The Secretary Problem with a Selection Committee: Do Conformist Committees Hire Better Secretaries?

Steve Alpern

ORMS Group, Warwick Business School, University of Warwick, Coventry CV4 7Al UK

steve.alpern@wbs.ac.uk

Vic Baston

Department of Mathematics, University of Southampton, Southampton SO17 1BJ UK

vic.baston@btinternet.com

Abstract

This paper analyses a variation of the secretary problem in which two selectors, with different fields of interest, each want to appoint one of the n candidates with as much expertise as possible in their field. Selectors simultaneously vote to accept or reject: Unanimous decisions are respected and candidates with a split decision are hired with probability p. Each candidate arrives with expertises x and y in the two fields, uniformly and independently distributed on [0,1] and observable to both selectors. If a candidate with expertise pair (x, y) is hired by unanimous decision, the payoffs to the selectors are simply x and y. However in order to model the level of conformity in the firm, we deduct a positive 'consensus cost' c from the utility of a selector who has rejected a candidate who is nevertheless hired.

We show (Theorem 1) that each stage game has a unique equilibrium in which there are two thresholds, $z < v$, and say selector I accepts candidate (x, y) if $x > v$ or $x > z$ and $y > v$. We show that for p and c sufficiently large, decisions are unanimous and that as the number n of candidates goes to infinity the equilibrium value of the game goes to the golden mean. We show that as the consensus cost c increases from 0, this hurts the selectors (Theorem 4) but helps the firm (Theorem 6), whose utility from hiring candidate (x, y) is a weighted average of x and y. Thus a little conformity is good for the firm.

1 Introduction

This paper represents an attempt to game theoretically model the notion of 'conformity' in a group and more specifically consensus-seeking on a committee. When a committee vote is about to ratify a positive decision, for example a high level appointment or the launch of a new product line, it is often considered desirable to do this by unanimous assent. (For negative decisions such as not to hire someone, unanimity is not considered important.) In particular there is pressure on a voter, who is not in favour of a positive decision about to be agreed, to conform to the majority view. Eriksson and Coultas (2009) refer to this pressure as a 'conformist-bias', and model it quite generally. Our approach here is to attach a 'consensus cost' $-c$ to a committee member's decision to defy a majority view which is nevertheless carried. This approach is somewhat more general than that of Alpern and Gal (2009) and Alpern et al (2010) where the nonconformist (who vetoes a decision which otherwise would be carried) is punished by depriving him of one veto in subsequent votes. (Roughly speaking, c could represent the value of the lost veto, though this might vary over time.) Experimental evidence in Roch (2007) shows that when rater teams (our committee) are required to reach consensus, they appear to show improved behavioural accuracy; in our model they are not required but merely encouraged to reach consensus.

There is an important asymmetry in our use of the consensus cost, motivated by an essential asymmetry in the drive towards a consensus. When committees pass a motion, or hire a candidate, they often like to announce that the decision was 'by unanimous consent'. Sometimes they tell a successful candidate that she was the unanimous choice of the committee, to increase the chance that she will feel wanted and accept the post. Often when a motion is adopted by majority but non unanimous vote, a second vote is carried out to see if the dissenters will agree to make the decision unanimous. However, when a candidate is not hired, she is never told that the decision was unanimous. For motions which are not passed, there is little advantage of a unanimous decision. For this reason we ascribe a cost to preventing a consensus to hire, but no cost to prevent a consensus not to hire.

The context in which we place the notion of conformity is a 'secretary problem' with a committee of selectors, rather than the usual single decision maker. This approach changes a stopping time problem to one of game theory. For simplicity we assume that the committee consists of two selectors, I and II, who value different qualities in the n secretaries that they are about to sequentially interview. Selector I cares about say typing skills, denoted by x; while II cares about IT skills, denoted by y. Each secretary comes with an observable skill level pair (x, y), with each component drawn independently from the uniform distribution on [0,1]. We assume that each selector cares only about one component but is sufficiently knowledgeable so that he observes both. We take the usual dynamics of the secretary problem, though we are concerned with utility rather than rank maximization: The n secretarial candidates come to the committee one at a time. For each one, the selectors vote simultaneously to accept or reject her. If both accept, she is hired; if both reject, the next one is interviewed. If there is a split decision, the case is adjudicated at a higher level, modelled as a coin flip which hires her with a known probability p. If the first $n-1$ candidates are rejected, the n-th one is automatically hired, regardless of quality. If a secretary with skill level pair (x, y) is eventually hired, the utility to say selector I is given by x if he voted to accept, and by $x - c$ if he voted to reject (and she was hired by adjudication after the other selector accepted her). Similarly for selector II.

The idea that a decision maker may take into account factors other than his sectional interests is not new and has already been used in analysing different types of decision making. Indeed there is a considerable body of literature involving the concepts of conformity and compliance. The classic experiments of Asch (1951) in the 1950s demonstrated that an individual's own opinions are influenced by those of a majority group. It is commonly accepted that social influence processes are subtle, indirect and outside of awareness and that they can arise in a wide variety of practical situations. Li et al (2001) stated that "each member becomes more cautious in casting the crucial vote than when he alone makes the decision based on his own information". Callander (2008) analysed symmetric majority rule voting equilibria when voters want to elect the better candidate and also want to vote for the winner. Moldovanu and Shi (2013) have obtained many results showing how acceptance standards vary with the amount of partisanship shown by the selectors.

Our main concern is the role played by conformity, as represented by the consensus cost c, in the equilibrium behaviour of the secretary game. For fixed n and p, the value of the game to the selectors dips, then increases, then is constant, as c increases from 0. The last part is easy to understand, as for large enough c a selector will never reject a candidate that the other player likes enough to accept, because the possibility of having to pay the cost c is simply too high. The value of the game to the firm who the secretary will work for, assumed to be a weighted average of her x and y values, also depends on c. It increases as c increases from 0, so we say that 'a little conformity is good for the firm'.

The paper is organized as follows. Section 2 formally presents our model. Section 3 gives the main results. Section 4 derives the unique equilibrium strategies for the stage game. Section 5 describes the iteration process by which the value v_n for n candidates can be derived from v_{n-1}. Section 6 illustrates the iteration process by numerically determining v_n as a function of c for $p = 1/2$. Section 7 analyses the limiting behaviour of the v_n and the optimal strategies as the number n of candidates goes to infinity. Section 8 shows that some properties of the value function established numerically for $p = 1/2$ in Section 6 are true for general p. In particular Theorem 4 says that a little conformity (small c, compared with $c = 0$) is bad for the selectors. Section 9 deals with the expected quality of the eventually hired secretary (what matters to the firm) and shows in Theorem 5 that this increases as c increases from 0, so for the firm a little conformity is a good thing.

2 The Model

In this section we describe our dynamic multistage game Γ_n in which two selectors eventually hire one of n successive candidate ' secretaries'. At the end of the section we give a brief discussion on some natural generalizations of the model.

In our model, each candidate successively arrives for an interview with abilities observable by both selectors valued at x and y in each of two areas, say administration and IT. While both selectors can observe x and y, Selector I only cares about (values) administrative ability x while Selector II only cares about y. What this means, roughly, is that if a candidate with ability pair (x, y) is hired, then the utility to I is x and to II is y. We say roughly, because we additionally specify that if a candidate with ability pair (x, y) is hired after being accepted by (say) I and rejected by II, then Selector II has utility $y - c$, where the given cost $c > 0$ is a penalty paid by II for preventing a consensus. Interpretations of c were discussed in the Introduction.

After each candidate (x, y) (we label them by their ability pair) is interviewed, each selector votes (simultaneously) to either accept or reject. Unanimous decisions are respected: doubly accepted candidates are hired; doubly rejected candidates are not hired (leading to consideration of the next candidate). If there is a mixed decision, with one selector voting to accept and the other voting to reject, the candidate is hired with a known probability p. If the first $n - 1$ candidates are not hired, then the nth one must be hired. So the parameters of the game are n, p, the consensus cost c, and the distribution of (x, y) . We assume that x and y are independent and uniformly distributed on [0,1]. The assumption of a uniform distribution for the skill level x is not arbitrary but rather fits naturally into the versions of the secretary problem where the rank of the candidate among the field is what matters. If some arbitrary measure x of skill level has a certain distribution, we can convert the problem using $\hat{x}$ to represent the fraction of secretaries whose skills lie below level x. Thus the utility to selector I of skill level x is given by $\hat{x}$, which by definition has the uniform distribution.

We begin our analysis of this multistage game which we denote by Γ_n by considering a generic stage game $M_r = M$, which describes the game that arises after a candidate arrives (with $r - 1$ to follow) with abilities (x, y) , assuming that a selector knows that he can expect a payoff of v if this candidate is not hired and the next stage game is played. Note that the selectors can expect the same payoff because the distribution of the abilities of the candidates in each field is the same and also the probability of acceptance if there is a split decision is the same for each selector. Thus the game is symmetric. The payoff matrix for this stage game of Γ_n is given by

$$M = \begin{array}{cc} & \begin{array}{cc} a & r \end{array} \\ \begin{array}{c} A \\ \\ R \end{array} & \begin{pmatrix} (x, y) & p(x, y - c) + p'(v, v) \\ & \\ p(x - c, y) + p'(v, v) & (v, v) \end{pmatrix} \end{array} \tag{1}$$

where

$$p' = 1 - p, \tag{2}$$

and A and a represent accept strategies and R and r represent reject strategies for selectors I and II respectively. The reason why we can replace the subsequent stage game (which would be M_{r-1} if $M = M_r$) by its value is that we will establish for these stage games that there is always a unique Nash equilibrium which is not Pareto dominated.

The model can encompass discounting by introducing a δ in front of the v's and asymmetries between the selectors by taking different v's, p's and c's for the individual selectors. The arguments used in Section 4 to establish the existence of a Nash equilibrium (Theorem 1) for our game are easily adaptable to these more general situations to show that they also have a Nash equilibrium. Furthermore it is not clear that discounting is appropriate for our model. If a quick appointment is deemed necessary, this can be allowed for in our model by making the probability of appointing a disputed candidate comparatively high. Also, in terms of the quality of the appointee, we will show that, in many cases, little benefit is gained by interviewing more than six candidates.

3 Main Results

In this section we provide details of our main results. The first result, Theorem 1, establishes that the game Γ_n has a unique solution. It says that a selector has two threshold values in his area of expertise which determines whether he will vote for a candidate who attains them. Attainment of the upper value earns unqualified approval whilst attainment of the lower but not the upper value earns approval only if the other selector has unqualified approval for the candidate. The remaining results detail the effects that are introduced by a consensus cost.

Theorem 1, which is proved in Section 4, establishes that the game Γ_n has a unique solution, which can be obtained by backwards induction. A selector i ($i = I, II$) has an equilibrium strategy for the stage game consisting of a pair of threshold values $0 \leq z \leq v$ with the following acceptance rule for an arriving candidate with known abilities (x, y):

- If her value to you (e.g. x if you are Selector I) is at least v, then accept; if the value to you is at most z, then reject.
- If her value to you is between z and v, accept only if the other selector's value of her is above the upper threshold value.

We label the upper threshold value by v because it is the expected utility E for a selector if the procedure moves to the next stage. (If $M = M_r$, then in the recursion $v = v_{r-1}$.) In general the lower threshold value involves not only the expected utility E but also the consensus cost c and the probability of the candidate being appointed if one selector votes for acceptance and the other for rejection. Taking into account the latter is intuitively reasonable because, if it is extremely likely that an arbitrated decision will favour the other selector, it will not be worth risking the consensus cost.

The other results are as follows:

- the selectors become less choosy over time in the sense that their threshold values z and v decrease as more candidates are rejected (Theorem 2);
- the effect of consensus cost varies; in some cases it has no effect whatsoever on threshold values, in others it has an influence in every interview except the one when there are no further candidates

available and sometimes it influences early decisions but not those when the number of remaining candidates starts getting small (Section 6 and Appendix 3);

- as the consensus cost level c increases from zero, the equilibrium value to a selector initially decreases (Theorem 4);

- in some circumstances, an increase in consensus cost can increase the equilibrium value to a selector (Section 6 and Theorem 3);

- if the value of the candidate to the firm is the weighted average of the values of the candidate in the two fields, a small amount of consensus cost can result in an increase in the expected quality of the appointee compared with the situation when there is no consensus cost (Theorem 5). Thus a small dose of consensus cost can mean that the firm is likely to hire slightly better qualified staff;

- (Theorem 2) when the number of candidates becomes very large (n goes to infinity), the (fixed) consensus cost is comparatively large and the probability p not too small, the equilibrium value (denoted by v_n) to the selector approaches the *golden mean* ϕ given by

$$\phi = (\sqrt{5} - 1)/2 \approx 0.61803. \tag{3}$$

However, when p is small and there are a very large number of candidates, a selector can afford to wait until an excellent candidate in his field is acceptable to the other selector (in this case the equilibrium value tends to 1). Thus the selectors can expect to appoint a candidate who is excellent in both fields.

To indicate how our paper relates to previous work on game variations of the secretary problem, we provide a brief survey of papers already in the literature. All of these papers assume that each selector is acting without constraint in the sense that decisions taken by him which may hurt the interests of the other selector do not result in any adverse consequences. Thus the concept of consensus cost is new. The paper by Mazalov et al (2002) is the one that is nearest in structure to ours but they only investigate a very special case. In Baston and Garnaev (2005) the interview process has similarities to ours but there are two positions to be filled (one in each area) and each selector is trying to appoint a good candidate in their particular area. An interesting variation of the secretary problem is given in Eriksson et al (2007) where employers search for a secretary at the same time as secretaries search for an employer. Cownden and Steinsaltz (2014) analyse the situation in which multiple players seek to employ secretaries from a common labour pool, and, for the two-person case, they compute how much players can gain through cooperation and also how the optimal strategy changes under a payoff structure that promotes spite.

Possibly the closest in spirit to our paper are the papers by Alpern and Gal (2009) and Alpern et al (2010) as they involve two selectors in a "one at a time" process making a decision whether or not to appoint an interviewee. However their selectors start out with a fixed number of vetoes and, if at least one of the selectors wants to appoint, a candidate is appointed unless the other selector uses one of his vetoes. Furthermore the selectors do not have equal powers as one selector, say selector I, has to announce his decision before the other and this gives selector I the opportunity to play strategically when selector II still has vetoes at his disposal. In particular selector I could vote to appoint a candidate that would otherwise be unacceptable to him safe in the knowledge that the candidate is also unacceptable to selector II who would have to use a veto to ensure the candidate is rejected. Another major difference is that our methods are applicable to any (known) number of candidates whereas they employ steady state methods which are only applicable to a large number of candidates.

4 Analysis of the Stage Game

In this section we show that the stage game given by the matrix M in (1) has a unique subgame perfect Nash equilibrium in which the selectors have, in general, an upper and lower threshold value for each stage. If the value (quality) of the candidate in the selector's area of expertise is above the upper threshold, the selector votes to accept the candidate. If the value of the candidate lies between the upper and lower thresholds, the selector accepts the candidate if and only if the candidate's value to the other selector is above the upper threshold value. If the candidate's value to a selector is below the lower threshold, the selector rejects the candidate.

To find the Nash equilibria of the bimatrix $M = M(v, v)$ given in (1) by

$$M = \begin{array}{c} \\ A \\ \\ R \end{array} \begin{array}{c} \begin{array}{cc} a & r \end{array} \\ \begin{pmatrix} (x, y) & p(x, y-c) + p'(v, v) \\ & \\ p(x-c, y) + p'(v, v) & (v, v) \end{pmatrix} \end{array}, \tag{4}$$

we find the best replies to the individual pure strategies. If selector II chooses the pure strategy accept (a), then it is strictly better for selector I to choose Accept (A) than Reject (R) if

$$x > p(x-c) + p'v \quad \text{giving} \quad x > v - \frac{pc}{p'}. \tag{5}$$

When it is positive, the right hand side of the second inequality of (5) plays an important role in our analysis so it is convenient to introduce the notation

$$\lambda = pc/p' \quad \text{and} \quad z = \max\{v - \lambda,\ 0\}. \tag{6}$$

Note that Accept (A) is strictly better than Reject (R) for selector I when selector II accepts (a) if and only if $x > z$, which we write as

$$A >_a R \iff x > z$$

In a similar fashion we have

$$A >_r R \iff px + p'v > v \iff x > v.$$

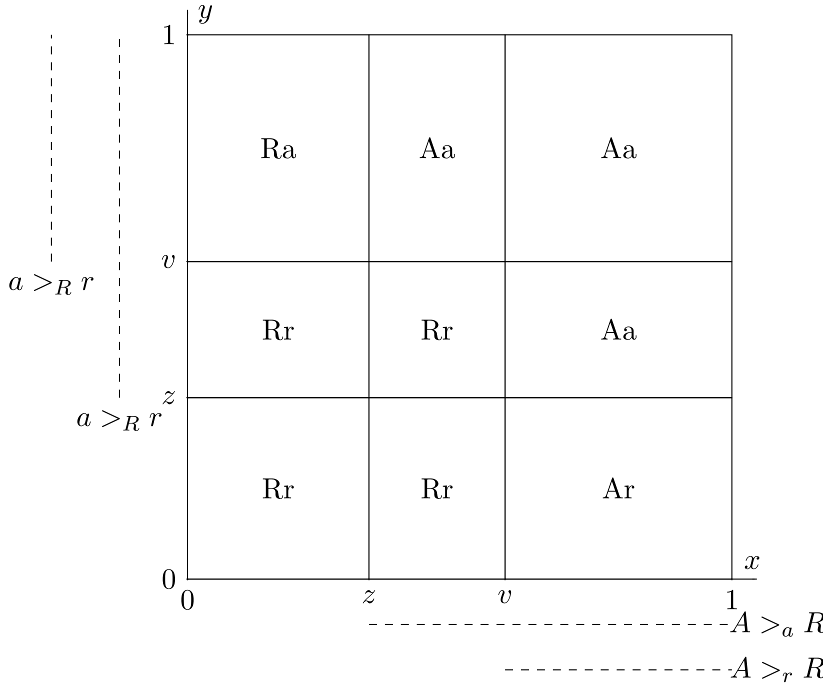

Figure 1: Equilibria in $x \times y$ space for $z > 0$.

Analagous results for the preferences of selector II are

$$a >_A r \iff y > z \qquad \text{and} \qquad a >_R r \iff y > v.$$

These considerations divide the $x \times y$ candidate space $[0,1]^2$ into nine rectangles as shown in Figure 1 where, except for the middle one, there is a unique Nash equilibrium which is written in the centre of the box. In the middle rectangle there is a second pure Nash equilibrium Aa in addition to the cited Rr. However both selectors prefer Rr to Aa; to see this, note that, in the Rr rectangle, the payoffs for Rr and Aa are (v, v) and (x, y) respectively and the inequalities $v \geq x$ and $v \geq y$ hold.

There is also a Nash equilibrium which is a mixture of Rr and Aa in the middle rectangle but this is also dominated by Rr for the same reasons. Hence it is reasonable to expect the selectors to agree on the Nash equilibrium Rr.

There are cases in which Figure 1 reduces to just four rectangles, namely when $z = 0$ or $z = v$. If

$z = 0$, a candidate is accepted if one of the selectors wants her so the effect for this stage is the same as taking $p = 1$. If $z = v$ a candidate is only accepted if both selectors want her so the effect for this stage is the same as taking $p = 0$. For both these cases the consensus cost does not come into play so a consensus cost will only be relevant for $p \in (0, 1)$.

We summarize our equilibrium of the stage game M when the distribution is continuous in the following theorem.

Theorem 1. *There is a unique subgame perfect Nash equilibrium of Γ_n given by (4) which can be characterized as follows:*

Consider any period in which the equilibrium value of entering the following period without having appointed a candidate is (v, v) where v lies in (0,1), then the equilibrium strategy for each selector is:

- *If your valuation of the candidate exceeds the given value v, ACCEPT:*
- *If your valuation is less than $z = \min\{v - \lambda, 0\}$ given by (6), REJECT;*
- *If your valuation lies between z and v, ACCEPT if the other interviewer's valuation exceeds v and REJECT otherwise.*

5 Value Iterations in the Game Γ_n

In this section we obtain a recursion formula for the value of Γ_n which, for fixed p, can be thought of as splitting into two cases; one in which the consensus cost is comparatively small, the other when the consensus cost is comparatively large. In the latter case a selector will vote for a candidate who attains the upper threshold in at least one of the two required qualities; in effect this means that the selectors act as though $p = 1$.

Because the qualities of the candidates are uniformly distributed in [0,1], each selector can expect to have a payoff of 1/2 when there is just one candidate. Thus when there are just two candidates, the selectors know that each can expect a payoff of 1/2 if the first candidate to be interviewed is not appointed; as a consequence they are effectively playing the stage game

$$
\begin{array}{ccc}
 & a & r \\
M_2 = \begin{array}{c} A \\ \\ R \end{array} & \multicolumn{2}{c}{\begin{pmatrix} (x, y) & p(x, y - c) + p'(1/2, 1/2) \\ & \\ p(x - c, y) + p'(1/2, 1/2) & (1/2, 1/2) \end{pmatrix}}
\end{array}
$$

where $p' = 1 - p$ as defined in (2).

Once the Nash equilibrium of this game has been found it can be substituted into the stage game corresponding to the case when there are three candidates and so on. This process gives rise to the recursive equilibrium equation

$$(v_n, v_n) = E[\text{eq. val. } M(v_{n-1}, v_{n-1})], \quad v_1 = 1/2 \tag{7}$$

where $M = M(v_{n-1}, v_{n-1})$ is

$$
\begin{array}{ccc}
 & a & r \\
M = \begin{array}{c} A \\ \\ R \end{array} & \multicolumn{2}{c}{\begin{pmatrix} (x, y) & p(x, y - c) + p'(v_{n-1}, v_{n-1}) \\ & \\ p(x - c, y) + p'(v_{n-1}, v_{n-1}) & (v_{n-1}, v_{n-1}) \end{pmatrix}}.
\end{array}
$$

In order to evaluate the equilibrium equation (7) we need to find a formula for the expected equilibrium payoff E as a function of p, c and v. The payoffs to selector I at the indicated equilibria are given in Figure 2.

We now calculate the expected payoffs of the selectors at equilibrium averaged over the $x \times y$ square $[0, 1] \times [0, 1]$. The contributions from the various rectangles making up the square are as follows:

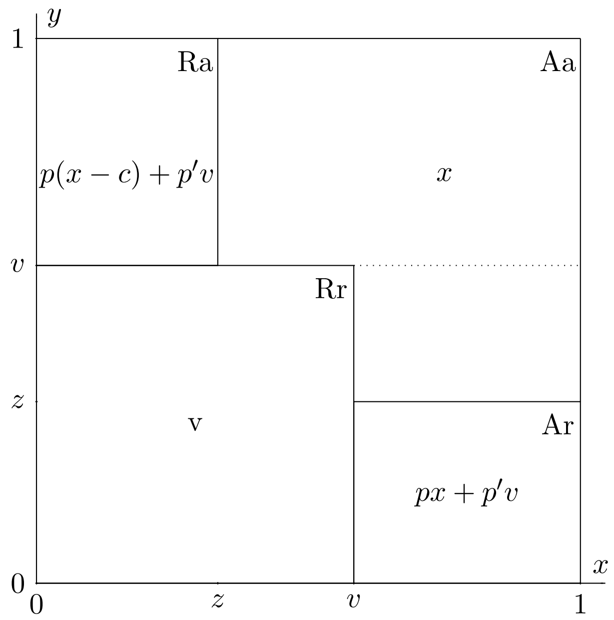

Figure 2: Equilibrium payoffs for selector I in $x \times y$ space.

$\int_{x=0}^{v}\int_{y=0}^{v} v\,dx\,dy = v^3$	for the rectangle with label Rr
$\int_{x=z}^{1}\int_{y=v}^{1} x\,dx\,dy$	for the top rectangle of area labelled Aa
$\int_{x=v}^{1}\int_{y=z}^{v} x\,dx\,dy$	for the bottom rectangle of area labelled Aa
$\int_{x=0}^{z}\int_{y=v}^{1}(p(x-c)+p'v)\,dx\,dy$	for the rectangle with label Ra
$\int_{x=v}^{1}\int_{y=0}^{z}(px+p'v)\,dx\,dy$	for the rectangle with label Ar

Summing these contributions gives

$$
\begin{aligned}
E(v) = v^3 + (1-v)\int_z^1 x dx + (v-z)\int_v^1 x\,dx \\
+ (1-v)z(2p'v - pc) + (1-v)p\int_0^z x\,dx + zp\int_v^1 x\,dx \\
= \frac{v^3 + 1 - z^2 - z - 3zv^3}{2} + 2zv - 2zvp - zpc \\
+ \frac{3zv^2p}{2} + zvpc + \frac{pz^2 - pz^2v + zp}{2} \\
= \frac{v^3 + 1 + z(1-v)(2p'v - pc - 1 + p)}{2}
\end{aligned} \tag{8}
$$

Although (8) gives a single expression for $E(v)$, the expression splits naturally into two parts, one in which $z = 0$ and the other when $z > 0$. The former arises when the consensus cost is comparatively large ($c \geq p'v/p$) and, in this case, the iteration does not explicitly involve either p or c. Because z given by (6) involves v, its value can change from one stage to the next and we will see later in Appendix B that there are values of c for which $z > 0$ in a stage game M_{r+1} but $z = 0$ in the stage game M_r.

From our analysis summarised in Theorem 1, the equilibrium strategy in Γ_n is completely determined by the sequence of values $v_i = v_i(p, c)$ $(i = 1, \ldots, n-1)$ as the only other strategic variable is z, which, from (6), is determined by v_i. Thus, by (8), we can iteratively compute all the v_i starting with the known value $v_1 = 1/2$ using the iteration $v_{i+1} = E(v_i)$.

6 An Illustrative Case

In order to get some idea of how the iterative process operates we look at the special case $p = 1/2$. Using (8) we can numerically calculate (approximate) all the v_i's as functions of c, including the limiting value v_∞, as shown in Figure 3 (where v_i is written as $v(i)$).

The plot fixes c and uses the iteration $v_n(c) = E(1/2, c, v_{n-1}(c))$ with $v_1(c) = 1/2$. In performing the iteration it is necessary to consider two cases, one when $z > 0$ and the other when $z = 0$. When

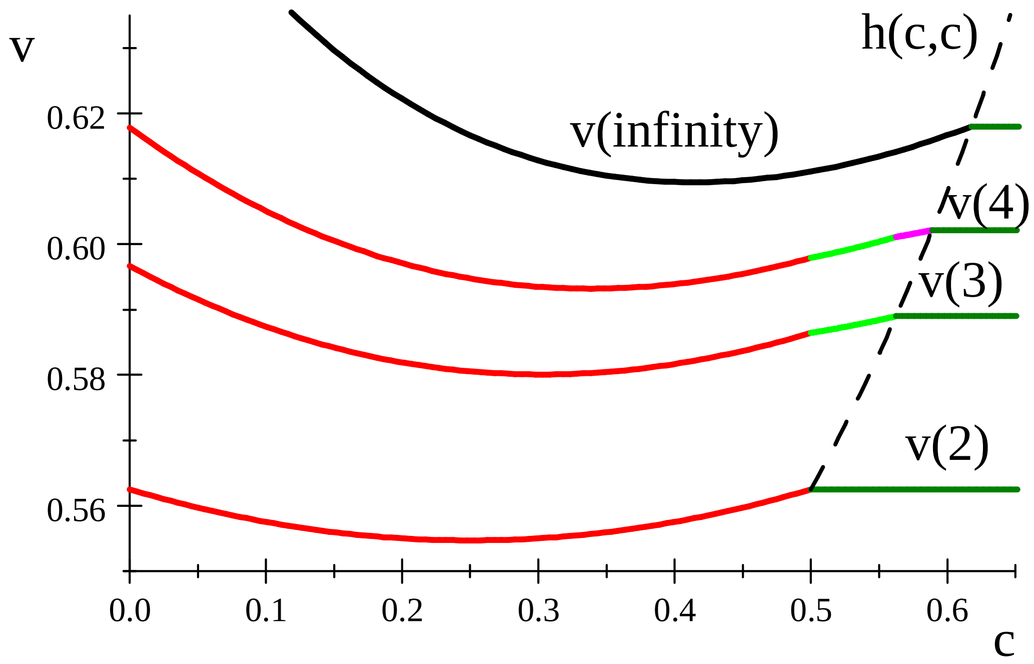

Figure 3: Computed iterations of E for $p = 1/2$.

$p = 1/2$ and $z > 0$, the iteration (8) becomes

$$E(v,c) \equiv h(c,v) \equiv (\frac{1}{2}v^3 - \frac{1}{2}(c-v)(v-1)\Big(\frac{1}{2}c - v + \frac{1}{2}\Big) + \frac{1}{2} \tag{9}$$

which is quadratic in c.

When $p = 1/2$ and $z = 0$, the iteration becomes

$$E(v) \equiv g(v) \equiv \frac{1+v^3}{2}, \quad \text{with} \quad g(1/2) = 9/16.$$

Because c is fixed over time, the dynamics take place on a vertical line, starting at $v_1 = 1/2$ (below the bottom of the figure).

For an example of the iteration process (9) when $c = 0$ the iteration formula reduces to $v_n = E(v_{n-1})$ with $E(v) = (3v^2 - v + 2)/4$ so the first four iterations are $v_1 = 1/2$, $v_2 = 9/16 = 0.5625$, $v_3 \approx 0.596680$, $v_4 \approx 0.617850$, $v_5 \approx 0.631841$ as can be seen in Figure 3 and then $v_6 \approx 0.641457$. The limiting value v_∞ is obtained by setting $E(v) = v$ in (8) which becomes $4v = 3v^2 - v + 2$ with relevant solution $v_\infty = 2/3$.

We carry out the first few interations algebraically, beginning with $v_2(c) = h(c, 1/2) = \frac{1}{8}c(c-\frac{1}{2})+\frac{9}{16}$ for $c \le 1/2$, and constant at $g(1/2) = 9/16$ for $c \ge 1/2$. For $c \le 1/2$, the next value function $v_3 = v_3(c)$ is given by the fifth degree polynomial

$$v_3(c) = h(c, v_2(c)) = \frac{3}{256}c^5 - \frac{1}{32}c^4 - \frac{13}{1024}c^3 + \frac{201}{1024}c^2 - \frac{115}{1024}c^2 + \frac{611}{1024}.$$

For c between 1/2 and $g(1/2) = 9/16$, it is given by the quadratic

$$v_3(c) = h(c, g(1/2)) = h(c, 9/16) = \frac{7}{64}c^2 - \frac{77}{1024}c + \frac{611}{1024}.$$

For $c \ge 9/16$, it is given by the constant $v_3(c) = g(g(1/2)) \approx 0.58899$. It follows that the left hand derivative of $v_3(c)$ at 1/2 is 1/6=0.625 while the right hand derivative of $v_3(c)$ at 1/2 is $35/1024 \approx 0.034$. Hence the function $v_3(c)$ is not convex on any interval containing $c = 1/2$. However $v_3(c)$ is convex on each of the intervals (0,1/2) and (1/2,9/16) as it has negative second derivative on each. The top curve v_∞ is given by $v_\infty = (2 + c + c^2)/(3 + 3c)$ for $c \le (\sqrt{5} - 1)/2 = \phi \approx 0.61803$ and equal to ϕ for $c \ge \phi$.

It is intuitively reasonable to think that the expected utility to a selector will go down as the consensus cost increases and this is what happens as c moves away from 0 in Figure 3; Theorem 4, proved in Section 8, says that the same happens for any $p \in [1/2, 1)$. However, after a certain value of c, the curve in Figure 3 actually starts to rise and continues to do so until it meets the horizontal line. Thus, when $p = 1/2$, the selectors *can actually benefit* from an increase in consensus cost. An explanation for this behaviour is that, as the consensus cost increases, although a selector accepts poorer qualified candidates in his own field when the other selector wants to appoint the candidate, the same is true with regard to the other selector. As the consensus cost increases there comes a time for each selector when the benefit from the acquiescence of the other selector becomes the dominant factor.

We now analyze the effect of a small increase in the consensus cost from c to $c + \delta$ on the payoff

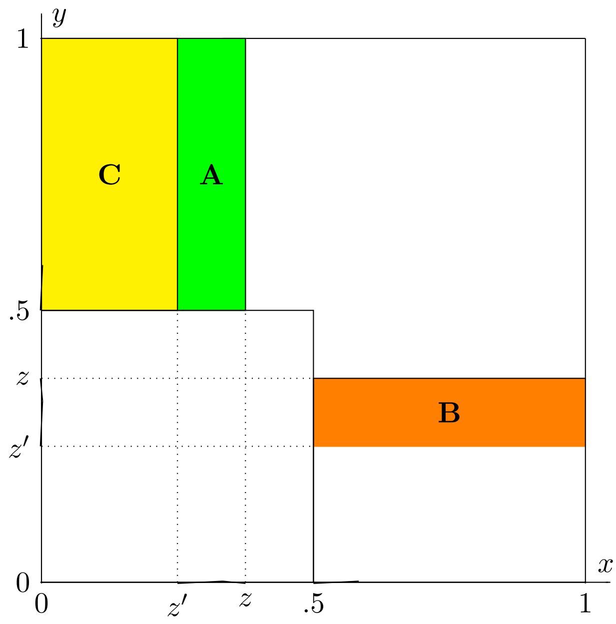

Figure 4: Payoff changes for selector I when c increases.

to the selectors. For the remainder of this section, we take $n = 2$ and consider the effect on selector I (though the final effect is the same on both). The candidates in (x, y) space for which a change will take place are in the three rectangles A, B, C in Figure 4. A change in c will not affect the upper threshold, which remains at $v = 1/2$ as this is the mean value of the candidates, and whoever appears in the next period will be hired. However the lower threshold goes from z to a lower value z' when c is increased by δ.

Candidates in regions A and B were originally above one selector's upper threshold and below the other's lower threshold, so originally they would have gone to adjudication. If hired, one of the selectors would have had to pay the consensus cost c for voting against them. With the smaller lower threshold, they are now hired definitely by unanimous assent. So in particular, selector I benefits from the fact that candidates in region B, with high x values, are now hired. The weaker candidates in region A are now definitely hired, but no consensus costs are paid for them; these two changes cancel out in utility terms, which can be seen from a calculation given below, or more simply by noting that if the utility change ΔA is not zero then selector I (for fixed c and fixed selector II strategy z) can improve his payoff by marginally (and unilaterally) changing his z, contradicting the Nash Equilbrium nature of z. The candidates in region C still go to adjudication with the same probability p of being hired as before. However if they are hired, the consensus cost that selector I has to pay is now higher. So to sum up, as c increases, selector I benefits from changes in region B but loses from C. We calculate these changes in utility for selector I, using the thresholds $z = 1/2 - (p/p')\,c$ and $z' = 1/2 - (p/p')\,(c+\delta)$ and neglecting terms of order δ^2, as

$$\begin{aligned}
\Delta &= \Delta A + \Delta B + \Delta C \\
\Delta &= 0 + \left(p\delta/p'\right) \int_{1/2}^{1} p'\,(x - 1/2)\ dx + (-pz\delta/2) \\
&= 0 + p\delta/8 - pz\delta/2 \\
&= (p\delta/8)\ (1 - 4z).
\end{aligned}$$

From this we see that the sign of Δ is the same as the sign of $1 - 4z$, which equals 0 for $1/4 = z \equiv 1/2 - (p/p')\,c$, or $c = \hat{c} = p'/\,(4p)$. Thus the value (in this case $v_2\,(c)$) decreases as c increases from 0 to $\hat{c}$ and then increases until $c = p'/\,(2p)$, after which it is constant as the consensus cost is never paid. Of course this result (albeit without an understanding of the reason) could be obtained by simple calculus for a given case like $n = 2$.

For those readers who would like to see a direct demonstration that $\Delta A = 0$ (without reference to the equilibrium property of z) note that the original mean payoff of $p\,(x - c) + p'\,(1/2)$ is replaced by a definite payoff of x, so

$$\begin{aligned}
\Delta A &= \int_{z'}^{z} \int_{1/2}^{1} \left(p'\,(x - 1/2) + pc\right)\ dy\ dx \\
&= \frac{1}{2} \int_{z'}^{z} \left(p'\,(x - 1/2) + pc\right)\ dx \\
&= \frac{1}{2} \left(p' \left(z^2 - (z')^2\right)/2 + \left(z - z'\right)\left(pc - p'/2\right)\right) \\
&= \frac{z - z'}{2} \left(p'(z + z')/2 + pc - p'/2\right) = \frac{z - z'}{2} \left(p'z - \delta p/p' + pc - p'/2\right) = -\frac{(z - z')\delta p}{2p'},
\end{aligned}$$

using the fact that $z = 1/2 - pc/p'$ in the final line.

7 Game Properties when there are a large number of candidates

In Figure 3 the graphs of the values v_r for a given number of applicants r have the same basic shape whatever the value of r. Furthermore the figure indicates that the limit curve is a fairly good approximation for the curve of v_r when $r \geq 6$. Although the figure pertains to the case $p = 1/2$, it is not unreasonable to expect that similar considerations apply to other values of p. However, to establish that this is so for general p, does not seem to be easy. In this section, we therefore concentrate on the case when there are a large number of candidates. In particular, a formula for the limit value of our game is obtained for all values of p and c which enables us to show that, for all $p \in (0, 1]$, the limit value has the same shape as the limit curve in Figure 3. It is also established that, when $\lambda = pc/p'$ is comparatively large, the selectors gain very little benefit in terms of the expected quality of the appointee by having more than six candidates available for interview.

For fixed values of p and c, intuition tells us that the expected value of the appointed candidate to each selector increases as the number of candidates increases with the result that the selectors can be more choosy when there are more candidates. This intuition is justified in the next theorem; because the proof is not totally straightforward, it is postponed to Appendix B.

Theorem 2. *For any fixed parameter values p and c, the values v_n (i.e. the equilibrium value of stage game Γ_n) obtained by the equilibrium equation $v_{n+1} = E(v_n),\ v_1 = 1/2$, are increasing in n and converge to a limit v_∞ given by*

$$v_\infty = \begin{cases} \phi & \text{if } \lambda = pc/p' \geq \phi, \\ V(p, c) & \text{if } \lambda = pc/p' \leq \phi. \end{cases}$$

where ϕ given by (3) is the golden mean $(= (\sqrt{5} - 1)/2 \approx 0.61803)$ and V is given by

$$V(p, c) = \frac{2H}{-B + \sqrt{B^2 - 4(1 - 2p)H}} \tag{10}$$

$$\text{where} \quad B = -2 + p - 3pc < 0 \quad \text{and} \quad H = 1 + \lambda p' + \lambda^2 p' > 0. \tag{11}$$

Theorem 2 tells us that, if p is not too close to zero and the consensus cost is comparatively high ($c \geq p'\phi/p$) we are in the first case so that an increase in the consensus cost has no effect on the quality of the appointee when there are a large number of candidates. In fact we now show that if $\lambda = pc/p' > \phi$, the value of the game with six interviewees differs from the value when there are an unlimited number of interviewees by about two per cent. If $\lambda > \phi$, $z = \max\{v - \lambda, 0\} = 0$ in (8) so, remembering that the golden mean ϕ satisfies $\phi^2 + \phi - 1 = 0$, we have

$$\begin{aligned}
\phi - E(v) = \phi - \frac{v^3+1}{2} &= \phi - v - \frac{(1-v)(1-v-v^2)}{2} \\
&= \phi - v - \frac{(1-v)(\phi^2+\phi-v-v^2)}{2} \\
&= (\phi - v)\Big(1 - \frac{(1-v)(\phi+v+1)}{2}\Big).
\end{aligned}$$

Now $1/2 \leq v < \phi$ so $1 - v > 1 - \phi$ and $\phi + v + 1 > 2v + 1 = 2$. Hence $\phi - E(v_r) < \phi(\phi - v_{r-1}) < \phi^2(\phi - v_{r-2}) < \cdots < \phi^{r-1}(\phi - 1/2)$. Thus

$$\frac{E(v_6)}{\phi} = 1 - \frac{\phi - E(v_6)}{\phi} \geq 1 - \frac{\phi^5(\phi - 1/2)}{\phi} = 0.982$$

so $E(v_6)$ is within 2% of ϕ.

Each curve in Figure 3 starts from $c = 0$ by steadily going down, then bottoming out and then steadily rising until it meets the line where the value is constant. The next theorem says that this is the case for the limit curve when $p \in (0,1]$ and not just when $p = 1/2$. As the proof is rather technical, it is postponed to Appendix C.

Theorem 3. *Let $p \in (0,1]$ be fixed, then the limiting value $v_\infty = V(p,c)$ given by (10) is strictly convex in c for $c \in [0, p'\phi/p]$. For fixed p the function V has a negative derivative at $c = 0$ and a positive left-hand derivative at $c = p'\phi/p$.*

8 Further Game Properties

In this section we show that some properties of the value functions v_n shown for $p = 1/2$ in Figure 3 remain true for all $p \in (0,1)$ while another property remains true only for $p \geq 1/2$.

One observation from Figure 3 is that the functions $v_n(c)$ become constant for sufficiently large c, and this remains true for general p.

Proposition 1. *For any $p \in (0,1)$ and $c > p'/p$, the value function $v_n(c)$ is constant and is the same as the value of the game when $p = 1$.*

Proof. First consider the game when $p = 1$, where approval of a single selector is sufficient for a candidate to be hired. In this case it makes no sense to reject a candidate that the other selector wants to hire, so the social cost will never be paid. It follows that the cost c will not enter the value function v_n, which will just depend on n. Now let $p \in (0,1)$ be arbitrary but suppose that $c \geq p'/p$. It follows that $z = \max\{v - pc/p', 0\} = 0$ because the value cannot exceed one. Thus by (8), v_n is obtained using the iteration $E(v) = \left(v^3 + 1\right)/2$ starting with $v_1 = 1/2$, and hence the game value is independent of c and equal to the value when $p = 1$. □

In Figure 3 the right-hand slope of $v_n(c)$ at $c = 0$ gets steeper as n increases, which means that when $p = 1/2$ a small consensus cost has relatively more influence on the selector's expected value when there are more candidates. This is extended to the case $p \geq 1/2$ in Theorem 4, which is proved in Appendix D.

Theorem 4. *If $n \geq 2$, $v_n'(0)$ is negative for fixed $p \in (0,1)$ and it is a strictly decreasing function of n satisfying $v_n'(0) \to v_\infty - 1$ for all $p \in [1/2, 1)$,*

However this result is not true for all $p < 1/2$ because for p sufficiently near 0 we have the opposite inequality $\left|v_{n+1}'(0)\right| < \left|v_n'(0)\right|$ which is inequality (31) (see Appendix D).

9 The Quality of the Candidate for the Firm

In the analysis up to now we have looked at our game from the standpoint of the selectors. However the selectors' interests do not necessarily completely coincide with those of the firm. In this section we show that, under some natural assumptions concerning the interests of the firm, it can actually benefit from a small but non-zero cohesion cost in the sense that the appointee will have the best expected quality when there is a relatively small non-zero consensus cost.

In order to take the standpoint of the firm we must make an assumption about the value to the firm of appointing a candidate of type (x, y). In contrast to the selectors the firm may be indifferent to any consensus cost that might be incurred. Secondly it may not be neutral between the areas of expertise; it was mentioned in the Introduction that a preference for one of the values might be reflected by making the probabilities of a candidate being appointed on a split vote of the selectors depend on which selector voted for acceptance.

In our analysis the only characteristic of the distribution of the firm's utility that will be examined is the expected value. If the value to the firm of an appointee of type (x, y) is a linear function of x and y, the expected value to the firm is the same whatever form the linear function takes and we can assume that it is of the form $(x+y)/2$. We make this assumption in this section. For $c = 0$, and c large enough, the expected values of the firm and the selectors coincide but the question arises as to what happens for the intermediate values of c. For these values of c, the expected value to the firm, which we denote by q_n will, in general, be higher than v_n, because q_n does not involve the consensus cost when a candidate is appointed on a split decision. In contrast to the individual selector, our analysis shows that the firm actually benefits from a small consensus cost in the sense that the firm can expect to appoint a slightly better quality candidate as c increases from zero. However the quality soon reaches a maximum and then declines until it reaches a constant value which coincides with that of the selectors.

The principles used in the analysis of v_n in Section 5 can be applied here to obtain Figure 5 (which corresponds to Figure 2). The differences between the two figures are that, in Figure 5, the values in the boxes no longer involve c explicitly and have v's replaced by q's. Thus, in any period r, the value of q_r depends on the strategic variables v_{r-1} and z_{r-1} for that period and also on the expected value

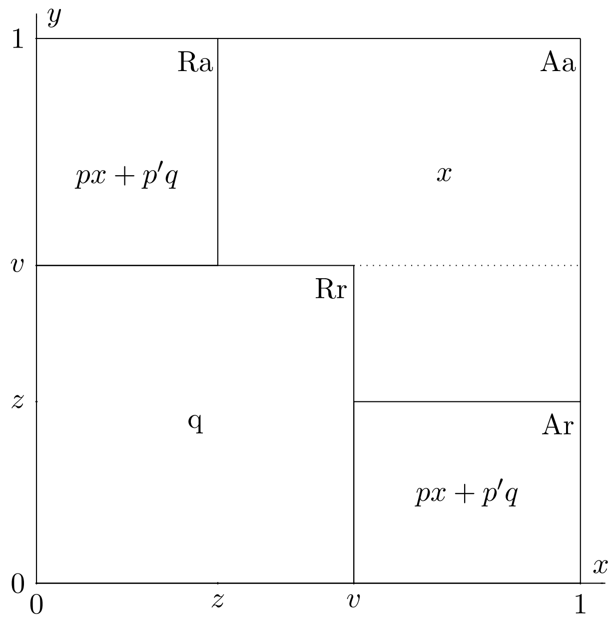

Figure 5: Expected payoffs for the firm in $x \times y$ space.

q_{r-1} of the eventually hired candidate if no one is hired in period r. Although the consensus cost no longer occurs explicitly, it occurs implicitly via v_{r-1} and q_{r-1}. We now calculate the formula for the F which gives $q_r = F(q_{r-1}, v_{r-1})$.

$$
\begin{aligned}
F(q,v) &= v^2 q + (1-v)\int_z^1 x\,dx + (v-z)\int_v^1 x\,dx + (1-v)zq \\
&+ \frac{1-v}{2}\int_0^z x\,dx + \frac{z}{2}\int_v^1 x\,dx \\
&= v^2 q + (1-v)(\frac{1}{2} - \frac{1}{2}z^2) + (v-z)(\frac{1}{2} - \frac{1}{2}v^2) \\
&+ (1-v)zq + \frac{1}{4}(1-v)z^2 + \frac{1}{4}z(1-v^2) \\
&= \frac{1 + 2qv^2 - v^3}{2} + \frac{z(1-v)(4q - (1+v+z))}{4}
\end{aligned}
\tag{12}
$$

Thus the expected quality of the eventually hired candidate when there is equilibrium play in Γ_n, called q_n, is given by $q_n = F(v_{n-1}, q_{n-1}), \quad q_1 = v_1 = 1/2$. To obtain an idea of the sort of properties

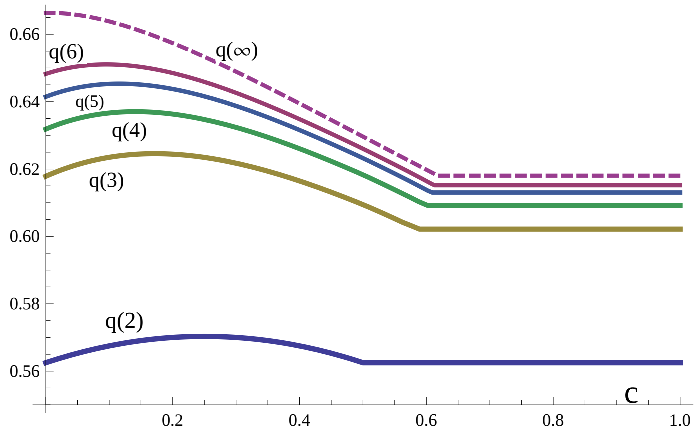

Figure 6: Computed iterations of F when $p = 1/2$.

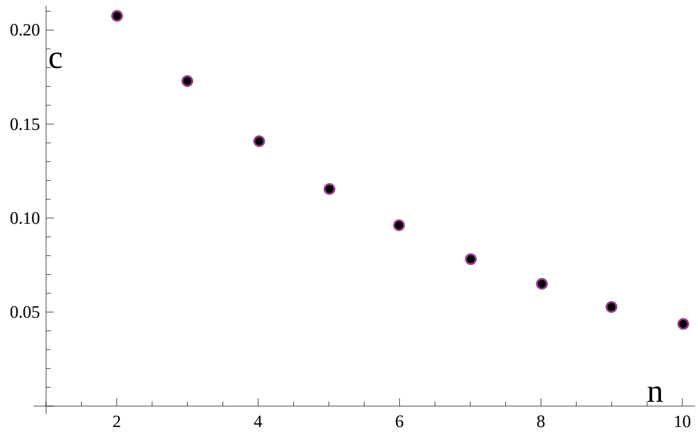

Figure 7: Optimal c for committee size $n = 2, \ldots, n$ when $p = 1/2$.

that q_n has, we now consider the game $p = 1/2$. Figure 6 shows the values of q_n as a function of c. We see that for c in an interval (which depends on n) with zero as left end point, the function q_n is increasing. This interval shrinks as n gets larger, with its length appearing to go to zero as n goes to infinity. In particular, this suggests that increasing the consensus cost c from zero increases the expected quality of the candidate that is eventually hired - a good thing for the firm. However Figure 6 indicates that the benefit from a small value of c declines as the number of candidates increases. Figure 7 plots the values of c giving the optimal values of q_n for $n = 2, \ldots, 8$; it reinforces the impression given by Fig 6 that the impact of a small cohesion cost for the firm is likely to be more significant when there are a small number of candidates. Whereas Figure 3 indicated that $v_n'(0)$ tends to get steadily steeper as n

increases, Figure 6 suggests that the behaviour of $q'_n(0)$ could be more complex as the slopes at $c = 0$ of $q_3,\ q_4,\ q_5$ and q_6 seem to be quite close to each other and steeper than that of q_2.

However, for large values of n, the slope of q_n at $c = 0$, appears to flatten out. Thus, in contrast to Figure 3, q_∞ does not provide a very good approximation for the cases when there are five or six candidates. However the following theorem which is proved in Appendix E tells us that the slopes of the curves for the q_n at $c = 0$ behave in an orderly way when $p = 1/2$.

Theorem 5. *Let $p = 1/2$. For all positive integers $n \geq 2$, $q'_n(c)$ is positive at $c = 0$. Furthermore, as n increases, $q'_n(0)$ initially increases and then decreases tending to zero as n tends to ∞.*

10 Conclusions.

In this paper we have developed a model for a selection procedure involving two selectors who interview candidates on a ‘ one at a time’ basis. The selectors have different areas of expertise and each is only interested in the candidate's ability in their particular area. The range of abilities of the candidates in each area is assumed to be independent and uniformly distributed on [0,1] but any continuous distribution can be covered by this assumption by taking the candidate's quality x to a selector to be the fraction of inferior candidates in the appropriate population. If both selectors agree, their decision is respected. If not, there is a procedure which results in the candidate being appointed with a known probability p. A novel feature of our model is that, when a candidate is appointed after a split decision, the selector who favoured rejection pays a cost c which we call a consensus cost; interpretations of c are given in the Introduction.

We have seen that, when there is a consensus cost and the other selector's assessment of a candidate is sufficiently high, a selector can vote to accept that candidate even though he would have voted to reject the candidate if there were no consensus cost; thus consensus cost has a similar effect to non-informational conformity. A consensus cost can adversely affect the expected quality of the appointed candidate for a selector in two ways; one by inflicting a cost when the candidate is appointed on a split decision and secondly by a selector accepting a candidate he would otherwise reject. However a selector can benefit from the other selector accepting a candidate who would be rejected without a consensus cost so it is not obvious whether the benefits of consensus cost outweigh the disadvantages to a selector. For fixed $p \in (0, 1)$, we have shown that there are cases where, although the expected quality for a selector initially declines as the consensus cost increases from zero, there comes a point where it starts to rise and continues to do so until it reaches a point after which it remains constant; numerical data suggests that this is in fact true for all cases but we have not been able to prove it. This constant is less than the expected value when there is no consensus cost.

On the other hand, if we take the firm's measure of quality as the weighted average of the selectors' assessments, the firm can actually benefit from a small amount of consensus cost. The expected quality of the appointed candidate for the firm initially increases as the consensus cost increases but then declines until it reaches a point after which it remains a constant which is the same as that of the selectors. An interesting question from the firm's standpoint is what value should the probability of accepting the candidate take when there is a split vote by the selectors. If the priority is to have a person in post as quickly as possible, it is clear that they should accept the first candidate that is acceptable to at least one of the selectors. However, when there are a large number of candidates, they would do better to insist on unanimity if the primary interest is appointing as well qualified person as possible; even in this case there is often little benefit from having more than six candidates.

There are many interesting lines of further research which break the symmetry between the selectors. Thus giving the selectors different values of p could represent the firm favouring one area of expertise over the other whilst giving them different values of c could represent that one selector is less dominant (more conformist) than the other.

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APPENDIX

A Properties of $E(v)$.

In this section we analyse the properties of $E(v)$ given by (8). Because (8) takes a simple form when $z = 0$ it is useful to distinguish the cases $z = 0$ and $z > 0$ so we write

$$E^0(v) = \frac{v^3+1}{2} \qquad \text{when} \qquad z = 0 \tag{13}$$

and

$$E^+(v) = \frac{v^3 + 1 + z(1-v)(2p'v - pc - 1 + p)}{2} \qquad \text{when} \quad z > 0. \tag{14}$$

We now obtain lemmas that assure us that $E^0(v) > v$ and $E^+(v) > v$ for the range of values of v in which we are interested.

Lemma 1. *If $z = 0$ and $1/2 \leq v \leq \phi$ where $\phi \approx 0.61803$ is the golden mean given by (3), then $v \leq E(v) = E^0(v) \leq \phi$.*

Proof. First note that ϕ satisfies $\phi^2 + \phi - 1 = 0$. We have

$$E^0(v) - v = \frac{v^3 - 2v + 1}{2} = \frac{(1-v)(1-v-v^2)}{2} \geq \frac{(1-\phi)(1-\phi-\phi^2)}{2} = 0$$

so $E^0(v) \geq v$. Further

$$E^0(v) \leq \frac{\phi^2\phi + 1}{2} = \frac{\phi(1-\phi)+1}{2} = \frac{(\phi - \phi^2 + 1)}{2} = \frac{\phi - (1-\phi) + 1}{2} = \phi.$$

□

Now consider the case $z = v - pc/p' = v - \lambda > 0$. Substituting for z into (14) and simplifying we get

$$E(v) = E^+(v) = v + \frac{1-v}{2} Q_{(p,c)}(v) \tag{15}$$

where Q is the "quadratic" in v with constant coefficients B and H given by

$$Q = Q_{(p,c)}(v) = (1-2p)v^2 + Bv + H \tag{16}$$

$$B = -2 + p - 3pc < 0 \quad \text{and} \quad H = 1 + \lambda p' + \lambda^2 p' > 0. \tag{17}$$

The following lemma gathers together the properties of Q that we need.

Lemma 2. *If $\lambda \leq 1$, Q has a unique root, $V = V(p,c)$, in (0,1) given by*

$$V = \frac{2H}{-B + \sqrt{B^2 - 4(1-2p)H}} \tag{18}$$

and satisfies $Q(v) > 0$ for $v \in [0, V)$ and $Q(v) < 0$ for $v \in (V, 1]$.

When $p \neq 1/2$ Q has a second root, $U = U(p,c) \notin [0,1]$, which satisfies

$$(1-2p)(V+U) = -B \tag{19}$$

$$(1-2p)(1-v)(V-U) \leq (1-2p)(1-v)(v-U) < 0 \quad \textit{for} \quad v \in [1/2, V) \tag{20}$$

and

$$(1-2p)(V-U) = -\sqrt{B^2 - 4(1-2p)H}. \tag{21}$$

Proof. Let $\lambda \leq 1$, then $Q(0) > 0$ and $Q(1) = p - 2\lambda p' + \lambda^2 p' = -p' + p'(1-\lambda)^2 < 0$ so Q has a root, $V(p,c)$, in (0,1); routine calculations show that V satisfies (18).

When $p \neq 1/2$ Q is a quadratic so that it has a second root, $U(p,c)$, and (19) represents the standard sum of the roots formula. If $p < 1/2$, Q is convex so has a root greater than 1 because $Q(1) < 0$; if $p > 1/2$, Q is concave so has a root less than 0 because $Q(0) > 0$. Thus $U \notin [0,1]$ and $(1-2p)(V-U) \leq (1-2p)(v-U) < 0$ for $v \in [1/2, V]$.

When $p \neq 1/2$, an alternative expression for V in (18) is

$$V = \frac{2H(-B - \sqrt{B^2 - 4(1-2p)H})}{4(1-2p)H} = \frac{-B - \sqrt{B^2 - 4(1-2p)H}}{2(1-2p)}. \tag{22}$$

Now (19) gives $U = -V - B/(1-2p)$ so

$$(1-2p)(V-U) = 2(1-2p)V + B = -\sqrt{B^2 - 4(1-2p)H}.$$

$\square$

The next lemma gives a condition which ensures that the value of the game increases from stage n to stage $n+1$ when the iteration uses E^+.

Lemma 3. *Let V be given by (18). If $z > 0$, $\lambda < V$ and $\max\{1/2, \lambda\} \le v \le V$ then $E(v) = E^+(v)$ and $v \le E^+(v) \le V$.*

Proof. First assume $p = 1/2$. Because $Q(0) > 0$ and Q has just one root in [0,1], $Q > 0$ in $[0, V)$ so $E^+(v) \ge v$ follows from (15).

Now assume $p \ne 1/2$. Using (15), (16), (19) and $Q(V) = 0$,

$$
\begin{aligned}
V - E^+(v) &= V - v - \frac{(1-v)(Q(v) - Q(V))}{2} \\
&= (V - v)(1 + \frac{(1-v)((1-2p)(V+v) + B)}{2}) \\
&= (V - v)(1 + \frac{(1-v)(1-2p)(v-U)}{2}) \\
&= (V - v)(1 + g/2) \qquad (23)
\end{aligned}
$$

where $g = (1-v)((1-2p)(v-U)$.

By (20) $g < 0$ so $E^+(v) \le V$ if $|g| < 2$. To prove $|g| < 2$, using (17) and simplifying, we have

$$
\begin{aligned}
B^2 - 4(1-2p)H &= \frac{p^2c^2(5-p)}{1-p} + 2pc(4+p) + p(4+p) \qquad (24) \\
&\le (1-p)(5-p) + 2(1-p)(4+p) + p(4+p) \\
&= 13 - 8p + 5p^2 < 13. \qquad (25)
\end{aligned}
$$

By (20) and (21),

$$
|g| < |(1-2p)(v-U)/2| \le |(1-2p)(V-U)/2| = \sqrt{B^2 - 4(1-2p)H}\,/2 \qquad (26)
$$

From (26) and (25), $|g| < \sqrt{13}/2 < 2$. Hence $E^+(v) < V$ by (23) $\square$

We require the following result in the proof of Theorem 2.

Lemma 4. *Suppose $1/2 \le v < \lambda < \phi$ and $E^0(v) \ge \lambda$, then $E^0(v) < V$.*

Proof. Using (16) and (17) with $\lambda = pc/p'$, $Q(\lambda) = -\lambda^2 - \lambda + 1$. The golden mean ϕ is a root of the quadratic $-x^2 - x + 1$ which has a (positive) maximum at $-1/2$. Thus $-x^2 - x + 1$ is positive in $[0, \phi)$.

In particular $Q(\lambda) > 0$ because $0 < \lambda < \phi$. But we know, by Lemma 2, that $Q(v) > 0$ for $v \in [0, V)$ and $Q(v) < 0$ for $v \in (V, 1]$ so $\lambda < V$.

Now $1/2 \leq v < \lambda < \phi$ so, by Lemma 1 and the fact that $E^0(v)$ is increasing in [0,1], $E^0(v) < E^0(\lambda) < \phi$. But $E^0(\lambda) = E^+(\lambda)$ so that, by Lemma 3, $\lambda < E^+(\lambda) < V$ so $E^0(v) < V$. □

B Proof of Theorem 2

The proof of Theorem 2 divides into three cases and, as mentioned in section 7, it is not entirely straightforward. We therefore give three examples to illustrate the cases. Recall from (13) and (14) that $E(v)$ can take on two distinct forms, $E^0(v)$ and $E^+(v)$ depending respectively on whether $z = 0$ or $z > 0$. In the first example E^+ is used in every iteration, in the second E^0 is used in every iteration and, in the third, iterations start by using E^0 but switch to using E^+.

The three examples are:

(i) $p = 1/4, \;\; c = 1/3$; (ii) $p = 3/4, \;\; c = 1/3$; iii) $p = 4/5 \;\; c = 1/7$.

For (i), $\lambda = 1/9 < 1/2$ so $z_1 > 0$ because $v_1 = 1/2$. Repeated use of Lemma 3 gives $v_n \geq 1/2 > \lambda$ so $z_n > 0$ for all n. Thus E^+ is used in all iterations to calculate v_n. In this example the consensus cost factor occurs explicitly in the formula at each iteration. Thus a small change in the value of c affects the iterated values whatever the number of candidates available for consideration.

For (ii), $\lambda = 1 > \phi > 1/2$ so $z_1 = 0$. Repeated use of Lemma 1 gives $v_n \leq \phi < \lambda$ so $z_n = 0$ for all n. Thus E^0 is used in all iterations to calculate v_n. In contrast to (i) the consensus cost factor does not occur explicitly in the iterative equation. Thus a small change in the value of c will have no effect on the iterative values whatever the number of candidates available for consideration.

For (iii), $\lambda = 4/7 > 1/2$ so $z_1 = 0$ and E^0 is used to give $v_2 = 9/16$. Because $4/7 > 9/16$, we also have to use E^0 to calculate v_3 which gives $v_3 = 4825/8192 \approx 0.58899 > 4/7 = \lambda$. Thus $z_3 > 0$ so we use E^+ to calculate v_4. We want to apply Lemma 3 to assert that $v_4 > v_3$ but to do so we need $v_3 < V(4/5, 1/7)$. For this particular case the inequality holds because $V \approx 0.6528$ so Lemma 3 ensures that the iterations for v_n for $n \geq 3$ are given by E^+ and that the v_n are increasing. This example illustrates that the consensus cost factor may not be a significant influence when there are a small number of candidates but can play a role when there are a comparatively large number. In particular it is significant for the initial interviews but not once a particular number of applicants have been rejected.

Notice that the quantity that determines the behaviour in these examples is λ rather than c so that, when $p = 1/2$, the analysis in (i), (ii) and (iii) covers the cases $c = 1/9$, $c = 1$ and $c = 4/7$ respectively.

It turns out that the observations hold generally as we see in the proof of the following theorem.

Theorem 2. *For any fixed parameter values p and c, the values $v_n = eq.val.\Gamma_n$ obtained by the equilibrium equation $v_{n+1} = E(v_n),\ v_1 = 1/2$ are increasing in n and converge to a limit v_∞ given by*

$$v_\infty = \begin{cases} \phi & \text{if } \lambda \geq \phi, \\ V & \text{if } \lambda \leq \phi. \end{cases}$$

where λ, the golden mean ϕ and V are given by (6), (3) and (18) respectively.

Proof. The proof is divided into three cases

(i) $pc \geq (1-p)\phi$, (ii) $1/2 < \lambda < \phi$, and (iii) $\lambda \leq 1/2$.

(i) Suppose $\lambda \geq \phi$, then $1/2 < \lambda$ so (6) gives $z = 0$ and the first iteration is given by $v_2 = E^0(1/2) < \lambda$ by Lemma 1. The same argument shows that at each subsequent stage $v_{i+1} = E^0(v_i)$ and Lemma 1 tells us that (v_n) is an increasing sequence which is bounded above by ϕ. Thus it has a limit $\nu \in [0, \phi]$ as $n \to \infty$ which satisfies $v = E^0(v) = (v^3+1)/2$; because $v^3 - 2v + 1 = (v-1)(v^2+v-1)$, it has just one root in $[0, \phi]$, namely ϕ.

(ii) Suppose $1/2 < \lambda < \phi$, then the iterative process begins the same way as in (i) taking $v_2 = E^0(1/2)$. However there must come a stage m such that $v_m < \lambda$ but $v_{m+1} = E^0(v_m) \geq \lambda$ so that the iterative step $v_{m+2} = E(v_{m+1})$ is given by $v_{m+1} = E^+(v_m)$. By Lemma 4 $v_{m+1} < V$ so we can apply Lemma 3 to obtain $E(v_{m+1}) < v_{m+2} < V$. Using Lemma 3 it is now easy to see that all subsequent iterations have $E(v_r) = E^+(v_r)$ and that v_r is an increasing sequence bounded above by V. Thus v_r tends to a limit $v_\infty \leq V < 1$ satisfying $E^+(v_\infty) = v_\infty$; by (15), this implies $Q(v_\infty) = 0$. But $Q(v) = 0$ has precisely one root, V, in [0,1] so $v_\infty = V$.

(iii) Suppose $\lambda \leq 1/2$, then $v_2 = E(1/2) = E^+(1/2)$ and it follows from Lemma 3 that all subsequent iterations are given by $E(v) = E^+(v)$. The argument in (ii) shows that the iterations have limit V. $\square$

C Proof of Theorem 3

Proof. When $p = 1/2$, (18) becomes

$$V(1/2, c) = H/(-B) = \frac{c}{3} + \frac{2}{3(1+c)}.$$

The second derivative of $V(1/2, c)$ w.r.t. c is positive so $V(1/2, c)$ is strictly convex. Furthermore $v(1/2, c)$ has derivative less than zero at $c = 0$ and derivative greater than zero at $c = \phi \approx 0.61803$. Hence the result holds for $p = 1/2$.

Thus assume $p \neq 1/2$. Let $f(c) = B^2 - 4(1-2p)H$, then $f(c)$ is a quadratic $\alpha c^2 + 2\beta c + \gamma$ in c where

$$\begin{aligned}
\alpha &= 9p^2 - \frac{4(1-2p)p^2}{1-p} = \frac{p^2(5-p)}{1-p} \\
\beta &= 3p(2-p) - 2(1-2p)p = p(4+p) \\
\gamma &= (2-p)^2 - 4(1-2p) = p(4+p).
\end{aligned}$$

so

$$(1-p)(\alpha\gamma - \beta^2) = p^2(4+p)(p(5-p) - (4+p)(1-p)) = p^2(4+p)(8p-4)$$

It is easy to show that $\sqrt{\alpha c^2 + 2\beta c + \gamma}$ is strictly convex (concave) in c if $\alpha\gamma - \beta^2 > 0$ ($\alpha\gamma - \beta^2 < 0$). Thus $\sqrt{f(c)}/(2p-1)$ is strictly convex. As in (22), V can be written as

$$V(p, c) = \frac{-B - \sqrt{B^2 - 4(1-2p)H}}{2(1-2p)} = \frac{2 - p + 3pc}{2(1-2p)} + \frac{\sqrt{f(c)}}{2(2p-1)}. \tag{27}$$

The first term is linear in c and the second strictly convex so v_∞^+ is strictly convex.

Now V is a solution of $E^+(v) = v$ so a solution of $Q = 0$ where Q is defined in (16). Differentiating $Q(V) = 0$ implicitly with respect to c gives

$$0 = \frac{dQ}{dc} = 2(1-2p)V\frac{dV}{dc} + V\frac{dB}{dc} + B\frac{dV}{dc} + \frac{dH}{dc}$$

so using (17) and (22)

$$-\sqrt{B^2 - 4(1-p)H}\frac{dV}{dc} = (2(1-2p)V + B)\frac{dV}{dc} = 3pV - p - 2p\lambda. \tag{28}$$

Hence dV/dc has the opposite sign to $3pV - p - 2\lambda$.

Because $V \geq 1/2,\ 3pV - p > 0$ so $dV/dc < 0$ when $c = 0$.

When $c = p'\phi/p,\ V = \phi$ so that $3pV - p - 2\lambda = 3p\phi - p - 2\phi < 2p\phi - 2\phi \leq 0$ so $dV/dc > 0$ at $c = p'\phi/p$. □

D Proof of Theorem 4

Before proving the theorem, we do some preliminary analysis.

For c sufficiently near zero, v_{r+1} is given by $v_{r+1} = E^+(v_r)$ where E^+ is given by (14). Thus, by (14) and (6)

$$2(v_{n+1} - v_n) = (1 - v_n)(v_n^2(1 - 2p) + v_n(-2 + p - 3pc) + 1 + pc + p^2c/p')$$

so, on differentiating w.r.t. c, we obtain

$$\begin{aligned} 2(v'_{n+1} - v'_n) = (1 - v_n)(2v_nv'_n(1 - 2p) + v'_n(-2 + p - 3pc) - 3pv_n + p(1 + 2p/p')) \\ - v'_n(v_n^2(1 - 2p) + v_n(-2 + p - 3pc) + 1 + pc + p^2c/p' \end{aligned}$$

and, after putting $c = 0$ and some simplification,

$$\begin{aligned} 2v'_{n+1}(0) &= v'_n(0)(v_n^2(0)(-3 + 6p) + 6p'v_n(0) - p') + p(3v_n(0) - 1)(v_n(0) - 1) \\ &= v'_n(0)((J_n + pK_n) + p(3v_n(0) - 1)(v_n(0) - 1)) \end{aligned} \tag{29}$$

where

$$J_n = 2 - 3(v_n(0) - 1)^2 \quad \text{and} \quad K_n = 6(v_n(0) - 1/2)^2 - 1/2. \tag{30}$$

When p is sufficiently small, we therefore have that $2v'_{n+1}(0) \approx J_nv'_n(0)$ where $J_n < 2$ so that

$$|v'_{n+1}(0)| < |v'_n(0)| \quad \text{for } p \text{ sufficiently small.} \tag{31}$$

We now prove the theorem.

Proof. Now $J_n + pK_n$ defined in (30) is a strictly increasing function of $v_n(0)$ for $v_n(0) \in [1/2, 1]$. When $v_n(0) = 1/2$, $J_n + pK_n = 5/4 - 1/2 = 3/4$ so $J_n + pK_n > 3/4$ for $v_n(0) \in [1/2, 1]$. Further $(3v_n(0) - 1)(v_n(0) - 1)$ is strictly decreasing for $v_n(0) \in [1/2, 2/3]$ and non-positive for $v_n(0) \in [1/2, 1]$. Thus $v_n'(0) < 0$ if $v_{n-1}'(0) < 0$ by (29). However $v_1'(0) = 0$ and $v_2'(0) = -p/4 < v_1'(0)$ so $v_n'(0) < 0$ for $n \geq 2$.

We now prove that, for $p \in [1/2, 1)$, $v_n'(0)$ is a decreasing function of n by induction. Suppose that $v_r'(0) < v_{r-1}'(0) \leq 0$, then, by (29),

$$\begin{aligned}
2v_{r+1}'(0) &= v_r'(0)((J_r + pK_r) + p(3v_r(0) - 1)(v_r(0) - 1) \\
&\leq v_r'(0)(J_{r-1} + pK_{r-1}) + p(3v_{r-1}(0) - 1)(v_{r-1}(0) - 1) \\
&< v_{r-1}'(0)((J_{r-1} + pK_{r-1}) + p(3v_{r-1}(0) - 1)(v_{r-1}(0) - 1) = 2v_r'(0).
\end{aligned}$$

Thus, for $p \in [1/2, 1)$, $v_n'(0)$ is a strictly decreasing function of n. We now show by induction that $v_n'(0) > -1$ for all n. The result holds for $n = 1$ and $n = 2$ so suppose it holds for $n = r$, then

$$\begin{aligned}
2v_{r+1}'(0) &= v_r'(0)((J_r + pK_r) + p(3v_r(0) - 1)(v_r(0) - 1) \\
&\geq -((J_{r-1} + pK_{r-1}) + p(3v_{r-1}(0) - 1)(v_{r-1}(0) - 1) \\
&> v_{r-1}'(0)((J_{r-1} + pK_{r-1}) + p(3v_{r-1}(0) - 1)(v_{r-1}(0) - 1) = 2v_r'(0).
\end{aligned}$$

Thus $v_n'(0)$ is bounded below by -1 and therefore tends to a limit as $n \to \infty$. This limit, L say, must be a fixed point of (29) so it must satisfy

$$\left(v_n^2(0)(-3 + 6p) + 6v_n(0)(1 - p) - 3 + p\right)L + p(3v_\infty(0) - 1)(v_\infty(0) - 1) = 0.$$

Now $v_\infty(0)$ is a solution of $(1-2p)x^2+(-2+p)x+1 = 0$ so, on substituting $(-3+6p)v_\infty^2 = (-6+3p)v_\infty+3$ and simplifying, we obtain $L = -1 + v_\infty$. □

E Proof of Theorem 5

Proof. The proof is by induction on the period n. For $n = 1$, we have already observed that any candidate that appears is hired, so $q_1(c) = 1/2$ for all c so $q_1'(0) = 0$. We assume that $q_n'(0) \geq 0$ and establish that $q'_{n+1}(0) > 0$. Taking $p = 1/2$ (and hence $z = v - c$) in the recursive formula (12), we obtain using the simplifying notation $q = q_n(c)$, $v = v_n(c)$

$$
\begin{aligned}
4q_{n+1} &= 2(1 + 2qv^2 - v^3) + z(1 - v)(4q - (1 + v + z)) \text{ and setting } z = v - c, \\
&= 2(1 + 2qv^2 - v^3) + (v - c)(1 - v)(4q - (1 + 2v - c)). \\
&= 2 + 4q(v + vc - c) - v^2(1 + 3c) - v(1 - 2c - c^2) - c^2 + c
\end{aligned}
$$

Differentiating with respect to c and setting $c = 0$ gives

$$
\begin{aligned}
4q'_{n+1}(0) = 4q'(0)v(0) + v'(0)(4q(0) - 2v(0) - 1) \\
+ 4q(0)v(0) - 4q(0) - 3v^2(0) + 2v(0) + 1
\end{aligned}
$$

Because $v(0) = q(0)$, we have

$$4q'_{n+1}(0) = 4v(0)q'(0) + (2v(0) - 1)v'(0) + (1 - v(0))^2 \tag{32}$$

By Theorem 2 $v(0) = v_n(0)$ is an increasing function of n which tends to $2/3$ as $n \to \infty$. By Theorem 4 $v'_n(0) < 0$ and tends to $-1/3$. Hence final two terms are strictly decreasing functions of n and, as $n \to \infty$, their sum tends to zero. Thus, if $q'_n(0) \geq 0$, then $q'_{n+1}(0) > 0$. But $q'_1(0) = 0$ so $q'_n(0) > 0$ for $n \geq 2$.

By (13) $v_2(0) = 9/16$ so, for $n \geq 2$, $(1 - v_n(0))^2 \leq (7/16)^2$. The second term of (32) is negative so (32) gives $4q'_{n+1}(0) \leq (7/16)^2 + (8/3)q'_n(0)$. Routine calculations show that $q'_{n+1}(0) < 1/6$ if $q'_n(0) < 1/6$. But, using (32), $q'_2(0) = 1/16$ so $q'_n(0) < 1/6$ for $n \geq 2$. Note that (32) can be written in the form

$$4q'_{n+1}(0) = (v_n(0) - 2/3)^2 + 5/9 + (2v_n(0) - 1)v'_n(0) + v_n(0)(4q'_n(0) - 2/3).$$

The first and third terms are strictly decreasing functions of n. Because $q'_n(0) < 1/6$ for $n \geq 2$, the third term is negative for $n \geq 2$ and

$$v_n(0)(4q'_n(0) - 2/3) < v_{n-1}(0)(4q'_{n-1}(0) - 2/3) \quad \text{if} \quad q'_n(0) < q'_{n-1}(0).$$

Hence $q'_{n+1}(0) < q'_n(0)$ if $q'_n(0) < q'_{n-1}(0)$.

Now $q'_\infty(0) = \lim_n q'_n(0)$ is given by taking $q'_{n+1}(0) = q'(0)$ in (32) so

$$
\begin{aligned}
q'_\infty(0) =& 4v_\infty(0)q'_\infty(0) + (2v_\infty(0) - 1)v'_\infty(0) + (1 - v_\infty(0))^2 \\
&= 8q'_\infty(0)/3 + (1/3)(-1/3) + 1/9
\end{aligned}
$$

giving $q'_\infty(0) = 0$.

Because $q'_2(0) = 1/16 > 0 = q'_1(0)$, it follows that $q'_n(0)$ initially increases but then decreases.

$\square$