Coloring square-free Berge graphs

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Abstract

We consider the class of Berge graphs that do not contain a chordless cycle of length 4. We present a purely graph-theoretical algorithm that produces an optimal coloring in polynomial time for every graph in that class.

Keywords: Berge graph, square-free, coloring, algorithm

1 Introduction

A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$, where $\chi(H)$ is the chromatic number of H and $\omega(H)$ is the maximum clique size in H. In a graph G, a *hole* is a chordless cycle with at least four vertices and an *antihole* is the complement of a hole. We say that graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G. A graph G is F*-free* if it does not contain F, and for a family of graphs $\mathcal{F}$, G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$. Berge [2, 3, 4] introduced perfect graphs and conjectured that a graph is perfect if and only if it does not contain an odd hole or an odd antihole. A *Berge graph* is any graph that contains no odd hole and no odd antihole. This famous question (the Strong Perfect Graph Conjecture) was solved by Chudnovsky, Robertson, Seymour and Thomas [7]: *Every Berge graph*

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is perfect. Moreover, Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [6] devised a polynomial-time algorithm that determines if a graph is Berge.

It is known that one can obtain an optimal coloring of a perfect graph in polynomial time due to the algorithm of Grötschel, Lovász and Schrijver [12]. This algorithm however is not purely combinatorial and is usually considered impractical. No purely combinatorial algorithm exists for coloring all Berge graphs optimally and in polynomial time.

The *length* of a chordless path or cycle is the number of its edges. We let C_k denote the hole of length k ($k \geq 4$). The graph C_4 is also referred to as a *square*. A graph is *chordal* if it is hole-free. It is well-known that chordal graphs are perfect and that their chromatic number can be computed in linear time (see [11]).

Alekseev [1] proved that the number of maximal cliques in a square-free graph on n vertices is $O(n^2)$. Moreover it is known that one can list all the maximal cliques in a graph G in time $O(n^3K)$, where K is the number of maximal cliques; see [19, 17] among others. It follows that finding $\omega(G)$ (the size of a maximum clique) can be done in polynomial time for any square-free graph, and in particular finding $\chi(G)$ can be done in polynomial time for a square-free Berge graph. Moreover, Parfenoff, Roussel and Rusu [18] proved that every square-free Berge graph has a vertex whose neigbhorhood is chordal, which yields another way to find all maximal cliques in polynomial time. However getting an exact coloring of a square-free Berge graph is still hard, and this is what we do. The main result of this paper is a purely graph-theoretical algorithm that produces an optimal coloring for every square-free Berge graph in polynomial time.

Theorem 1.1 *There exists an algorithm which, given any square-free Berge graph G on n vertices, returns a coloring of G with $\omega(G)$ colors in time $O(n^9)$.*

A *prism* is a graph that consists of two vertex-disjoint triangles (cliques of size 3) with three vertex-disjoint paths P_1, P_2, P_3 between them, and with no other edge than those in the two triangles and in the three paths. Note that if two of P_1, P_2, P_3 have lengths of different parities, then their union induces an odd hole. So in a Berge graph, the three paths of a prism have the same parity. A prism is *even* (resp. *odd*) if these three paths all have even length (resp. all have odd length).

Let $\mathcal{A}$ be the class of graphs that contain no odd hole, no antihole of length at least 6, and no prism. This class was studied in [16], where purely graph-theoretical algorithms are devised for coloring and recognizing graphs in that class. In particular:

Theorem 1.2 ([16]) *There exists an algorithm which, given any graph G in class $\mathcal{A}$ on n vertices, returns a coloring of G with $\omega(G)$ colors and a clique of size $\omega(G)$, in time $O(n^9)$.*

Note that every antihole of length at least 6 contains a square; so a square-free graph contains no such antihole.

Since Theorem 1.2 settles the case of graphs that have no prism, we may assume for our proof of Theorem 1.1 that we are dealing with a graph that contains a prism. The next sections focus on the study of such graphs. We will prove that whenever a square-free Berge graph G contains a prism, it contains a cutset of a special type, and, consequently, that G can be decomposed into two induced subgraphs G_1 and G_2 such that an optimal coloring of G can be obtained from optimal colorings of G_1 and G_2.

Note that results from [15] show that finding an induced prism in a Berge graph can be done in polynomial time but that finding an induced prism in general is NP-complete.

In [14], it was proved that when a square-free Berge graph contains no odd prism, then either it is a clique or it has an "even pair", as suggested by a conjecture of Everett and Reed (see [9]). However, this property does not carry over to all square-free Berge graphs; indeed it follows from [13] that the line-graph of any 3-connected square-free bipartite graph (for example the "Heawood graph") is a square-free Berge graph with no even pair.

We finish this section with some notation and terminology. In a graph G, given a set $T \subset V(G)$, a vertex of $V(G) \setminus T$ is *complete to* T if it is adjacent to all vertices of T. A vertex of $V(G) \setminus T$ is *anticomplete to* T if it is non-adjacent to every vertex of T. Given two disjoint sets $S, T \subset V(G)$, S is *complete* to T if every vertex of S is complete to T, and S is *anticomplete* to T if every vertex of S is anticomplete to T. Given a path or a cycle, any edge between two vertices that are not consecutive along the path or cycle is a *chord*. A path or cycle that has no chord is *chordless*.

The *line-graph* of a graph H is the graph $L(H)$ with vertex-set $E(H)$ where $e, f \in E(H)$ are adjacent in $L(H)$ if they share an end in H.

In a graph J, *subdividing* an edge $uv \in E(J)$ means removing the edge uv and adding a new vertex w and two new edges uw, vw. Starting with a graph J, the effect of repeatedly subdividing edges produces a graph H called a *subdivision* of J. Note that $V(J) \subseteq V(H)$.

Lemma 1.3 *Let G be square-free. Let K be a clique in G, possibly empty. Let $X_1, X_2, \dots, X_k$ be pairwise disjoint subsets of $V(G)$, also disjoint from K, such that X_i is complete to X_j for all $i \neq j$, and let $X = \bigcup_i X_i$. Suppose that for every v in K, there is an integer i so that v is complete to $X \setminus X_i$. Then there is an integer i such that $(K \cup X) \setminus X_i$ is a clique in G.*

Proof. First observe that there exists an integer j such that $X \setminus X_j$ is a clique, for otherwise two of $X_1, \dots, X_k$ are not cliques and their union contains a square. Hence if K is empty, the lemma holds.

Now we claim that K is complete to at least $k-1$ of the X_i's. For suppose on the contrary that K is not complete to any of X_1 and X_2. Then there are vertices $v_1, v_2 \in K$, $x_1 \in X_1$, $x_2 \in X_2$ such that for $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, v_i adjacent to x_i and non-adjacent to x_j. By the assumption, $v_1 \neq v_2$. Then $\{v_1, x_1, x_2, v_2\}$ induces a square, contradiction. Hence there exists an index h such that K is complete to $X \setminus X_h$.

Suppose that the lemma does not hold. Then $j \neq h$ and there are vertices $x, x' \in X_j$, $v \in K$, $w \in X_h$ such that x and x' are non-adjacent and v and w are non-adjacent. Then $\{x, v, x', w\}$ induces a square, contradiction. This proves the lemma. $\square$

In a graph G, we say (as in [7]) that a vertex v can be *linked* to a triangle $\{a_1, a_2, a_3\}$ (via paths P_1, P_2, P_3) when: the three paths P_1, P_2, P_3 are mutually vertex-disjoint; for each $i \in \{1, 2, 3\}$, a_i is an end of P_i; for all $i, j \in \{1, 2, 3\}$ with $i \neq j$, $a_i a_j$ is the only edge between P_i and P_j; and v has a neighbor in each of P_1, P_2, P_3.

Lemma 1.4 ((2.4) in [7]) *In a Berge graph, if a vertex v can be linked to a triangle $\{a_1, a_2, a_3\}$, then v is adjacent to at least two of a_1, a_2, a_3.*

2 Good partitions

In a graph G, a *triad* is a set of three pairwise non-adjacent vertices.

A *good partition* of a graph G is a partition (K_1, K_2, K_3, L, R) of $V(G)$ such that:

(i) L and R are not empty, and L is anticomplete to R;

(ii) $K_1 \cup K_2$ and $K_2 \cup K_3$ are cliques;

(iii) In the graph obtained from G by removing all edges between K_1 and K_3, every chordless path with one end in K_1, the other in K_3, and interior in L contains a vertex from L that is complete to K_1.

(iv) Either K_1 is anticomplete to K_3, or no vertex in L has neighbors in both K_1 and K_3;

(v) For some $x \in L$ and $y \in R$, there is a triad of G that contains $\{x, y\}$.

Theorem 2.1 *Let G be a square-free Berge graph. If G contains a prism, then G has a good partition.*

The proof of this theorem will be given in the following sections, depending on the presence in G of an even prism (Theorem 4.2), an odd prism (Theorem 5.2), or the line-graph of a bipartite subdivision of K_4 (Theorem 6.1).

In the rest of this section we show how a good partition can be used to find an optimal coloring of the graph.

Lemma 2.2 *Let G be a square-free Berge graph. Suppose that $V(G)$ has a good partition (K_1, K_2, K_3, L, R). Let $G_1 = G \setminus R$ and $G_2 = G \setminus L$, and for $i = 1, 2$ let c_i be an $\omega(G_i)$-coloring of G_i. Then an $\omega(G)$-coloring of G can be obtained in polynomial time.*

Proof. Since $K_1 \cup K_2$ is a clique, by permuting colors we may assume that $c_1(x) = c_2(x)$ holds for every vertex $x \in K_1 \cup K_2$.

Say that a vertex u in K_3 is *bad* if $c_1(u) \neq c_2(u)$, and let B be the set of bad vertices. If $B = \emptyset$, we can merge c_1 and c_2 into a coloring of G and the lemma holds. Therefore let us assume that $B \neq \emptyset$. We will show that we can produce in polynomial time a pair (c'_1, c'_2) of $\omega(G)$-colorings of G_1 and G_2, respectively, that agree on $K_1 \cup K_2$ and have strictly fewer bad vertices than (c_1, c_2). Repeating this argument at most $|B|$ times will prove the lemma.

For each $h \in \{1, 2\}$ and for any two distinct colors i and j, let $G_h^{i,j}$ be the bipartite subgraph of G_h induced by $\{v \in V(G_h) \mid c_h(v) \in \{i, j\}\}$; and for any vertex $u \in K_3$, let $C_h^{i,j}(u)$ be the component of $G_h^{i,j}$ that contains u.

Let $u \in B$, with $i = c_1(u)$ and $j = c_2(u)$. Then $C_h^{i,j}(u) \cap K_2 = \emptyset$ for each $h \in \{1, 2\}$ because u is complete to K_2. Say that u is *free* if $C_h^{i,j}(u) \cap K_1 = \emptyset$ holds for some $h \in \{1, 2\}$. In particular u is free whenever colors i and j do not appear in K_1.

$$\text{We may assume that there is no free vertex.} \tag{1}$$

Suppose that u is a free vertex, with $C_1^{i,j}(u) \cap K_1 = \emptyset$ say. Then we swap colors i and j on $C_1^{i,j}(u)$. We obtain a coloring c'_1 of G_1 where the color of every vertex in $K_1 \cup K_2$ is unchanged, by the definition of a free vertex; so c'_1 and c_2 agree on $K_1 \cup K_2$. For all $v \in K_3 \setminus B$ we have $c_1(v) \neq i$, because $c_1(u) = i$, and $c_1(v) \neq j$, because $c_1(v) = c_2(v) \neq c_2(u) = j$; so the color of v is unchanged. Moreover we have $c'_1(u) = j = c_2(u)$, so c'_1 and c_2 agree on u. Hence the pair (c'_1, c_2) has strictly fewer bad vertices than (c_1, c_2). Thus (1) holds.

Choose w in B with the largest number of neighbors in K_1. Then:

$$\text{Every vertex } u \in B \text{ satisfies } N(u) \cap K_1 \subseteq N(w) \cap K_1. \tag{2}$$

For suppose that some vertex $x \in K_1$ is adjacent to u and not to w. By the choice of w there is a vertex $y \in K_1$ that is adjacent to w and not to u. Then $\{x, y, u, w\}$ induces a C_4, a contradiction. Thus (2) holds.

Up to relabelling, let $c_1(w) = 1$ and $c_2(w) = 2$. By (1) w is not a free vertex, so $C_1^{1,2}(w) \cap K_1 \neq \emptyset$ and $C_2^{1,2}(w) \cap K_1 \neq \emptyset$. Hence for some $i \in \{1, 2\}$ there is a chordless path $P = w\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}a$ in $C_1^{1,2}(w)$, with $k \geq 1$, $p_1 \in K_3 \cup L$, $p_2, \ldots, p_k \in L$ and $a \in K_1$ with $c_1(a) = i$; and for some $i' \in \{1, 2\}$ there is a chordless path $Q = w\text{-}q_1\text{-}\cdots\text{-}q_\ell\text{-}a'$ in $C_2^{1,2}(w)$, with $\ell \geq 1$, $q_1 \in K_3 \cup R$, $q_2, \ldots, q_\ell \in R$ and $a' \in K_1$ with $c_2(a') = i'$. It follows that at least one of the colors 1 and 2 appears in K_1. We claim that:

$$\text{Exactly one of the colors 1 and 2 appears in } K_1. \tag{3}$$

For suppose that there are vertices $a_1, a_2 \in K_1$ with $c_1(a_1) = 1$ and $c_1(a_2) = 2$. We know that w is anticomplete to $\{a_1, a_2\}$. Since P is bicolored by c_1, it cannot contain a vertex complete to $\{a_1, a_2\}$; so, by assumption (iii), P does not meet L. This implies that $P = w\text{-}p_1\text{-}a_1$ and $p_1 \in K_3$. Then $c_2(p_1) \neq 2$, because

$c_2(w) = 2$, and so $p_1 \in B$; but then (2) is contradicted since w is non-adjacent to a_1. Thus (3) holds.

By (3) we have $i = i'$ and $a = a'$. Let $j = 3 - i$. Note that if $i = 1$ then P has even length and Q has odd length, and if $i = 2$ then P has odd length and Q has even length. So P and Q have different parities. If $p_1 \in L$ and $q_1 \in R$, then $V(P) \cup V(Q)$ induces an odd hole, a contradiction. Hence,

$$\text{At least one of } p_1 \text{ and } q_1 \text{ is in } K_3. \tag{4}$$

We claim that:

$$\text{There is no vertex } y \text{ in } K_3 \text{ such that } c_1(y) = 2 \text{ and } c_2(y) = 1. \tag{5}$$

Suppose that there is such a vertex y. If $p_1 \in K_3$ and $q_1 \in K_3$, then $p_1 = y = q_1$ and $(V(P) \cup V(Q)) \setminus \{w\}$ induces an odd hole, a contradiction. So, by (4), exactly one of p_1 and q_1 is in K_3. Suppose that $p_1 \in K_3$ and $q_1 \in R$. So $p_1 = y$. If p_1 has no neighbor on $Q \setminus w$, then $V(P) \cup V(Q)$ induces an odd hole. So suppose that p_1 has a neighbor on $Q \setminus w$. Then there is a chordless path Q' from p_1 to a' with interior in $Q \setminus w$, and since it is bicolored by c_2 the parity of Q' is different from the parity of Q. Then $(V(P) \setminus \{w\}) \cup V(Q')$ induces an odd hole, a contradiction. When $p_1 \in L$ and $q_1 \in K_3$ the proof is similar. Thus (5) holds.

$$p_1 \notin K_3. \tag{6}$$

For suppose that $p_1 \in K_3$. We have $c_2(p_1) \neq 1$ by (5) and $c_2(p_1) \neq 2$ because $c_2(w) = 2$. Hence let $c_2(p_1) = 3$. So color 3 does not appear in K_2.

Suppose that color 3 does not appear in K_1. Then, by (3), $C_2^{j,3}(p_1) \cap K_1 = \emptyset$. We swap colors j and 3 on $C_2^{j,3}(p_1)$. We obtain a coloring c_2' of G_2 such that the color of all vertices in $K_1 \cup K_2$ is unchanged, so c_2' agrees with c_1 on $K_1 \cup K_2$. For every vertex v in $K_3 \setminus B$ we have $c_2(v) \neq 3$, because $c_2(p_1) = 3$, and $c_2(v) \notin \{1, 2\}$, because $c_2(v) = c_1(v)$ and $\{1, 2\} = \{c_1(w), c_1(p_1)\}$; so the color of v is unchanged. Moreover, $c_2'(p_1) = j$. If $j = 1$, the pair (c_1, c_2') contradicts (5) (note that in this case the color of w is unchanged). If $j = 2$, then $c_2'(p_1) = c_1(p_1)$, so the pair (c_1, c_2') has strictly fewer bad vertices than (c_1, c_2). Therefore we may assume that there is a vertex a_3 in K_1 with $c_1(a_3) = 3$.

Vertex p_1 is not adjacent to a_3 because $c_2(p_1) = c_2(a_3)$, and p_1 is not adjacent to a by (2) and because w is not adjacent to a. This implies $k \geq 2$, so the path $P \setminus w$ meets L. Assumption (iii) implies that $P \setminus w$ contains a vertex that is complete to $\{a, a_3\}$, and since P is chordless, that vertex is p_k.

Suppose that a_3 has a neighbor p_g on $P \setminus \{w, p_k\}$, and choose the smallest such integer g. We know that $g \geq 2$. The chordless path p_1-$\cdots$-p_g-a_3 meets L, but it contains no vertex that is complete to $\{a, a_3\}$ because a has no neighbor on $P \setminus p_k$, so assumption (iii) is contradicted. Therefore a_3 has no neighbor on $P \setminus \{w, p_k\}$.

Suppose that $i = 1$. Then P has even length, and by (3) color 2 does not appear in K_1. If w is adjacent to a_3, then since $k \geq 2$, we see that $(V(P) \setminus$

$\{a\}) \cup \{a_3\}$ induces an odd hole. So w is non-adjacent to a_3. Hence $\{a, a_3\}$ is anticomplete to $\{w, p_1\}$. Since, by (1), p_1 is not a free vertex, and color 2 does not appear in K_1, there is a chordless path S between p_1 and a_3 in $C_2^{2,3}(p_1)$, and S has even length because $c_2(p_1) = c_2(a_3)$. If w has a neighbor in $S \setminus p_1$, then there is a chordless path S' from w to a_3 with interior in $S \setminus p_1$, and S' has odd length since it is bicolored by c_2; but then $V(P \setminus a) \cup V(S')$ induces an odd hole. So w has no neighbor in $S \setminus p_1$, and in particular $w \notin V(S)$. Then $V(P \setminus \{a, w\}) \cup V(S)$ induces an odd hole, a contradiction.

Now suppose that $i = 2$. Since, by (1), p_1 is not a free vertex, there is a chordless path T from p_1 to $\{a, a_3\}$ in $C_1^{2,3}(p_1)$. Since T is bicolored by c_1 it cannot contain a vertex that is complete to $\{a, a_3\}$, so assumption (iii) implies that T does not meet L. So we have $T = p_1$-x-a for some vertex x in K_3 with $c_1(x) = 3$. We have $c_2(x) \neq 3$ because $c_2(p_1) = 3$; so $x \in B$. But the fact that a is adjacent to x and not to w contradicts (2). Thus (6) holds.

By (4) and (6) we have $p_1 \notin K_3$ and $q_1 \in K_3$. In particular, P meets L. We have $c_1(q_1) \neq 1$ because $c_1(w) = 1$, and $c_1(q_1) \neq 2$ by (5). Hence let $c_1(q_1) = 3$.

$$\text{Color 3 appears in } K_1. \tag{7}$$

Assume the contrary. Then, by (3), $C_1^{j,3}(q_1) \cap K_1 = \emptyset$. We swap colors j and 3 on $C_1^{j,3}(q_1)$. We obtain a coloring c_1' of G_1 such that the color of every vertex in $K_1 \cup K_2$ is unchanged, so c_1' agrees with c_2 on $K_1 \cup K_2$. Also every vertex v in $K_3 \setminus B$ satisfies $c_1'(v) = c_1(v)$. Moreover, $c_1'(q_1) = j$. If $j = 1$, then $c_1'(q_1) = c_2(q_1)$, so the pair (c_1', c_2) has strictly fewer bad vertices than (c_1, c_2). If $j = 2$, the pair (c_1', c_2) contradicts (5) (note that in this case the color of w is unchanged). Thus we may assume that (7) holds.

By (7) there is a vertex a_3 in K_1 with $c_1(a_3) = 3$. By (2), q_1 is anticomplete to $\{a, a_3\}$. Vertex q_1 has a neighbor in $P \setminus w$, for otherwise $V(P) \cup V(Q)$ induces an odd hole. So there is a chordless path P' from q_1 to a with interior in $P \setminus w$, and P' meets L because it contains p_k. By assumption (iii), P' contains a vertex that is complete to $\{a, a_3\}$, and since P is chordless that vertex is p_k.

Suppose that $C_1^{i,3}(q_1) \cap K_1 = \emptyset$. Then we swap colors i and 3 on $C_1^{i,3}(q_1)$. We obtain a coloring c_1' of G_1 such that the color of every vertex in $K_1 \cup K_2$ is unchanged, so c_1' agrees with c_2 on $K_1 \cup K_2$. Also every vertex v in $K_3 \setminus B$ satisfies $c_1'(v) = c_1(v)$. Moreover, $c_1'(q_1) = i$. If $i = 1$, the pair (c_1', c_2) has strictly fewer bad vertices than (c_1, c_2). If $i = 2$, then (c_1', c_2) contradicts (5) (note that in this case the color of w is unchanged). Therefore we may assume that $C_1^{i,3}(q_1) \cap K_1 \neq \emptyset$.

Let Z be a chordless path from q_1 to $\{a, a_3\}$ in $C_1^{i,3}(q_1)$. Since Z is bicolored by c_1, no vertex of Z can be complete to $\{a, a_3\}$, and so assumption (iii) implies that Z does not meet L. This means that either $i = 1$ and $Z = q_1$-w-a_3, or $i = 2$ and $Z = q_1$-z-a_3 for some z in K_3 with $c_1(z) = 2$. In either case, K_1 is not anticomplete to K_3, so asumption (iv) implies that p_k is non-adjacent to q_1 and $k \geq 2$.

If a_3 has a neighbor in $P \setminus \{w, p_k\}$, then (since q_1 also has a neighbor in $P \setminus \{w, p_k\}$) there is a chordless path from q_1 to a_3 with interior in $V(P) \setminus \{w, p_k\}$, so, by (iii), that path must contain a vertex that is complete to $\{a, a_3\}$; but this is impossible because a has no neighbor in $P \setminus p_k$. So a_3 has no neighbor in $P \setminus \{w, p_k\}$.

Now if $i = 1$, then w is adjacent to a_3, P has even length, and hence $(V(P) \setminus \{a\}) \cup \{a_3\}$ induces an odd hole. So $i = 2$, and $Z = q_1$-z-a_3 with $z \in K_3$ and $c_1(z) = 2$. Vertex p_k is non-adjacent to z by assumption (iv). The path z-w-P-p_k has odd length, and it is bicolored by c_1, so it contains a chordless odd path P'' from z to p_k. But then $V(P'') \cup \{a_3\}$ induces an odd hole. This completes the proof. $\square$

3 Prisms and hyperprisms

In a graph G let R_1, R_2, R_3 be three chordless paths that form a prism K with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. A vertex of $V(G) \setminus K$ is a *major neighbor* of K if it has at least two neighbors in $\{a_1, a_2, a_3\}$ and at least two neighbors in $\{b_1, b_2, b_3\}$. A subset X of $V(K)$ is *local* if either $X \subseteq \{a_1, a_2, a_3\}$ or $X \subseteq \{b_1, b_2, b_3\}$ or $X \subseteq V(R_i)$ for some $i \in \{1, 2, 3\}$.

If F, K are induced subgraphs of G with $V(F) \cap V(K) = \emptyset$, any vertex in K that has a neighbor in F is called an *attachment* of F in K, and whenever any such vertex exists we say that F *attaches* to K.

Here are several theorems from the Strong Perfect Graph Theorem [7] that we will use.

Theorem 3.1 ((7.4) in [7]) *In a Berge graph G, let R_1, R_2, R_3 be three chordless paths, of even lengths, that form a prism K with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Assume that R_1, R_2, R_3 all have length at least 2. Let R'_1 be a chordless path from a'_1 to b_1, such that R'_1, R_2, R_3 also form a prism. Let y be a major neighbor of K. Then y has at least two neighbors in $\{a'_1, a_2, a_3\}$.*

Theorem 3.2 ((10.1) in [7]) *In a Berge graph G, let R_1, R_2, R_3 be three chordless paths that form a prism K with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in K is not local. Assume no vertex in F is major with respect to K. Then there is a path f_1-$\ldots$-f_n in F with $n \geq 1$, such that (up to symmetry) either:*

1. *f_1 has two adjacent neighbors in R_1, and f_n has two adjacent neighbors in R_2, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, and (therefore) G has an induced subgraph which is the line graph of a bipartite subdivision of K_4, or*

2. *$n \geq 2$, f_1 is adjacent to a_1, a_2, a_3, and f_n is adjacent to b_1, b_2, b_3, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or*

3. *$n \geq 2$, f_1 is adjacent to a_1, a_2, and f_n is adjacent to b_1, b_2, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, or*

4. *f_1 is adjacent to a_1, a_2, and there is at least one edge between f_n and $V(R_3)\setminus\{a_3\}$, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)\setminus \{a_3\}$.*

A *hyperprism* is a graph H whose vertex-set can be partitioned into nine sets:

$$\begin{array}{ccc} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{array}$$

with the following properties:

- Each of $A_1, A_2, A_3, B_1, B_2, B_3$ is non-empty.
- For distinct $i, j \in \{1, 2, 3\}$, A_i is complete to A_j, and B_i is complete to B_j, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_j \cup B_j \cup C_j$.
- For each $i \in \{1, 2, 3\}$, every vertex of $A_i \cup B_i \cup C_i$ belongs to a chordless path between A_i and B_i with interior in C_i.

For each $i \in \{1, 2, 3\}$, any chordless path from A_i to B_i with interior in C_i is called an *i-rung*. The triple (A_i, C_i, B_i) is called a *strip* of the hyperprism. If we pick any i-rung R_i for each $i \in \{1, 2, 3\}$, we see that R_1, R_2, R_3 form a prism; any such prism is called an *instance* of the hyperprism. If H contains no odd hole, it is easy to see that all rungs have the same parity; then the hyperprism is called even or odd accordingly.

Let G be a graph that contains a prism. Then G contains a hyperprism H. Let $(A_1, \dots, B_3)$ be a partition of $V(H)$ as in the definition of a hyperprism above. A subset $X \subseteq V(H)$ is *local* if either $X \subseteq A_1 \cup A_2 \cup A_3$ or $X \subseteq B_1 \cup B_2 \cup B_3$ or $X \subseteq A_i \cup B_i \cup C_i$ for some $i \in \{1, 2, 3\}$. A vertex x in $V(G) \setminus V(H)$ is a *major neighbor* of H if x is a major neighbor of some instance of H. The hyperprism H is *maximal* if there is no hyperprism H' such that $V(H)$ is strictly included in $V(H')$.

Lemma 3.3 *Let G be a Berge graph, let H be a hyperprism in G, and let M be the set of major neighbors of H in G. Let F be a component of $G \setminus (V(H) \cup M)$ such that the set of attachments of F in H is not local. Then one can find in polynomial time one of the following:*

- *A chordless path P, with $\emptyset \neq V(P) \subseteq V(F)$, such that $V(H) \cup V(P)$ induces a hyperprism.*
- *A chordless path P, with $\emptyset \neq V(P) \subseteq V(F)$, and for each $i \in \{1, 2, 3\}$ an i-rung R_i of H, such that $V(P) \cup V(R_1) \cup V(R_2) \cup V(R_3)$ induces the line-graph of a bipartite subdivision of K_4.*

Proof. When H is an even hyperprism, the proof of the lemma is identical to the proof of Claim (2) in the proof of Theorem 10.6 in [7], and we omit it. When H is an odd hyperprism, the proof of the lemma is similar to the proof of Claim (2), with the following adjustments: the case when the integer n in that proof is even and the case when n is odd are swapped, and the argument on page 126 of [7], lines 16–18, is replaced with the following argument:

> Suppose that f_n is not adjacent to b_1; so f_1 is adjacent to b_1, $n \geq 2$, and f_n is adjacent to a_2. Let R_3 be any 3-rung, with ends $a_3 \in A_3$ and $b_3 \in B_3$. Then a_1b_1 is an edge, for otherwise f_1-a_1-R_1-b_1-f_1 is an odd hole; and f_1 has no neighbor in $\{a_3, b_3\}$, for otherwise we would have $n = 1$. Likewise, a_2b_2 is an edge, and f_n has no neighbor in $\{a_3, b_3\}$. But then $V(R_1) \cup V(R_2) \cup V(R_3) \cup \{f_1, \ldots, f_n\}$ induces the line-graph of a bipartite subdivision of K_4, a contradiction.

This completes the proof of the lemma. $\square$

4 Even prisms

We need to analyze the behavior of major neighbors of an even hyperprism. The reader may note that in the following theorem we are not assuming that the graph is square-free.

Theorem 4.1 *Let G be a Berge graph that contains an even prism and does not contain the line-graph of a bipartite subdivision of K_4. Let H be an even hyperprism in G, with partition $(A_1, \ldots, B_3)$ as in the definition of a hyperprism, and let x be a major neighbor of H. Then either:*

- *x is complete to at least two of A_1, A_2, A_3 and at least two of B_1, B_2, B_3, or*
- *$V(H) \cup \{x\}$ induces a hyperprism.*

Proof. Since x is a major neighbor of H, there exists for each $i \in \{1, 2, 3\}$ an i-rung W_i of H such that x is a major neighbor of the prism K_W formed by W_1, W_2, W_3. Suppose that the first item does not hold; so, up to symmetry, x has a non-neighbor $u_1 \in A_1$ and a non-neighbor $u_2 \in A_2$. For each $i \in \{1, 2\}$ let U_i be an i-rung with end u_i, and let U_3 be any 3-rung. Then x is not a major neighbor of the prism K_U formed by U_1, U_2, U_3. We can turn K_W into K_U by replacing the rungs one by one (one at each step). Along this sequence there are two consecutive prisms K and K' such that x is a major neighbor of K and not a major neighbor of K'. Since K and K' are consecutive they differ by exactly one rung. Let K be formed by rungs R_1, R_2, R_3, where each R_i has ends $a_i \in A_i$ and $b_i \in B_i$ ($i = 1, 2, 3$), and let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$; and let K' be formed by P_1, R_2, R_3 for some i-rung P_1. Let P_1 have ends $a'_1 \in A_1$ and $b'_1 \in B_1$, and let $A' = \{a'_1, a_2, a_3\}$ and $B' = \{b'_1, b_2, b_3\}$.

Let $\alpha = |N(x) \cap A|$, $\beta = |N(x) \cap B|$, $\alpha' = |N(x) \cap A'|$, $\beta' = |N(x) \cap B'|$. We know that $\alpha \geq 2$ and $\beta \geq 2$ since x is a major neighbor of K, and $\min\{\alpha', \beta'\} \leq 1$ since x is not a major neighbor of K'. Moreover, $\alpha' \geq \alpha - 1$ and $\beta' \geq \beta - 1$ since K and K' differ by only one rung. Up to the symmetry on A, B, these conditions imply that the vector $(\alpha, \beta, \alpha', \beta')$ is equal to either $(3, 2, 3, 1)$, $(3, 2, 2, 1)$, $(2, 2, 2, 1)$ or $(2, 2, 1, 1)$. In either case we have $\beta = 2$ and $\beta' = 1$, so x is adjacent to b_1, non-adjacent to b'_1, and adjacent to exactly one of b_2, b_3, say to b_3.
Suppose that (α', β') is equal to $(3, 1)$ or $(2, 1)$. We can apply Theorem 3.2 to K' and $F = \{x\}$, and it follows that x satisfies item 4 of that theorem, so x is adjacent to a'_1, a_2, b_3 and has no neighbor in $V(K') \setminus (\{a'_1, a_2\} \cup V(R_3))$. But then $V(R_2) \cup \{x, b_3\}$ induces an odd hole, a contradiction. So we may assume that $(\alpha, \beta, \alpha', \beta') = (2, 2, 1, 1)$, which restores the symmetry between A and B. Since $\alpha = 2$ and $\alpha' = 1$, x is adjacent to a_1, non-adjacent to a'_1, and adjacent to exactly one of a_2, a_3. If x is adjacent to a_2, then K' and $\{x\}$ violate Theorem 3.2. So x is adjacent to a_3 and not to a_2, and Theorem 3.2 implies that x is a local neighbor of K' with $N(x) \cap K' \subseteq V(R_3)$, so x has no neighbor in P_1 or R_2. Then we claim that:

$$\text{For every 1-rung } Q_1\text{, the ends of } Q_1 \text{ are either both adjacent to } x \text{ or both non-adjacent to } x. \tag{1}$$

For suppose the contrary. Then x is not a major neighbor of the prism formed by Q_1, R_2, R_3, and consequently that prism and the set $F = \{x\}$ violate Theorem 3.2. So (1) holds.

Let $A'_1 = A_1 \setminus N(x)$ and $A''_1 = A_1 \cap N(x)$, and $B'_1 = B_1 \setminus N(x)$ and $B''_1 = B_1 \cap N(x)$. By (1), every 1-rung is either between A'_1 and B'_1 or between A''_1 and B''_1. Let C'_1 be the set of vertices of C_1 that lie on a 1-rung whose ends are in $A'_1 \cup B'_1$, and let C''_1 be the set of vertices of C_1 that lie on a 1-rung whose ends are in $A''_1 \cup B''_1$. By (1), C'_1 and C''_1 are disjoint and there is no edge between $A'_1 \cup C'_1 \cup B'_1$ and C''_1 or between $A''_1 \cup C''_1 \cup B''_1$ and C'_1.

Pick any 1-rung P'_1 with ends in $A'_1 \cup B'_1$. Then Theorem 3.2 implies (just like for P_1) that x is a local neighbor of the prism formed by P'_1, R_2, R_3, so x has no neighbor in P'_1. It follows that:

$$x \text{ has no neighbor in } A'_1 \cup C'_1 \cup B'_1. \tag{2}$$

Moreover, we claim that:

$$A'_1 \text{ is complete to } A''_1\text{, and } B'_1 \text{ is complete to } B''_1. \tag{3}$$

For suppose on the contrary, up to relabelling vertices and rungs, that a'_1 and a_1 are non-adjacent. Then, by (2), $V(P_1) \cup \{x, a_1, a_2, b_3\}$ induces an odd hole. Thus (3) holds.

Let $A'_2 = A_2 \setminus N(x)$, $A''_2 = A_2 \cap N(x)$, $B'_2 = B_2 \setminus N(x)$ and $B''_2 = B_2 \cap N(x)$. Let C'_2 be the set of vertices of C_2 that lie on a 2-rung whose ends are in $A'_2 \cup B'_2$, and let C''_2 be the set of vertices of C_2 that lie on a 1-rung whose ends are in

$A''_2 \cup B''_2$. By the same arguments as for the 1-rungs, we see that every 2-rung is either between A'_2 and B'_2 or between A''_2 and B''_2, that C'_2 and C''_2 are disjoint and that there is no edge between $A'_2 \cup C'_2 \cup B'_2$ and C''_2 or between $A''_2 \cup C''_2 \cup B''_2$ and C'_2. Also x has no neighbor in $A'_2 \cup C'_2 \cup B'_2$, and A'_2 is complete to A''_2, and B'_2 is complete to B''_2. Note that, since xa'_1 and xa_2 are not edges, the sets A'_1, B'_1, C'_1, A'_2, B'_2, C'_2 are all non-empty. It follows that the nine sets

$$\begin{array}{ccc} A'_1 & C'_1 & B'_1 \\ A'_2 & C'_2 & B'_2 \\ A''_1 \cup A''_2 \cup A_3 & C''_1 \cup C''_2 \cup C_3 \cup \{x\} & B''_1 \cup B''_2 \cup B_3 \end{array}$$

form a hyperprism. So the second item of the theorem holds. □

Theorem 4.2 *Let G be a square-free Berge graph that contains an even prism and does not contain the line-graph of a bipartite subdivision of K_4. Then G has a good partition.*

Proof. Let H be a maximal even hyperprism in G, with partition $(A_1, \ldots, B_3)$ as in the definition of a hyperprism. Let M be the set of major neighbors of H. Let Z be the set of vertices of the components of $V(G) \setminus (V(H) \cup M)$ that have no attachment in H. By Lemma 3.3 every component of $G \setminus (V(H) \cup M \cup Z)$ attaches locally to H. For each $i = 1, 2, 3$, let F_i be the union of the vertex-sets of the components of $G \setminus (V(H) \cup M \cup Z)$ that attach to $A_i \cup B_i \cup C_i$. Let F_A be the union of the vertex-sets of the components of $G \setminus (V(H) \cup M \cup Z \cup F_1 \cup F_2 \cup F_3)$ that attach to $A_1 \cup A_2 \cup A_3$, and define F_B similarly. By Lemma 3.3 the sets F_1, F_2, F_3, F_A, F_B are well-defined and are pairwise anticomplete to each other, and $V(G) = V(H) \cup M \cup Z \cup F_1 \cup F_2 \cup F_3 \cup F_A \cup F_B$.

By Theorem 4.1, every vertex in M is complete to at least two of A_1, A_2, A_3 and at least two of B_1, B_2, B_3.

Suppose that M contains non-adjacent vertices x, y. By Theorem 4.1, x and y have a common neighbor a in A and a common neighbor b in B. Then $\{x, y, a, b\}$ induces a square, a contradiction. Therefore M is a clique. By Lemma 1.3, $M \cup A_i$ is a clique for at least two values of i, and similarly $M \cup B_j$ is a clique for at least two values of j. Hence we may assume that both $M \cup A_1$ and $M \cup B_1$ are cliques.

Define sets $K_1 = A_1$, $K_2 = M$, $K_3 = B_1$, $L = A_2 \cup B_2 \cup C_2 \cup F_2 \cup A_3 \cup B_3 \cup C_3 \cup F_3 \cup F_A \cup F_B$ and $R = V(G) \setminus (K_1 \cup K_2 \cup K_3 \cup M)$. (So $R = C_1 \cup F_1 \cup Z$.) Every chordless path from K_3 to K_1 that meets L contains a vertex from $A_2 \cup A_3$, which is complete to K_1. Moreover, since H is an even hyperprism, K_1 is anticomplete to K_3, and the sets C_1, C_2, C_3 are non-empty, so, picking any vertex $x_i \in C_i$ for each $i \in \{1, 2, 3\}$, we see that $\{x_1, x_2, x_3\}$ is a triad with a vertex in L and a vertex in R. So (K_1, K_2, K_3, L, R) is a good partition of $V(G)$. □

5 Odd prisms

Now we analyze the behavior of major neighbors of an odd hyperprism. The following theorem is the analogue of Theorem 4.1, but here we need the assumption that the graph is square-free and the proof is different.

Theorem 5.1 *Let G be a square-free Berge graph. Let H be an odd hyperprism in G, with partition $(A_1, \dots, B_3)$ as in the definition of a hyperprism, and let m be a major neighbor of H. Then either:*

- *m is complete to at least two of A_1, A_2, A_3 and at least two of B_1, B_2, B_3, or*
- *$V(H) \cup \{m\}$ induces a hyperprism.*

Proof. We first observe that:

$$\text{Every rung of } H \text{ contains a neighbor of } m. \tag{1}$$

For suppose the contrary. Let a_1-P_1-b_1 be a rung that contains no neighbor of m. Suppose m has neighbors p and q such that $p \in A_2 \cup A_3$, $q \in B_2 \cup B_3$, and p is non-adjacent to q. Then p-a_1-P_1-b_1-q-m-p is an odd hole, contradiction. Hence, since m is major, we may assume that m is anticomplete to $A_3 \cup B_3$ and has neighbors a_1', b_1', a_2, b_2 such that $a_1' \in A_1$, $b_1' \in B_1$, $a_2 \in A_2$, $b_2 \in B_2$, and a_2 and b_2 are adjacent. Pick any $a_3 \in A_3$ and $b_3 \in B_3$. Then $a_1'b_1'$ is not an edge, for otherwise $\{a_1', b_1', a_2, b_2\}$ induces a square; and similarly $a_1'b_1$, a_1b_1' and a_3b_3 are not edges. If a_1' is adjacent to a_1, then a_1'-a_1-P_1-b_1-b_2-m-a_1' is an odd hole, a contradiction. Similarly, if b_1' is adjacent to b_1, there is an odd hole. Hence a_1a_1' and b_1b_1' are not edges. But then a_1'-a_3-a_1-P_1-b_1-b_3-b_1'-m-a_1' is an odd hole. This proves (1).

Suppose, up to symmetry, that m does not satisfy the property of being complete to at least two of B_1, B_2, B_3. So we may assume that m is not complete to B_1, not complete to B_2, and (up to reversing B_2 and B_3), that m has a neighbor in B_3. Let $b_2 \in B_2$ be a non-neighbor of m, and $b_3 \in B_3$ a neighbor of m.

$$\text{Let } b_1 \text{ be a non-neighbor of } m \text{ in } B_1, \text{ and let } a_1\text{-}P_1\text{-}b_1 \text{ be a rung through } b_1. \text{ Then } m \text{ is adjacent to } a_1 \text{ and anticomplete to } P_1 \setminus a_1. \tag{2}$$

Let a_2-P_2-b_2 be a rung through b_2. By (1), m has a neighbor c_1 in $P_1 \setminus b_1$ and a neighbor in $P_2 \setminus b_2$. If c_1 can be chosen different from a_1, then we can link m to $\{b_1, b_2, b_3\}$ via $P_1 \setminus a_1$, P_2 and m-b_3, a contradiction to Lemma 1.4. So it must be that the only neighbor of m in P_1 is a_1. This proves (2). Note that the analogue of (2) also holds for B_2.

For $i = 1, 2, 3$, let $A_i^* = \{x \in A_i \mid x \text{ has a neighbor in } B_i\}$ and $B_i^* = \{x \in B_i \mid x \text{ has a neighbor in } A_i\}$. So A_i^* and B_i^* are either both empty or both non-empty. Moreover, since G is square-free, $A_i^* \cup B_i^*$ is non-empty for at most one value of i.

$$\text{For each } i, m \text{ is complete to } A_i^* \cup B_i^*. \tag{3}$$

For suppose the contrary. Then there are vertices $u_i \in A_i^*$ and $v_i \in B_i^*$ such that u_iv_i is an edge and m has a non-neighbor in $\{u_i, v_i\}$. Since $m \in M$, it has a neighbor a in $(A_1 \cup A_2 \cup A_3) \setminus A_i$ and a neighbor b in $(B_1 \cup B_2 \cup B_3) \setminus B_i$. Then the subgraph induced by $\{m, a, b, u_i, v_i\}$ contains a square or a C_5, a contradiction. Thus (3) holds.

For every symbol X in $\{A, B, C\}$ there is a partition of X_1 into two sets X_1' and X_1'' such that:
– A_1' is complete to A_1'', and B_1' is complete to B_1'';
– A_1' is anticomplete to B_1'; (4)
– C_1' is anticomplete to $A_1'' \cup B_1'' \cup C_1''$,
– C_1'' is anticomplete to $A_1' \cup B_1' \cup C_1'$;
– m is complete to A_1' and anticomplete to $B_1' \cup C_1'$.

Pick rungs a_2-P_2-b_2 and a_3-P_3-b_3 containing b_2 and b_3 respectively. By (2), m is adjacent to a_2.

Let $B_1' = \{y \in B_1 \setminus B_1^* \mid y$ is non-adjacent to m and there exists a rung from $A_1 \setminus A_1^*$ to $y\}$, and let $A_1' = \{x \in A_1 \setminus A_1^* \mid$ there is a rung from x to $B_1'\}$. Let $C_1' = \{z \in C_1 \mid z$ lies on a rung between B_1' and $A_1'\}$. So m is anticomplete to B_1' and, by (2), m is complete to A_1' and anticomplete to C_1'. Let $B_1'' = B_1 \setminus B_1'$, $A_1'' = A_1 \setminus A_1'$, and $C_1'' = C_1 \setminus C_1'$. Let Q be any rung with ends $x \in A_1'$ and $y \in B_1'$. We prove six claims (a)–(f) as follows.

(a) *B_1'' is anticomplete to $A_1' \cup C_1'$.*
We know that B_1'' is anticomplete to A_1' since $A_1' \subseteq A_1 \setminus A_1^*$. Suppose up to relabelling that some vertex b_1 in B_1'' has a neighbor in $Q \setminus \{x, y\}$. Then there is a rung Q' from x to b_1, with interior in $Q \setminus \{x, y\}$, of length at least 3. The definition of B_1' and the existence of Q' implies that b_1 is adjacent to m; but then $V(Q') \cup \{m\}$ induces an odd hole. Since this holds for all Q, claim (a) is established.

(b) *B_1'' is complete to B_1'.*
Suppose, up to relabelling, that some b_1 in B_1'' is not adjacent to y. By (a) b_1 has no neighbor in Q. Then b_1 is non-adjacent to m, for otherwise x-Q-y-b_2-b_1-m-x induces an odd hole. Pick a rung a_1-P_1-b_1. By (3), $b_1 \notin B_1^*$, hence, by the definition of B_1', we have $a_1 \in A_1^*$, and so a_1 has a neighbor $b_1^* \in B_1^*$. If b_1^* is not adjacent to y, then, by the same argument as for b_1 it follows that b_1^* is not adjacent to m, which contradicts (3). Therefore b_1^* is adjacent to y and, by (3), to m. By (a) b_1^* has no neighbor in $Q \setminus y$. We know that a_1 is not adjacent to y since $y \notin B_1^*$. Moreover a_1 has no neighbor in $Q \setminus x$, for otherwise we can link a_1 to $\{b_3, y, b_1^*\}$ via a_3-P_3-b_3, $Q \setminus x$ and a_1-b_1^*, a contradiction to Lemma 1.4. Then xa_1 is an edge, for otherwise x-Q-y-b_1^*-a_1-a_3-x is an odd hole. There is no edge between Q and P_1 except a_1x, for otherwise there would be a rung from x to b_1, implying $b_1 \in B_1'$. But then b_1-P_1-a_1-x-Q-y-b_3-b_1 is an odd hole. Thus B_1'' is complete to y, and since this holds for all Q, the claim is established.

(c) *There is no rung from A_1' to B_1''.*
For suppose on the contrary, up to relabelling, that x-P_1-b_1 is such a rung, with

$b_1 \in B_1''$. We know that b_1 is non-adjacent to x (because $x \notin A_1^*$), and b_1 is adjacent to m, for otherwise the existence of P_1 would imply $b_1 \in B_1'$. By (a) and (b), b_1 is adjacent to y and anticomplete to $Q \setminus y$. Since $V(Q) \cup V(P_1)$ does not induce an odd hole, there are edges between $P_1 \setminus \{x, b_1\}$ and $Q \setminus x$. Since x-P_1-b_1-m-x is not an odd hole, m has neighbors in $P_1 \setminus \{x, b_1\}$. It follows that there is a path S from m to y with interior in $(P_1 \cup Q) \setminus \{x, b_1\}$. But now we can link m to $\{y, b_2, b_3\}$ via S, P_2 and m-b_3, a contradiction.

(d) *There is no rung between A_1'' and B_1'.*
For suppose up to relabelling that a_1-P_1-y is such a rung. By the definition of B_1' we have $a_1 \in A_1^*$, so a_1 has a neighbor $b_1^* \in B_1^*$. By (a) and (b), b_1^* is adjacent to y and anticomplete to $Q \setminus y$. Then a_1 has no neighbor in $Q \setminus x$, for otherwise we can link a_1 to $\{y, b_1^*, b_2\}$ via $Q \setminus x$, a_1-b_1^* and a_2-P_2-b_2. Moreover a_1 is adjacent to x, for otherwise a_1-b_1^*-y-Q-x-a_2-a_1 is an odd hole. Since $V(Q) \cup V(P_1)$ does not induce an odd hole, there is an edge between $Q \setminus y$ and $P_1 \setminus \{a_1, y\}$. Since b_1^*-y-P_1-a_1-b_1^* cannot be an odd hole, b_1^* has a neighbor in $P_1 \setminus \{a_1, y\}$. But this implies the existence of a rung between x and b_1^*, which contradicts (c).

(e) *A_1'' is complete to A_1'.*
Suppose on the contrary that some a_1 in A_1'' is non-adjacent to x. Let a_1-P_1-b_1 be a rung. By (d), $b_1 \in B_1''$, and by (a)–(d), the only edge between P_1 and Q is b_1y. Then a_1-P_1-b_1-y-Q-x-a_3-a_1 is an odd hole, a contradiction.

(f) *C_1' is anticomplete to $A_1'' \cup C_1'' \cup B_1''$, and C_1'' is anticomplete to $A_1' \cup C_1' \cup B_1'$.*
Indeed, in the opposite case there is a rung that violates (c) or (d).

It follows from claims (a)–(f) that all the properties described in (4) are satisfied. So (4) holds.

By (4), (A_1, C_1, B_1) breaks into two strips (A_1', C_1', B_1') and (A_1'', C_1'', B_1'') such that m is complete to A_1' and anticomplete to $B_1' \cup C_1'$. Likewise, the strip (A_2, C_2, B_2) breaks into two strips (A_2', C_2', B_2') and (A_2'', C_2'', B_2'') such that m is complete to A_2' and anticomplete to $B_2' \cup C_2'$. Then, using the properties described in (4), we obtain a hyperprism:

$$\begin{array}{ccc} A_1' & C_1' & B_1' \\ A_2' & C_2' & B_2' \\ A_3 \cup A_1'' \cup A_2'' \cup \{m\} & C_3 \cup C_1'' \cup C_2'' & B_3 \cup B_1'' \cup B_2''. \end{array}$$

So the second item of the theorem holds. □

Theorem 5.2 *Let G be a square-free Berge graph that contains an odd prism and does not contain the line-graph of a bipartite subdivision of K_4. Then G admits a good partition.*

Proof. Since G contains an odd prism, it contains a maximal odd hyperprism $(A_1, C_1, B_1, A_2, C_2, B_2, A_3, C_3, B_3)$ which we call H. Let M be the set of major neighbors of H. Let Z be the set of vertices of the components of $V(G) \setminus (V(H) \cup M)$ that have no attachment in H. Since H is maximal, by Lemma 3.3 there is a partition of $V(G) \setminus (V(H) \cup M \cup Z)$ into sets F_1, F_2, F_3, F_A, F_B such that:

- For $i = 1, 2, 3$, $N(F_i) \subseteq A_i \cup C_i \cup B_i \cup M$;
- $N(F_A) \subseteq A_1 \cup A_2 \cup A_3 \cup M$ and $N(F_B) \subseteq B_1 \cup B_2 \cup B_3 \cup M$;
- The sets $Z, F_1, F_2, F_3, F_A, F_B$ are pairwise anticomplete to each other.

We observe that:

At least two of A_1, A_2, A_3 are cliques, and at least two of B_1, B_2, B_3 are cliques. (1)

This follows directly from Lemma 1.3 (with $K = \emptyset$).

Since H is maximal, Theorem 5.1 implies that:

Every vertex in M is complete to at least two of A_1, A_2, A_3 and at least two of B_1, B_2, B_3. (2)

We claim that:

$$M \text{ is a clique.} \tag{3}$$

Suppose that there are non-adjacent vertices m_1, m_2 in M. By (2), m_1 and m_2 have a common neighbor in $A_1 \cup A_2 \cup A_3$. Therefore let a_1 be a common neighbor of m_1 and m_2 in A_1. If m_1 and m_2 are both complete to B_2 or both complete to B_3, then they have a common neighbor $b \in B_2 \cup B_3$ and $\{m_1, m_2, a_1, b\}$ induces a square, a contradiction. So we may assume up to symmetry that m_1 is not complete to B_2, so it is complete to $B_1 \cup B_3$, and consequently that m_2 is not complete to B_3, and so it is complete to $B_1 \cup B_2$. Pick a non-neighbor b_2 of m_1 in B_2 and a non-neighbor b_3 of m_2 in B_3. Then $\{m_1, m_2, a_1, b_2, b_3\}$ induces a C_5, a contradiction. This proves (3).

$M \cup A_i$ is a clique for at least two values of i, and similarly $M \cup B_j$ is a clique for at least two values of j. (4)

This follows directly from (2), (3) and Lemma 1.3. Thus (4) holds.

Since G is square-free, we may assume, up to symmetry, that A_1 is anticomplete to B_1 and that A_2 is anticomplete to B_2, and so $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$. Pick any $x_1 \in C_1$, $x_2 \in C_2$ and $a_3 \in A_3$. So $\{x_1, x_2, a_3\}$ is a triad τ.

By (4) there is at least one integer h in $\{1, 2, 3\}$ such that both $M \cup A_h$ and $M \cup B_h$ are cliques.

Suppose that $h = 1$. Set $K_1 = A_1$, $K_2 = M$, $K_3 = B_1$, $L = A_2 \cup B_2 \cup C_2 \cup F_2 \cup A_3 \cup B_3 \cup C_3 \cup F_3 \cup F_A \cup F_B$ and $R = V(G) \setminus (K_1 \cup K_2 \cup K_3 \cup L)$. (So $R = C_1 \cup F_1 \cup Z$.) We observe that K_1 is anticomplete to K_3, every chordless path from K_3 to K_1 with interior in L contains a vertex from $A_2 \cup A_3$, which is complete to K_1, and τ is a triad with a vertex in L and a vertex in R. Thus (K_1, K_2, K_3, L, R) is a good partition of $V(G)$. The same holds if $h = 2$.

Now we may assume that $h = 3$, and, up to symmetry, that $M \cup A_1$ and $M \cup B_2$ are cliques. Set $K_1 = B_2 \cup B_3$, $K_2 = M$, $K_3 = A_1 \cup A_3$, $L = B_1 \cup C_1 \cup F_1 \cup F_B$, and $R = A_2 \cup C_2 \cup F_2 \cup C_3 \cup F_3 \cup F_A \cup Z$. We observe that in the graph obtained from G by removing all edges with one end in K_3 and one in K_1 every chordless path from K_3 to K_1 with interior in L goes through B_1, which is complete to K_1. Moreover, every vertex in $C_1 \cup F_1$ is anticomplete to K_1 and every vertex in $B_1 \cup F_B$ is anticomplete to K_3, so no vertex in L has a neighbor in each of K_1, K_3. Also τ is a triad with a vertex in L and a vertex in R. Thus (K_1, K_2, K_3, L, R) is a good partition of $V(G)$. This completes the proof of the theorem. □

6 Line-graphs

The goal of this section will be to prove the following decomposition theorem.

Theorem 6.1 *Let G be a square-free Berge graph, and assume that G contains the line-graph of a bipartite subdivision of K_4. Then G admits a good partition.*

Before proving this theorem, we need some definitions from [7].

In a graph H, a *branch* is a path whose interior vertices have degree 2 and whose ends have degree at least 3. A *branch-vertex* is any vertex of degree at least 3.

In a graph G, an *appearance* of a graph J is any induced subgraph of G that is isomorphic to the line-graph $L(H)$ of a bipartite subdivision H of J. An appearance of J is *degenerate* if either (a) $J = H = K_{3,3}$ (the complete bipartite graph with three vertices on each side) or (b) $J = K_4$ and the four vertices of J form a square in H. Note that a degenerate appearance of a graph contains a square. An appearance $L(H)$ of J in G is *overshadowed* if there is a branch B of H, of length at least 3, with ends b_1, b_2, such that some vertex of G is non-adjacent in G to at most one vertex in $\delta_H(b_1)$ and at most one in $\delta_H(b_2)$, where $\delta_H(b)$ denotes the set of edges of H (vertices of $L(H)$) of which b is an end.

An *enlargement* of a 3-connected graph J (also called a *J-enlargement*) is any 3-connected graph J' such that there is a proper induced subgraph of J' that is isomorphic to a subdivision of J.

To obtain a decomposition theorem for graphs containing line graphs of bipartite graphs, we first thicken the line graph into an object called a strip system, and then study how the components of the rest of the graph attach to the strip system.

Let J be a 3-connected graph, and let G be a Berge graph. A *J-strip system* (S, N) *in G* means

- for each edge uv of J, a subset $S_{uv} = S_{vu}$ of $V(G)$,
- for each vertex v of J, a subset N_v of $V(G)$,
- $N_{uv} = S_{uv} \cap N_u$,

satisfying the following conditions (where for $uv \in E(J)$, a *uv-rung* means a path R of G with ends s, t, say, where $V(R) \subseteq S_{uv}$, and s is the unique vertex of R in N_u, and t is the unique vertex of R in N_v):

- The sets S_{uv} $(uv \in E(J))$ are pairwise disjoint;
- For each $u \in V(J)$, $N_u \subseteq \bigcup_{uv \in E(J)} S_{uv}$;
- For each $uv \in E(J)$, every vertex of S_{uv} is in a uv-rung;
- For any two edges uv, wx of J with u, v, w, x all distinct, there are no edges between S_{uv} and S_{wx};
- If uv, uw in $E(J)$ with $v \neq w$, then N_{uv} is complete to N_{uw} and there are no other edges between S_{uv} and S_{uw};
- For each $uv \in E(J)$ there is a special uv-rung such that for every cycle C of J, the sum of the lengths of the special uv-rungs for $uv \in E(C)$ has the same parity as $|V(C)|$.

The vertex set of (S, N), denoted by $V(S, N)$, is the set $\bigcup_{uv \in E(J)} S_{uv}$.

Note that N_{uv} is in general different from N_{vu}. On the other hand, S_{uv} and S_{vu} mean the same thing.

The following two properties follow from the definition of a strip system:

- For distinct $u, v \in V(J)$, we have $N_u \cap N_v \subseteq S_{uv}$ if $uv \in E(J)$, and $N_u \cap N_v = \emptyset$ if $uv \notin E(J)$.
- For $uv \in E(J)$ and $w \in V(J)$, if $w \neq u, v$ then $S_{uv} \cap N_w = \emptyset$.

In 8.1 from [7] it is shown that for every $uv \in E(J)$, all uv-rungs have lengths of the same parity. It follows that the final axiom is equivalent to:

- For every cycle C of J and every choice of uv-rung for every $uv \in E(C)$, the sums of the lengths of the uv-rungs has the same parity as $|V(C)|$. In particular, for each edge $uv \in E(J)$, all uv-rungs have the same parity.

By a *choice of rungs* we mean the choice of one uv-rung for each edge uv of J. By the above points and since G is odd-hole-free, it follows that for every choice of rungs, the subgraph of G induced by their union is the line-graph of a bipartite subdivision of J.

We say that a subset X of $V(G)$ *saturates* the strip system if for every $u \in V(J)$ there is at most one neighbor v of u such that $N_{uv} \not\subseteq X$. A vertex v in $V(G) \setminus V(S, N)$ is *major* with respect to the strip system if the set of its neighbors saturates the strip system. A vertex $v \in V(G) \setminus V(S, N)$ is *major with respect to some choice of rungs* if the J-strip system defined by this choice of rungs is saturated by the set of neighbors of v.

A subset X of $V(S, N)$ is *local* with respect to the strip system if either $X \subseteq N_v$ for some $v \in V(J)$ or $X \subseteq S_{uv}$ for some edge $uv \in E(J)$.

Lemma 6.2 *Let G be a Berge graph, let J be a 3-connected graph, and let (S, N) be a J-strip system in G. Assume moreover that if $J = K_4$ then (S, N) is non-degenerate and that no vertex is major for some choice of rungs and* *non-major for another choice of rungs. Let $F \subseteq V(G) \setminus V(S,N)$ be connected and such that no member of F is major with respect to (S,N). If the set of attachments of F in $V(S,N)$ is not local, then one can find in polynomial time one of the following:*

- *A chordless path P, with $\emptyset \neq V(P) \subseteq V(F)$, such that $V(S,N) \cup V(P)$ induces a J-strip system.*
- *A chordless path P, with $\emptyset \neq V(P) \subseteq V(F)$, and for each edge $uv \in E(J)$ a uv-rung R_{uv}, such that $V(P) \cup \bigcup_{uv \in E(J)} R_{uv}$ is the line-graph of a bipartite subdivision of a J-enlargement.*

Proof. The proof of this lemma is essentially the same as the proof of Theorem 8.5 in [7]. In 8.5 there is an assumption that there is no overshadowed appearance of J; but all that is used is that no vertex is major for some choice of rungs of (S,N) and non-major for another. $\square$

We say that a K_4-strip system (S,N) in a graph G is *special* if it satisfies the following properties, where for all $i,j \in [4]$, O_{ij} denotes the set of vertices in $V(G) \setminus V(S,N)$ that are complete to $(N_i \cup N_j) \setminus S_{ij}$ and anticomplete to $V(S,N) \setminus (N_i \cup N_j \cup S_{ij})$:

(a) $N_{13} = N_{31} = S_{13}$ and $N_{24} = N_{42} = S_{24}$.

(b) Every rung in S_{12} and S_{34} has even length at least 2, and every rung in S_{14} and S_{23} has odd length at least 3.

(c) O_{12} and O_{34} are both non-empty and complete to each other.

(d) If some vertex of $V(G) \setminus (V(S,N) \cup O_{12} \cup O_{34})$ is major with respect to some choice of rungs in (S,N), then it is major with respect to (S,N). In particular, there is no overshadowed appearance of (S,N) in $G \setminus (M \cup O_{12} \cup O_{34})$, where M is the set of vertices that are major with respect to (S,N).

(e) There is no enlargement of (S,N) in $G \setminus (O_{12} \cup O_{34})$, and (S,N) is maximal in $G \setminus (O_{12} \cup O_{34})$.

Lemma 6.3 *Let G be Berge and square-free, and let J be a 3-connected graph. Let (S,N) be a J-strip system in G. Let $m \in V(G) \setminus V(S,N)$. If m is major for some choice of rungs in (S,N), then one of the following holds:*

1. *There is a J-enlargement with a non-degenerate appearance in G (and such an appearance can be found in polynomial time).*
2. *There is a J-strip system (S',N') such that $V(S,N) \subset V(S',N')$ with strict inclusion (and (S',N') can be found in polynomial time).*
3. *m is major with respect to (S,N).*
4. *G has a special K_4-strip system.*

Proof. Let m be major for some choice of rungs in (S, N). Suppose that there is no J-enlargement with a non-degenerate appearance in G, and (S, N) is maximal in G, and that m is not major with respect to (S, N). Let X be the set of neighbors of m. Let M be the set of vertices of $V(G) \setminus V(S, N)$ that are major with respect to (S, N). Let M^* be the set of vertices of $V(G) \setminus V(S, N)$ that are major with respect to some choice of rungs. So $m \in M^* \setminus M$.

As noted earlier, every degenerate appearance of any 3-connected graph contains a copy of square, so G contains no degenerate appearances of any 3-connected graph. Hence, by 8.4 in [7], we must have $J = K_4$. Let $V(J) = \{1, 2, 3, 4\}$. Since m is major with respect to some choice of rungs and not major with respect to the strip system, we may choose rungs R_{ij}, R'_{ij} ($i \neq j \in \{1, 2, 3, 4\}$) forming line graphs $L(H)$ and $L(H')$ respectively, so that X saturates $L(H)$ but not $L(H')$. Moreover, we may assume that $R_{ij} \neq R'_{ij}$ if and only if $\{i, j\} = \{1, 2\}$.

Let the ends of each R_{ij} be r_{ij} and r_{ji}, where $\{r_{ij} \mid j \in \{1, 2, 3, 4\} \setminus \{i\}\}$ is a triangle T_i for each i. Similarly, let the ends of each R'_{ij} be r'_{ij} and r'_{ji}, where $\{r'_{ij} \mid j \in \{1, 2, 3, 4\} \setminus \{i\}\}$ is a triangle T'_i for each i.

Since X saturates $L(H)$, it has at least two members in each of $T_1, \ldots, T_4$, and since X does not saturate $L(H')$, there is some T'_i that contains at most one member of X. Since $T_3 = T'_3$ and $T_4 = T'_4$ we may assume that $|X \cap T_1| = 2$ and $|X \cap T'_1| = 1$, so $r_{12} \in X$, $r'_{12} \notin X$, and exactly one of r_{13}, r_{14} is in X, say $r_{13} \in X$ and $r_{14} \notin X$. Also, at least two vertices of T_3 are in X, and similarly for T_4, so there are at least two branch-vertices of H' incident in H' with more than one edge in X. By 5.7 in [7] applied to H', we deduce that 5.7.5 in [7] holds, so (since odd branches of H' correspond to even rungs in $L(H')$ and vice-versa) there is an edge ij of J such that

$$R'_{ij} \text{ is even and } [X \cap V(L(H'))] \setminus V(R'_{ij}) = (T'_i \cup T'_j) \setminus V(R'_{ij}). \tag{1}$$

In particular, T'_i and T'_j both contain at least two vertices in X, so $i, j \geq 2$. Since $r_{13} \in X$, it follows that one of i, j is equal to 3, say $j = 3$, and so $r_{13} = r_{31}$, in other words R_{13} has length 0. Hence $i \in \{2, 4\}$. We claim that:

$$i = 4. \tag{2}$$

For suppose that $i = 2$. By (1) R_{23} is even and $[X \cap V(L(H'))] \setminus V(R_{23}) = \{r'_{21}, r_{24}, r_{31}, r_{34}\}$. Since at least two vertices of T_4 are in X it follows that $r_{42} = r_{24}$ and $r_{43} = r_{34}$ (and $r_{41} \notin X$). Hence R_{24} and R_{34} both have length 0, and since R_{23} is even this is a contradiction to the last axiom in the definition of a strip system. Thus (2) holds.

Therefore we have $i = 4$ and $j = 3$. So (1) translates to:

$$R_{34} \text{ is even and } [X \cap V(L(H'))] \setminus V(R_{34}) = \{r_{31}, r_{32}, r_{41}, r_{42}\}. \tag{3}$$

This implies that $V(R'_{12}) \cap X = \emptyset$; moreover, if $r_{23} \in X$ then $r_{23} = r_{32}$, and similarly if $r_{24} \in X$ then $r_{24} = r_{42}$.

One of R_{23}, R_{24} has length 0, the other has odd length, R_{14} has odd length, and $r_{21} \in X$. (4)

Since the path r_{32}-R_{23}-r_{23}-r_{24}-R_{24}-r_{42} can be completed to a hole via r_{42}-r_{43}-R_{34}-r_{34}-r_{32}, the first path is even, and so exactly one of R_{23}, R_{24} is odd. Since the same path can be completed to a hole via r_{42}-r_{41}-R_{14}-r_{14}-r_{13}-r_{32}, it follows that R_{14} is odd. Since one of R_{23}, R_{24} is odd, they do not both have length 0, and hence at most one of r_{23}, r_{24} is in X. On the other hand, since X saturates $L(H)$, the triangle T_2 has at least two vertices from X; it follows that $r_{21} \in X$ and that exactly one of r_{23}, r_{24} is in X (in other words exactly one of R_{23}, R_{24} has length 0). Thus (4) holds.

$$R_{12} \text{ has length } 0. \tag{5}$$

For suppose that $r_{21} \neq r_{12}$. If r_{21} has a neighbor in R'_{12}, then m can be linked onto the triangle T'_1 via R'_{12}, R_{14} and m-r_{13}, a contradiction. Hence r_{21} has no neighbor in R'_{12}. Then from the hole m-r_{21}-r_{24}-r'_{21}-$R'_{1,2}$-r'_{12}-r_{13}-m, we deduce that the rungs R_{12} and R'_{12} are odd. But then either m-r_{21}-r_{23}-r'_{21}-R'_{12}-r'_{12}-r_{14}-R_{14}-r_{41}-m or m-r_{21}-r_{24}-r'_{21}-R'_{12}-r'_{12}-r_{14}-R_{14}-r_{41}-m is an odd hole, contradiction. Thus (5) holds. It follows that every 12-rung (in particular R'_{12}) has even length.

$$R_{24} \text{ has length 0 and } R_{23} \text{ has odd length.} \tag{6}$$

For suppose the contrary. As shown above, this means that R_{23} has length 0 and R_{24} has odd length. Then R_{24}, R_{12} and R_{14} contradict the last axiom in the definition of a strip system (the parity condition). Thus (6) holds. So $r_{24} = r_{42}$ and $r_{23} \neq r_{32}$ (and hence $r_{23} \notin X$).

$$\text{Every 34-rung has non-zero even length.} \tag{7}$$

By (3) R_{34} has even length, so every 34-rung has even length. If some 34-rung has length zero, then its unique vertex x is such that $\{x, r_{42}, r_{21}, r_{13}\}$ induces a square, a contradiction. Thus (7) holds.

For $i \neq j$, let O_{ij} be the set of vertices that are not major with respect to $L(H')$ and are complete to $(T'_i \cup T'_j) \setminus R'_{ij}$. In particular, r_{12} ($= r_{21}$) is in O_{12} and m is in O_{34}, so O_{12} and O_{34} are non-empty. Every vertex in $M^* \setminus M$ is complete to $\{r_{13}, r_{32}, r_{42}, r_{41}\}$ and has no other neighbor outside of R_{34} in $L(H)$. Moreover, since G is square-free, every such vertex is adjacent to every 12-rung of length 0.

For $\{i,j\} \notin \{\{1,2\},\{3,4\}\}$ and for every rung R in S_{ij} let $L(H_1)$ (resp. $L(H'_1)$) be the graph obtained from $L(H)$ (resp. $L(H')$) by replacing R_{ij} with R. Then m is major with respect to $L(H_1)$ and minor with respect to $L(H'_1)$. (8)

Clearly m is minor with respect to $L(H'_1)$. Suppose it is also minor with respect to $L(H_1)$. Then by symmetric argument applied to $L(H)$ and $L(H_1)$, it follows that R is of even length. So we may assume that $\{i,j\} = \{1,3\}$. But then R_{24} must be of non-zero length, a contradiction. Thus (8) holds.

By (8) all rungs in S_{13} and S_{24} have length 0, and all rungs in S_{23} and S_{14} are odd. Also, $M^* \setminus M$ is complete to $N_{13} \cup N_{32} \cup N_{42} \cup N_{41}$ and to every zero-length rung in S_{12} and has no other neighbor in $V(S, N) \setminus S_{34}$. Thus $M^* \setminus M \subseteq O_{34}$; and conversely, since O_{34} is complete to R_{12} (for otherwise $O_{34} \cup \{r_{12}, r_{24}, r_{13}\}$ would contain a square), we deduce that $O_{34} = M^* \setminus M$. We observe that if R is any 14-rung or 23-rung, then R has length at least 3, for otherwise R has length 1 and $V(R) \cup \{r_{21}, m\}$ induces a square.

Let (S', N') be the strip system obtained from (S, N) by replacing S_{12} with $S_{12} \setminus O_{12}$. It follows from the definition of (S', N') and the facts above that items (a)–(c) of the definition of a special K_4-strip system hold. Since only S_{12} and S_{34} have even non-zero rungs, we deduce that item (d) in that definition also holds.

Finally suppose that (S', N') is not maximal in $G \setminus (O_{12} \cup O_{34})$. Since there is no J-enlargement of (S, N) and (S, N) is maximal, there exists an appearance (S'', N'') of J that contains (S', N'), and we may assume that (S'', N'') is obtained from (S', N') by adding one rung R. If $R \in S''_{12}$, then (S'', N'') is an enlargement of (S, N), a contradiction. So $R \notin S''_{12}$, and we do not get a J-enlargement by adding $O_{12} \cap S_{12}$ to S''_{12}. Therefore, there is $r \in O_{12} \cap S_{12}$ such that we do not get a J-enlargement or a larger strip by adding r to S''_{12}. By 5.8 of [7], r is major with respect to an appearance of J using the new rung, and minor otherwise. So $R \in S''_{34}$, $|V(R)| = 1$ and $V(R) \subseteq O_{34}$, a contradiction. Thus, (S, N) is a special K_4-strip system in G, and outcome 4 of the theorem holds. $\square$

We now focus on the case of a special K_4-strip system.

Lemma 6.4 *Let G be a C_4-free Berge graph and (S, N) be a special K_4-strip system in G, with the same notation as in the definition. Let M be the set of vertices that are major with respect to (S, N). Let $X_1 \in \{N_{12}, N_1 \setminus N_{12}\}, X_2 \in \{N_{21}, N_2 \setminus N_{21}\}$ and $X = X_1 \cup X_2$. Let $A = S_{12} \setminus X$ and $B = V(S, N) \setminus (S_{12} \cup X)$. Let $F \subseteq V(G) \setminus (V(S, N) \cup M \cup O_{12})$ be connected. Then F has attachments in at most one of A and B.*

Proof. Suppose for the sake of contradiction that F has attachments in both A and B. We may assume that $|F|$ is minimal under this condition. Then F forms a path with ends f_1, f_2 such that f_1 has attachments in A, f_2 has attachments in B, and there are no other edges between F and $A \cup B$.

Let Y be the set of attachments of F in $V(S, N)$. Suppose that Y is local with respect to (S, N). Then, as F has attachments in both $A \subseteq S_{12}$ and $B \subseteq V(S, N) \setminus S_{12}$, it follows that either $Y \subseteq N_1$ or $Y \subseteq N_2$. We may assume without loss of generality that $Y \subseteq N_1$. Then $N_1 \cap A$ is non-empty, so $N_{12} \not\subseteq X$, and $N_1 \cap B$ is non-empty, so $N_1 \setminus N_{12} \not\subseteq X$, a contradiction. Hence Y is not local in (S, N).

Suppose that $F \cap O_{34} \neq \emptyset$. Then $f_2 \in O_{34}$. Let (S', N') be the strip system obtained from (S, N) by adding O_{34} to S_{34}. Then $F \setminus f_2$ has non-local attachments in (S', N'), and no vertex of $F \setminus \{f_2\}$ has neighbors in B. Let

$L(H)$ be the line graph formed by some choice of rungs in (S', N'), where f_2 is the rung chosen from $S_{3,4}$, and the rung from $S_{1,2}$ contains a neighbor of f_1. Apply 5.8 of [7]. Since no vertex of $F \setminus \{f_2\}$ has a neighbor in $B \setminus \{f_2\}$, none of the outcomes are possible, a contradiction. This proves that $F \cap O_{34} = \emptyset$. So $F \subseteq V(G) \setminus (V(S,N) \cup M \cup O_{12} \cup O_{34})$. By Lemma 6.3, (S,N) is maximal in $G \setminus (O_{12} \cup O_{34})$, and no vertex of $V(G) \setminus (V(S,N) \cup M \cup O_{12} \cup O_{34})$ is major or overshadowing with respect to (S,N), a contradiction to Lemma 6.2. This proves the theorem. $\square$

Lemma 6.5 *Let G be a C_4-free Berge graph and (S,N) be a special K_4-strip system in G, with the same notation as in the definition. Let M be the set of vertices that are major with respect to (S,N). Then:*

(1) *$O_{12} \cup M$ and $O_{34} \cup M$ are cliques; and*

(2) *there is an integer k such that $O_{12} \cup M \cup (N_1 \setminus N_{1k})$ is a clique, and similarly there is an integer ℓ such that $O_{12} \cup M \cup (N_2 \setminus N_{2\ell})$ is a clique.*

Proof. Suppose that (1) does not hold. Then there are non-adjacent vertices x_1, x_2 in $O_{12} \cup M$, say. If $x \in O_{12}$, then by Lemma 6.3 x is complete to N_{1k} for all $k \neq 2$, and complete to $N_{2\ell}$ for all $\ell \neq 1$. If $x \in M$, then x is complete to N_{1k} for all but at most one k, and complete to $N_{2\ell}$ for all but at most one ℓ. Hence there exist k, ℓ so that $\{x_1, x_2\}$ is complete to $N_{1k} \cup N_{2\ell}$, so for every $u \in N_{1k}$ and $v \in N_{2\ell}$, $\{x_1, u, x_2, v\}$ induces a square, contradiction. This proves (1).

By definition, for every $x \in O_{12} \cup M$ there are indices k and ℓ so that x is complete to $(N_1 \setminus N_{1k}) \cup (N_2 \setminus N_{2\ell})$. Hence (2) follows from (1) by a direct application of Lemma 1.3. $\square$

Lemma 6.6 *Let G be a C_4-free Berge graph. If G has a special K_4-strip system, then it has a good partition.*

Proof. Let (S,N) be a special K_4-strip system of G, with the same notation as above. Let M be the set of vertices that are major with respect to (S,N). There are vertices $t_{12} \in S_{12} \setminus (N_{12} \cup N_{21})$, $t_{34} \in S_{34} \setminus (N_{34} \cup N_{43})$ and $t_{13} \in S_{13}$, and hence $\{t_{12}, t_{34}, t_{13}\}$ is a triad τ.

Suppose that both $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ and $(N_2 \setminus N_{21}) \cup M \cup O_{12}$ are cliques. Let $K_1 = N_1 \setminus N_{12}$, $K_2 = O_{12} \cup M$, and $K_3 = N_2 \setminus N_{21}$. By Lemma 6.4, $K_1 \cup K_2 \cup K_3$ is a cutset. Let L be the union of those components of $G \setminus (K_1 \cup K_2 \cup K_3)$ that contain vertices of S_{12}, and let $R = V(G) \setminus (L \cup K_1 \cup K_2 \cup K_3)$. Then K_1 is anticomplete to K_3, and every chordless path from K_3 to K_1 with interior in L contains a vertex of N_{12}, which is complete to K_1, and τ is a triad that contains a vertex of L and a vertex of R. So (K_1, K_2, K_3, L, R) is a good partition of $V(G)$.

Now assume, up to symmetry, that $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ is not a clique. By Lemma 6.5, $N_{12} \cup M \cup O_{12}$ is a clique. Also, at least one of $N_{21} \cup M \cup O_{12}$ and $(N_2 \setminus N_{21}) \cup M \cup O_{12}$ is a clique. If the former is a clique, let $X = N_{21}$, and otherwise let $X = N_2 \setminus N_{21}$. Set $K_1 = N_{12}$, $K_2 = M \cup O_{12}$, and $K_3 = X$. By Lemma 6.4, $K_1 \cup K_2 \cup K_3$ is a cutset. Let L be the component of $G \setminus (K_1 \cup K_2 \cup K_3)$ that contains $N_1 \setminus N_{12}$ (note that $N_1 \setminus N_{12}$ is connected because N_{13} is complete to N_{14}), and let $R = V(G) \setminus (L \cup K_1 \cup K_2 \cup K_3)$. Then K_1 is anticomplete to K_3, and every path from K_3 to K_1 with interior in L contains a vertex of $N_1 \setminus N_{12}$, which is complete to K_1, and τ is a triad that contains a vertex of L and a vertex of R. So (K_1, K_2, K_3, L, R) is a good partition of $V(G)$. □

Now we can give the proof of Theorem 6.1.

Proof. Since G contains the line-graph of a bipartite subdivision of K_4, there is a 3-connected graph J such that G contains an appearance of J, and we choose J maximal with this property. Hence G contains the line-graph $L(H)$ of a bipartite subdivision H of J. Then there exists a J-strip system (S, N) such that $V(S, N) \subseteq V(G)$, and we choose $V(S, N)$ maximal. Let M be the set of vertices in $V(G) \setminus V(S, N)$ that are major with respect to the strip system (S, N). We observe that:

$$M \text{ is a clique.} \tag{1}$$

Suppose that m, m' are non-adjacent vertices in M. Let B be a branch of H, and let u, v be its ends. Since there is no triangle in H, there exist a neighbor u' of u and a neighbor v' of v in H such that $N_{uu'}$ and $N_{vv'}$ are complete to M and anticomplete to each other. Then $\{m, m', u', v'\}$ induces a square. This proves (1).

> For every branch vertex u in H, there is a branch vertex v in H such that $M \cup (N_u \setminus N_{uv})$ is a clique. (2)

It follows from (1) that M is a clique, and by the definition of major vertices, for every $m \in M$ and every branch vertex u there is a branch vertex v such that m is complete to $N_u \setminus N_{uv}$. Hence (2) follows by a direct application of Lemma 1.3.

If some vertex of $V(G) \setminus V(S, N)$ is major with respect to some choice of rungs but not with respect to the strip system, then by Lemma 6.3 G has a special K_4-strip system, and by Lemma 6.6 G has a good partition, so the theorem holds. Therefore we may assume that every vertex of $V(G) \setminus V(S, N)$ that is major with respect to some choice of rungs is major with respect to the strip system. By Lemma 6.2 (or Theorem 8.5 from [7]), every component of $V(G) \setminus (V(S, N) \cup M)$ attaches locally to $V(S, N)$.

> For every strip S_{uv} there exists a triad $\{t, t', t''\}$ in G such that $t \in S_{uv}$ and $t', t'' \in V(S, N) \setminus (S_{uv} \cup N_u \cup N_v)$. (3)

For every strip S_{xy} let R_{xy} be an xy-rung, with endvertices $r_{xy} \in N_{xy}$ and $r_{yx} \in N_{yx}$. Suppose that $r_{uv} \neq r_{vu}$. Since J is 3-connected, there is a cycle C in J that contain u and not v. In G let $C' = \bigcup_{xy \in E(C)} R_{xy}$. Then C' is an even hole, of length at least 6, so it has two non-adjacent vertices t', t'' that are not in N_u. Then $\{r_{vu}, t', t''\}$ is the desired triad. Now suppose that $r_{uv} = r_{vu}$. There is a cycle C in J that contains u and v. In G let $C' = \bigcup_{xy \in E(C)} R_{xy}$. Then C' is an even hole, of length at least 6, so it has three non-adjacent vertices including r_{uv}. Then these vertices form the desired triad. So (3) holds.

For every strip S_{uv}, let S^*_{uv} denote the union of S_{uv} with the components of $G \setminus V(S,N)$ that attach in S_{uv} only, and let $T_{uv} = N_u \cap N_v$ $(= N_{uv} \cap N_{vu})$. Note that T_{uv} is complete to $N_u \setminus N_{uv}$ and to $N_v \setminus N_{vu}$. Moreover we observe that:

$$M \cup T_{uv} \text{ is a clique.} \tag{4}$$

Suppose that $M \cup T_{uv}$ has two non-adjacent vertices a, b. By (2), and since every branch vertex in H has degree at least 3, M is complete to at least one vertex $n_u \in N_u \setminus N_{uv}$, and similarly to at least one vertex $n_v \in N_v \setminus N_{vu}$. By (1) at least one of a, b is in T_{uv}, say $a \in T_{uv}$. Since edges in H that correspond to a, n_u and n_v cannot induce a triangle (as H is bipartite), it follows that n_u and n_v are not adjacent. Then $\{a, b, n_u, n_v\}$ induces a square, a contradiction. So (4) holds.

Let us say that a strip S_{uv} is *rich* if $S_{uv} \setminus T_{uv} \neq \emptyset$.

$$\text{If } (S,N) \text{ has a rich strip, the theorem holds.} \tag{5}$$

Let S_{uv} be a rich strip in (S,N). First suppose that both $M \cup (N_u \setminus N_{uv})$ and $M \cup (N_v \setminus N_{vu})$ are cliques. Hence, by (4) and the definition of T_{uv}, both $M \cup (N_u \setminus N_{uv}) \cup T_{uv}$ and $M \cup (N_v \setminus N_{vu}) \cup T_{uv}$ are cliques. Let $K_1 = N_u \setminus N_{uv}$, $K_2 = M \cup T_{uv}$, $K_3 = N_v \setminus N_{vu}$, let L consist of $S^*_{uv} \setminus T_{uv}$ together with those components of $G \setminus V(S,N)$ that attach only to N_u and those that attach only to N_v, and let $R = V(G) \setminus (K_1 \cup K_2 \cup K_3 \cup L)$. Then every chordless path from K_3 to K_1 with interior in L contains a vertex of N_{uv}, which is complete to K_1, and no vertex of L has both a neighbor in K_1 and a neighbor in K_3; moreover, by (3) there is a triad $\{t, t', t''\}$ with $t \in S_{uv}$ and $t', t'' \in V(S,N) \setminus (S_{uv} \cup N_u \cup N_v)$, so this is a triad with a vertex (namely t) in L and a vertex in R; so (K_1, K_2, K_3, L, R) is a good partition of $V(G)$.
Therefore we may assume that $M \cup (N_u \setminus N_{uv})$ is not a clique, and so $M \cup N_{uv}$ is a clique. If $M \cup (N_v \setminus N_{vu})$ is a clique, let $K_1 = N_{uv} \setminus T_{uv}$, $K_2 = M \cup T_{uv}$, $K_3 = N_v \setminus N_{vu}$, let R consist of $S^*_{uv} \setminus N_u$ together with those components of $G \setminus V(S,N)$ that attach only to N_v, and let $L = V(G) \setminus (R \cup K_1 \cup K_2 \cup K_3)$. Then K_1 is anticomplete to K_3, and every chordless path from K_3 to K_1 with interior in L contains a vertex of $N_u \setminus N_{uv}$, which is complete to K_1; moreover, by (3) there is a triad $\{t, t', t''\}$ with $t \in S_{uv}$ and $t', t'' \in V(S,N) \setminus (S_{uv} \cup N_u \cup N_v)$, so this is a triad with a vertex in L and a vertex (namely t) in R; So (K_1, K_2, K_3, L, R) is a good partition of $V(G)$.
Therefore we may assume that for every rich strip S_{xy}, both $M \cup N_{xy}$ and $M \cup N_{yx}$ are cliques, and neither of $M \cup (N_x \setminus N_{xy})$ and $M \cup (N_y \setminus N_{yx})$ is a clique. Hence, regarding S_{uv}, there is an edge uw in J such that $M \cup N_{uw}$ is not a clique. Then S_{uw} is not rich, and hence $S_{uw} = T_{uw} = N_{uw}$. By (4) $M \cup T_{uw} = M \cup N_{uw}$ is a clique, a contradiction. So (5) holds.

By (5) we may assume that there is no rich strip in (S, N). It follows that for every $uv \in E(J)$ we have $S_{uv} = T_{uv}$, which is a clique by (4). Consequently N_u is a clique for every u, and by (4), $M \cup N_u$ is a clique for every u. Let S_{uv} be a strip. By (3) there is a triad $\{t, t', t''\}$ with $t \in S_{uv}$ and $t', t'' \in V(S, N) \setminus (S_{uv} \cup N_u \cup N_v)$. Let $K_1 = N_u \setminus S_{uv}$, $K_2 = M$, $K_3 = N_v \setminus S_{uv}$, let L consist of S^*_{uv} together with the components of $G \setminus V(S, N)$ that attach only to N_u and only to N_v, and let $R = V(G) \setminus (K_1 \cup K_2 \cup K_3 \cup L)$. Then K_1 is anticomplete to K_3 (since there is no triangle in H), and every chordless path from K_3 to K_1 with interior in L contains a vertex of S_{uv}, which is complete to K_1, and $\{t, t', t''\}$ is a triad with a vertex in L and a vertex in R. So (K_1, K_2, K_3, L, R) is a good partition of $V(G)$. This concludes the proof. □

7 Algorithmic aspects

Assume that we are given a graph G on n vertices. We want to know if G is a square-free Berge graph and, if it is, we want to produce an $\omega(G)$-coloring of G. We can do that as follows, based on the method described in the preceding sections. We can first test whether G is square-free in time $O(n^4)$. Therefore let us assume that G is square-free.

Let $\mathcal{A}$ be the class of graphs that contain no odd hole, no antihole of length at least 6, and no prism (sometimes called "Artemis" graphs). There is an algorithm, "Algorithm 3" in [16], of time complexity $O(n^9)$, which decides whether the graph G is in class $\mathcal{A}$ or not, and, if it is not, returns an induced subgraph of G that is either an odd hole, an antihole of length at least 6, or a prism. If the first outcome happens, then G is not Berge and we stop. The second outcome cannot happen since G is square-free. Therefore we may assume that G is Berge and that the algorithm has returned a prism K. We want to extend K either to a maximal hyperprism or to the line-graph of a bipartite subdivision of K_4. We can do that as follows. Let K have rungs R_1, R_2, R_3, where, for each $i = 1, 2, 3$, R_i has ends a_i, b_i, such that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles.

- Intially, for each $i \in \{1, 2, 3\}$ let $A_i = \{a_i\}$, $B_i = \{b_i\}$ and $C_i = V(R_i) \setminus \{a_i, b_i\}$. Let $V(H) = V(K)$.
- Let M be the set of major neighbors of H.
- If there is a component F of $G \setminus (H \cup M)$ whose set of attachments on H is not local, then by Lemma 3.3, one of the following occurs (and can be found in polynomial time):
  - (i) There is a chordless path P in F such that $V(H) \cup V(P)$ induces a larger hyperprism H'; or
  - (ii) There are three rungs R_1, R_2, R_3 of H, one in each strip of H, and a chordless path P in F, such that $V(R_1) \cup V(R_2) \cup V(R_3) \cup V(P)$ induces the line-graph of a bipartite subdivision of K_4.

Assume that outcome (ii) never happens. Whenever outcome (i) happens, we start again from the larger hyperprism that has been found. Note that outcome (i) can happen only n times, because at each time we start again with a strictly larger hyperprism. So the procedure finishes with a maximal hyperprism. Then we can find a good partition of G as explained in Theorem 4.2 or 5.2, decompose G along that partition, and color G using induction as explained in Lemma 2.2.

Remark: Since a hyperprism may have exponentially many rungs, we need to show how we can determine in polynomial time the set M of major neighbors of a hyperprism H in a graph G without listing all the rungs of H. It is easy to see that a vertex x in $V(G) \setminus V(H)$ is a major neighbor of H if and only if one of the following two situations occurs:

- For at least two distinct values of $i \in \{1, 2, 3\}$, there exists an i-rung R_i such that x is adjacent to both ends of R_i, or
- For a permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, there exists an i-rung R_i such that x adjacent to both ends of R_i and x has a neighbor in A_j and a neighbor in B_k.

So it suffices to test, for each $i \in \{1, 2, 3\}$, whether there exists an i-rung such that x is adjacent to both its ends. This can be done as follows. For every pair $u_i \in A_i$ and $v_i \in B_i$, test whether there is a chordless path between u_i and v_i in the subgraph induced by $C_i \cup \{u_i, v_i\}$. If there is any such path R_i, then record it for the pair $\{u_i, v_i\}$, and for every vertex x in $V(G) \setminus V(H)$ record whether x is adjacent to both u_i and v_i or not. This takes time $O(n^4)$ ($O(n^2)$ for each pair $\{u_i, v_i\}$). So the whole procedure of growing the hyperprism and determining the set M of its major neighbors takes time $O(n^4)$.

Now assume that outcome (ii) happens, and so G contains the line-graph of a bipartite subdivision of K_4. So G contains the line-graph of a bipartite subdivision of a 3-connected graph J, and we want to grow J and the corresponding J-strip system (S, N) to maximality. We can do that as follows.

- Intially, let (S, N) be the strip system equal to the line-graph of a bipartite subdivision of K_4 found in outcome (ii).
- Let M be the set of vertices in $V(G) \setminus V(S, N)$ that are major on some choice of rungs of (S, N). (Determining M can be done with the same arguments as in the remark above concerning the set of major neigbhors of a hyperprism, and we omit the details.)
- If there is a component F of $G \setminus (V(S, N) \cup M)$ whose set of attachments on H is not local, then by Lemma 6.2, one of the following occurs (and can be found in polynomial time):
  - A chordless path P, with $\emptyset \neq V(P) \subseteq V(F)$, such that $V(S, N) \cup V(P)$ induces a J-strip system, or

- – A chordless path P, with $\emptyset \neq V(P) \subseteq V(F)$, and for each edge $uv \in E(J)$ a uv-rung R_{uv}, such that $V(P) \cup \bigcup_{uv \in E(J)} R_{uv}$ is the line-graph of a bipartite subdivision of a J-enlargement.

- If some vertex in M is not major on some choice of rungs of (S, N), then, by Lemma 6.3, we can either find a larger strip system of the special case described in item (iv) of that lemma.

  In either case, whenever we find a larger strip system we start again with it. This will happen at most n times. So the procedure finishes with a maximal strip system. Similarly to the case of the hyperprism, the whole procedure of growing the strip system and determining the set M of its major neighbors takes time $O(n^4)$. Then we can find a good partition of G as explained in Theorem 6.1, decompose G along that partition, and color G using induction as explained in Lemma 2.2.

Complexity analysis. Whenever G contains a prism, we have shown that G has a partition into sets K_1, K_2, K_3, L, R such that $K_1 \cup K_2$ and $K_2 \cup K_3$ are cliques, with L and R non-empty, and L is anticomplete to R. Then G is decomposed into the two proper induced subgraphs $G \setminus L$ and $G \setminus R$. These subgraphs themselves may be decomposed, etc. This can be represented by a decomposition tree T, where G is the root, and the children of every non-leaf node G' are the two induced subgraphs into which G' is decomposed. Every leaf is a subgraph that contains no prism.

Let us consider the triads of G. By item (v) of a good partition, there exists a triad τ_G that has at least one vertex from each of L, R; we label G with τ_G. Since the cutset $K_1 \cup K_2 \cup K_3$ is the union of two cliques it contains no triad, and so no triad of G is in both $G \setminus L$ and $G \setminus R$; moreover τ_G itself is in none of these two subgraphs. Consequently every triad of G can be used as the label of at most one non-leaf node of T. So T has at most n^3 non-leaf nodes. Since every node has at most two children, the number of leaves is at most $2n^3$, and the total number of nodes of T is at most $3n^3$.

Testing if G is Berge takes time $O(n^9)$; this is done only once, at the first step of the algorithm, as a subroutine of testing whether G is in class $\mathcal{A}$. At any decomposition node of T different from the root we already know that we have a Berge graph (an induced subgraph of G), so we need only test whether the graph contains a prism; this can be done in time $O(n^5)$ with "Algorithm 2" from [16]. The complexity of coloring a leaf is $O(n^6)$ since it contains no prism [16]. The coloring algorithm described in the Lemma 2.2 involves only a few bichromatic exchanges, so its complexity is small. The complexity of growing a hyperprism (once a prism is known) or a strip structure is also negligible in comparison with the rest. So the total complexity of the algorithm is $O(n^9) + O(n^3) \times O(n^6) = O(n^9)$ (proving Theorem 1.1).

An alternative algorihm

The goal of this section is to present another algorithm which we feel is conceptually simpler. Instead of looking for hyperprisms or strip systems, this algorithm will search for a good partition directly, and, whenever it finds one, decompose the graph along that partition.

Let $k(G)$ denotes the number of maximal cliques in a graph G. Farber [10] and independantly Alexeev [1] showed that there are $O(n^2)$ maximal cliques in any square-free graph on n vertices. Tsukiyama, Ide, Ariyoshi and Shirakawa [19] gave an $O(nmk(G))$ algorithm for generating all maximal cliques of a graph G, and Chiba and Nishizeki [5] improved this complexity to $O(\sqrt{m+n}\ mk)$. This leads to the following.

Theorem 7.1 *The maximal cliques of a square-free graph can be generated in time $O(\sqrt{m+n}\ mn^2)$ and there are $O(n^2)$ of them.*

An *almost good partition* in a graph G is a partition $K = (K_1, K_2, K_3, L, R)$ of $V(G)$ into five sets K_1, K_2, K_3, L, R that satisfy items (i), (ii) and (v) of the definition of a good partition. Thus a good partition in a graph G is an almost good partition that also satisfies items (iii) and (iv) of the definition of a good partition.

A *frame* in a graph G is a 6-tuple $(Q_1, Q_3, x, y, C_1, C_3)$ such that:

- x and y are distinct vertices of G that are contained in a triad of G.
- Q_1 and Q_3 are maximal cliques of $G \setminus \{x, y\}$.
- $|C_1| \leq 1$ and $|C_3| \leq 1$.
- $C_1 \subseteq Q_1 \setminus Q_3$ and $C_3 \subseteq Q_3 \setminus Q_1$.

It is not easy to enumerate all possible good partitions of a square-free graph, while by Theorem 7.1 it is possible to enumerate in polynomial time all frames. A frame should be thought of as "something that may lead to a good partition". Our algorithm roughly works as follows: it enumerates all possible frames, and tries to refine each one until a good partition is found. Let us make this formal. For any set $A \subseteq V(G)$ and vertex v, let $N_A(v)$ denote the set of vertices in A that are adjacent to v.

Let $Q = (Q_1, Q_3, x, y, C_1, C_3)$ be a frame. We set $Q'_1 = Q_1 \setminus Q_3$ and $Q'_3 = Q_3 \setminus Q_1$ (and we will use this notation in what follows every time a frame is considered). Note that Q'_1 and Q'_3 are disjoint cliques (possibly empty). In a square-free graph, the edges between Q'_1 and Q'_3 have a special structure that we now explain. When a, a' are two vertices of Q'_1, it must be that either $N_{Q'_3}(a) \subseteq N_{Q'_3}(a')$ or $N_{Q'_3}(a') \subseteq N_{Q'_3}(a)$, for otherwise G contains a square. It follows that all vertices of Q'_1 can be ordered by decreasing neighborhood in Q'_3, that is as $a_1 > a_2 > \cdots > a_r$, in such a way that: for all $1 \leq i \leq j \leq r$ we have $N_{Q'_3}(a_j) \subseteq N_{Q'_3}(a_i)$. Similarly, all vertices of Q'_3 can be ordered by non-increasing neighborhood in Q_1', that is as $b_1, b_2, \dots b_s$, in such a way that for all $1 \leq i \leq j \leq s$ we have $N_{Q_1'}(b_j) \subseteq N_{Q_1'}(b_i)$. Vertices a_1 and b_1 in these orderings will be refered to as *maximal* vertices in Q_1' and Q_3' respectively.

It is easy to see that for any subsets K_1, K_3 of Q_1', Q_3' respectively, the orders obtained on K_1 and K_3 by simply comparing vertices according to their order in Q_1', Q_3' satisfy the same properties, namely: for all $a, a' \in K_1$, if $a < a'$ we have $N_{K_3}(a) \subseteq N_{K_3}(a')$ and for all $b, b' \in K_3$, if $b < b'$ we have $N_{K_1}(b) \subseteq N_{K_1}(b')$.

Let $K = (K_1, K_2, K_3, L, R)$ be a good partition in a graph G and $Q = (Q_1, Q_3, x, y, C_1, C_3)$ be a frame in G. We say that Q is a *frame for* K whenever the following hold:

- $K_1 \subseteq Q_1', K_2 = Q_1 \cap Q_3, K_3 \subseteq Q_3', x \in L$ and $y \in R$.
- If $K_1 = \emptyset$, then $C_1 = \emptyset$. Otherwise, $C_1 = \{c_1\}$, where c_1 is a maximal vertex in the ordering of K_1.
- If $K_3 = \emptyset$, then $C_3 = \emptyset$. Otherwise, $C_1 = \{c_3\}$, where c_3 is a maximal vertex in the ordering of K_3.

(Note that whenever there exists an edge between K_1 and K_3 the vertices c_1 and c_3 exist and are adjacent).

Our goal now is to prove that if a graph G has a good partition, then some good partition of G has a frame. Say that a good partition (K_1, K_2, K_3, L, R) is *optimal* if K_2 is maximal, and subject to this, K_3 is maximal, in other words, if there is no good partition $(K_1', K_2', K_3', L', R')$ such that:

- $K_2 \subsetneq K_2'$, or
- $K_2 = K_2'$ and $K_3 \subsetneq K_3'$.

Lemma 7.2 *Let $K = (K_1, K_2, K_3, L, R)$ be an optimal good partition of a graph G, and let $x \in L$ and $y \in R$ be such that x and y are contained in a triad of G. If u is any vertex in $V(G) \setminus (K_2 \cup \{x, y\})$, then u is not complete to $(K_1 \cup K_2 \cup K_3) \setminus \{u\}$. Moreover, if $u \in R$ then u is not complete to $K_2 \cup K_3$.*

Proof. Suppose for a contradiction that u is complete to $(K_1 \cup K_2 \cup K_3) \setminus \{u\}$. We shall prove that there exists a good partition $(\cdot, K_2 \cup \{u\}, \cdot, \cdot, \cdot)$, a contradiction to the maximality of K_2. We now break into cases according to where u is.

If $u \in R$, then $(K_1, K_2 \cup \{u\}, K_3, L, R \setminus \{u\})$ is a good partition of G. Indeed, since $u \neq y$, we have $R \setminus \{u\} \neq \emptyset$. All the other requirements are inherited from the fact that (K_1, K_2, K_3, L, R) is a good partition.

If $u \in L$, then let L_x be the component of $L \setminus \{u\}$ that contains x. We claim that $(K_1, K_2 \cup \{u\}, K_3, L_x, R \cup (L \setminus L_x))$ is a good partition of G. Indeed, $u \neq x$, so we have $L_x \neq \emptyset$ and L_x is clearly connected. Conditions (iii) and (iv) are satisfied since moving $L \setminus L_x$ from L to R cannot perturb these conditions. All the other requirements are inherited from the fact that (K_1, K_2, K_3, L, R) is a good partition.

If $u \in K_1$, then $(K_1 \setminus \{u\}, K_2 \cup \{u\}, K_3, L, R)$ is a good partition of G. Again, moving a vertex from K_1 to K_2 cannot perturb conditions (iii) and (iv). All the other requirements are inherited from the fact that (K_1, K_2, K_3, L, R) is a good partition.

If $u \in K_3$, then $(K_1, K_2 \cup \{u\}, K_3 \setminus \{u\}, L, R)$ is a good partition of G. The proof is similar to the one above. This proves the first assertion of the theorem.

We now prove the second assertion. Suppose for a contradiction that $u \in R$ is complete to $K_2 \cup K_3$. Then $(K_1, K_2, K_3 \cup \{u\}, L, R \setminus \{u\})$ is a good partition, a contradiction to the maximality of K_3. Indeed, since $u \in R$, u has no neighbor in L, so moving u from R to K_3 will not perturb conditions (iii) and (iv). $\square$

Lemma 7.3 *If K is an optimal good partition of a square-free graph G, then there exists a frame Q for K.*

Proof. Let $K = (K_1, K_2, K_3, L, R)$. So there exist a vertex $x \in L$ and a vertex $y \in R$ that are contained in a triad of G, and there exists a maximal clique Q_1 of $G \setminus \{x, y\}$ that contains $K_1 \cup K_2$ and a maximal clique Q_3 of $G \setminus \{x, y\}$ that contains $K_3 \cup K_2$. For $i = 1, 3$, if $K_i = \emptyset$, set $C_i = \emptyset$, and otherwise consider a maximal vertex c_i of K_i (for the ordering defined in Q'_i), and set $C_i = \{c_i\}$. We claim that $Q = (Q_1, Q_3, x, y, C_1, C_3)$ is a frame for K. All the conditions in the definition of a frame for a good partition are trivially satisfied except one: $K_2 = Q_1 \cap Q_3$. So let us check that one.

Obviously, $K_2 \subseteq Q_1 \cap Q_3$, so suppose for a contradiction that there exists $u \in (Q_1 \cap Q_3) \setminus K_2$. Note that $u \notin \{x, y\}$. So, $u \in V(G) \setminus (K_2 \cup \{x, y\})$ and u is complete to $(K_1 \cup K_2 \cup K_3) \setminus \{u\}$, a contradiction to Lemma 7.2. $\square$

Lemma 7.4 *Let K be an optimal good partition of a square-free graph G and Q a frame for K. For $i \in \{1, 3\}$ the following hold.*

- *If $C_i = \emptyset$, then for all $d \in Q'_i$, $d \in L \cup R$.*
- *If $C_i = \{c_i\}$, then for all $d \in Q'_i$, if $d > c_i$ then $d \in L \cup R$.*

Proof. First observe that if $d \in Q'_i$ then $d \notin K_2 \cup K_{4-i}$ because $K_2 = Q_1 \cap Q_3$ and $K_{4-i} \subseteq Q'_{4-i}$.

If $C_i = \emptyset$, then a vertex $d \in Q'_1$ cannot be in K_1, so it is in $L \cup R$ by the remark above.

If $C_i = \{c_i\}$, if $d \in Q'_i$ and $d > c_i$, then by the maximality of c_i, we have $d \notin K_i$. So, $d \in L \cup R$. $\square$

Lemma 7.5 *Let G be a square-free graph. Let $K = (K_1, K_2, K_3, L, R)$ be an optimal good partition of G such that $K_1, K_3 \neq \emptyset$, let $Q = (Q_1, Q_3, x, y, C_1, C_3)$ be a frame for K, and let $K' = (K'_1, K'_2, K'_3, L', R')$ be an almost good partition of G such that $x \in L' \subseteq L$, $y \in R'$, $K_1 \subseteq K'_1 \subseteq Q'_1$, $K_2 = K'_2 = Q_1 \cap Q_3$ and $K_3 \subseteq K'_3 \subseteq Q'_3$. Let $C_1 = \{c_1\}$ and $C_3 = \{c_3\}$, and suppose that c_1 is maximal in K'_1 and c_3 is maximal in K'_3. Let R'_y be the component of $G[R']$ that contains y. Then the following properties hold.*

(1) *Suppose that K' does not satisfy condition (iii) of the definition of a good partition, so there exist u in K'_3, v in K'_1, and letting $G' = G \setminus uv$ if $uv \in E(G)$ and otherwise $G' = G$, there exists a chordless path $P = uu' \dots v'v$ of length at least 2 in G' with interior in L', that contains no vertex complete to K'_1. We suppose that P is such a path of minimal length. Then:*

   If $c_1 \in N(v')$, then $(K'_1 \setminus N(v')) \cap K_1 = \emptyset$. If $c_1 \notin N(v')$, then $N(v') \cap K_1 = \emptyset$.

(2) *Suppose that K' satisfies condition (iii) but does not satisfies condition (iv). So, there exists an edge between K'_1 and K'_3 and some vertex $u \in L'$ has neighbors in both K'_1 and K'_3. Then $N(u) \cap K_3 = \emptyset$.*

Proof. Let us prove (1). Suppose for a contradiction $(K'_1 \setminus N(v')) \cap K_1 \neq \emptyset$ and $N(v') \cap K_1 \neq \emptyset$.

If u' has a neighbor w in K_3, then $\{w\} \cup V(P)$ contains a path from w to v' and then to K_1, which contradicts condition (iii) (for K) of the definition of a good partition. So, u' has no neighbor in K_3. In particular, $u \notin K_3$. We have $u \notin K_2 \cup K_1$ because $u \in Q'_3$. If $u \in R$, then u is complete to $K_2 \cup K_3$, which contradicts Lemma 7.2. Hence, we have $u \in L$.

We claim that u has no neighbor in K_1. Otherwise, from the maximality of c_1, we see that u is adjacent to c_1. Also, u is adjacent to c_3 because K'_3 is a clique. Note that because of their maximality, c_1 and c_3 are adjacent since u has a neighbor in K'_1. So, there are edges between K_1 and K_3, and by condition (iv) in the definition of a good partition, u yields a contradiction. This proves our claim. Now, u has a neighbor w in K_3 (because K'_3 is a clique), so $\{w\} \cup V(P)$ contains a path that contradicts condition (iii) of the definition of a good partition.

Let us prove (2). Since there are edges between K'_1 and K'_3, c_1 and c_3 are adjacent. Vertex u is the unique internal vertex of a path from K'_3 to K'_1, and since the almost good partition K' satisfies condition (iii), u is complete to K'_1, and therefore to K_1. Because of condition (iv) (for K), we do have $N(u) \cap K_3 = \emptyset$. □

We are now ready to describe the algorithm. The algorithm will enumerate a polynomial number of partitions of $V(G)$ into five sets, and stop whenever such a 5-tuple is a good partition. The details will be described below, but from this informal description, it is clear that the algorithm gives the correct answer when G has no good partition. So, when we prove the correctness of the algorithm, it can be assumed that G contains a good partition.

The algorithm enumerates all frames $Q = (Q_1, Q_3, x, y, C_1, C_3)$. For each of them, we set $K'_1 = Q_1 \setminus Q_3$, $K'_2 = Q_1 \cap Q_3$, $K'_3 = Q_3 \setminus Q_1$. The sets K'_1, K'_2 and K'_3 are our "working sets". Throughout the process, they can only decrease until they form the cutset of a good partition, or until a failure is reported.

We now describe a subroutine that we call the *connectivity update*. For a given triple (K'_1, K'_2, K'_3), this operation consists of computing the component

L' of $G \setminus (K'_1 \cup K'_2 \cup K'_3)$ that contains x, setting $R' = V(G) \setminus (K'_1 \cup K'_2 \cup K'_3 \cup L')$, and computing the component R'_y of $G[R']$ that contains y. In the case when $R'_y = L'$ (so, x and y are in the same component), the connectivity update subroutine reports a failure. Then, the algorithm must test a further frame, and if for all frames a failure is reported, the algorithm answers that G has no good partition.

We now explain what is done for each enumerated frame Q. We refer to the following invariant that holds under the assumption that K is an optimal good partition for which Q is a frame: this invariant is the fact that throughout the following procedure we always have $K_1 \subseteq K'_1$, $K_2 = K'_2$, $K_3 \subseteq K'_3$, $x \in L'$ and $y \in R'_y$.

Step 1: For $i = 1, 3$, do the following. If $C_i = \emptyset$, then set $K'_i = \emptyset$. Otherwise, for every vertex $u > c_i$ in K'_i, set $K'_i \leftarrow K'_i \setminus \{u\}$. Once all this is done, perform the connectivity update (and recall that if a failure is reported, the algorithm immediatly goes to next frame, so in what follows, we assume that no failure is reported). Note that in the case where $K'_1 = \emptyset$ or $K'_3 = \emptyset$, a good partition is already detected, so the algorithm stops. We may therefore assume that $K'_1 \neq \emptyset$ and $K'_3 \neq \emptyset$. Note also that by Lemma 7.4, the invariant is preserved through this step.

Step 2: The goal of this step is to handle condition (iii). We mark all vertices of L' that have a neighbor in K_3 with the mark "start", and all the vertices of L' that have a neighbor in K'_1 and are not K'_1-complete with mark "bad arrival". Let L'' be the set obtained from L' by removing all the K_1-complete vertices. We perform a breadth-first-search in L'' from "start" to "bad arrival". This will either detect a path P violating (iii) or certify that no such path exists.

If a path $P = uu' \dots v'v$ is detected, then we choose a shortest such path (with breadth-first search). If $c_1 \in N(v')$, then we set $K'_1 \leftarrow K'_1 \cap N(v')$. If $c_1 \notin N(v')$, then we set $K'_1 \leftarrow K'_1 \setminus N(v')$. We then perform the connectivity update, and if no failure is reported, we repeat Step 2 until condition (iii) is satisfied. Note that by Lemma 7.5 (1) the invariant is preserved through this step.

Step 3: The goal of this step is to handle condition (iv). Because of the previous step, we know that condition (iii) is satisfied. The algorithm checks whether condition (iv) is satisfied or finds a vertex u with neighbors in both K'_1 and K'_3. We then set $K'_3 \leftarrow K'_3 \setminus N(u)$. We then perform the connectivity update, and if no failure is reported, we repeat Step 3, until condition (iv) is satisfied. Note that by Lemma 7.5 (2), the invariant is preserved through this step.

We claim that this algorithm detects a good partition when there is one. So suppose there is an optimal good partition K. At some point, the algorithm considers a frame Q for it (that exists by Lemma 7.3), and defines the sets K'_1, K'_2, K'_3. The key to the proof is that during all the process, K'_1 and K'_3 always decrease while the invariant is preserved.

Let us now compute the complexity of this algorithm. By Theorem 7.1, the enumeration of all frames takes time $O(n^8)$: the list of all maximal cliques is established in time $O(n^5)$, and then is suffices to enumerate all vertices x, y, c_1, c_3 and all pairs of maximal cliques. Step 1 can be performed in linear time. Step 2 requires a breadth-first search, which can be computed in time $O(n^2)$, and it is done at most $O(n)$ times. Step 3 is similar. This leads to a global complexity of $O(n^{11})$ for finding a good partition (and the complexity of $\mathcal{O}(n^{14})$ for coloring G).

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