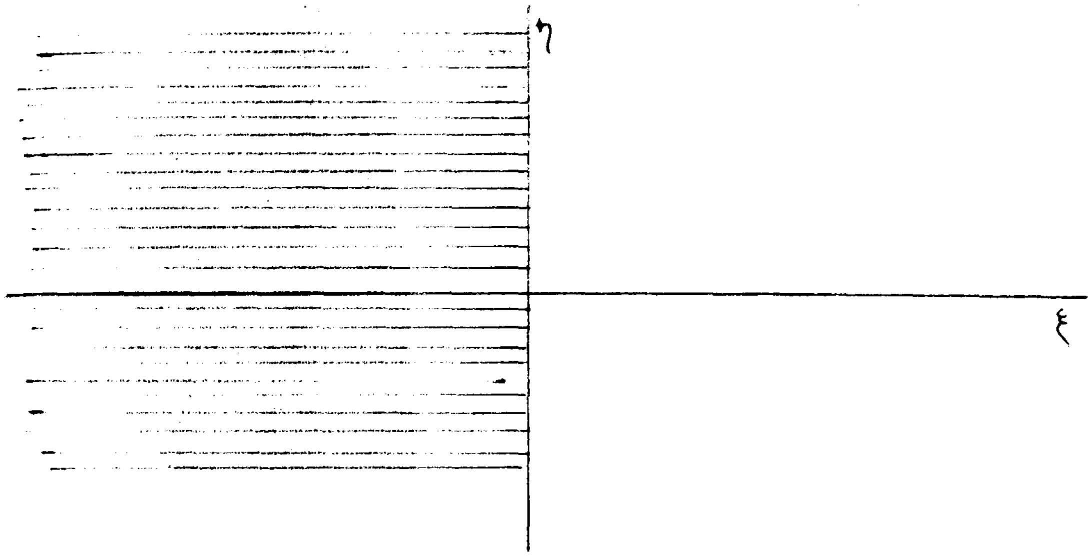

Figure 2

If $x \in$ (η-axis) and X is a real analytic vectorfield in B defined on a connected neighbourhood of x, then

$$X = \alpha\frac{\partial}{\partial\xi} + \beta\frac{\partial}{\partial\eta}$$

where β is a real analytic function vanishing for $\xi < 0$. Hence $\beta = 0$, and it is easily seen that B is an example of a <u>non-integrable involutive</u> real analytic distribution on R^2.

§9. NEAT LEAVES

Throughout this section, B is an integrable C^1 distribution on the C^2 manifold M and σ is the C^1 structure which makes (M,σ) into an integral manifold of B (§7). Unless otherwise stated, the words 'differentiable', 'diffeomorphism' etc. refer to the class C^1.

9.1 LOCAL AUTOMORPHISMS. By a local automorphism of B we understand a bijection of an open subset of M onto another open subset of M which is a local diffeomorphism for both M and (M,σ). The set of local automorphisms of B is denoted by Loc Aut B.

LEMMA 9.1.1. A local diffeomorphisms ϕ of M belongs to Loc Aut (B) if and only if

(9.1.a) $$\phi^*(x)B_x = B_y \quad \text{whenever} \quad \phi(x) = y.$$

PROOF. Assume that ϕ satisfies (9.1.a). It is clearly sufficient to show that ϕ is differentiable relatively to σ. If x belongs to the domain of ϕ and Z is an integral manifold of B which contains x and is contained in the domain of ϕ, then $\phi(Z)$, with the differentiable structure defined by the bijection $\phi| : Z \to \phi(Z)$, is an immersed submanifold of M passing through $y = \phi(x)$. The result now follows from Theorem 1 (5).

Let θB denote the set of all the local diffeomorphisms ϕ of M such that $\phi : x \to X^t.x$ for some differentiable vectorfield X in B and some $t \in R$, and let ΨB denote the set which consists of id_M and all the finite compositions of members of θB.

LEMMA 9.1.2. ΨB is contained in Loc Aut B and closed under the operations of restricting the domain, composition, and taking an inverse. Two points x and y in M belong to the same connected component of (M,σ) if and only if $y = \phi(x)$ for some $\phi \in \Psi B$.

This follows at once from Theorem 1 ((3) and (7)) and from Lemma 9.1.1.

We define the normaliser $N\Psi B$ of ΨB as the set of all local automorphisms $\phi \in$ Loc Aut B such that

(9.1.b) $\qquad \phi\Psi B\phi^{-1} \subset \Psi B \quad$ and $\quad \phi^{-1}\Psi B\phi \subset \Psi B$.

These inclusions are to be understood as follows: If $\psi \in \Psi B$ and both the domain and the range of ψ are included in the domain of ϕ, then $\phi\psi\phi^{-1}$ is in ΨB; if both the domain and the range of ψ are in the range of ϕ, then $\phi^{-1}\psi\phi$ is in ΨB (Fig. 3).

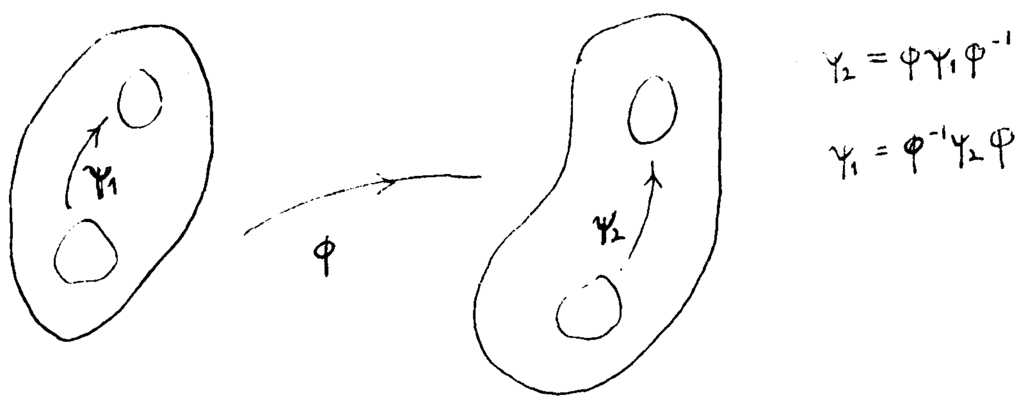

Figure 3

LEMMA 9.1.3.

$$\Psi B \cup (\text{Loc Aut } B \cap \text{Diff}^2 M) \subset N\Psi B.$$

PROOF. It is clear that $\Psi B \subset N\Psi B$. If $\phi \in \text{Loc Aut } B \cap \text{Diff}^2 M$ and X is a C^1 vectorfield in B, then $\phi \circ X^t \circ \phi^{-1} = Y^t$, where $Y(\xi) = \phi^*(\phi^{-1}(\xi)).X(\phi^{-1}(\xi)) \in B_\xi$. Hence $\phi \theta B \phi^{-1} \subset \Psi B$ and it is easily deduced that $\phi \Psi B \phi^{-1} \subset \Psi B$ (there are no problems with the domains because domain ϕ = range ϕ = M).

LEMMA 9.1.4. Let $\sim$ be the equivalence relation given by the partition of M into the connected components of (M,σ). If $\phi \in N\Psi B$, then ϕ <u>respects</u> $\sim$ (i.e. $x \sim y$ and x and y $\in$ domainϕ implies $\phi(x) = \phi(y)$).

PROOF. Let $\phi \in N\Psi B$ and let x and y lie in domain ϕ. If $x \sim y$, then $x = \psi(y)$ for some $\psi \in \Psi B$. We may assume that the domain and range of ψ are contained in the domain of ϕ, so that $\phi\psi\phi^{-1} \in \Psi B$ and $\phi(x) = \phi\psi\phi^{-1}(\phi(y))$, which proves that $\phi(x) \sim \phi(y)$.

9.2. NEAT LEAVES. A <u>leaf</u> is a connected component of (M,σ). A C^1 box (ψ,U,W) of M is <u>admissible</u> if $\psi(-,w) : U^1 \to (M,\sigma)$ is differentiable for every $w \in W^1$ and if the slice $\psi(-,0) : U^1 \to (M,\sigma)$ is a local chart. An admissible box is <u>neat</u> if $\psi(-,w)$ is a local chart for (M,σ) whenever $\psi(0,w)$ belongs to the same leaf as $\psi(0,0)$ (Fig. 4, p. 68).

A point $x \in M$ is <u>neat</u> if there exists a neat box (ψ,U,W) such that $\psi(0,0) = x$.

LEMMA 9.2.1. a) If $\phi \in \mathrm{Loc\,Aut}\,B$ and (ψ,U,W) is an admissible box such that $\psi(U^1 \times W^1) \subset \mathrm{dom}\,\phi$, then $(\phi \circ \psi,U,W)$ is an admissible box.

b) If, in addition, $\phi \in N\Psi B$ and (ψ,U,W) is neat, then $(\phi \circ \psi,U,W)$ is neat.

This follows at once from the definitions and Lemma 9.1.4.

If now x and y belong to the same leaf, then $x = \phi(y)$ for some $\phi \in \Psi B$. Since $\Psi B \subset N\Psi B$, we have the following result.

COROLLARY 9.2.1. If a leaf L of (M,σ) contains a neat point, then every point of L is neat.

Such leaves will be referred to as <u>neat leaves</u>.

LEMMA 9.2.2. a) Every finite dimensional or finite codimensional leaf is neat.

b) If B is a <u>regular</u> distribution, then every leaf of (M,σ) is neat.

To prove b), set

$$\delta(u,w) = \psi^*(0,0)^{-1}f^{-1}(\psi(u,w))\psi^*(u,w),$$

where f is some regular stem of B at $x = \psi(0,0)$ and we assume that $\psi(U^1 \times W^1) \subset \mathrm{domain}\,f$. Then $\delta(u,w) \in GL(E|U \times 0)$, $E = U \times W$, and so, by Proposition 1.2, $\delta(u,w) \in GL_o(E|U \times 0)$ for every $(u,w) \in U^1 \times W^1$. Hence $D_1\psi(u,w).U = f(\psi(u,w))\psi^*(0,0)\delta(u,w).(U \times 0) = f(\psi(u,w))\psi^*(0,0)(U \times 0) =$ $= f(\psi(u,w))B_x = B_{\psi(u,w)}$. (This argument shows that <u>every</u> admissible box of a regular distribution is neat.)

9.3. NEAT SUBMANIFOLDS. An immersed submanifold L of M is <u>neat</u> if, for every $x \in L$, there exists a C^1 box (ψ,U,W) of M such that

(i) $\psi(0,0) = x$;

(ii) $L \cap \psi(U^1 \times W^1) = \psi(U^1 \times \ell)$, where $\ell = \{w \in W^1 : \psi(0,w) \in L\}$; and

(iii) for every $w \in \ell$, $\psi(-,w) : U^1 \to L$ is a local chart for L.

For example, the union of all the leaves of (M,σ) of a given finite dimension (or codimension) is a neat submanifold of M, and so is (M,σ) if B is a regular distribution, or any single neat leaf of (M,σ) in general.

A C^1 box (ψ, U,W) which satisfies the conditions (ii) and(iii) above is called a <u>neat box for L</u>.

PROPOSITION 9.3.1. Let L be a neat submanifold of M and let $\psi : N \to M$ be a continuous mapping such that $\psi(N) \subset L$.

(a) If $\psi : N \to M$ is a differentiable mapping between manifolds and if, for every $\xi \in N$, $\psi^*(\xi)T_\xi N \subset T_{\psi(\xi)}L$, then $\psi : N \to L$ is differentiable.

(b) If $\psi : N \to M$ is a differentiable mapping between manifolds and L is separable, then $\psi : N \to L$ is differentaible.

(c) More generally, if N is a locally connected topological space and L is separable, then $\psi : N \to L$ is continuous.

The proof follows the same lines as the proof of Lemma 3.1 in Part One and is therefore omitted. Note that the assertion (b) follows from (c).

The next proposition is probably a special case of a more general result.

PROPOSITION 9.3.2. Let L be a connected neat submanifold of M. If M is paracompact and modelled on a separable Banach space E, then L is separable*).

LEMMA 9.3.3. (See [1], Ch. 111, §9, Lemma 1.) If a topological space T admits a locally countable covering by open separable subsets, then each connected component of T is separable.

PROOF OF PROPOSITION 9.3.2. Since M is paracompact and locally separable, each connected component of M is separable by Lemma 9.3.3. Hence, there exist a countable family of boxes (ψ_n, U_n, W_n) for L such that $L \subset \bigcup H_n$, where $H_n = \psi_n(U_n^1, W_n^1)$. Let $S_{nw} = \psi_n(U_n^1, w)$ be a slice of H_n. It is easily checked that, if $w \in \ell_n =$ $= \{w \in W_n^1 : \psi_n(0,w) \in L\}$, then each connected component of the open subset $S_{nw} \cap H_m$ of S_{nw} is contained in a slice of ψ_m. Since S_{nw} is separable, we see that S_{nw} meets $S_{m\bar{w}}$ for at most countably many $\bar{w} \in \ell_m$. Hence the family $(S_{nw} : n \in N, w \in \ell_n)$ is a locally countable cover of L and the assertion follows from Lemma 9.3.3.

9.4. AN UNSOLVED PROBLEM. A leaf of (M,σ) is <u>wild</u> if it contains no neat points.

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*) by 'separable' we mean: with a countable basis of open sets.

QUESTION 9.4.1. Do wild leaves exist? More precisely: does there exist an integrable C^1 distribution B on a (separable) C^2 manifold M such that the corresponding foliation of M has a wild leaf?

The author made several attempts to construct such a foliation, but failed. The examples below illustrate the difficulties encountered.

LEMMA 9.4.2. Let H be a separable Hilbert space and let $(A_n)_{n \in Z}$ be a doubly infinite sequence of members of GL(H). There exists a C^∞ function $\gamma : R \to GL(H)$ such that

(i) $f(n) = A_n$ for all $n \in Z$; and

(ii) f is constant on each of the intervals $[n - \frac{1}{3}, n + \frac{1}{3}]$.

PROOF. Since GL(H) is contractible [15], there exists a continuous path $[0,1] \to GL(H)$ joining any two elements. Since GL(H) is an open subset of the Banach space End(H), this path can be replaced by a broken straight line with finitely many segments. The corners can be smoothed off in the 2 dimensional space spanned by the two adjacent edges. Property (ii) follows on re-parametrization.

From now on, $H = \ell_2(Z)$ is the space of all doubly infinite real sequences $(x_n)_{-\infty}^{\infty}$ such that $\Sigma x_n^2 < \infty$, F is the closed subspace of H given by the equations $x_n = 0$ for $n \geq 1$, and S is the right shift on H:

$$(Sx)_n = x_{n-1}.$$

We note that $S : H \to H$ is an isometric isomorphism.

EXAMPLE 9.4.2. We take

$$L = \phi(R \times F),$$

where

$$\phi : R \times F \to N \times H : (x,v) \to (\theta(x), \gamma(x).v),$$

N is a manifold, $\theta : R \to N$ is an immersion and $\gamma : R \to GL(H)$ is a differentiable function. It is easily checked that ϕ is an immersion, so that L is an immersed submanifold of $N \times H$.

(9.4.2.a) Chose $N = R^2$, $\gamma(t) = id_H$ for $-\frac{1}{3} \le t \le \frac{1}{3}$, $\gamma(t) = S$ for $\frac{2}{3} \le t \le \frac{4}{3}$ and $\theta : R \to R^2$ as in Fig. 4.

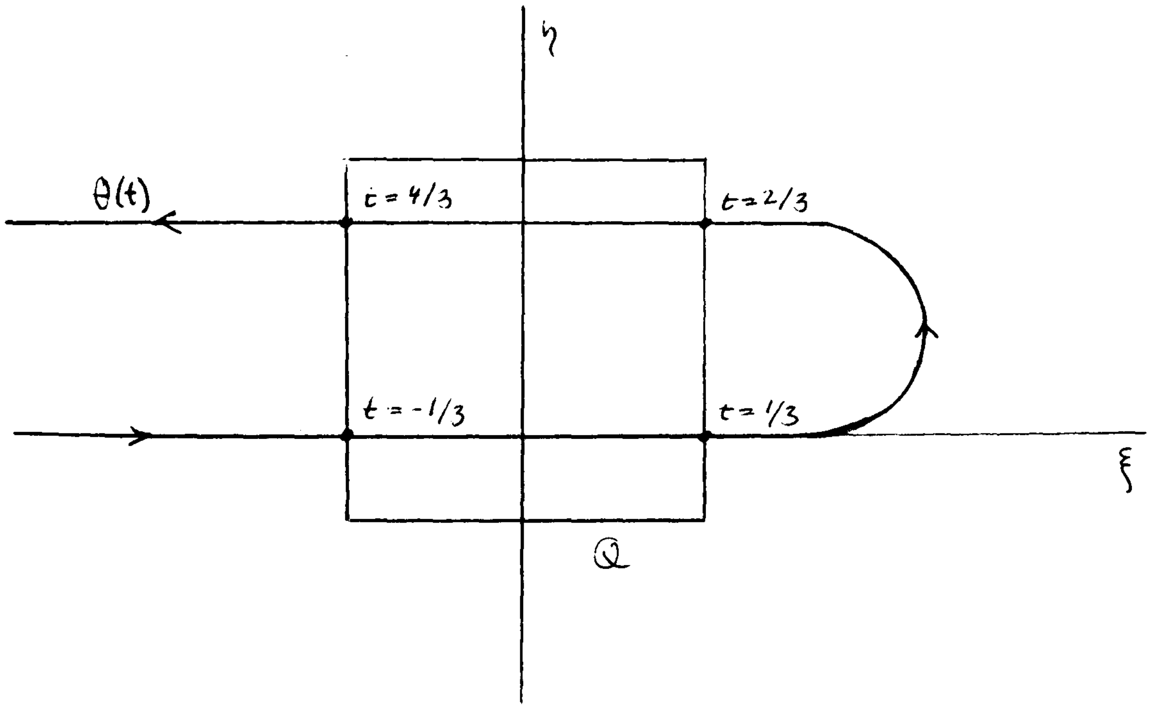

Figure 4

It is easily checked that ϕ is then an embedding and that L is a leaf of the distribution B given by $B_x = T_x L$ for $x \in L$ and $B_x = T_x M = T_x(R^2 \times H)$ for $x \notin L$.

An admissible box with the range $Q \times H$, where Q is the square indicated in Fig. 4 is obtained from the identity mapping of $R^2 \times H$, with (ξ-axis) $\times F$ for the first coordinate and (η-axis) $\times F^{\perp}$ for the second coordinate. It is clear that this box is not neat and that it restricts to a neat box on a neighbourhood of $\phi(0,0)$.

(9.4.2.b) Take $N = R^2$, $\gamma(t) = id_H$ for $t \leq 0$, $\gamma(n) = S^n$ for $n \geq 1$, and $\gamma(t)$ constant on each of the intervals $[n - \frac{1}{3}, n + \frac{1}{3}]$. Let $\theta : R \to R^2$ be as in Figure 5.

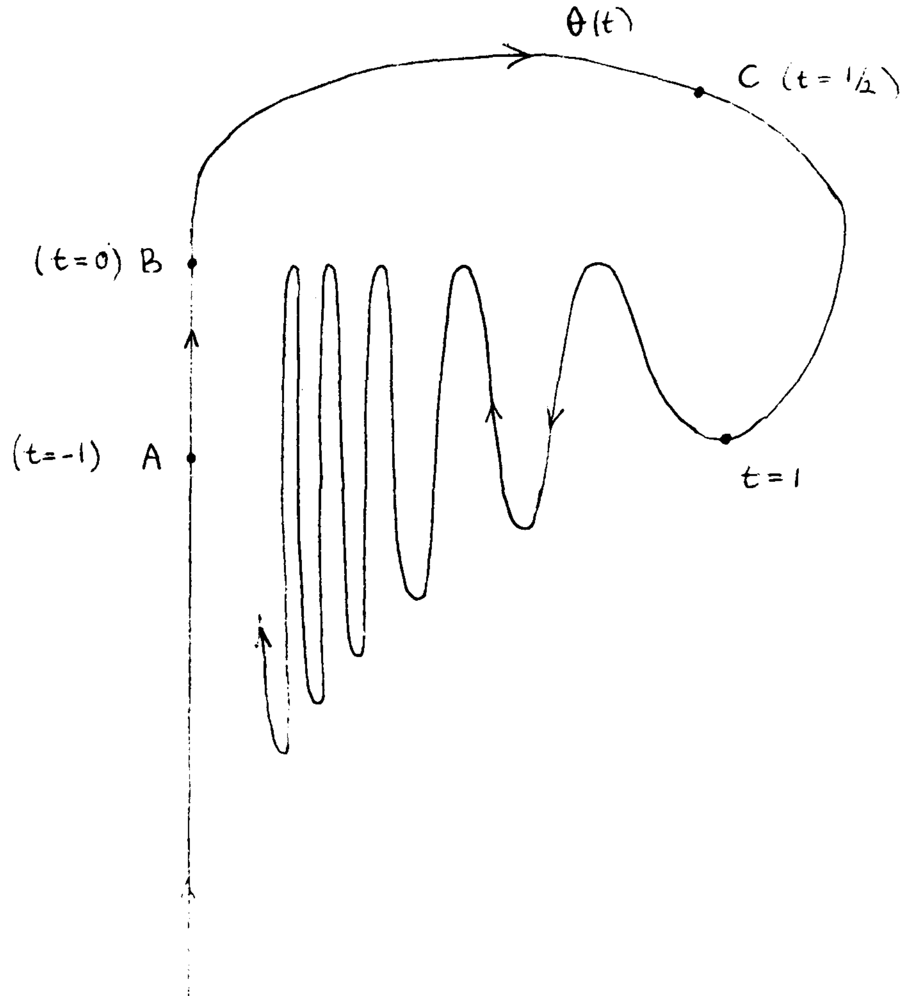

Figure 5

Using the identity of $R^2 \times H$, we can define an admissible box at $A = \phi(-1,0)$ which does <u>not</u> restrict to a neat box for L. Note, however, that L is not a leaf of a foliation with singularities. (C is a neat point and if L were a leaf of a foliation this would imply that <u>every</u> point of L is neat (Corollary 9.2.1). There are no admissible charts at B.)

(9.4.2.c) The attempts to use N = two-torus, $\theta : R \to N$ an integral curve of a fixed irrational flow, and a suitable $\gamma : R \to GL(H)$ failed for the following reasons: 1) the differentiability of the distribution B demands that <u>every</u> $x \in N \times H$ has a neighbourhood Ω such that B_y is 'larger' than B_x for $y \in \Omega$; 2) Proposition 1.2.

EXAMPLE 9.4.3. Let $H_o = \bigcup_{n=1}^{\infty} S^nF$ and let the distribution B on H be defined by taking $B_x = S^nF$ for $x \in S^nF \backslash S^{n-1}F$ and $B_x = 0$ if $x = 0$ or if $x \in H \backslash H_o$. (The projection $S^{n+2}F \to S^{n+2}F/S^nF \cong R^2$ takes B into the distribution $\tilde{B}$ illustrated in Fig. 6.)

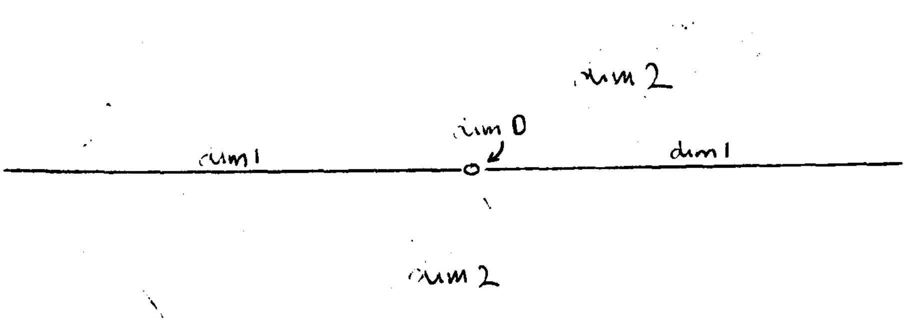

Figure 6

It is clear that B is integrable. Note, however, that B is <u>not</u> differentiable. (H_o is a dense subset of H of the first category, so we are in a similar difficulty as with the flow-line of the irrational flow on the torus in (9.4.2.c).)

We now construct an integrable distribution on $T \times H$, where T is the circle. Consider $[0,1] \times H$ as a subspace of $R \times H$ and let

$$\tilde{B}(t,x) = V(t,x) + \gamma_t B(\gamma_t^{-1}x),$$

where $\gamma : [0,1] \to GL(H)$ is a C^∞ function such that $\gamma_t = id_H$ for $0 \le t \le \frac{1}{3}$, $\gamma(t) = S$ for $\frac{2}{3} \le t \le 1$, and $V(t,x)$ is the one-dimensional subspace of $R \times H$ spanned by the vector $(1,\dot{\gamma}(t)x)$ (Fig. 7)

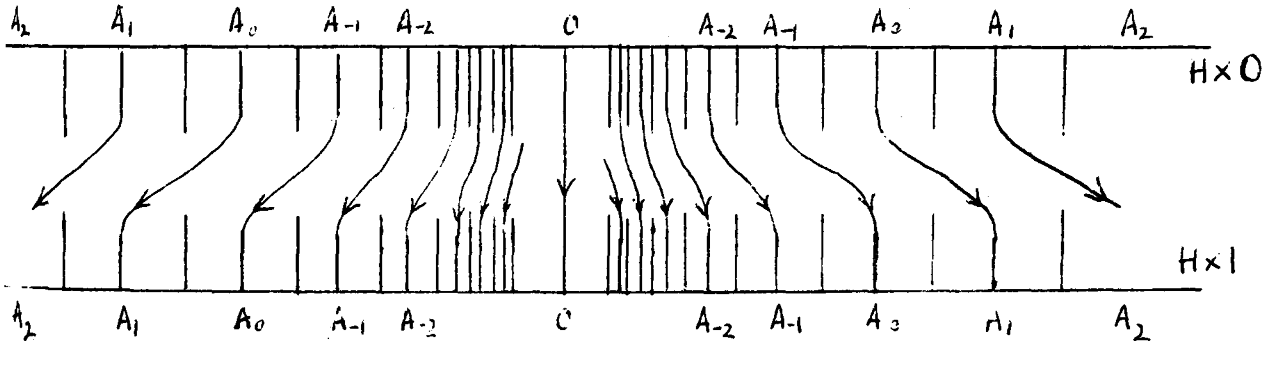

Figure 7

Note that $\tilde{B}(t,x) = R \times B_x$ for $0 \le t \le \frac{1}{3}$ and for $\frac{2}{e} \le t \le 1$, so that $\tilde{B}$ defines an integrable distribution $\hat{B}$ on $T \times H$. If ℓ is the point of T obtained by identifying the endpoints of the interval $[0,1]$, then, clearly, $\{\ell\} \times (H_o \setminus \{0\})$ is contained in a single wild leaf of $\hat{B}$.

PART THREE

Integrability and irreducibility of systems of vectorfields

§1. ON INTEGRABILITY OF SYSTEMS OF VECTORFIELDS

1.1 GENERALITIES. Let S be a set of smooth vectorfields on a paracompact finite-dimensional manifold M. (For simplicity, we assume that M and the vectorfields in S are of the class C^∞ or C^ω.) Recall that the accessible sets of S (or orbits in the terminology of [10]) are the equivalence classes of the relation 'x and y can be joined by finitely many (unoriented) pieces of integral curves of vectorfields in S'. It is proved in Part One and in [10] that the accessible sets of S are immersed submanifolds of M.

S is said to be homogeneous if every vectorfield in S respects the distribution $B(S) = (S(x) : x \in M)$, where $S(x)$ is the vector subspace of T_xM spanned by the values at x of the vectorfields in S.

THEOREM 1. The following conditions are equivalent.

(a) For every $x \in M$, there exists an integral manifold of $B(S)$ which contains the point x.

(b) S is homogeneous.

(c) S spans the tangent spaces of its accessible sets.

We say that S is integrable if it satisfies either of the conditions in Theorem 1. The non-trivial step in the proof of Theorem 1 is the proof of (b) $\Rightarrow$ (c), given in Part One and in [10]. The assertion (a) $\Rightarrow$ (b) follows from Theorem 1 in Part Two. If the integral manifolds in question are at least of the class C^2, then (a) $\Rightarrow$ (b) can be deduced from the existence and uniqueness theorem for ordinary differential equations and the following lemma.

LEMMA 1.1. A vectorfield X on M respects a distribution B
d only if, for every x in the domain of X, there exists
such that

$$(X^t)*(x).B_x = B_y$$

ver $|t| < \varepsilon$ and $X^t.x = y$.

PROOF. Let $x_t = X^t.x$, $\gamma_t = (X^t)*(x)$ and $\gamma_{ts} = (X^t)*(x_s)$.
be the domain of the integral curve $t \to x_t$ and let

$$I_o = \{t \in I : \gamma_t B_x = B_{x_t}\} .$$

$\gamma_{t+s} = \gamma_{ts}\gamma_s$, I_o is easily shown to be both open and closed

REMARKS. 1) If, for every $x \in$ domain X,

a) $$X^t.x = y \Rightarrow (X^t)*(x).B_x \subset B_y,$$

X respects B. This follows at once from (1.1.a) since
$= x$ and $(X^{-t})*(y)$ is the inverse of $(X^t)*(y)$.

2) Consider the following property of X:

b) For every $x \in$ domain X, there exists $\varepsilon > 0$ such that $|t| < \varepsilon$ and $X^t.x = y$ implies $(X^t)*(x).B_x \subset B_y$.

ext example shows that (1.1.b) does not imply that X respects
istribution B.

EXAMPLE 1. Let $M = R^2$ and let B be the distribution spanned
e vectorfields $\partial/\partial\xi$ and $\xi.\partial/\partial\eta$ (cf. Fig. 1).

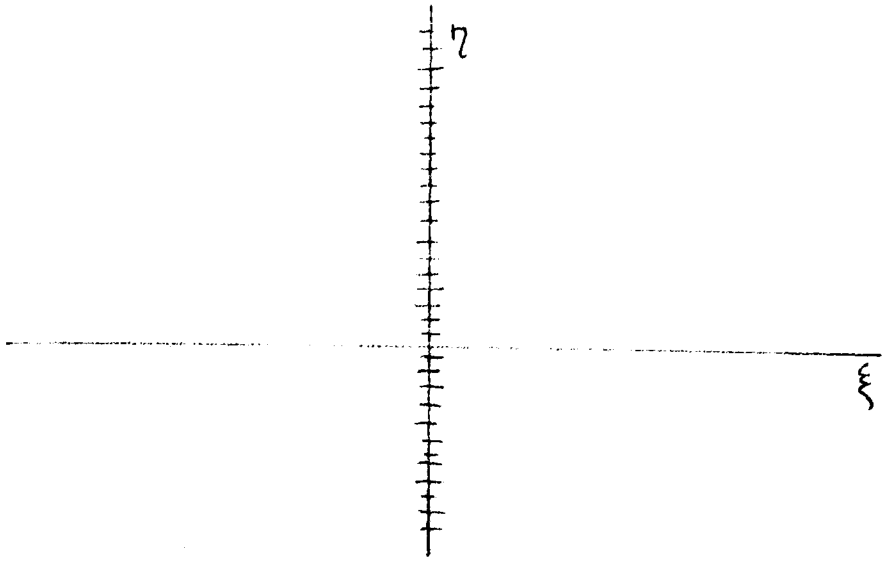

Figure 1.

X be an arbitrary vectorfield such that $X(x) \in B_x$ for every domain X. It is easily checked that X satisfies the ition (1.1.b) at every $x \in R^2$. (If $x \in \eta$-axis and $X(x) \neq 0$ this follows from $(X^t)^*(x).X(x) = X(y) \in B_y$, where $y = X^t.x$.) no vectorfield which moves a point x on the η-axis onto η-axis can respect B since then $\dim(X^t)^*(x).B_x = \dim B_x = 1$ $\dim B_y = 2$.

CONDITION L FOR S. Recall [6] that S is <u>locally of finite</u> <u>at x</u> (Lx) if there exist finitely many vectorfields $X_1, X_2, \ldots, X_p$ S such that

1) The vectors $X_i(x)$ $(i = 1,\ldots,p)$ span $S(x)$;
2) For every $Y \in S$, there exists a neighbourhood Ω of x and the continuous real-valued function λ_{ij} defined on Ω such that

$$[Y,X_i](y) = \sum_{j=1}^{p} \lambda_{ij}(y)X_j(y)$$

for every $y \in \Omega$.

is said to be locally of finite type (L) if it is locally of inite type at every x in M.

It is claimed in [6] *) that every set S of smooth ectorfields which is locally of finite type is homogeneous, and herefore integrable. However, the proofs given show only hat the vectorfields in S satsify (1.1.b) (with $B_x \equiv S(x)$). he next example shows that, in fact, L for S does <u>not</u> imply hat S is integrable.

EXAMPLE 2. Let $M = R^2$ and let S be the set of all vectorfields of the form

$$\partial/\partial\xi + \phi(\xi,\eta).\partial/\partial\eta \ ,$$

where ϕ is an arbitrary function such that $\phi(0,0) = 0$ and $\partial\phi/\partial\xi = 0$ in some neighbourhood of the origin depending on ϕ.

If $x \neq 0$, then there exists a neighbourhood Ω of x and a vectorfield X_1 in S such that $X_1 = \partial/\partial\eta$ on Ω. Taking $X_2 = \partial/\partial\xi \in S$, it is easy to see that S satisfies the condition (Lx) with X_1,X_2 and the same Ω for every Y in S. If, on the other hand, x is the origin, we may take $\{X_1,X_2,\ldots,X_p\} = \{\partial/\partial\xi\}$, as

$$[\partial/\partial\xi + \phi.\partial/\partial\eta,\ \partial/\partial\xi] = -\partial\phi/\partial\xi.\partial/\partial\eta = 0$$

in a sufficiently small neighbourhood of 0. This shows that S is locally of finite type, and it is clear that S is not integrable.

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*) And repeated in [10],[11] and several other places.

REMARK. We say that S is <u>involutive</u> if $[X,Y] \in S$ whenever and Y are in S. Lobry [6] proves that an involutive set of ɜal-analytic vectorfields on a real-analytic manifold M is ɔcally of finite type. He then comes to the (false) conclusion nat every involutive set of real-analytic vectorfields is ntegrable (cf. Part Two, Theorem 5 and Example 8.5.2.).

.3. CONDITION L for $\tilde{S}$. Let $\tilde{S}$ denote the set of all smooth ectorfields X on M such that $X(x) \in S(x)$ for every x in the omain of X. We observe that if S is defined as in Example 2, hen $\tilde{S}$ is <u>not</u> locally of finite type. It is therefore natural o ask the following question.

QUESTION 1.3. Assume that $\tilde{S}$ is locally of finite type. Does it follow that S is integrable?

Example 8.5.2 in Part Two shows that the answer is NO in the real-analytic case (unless the vectorfields in S be locally everywhere defined). On the other hand, Theorem 2 below shows the answer is YES in the C^∞ case. The example in (1.5) shows that this is only a sufficient condition: S may be integrable even if $\tilde{S}$ is not locally of finite type.

1.4. CONDITION K. Let $x \in M$. We say that S satisfies the condition (Kx) if there exist finitely many vectorfields $X_1, X_2, \ldots, X_p$ in S, defined on a neighbourhood Ω of x, and continuous*) functions $\lambda_{ijk} : \Omega \to R$ such that

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*) It is sufficient to assume that λ_{ijk} are bounded measurable.

(K1) $X_i(x)$ span $S(x)$ and, for every $y \in \Omega$,

$$[X_i,X_j](y) = \sum_{k=1}^{p} \lambda_{ijk}(y)X_k(y), \quad 1 \leq i,\ j \leq p;$$

(K2) for every X in S, such that $x \in$ domain X, there exists $\varepsilon > 0$ and continuous function $\lambda_{ij} : [-\varepsilon,\varepsilon] \to R$ such that, for every $t \in [-\varepsilon,\varepsilon]$, and every i, $1 \leq i \leq p$,

$$[X,X_i](x_t) = \sum_{i=1}^{p} \lambda_{ij}(t)X_j(x_t),$$

where $x_t = X^t.x$.

It is clear that (Lx) $\Longrightarrow$ (Kx). We say that S satisfies :ondition K, if it satisfies the condition (Kx) for every $x \in M$.

THEOREM 2. Let S be a set of C^∞ vectorfields on a C^∞ manifold 1 let $\tilde{S}$ be defined as in 1.3. Then S is integrable if and if $\tilde{S}$ satisfies the condition K.

In particular, S is integrable if $\tilde{S}$ is locally of finite type.

LEMMA 1.4.1. Assume that S satisfies the condition (Kx) and iimS(x) = d. Then there exists a box (ψ,U,W) of M such that

(a) $\psi(0,0) = x$ and $\dim U = d$;

(b) $\psi^*(u,w).(U \times 0) \subset S(\psi(u,w))$ for every (u,w) in the domain of ψ.

(c) If Y is an arbitrary vectorfield in S such that $x \in$ domain Y and if $\sigma(t) = Y^t.x$, then there exists $\varepsilon > 0$ such that $\sigma(t) \in \psi(U,0)$ for $|t| < \varepsilon$.

PROOF. Assume that the vectorfields $X_1,\dots,X_p$ satsify the condition (K1) and (K2) on a neighbourhood Ω of x, and let $S^* = \{X_1|_\Omega,\dots X_p|_\Omega\}$. It follows at once from Theorem 6, Part One, that S^* is homogeneous (and hence integrable) on Ω. Since $S^*(x) = S(x)$ and $S^*(y) \subset S(y)$ for every y in Ω, there exists a box (ψ,U,W) of M which satisfies the conditions (a) and (b) (see Theorem 5 and its Corollary in Part One). Moreover, we may assume that $\psi(U,0)$ is a neighbourhood of x in L^*, where L^* is the accessible set of S^* through the point x.

Let now $\sigma(t) = Y^t.x$ for some vectorfield Y in S such that $x \in$ domain Y and let $\gamma^t = (Y^t)^*(x)$. By Theorem 2, Part Two (or by the proof of Theorem 6, Part One), (K2) implies that, for $|t| \le \varepsilon$.

$$\gamma^t S^*(x) = S^*(\sigma(t)).$$

Hence, for $|t| \le \varepsilon$,

(1) $\dim S^*(\sigma(t)) = \dim S^*(x) = k$ and

(2) $\dot\sigma(t) = Y(\sigma(t)) = \gamma^t Y(x) \in S^*(\sigma(t))$.

Let L be the union of the k-dimensional accessible sets of S^*. Then L is a neat submanifold of Ω (Part Two, §9.3) and the tangent spaces of L are spanned by S^* because S^* is homogeneous. By (1), $\sigma(t) \in L$ for $|t| \le \varepsilon$ and so, by (2) and Proposition 9.3.1, Part Two

$$\sigma :]-\varepsilon,\varepsilon[\to L$$

is differentiable. Since L^* is the connected component of x in L, $\sigma(t) \in L^*$ for $|t| < \varepsilon$, whence follows the assertion (c) of our lemma.

LEMMA 1.4.2. If $\tilde{S}$ satisfies the condition K, then $\dim S(x)$ remains constant along the integral curves of vectorfields in S.

PROOF. Let $\sigma(t) = X^t.x$ for some X in S and some $x \in$ domain X. Let $t_1 < t_2$ and assume that $\dim S(\sigma(t_1)) < \dim S(\sigma(t_2))$. Let $\ell = \max\{\dim S(\sigma(t)) : t_1 \leq t \leq t_2\}$. The set $I_o = \{t \in [t_1,t_2] : \dim S(\sigma(t)) = \ell\}$ is relatively open in I. Let J be a connected component of I_o nad let t_o be the left-hand end-point of J. It is easily seen that $t_o \notin I_o$. Without loss of generality, we assume that $t_o = 0$, so that, for some $\varepsilon > 0$ and all t, $0 < t < \varepsilon$,

$$\dim S(x) = k < \ell = \dim S(\sigma(t)).$$

Let now (ψ,U,W) satisfy the conditions of Lemma 1.4.1 with $\tilde{S}$ in the place of S, and let $Z = \psi(U^1,0)$. Using ψ to identify a neighbourhood of x in M with an open subset Ω of R^n, we may assume that

(1) x is the origin in R^n;

(2) Z is the open unit ball in a k-dimensional subspace E of R^n;

(3) every vectorfield $Y : \Omega \to E$ belongs to $\tilde{S}$;

(4) If Y is a vectorfield in $\tilde{S}$ and $x \in$ domain Y, then there exists $\delta > 0$ such that, for $|t| < \delta$,

$$Y^t.x \in E.$$

Let P be the orthogonal projection of R^n onto E and let $Q = \mathrm{id} - P$ be the projection onto $E^\perp$. If $Y : \Omega \to R^n$ is an arbitrary vectorfield, then $PY \in \tilde{S}$ by (3). If $Y \in \tilde{S}$, then $QY = Y - PY \in \tilde{S}$. Using this, the fact that $\dim S(\sigma(t)) = \ell > \dim E$ for $0 < t < \varepsilon$, and suitable 'bump functions', it is easy to construct a vectorfield Y in $\tilde{S}$ such that

(5) $Y(y) \perp E$ for all $y \in \Omega$ and,

(6) for every $\delta > 0$, there exists t such that $0 < t < \delta$ and $Y(\sigma(t)) \neq 0$.

Let now $u(t) = (PX + Y)^t.x$. By (4), there exists $\delta > 0$ such that

(7) $u(t) \in E$ and $\sigma(t) \in E$ for $|t| < \delta$. Hence $\dot{u}(t) = $
$= PX(u(t)) + Y(u(t)) \in E$ and so

(8) $Y(u(t)) = 0$ and

(9) $\dot{u}(t) = PX(u(t))$ for $|t| < \delta$.

Since $\dot{\sigma}(t) = X(\sigma(t)) \in E$, we have $X(\sigma(t)) = PX(\sigma(t))$ and

(10) $\dot{\sigma}(t) = PX(\sigma(t))$ for $|t| < \delta$.

By (9) and (10), $\sigma(t) = u(t)$ for $|t| < \delta$ and so, by (8), $Y(\sigma(t)) = 0$ for $|t| < \delta$, in contradiction with (6).

PROOF OF THEOREM 2. Assume that $\tilde{S}$ satisfies the condition K. Let X be a vectorfield in S, $x \in$ domain X, $\sigma(t) = X^t.x$, and $\gamma^t = (X^t)^*(x)$. It follows easily from (K2) and from Theorem 6, Part One, that there exists $\varepsilon > 0$ such that, for $|t| < \varepsilon$,

$$\gamma^t S(x) \subset S(\sigma(t)).$$

Hence, by Lemma 1.4.2, $\gamma^t S(x) = S(\sigma(t))$ for $|t| < \varepsilon$. By Lemma 1.1, X respects the distribution $(S(x) : x \in M)$, which proves that S is homogeneous and, therefore, integrable.

Conversely, assume that S is integrable and let L be the ssible set of S through the point x. By the Corollary of rem 5, Part One, there exists a coordinate system $\xi_2,\ldots,\xi_n)$ on a neighbourhood Ω_M of x in M such that the orfields $X_i = \partial/\partial\xi_i$ belong to $\tilde{S}$ for $1 \le i \le k$ and span $T_y L$ $y \in \Omega_L$, where Ω_L is some neighbourhood of x in L. If X vectorfield in $\tilde{S}$ and $x \in$ domain X, then there exists 0 such that $x_t = X^t.x \in \Omega_L$ for $|t| < \varepsilon$. Hence, for any torfield X in $\tilde{S}$ defined on a neighbourhood of x, $|t| < \varepsilon$ lies

$$X(x_t) = \sum_{i=1}^{p} \lambda_i(t) X_i(x_t)$$

h some differentiable functions $\lambda_i :]-\varepsilon,\varepsilon[\to R$. Since S is egrable, $\tilde{S}$ is involutive and (*) applies in particular to torfields $[X,X_j]$, $1 \le j \le p$,-which proves that $\{X_1,X_2,\ldots,X_k\}$ isfy the condition (K2). The condition (K1) follows at once m $[X_i,X_j] = 0$ $(1 \le i,j \le p)$.

AN EXAMPLE. In Part One, Example 5.4, we have shown that integrable system of vectorfields need not be locally of ite type. We shall now show that the integrability of S does t imply that $\tilde{S}$ (as defined in 1.3) is locally of finite type.

Let $\phi : R \to R$ be the C^∞ function defined by $\phi(\xi) = e^{-1/\xi}$ r $\xi > 0$ and $\phi(\xi) = 0$ for $\xi \le 0$. Let the vectorfield X_2 on be defined by

$$X_2(\xi_1,\xi_2) = \phi(\xi_2).\partial/\partial\xi_2,$$

d let $X_1 = \partial/\partial\xi_1$ and $S = \{X_1,X_2\}$. Let $\tilde{S}$ be the set of all C^∞ ctorfields in the distribution $(S(x) : X \in R^2)$ (Fig. 2).

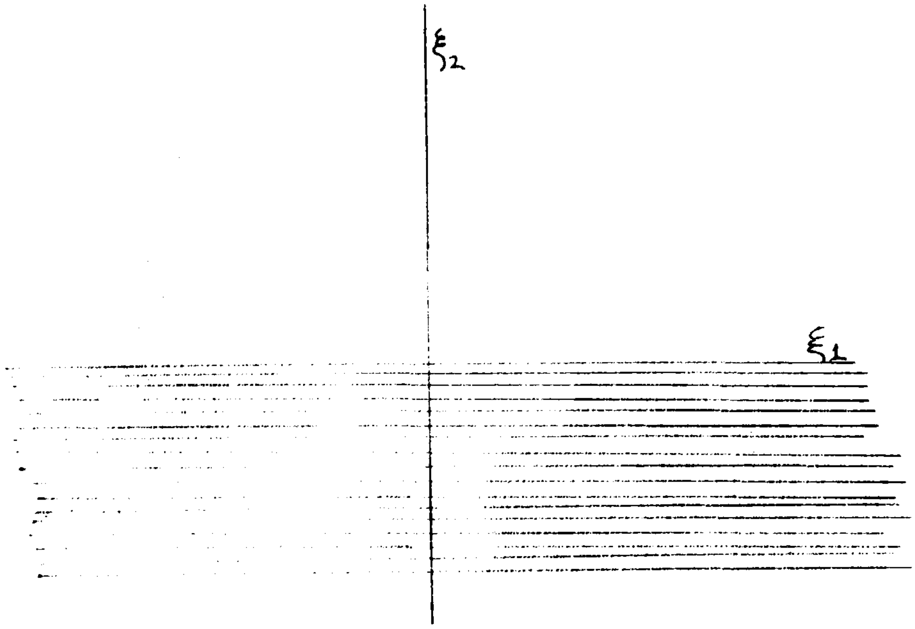

Figure 2.

To prove that $\tilde{S}$ is not locally of finite type, we argue by contradiction and assume that the vectorfields $Y_1,\dots,Y_p$ in $\tilde{S}$ satisfy the conditions (L.1) and (L.2) at the origin, with $\tilde{S}$ in the place of S. Let

$$Y_i = \alpha_i . \partial/\partial\xi_1 + \beta_i . \partial/\partial\xi_2,$$

where $\beta_i(\xi_1,\xi_2) = 0$ for $\xi_2 < 0$. We may assume that the C^∞ function α_1 is > 0 on a neighbourhood of the origin and introduce a local diffeomorphism

$$\psi : (R^2,0) \to (R^2,0) : (\xi_1,\xi_2) \to Y_1^{\xi_1}.(0,\xi_2).$$

Since ψ maps the upper half-plane ($\xi_2 > 0$), ξ_1-axis and lower half-plane into themselves and since the conditions (L.1) and (L.2) are invariant under C^∞ changes of coordinates, we may assume $Y_1 = \partial/\partial\xi_1$.

Let $\theta : R \to R$ be an arbitrary C^∞ function such that $\theta(\xi) = 0$ for $\xi \le 0$ and let the vectorfield X on R^2 be defined by

$$X(\xi_1,\xi_2) = e^{\xi_1}.\theta(\xi_2).\frac{\partial}{\partial\xi_2} .$$

Then $X \in \tilde{S}$, $[Y_1,X] = X$ and so (L.2) implies that

$$e^{\xi_1}\theta(\xi_2) = \sum_{j=1}^{p} \lambda_j(\xi_1,\xi_2)\beta_j(\xi_1,\xi_2)$$

for all ξ_1 and ξ_2 in a sufficiently small neighbourhood of the origin (depending on the function θ). But this is absurd, since, on setting $\xi_1 = 0$, it contradicts the following lemma.

LEMMA 5.3. Let A be the ring of germs at the origin of the continuous function $R \to R$. Let B be the subring of A generated by the C^∞ functions $\theta : R \to R$ such that $\theta(\xi) = 0$ for $\xi < 0$. Then B is not contained in any ideal of A generated by finitely many members of B.

PROOF No. 1. Let S be the collection of all C^∞ vectorfields

$$X = \alpha.\partial/\partial\xi_1 + \beta.\partial/\partial\xi_2$$

on R^2 such that $\beta(\xi_1,\xi_2) = 0$ for $\xi_1 \le 0$ (cf. Example 8.5.2 , Part Two). Then $S = \tilde{S}$. If B were contained in some ideal of A generated by finitely many members of B, then $\tilde{S}$ would be locally of finite type and therefore, by Theorem 2, integrable, which is absurd.

PROOF No. 2. Let B^* be the set of all the C^∞ function $\theta : R \to R$ such that $\theta(t) = 0$ for $t \leq 0$. If B is contained in some ideal of A which is generated by finitely many members of B, then there exists functions $\beta_1, \beta_2, \ldots, \beta_p$ in B^* such that, given any $\theta \in B^*$, there exist $\varepsilon > 0$ and continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that, for $|t| < \varepsilon$,

$$\theta(t) = \sum_{i=1}^{p} \lambda_i(t)\beta_i(t).$$

In particular, there exists $\varepsilon > 0$ and continuous functions λ_{ij} such that, for $|t| < \varepsilon$,

$$\dot{\beta}_i(t) = \sum_{i=1}^{p} \lambda_{ij}(t)\beta_j(t).$$

As this is a homogeneous system of linear differential equations and as $\beta_i(0) = 0$ for $1 \leq i \leq p$, we have $\beta_i(t) = 0$ for all t, $|t| \leq \varepsilon$ and all i, $1 \leq i \leq p$, which is absurd.

§2. ALMOST ALL PAIRS OF VECTORFIELDS ARE IRREDUCIBLE

.1 INTRODUCTION. A set S of vectorfields on M is <u>irreducible</u> if the ccessible sets of S coincide with the connected components of M. By heorem 5, Part One, this is equivalent to saying that $\bar{S}(x) = T_xM$ for very x in M.

In [18], Lobry probed that the set of irreducible pairs of C^k ectorfields on a C^∞ manifold M is C^k-generic for $k \geq n^2 + n$. Later 19], he improved this to $k \geq 2n$. We shall show*) in this section hat the set of irreducible pairs of C^k vectorfields on a C^{k+1} anifold is C^k-generic for all $k \geq 1$.

Lobry's proof can be roughly described as follows. Consider a differentiable function

$$\rho : A \times M \to Q,$$

where A is the space of pairs of vectorfields and Q is a manifold, and a stratified subset W of Q such that

$$\rho_a^{-1}(W) = \phi \Longrightarrow a \text{ is irreducible},$$

where $\rho_a = \rho(a,-) : M \to Q$. If now ρ is transversal to W and $\operatorname{codim} W > \dim M$, Thom's transversality theorem implies that almost every pair a is irreducible.

Our proof follows a similar pattern, with the difference that the results of Part One allow us to take a simpler ρ, Q and W. We also go through the details of the transversality argument, so that our proof is more self-contained than Lobry's (and, in particular, independent of [20]).

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*) Héctor Sussmann tells me that he has recently obtained some similar I do not know his methods, but he will no doubt publish his

For the sake of simplicity, we assume that k is finite and that M is a compact manifold. A similar, but a little more involved, proof shows that the assertion of Theorem 2.6 holds for any separable finite-dimensional C^{k+1} manifold M and for $1 \le k \le \infty$, with the Whitney C^k topology on the space of C^k vectorfields on M.

2.2. A TRANSVERSALITY THEOREM. Recall that by an immersion we always mean a split immersion. By a submanifold of a differentiable manifold Q we mean an immersed submanifold such that the inclusion mapping is an embedding. A subset W of Q is of codimension $\ge c$ if it is contained in a countable union of submanifolds of codimension $\ge c$:

(2.2.a) $W \subset \bigcup_n W_n$, W_n a submanifold of Q, codim $W_n \ge c$, $n = 1,2,\ldots$

The next result is stated in such generality as we need in what follows; the proof is adapted from [17], §§18 and 19.

PROPOSITION 2.2. Let A, M and Q be C^r manifolds and let

$$\alpha : A \times M \to Q \text{ and } \beta : A \times M \to Q$$

be C^r functions. Let W be a subset of Q of codimension $\ge c$ and put

$$A_W = \{a \in A : \alpha_a^{-1}W \cap \beta_a^{-1}W \text{ is of codimension} \ge c\},$$

where $\alpha_a = \alpha(a,-)$ and $\beta_a = \beta(a,-) : M \to Q$. Assume that

(1) M has finite dimension d;

(2) A and M are second countable;

(3) $r > \max(0,d-c)$;

(4) for every $(a,x) \in A \times M$, at least one of the derivatives

$$\alpha^*(a,x) : T_a A \times T_x M \to T_{\alpha(a,x)} Q$$

l

$$\beta^*(a,x) : T_a A \times T_x M \to T_{\beta(a,x)} Q$$

split surjective.

Then A_W is a residual*) subset of A.

COROLLARY 2.2. If $c \geq d+1$, we can take $r = 1$ and

$$A_W = \{a \in A : \alpha_a^{-1} W \cap \beta_a^{-1} W = \phi\}.$$

PROOF OF PROPOSITION 2.2. Let W_n satisfy the condition (2.2.a) d put

$$L_n = W_n \times Q, \quad R_n = Q \times W_n \text{ and } Z_n = W_n^2 = L_n \cap R_n.$$

:t $f = (\alpha, \beta) : A \times M \to Q \times Q$ and let

$$A_n = \{a \in A : \operatorname{codim} f_a^{-1}(Z_n) \geq c\}.$$

; $\bigcap_{n=1}^{\infty} A_n \subset A_W$, it is sufficient to show that each A_n is residual. rom now on, n is assumed to be fixed.

Since $A \times M$ is second countable, the hypothesis (4) implies that here exist a countable open cover $(H_m)_{m=1}^{\infty}$ of $A \times M$ such that, for very m, at least one of the functions

$$\alpha | H_m \quad \text{or} \quad \beta | H_m$$

s a submersion and therefore $f|H_m$ is tranversal to at least one f the manifolds L_n and R_n. Let $\Lambda = f^{-1}(L_n)$, $P = f^{-1}(R_n)$ and let

$$Q_m = \begin{cases} \Lambda \cap H_m & \text{if } f|H_m \text{ is transversal to } L_n \\ P \cap H_m & \text{if } f|H_m \text{ is not transversal to } L_n. \end{cases}$$

:) We say that a set is residual if it <u>contains</u> a countable intersection ›f open dense sets.

Then Q_m is a submanifold of $A \times M$ of codimension $q \geq c$. Let $p : A \times M \to A$ be the coordinate projection and let $p_m = p|Q_m$. By ([17], Lemma 19.3), $p_m : Q_m \to A$ is a Fredholm map of constant index $d - q \leq d - c < r$. Since Q_m is embedded in $A \times M$, it is second countable and so, by Smale's density theorem ([17], §16) the set B_m of regular values of p_m is residual. If now $a \in B_m$, then the mapping

$$\theta_a : M \to A \times M : x \to (a,x)$$

is transversal to Q_m and so $\theta_a^{-1}(Q_m)$ is a submanifold of M of codimension $q \geq c$. Since $f^{-1}(Z_n) \subset \bigcup_{m=1}^{\infty} Q_m$ and $f_a = f \circ \theta_a$, we have $f_a^{-1}(Z_n) \subset \bigcup_{m=1}^{\infty} \theta_a^{-1}(Q_m)$ and thus $A_n \supset \bigcap_{m=1}^{\infty} B_m$, Q.E.D.

2.3. A TRANSVERSALITY LEMMA.

We need the following result.

LEMMA 2.3. Let E be a separable Banach space, M a compact n-dimensional C^1 manifold and Q a finite dimensional C^k vector bundle over M, where $k = 0$ or 1. Let W be a closed subset of Q; if $k = 1$, assume that $\operatorname{codim} W \geq n + 1$. Let

$$\rho : E^2 \times M \to Q$$

be a C^k mapping and

$$E^2_W = \{(a,b) \in E^2 : \rho_{ab}^{-1} W \cap \rho_{ba}^{-1} W = \phi\},$$

where $\rho_{ab} = \rho(a,b,-) : M \to Q$. Assume that:

(1) for every $a \in E$, $a^{\#} : M \times E \to Q : (x,b) \to \rho(a,b,x)$ is a vector-bundle morphism;

(2) there exists an open dense subset A of E^2 such that, for every $(a,b,x) \in A \times M$, at least one of the linear functions $a_x^{\#} : E \to Q_x$ and $b_x^{\#} : E \to Q_x$ is surjective.

Then E_W^2 is an open subset of E^2. If $k = 1$, then E_W^2 is open dense.

PROOF. The openness of E_W^2 follws at once from the fact that W is closed and the function $E^2 \to C^0(M,Q^2) : (a,b) \to (\rho_{ab},\rho_{ba})$ is continuous for the compact-open topology on $C^0(M,Q^2)$. The fact that, for $k = 1$, E_W^2 is residual follows at once from Proposition 2.2 on setting $\alpha = \rho|A \times M$, $\beta(a,b,x) = \alpha(b,a,x)$ and observing that, locally, the derivative of α at (a,b,x) is given by the matrix

$$\begin{pmatrix} * & a_x^{\#} & * \\ 0 & 0 & id_M \end{pmatrix}$$

where the columns represent the partial derivatives of α with respect to the first, second and third coordinates, the top row corresponds to the 'fibre coordinate' of Q and the bottom row to the 'base' coordinate.

2.4. A STRATIFIED SET. The following result is elementary and the proof is given here only for the sake of completeness.

LEMMA 2.4. Let T be an n-dimensional C^1 vector-bundle over a manifold M and let $Q = \bigoplus^p T$ be the Whitney sum of p copies of T, $p \geq n$. For each $x \in M$, let W_x be the subset of $Q_x = T_x^p$ consisting of those p-tuples $(v_1, v_2, \ldots, v_p) \in Q_x$ which do <u>not</u> contain a basis of T, and let

$$W = \bigcup_{x \in M} W_x.$$

Then W is a closed subset of Q of codimension $p - n + 1$. More precisely, there exists a finite sequence (V_i) of submanifolds of M such that

$$W = \bigcup_{i=1}^{\tau} V_i ,$$

$\tau = \sum_{k=0}^{n-1} \binom{p}{k}$, and there are exactly $\binom{p}{k}$ V_i's of codimension $(p-k)(n-k)$ for each k, $0 \le k \le n - 1$.

PROOF. It is clear that, locally, W is a product of an open subset of M with W_x and so it is sufficient to prove the lemma if M is a point and T is a single n-dimensional vector space.

Let now α be a subset of $\{1,2,\ldots,p\}$ and let V_α be the set of all p-tuples $(v_1,v_2,\ldots,v_p)$ in T^p such that the vectors $\{v_i : i \in \alpha\}$ form a basis of the vector space spanned by $\{v_1,v_2,\ldots,v_p\}$. It is clear that

$$W = \bigcup_{0 \le |\alpha| \le n-1} V_\alpha$$

and so it is sufficient to show that each V_α is a submanifold of T^p of codimension $(p-|\alpha|)(n-|\alpha|)$.

To see this, assume that $\alpha = \{1,2,\ldots,k\}$ and let A be the set of all linearly independent k-tuples (k-frames) in T^k. Then A is an open subset of T^k and V_α is the kernel of the vector-bundle morphism

$$\theta : A \times T^{p-k} \to A \times \bigoplus^{p-k} \bigwedge^{k+1} T,$$

where, for each $a = (a_1, a_2, \dots, a_k) \in A$, we put $\bar{a} = a_1 \wedge a_2 \wedge \dots \wedge a_k$ and

$$\theta_a(v_{k+1}, v_{k+2}, \dots v_p) = (\bar{a} \wedge v_{k+1}, \bar{a} \wedge v_{k+2}, \dots, \bar{a} \wedge v_p).$$

The result now follws at once from the fact that $\operatorname{rank}\theta_a = (p-k)(n-k)$ for every $a \in A$.

2.5. DIFFERENTIABILITY. From now on, M is an n-dimensional compact C^{k+1} manifold and k is a finite integer ≥ 1. Let $E = \Gamma^k$ be the vector space of C^k vectorfields on M together with its C^k-topology. Fixing a Finsler structure on the k-jet bundle of sections of TM and taking the corresponding sup-norm we make E into a separable Banach space ([17], Theorem 12.2).

LEMMA 2.5. The mapping

$$\phi : E \times E \times M \times R \to TM,$$
$$(X, Y, x, t) \to ((X^t)^*(x))^{-1} . Y(X^t . x)$$

is of the class C^{k-1}.

PROOF. Let $v \in T_xM$ and let

$$u(t) = u(t, X, v) = (X^t)^*(x) . v \ .$$

Then u is the integral curve of the differential equation

$$\dot{u}(t) = \xi(u(t), X), \quad u(0) = v \tag{1}$$

on TM, where the parametrized vectorfield

$$\xi : TM \times E \to T(TM)$$

is defined by the equation $\xi(u, X) = \omega TX(u)$ and $\omega : T^2M \to T^2M$ is the involution which 'interchanges the second and third coordinates' (cf. [17], p. 17).

Locally, the equation (1) is given by the system of 2n equations

$$\dot{x} = X(x) \ ,$$
$$\dot{u} = DX(x).u \ .$$

Since

$$\xi \ : \ TM \times \Gamma^k \overset{\delta}{\to} TM \times \Gamma^{k-1} \overset{ev}{\to} T^2M \overset{\omega}{\to} T^2M,$$

where $\delta = id \times T$, ξ is of the class C^{k-1} ([17], Theorem 12.3). By the theorem on dependence of the solutions of differential equations on initial conditions and parameters [3],

$$u \ : \ R{\times}E{\times}TM \to TM$$
$$(t,X,v) \to u(t,X,v)$$

is of the class C^{k-1}. Similarly, we have C^k-functions

$$\lambda \ : \ E{\times}R{\times}M \to M \ : \ (X,t,x) \to X^t.x$$

and

$$\mu \ : \ E{\times}(E{\times}R{\times}M) \overset{id\times\lambda}{\to} E \times M \overset{ev}{\to} M,$$
$$(Y, \ X,t,x) \longrightarrow Y(X^t.x),$$

and so it is sufficient to note that

$$\phi(X,Y,x,t) = u(-t,X,\mu(Y,X,t)).$$

2.6. RESULT. We are now in a position to prove the following theorem.

THEOREM 2.6. Let $1 \leq k < \infty$ and let M be a compact C^{k+1} manifold. Let E be the Banach space of C^k vectorfields on M and let $P \subset E^2$ be the set of all irreducible pairs of C^k vectorfields on M. Then P contains an open dense subset of E^2.

PROOF. Let $T = TM$, $Q = \bigoplus^{2n} T$ be the Whitney sum of 2n copies of T and

$$W = \bigcup_{x \in M} W_x,$$

where W_x consists of those 2n-tuples $(v_1, v_2, \ldots v_{2n}) \in Q_x = T_x^{2n}$ which do <u>not</u> contain a basis of T_x. By Lemma 2.4, W is a closed subset of Q of codimension $n+1$.

Let ϕ be the mapping in Lemma 2.5,

$$\phi : E^2 \times M \times R \to T : (X,Y,x,t) \to ((X^t)^*(x))^{-1}.Y(X^t.x),$$

and let

$$\phi_k = \phi(-,-,-,k) : E^2 \times M \to T.$$

The range of the C^{k-1} mapping

$$\psi = (\phi_0, \phi_1, \ldots, \phi_{2n-1}) : E^2 \times M \to \prod^{2n} T = T^{2n}$$

is contained in the C^{k-1} submanifold Q of T^{2n} and so ψ defines a C^{k-1} mapping ρ such that the diagram

$$\begin{tikzcd} E^2 \times M \arrow[rr, "\psi"] \arrow[dr, "\rho"'] & & T^{2n} \\ & Q \arrow[ur, "\text{inclusion}"'] & \end{tikzcd}$$

commutes. It is easily seen that, for every $X \in E$,

$$X^{\#} : M \times E \to Q : (x,Y) \to \rho(X,Y,x)$$

is a vector-bundle morphism. Let now $X \in E$ and $x \in M$ be fixed and let

(1) $$x_t = X^t.x \quad \text{and} \quad \gamma_t = ((X^t)^*(x))^{-1}.$$

Then

$$X_x^{\#} : E \to Q_x : Y \to (Y(x), \gamma_1 Y(x_1), \ldots \gamma_{2n-1} Y(x_{2n-1})).$$

If $X(x) \neq 0$, then the points $x, x_1, \ldots, x_{2n-1}$ are mutually distinct and so $X_x^{\#} : E \to Q_x$ is surjective.

Let $A \subset E^2$ be the set of pairs (X,Y) such that, for every $x \in M$, at least one of the vectors $X(x)$, $Y(x)$ is non-zero. A simple transversality argument applied to the evaluation map

$$E^2 \times M \to \bigoplus^2 T$$

shows that A is an open dense subset of E^2. If $(X,Y) \in A$ then, for every $x \in M$, at least one of the functions

$$X_x^\# : E \to Q_x, \quad Y_x^\# : E \to Q_x$$

is surjective. We claim that

$$E_W = \{(X,Y) \in E^2 : \rho_{XY}^{-1} W \cap \rho_{YX}^{-1} W = \phi\}$$

is an open dense subset of E . For $k \geq 2$ this follows at once from Lemma 2.3. If $k = 1$, then Lemma 2.3 implies that E_W is open; the density of E_W follows from the result for $k = 2$ and from the fact that the space Γ^2 of C^2 vectorfields on M is dense in $E = \Gamma^1$. (Note that we may assume without loss of generality that M is a C^∞ manifold; cf. [15], p. 15.)

If $S = \{X,Y\}$ and $(X,Y) \in E_W$, then, for every $x \in M$, the collection of vectors

$$Y(x),\ \gamma_1 Y(x_1), \ldots, \gamma_{2n-1} Y(x_{2n-1}),$$
$$X(x),\ \delta_1 X(y_1), \ldots, \delta_{2n-1} X(y_{2n-1})$$

(where x_t and γ_t are as in (1), $y_t = Y^t.x$ and $\delta_t = ((Y^t)*(x))^{-1}$) contains a basis of $T_x M$. Hence $\bar{S}(x) = T_x M$ for every $x \in M$ and, by Theorem 5 of Part One, (X,Y) is an irreducible pair of vectorfields.

REFERENCES

PART ONE

1. CLAUDE CHEVALLEY, Theory of Lie Groups, Princeton University Press, Princeton, New Jersey 1946.

2. W.L. CHOW, Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung, Math. Ann., 117 (1939), pp. 98-105.

3. J. DIEUDONNÉ, Foundations of Modern Analysis, Academic Press, New York and London, 1960.

4. J. DIEUDONNÉ, Treastise on Analysis, Volume III, Academic Press, New York and London, 1972.

5. R. HERMANN, On the accessibility problem in control theory, Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York 1963, pp. 325-332.

6. C. LOBRY, Contrôlabilité des systèmes non linéaires, SIAM J. Control, 8 (1970), pp. 573-605.

7. T. NAGANO, Linear differential systems with singularities and application to transitive Lie algebras, J. Math. Soc. Japan 18 (1966), pp. 398-404.

8. H.J. SUSSMANN and V. JURDJEVIC, Controllability of Nonlinear Systems, J. Differential Equations 12 (1972), pp. 95-116.

9. H.J. SUSSMANN, Orbits of families of vector fields and integrability of systems with singularities, Bull. Amer. Math. Soc. 79 (1973), pp. 197-199.

10. H.J. SUSSMANN, Oribits of families of vector fields and integrability of distributions, to appear in Trans. Amer. Math. Soc.

11. P. STEFAN, Attainable sets are manifolds, duplicated notes, University College of North Wales, Bangor, June 1973.

12. P. STEFAN, A system of vectorfields which is locally of finite type is not necessarily integrable, preprint, Bangor 1973.

PART TWO

13. N. BOURBAKI, Variétés différentielles et analytiques, Fascicule de resultats, Hermann, Paris. (Paragraphes 1 à 7, 1967; Paragraphes 8 à 15, 1971.)

14. D.R.J. CHILLINGWORTH, A new proof of the Frobenius theorem for regular distributions on a Banach manifold, in a letter to the author, 1973.

15. N.H. KUIPER, Variétés Hilbertiennes, Aspects Géométriques, Lecture notes, Les Presses de l'Université de Montréal, 1971.

16. S. LANG, Introduction to Differentiable Manifolds, Wiley (Interscience), New York, 1962.

PART THREE

17. R. ABRAHAM and J. ROBBIN, Transversal Mappings and Flows, Benjamin, New York 1967.

18. C. LOBRY, Une propriété générique des couples de vecteurs, Czechoslovak Math. Journal 22 (1972), 230-237.

19. C. LOBRY, Thèse, L'Université Scientifique et Medicale de Grenoble, 1972.

20. J. MARTINET, Sur les singularités des formes différentielles, Ann. Inst. Fourier, 20 (1970) : 1, 95-178.

21. R. HERMANN, The Differential Geometry of Foliations II, Journal of Mathematics and Mechanics, 11 (1962), 303-315.