DYNAMICAL PROPERTIES

OF

ALGEBRAIC SYSTEMS

A STUDY IN CLOSED GEODESICS

by

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SUMMARY

To a great extent, rigidity theory is the study of boundaries of semisimple groups. Here we investigate the action of a lattice on such a boundary. While we can construct topological factors for real rank 1 groups we show the nonexistence of such factors in higher rank for some cases.

We also study the geodesic flow on a compact locally symmetric manifold of the noncompact type. We calculate metric and topological entropies and see that the Liouville measure is a measure of maximal entropy. This leads to a study of compact maximal flats. We give a new proof of their density in the space of all flats. We prove specification and expansiveness theorems for the geodesic flow and apply them to determine a growth rate for compact maximal flats. Finally, we give an example of a space with infinitely many closed singular geodesics.

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I very much appreciate the wonderful opportunities of learning at both the University of Maryland and Warwick.

Finally, it was a pleasure to have been guest of the Institut des Hautes Etudes early one summer. In fact, it was there that I started to learn and think about rigidity.

DECLARATION

Chapter I, Section 1 and part of section 2 were published in concise form in my paper "Lattices Acting on Boundaries of Semisimple Groups", Ergodic Theory and Dynamical Systems (1981), 1, 489 - 494.

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Introduction

We present this thesis in two parts. They deal with different specific problems. But at the core of both of them lies the study of closed geodesics.

Part I: In the first part, we investigate a problem of G. A. Margulis. He analyzed the group theoretical structure of a lattice Γ in a semi-simple group G by analyzing the action of Γ on the maximal boundary of G. More precisely, he proved

Theorem (G. A. Margulis): Let Γ be an irreducible lattice in a connected semisimple algebraic group G over some local field k and suppose that $\mathrm{rk}_k\, G \geq 2$. Then any Γ-equivariant measurable quotient of a boundary G/P, P a parabolic, is measurably isomorphic to G/P', $P' \supset P$ a parabolic (i.e. up to sets of measure 0).

He also showed that this theorem is false for $\Gamma = SL(2,\mathbb{Z})$ and $G = SL(2,\mathbb{R})$. More generally, it fails for all surface groups and some three-dimensional hyperbolic groups. The general situation in rank 1 seems to be unknown.

One may wonder whether this theorem holds true in the topological category rather than the measurable realm. More precisely, Margulis asked at the end of [Ma1]:

When does $SL(n,\mathbb{Z})$ acting on the projective space $\mathbb{P}^{n-1}$ have an equivariant topological Hausdorff factor? In particular, is there a dichotomy between $n \geq 3$ and $n = 2$?

R. J. Zimmer first proved in [Zi1] that for $n > 2$ there are no such quotients. His method of proof relied heavily on results of S. G. Dani and Raghavan ([Da2]). In [Sp1] we gave an entirely elementary argument and we also indicated how to construct a quotient for $n = 2$. This construction generalizes very nicely to an arbitrary lattice in a group of real rank 1, even to fundamental groups of visibility manifolds. The idea is to use the classical correspondence between geodesics and pairs of points on the boundary. Then one can employ the special properties of the endpoints of the lift of a closed geodesic to construct an equivalence relation that defines our quotient. We present this in detail in I Section 1.

In Section 2 we discuss the nonexistence of factors for $\Gamma = SL(n,Z)$ acting on Grassmannians. Basically it is a calculation. The key is that the isotropy group of a rational point is "big" in Γ. One might try to use this for more general split lattices.

For $\mathbb{P}^{n-1}$ we get a slightly better result: $SL(n,Z)$ acts minimally on $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$-diagonal. This obviously implies the non-existence of factors. This approach can't work in general as we prove in Section 3: for the maximal boundary $G/\mathcal{P}$ there are always Γ-invariant closed sets in $G/\mathcal{P} \times G/\mathcal{P}$ that are not G-invariant. Here Γ and G are arbitrary. The technique is the same as in Section 1: use a compact maximal flat to find special points on the boundary. Here we use Mostow's realization of $G/\mathcal{P}$ as points at infinity of a globally symmetric space.

The general case of the existence of factors remains open.

After the completion of this work Dani proved the non-existence of these factors in the higher rank case.

Part II: We study the geodesic flow on a locally symmetric manifold of the noncompact type. Our motivation is threefold.

(i) While the geodesic flow on a manifold of negative curvature is well understood not too much is known for arbitrary manifolds of nonpositive curvature. Ballmann's condition seems to be quite critical. It requires that no geodesic in the universal cover bound a flat half plane. If this condition is satisfied quite a lot of the usual theory can be pushed through using Pesin theory.

Locally symmetric spaces of higher rank clearly fail to satisfy this condition. We have tried to understand some of the difficulties caused by the presence of flats for these simple examples.

Finally, locally symmetric spaces seem to be prime examples of manifolds that do not satisfy Ballmann's condition in the sense that they may be building blocks for a general manifold of this type. Evidence for this is the Gromov-Eberlein theorem. It roughly says that one cannot perturb the metric on a locally symmetric manifold of rank ≥ 2 maintaining nonpositive curvature.

(ii) The geodesic flow on a general locally symmetric manifold is another example of a non Axiom A situation where we still have a lot of hyperbolicity. In particular, it is an example of an Anosov R^n-action . We do not know to what extent our results generalise to Anosov actions.

(iii) Locally symmetric spaces are very lovely as they are rich in structure and display many connections to number theory and representation theory in particular. We hope that'soft' dynamical methods like ours will shed some light on these areas.

More precisely we may call our investigations the study of the Liouville measure as a measure of maximal entropy. Let us review the Bowen-Margulis theory for the geodesic flow on a compact manifold of negative curvature.

There are two natural flow invariant measures:

1) the Liouville measure μ: this is the only smooth invariant measure for the geodesic flow arising from its Hamiltonian nature.

2) the measure of maximal entropy or Bowen-Margulis measure ν: there is a unique measure of maximal entropy for the geodesic flow due to its hyperbolicity. It can be obtained in two ways:

a) as Bowen showed it gives the equidistribution of the closed geodesics on M (cf. [Bo 1]).

b) Margulis on the other hand constructs it by exhibiting uniformly expanding and contracting measures along the stable and unstable manifolds of the geodesic flow (cf. [Ma 1]).

This measure is also the unique invariant measure for the horocycle foliation and can be obtained from the symbolic dynamics of the geodesic flow (cf. [Bo-Ma 1]).

Both of these measures are ergodic, even Bernoulli. Naturally one wonders when μ and ν coincide.

The classical examples of manifolds of negative curvature are the real rank one locally symmetric spaces M of the noncompact type. The unit tangent bundle T_1M is just a double coset space of a semisimple group G and the Liouville measure turns out to be Haar measure. Since the horocycle foliation is the orbit foliation of a maximal unipotent subgroup acting on T_1M Haar measure is invariant for it. By unique ergodicity we see that $\mu = \nu$. This was first proved in [Bo 2] in a somewhat less sophisticated way.

Moreover, one may conjecture that these are the only manifolds of negative curvature with $\mu = \nu$. For Riemann surfaces this was proved in [Ka 1].

Here we pursue these ideas in a different direction. Consider an arbitrary compact locally symmetric manifold M of the noncompact type. The geodesic flow fails to be ergodic if the rank is greater than 1. But the ergodic decomposition is readily obtained in terms of algebraic data (as in [Mau 1]). In particular, the ergodic components are double coset spaces of the group and embed smoothly into T_1M. Hence the topological entropy of the geodesic flow is defined on the ergodic components and we can compare it with the metric entropy for the Liouville measure. It turns out that they coincide and we have generalised part of Bowen's result:

the Liouville measure is a measure of maximal entropy on the ergodic components of the geodesic flow.

This follows quite easily from the observation that the sum of the positive Lyapunov exponents is constant everywhere, not just almost everywhere, by the 'homogeneity' of the ergodic component. We also calculate the exponents and the entropies on the ergodic components and the unit tangent bundle explicitly in terms of the root system. This constitutes Section 1 of part II.

The main problem now is to see which properties of the measure of maximal entropy carry over from the negative curvature case to our situation. Certainly, the Liouville measure is a Margulis measure: it contracts and expands uniformly along stable and unstable manifolds. As is well known, it is also the unique invariant measure for the horospherical foliation. It is not so clear however that the closed geodesics are equidistributed with respect to Liouville measure. Indeed, in higher rank there are uncountably many closed geodesics coming from flat tori and the question doesn't even make quite sense. But it leads us to the study of compact maximal flats in a locally symmetric space.

In Section 2 we discuss some basic properties of compact flats and give a new proof of Mostow's result that the compact flats are dense in the space of all flats. We use dynamics and in fact, we try to keep it as soft as possible.

The crucial point is that to 'close up' a flat one only has 'to close up' a regular geodesic in it. For this we can use a generalised Closing Lemma.

In Section 3 we briefly go back to closed geodesics. While most closed geodesics (in some sense) are going to lie in a compact maximal flat there may be some exceptional closed geodesics that are not contained in any higher dimensional compact flat. In fact, we show by way of example that this situation can arise. We do not know however whether such exceptional closed geodesics always exist.

In Section 4 we discuss Bowen's main technical tool, the specification theorem. We prove weak specification for the geodesic flow on anergodic component, i.e. we can shadow orbit segments by the orbit of some point though not necessarily a periodic point (as is the case for Anosov flows). Let us draw attention to the similarity of this with the specification properties of a nonhyperbolic toral automorphism. The crucial point is that the geodesic flow is an isometry on the centermanifolds .

In Section 5 finally we apply specification to get hold of the logarithmic growth of the maximal compact flats. Let us first recall the situation for negative curvature. The best result here is due to Margulis. Let $\nu(t)$ be the number of closed geodesics of length $\leq t$ (counted with multiplicities). Then $\nu(t) \sim e^{ht}/ht$ where h is the topological entropy of the geodesic flow (the determination of the constant in the denominator is due to Ch.Toll). In particular, the logarithmic growth rate is the topological entropy.

In the higher rank case it is not so clear how to count the maximal compact flats as there are various characteristics for a compact flat. In fact, we propose to study a mixed property: recall that the regular systol of a compact flat is the length of a shortest closed regular geodesic. Then we study the function

$$VS(t) = \sum_{\text{regsys} F \leq t} \text{vol } F \qquad ,$$

show that it is well defined and again show that its logarithmic growth rate is the topological entropy of the geodesic flow on the unit tangent bundle. The summation condition is of a technical nature but unfortunately necessary (as we show by way of example).

It would be interesting to determine the growth rate of quantities that just involve the systol or just the volume.

There are two other questions that we haven't quite answered yet:

a Is the measure of maximal entropy unique? At the moment we only know that it is unique if we assume that the measure is invariant under the center manifold foliation.

b Are the compact maximal flats equidistributed with respect to the Liouville measure (in a suitable sense) ? We hope that we have exhibited many enough properties of the Liouville measure to make this look plausible.

As far as the techniques are concerned we have been drawing heavily on Bowen's hyperbolic flow paper [Bo 1]. As can be expected, many of the details just work the same. The geometry usually is just ad hoc with no bigger underlying scheme.

Appendix: We include a brief review of the basic properties of semisimple groups in an appendix. The material is mainly from [He 1] and [Wa 1].

Chapter I

Section 1. Closed Geodesics and Factors of the Boundary

In this section we will consider the fundamental group Γ of a compact visibility manifold of non-positive curvature. We will first review the notion of a boundary B for these manifolds and then give our construction of a non-trivial topological quotient of Γ acting on B. The main reference on visibility manifolds is [Eb 1].

1.1: Points of Infinity

For any Riemannian manifold M we let $\langle\,,\rangle$ be the Riemannian structure and d be the Riemannian metric. For $p \in M$ we let $S(p)$ be the unit sphere in the tangent space M_p and we let SM be the unit tangent bundle. If $v,w \in S(p)$ the angle $\theta = \measuredangle_p(v,w)$ is the unique number $0 \leq \theta \leq \pi$ such that $\langle v,w\rangle = \cos\theta$. All our manifolds will be complete and for $v \in SM$ we let $\alpha_v\colon \mathbb{R} \to M$ be the geodesic such that $\alpha_v'(0) = v$. All geodesics will be parametrized by arc length. Finally, K will denote the sectional curvature.

Definition: A Hadamard manifold H is a complete simply connected Riemannian manifold of dimension $n \geq 2$ with sectional curvature $K \leq 0$. The most important feature of a Hadamard manifold is Cartan's:

Proposition 1: Any two points on a Hadamard manifold H are joined by a unique geodesic.

Proof: This is well known, see for example Theorem 19.2 of [Mi 1]. □

For Hadamard manifolds we can introduce a nice equivalence relation between geodesics:

Definition: Two geodesics α and β in a Hadamard manifold H are asymptotic if there exists a number $c > 0$ such that $d(\alpha t, \beta t) < c$ for all $t \geq 0$. The equivalence classes are called asymptote classes.

Remark: Clearly, this definition works for any complete Riemannian manifold. For Hadamard manifolds a point $p \in H$ lies on at most one geodesic in each asymptote class, i.e. "there is at most one geodesic joining a point $p \in H$ to a point at infinity". This follows easily from the "law of cosines": for any p,q and $r \in H$

$$d^2(p,q) \geq d^2(p,r) + d^2(q,r) - 2d(p,r)d(q,r) \cdot \cos \sphericalangle_r(p,q).$$

Proposition 2: Given a geodesic α and a point $p \in H$ there exists a unique geodesic β such that $\beta(0) = p$ and β is asymptotic to α.

Proof: This is consequence (4) after Definition 1.1 in [Eb 1]. □

Definition: A point at infinity of H is an asymptote class of geodesics of H. The collection of points at infinity of H is called the boundary B of H or boundary sphere of H. For a geodesic α of H, we let $\alpha(\infty)$ denote the asymptote class of α and $\alpha(-\infty)$ the asymptote class of the reverse curve $t \mapsto \alpha(-t)$. $\alpha(\infty)$ and $\alpha(-\infty)$ are called the endpoints of α.

In this terminology, the last two propositions may be restated as:

Any point p in a Hadamard manifold H can be joined uniquely to any point q in $H \cup B$ by a geodesic γ_{pq}.

Next, we want to put a topology on $H \cup B$. There are a few natural topologies on $H \cup B$ (cf. [Eb 1, §3]) but we will only be interested in the cone topology.

Definition: Let $v \in S(p) \subset H_p$ and let ε be a number, $0 < \varepsilon < \pi$. Then the set

$$C(v,\varepsilon) = \{b \in H \cup B: \measuredangle_p(v,\gamma_{pb}) < \varepsilon\}$$

is called the cone of vertex p and angle ε.

Proposition 3: There is a unique topology k at $H \cup B = \overline{H}$ such that

(1) H is dense and open in $\overline{H}$.

(2) k induces the original topology on H.

(3) For each $b \in B$ the set of cones containing b is a local basis for k at x.

We call k the cone topology on $H \cup B$.

Proof: This is Proposition 2.3 of [Eb 1]. □

The cone topology is admissible in the sense of [Eb 1], p.50. In particular, the following two properties hold:

a Geodesic extension property: for any α in H its asymptotic extension $\alpha: \mathbb{R} \cup \{\pm\infty\} \to H \cup B$ is continuous.

b Isometric extension property: if φ is any isometry of H then its asymptotic extension is a homeomorphism.

In particular, any group of covering transformations of a quotient of H will act on the boundary.

Finally, we describe the topology of B.

Proposition 4: B is homeomorphic to a sphere. H ∪ B is a topological cell. In fact, a homeomorphism from B to the sphere is given by: Let $p \in H$. To any $v \in S(p)$ associate the point at ∞: $\alpha_v(\infty)$.

Proof: This is Theorem 2.10 and Corollary 2.12 of [Eb 1]. □

Example: For n-dimensional hyperbolic space consider the unit ball model. Then the boundary sphere of the unit ball is clearly the boundary of hyperbolic space as defined above.

1.2. Visibility manifolds

Definition: A Hadamard manifold H satisfies Axiom 1 if for any two points $x \neq y$ in B there exists at least one geodesic joining them.

Notice that the geodesic joining two points on the boundary may not be unique. If uniqueness holds true we say that H satisfies Axiom 2. See Example 5.10 of [Eb 1] for a Hadamard manifold satisfying Axiom 1 but not Axiom 2.

Definition: A Hadamard manifold H satisfies the Visibility Axiom if for any point $p \in H$ and $\varepsilon > 0$ there exists a number $r = r(p,\varepsilon)$ such that any geodesic segment $G: [a,b] \to H$ with $d(p,G) \geq r$ makes an angle less than ε with p: $\measuredangle_p(\alpha_{p,G(a)}, \alpha_{p,G(b)}) < \varepsilon$.

Roughly speaking, H is a visibility manifold if distant geodesics look small.

<u>Proposition 5</u>: The following three properties are equivalent for a Hadamard manifold H:

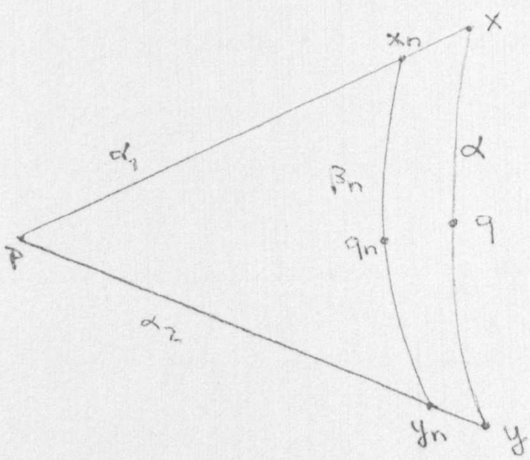
<u>1</u> H satisfies Axiom 1.

<u>2</u> H satisfies the Visibility Axiom.

<u>3</u> Let $\alpha_n: [a_n, b_n] \to H$ be a sequence of geodesics in H, $-\infty \le a_n < b_n \le \infty$. If $\alpha_n(a_n) \to x$ and $\alpha_n(b_n) \to y$ as $n \to \infty$ and $x \ne y$ then every α_n meets some compact set K of H. In particular, some subsequence of the α_n converges to a geodesic α joining x to y.

Proof: The equivalence of <u>2</u> and <u>3</u> and that <u>2</u> implies <u>1</u> are Proposition 4.4 of [Eb 1].

<u>1</u>⇒<u>2</u>: We first prove that no geodesic bounds a flat half plane: Suppose the contrary and pick a point p and two lines α_1, α_2 in a flat half plane. Let x and y be the end points of α_1 and α_2. Since H satisfies Axiom 1 we may pick a geodesic α joining x to y. Let $x_n(y_n)$ be points on $\alpha_1(\alpha_2)$ converging to $x(y)$ and let β_n be the geodesic segment joining x_n to y_n.



Recall that the square of the distance to α is a convex function and hence $d^2(\beta_n(t), \alpha)$ is convex in t (cf. [Bi 1] Proposition 2.1(2) and Theorem 4.1). Let q be the point on α closest to p and q_n on β_n a point on β_n such that its orthogonal projection (cf. [Bi 1], Lemma 3.2) onto α is q. Such a point clearly exists by continuity and because the orthogonal projections from x_n to α converge to x (the ones from y_n to y respectively).

Hence by choice of q_n and convexity we have:

$$d(q_n,q) = d(q_n,\alpha) \leq \max(d(x_n,\alpha),d(y_n,\alpha)) < C < \infty$$

since α_1 and α (α_2 and α) are asymptotes.

From this we see that

$$d(p,\beta_n) \leq d(p,q_n) \leq d(p,q) + d(q,q_n) \leq d(p,q) + C < \infty.$$

This is a contradiction since the β_n and p lie in a flat half plane. Now we can use Ballmann's (cf. [Ba 1], Lemma 2.2):

Lemma: Suppose a geodesic α in a Hadamard manifold H doesn't bound a flat half plane. Then there are neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of $\alpha(\infty)$ and $\alpha(-\infty)$ respectively such that for any $u \in U$ and $v \in V$ there exist a geodesic joining them. Moreover, any geodesic β joining u and v satisfies $d(\beta,\alpha(0)) < C$ where C only depends on U and V.

Indeed, suppose visibility fails. Then there is a point $p \in H$ and a sequence of geodesics α_n such that $d(p,\alpha_n) \to \infty$ as $n \to \infty$ and $\sphericalangle_p(\alpha_n(\infty),\alpha_n(-\infty)) \geq \varepsilon$ for some $\varepsilon > 0$. For a subsequence of the α_n, the endpoints $\alpha_n(\infty)$ and $\alpha_n(-\infty)$ will converge to x and y respectively. By Axiom 1 join x and y by a geodesic α. From what we have proved above and Ballmann's Lemma we see that

$$d(\alpha(0),\alpha_n) < C < \infty.$$

Therefore $d(p,\alpha_n) \leq d(p,\alpha(0)) + d(\alpha(0),\alpha_n)$ is bounded in contradiction to our assumptions. □

Note: That 1 ⇒ 2 seems to be due to B. O'Neill (cf. [Eb 1] p. 62). It was pointed out again in [Eb 2] p. 439, also Lemma 2.3a, but there seems to be no clear reference. Essentially we followed Eberlein's suggestions in [Eb 2], avoiding Busemann functions though.

Proposition 6: If a Hadamard manifold H has sectional curvature $K \leq C < 0$ then H is a visibility manifold.

Proof: This is Lemma 9.10 of [Bi 1]. □

In particular, any globally symmetric space of rank 1 (and no compact factors) is a visibility manifold. On the other hand we have:

Example: Any globally symmetric space of rank ≥ 2 (and no compact factors) violates the Visibility Axiom.

Proof: By definition, the rank is the maximal dimension a flat totally geodesic subspace can have. Clearly, visibility fails for flat n-space, $n \geq 2$.

Remark: We will see later on that even the higher rank locally symmetric spaces satisfy a suitable modification of the Visibility Axiom.

1.3: Axial isometries

Let us first recall the standard classification of isometries.

Definition: For any isometry φ, we let the displacement function g_φ be defined by:

$$g_\varphi(p) = d(p,\varphi p).$$

We call an isometry φ

<u>elliptic</u> if g_φ has minimum 0

<u>axial</u> if g_φ has positive minimum

<u>parabolic</u> if g_φ has no minimum.

<u>Proposition 1</u>: An isometry φ is axial iff φ translates a geodesic α, i.e., if $\varphi(\alpha(t)) = \alpha(t+t_0)$. α is called an <u>axis</u> of φ.

Proof: This is Proposition 10.9 of [Bi 1]. □

<u>Proposition 2</u>: Let H satisfy the Visibility Axiom. Then every non-elliptic isometry has at most two fixed points in B, the boundary: one if parabolic and two if axial.

Proof: This is Theorem 6.5 of [Eb 1]. □

<u>Proposition 3</u>: Let α be an axis of an isometry φ of H with endpoints x and y. If an isometry ψ fixes x and if ψ and φ generate a properly discontinuous group then ψ commutes with a power of φ, and in particular, ψ leaves y invariant.

Proof: This is Proposition 6.8 of [Eb 1]. □

This last result is the key to our construction.

1.4: <u>The Construction</u>

Let M be a manifold of non-positive curvature whose universal cover satisfies the Visibility Axiom (a visibility manifold for short). Assume that M has a closed geodesic

$\overline{\alpha}$. For example, if M is compact this holds true by the theorem of Lyusternik and Fet ([Fl 1] Theorem 5.7) (or simply because there is a closed geodesic in each free homotopy class and because M is not simply connected). Let $\Gamma = \pi_1(M)$. Then Γ acts properly discontinuously on the universal cover H of M. In particular, we have the

<u>Lemma 1</u>: Let α be a lift of $\overline{\alpha}$ to H. Let x and y be the endpoints of α in B (points at infinity of H) and let $\Gamma_x(\Gamma_y)$ be the isotropy subgroup in Γ of $x(y)$. Then $\Gamma_x = \Gamma_y$. Moreover, there is no $\delta \in \Gamma$ such that $\delta x = y$.

Proof: Clearly, α is an axis for some isometry $\gamma \in \Gamma$ (since $\overline{\alpha}$ is closed there is a $t_0 \in \mathbb{D}$ and elements $\gamma_t \in \Gamma$ such that $\alpha(t+t_0) = \gamma_t(\alpha(t))$. By proper discontinuity $\gamma_t = \gamma$ is constant). The first claim therefore is Proposition 3 of 1.3.

Suppose that $y = \delta x$ for some $\delta \in \Gamma$. Then $\delta\gamma\delta^{-1}$ has the geodesic through $(y, \delta y)$ as an axis. On the other hand, γ fixes y, hence δy by the first part of this lemma. Since any non-elliptic isometry of H has at most two fixed points we see that $\delta y = x$ (1.3 Proposition 2), i.e. δ permutes x and y. Since Γ is torsion free there are no elliptic elements in Γ. Since $H \cup B$ is a closed cell (1.1 Proposition 4) δ has a fixed point in $H \cup B$ by the Brouwer fixed point theorem. Since δ is non-elliptic it has a fixed point in B. We see that δ^2 has at least three fixed points in B in contradiction to 1.3 Proposition 2. □

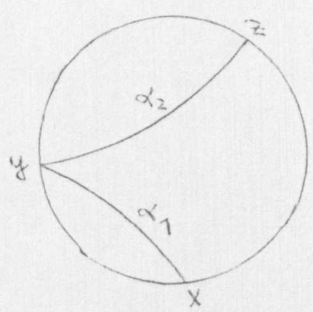
Now we can define our equivalence relation: Let $\overline{\alpha}$ be a closed geodesic in M as before. We let $\sim$ be the relation on B given by:

a $x \sim x$ for all $x \in B$

b $x \sim y$, $x \neq y$ iff x and y are endpoints of one and the same lift α to H of $\overline{\alpha}$.

Lemma 2: The relation $\sim$ is an equivalence relation.

Proof: We only have to check transitivity: Let $x \sim y$ and $y \sim z$. We have geodesics α_1 and α_2 joining x to y and y to z respectively that project to $\overline{\alpha}$ in M. Hence there is $\delta \in \Gamma$ such that $\alpha_2 = \delta\alpha_1$.



In particular, we must have either that a $\delta x = z$ and $\delta y = y$ or b $\delta x = y$ and $\delta y = z$. Both of these cases are impossible by Lemma 1 unless $x = z$. □

Finally, we use the visibility property in

Lemma 3: The relation $\sim$ is closed.

Proof: Suppose that $x_n \sim y_n$ and $x_n \to x$, $y_n \to y$ as $n \to \infty$. Pick axes α_n joining x_n to y_n. By the third equivalence in Proposition 5 of 1.2 the α_n converge to a geodesic α joining x to y. As all the α_n project to $\overline{\alpha}$ in M α has to project to $\overline{\alpha}$ in M. This means that $x \sim y$. □

This finishes our construction and we have

Proposition: Let M be a visibility manifold with a closed geodesic. In particular, M may be any compact visibility manifold. Then there exists a non-trivial $\pi_1(M)$-equivariant topological Hausdorff quotient of the boundary B of the universal cover of M.

Proof: Above we constructed a closed equivalence relation $\sim$ on B that was clearly Γ-invariant. Any closed equivalence relation on a compact Hausdorff space gives rise to a Hausdorff quotient (cf. [Vi 1] Proposition 2.1). Since $\sim$ is $\pi_1(M)$-invariant we can define an action of $\pi_1(M)$ on $B/\sim$. Since we identify only countably many points by $\sim$ $B/\sim$ cannot be a point. On the other hand, the axial isometry γ of the lift α of our closed geodesic $\overline{\alpha}$ has two fixed points on B which get identified in $B/\sim$. There are no more fixed points of γ in $B/\sim$ since any such fixed point corresponds to an axis of γ in H, but axes are unique. So B and $B/\sim$ cannot be equivariantly isomorphic. This proves that $B/\sim$ is non-trivial. □

It may be interesting to notice that $B/\sim$ is not even a manifold. This follows from a very general argument about branch points:

<u>Lemma 4</u>: Let M be a manifold and $\sim$ be a countable equivalence relation. (i.e. $\sim$ is trivial on $M - \Sigma$ where Σ is a countable set). Then $M/\sim$ is not a manifold.

Proof: Let $p\colon M \to M/\sim$ be the projection map and let $x \neq y \in M$, $x \sim y$. Let $z = p(x)$. Suppose $M/\sim$ is a manifold and let U be a coordinate chart about z. Let $S_r(t)$ be the sphere of radius r about t. Then $p\colon \bigcup_{r \neq r_i} S_r(x) \cup S_r(y) \to M/\sim$ is an injective continuous map where the $\{r_i\}$ are a countable exceptional set of radii. In particular, $p|_{S_r(x)}$ (resp. $p|_{S_r(y)}$) is a homeomorphism onto its image for $r \neq r_i$. For $r \neq r_i$ small enough, $p(S_r(x))$ separates U into two connected components by the Jordan-Brouwer separation theorem ([Spa1], Chapter 4 Theorem 15). As $r \to 0$ $p(S_r(x)) \to z$ by continuity. For any given r $p(S_{r'}(x))$ lies either inside or outside $p(S_r(x))$ $(r,r' \neq r_i)$. As $p(S_{r'}(x)) \to z$, $p(S_{r'}(x))$ lies inside $p(S_r(x))$ and in particular z lies inside $p(S_r(x))$. As the $p(S_{r'}(y)) \to z$ $p(S_{r'}(y))$ lies inside $p(S_r(x))$ and z lies inside $p(S_{r'}(y))$. Consider a radial line $\rho(t)$ connecting x to $S_r(x)$. By the Jordan-Brouwer separation theorem $p(\rho(t_0)) \in p(S_{r'}(y))$ for some t_0. Since this happens for uncountably many $r' \neq r_i$ there is a t_0 with $\rho(t_0) \notin S_{r_i}(x)$, all i, and $p(\rho(t_0)) \in p(S_{r'}(y))$ in contradiction to our assumption on the r_i's. □

In particular, this construction solves Margulis' question in the case of a lattice in any $\mathbb{R}$-rank 1 semisimple group of the non-compact type. Strictly speaking we had to assume that Γ has no torsion so that the locally symmetric space $\Gamma\backslash G/K$ is a manifold. An easy variation of our argument gives the same result for lattices with torsion.

Also notice that our factor is measurably trivial as we identified only countably many points. Therefore we may conclude this section with the

Problem: Does there exist a factor of $SL(2,\mathbb{Z})$ acting on S' that is non-trivial both topologically and measure-theoretically? Margulis constructs a non-trivial measure-theoretical quotient in Corollary 2.9.1 of [Ma 1].

Section 2. Nonexistence of Factors in Higher Rank

We mainly consider $SL(n,\mathbb{Z})$ acting on $\mathbb{P}^{n-1}$, $n > 2$, and show that there do not exist any $SL(n,\mathbb{Z})$-equivariant Hausdorff quotients of $\mathbb{P}^{n-1}$. This together with 1.4 for $n = 2$ answers Margulis precise question in [Ma 1]. This was first proven by Zimmer in [Zi 1] using different techniques. We can generalise the above result to some other Grassmannians using a result of Dani.

2.1: $SL(n,\mathbb{Z})$ acting on $\mathbb{P}^{n-1}$

Let $\Gamma = SL(n,\mathbb{Z})$ and $G = SL(n,\mathbb{R})$ for short. It follows from Hermite's and Siegel's work that Γ is a lattice in G (cf. [B-Hch] Theorem 9.4). As we discuss in A10.5 the projective space $\mathbb{P}^{n-1}$ is a boundary of G. We claim:

Proposition 1: All Hausdorff quotients of Γ acting on $\mathbb{P}^{n-1}$, $n > 2$, are trivial.

Note: For uniform lattices Γ in $SL(n,\mathbb{R})$ (i.e. G/Γ is compact) the same result is true by [Ve 2]. We first observe

Lemma 1: Let a group Γ act on a compact Hausdorff space M. If the diagonal action of Γ on $(M \times M\text{-diagonal})$ is minimal then all equivariant Γ-quotients of M are trivial.

Note: Recall that one calls an action *minimal* if every nonempty Γ-invariant closed set is the whole space.

Proof: Any Hausdorff quotient X of M is defined by a closed equivalence relation $R \subset M \times M$. By Γ-equivariance of X we conclude that R is Γ-invariant. By the minimality of Γ on $(M \times M\text{-diagonal})$ $\mathcal{R}$ is either $M \times M$ or just the diagonal and the quotient is trivial. □

Proposition 1 follows from the stronger

<u>Proposition 2</u>: The lattice $\Gamma = SL(n,\mathbb{Z})$ acts minimally on $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$- diagonal for $n > 2$.

Proof: On the level of $\mathbb{P}^{n-1}$ itself we first have

<u>Lemma 2</u>: For $n > 1$, Γ acts minimally on $\mathbb{P}^{n-1}$.

<u>Note</u>: This is completely general: any lattice Γ in any semisimple Lie group G without compact factors acts minimally on any boundary of G; in fact, [Mo 1] Lemma 8.5 says that $\overline{\Gamma \cdot P} = G$. Since for any $x \in G$, $x^{-1}\Gamma x$ is a lattice it is clear that $\Gamma \cdot x \cdot P$ is dense in G/P for any $x \in G$. Of course, the case at hand is standard and follows from elementary arguments.

Now the proof of Proposition 2 develops in two stages. For notation let $\overline{x} \in \mathbb{P}^{n-1}$ be the line through x for any $x \in \mathbb{R}^n$.

1) Let e_i be the standard basis of $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$, $\overline{x} \neq \overline{e}_1$. We claim that $\Gamma(\overline{e}_1,\overline{x})$ is dense in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$-diagonal: a typical element of the stabiliser subgroup Γ_0 of Γ at $\overline{e}_1$ looks like

$$\begin{pmatrix} * & * & \cdot & \cdot & \cdot & * \\ 0 & * & \cdot & \cdot & \cdot & * \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & * & \cdot & \cdot & \cdot & * \end{pmatrix} .$$

In particular, $SL(n-1,\mathbb{Z})$ embeds into Γ_0 in the obvious way. It suffices to prove that $\Gamma(\overline{e}_1,\overline{y})$ is dense in

$\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$-diagonal for some $\overline{y}$ in the closure of $\Gamma_0(\overline{x})$. By Lemma 2 and the above we may assume that the coordinates $x_2,\ldots,x_n$ of x are linearly independent over $\mathbb{Q}$. Let $\overline{y} \neq \overline{z}$ be two lines in $\mathbb{P}^{n-1}$ and V,W neighbourhoods of them. By Lemma 2 there is a $\gamma \in \Gamma$ such that $\gamma(\overline{e}_1) \in W$. Hence it suffices to find a $\gamma_0 \in \Gamma_0$ such that $\gamma_0(\overline{x}) \in \gamma^{-1}(V)$. For some $t \in \mathbb{R}^n$ let $\overline{t} = \gamma^{-1}(\overline{y})$. We can find $\gamma_1 \in SL(n-1,\mathbb{Z})$ such that $\gamma_1(\overline{0,x_2,\ldots,x_n})$ is close to $(\overline{0,t_2,\ldots,t_{n-1}})$ by Lemma 2. Let $x_2',\ldots,x_n'$ be a choice of coordinates for $\gamma_1(\overline{0,x_2,\ldots,x_n})$. We may assume that $t_2,\ldots,t_n$ are close to $x_2',\ldots,x_n'$. Clearly, the $x_2',\ldots,x_n'$ are linearly independent over $\mathbb{Q}$. As the group generated by $x_2',\ldots,x_n'$ is dense in $\mathbb{R}$ we can find a

$$\gamma_2 = \begin{pmatrix} 1 & m_2 & \cdots & m_n \\ 0 & & & \\ \vdots & & \text{id} & \\ 0 & & & \end{pmatrix} \in \Gamma_0$$

such that $x_1' + m_2x_2' + \ldots + m_nx_n'$ is close to t_1. Since γ_2 doesn't change the other coordinates we have proved our first claim.

2) Consider any two lines $\overline{y} \neq \overline{z}$. We claim that the closure of their Γ-orbit contains $(\overline{x},\overline{e}_1)$ or $(\overline{e}_1,\overline{x})$. We consider two cases:

a) $\overline{z}$ is rational. Our claim follows by the well known:

<u>Lemma 3</u>: All rational lines lie on the $SL(n,\mathbb{Z})$-orbit of $\overline{e}_1$.

Note: One may phrase this in terms of number theory: all coprime n-tuples of integers lie in the $SL(n,\mathbb{Z})$-orbit of $(1,0,\ldots,0)$.

Proof: For $n = 2$ a rational line is represented by a pair of coprime integers (p,q). There are integers p_1 and q_1 such that $pp_1 - qq_1 = 1$. Clearly,

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p & q_1 \\ q & p_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is in the $SL(2,\mathbb{Z})$ orbit of $(1,0)$.

For $n > 2$ let $(m_1,\ldots,m_n)$ be a point on a given line ℓ with integer entries. Then ℓ lies in the plane spanned by $(m_1,\ldots,m_{n-1},0)$ and $\bar{e}_n$. By induction pick $\gamma \in SL(n-1,\mathbb{Z})$ such that

$$\gamma\bar{e}_1 = \overline{(m_1,\ldots,m_{n-1},0)}.$$

Then $\gamma^{-1}(\ell)$ lies in the plane spanned by $\bar{e}_1$ and $\bar{e}_n$ and we can apply the result for $n = 2$. □

b) $\bar{z}$ is irrational. Then there is i and j such that z_i and z_j are rationally independent, say $i = 2$, $j = 3$. In particular, $\mathbb{Z}\, z_2 + \mathbb{Z}\, z_3$ is dense in $\mathbb{R}$. Hence there are matrices

$$\gamma_n = \begin{pmatrix} 1 & m_2^n & m_3^n & 0 \ldots 0 \\ 0 & & & \\ \vdots & & \text{id} & \\ 0 & & & \end{pmatrix} \in \Gamma$$

such that $\gamma_n \cdot z \to (0,z_2,\ldots,z_n)$ as $n \to \infty$ while

$$\gamma_n y = (1 + m_2^n y_2 + m_3^n y_3, y_2, y_3, \ldots).$$

If $(\overline{z_2,z_3}) \neq (\overline{y_2,y_3})$ then

$$\frac{m_2^n y_2 + m_3^n y_3}{m_2^n z_2 + m_3^n z_3} = \frac{y_2}{z_2} + m_3^n \frac{y_3 - z_3 y_2 z_2^{-1}}{m_2^n z_2 + m_3^n z_3} \longrightarrow \pm\infty$$

since the denominator stays bounded and w.l.o.g. $m_3^n \to \pm\infty$. Otherwise $y_3 - z_3 y_2 z_2^{-1} = 0$ and $(\overline{z_2,z_3}) = (\overline{y_2,y_3})$. Hence $\gamma_n \cdot \overline{y} \to (\overline{1,0,\ldots,0})$ and in this case we are done. If $(\overline{z_2,z_3}) = (\overline{y_2,y_3})$ we can still pick γ_n as above. Let a be so that $a \cdot (z_2,z_3) = (y_2,y_3)$. Then $\gamma_n \cdot y \to (y_1 - az_1, y_2, \ldots)$. If $(\overline{z_1,z_2,z_3}) \neq (\overline{y_1,y_2,y_3})$ then $y_1' \overset{\text{def}}{=\!=} y_1 - az_1 \neq 0$. Notice that y_2 and y_3 are rationally independent. Hence one of (y_1',y_2) or (y_1',y_3) is rationally independent, say the first. Since $(\overline{y_1',y_2}) \neq (\overline{0,z_2})$ we can apply the previous argument to (y_1',y_2). Instead of z_1 we could have used any $z_i, i > 3$, before. Hence we only have to deal with the case that $(\overline{z_2,z_3,z_i}) = (\overline{y_2,y_3,y_i})$ for all i. Obviously $\overline{z} = \overline{y}$ in this case. □

2.2: SL(n,ℤ) acting on Grassmannians

The argument of 2.1 generalises to some other Grassmannians. Again we let $\Gamma = SL(n,\mathbb{Z})$ and $G = SL(n,\mathbb{R})$.

Proposition 1: All Hausdorff factors of the action of $SL(n,\mathbb{Z})$ on $G_{k,n}$, the Grassmannian of k-planes in n-space, are trivial if $k < \frac{n}{2}$.

Proof: Let $\sim$ be a closed Γ-invariant equivalence relation.

We consider $\sim$ as a subset of $G_{k,n} \times G_{k,n}$. If $\sim$ $\neq$ diagonal we will show that $\sim$ is all of $G_{k,n} \times G_{k,n}$.

We need a result of Dani:

Lemma: Let $f_1,\ldots,f_{n-1}$ be linearly independent vectors in $\mathbb{R}^n$. Then there exist linearly independent rational vectors $a_1,\ldots,a_{n-1}$ in $\mathbb{R}^n$ (with respect to the canonical basis $\{e_i\}$), nonzero scalars $\lambda_1,\ldots,\lambda_{n-1}$ in $\mathbb{R}$ and a sequence $\{\gamma_j\}$ in Γ such that

$$\gamma_j(f_1 \wedge \cdots \wedge f_k) \to \lambda_k(a_1 \wedge \cdots \wedge a_k) \quad \text{as} \quad j \to \infty$$

for all $k = 1,2,\ldots,n-1$.

Proof: This is [Da 1] Corollary 9.8. Even though the linear independence of the a_i is not explicitly stated it is contained in the proof. □

Next recall the correspondence between k-planes and simple products $f_1 \wedge \cdots \wedge f_k$ in $\wedge_{k,n}$: by choosing a basis for a k-plane we get a point in $\wedge_{k,n}$ which is well defined up to multiplication by a scalar. Conversely, each simple product $f_1 \wedge \cdots \wedge f_k$ determines the plane spanned by $f_1,\ldots,f_k$.

Suppose that $P \neq Q$ but $P \sim Q$ where $P,Q \in G_{k,n}$. Let $\ell = \dim P \cap Q$. Then we may represent P in $\wedge_{k,n}$ by $r_1 \wedge \cdots \wedge r_\ell \wedge p_1 \wedge \cdots \wedge p_{k-\ell}$ and $\mathbb{Q}$ by $r_1 \wedge \cdots \wedge r_\ell \wedge q_1 \wedge \cdots \wedge q_{k-\ell}$. Since $k < \frac{n}{2}$ we may apply the lemma to the product $r_1 \wedge \cdots \wedge r_\ell \wedge p_1 \wedge \cdots \wedge p_{k-\ell} \wedge q_1 \wedge \cdots \wedge q_{k-\ell}$. We find $\gamma_i \in \Gamma$ and planes S and T such that

$$\gamma_i(P) \to S \quad \text{and} \quad \gamma_i(Q) \to T \qquad \text{as} \quad i \to \infty.$$

and such that S and T are represented by rational vectors. Since the rational vectors a_i in the lemma are rationally independent we see that $\dim(S \cap T) = \ell$. Moreover $S \sim T$ as $\sim$ is closed.

We claim that any two rational k-planes are translates of each other under Γ: by 2.1 Lemma 3 a rational vector a_1 in a rational plane S given by $a_1, \ldots, a_k$ is a translate of e_1 (where $\{e_i\}$ is a canonical basis). Assuming that S contains e_1 we can replace $a_2, \ldots, a_k$ by vectors in the span of $e_2, \ldots, e_n$. This starts an induction after which S is given by $e_1, \ldots, e_k$.

Applying this to our rational planes S and T above we may assume that S is spanned by $e_1, \ldots, e_k$. Let Γ_0 be the stabiliser of S in Γ. Notice that an element of Γ_0 has the form:

$$k \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right).$$

The intersection $T \cap S$ defines a plane in T. Any element γ of $SL(k,\mathbb{Z})$ embeds into Γ_0 by way of the first quadrant in the matrix expression above. Hence with $T \sim S$ we also have $\gamma(T) \sim S$ by the Γ-invariance of $\sim$. By the note after 2.1 Lemma 2 $SL(k,\mathbb{Z})$ acts minimally on any Grassmannian of S. As $\sim$ is closed we see that $S \sim R$ where $R \cap S$ is spanned by $e_1, \ldots, e_\ell$. Let R be spanned by $e_1, \ldots, e_\ell$, $s_1, \ldots, s_{k-\ell}$. Let S_1' be the orthogonal projection of S_1 onto the span of $e_{k+1}, \ldots, e_n$.

There exists a sequence $\gamma_n \in \Gamma_0$ of the form

$$\gamma_n = k\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \gamma_n' \end{pmatrix}$$

such that $\|\gamma_n'(S_1')\| \to \infty$. Hence $\lim \gamma_n R$ is spanned by $e_1, \ldots, e_\ell$, $a_1, \ldots, a_{k-\ell}$ where $a_1 = a_1'$. Similarly, we may get a plane B such that B is spanned by $e_1, \ldots, e_\ell$, $b_1, \ldots, b_{k-\ell}$ where $b_i = b_i'$ and $B \sim S$.

Since $SL(n-k, \mathbb{Z})$ acts minimally on $G_{k-\ell, n-k}$ we see that $S \sim U$ where U is any plane spanned by $e_1, \ldots, e_\ell$, $u_1, \ldots, u_{k-\ell}$ where u_i is a vector contained in the span of $e_{k+1}, \ldots, e_n$. Clearly, $S \sim U$ if U intersects S in an ℓ-dimensional rational plane and if U is transversely orthogonal to S (i.e. U is in the span of $U \cap S$ and $e_{k+1}, \ldots, e_n$). Since $\sim$ is closed and the rational planes are dense we see that $S \sim U$ whenever $\dim S \cap U = \ell$ and U is transversely orthogonal to S. Clearly we may allow S to be an arbitrary rational plane and hence $S \sim U$ whenever $\dim S \cap U = \ell$ and U is transversely orthogonal to S.

Next let F_1, F_2 be two k-planes such that $\dim F_1 \cap F_2 = k - 1$. Then there clearly exists S that has an ℓ-dimensional intersection with both F_1 and F_2 and is transversely orthogonal to both of them. Hence $S \sim F_1, F_2$ and by transitivity $F_1 \sim F_2$. Given any two k-planes F_1, F_n it is easy to find a chain of k-planes F_i, $i=2,\ldots,n-1$ such that $\dim F_i \cap F_{i+1} = k - 1$. Hence $F_1 \sim F_n$ and $\sim$ is everything. $\square$

Section 3. Nonminimality of Lattices Acting on Boundaries

When we proved the nonexistence of factors of the projective space in the last section, our main tool was to prove the minimality of $SL(n,\mathbb{Z})$ acting on $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$-diagonal. For a general boundary B we cannot hope for the same for the stupid reason that not even G acts minimally on $B \times B$-diagonal: consider the Grassmannian of 2-planes in n-space. Then pairs of planes that intersect nontrivially certainly are an $SL(n,\mathbb{R})$-orbit.

This suggests the

Definition: Let $\Gamma \subset G$ be a subgroup and let X be a G-space. We call the action of Γ on X G-minimal if all Γ-invariant closed sets in X are G-invariant.

We will prove that no lattice acts G-minimally on $G/P \times G/P$ if P is a minimal parabolic. The main idea is to replace closed geodesics in the construction in Section 1 by compact flats. We need to recall the geometric realization of G/P as "points at infinity" of flats. This material is from [Mo 1] and will be developed in 3.1-3.3. See also Chapter II, Section 3.

3.1 Flats

Definition: A flat F in a globally symmetric space H is a totally geodesic subspace of H of sectional curvature 0. In the following, we assume that $H = G/K$ is of the non-compact type. Let 0 be a point of a flat F such that K is the isotropy group of 0 in G. Then F is carried over into a vector subspace of $\mathfrak{p}$ by the inverse of the exponential map since F is totally geodesic. Since F is flat $F = \mathrm{Exp}\,\mathfrak{a}$ where $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{p}$. In particular, a maximal flat corresponds to a maximal abelian subalgebra of $\mathfrak{p}$ and the rank r of G is the maximal dimension of a flat of G/K

(cf. A8.4 Definition). Clearly, $\exp\mathfrak{a} \subset G$ stabilises F. On the other hand, let G_F be the stabiliser of F. Then we have:

Lemma 1: The polar part of each element of G_F is contained in $\exp\mathfrak{a}$ which acts simply transitively on F. If F is maximal then G_F=normaliser $(\exp\mathfrak{a})$. Let $\exp\mathfrak{a} = \text{pol}\, F$, the *polar subgroup of* F. The map $F \to \text{pol}\, F$ is a bijection between the set of all maximal flats and the set of all maximal polar subgroups.

Note: We call a subgroup H of G *polar* if $h = \text{pol}\, h$ for all $h \in H$.

Proof: This is [Mo 1], Lemma 5.1. □

Recall from A2.2 that Cartan subalgebras arise as centralisers of regular elements. In a similar vein, we define:

Definition: We call $g \in G$ *polar regular* if

$$\dim \text{centraliser}(\text{pol}\, g) \leq \dim \text{centraliser}(\text{pol}\, h)$$

for all $h \in G$.

It is easy to see that g is polar regular if $\text{pol}\, g$ lies off the walls of the Weyl chambers in a maximal abelian subgroup of some $\exp\mathfrak{p}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition. More important for us is

Lemma 2: Let rank $G = r$. Then

i) a polar regular element of G is semisimple.

ii) a polar regular element stabilises a unique r-flat in G/K.

iii) if the polar regular element g stabilises the r-flat F then centraliser of g stabilises F and acts transitively on F. Moreover, $g \cdot x = (\text{pol}\, g)\, x$ for all $x \in F$.

Proof: This is [Mo 1] Lemma 5.2. □

The next fact is a generalisation of the uniqueness of the geodesic joining two points at infinity in negative curvature.

Lemma 3: Any two maximal flats that are a bounded distance apart coincide.

Proof: This is [Mo 1] Lemma 5.4. □

3.2 Lattices and Compact Flats

By G we will always denote a connected semisimple group without compact factors, of rank r. Let us first recall the

Definition 1: A discrete subgroup Γ of G is called a *lattice* if G/Γ has finite volume.

Recall from A12 that $_\Gamma\backslash G/_K$ is a locally symmetric space if Γ is torsion free. On the other hand, it is well known that any lattice has a torsion free subgroup of finite index (cf. [Ra 1], Corollary 6.13). In this sense, this section is a direct generalization of our investigations in 2.1.

We are mainly interested in the existence of compact flats in $_\Gamma\backslash G/_K$. We first need the

Definition 2: Let $\wedge(g)$ be the representation of $g \in G$ on the exterior algebra of $\wedge\mathfrak{g}$. We call g $\mathbb{R}$-*hyperregular* if the number of eigenvalues of modulus 1 (counted with multiplicities) is as small as possible and -1 is not an eigenvalue of $\wedge(g)$.

Lemma 1: Every $\mathbb{R}$-hyperregular element is polar regular.

Proof: This is Remark 1.2 of [Pr-Ra 1]. □

More important is

Lemma 2: If Γ is a lattice in G and $\gamma \in \Gamma$ is $\mathbb{R}$-hyperregular *or* if Γ is cocompact and γ regular then centraliser γ/centraliser $\gamma \cap \Gamma$ is compact.

Proof: If G/Γ is not compact, this is [Pr-Ra] Theorem 1.14. If G/Γ is compact, this is [Mo 1] Lemma 8.1. Notice that we don't need any assumptions on γ in the latter case. □

The existence of $\mathbb{R}$-hyperregular elements in Γ is established by Mostow. In fact, we get the stronger

<u>Proposition 1:</u> The set of Γ-compact r-flats in G/K (i.e. flats F such that ${}_\Gamma\backslash\Gamma\cdot F$ is compact) is dense in the set of all r-flats in G/K.

Proof: This is [Mo 1] Lemma 8.3' or Chapter II, Section 3 of this thesis. □

Clearly, any compact r-flat in ${}_\Gamma\backslash G/_K$ is covered by a torus. Hence Γ contains abelian subgroups of rank r. In more detail, we have the

<u>Proposition 2:</u> a) If $\gamma \in \Gamma$ is $\mathbb{R}$-hyperregular, then centraliser $\gamma \cap \Gamma$ contains an abelian subgroup of rank r and finite index.

b) Any abelian subgroup of B semisimple elements of Γ has rank at most r. Any such group contains an $\mathbb{R}$-hyperregular element. In fact, all elements are $\mathbb{R}$-hyperregular except for those lying in a finite union of subgroups of rank less than r.

Proof: This is [Mo 1] 11.1' and 11.2'. □

3.3 <u>The Maximal Boundary from a Geometric Point of View</u>

The geometric boundary of a Euclidean space is not very interesting as the geodesics joining two points at are not unique. Moreover, any two such geodesics differ in only a trivial way as they are parallel. In a higher rank globally symmetric space H we have flat s. We would like to replace the geometric boundary of H by a smaller boundary that reflects only the non-Euclidean aspect of the geometry of H.

Given a maximal flat F and a geodesic ray c(t) we have 'zones of stability' $U \subset F(\infty)$, $c(\infty) \in U$ in the following sense: Let F' be a second flat and suppose that for some ray $c'(t) \subset F'$ we have $c'(\infty) = c(\infty)$. Then for any $c_1(t) \subset F$ with $c_1(\infty) \in U$ there is a $c_1'(t) \subset F'$ with $c_1(\infty) = c_1'(\infty)$. Moreover, $d(c_1(t), c_1'(t) \to 0$ as $t \to \infty$ for a suitable choice of $c_1'(t)$. These "zones of stability' are precisely the points at ∞ of the various open Weyl chambers of F (recall that $F = \mathrm{Exp}\,\mathfrak{a}$ so that we may call the exponentials of the Weyl chambers of μ the Weyl chambers of F). Hence, we may identify these 'zones of stability' to one point and forget about the points at infinity of the walls of the Weyl chambers.

On the basis of these considerations, we make the following

Definition: We call the exponentials of the open (closed) Weyl chambers of a maximal polar subalgebra $\mathfrak{a}$ the *open (closed) Weyl chambers* of $F = \mathrm{Exp}\,\mathfrak{a}$. We call two Weyl chambers C, C' of flats F, F' *asymptotic* if they are only a finite distance apart. We let X_0 be the collection of Weyl chambers modulo asymptoticity. By $[C]$, we will denote the equivalence class of a chamber C.

Lemma 1: We have $X_0 = G/P$ where P is a minimal parabolic.

Proof: This is [Mo 1] Lemma 4.1. □

Naturally, X_0 now carries the topology of the homogeneous space G/P. One may describe it geometrically as a '*cone topology*': let $P = M\cdot A\cdot N_+$ be a Langlands decomposition. Then $G/P = K/M$. Let $0 = 1\cdot K$ in G/K. Then any chamber C is asymptotic to a unique chamber C' that passes through 0: as $G = K\cdot P$ (immediate from $G = K\cdot A\cdot N^+$ and $P = M\cdot A\cdot N^+$) K acts transitively on G/P. Hence, we may assume that P stabilises $[C]$. Pick $g \in G$ such that $g\cdot C$ contains 0. Write $g = k\cdot p$. Then $k^{-1}gC = p\cdot C$ is asymptotic to C and contains 0. Given two chambers C, C' containing 0 there is $m \in K$ such that $m\cdot C' = C$. Since $m\cdot[C] = [C]$, $m \in P$ hence $m \in M$ where $P = MAN$ is the isotropy group of $[C]$. Therefore, m leaves

$C = A\,0$ invariant and $C = C'$. For two asymptoty classes $[C_1]$, $[C_2]$ let C_1, C_2 be the representatives containing 0. We may talk about the "angle" C_1, C_2 subtend at 0. Clearly, writing $C_2 = kC_2$ for $k \in K$ this angle is small iff $k \cdot M$ is close to $1 \cdot M$ in K/M.

Recall the notion of Hausdorff distance: Let X be a metric space, $A, B \subset X$ subsets. Then

$$hd(A,B) = \inf\{u \leq \infty \mid \text{for all } x \in A \text{ there is } y \in B \text{ such that } d(x,y) \leq u \text{ and for all } y \in B \text{ there is } x \in A \text{ such that } d(x,y) \leq u\}.$$

Let S be a geodesic ray in the closure of a chamber C of a flat F. Under the exponential map S corresponds to a vector v in the maximal polar subalgebra $\mathfrak{a}$ such that $\exp \mathfrak{a} = F$. Let Θ be the set of all those roots that vanish on v and let $P(S)$ be the parabolic containing P defined by Θ (where P stabilises $[C]$). Then we have the

<u>Lemma 2</u>: For S a geodesic ray and $g \in G$. Then

a) $hd(S, gS) < \infty$ iff $g \in P(S)$.

b) $d(S, gS) = 0$ iff $g \in R_u P(S)$.

c) If S' is another geodesic ray, then
$hd(S, S') < \infty$ iff $S' = gS$ and $g \in P(S)$.

Proof: These are Lemma 7.1 and 7.2 of [Mo 1]. □

Notice that this implies our claim above on "zones of stability': As indicated, we let the U_i be the points at infinity of the open Weyl chambers. Since any two flats are translates of each other, $F' = g \cdot F$ (cf. A5.2 Proposition 1). If α and α' are asymptotic, $\alpha' = g_0 \cdot \alpha$ for $g_0 \in P(\alpha)$ by Lemma 2. Since α lies in an open Weyl chamber $P(\alpha) = P$ is minimal. Also, g can be taken to be g_0 since α is regular. As β lies in the same Weyl chamber $P = P(\beta)$.

So $g\beta$ lies a finite distance apart from β and also $g\beta \subset F'$.

Note: It might be interesting to try to understand "zones of stability' for artibrary Hadamard manifolds and to try to define a smaller, more manageable boundary as above.

3.4 The Nonminimality

Unless otherwise stated, G will be a connected semisimple group of rank r without compact factors, $\Gamma \subset G$ will always be a lattice. Let M be the locally symmetric space ${}_{\Gamma}\backslash G/_{K}$. The maximal boundary $B = G/P$ will be interpreted as the collection of asymptote classes of maximal flats in G/K. We will construct a Γ-invariant closed set E in $B \times B$ that is not G-invariant.

To find E, we pick a compact r-flat $\bar{F}$ in M and a lift F of it in G/K such that F is stabilised by an $\mathbb{R}$-hyperregular element $\gamma \in \Gamma$ (cf. 3.2 Proposition 2). For each chamber C of F, let P be the parabolic stabilising the asymptote class $[C]$ of F. Fix C and let $\mathcal{D}$ be the chamber opposite to C. Let E be the closure of the Γ-orbit of the pair $([C],[\mathcal{D}])$.

The next lemma generalizes the visibility property to the higher rank case:

Lemma 1: Any two points $p_1, p_2 \in B$ can be "joined" by a maximal flat F. More precisely, F has chambers C_1, C_2 such that $[C_i] \in p_i$. If C_1 and C_2 are opposite chambers then F is the unique flat joining p_1 and p_2.

Proof: Let P_1, P_2 be the isotropy subgroups of p_1, p_2 and choose an $x \in G$ such that $P_2 = x^{-1}P_1x$. Write $x = p_1' \cdot w \cdot p_1$ as in the Bruhat decomposition for P_1 and a maximal polar subgroup A

of P (we may assume that P_1 is a standard parabolic for A). As w normalises A, the polar subgroup $A' = p_1^{-1}Ap_1$ is contained both in P_1 and P_2. Clearly, both P_1 and P_2 are standard with respect to A', i.e., there are chambers C_1 and C_2 of A' such that P_1 and P_2 are the associated parabolic subgroups.

Let C_1 and C_2 be opposite, i.e., for some order of the root system C_1 is defined by the positive roots and C_2 by the negative roots. Hence, P_1 and P_2 are opposite parabolics. We see that being opposite is an intrinsic property of the points at infinity. Also, P_1 and P_2 intersect in their common Levi subgroup L. Let $L = M \cdot A$ be the decomposition into the compact and split parts (cf.A 10.3). Then A is the polar subgroup corresponding to the flat F. As any other flat F' joining p_1 and p_2 represents p_1 and p_2 by opposite chambers, A also is the polar subgroup corresponding to F'. By 3.1 Lemma 1, F and F' coincide. □

Lemma 2: G acts transitively on pairs of opposite points in G/P.

Proof: Let (p_1,p_2) and (q_1,q_2) be pairs of opposite points. Let F_1 and F_2 be the unique flats joining them. As any two flats are translates of each other (cf.A 5.2 Proposition 1(i)), there is $g \in G$ with $gF_1 = F_2$. Then (gp_1,gp_2) are pairs of opposite points joined by F_2. Use an element of the Weyl group to translate (gp_1,gp_2) to (q_1,q_2) and recall that it has a representative in G. □

Lemma 3: There are only countably many pairs of opposite points in E.

Proof: First, note that the flat joining opposite points depends continuously on these points: let (r,q) and (r',q') be pairs of opposite points close to each other. Then there is $g \in G$ close to the identity such that $gr = r'$. Clearly, if F and F' are the flats joining these points, $g^{-1}F'$ is close to F' and we may assume that $r = r'$. There is g' close to 1 in G such that $g' \cdot q = q'$. Decompose $g' = p_1 w p_2 w^!$ as in the decomposition $G = \bigcup PwPw^!$ with respect to the Weyl group of the flat F (derived from the Bruhat decomposition A 10.1), where $w^!$ is the element of the Weyl group that sends the positive roots to the negative roots. One sees that $w = w^!$ since otherwise $g \cdot q$ cannot be opposite to r. Since $w^! P w^! = \bar{P}$, the opposite parabolic, $g' \in P \cdot \bar{P}$. The map $P \times N^- \to G$ given by $(p, \bar{n}) \to p \cdot \bar{n}$ where N^- is the nilpotent radical of $\bar{P}$ is injective, open and continuous (cf. [Wa 1], the proof of Proposition 1.2.3.5). Hence, $g' = p \cdot \bar{n}$ with p close to the identity. As $q' = g'q = p \cdot q$, we see that $p \cdot F$ is the unique flat joining r to q' and $p \cdot F$ is close to F.

Suppose there are uncountably many pairs of opposite points in E. Then there is a nonconstant sequence of such points (x_n, y_n) converging to $(x,y) \in E$ where x and y are opposite. Let F be the unique flat through (x,y) and F_n the flat through (x_n, y_n). By the above, $F_n \to F$. As all the F_n cover the compact flat $\bar{F}$ in M, so does F. Pick a point p in F and a fundamental set $\mathcal{F}$ for Γ containing p. As $F_n \to F$, $F_n \cap \mathcal{F}$ is open in F_n for big n. As F_n and F project to $\bar{F}$, we see that $F_n \cap \mathcal{F} = F \cap \mathcal{F}$ is open on F. By analyticity of F_n and F, it is clear that $F = F_n$. This is the final contradiction. □

Lemmata 2 and 3 show that E is not G-invariant. By construction E is closed and Γ-invariant. Hence, we have shown the

<u>Proposition:</u> Let G and Γ be as above. Then Γ does not act G-minimally on $G/\mathbf{P} \times G/\mathbf{P}$ for P a minimal parabolic.

Chapter II

Section 1. The Geodesic Flow on Locally Symmetric Spaces

We will describe the geodesic flow φ_t on locally symmetric spaces in algebraic terms. This is originally due to Mautner in [Ma 1]. Then we calculate the topological and the metric entropy of φ_t and show that the Liouville measure is a measure of maximal entropy for φ_t on each ergodic component. This extends a result of Bowen (cf. [Bo 1], Theorem 3.1) in the rank one case.

1.1: The Action of G on the Unit Tangent Bundle of a Globally Symmetric Space

The decomposition of the unit tangent bundle $T_1(H)$ of $H = G/K$ by orbits of G as well as the appropriate decomposition of the Liouville measure are developed. We recall that every globally symmetric space H looks like G/K where G is semisimple and K is a maximal compact subgroup of G. Henceforth we assume that G does not have compact factors.

Given a point $(g \cdot K, X) \in T_1(H)$ we can translate it by g to $0 = 1 \cdot K \in G/K$. Let $(g \cdot K, X) = g \cdot (0, Y)$ for some $Y \in T_{1,0}(H)$. Recall from A12.1 that we may identify $T_{1,0}(H)$ with the unit sphere $\mathfrak{p}_1$ of $\mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Since K stabilises 0 and $\mathfrak{p} = \mathrm{Ad}(K) \cdot \mathfrak{a}$ by A5.2 Proposition 1 we may translate $(0, Y)$ to $(0, H_1)$ by some $k \in K$ where $H_1 \in \mathfrak{a}$. Since the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{a})$ acts transitively on the Weyl chambers (cf. A6.3 Theorem 1) and since each element of W is represented by an element of K (cf. A8.6) we may translate $(0, H_1)$ to some $(0, H)$

where H lies in the closure of the positive Weyl chamber for some fixed order of the root system of $(\mathfrak{g},\mathfrak{a})$ (recall that a Weyl chamber C is called positive if all positive roots take on positive values on C). This representation is unique: Suppose the contrary and let $g(0,H) = (0,H')$ where both H and H' lie in $\bar{C}$. Hence $g \in k$ as g fixes 0 and $gH = H'$. By A8.6 Proposition 2 there is a $w \in W$ such that $wH = H'$. By A6.3 Theorem 1 each W-orbit meets $\bar{C}$ exactly once and $H = H'$.

We summarize this discussion in

Lemma 1: The elements $\bar{C}_1$ of norm 1 in the closure of the positive Weyl chamber C viewed as a subset of the unit tangent bundle at 0 is a fundamental set for the action of G on T_1H, i.e. each orbit of G intersects $\bar{C}_1$ exactly once.

Next, we describe the Liouville measure μ on T_1H in terms of the Haar measure on G and see how μ decomposes under the action of G.

We first describe the canonical metric on T_1H:

Let $\pi:T_1H \to H$ be the projection and $K:TTH \to TH$ the connector map for the Riemannian connection on H. Then the canonical metric on TH is given by $\langle\xi,\eta\rangle = \langle d\pi\xi, d\pi\eta\rangle + \langle K\xi,K\eta\rangle$. The canonical metric on T_1H is given by restriction.

Now we define the Liouville measure μ as the volume induced by this metric. Let us note that μ is invariant under the geodesic flow ϕ_t introduced in the next section (cf. [Besl], p. 51 and [Bel], pp. 161 ff). As the canonical metric is invariant under $d\phi$ for any isometry ϕ of H

it is clear that μ is invariant under the action of G on $T_1 H$. Finally we can write $d\mu = d\sigma \otimes dx$ where $d\sigma$ is the canonical measure of the unit sphere $T_{o,1}$ in the Euclidean space $T_o H$ with the metric given by the Riemannian metric on H (cf. [Besl] p.52).

To decompose μ along the orbits of G we first normalize Haar measure on the orbits:

On the Lie algebra $\mathfrak{g}$ we have a canonical positive definite metric defined by

$$B_\theta(X,Y) = -B(X,\theta Y).$$

(this is clear since for $X = Y \in \mathfrak{k}$ $B_\theta(X,X) = -B(X,X) > 0$ and for $X = Y \in \mathfrak{p}$ $B_\theta(X,X) = -B(X,-X) > 0$ (cf. A8.3 Proposition 1). On G we can normalise Haar measure by requiring that the volume of a hypercube determined by an orthonormal basis with respect to B_θ have measure 1. This doesn't depend on the choice of orthonormal basis since orthogonal matrices have determinant ± 1. One can check that this is also independent of the choice of Cartan involution θ (basically since any two Cartan involutions are conjugate). On the orbits themselves we simply choose the Haar measure that comes from the normalised Haar measure on G. We denote the measure on the orbit of $H \in \overline{C}$ by μ_H. Furthermore, we let λ be Lebesque measure on $\mathfrak{a}$ restricted to $\overline{C}_1$.

Since μ, μ_H and λ are all smooth measures we can decompose μ into

$$\mu = \int_{H \in \overline{C}_1} f(H) \mu_H d\lambda(H)$$

where f(H) is smooth weighting function.

Without further work we get

Lemma 2: Measure theoretically, T_1H splits into a direct product

$$T_1 H = G/M \times C_1$$

where M = centraliser of $\mathfrak{a}$ in K.

Proof: With respect to Lebesque measure the open Weyl chamber C has full measure in its closure. For $H \in C$ the isotropy group of $(1 \cdot K, H) \in TH$ is the centraliser $H \cap K$. Clearly,

$$\text{centraliser } H = \sum_{\substack{\alpha\in\Sigma \\ \alpha(H)=0}} \mathfrak{g}^{\alpha} + \mathfrak{a} + \mathfrak{m} = \mathfrak{a} + \mathfrak{m}$$

(cf. A8.3 Proposition) where Σ is the root system of $(\mathfrak{g}, \mathfrak{a})$. In particular, the isotropy group of $(1 \cdot K, H)$ is M and our claim is clear. □

Now we determine the weighting function $f(H)$ explicitly:

Lemma 3: Up to a constant multiple

$$f(H) = \prod_{\alpha\in\phi_+} \alpha(H)$$

where ϕ_+ are the nonimaginary positive roots (cf. A7.1).

Proof: We first decompose μ along the fibers of the unit tangent bundle over H:

$$\mu = \int_H \mu_x dx$$

where dx is the Riemannian volume on H and μ_x is the fiber measure.

Since $d\mu = d\sigma \otimes dx$ we see that $\mu_{1.K}$ is the Riemannian volume on $T_{1,1.K} = p$ given by restriction of the Cartan-Killing form B to the unit sphere p_1 of p.

Now consider the decomposition of $T_1 H$ into orbits of G. On $T_{1,1\cdot K} H = \mathfrak{p}$ this induces the decomposition into K-orbits of $\mathfrak{p}$. We claim that $\mu_1 = \int_{\bar{C}_1} f(H)\mu_{1,H} \times d\lambda(H)$ where $\mu_{1,H}$ is Haar measure on the K-orbit through $(1\cdot K,H)$. Again we normalise $\mu_{1,H}$ by comparing it with the Cartan-Killing form. Note that the Cartan-Killing form of $\mathfrak{k}$ is the restriction of the Cartan-Killing form on $\mathfrak{g}$ to $\mathfrak{k}$.

Indeed, we do have $\mu_1 = \int_{\bar{C}_1} g(H)\mu_{1,H} d\lambda(H)$ for some weighting function $g(H)$. Hence

$$\mu = \int_H x_* \mu_1 dx = \int_H \int_{\overline{\mathcal{C}}_1} g(H) x_*(\mu_{1,H}) d\lambda(H) dx$$

$$= \int_{\overline{\mathcal{C}}_1} g(H) \left(\int_H x_*(\mu_{1,H}) dx \right) d\lambda(H)$$

$$= \int_{\overline{\mathcal{C}}_1} g(H) \mu_H d\lambda(H).$$

because by our normalizations $\mu_H = \int_H x_*(\mu_{1,H}) dx$. We deduce that $g(H) = f(H)$ and derive the following description: The weighting function $f(H)$ is the volume of the orbit $AdK(H)$ in the Euclidean metric on $\mathfrak{p}$ defined by the Cartan-Killing form B.

We have the following commutative diagram where Exp is the exponential map: $\mathfrak{p} = T_{1 \cdot K} H \to H$ in the differential-geometric sense and $\exp: \mathfrak{g} \to G$ is the exponential map from group theory: For $H \in \overline{\mathcal{C}}$

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\ \mathrm{Exp}\ } & G/K \\ \uparrow & & \uparrow \\ AdK(H) & \xrightarrow{\ \mathrm{Exp}\ } & K \cdot \exp H \cdot K/K. \end{array}$$

This allows us to compute the volume of $AdK(H)$ from the volume of $K \exp H \cdot K/K$ (Lemma 5) and knowledge of the derivative of the exponential map (Lemmata 6 and 7).*

Lemma 5: The volume of $K \cdot \exp H \cdot K/K$ is (up to a constant independent of H)

$$m(\exp H) = \prod_{\alpha \in \phi_+} e^{\alpha(H)} - e^{-\alpha(H)} = 2^{\operatorname{card} \phi_+} \prod_{\alpha \in \phi_+} \sinh \alpha(H).$$

* Notice that $\det(\mathrm{Exp}|_{AdK(H)}) = d(\mathrm{Exp})|_{AdK(H)}$ since Exp is an isometry on $\mathfrak{p}$ and $\mathfrak{p}$ is transversal to $AdK(H)$.

Proof: This is clear from the formula:

$$\int_G f(x)d_G(x) = \int_K\int_C\int_K f(k_1 \exp H\ k_2)m\ (\exp H)dk_1 d\lambda(H)dk_2$$

where f is a continuous compactly supported function and $d_G(x)$ and dk are Haar measures on G and K respectively. This can be found in [Wa 1], vol. 2, 8.1.3.1, p. 68. Of course one only has to check that the measure defined by this formula is G-invariant. The commutation relations account for the m weighting function. □

Lemma 6: The differential of the exponential map Exp is:

$$d\mathrm{Exp}_X = d\tau(\exp X)_{1\cdot K} \cdot \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(\mathrm{ad}X)^{2n}\Big|_{\mathfrak{p}}$$

where $\tau(g): H \to H$ is the mapping $xk \to gxk$.

Proof: This is [He 1], Chapter IV, Theorem 4.1. □

Now we can calculate the determinant of $d\mathrm{Exp}_H$, $H \in \mathfrak{a}$, with respect to the Riemannian volumes determined by the Cartan-Killing form B on both $\mathfrak{p}$ and TH.

Lemma 7: $\det d\mathrm{Exp}_H = \prod_{\alpha\in\phi_+} \frac{\sinh(\alpha(H))}{\alpha(H)}$.

Proof: Since left translations on H are isometries we have to calculate $\det \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(\mathrm{ad}H)^{2n}\Big|_{\mathfrak{p}}$. Instead we complexify and calculate $\det \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(\mathrm{ad}H)^{2n}\Big|_{\mathfrak{p}_{\mathbb{C}}}$. For $\mathfrak{p}_{\mathbb{C}}$ we have the decomposition (cf. A8.5 Proposition)

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} + \sum_{\alpha\in\phi_+} \mathbb{C}(X_\alpha - \theta X_\alpha).$$

which is the eigenspace decomposition for $(\mathrm{adH})^2$: namely,

$$(\mathrm{adH})^2|_{\mathfrak{a}_{\mathbb{C}}} = 0$$

$$(\mathrm{adH})^2(X_\alpha - \theta X_\alpha) = \alpha(H)^2(X_\alpha - X_\alpha).$$

because H is fixed by θ. Now the lemma is obvious. □

In particular we see that this determinant is constant along AdK-orbits. By change of variables and Lemmata 5 and 7 this finishes the proof of Lemma 3. □

1.2: The Geodesic Flow

We describe the geodesic flow on a finite volume locally symmetric space in algebraic language. It turns out that the ergodic components are given by the G-orbits on $T_1 H$. Most of this is due to Mautner in [Mau 1]. Mainly we parametrize the ergodic components differently and deal with the case of a reducible lattice in all detail. First we recall the definition of the geodesic flow for an arbitrary Riemannian manifold M. We will not explain any of the standard terms of differential geometry (cf. [He 1], [Ko-No 1]). Let $(x,X) \in T_1 M$. There is a unique geodesic α passing through x in the direction of X. In terms of the exponential map $\mathrm{Exp}_x\colon T_x M \to M$ α is given by $\mathrm{Exp}\, tX$. Moreover, this gives the unit speed parametrization of α. Now we can define the geodesic flow φ_t by:

$$\varphi_t(x,X) = (\mathrm{Exp}_x tX, \frac{d}{ds} \mathrm{Exp}_x sX\Big|_{s=t}).$$

It is clear that $\varphi_t(x,X) \in T_1 M$ again.

Now consider a globally symmetric space $\mathcal{H} = G/K$ as above. We assume that H does not have any flat or compact factors (mainly for simplicity of the exposition). For $H \in \overline{\mathcal{C}}_1$ we calculate the orbit of the geodesic flow through $(0,H) \in T_1 H$: By definition $\varphi_t(0,H) = (\mathrm{Exp}\, tH, \frac{d}{ds} \mathrm{Exp}\, sH|_{s=t})$ where $\mathrm{Exp} = \mathrm{Exp}_{1 \cdot K}: \mathfrak{p} \to \mathcal{H}$. Using the chain rule and 1.1 Lemma 6

$$\frac{d}{ds} \mathrm{Exp}\, sH\Big|_{s=t} = d\mathrm{Exp}_{tH}\left(\frac{d}{ds}(sH)\Big|_{s=t}\right)$$

$$= d\mathrm{Exp}_{tH}(H) = d\tau(\exp tH)_0 \cdot \sum_{n=0}^{\infty} \frac{(\mathrm{ad}\, tH)^{2n}}{(2n+1)!}(H)$$

$$= d\tau(\exp tH)_0(H).$$

As $d\tau(\exp tH)_0$ is the action of $\exp tH$ on $T_1\mathcal{H}$ restricted to the unit tangent space at $1 \cdot K$ we finally arrive at the

Lemma 1: $\varphi_t(0,H) = (\exp tH)(0,H)$.

As a corollary of this we have the

Lemma 2: The geodesic flow fixes every G-orbit on $T_1\mathcal{H}$. Moreover, if we write $G(0,H) = G/Z(H) \cap K$ as a homogeneous space where $Z(H)$ is the centraliser of H the geodesic flow on $G(0,H)$ is given by:

$$g(Z(H) \cap K) \mapsto g \exp tH(Z(H) \cap K).$$

Proof: For any manifold M the geodesic flow φ_t commutes with the differentials of isometries of M as is obvious from the definition. Let (x,X) lie on the G-orbit of $(0,H)$: $(x,X) = g(0,H)$ for some $g \in G$. Then

$\varphi_t(x,X) = g\varphi_t(0,H) = g \cdot (\exp tH) \cdot (0,H)$ lies in the same orbit. This formula also proves the second claim of the lemma. □

Now we want to study the geodesic flow on locally symmetric spaces M of finite volume. Recall from A12 that the universal cover of M is a globally symmetric space $H = G/K$. In fact, we can write $M = \Gamma\backslash H$ where is a torsion free lattice in G.

Let $p: H \to M$ be the covering projection. Then it is clear that p intertwines the geodesic flows on $T_1 H$ and $T_1 M$. Also $T_1 M = \Gamma\backslash T_1 H$ is "foliated" by the G-orbits on $T_1 H$ factored out by Γ on the left (as the dimension of the G-orbit changes as $H \in \overline{C}_1$ moves out to a wall this is only a foliation with singularities). The leaves are $\Gamma\backslash G/Z(H) \cap K$. In particular, all the leaves are smooth submanifolds of $T_1 M$ of finite volume. As the geodesic flow φ_t acts on $G/Z(H) \cap K$ by right translations (Lemma 2) we see that the geodesic flow φ_t^M of M restricted to $\Gamma\backslash G/Z(H) \cap K$ acts by

$$\varphi_t^M(\Gamma \cdot g \cdot (Z(H) \cap K)) = \Gamma \cdot g \cdot \exp tH \cdot (Z(H) \cap K).$$

At this point Mautner proved a lemma on the ergodicity of the geodesic flow on most of these orbits. Later on C.C. Moore generalised this so called Mautner's Lemma to the

<u>Theorem 1</u>: Let G be a non-compact connected semisimple group with finite center but without compact factors and let $\Gamma \subset G$ be an irreducible lattice (cf. A12). Then a subgroup H of G is ergodic on $\Gamma\backslash G$ iff $\overline{H}$ is not compact.

Proof: The easiest proof of this theorem is in [H-M1]. There one first proves that for any unitary representation of G the "matrix coefficients vanish at infinity". Moore's theorem then becomes an obvious corollary (cf. also [Zi 2], the discussion of Theorems 2.3 and 2.4.). □

We want to apply the preceding theorem to our situation. For simplicity we assume first that Γ is irreducible in G. Let $\psi_t^H: \Gamma\backslash G \to \Gamma\backslash G$ be right translation by $\exp tH$. By Lemma 2 we obtain the commutative diagram:

$$\begin{array}{ccc} \Gamma\backslash G & \xrightarrow{\psi_t^H} & \Gamma\backslash G \\ \downarrow & & \downarrow \\ \Gamma\backslash G/Z(H)\cap K & \xrightarrow{\varphi_t^M} & \Gamma\backslash G/Z(H)\cap K . \end{array}$$

By Moore's theorem ψ_t^H is ergodic, in particular φ_t on $\Gamma\backslash G/Z(H) \cap K$ is also ergodic with respect to Haar measure or conditional Liouville measure.

In general we decompose Γ into irreducible lattices as in A12, Proposition 3. As in the proof of Lemma 3 we may assume that G is the adjoint group. In particular, $G = \Pi G_i$ is a direct product of adjoint groups and w.l.o.g.* $\Gamma = \Pi\Gamma_i$ where $\Gamma_i = \Gamma \cap G_i$ (cf. A12, Proposition 3). We see that $M = \Pi M_i$ is a Riemannian product of the spaces $M_i = \Gamma_i\backslash H_i$ where H_i is the globally symmetric space associated to G_i. Furthermore, any $H \in \overline{C}$ splits up into a sum $H = \sum_{i=1}^{n} H_i$ where H_i belongs to the positive Weyl chamber or its closure of some G_i. Hence $G(o,H) = \Pi G_i(o,H_i)$.

*as any Γ lies in between special Γ's all our claims follow easily from the special case.

If all the $H_i \neq 0$ then φ_t restricted to $\Gamma_i \backslash G_i(o,H_i)$ is ergodic for all t by Moore's theorem. The ergodicity of all single transformations of a flow implies the weak mixing of the flow as is clear from spectral theory: By [Fu 1] Theorem 4.30 weak mixing is equivalent to the non-existence of eigenfunctions. If f is an eigenfunction: $\varphi_t f = e^{2\pi i\lambda t} f$ then f is invariant under $\varphi_{\frac{1}{\lambda}}$ and hence constant. By [Fu 1] Proposition 4.4 any product of φ_t restricted to $\Gamma_i \backslash G_i(o,H_i)$ with an ergodic flow is ergodic. Hence all G-orbits $G(o,H)$ with none of the $H_i = 0$ are ergodic components.

If some H_i are 0 write $G = G_0 \times G'$ where the G_0-component of H is 0 and G' is the product of the remaining factors. Then the previous argument applies to the G'-factor. As right translation by $\exp tH$ doesn't affect the G_0-coordinate it is clear that the G-orbit of (o,H) splits up into G'-orbits each of which is an ergodic component.

We may summarize this discussion as follows:

<u>Theorem 2 (Mautner)</u>: Let $G = \prod_{i=1}^{n} G_i$, $\Gamma \supset \prod_{i=1}^{n} \Gamma_i$ as above. for any $H \in \overline{C}_1$ such that no $H_i = 0$ $\Gamma\backslash G(o,H)$ is an ergodic component of φ_t. If some $H_i = 0$ write $G = G_0 \times G'$ as above. Then all $\Gamma'\backslash G'(g_0 \cdot k,H)$ are ergodic components for φ_t where $g_0 \in G_0$ is arbitrary and Γ' is the intersection of Γ with G'.

<u>Corollary</u>: Almost all G-orbits on $T_1 H$ are ergodic components of φ_t.

Proof: Obvious. □

1.3: Lyapunov Exponents and Entropy

We will calculate the Lyapunov exponents of the geodesic flow and determine the metric and topological entropy. In more detail, we first recall the definition and fundamental properties of the Lyapunov exponents of a flow. We introduce Jacobi fields and relate them to the double tangent space. This reduces the calculation of the Lyapunov exponents to the determination of the asymptotic exponential growth rates of the Jacobi fields. These growth rates define a filtration of the space of Jacobi fields. To determine it we use a suitable basis of the Jacobi fields given explicitly in terms of the root system. From this and the general properties of the Lyapunov exponents we can then calculate the metric and topological entropy. In particular, this will prove that the Liouville measure is a measure of maximal entropy on the ergodic components.

We first recall the definition of the Lyapunov exponents of a diffeomorphism or flow on a compact manifold N. We fix a Riemannian metric on N. As will be clear the Lyapunov exponents are independent of the metric we choose. We let $\| \|$ denote the norm induced by the metric on each tangent space of N. Then we have the

<u>Definition 1</u>: The upper Lyapunov exponent for a C^1-flow φ_t on N is the function $\chi^+ : TN \to \mathbb{R}$ defined by:

$$\chi^+(v) = \overline{\lim_{t\to\infty}} \frac{\log\|d\varphi_t v\|}{t} .$$

The upper Lyapunov exponents have the following properties:

a On each tangent space T_xN there are at most $\dim N$ many values of χ^+, say $\chi_1(x) < \chi_2(x) < \cdots < \chi_{r(x)}(x)$.

b There is a filtration

$$L_1(x) < L_2(x) < \cdots < L_{r(x)}(x) = T_xN$$

given as follows:

$$\text{if } v \in L_i(x) \setminus L_{i-1}(x) \text{ then } \chi^+(v) = \chi_i(x).$$

We call the $\chi_i(x)$ the upper Lyapunov exponents of φ_t at the point x and say that $\chi_i(x)$ has multiplicity $\dim L_i(x) - \dim L_{i-1}(x)$.

c Let μ be a φ_t-invariant probability measure. Then for μ-almost every point x the

$$\lim_{t\to\infty} \frac{\log\|d\varphi_t(v)\|}{t}$$

exists for all $v \in T_xN$.

For such an x we simple speak of the Lyapunov exponents at x.

Moreover, for μ-almost every x the Lyapunov exponents $\overline{\chi}_i(x)$ of the inverse flow $\varphi_{-t}(x)$ are simply $-\chi_{r(x)-i}(x)$ with the same multiplicities.

d The upper Lyapunov exponents are measurable functions invariant under φ_t. In particular, if μ is an ergodic φ_t-invariant measure then the Lyapunov exponents are constant μ-almost everywhere.

e (Pesin's entropy formula). Let $\chi(x) = \sum_{\chi_i(x)>0} \chi_i(x)(\dim L_i(x) - \dim L_{i-1}(x))$ be the sum of the positive Lyapunov exponents. Then the metric entropy $h_\mu(\varphi_1)$ for any C^2-flow φ_t and any smooth φ_t-invariant probability measure μ is given by:

$$h_\mu(\varphi_1) = \int_N \chi(x)d\mu.$$

f For any C^1-flow φ_t and any φ_t-invariant probability measure μ we have

$$h_\mu(\varphi_1) \leq \int_N \chi(x)d\mu.$$

This was first due to Margulis for the case of a smooth measure. Then Ruelle gave this generalization in [Ru 1].

General references for this material are [Pe 1,2] and [M 1].

In the case of the geodesic flow the most convenient way to calculate the Lyapunov exponents is to use Jacobi fields. We recall the

Definition 2: A Jacobi field Y along a geodesic c is a vector field along c satisfying the so called Jacobi equation

$$Y'' + R_{XY}X = 0$$

where $X = \dot{c}(t)$, $'$ is covariant differentiation along c and R is the curvature tensor.

Note: Geometrically, Jacobi fields come about by variations of the geodesic c, i.e. let c^s be a one-parameter family of geodesics such that $c = c^0$. Then $Y(t) = \frac{d}{ds} c^s(t)\Big|_{s=0}$ is a Jacobi field.

Let $v \in TM$ and let c be the geodesic on M defined by v. Then we have an isomorphism between $T_v(TM)$ and Jacobi fields along c as follows: for any Jacobi field $Y(t)$ along c let $\xi = (v, Y(0), Y'(0))$ be the corresponding point in $T_v(TM)$. Moreover, one finds that

$$\|d\varphi_t(\xi)\|^2 = \|Y(t)\|^2 + \|Y'(t)\|^2.$$

(cf. [Eb 3]).

In the following we will mainly consider Jacobi fields perpendicular to c as we are only interested in the exponential growth rate of $d\varphi_t(\xi)$.

For locally symmetric manifolds we can write down the Jacobi fields explicitly. Recall that we can identify the tangent space at the identity with $\mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition. We first have to describe a basis for $\mathfrak{p}$ in terms of the root structure. Recall from A7.4 that the real structure of $\mathfrak{g}$ induces an automorphism σ of the root system ϕ of $\mathfrak{g}_{\mathbb{C}}$. The Cartan involution θ induces another automorphism of ϕ given by $\alpha^\theta(H) = \alpha(\theta H)$ for $H \in \mathfrak{h}_{\mathbb{C}}$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ as in 7.4. Then we have the

Lemma 1: Let $\phi_+ = \{\alpha > 0 | \alpha \neq \alpha^\theta\}$ and $\phi_- = \{\alpha > 0 | \alpha = \alpha^\theta\}$ (cf. A7.1). Then the following properties hold:

(i) If $\alpha \in \phi_+$ then $-\alpha^\theta \in \phi_+$ and $\alpha^\sigma \in \phi_+$.

(ii) If $\alpha \in \phi_-$ then $\alpha^\theta = \alpha$, $\alpha^\sigma = -\alpha$ and

$\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} \subset \mathfrak{k}_{\mathbb{C}}$.

Proof: This is [He 1] Chapter VI Lemma 3.3. □

Given a Weyl basis $X_\alpha, \alpha \in \phi$ for $\mathfrak{g}_{\mathbb{C}}$ we let $\sigma X_\alpha = k_\alpha X_{\alpha^\sigma}$ for $k_\alpha \in \mathbb{C}$.

<u>Lemma 2</u>: We can choose $E_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \phi$ that satisfy the properties:

(i) $[E_\alpha, E_{-\alpha}] = H_\alpha$

(ii) If $\alpha = \alpha^\sigma$ then $\sigma E_\alpha = E_\alpha \in \mathfrak{g}$. We call such a root <u>real</u>

(iii) If $\alpha \in \phi_+$ and $\alpha \neq \alpha^\sigma$ then $\sigma E_\alpha = E_{\alpha^\sigma}$.

Proof: Let τ be the complex conjugation over the compact real form $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Let X_α be a Weyl basis with respect to τ, i.e. $\tau X_\alpha = -X_{-\alpha}$ (cf. [Wa 1], vol. 1, p. 25). Since $\theta = \sigma\tau = \tau\sigma$ we see that $\overline{k}_\alpha = k_{-\alpha}$.

Next notice that $|k_\alpha| = 1$ for α real since

$$X_\alpha = \sigma^2 X_\alpha = \sigma(k_\alpha X_\alpha) = \overline{k}_\alpha k_\alpha X_\alpha.$$

Then for α real let $a_\alpha \in \mathbb{C}$ such that $a_\alpha^2 = k_\alpha$ and let $a_{-\alpha} = \overline{a}_\alpha$. Then $a_{-\alpha}^2 = k_{-\alpha}$ by the above. For α real let $E_\alpha = a_\alpha X_\alpha$. Then (i) and (ii) are clearly satisfied.

If $\alpha \in \phi_+$, $\alpha \neq \alpha^\sigma$ then α^σ is a positive root by Lemma 1. We may let $E_\alpha = X_\alpha$, $E_{-\alpha} = X_{-\alpha}$, $E_{\alpha^\sigma} = \sigma X_\alpha$, $E_{-\alpha^\sigma} = \sigma X_{-\alpha}$. Then $[E_{\alpha^\sigma}, E_{-\alpha^\sigma}] = \sigma[E_\alpha, E_{-\alpha}] = \sigma H_\alpha = H_{\alpha^\sigma}$ and (i) holds true. □

For $\alpha \in \phi_+$ let

$$T_\alpha = (E_\alpha + \sigma E_\alpha) - \theta(E_\alpha + \sigma E_\alpha).$$

and

$$W_\alpha = i(E_\alpha - \sigma E_\alpha) - \theta(i(E_\alpha - \sigma E_\alpha)).$$

It is clear that $W_\alpha = 0$ for α real and that $T_\alpha = T_{\alpha^\sigma}$ and $W_\alpha = W_{\alpha^\sigma}$. In particular the non-zero T_α, W_α are in 1-1 correspondence with the set of pairs (α, α^σ), $\alpha \in \phi_+$.

Lemma 3: For all $\alpha \in \phi_+$ both $T_\alpha \in \mathfrak{p}$ and $W_\alpha \in \mathfrak{p}$. Moreover, the set $\{T_\alpha, W_\alpha | W_\alpha \text{ non-zero}\}$ is linearly independent over $\mathbb{R}$ and together with $\mathfrak{a}$ generates $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{a} + \sum (\mathbb{R}T_\alpha + \mathbb{R}W_\alpha).$$

where the summation is over one representative of each pair (α, α^σ) for $\alpha \in \phi_+$.

Proof: To see that $T_\alpha \in \mathfrak{p}$ we have to check that

(a) $\sigma T_\alpha = T_\alpha$: this is clear since $\sigma\theta = \theta\sigma$

and

(b) $\theta T_\alpha = -T_\alpha$: this is obvious.

The same remarks apply to W_α.

By Lemma 1 the T_α, W_α are linearly independent if $T'_\alpha = (E_\alpha + \sigma E_\alpha)$ and $W'_\alpha = i(E_\alpha - \sigma E_\alpha)$ are linearly independent (since these are the projections of T_α, W_α to $\sum_{\alpha>0} \mathfrak{g}^\alpha$). Any dependence between T'_α and W'_α clearly is of the form: $aT'_\alpha + bW'_\alpha = 0$ for some particular $\alpha \in \phi_+$. It is obvious that there is no dependence over $\mathbb{R}$.

Recall from A8.5 that

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} + \sum_{\alpha\in\phi_+} \mathbb{C}(E_\alpha - \theta E_\alpha).$$

As $T_\alpha - iW_\alpha = 2(E_\alpha - \theta E_\alpha)$ and $T_\alpha + iW_\alpha = 2(E_{\alpha^\sigma} - E_{\alpha^\sigma})$ the last claim becomes obvious. □

Now we can describe a canonical set of Jacobi fields. As always let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ contained in $\mathfrak{h}$.

Lemma 4: Let $H \in \mathfrak{a}$. The space of Jacobi fields along the geodesic $\operatorname{Exp} tH$ admits the following basis:

$$T_\alpha(t) = \exp(tH)_*(e^{\alpha(H)t}T_\alpha)$$

$$T_{-\alpha}(t) = \exp(tH)_*(e^{-\alpha(H)t}T_\alpha)$$

$$W_\alpha(t) = \exp(tH)_*(e^{\alpha(H)t}W_\alpha)$$

$$W_{-\alpha}(t) = \exp(tH)_*(e^{-\alpha(H)t}W_\alpha)$$

$$H_\beta(t) = \exp(tH)_*(H_\beta)$$

$$H_{-\beta}(t) = \exp(tH)_*(tH_\beta)$$

where the indices α run over one representative of each pair (α,α^σ) with $\alpha \in \phi_+$ as above and the β are a set ψ of simple roots for the root system Σ of the pair $(\mathfrak{g},\mathfrak{a})$ (cf. A7.4).

Proof: First we check that our vector fields are Jacobi fields. We only check $T_{-\alpha}$ and $H_{-\beta}$ as the others are perfectly similar:

First recall that parallel translation along $\operatorname{Exp} tH$ is given by $(\exp tH)_*$. This is [He 1] Chapter IV Theorem

3.3 (iii). Then for any curve $Z(t) \in \mathfrak{p}$ we claim that

$$((\exp tH)_* Z(t))'\Big|_{t=s} = (\exp sH)_* \frac{d}{dt} Z(t)\Big|_{t=s}).$$

In fact, from [He 1] Chapter I Theorem 7.1 we see that

$$((\exp tH)_*^! Z(t))'\Big|_{t=s} = \lim_{\Delta t\to 0} \frac{1}{\Delta t}((\exp(-\Delta t\, H))_*(\exp(s+\Delta T)H)_* Z(s+\Delta t)$$

$$- (\exp sH)_* Z(s)) = (\exp sH)_* (\frac{d}{dt} Z(t)\Big|_{t=s}).$$

$\underline{a}$ Check of the Jacobi equation for $T_{-\alpha}(t)$:

By the above covariant differentiation gives

$$(T_{-\alpha})''(t) = (e^{tH})_*(\alpha(H)^2 e^{-t\alpha(H)} T_\alpha) = \alpha(H)^2 T_{-\alpha}(t).$$

To evaluate the curvature tensor we recall Theorem 4.2 from [He 1], Chapter IV:

Lemma 5: At the point $0 \in G/K$ and $X,Y,Z \in \mathfrak{p}$ we have

$$R_{X,Y}Z = -[[X,Y],Z].$$

In our case we find that at the origin

$$R_{H,T_\alpha}H = -[[H,T_\alpha],H] = -\alpha(H)^2 T_\alpha.$$

This is clear from the definition of T_α and since H is fixed by both σ and θ. Since e^{tH} is an isometry $(e^{tH})_*$ commutes with the curvature tensor. As $T_{-\alpha}(0) = T_\alpha$ we get

$$R_{(\exp tH)_*(H),T_{-\alpha}(t)} \exp(tH)_*(H) = -\alpha(H)^2 T_{-\alpha}(t).$$

Comparison with the expression from the covariant differentiation proves our claim.

<u>b</u> Check of the Jacobi equation for $H_{-\beta}(t)$:

Clearly we have

$$(H_{-\beta})''(t) = \exp(tH)_*(0) = 0$$

On the other hand,

$$R_{H,H_{-\beta}(0)}H = -[[H,0],H] = 0$$

as $H_{-\beta}(0)$ is 0. Our claim is clear.

That the given Jacobi fields are linearly independent is clear from Lemma 3 and the obvious fact that the $H_\beta(t)$ and $H_{-\beta}(t)$ are linearly independent.

Again from Lemma 3 it is clear that the dimension of the space generated by the given Jacobi fields is $2 \cdot \dim \mathfrak{p} = 2 \dim H = \dim T_{(0,H)}TH = \dim\{$Jacobi fields along Exp tH$\}$. □

Now we can calculate the Lyapunov exponents of a Jacobi field and determine the filtration they define

<u>Lemma 6</u>: Let $J(t)$ be a Jacobi field along Exp tH for $H \in C$, the positive Weyl chamber. Then

$$J(t) = \sum_{\pm(\alpha,\alpha^\sigma)\in\phi_+} a_\alpha T_\alpha(t) + b_\alpha W_\alpha(t) + \sum_{\pm\beta\in\psi} C_\beta H_\beta$$

where the indexing is as above. Let $\alpha_J(H)$ be the biggest of the numbers $\{\alpha(H))|\alpha$ as above$\}$. Then the Lyapunov exponent of $J(t)$ is

$$\chi^+(J) = \begin{cases} \alpha_J(H) & \text{if } \alpha_J(H) > 0 \\ 0 & \text{if } \alpha_J(H) < 0 \text{ and some} \\ & \qquad < \beta \neq 0 \\ \alpha_J(H) & \text{otherwise.} \end{cases}$$

<u>Note</u>: By the Lyapunov exponent of a Jacobi field J we mean the Lyapunov exponent of the corresponding vector ξ in $T_{(0,H)}T\mathcal{H}$.

Proof: Recall from our discussion of Jacobi fields that

$$\|d\varphi_t(\xi)\|^2 = \|J(t)\|^2 + \|J'(t)\|^2$$

where ξ corresponds to J in $T_{(0,H)}T\mathcal{H}$. To calculate $\|J(t)\|^2$ and $\|J'(t)\|^2$ notice that any two T_α, T_β and W_α, W_β are orthogonal $(\beta \neq \alpha, \alpha^\sigma)$ since this is true for the E_α, E_β's with respect to the Cartan-Killing form (cf. A3.2 Theorem (ii)). For a given α we have

$$\lim_{t\to\infty} \frac{1}{t} \log \|a_\alpha T_\alpha(t) + b_\alpha W_\alpha(t)\| = \lim_{t\to\infty} \frac{1}{t} \log e^{\alpha(H)t} \cdot \|a_\alpha T_\alpha + b_\alpha W_\alpha)\|$$

$$= \alpha(H)$$

since $\exp tH$ is an isometry. Similarly we find that

$$\lim_{t\to\infty} \frac{1}{t} \log \|a_\alpha T'_\alpha(t) + b_\alpha W'_\alpha(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \alpha(H) e^{\alpha(H)t} \|a_\alpha T_\alpha + b_\alpha W_\alpha\|$$

$$= \alpha(H).$$

Finally it is clear that the H_β's contribute 0 growth and that they are orthogonal to the rest of the space. Putting all these facts together proves the lemma. □

In particular, the Lyapunov exponents are defined everywhere since any point in $T_1 H$ is an isometric translate of $(0,H)$ for some $H \in \mathfrak{C}$.

We really want to calculate the Lyapunov exponents and the entropies on an ergodic component of φ_t. The next lemma describes the Jacobi fields tangent to an orbit of G.

Lemma 7: The tangent space $T_{(0,H)}(\{0\} \times \mathfrak{a})$ is orthogonal to $G \cdot (0,H)$ and corresponds to the Jacobi fields of the form $\sum_{-\beta \in \psi} a_\beta H_\beta(t)$.

Proof: Since $H_\beta(0) = 0$ for $\beta < 0$ the specified Jacobi fields are tangent to $\{0\} \times \mathfrak{a}$. By a dimension argument they also span $T_{(0,H)} | \{0\} \times \mathfrak{a})$.

Take any curve $\exp tX \in G$, $X \in \mathfrak{g}$. Then

$$\frac{d}{dt}((\exp tX)_* H) = [X,H] + \mathfrak{k} \in T(\{0\} \times \mathfrak{p})$$

as follows from the Campbell-Hausdorff-Dynkin formula (cf. [Va 1], Chapter I). Since $B([X,H],H) = B(X,[H,H]) = 0$ (as follows from the invariance of B under inner automorphisms) $G \cdot (0,H)$ is orthogonal to $T_{(0,H)}(\{0\} \times \mathfrak{a})$. □

Also we really have to work with $T_{(0,H)} T_1 H$ for $H \in C_1$. For a Jacobi field $J(t)$ along $\mathrm{Exp}\, tH$ this means that $J'(0) \perp H$. Expressing $J(t)$ as in Lemma 6 this is equivalent with

$$\sum_{-\beta \in \psi} c_\beta H_\beta \perp H.$$

By Lemma 7 this condition holds true for all $J(t)$ tangent to $G(0,H)$. Also, if we just want to calculate $\chi(0,H)$, the sum of the positive Lyapunov exponents on $G(0,H)$, by Lemma 7 and the above we may just as well add the positive Lyapunov exponents of all Jacobi fields along $\operatorname{Exp} tH$. By Lemma 6 it is clear that the $\chi_i(0,H)$ are just 0 and the $\alpha(H)$ for $\pm\alpha \in \phi_+$ with multiplicity the multiplicity of $\alpha(H)$ as $\pm\alpha$ runs over ϕ_+. In particular, we obtain the

Lemma 8: The sum of the positive Lyapunov exponents at $(0,H)$ for $H \in C_1$ is

$$\chi(0,H) = \sum_{\alpha\in\phi_+} \alpha(H).$$

Moreover, if $g \in G$ then $\chi(g(0,H)) = \chi(0,H)$.

Proof: The first claim follows from the discussion above. For the latter just notice that g is the differential of an isometry. □

Finally, we obtain the

Proposition: For a uniform lattice Γ in G,Γ without torsion, the metric and topological entropy of the geodesic flow on $\Gamma\backslash G\,(0,H)$ are equal for any $H \in C_1$ and given by

$$h_H = h_{\mu,H} = \sum_{\alpha\in\phi_+} \alpha(H) = 2\rho(H)$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$. Furthermore, we find expressions for the topological and metric entropy of the geodesic flow on $T_1(\Gamma\backslash G/K)$ as follows:

$$h = 2\|\rho\| = 2\max_{H\in C_1} \rho(H)$$

and

$$h_\mu = \int_{H \in C_1} 2\,\rho(H) \cdot \prod_{\alpha \in \phi_+} \alpha(H)\,d\lambda_1(H)$$

where λ_1 is a Lebesgue measure on C_1 such that $\prod_{\alpha \in \phi_+} \alpha(H)\,d\lambda_1(H)$ is a probability measure. In particular, $h = h_\mu$ iff G has real rank 1.

Proof: Pesin's formula proves that $h_{\mu,H} = 2\rho(H)$. On the other hand, $h_H = \sup h_\nu$ where ν runs over all the probability measures on $\Gamma\backslash G(0,H)$. By Ruelle's inequality (cf. Property f of the Lyapunov exponents) we see that

$$h_\nu \leq \int_{G(0,H)} \chi(x)\,d\nu = 2\rho(H)$$

since $\chi(x)$ is constant on $G(0,H)$. This proves that

$$h_H = 2\rho(H).$$

That $h = 2\|\rho\|$ is clear from generalities: the topological entropy is always the supremum over the topological entropies of a decomposition into invariant submanifolds. A similar generality proves the claim on h_μ.

Finally notice that C_1 consists of one point iff G has real rank 1. In this case, certainly $h = h_\mu$. On the other hand if the rank is not 1 then C_1 is a submanifold of a sphere of positive dimension. Notice that $\rho(H)$ is a linear functional positive on C_1. It is elementary that the average of $\rho(H)$ with respect to $\prod_{\alpha \in \phi_+} \alpha(H)\,d\lambda_1(H)$ is strictly smaller than ρ. □

<u>Corollary</u>: Both topological and metric entropy only depend on the universal cover in the locally symmetric case.

Proof: This is clear from our expressions. For the topological entropy h this is well known as h is the exponential growth rate of the volume of balls in the universal cover (cf. [Man 1]). □

Note: 1 To avoid further complications we didn't discuss the case where the ergodic component is not a G-orbit (cf. 1.2 Theorem 2). Certainly the topological entropy and metric entropy are equal and in fact are

$$2\rho(H).$$

This is clear as all the ergodic components $\Gamma'\backslash G'(g_0 \cdot k, H)$ are isometric, simply since $g_0 \in G_0$ commutes with G'. Hence the topological and metric entropies are independent of the particular $g_0 \in G_0$.

2 In rank 1 the geodesic flow is Anosov. Hence there is a unique measure of maximal entropy ν. That $\nu = \mu$ was first proved by Bowen in [Bo 2]. This particular case can be proven very easily (up to some hard dynamics) as follows: By [Bo-Ma 1] one knows that the horocycle foliation is uniquely ergodic with invariant measure the measure of maximal entropy for the geodesic flow. On the other, it is clear that Haar measure is invariant under the horocycle foliation in the locally symmetric case.

Also it may be of some interest to calculate the entropy in the rank 1 case explicitly. The following table first appeared in [Ka 1].

Symmetric Space	Dimension	Maximal Sectional Curvature	Entropy
Real hyperbolic n-space	n	$-k^2$	$(n-1)k$
Complex hyperbolic n-space	2n	$-k^2$	$2nk$
Quaternionic hyperbolic n-space	4n	$-k^2$	$(4n+2)k$
Cayley plane	16	$-k^2$	$22k$.

This follows easily from our Proposition: First note that $\overline{C}_1$ is just a point since G has real rank 1. Hence $h = h_\mu = 2\rho$ which we can compute explicitly from the Satake diagrams of these groups (cf. [He 1]), pp. 532ff). On the other hand, it is clear from Lemma 5 that the maximal curvature K is given by the unique root $\alpha \in \Sigma$ such that $\frac{1}{2}\alpha \notin \Sigma$ (by rank 1, Σ has at most two positive roots: $\alpha_1 \geq \alpha$).

Section 2. Compact Maximal Flats

For a compact manifold of negative curvature the closed geodesics correspond to the free homotopy classes in a one-to-one way as is well-known. The main point is that the energy functional E on the space of closed curves is strictly convex. If we allow some 0 curvature E is only convex and hence the free homotopy classes correspond to continuous families of closed geodesics. For example, in a compact locally symmetric space of rank $r > 1$ we have compact flat r-tori and hence $(r-1)$-dimensional families of closed geodesics. In this section we will first see that this is generic. Then we will study compact r-flats and their equidistribution.

We will consider a finite volume locally symmetric space $M = \Gamma\backslash G/K$ of the non-compact type. Let r be the rank of M. Any geodesic α is contained in an r-flat.

Definition: We call α a regular geodesic if α lies in an open Weyl chamber of F.

Notice that this does not depend on the choice of the flat F. In fact, let $\alpha = \mathrm{Exp}\, tH$ pass through $\Gamma \cdot 0$ where $H \in \mathfrak{p}$. Then α is regular iff H is polar regular (cf. I, 3.1 Definition and the remark thereafter). Moreover, a regular geodesic α lies on a unique r-flat F: We argue in the universal cover $H = G/K$ and pick the lift $\tilde{F}$ of F through $\tilde{F} = \mathrm{Exp}\, \mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{p}$ is abelian. Then H lies in $\mathfrak{a}$ hence $\exp tH$ leaves $\tilde{F}$ invariant. By I, 3.1, Lemma 2(ii) $\tilde{F}$ is unique.

Clearly the regular geodesics form an open dense subset of the set of all geodesics.

Proposition 1: Any closed regular geodesic α is contained in an $(r-1)$-dimensional family of closed geodesics. If M is compact α lies in a unique compact r-flat.

Note: Eberlein has informed us that he has a purely geometric proof of this proposition. Our argument is algebraic.

Proof: That F is unique is clear by the above. On the other hand let $\gamma \in \Gamma$ be an axial isometry for the lift of $\tilde{\alpha}$ through 0: $\gamma(\tilde{\alpha}(T)) = \tilde{\alpha}(t+t_0)$. Let $\tilde{F}$ be a lift through 0 by the above $\gamma \cdot \tilde{F} = \tilde{F}$. Hence F is covered by a cylinder and the first claim is clear.

Suppose M is compact. Clearly $\mathrm{pol}\,\gamma = \exp tH$ for some $t \in \mathbb{R}$ where $\alpha(t) = \mathrm{Exp}\, tH$. Hence γ is polar regular and by 2.3 Lemma 2, F is compact. □

Note: 1 I do not know whether α lies in a compact r-flat in the cofinite volume case. The usual criterion for the compactness of a flat is the $\mathbb{R}$-hyperregularity of γ while we only know that γ is regular.

2 Suppose G is algebraic. Then we have Borel's density theorem (cf. [Ra 1], Theorem 5.5) a lattice Γ is Zariski dense in G. As the regular as well as the $\mathbb{R}$-hyperregular elements form a Zariski dense open subset of G it is clear that a generic free homotopy class with respect to the Zariski topology on Γ corresponds to a compact r-flat.

For a locally symmetric space of rank 1 Bowen proved in [Bo 1] and [Bo 2] that the closed geodesics are equidistributed with respect to the Liouville measure. For higher rank we expect something similar. Since there are uncountably many closed geodesics equidistribution is not well defined and we have to consider continuous families instead. As we will use dynamical arguments we will henceforth assume that M is compact. Then the Proposition says that only the $(r-1)$-dimensional families are dense in the space of all geodesics. Since the $(r-1)$-dimensional families all lie in compact r-flats and are equidistributed there with respect to the Lebesgue measure we really want to study the equidistribution of the compact r-flats.

First, we want to replace the whole unit tangent bundle by a generic ergodic component of the geodesic flow φ_t, i.e., by $E = \Gamma\backslash G/Z(H)_1K$ where H is polar regular, H in C_1. Let $A = \exp \mathfrak{a}$ be a maximal polar subgroup. Notice that A acts on E. Let F be the orbit foliation of this action.

Lemma 1: The compact r-flats are in one-to-one correspondence with the compact leaves of F.

Proof: Suppose $F \subset M$ is a compact r-flat. Recall from A5.2 Proposition 1(i) that in the universal cover $G/K = H$ of M all r-flats are translates of each other. Hence $F = \Gamma \cdot g \cdot A$ for some $g \in G$ and $\Gamma \cdot g \cdot A \cdot |Z(G)_\cap K)|$ is compact in E.

Vice versa since $Z(H)_\cap K$ is compact a compact leaf $\Gamma \cdot g \cdot A \cdot (A(H)_\cap K)$ gives rise to the compact flat $F = \Gamma \cdot g \cdot A$. □

Clearly φ_t leaves F invariant. Recall the

Definition 1: A C^1-diffeomorphism $f : M \to M$ is normally hyperbolic to a C^1 foliation L if f preserves L and

Tf is normally hyperbolic over TL: there is a decomposition

$$TM = N^u \oplus TL \oplus N^s \quad \text{and} \quad Tf = N^u f \oplus Lf \oplus N^s f$$

such that for any point $p \in M$

$$\inf m(N_p^u f) > 1 \qquad \sup \| N_p^s f \| < 1$$

$$\inf m(N_p^u f) \| L_p f \|^{-1} > 1 \qquad \sup \| N_p^s f \| m(L_p f)^{-1} < 1$$

where m is the minimum norm of a linear operator and $\| \ \|$ is the sup norm for some metric d of M.

This definition is taken from [Hi-Pu-Sh 1], p. 116.

In our case we have the

Lemma 2: The geodesic flow φ_t on E is normally hyperbolic to F.

Proof: This is obvious from our discussion in 1.3. □

We need the

Definition 2: An ε-pseudo orbit of $f: M \to M$ is a sequence $p_n \in M$, $n \in \mathbb{Z}$ such that $d(fp_n, p_{n+1}) < \varepsilon$. We say that a pseudo orbit respects a foliation if $f(p_n)$ and p_{n+1} lie in the same leaf of the foliation.

Finally we arrive at the following generalization of Bowen's shadowing lemma.

Lemma 3: Suppose f is normally hyperbolic to a foliation F. Given $v > 0$ there exists δ such that any δ-pseudo orbit $\{x_n\}$ of f can be v-shadowed by a v-pseudo orbit $\{y_n\}$ for f which respects F. Moreover, φy_n and y_{n+1} lie in an ε-ball of a leaf of F.

Note: That $\{y_n\}$ v-shadows $\{x_n\}$ means that $d(x_n,y_n) < v$.

Proof: This is [Hi-Pu-Sh 1] (7a.2). That normal hyperbolicity implies that $(f,\mathcal{F})$ have local product structure is clear (cf. also p. 132 of [Hi-Pu-Sh 1].) The last claim emerges in the proof in [Hi-Pu-Sh 1]. □

We also have the

Definition 3: Suppose f leaves $\mathcal{F}$ invariant. We call $(f,\mathcal{F})$ expansive if there exists a constant $\varepsilon > 0$ such that: if $F_1 \neq F_2$ are leaves of $\mathcal{F}$ and $x_1,x_2 \in F_1,F_2$ respectively then $d(f^n x_1, f^n x_2) > \varepsilon$ for some $n \in \mathbb{Z}$.

This clearly generalizes the usual notion of expansiveness.

We have the

Lemma 4: The geodesic flow φ_t on E is expansive with respect to $\mathcal{F}$.

Proof: This is obvious from the normal hyperbolicity. □

From [Hi-Pu-Sh 1] or in our case by direct inspection we have strong stable and unstable manifolds $W^{uu}(p)$ and $W^{ss}(p)$ for any point p. Moreover, in a small enough neighborhood of a point x the foliations $\mathcal{F}$, $\{W^{uu}\}$ and $\{W^{ss}\}$ are transverse and 'span' the neighborhood. More precisely, we have the

Lemma 5: For any x there is a neighborhood B of x such that for any $p \in B$ there are unique points z^u and s^s in B such that x and z^u lie on the same leaf of $\mathcal{F}$, $z^s \in W^{uu}(z^u)$ and $p \in W^{ss}(z^s)$. We call B a box and z^u and z^s the canonical coordinates of p with respect to x. The canonical coordinates are continuous in p.

Proof: This follows easily from the transversality of the three foliations. □

The next lemma permits us to replace pseudo orbits by compact flats: We let ε be an expansion constant and assume that the ball of radius 100ε is contained in a box about any point $x \in T_1M$. Since T_1M is compact this is clearly possible.

Lemma 6. Let $0 < v < \delta/3$. For small enough δ any orbit $x, \varphi_1 x, \ldots, \varphi_n x$ such that $d(x, \varphi_n x) < \delta$ is v-shadowed by a v-pseudo orbit $\{y_i\}$ that is contained in a compact maximal flat.

Proof: Let $x_i = \varphi_{i \bmod (n+1)} x$. Then $\{x_i\}$ is a δ-pseudo orbit and for small enough δ there is a v-pseudo orbit $\{y_i\}$ that v-shadows $\{x_i\}$ and is contained in a leaf F of F. Let z_1^s, z_1^u be the canonical coordinates of y_{n+1} with respect to y_1. Also let $y_{i+1} = \varphi_i \cdot h_{i+1} y_i$ where $h_{i+1} \in A$ and $\|h_{i+1} - 1\| \le v$. Then $z_2^s = \varphi_1 \cdot h_2 \cdot z_1^s, z_2^u = \varphi_1 h_2 z_1^u$ are the canonical coordinates for $\varphi_1 h_2 \cdot y_{n+1}$ since h_2 commutes with φ_1 and hence leaves the stable and unstable foliations invariant. Since $\varphi_1 h_2 y_{n+1}$ and y_{n+2} lie on F and are close on F (in fact, they are at most $2v$ apart) $\varphi_1 h_2 z_1^s$ and $\varphi_1 h_2 z_1^u$ are the canonical coordinates of y_{n+2} with respect to y_2. Notice that

$$d(y_{n+2}, y_2) \le d(y_{n+2}, x_2) + d(x_2, y_2) \le 2v.$$

Hence the canonical coordinates z_2^s, z_2^u are close to y_2, say they are at most $f(2v)$ apart from y_2 for some functions f with $f(\xi) \to 0$ as $\xi \to 0$.

In general we see that we have canonical coordinates z_i^s, z_i^u for y_{n+i} with respect to y_i such that

$$z_{i+1}^s = \varphi_1 \cdot h_i z_i^s \quad \text{and} \quad z_{i+1}^u = \varphi_1 \cdot h_i z_i^u$$

where $H_i \in A$ and $\|h_i\| \le v$. Moreover z_i^s and z_i^u lie in a $f(2v)$-ball about y_i.

Suppose that $z_i^s \ne z_i^u$. Then $z_i^s = \operatorname{Exp} j$ where j is in the unstable part of the tangent bundle of z_1^u. Decompose j as in 1.3, Lemma 6 and let $\beta \in \phi_+$ index a non-trivial component of j. As $z_i^{s,u} = \varphi_{i-1} \prod_{j=1}^{i-1} h_j z_i^{s,u}$ the distance between z_i^u and z_i^s expands at least by

$$\exp \beta(\log(\varphi_{i-1} \prod_{j=1}^{i-1} h_j)) \ge \exp(i-1)(\beta(H)-v)$$

where $\varphi_1 = \exp H$ and we used the conservative estimate $\|\log h_j\| \le v$. As H is regular and fixed $\beta(H) > 0$. So for small v, z_i^u and z_i^s diverge in contradiction to the fact that $z_i^{s,u}$ lie in an $f(2v)$-ball about y_i. Hence $z_1^u = z_1^s$. By a similar argument for φ_{-i} we see that $y_{n+1} = z_1^s$. Hence y_1 and y_{n+1} lie close to each other on F, so $y_1 = (h \cdot \varphi_n \cdot \prod_{i=1}^{n} h_i) y_1$ where $\|h-1\| < f(2v)$. It is clear that for v small enough

$$H' = \log(h \cdot \varphi_n \cdot \prod_{i=1}^{n} h_i) \in \mathfrak{a}$$

is regular and lies in the positive Weyl chamber. This means that we have produced a closed orbit for $\exp tH'$ acting on $E = \Gamma\backslash G/z(H')_\cap k$. By Proposition 1 this regular closed geodesic lies in a compact maximal flat. □

An easy consequence of this is Mostow's theorem (cf. I3.3 Proposition 1):

Proposition 2: The compact r-flats in M are dense in the space of all flats.

Note: We would like to mention that P. Eberlein has a geometric proof of this (unpublished). Mostow's proof is algebraic.

Proof: By Lemma 1 (and its proof) it is sufficient to see that the compact leaves of $\mathcal{F}$ are dense in the leaves for some generic ergodic component $\mathcal{E}$. By the Poincaré recurrence theorem for almost every leaf F we can find an orbit of φ_t in F that returns very close to itself. This is we find $x \in F$ such that $d(\varphi_{n+1}x, x) < \delta$ for some given small δ and some $n \in \mathbb{N}$. By Lemma 6 there is a v-shadow that lies in a compact leaf F'. From the proof of Lemma 6 F' contains a regular geodesic α' close to $\alpha = \varphi_t x$. Since regular geodesics determine the flat they lie in uniquely it is clear that F and F' are close. □

Note: One may notice that our proof is not completely dynamic in as far as we use Proposition 1 which is a simple consequence of Selberg's lemma. The main point in Mostow's proof is to produce $\mathbb{R}$-hyperregular elements in Γ. This is exactly where we use dynamics instead.

Section 3. Singular Closed Geodesics

So far we have studied the relation between regular closed geodesics and maximal flats. Now we want to study singular closed geodesics. We first show by way of example that singular closed geodesics that are not contained in any higher dimensional compact flat exist for some higher rank locally symmetric spaces. The main tool is a theorem of Prasad and Ragunathan. Let us first recall the

Definition: Let σ be the representation of G on $\Lambda \mathfrak{g}$. We call $g \in G$ *hyperregular* if the multiplicity of 1 as an eigenvalue of $\sigma(g)$ is as small as possible.

Now we have the

Theorem: Let G be a semisimple Lie group without compact factors. Let Γ be an irreducible lattice in G and let H be any non-compact Cartan subgroup of G. Then Γ contains a hyperregular element conjugate to some element in H^0.

Note: Here we let H^0 be the component of the identity of H.

Proof: This is [Pr-Ra 1], Theorem 2.7. □

Recall that any Cartan subgroup H has a decomposition $H = H_K \cdot H_{\mathfrak{p}}$ where $H_K = H \cap K$ and $H_{\mathfrak{p}} = H \cap \exp\mathfrak{p}$ for a suitable choice of Cartan decomposition.

Suppose that H is a Cartan subgroup of some semisimple group G (no compact factors) such that $\dim H_{\mathfrak{p}} = 1$. Let $\Gamma \subset G$ be a *net* irreducible lattice in G, i.e. no element of Γ has a nontrivial root of unity as an eigenvalue for the adjoint representation.

By the theorem and conjugating H if necessary H contains a hyperregular element $\gamma \in \Gamma$. Clearly γ gives rise to a closed geodesic α. In fact, α is covered by $H_{\mathfrak{p}} \cdot 0$ where 0 is the fix point of K. Suppose α is contained in a compact flat

F with $\dim F > 1$. W.l.o.g. suppose $\dim F = 2$. Consider lifts $\tilde{\alpha} \subset \tilde{F}$ in the universal cover G/K of ${}_{\Gamma}\backslash G/_{K}$. There are two elements $\gamma_1, \gamma_2 \in \Gamma$ that translate F and form an abelian group of rank 2: in fact, F is covered by a compact torus T. Let $\gamma_1, \gamma_2 \in \Gamma$ be generators of $\pi_1 T$. Then we find that $\gamma_1\gamma_2 0 = \gamma_2\gamma_1 0$. Since Γ is torsion free (as Γ is net) γ_1 and γ_2 commute and our claim is clear. Next we claim that some power γ^k of γ commutes with γ_1 and γ_2: indeed, let $0 \in \tilde{\alpha}$ as above. Then for $k \in \mathbb{Z}$, $\gamma^k 0 \in \tilde{F}$ as $\tilde{\alpha} \subset \tilde{F}$. By compactness of F there are integers n_k and m_k and a constant $c > 0$ such that

$$d(\gamma_1^{n_k}\gamma_2^{m_k}0, \gamma^k 0) < c$$

for all k. As Γ is torsion free, Γ acts properly discontinuously on G/K. Hence, there are integers k, n and m such that $\gamma^k = \gamma_1^n \cdot \gamma_2^m$. In particular, γ^k commutes with γ_1 and γ_2. As Γ is net, γ^k is still hyperregular and clearly the centraliser of γ^k is H. Hence γ_1 and γ_2 lie in H. In particular γ_1 and γ_2 translate the geodesic $\tilde{\alpha} = \operatorname{Exp} H_{\mathfrak{a}}$. As Γ is discrete there are integers ℓ_1 and ℓ_2 such that $\gamma_1^{\ell_1}\gamma_2^{\ell_2} 0 = 0$ where $0 \in \tilde{\alpha}$ is as above. As Γ is torsion free $\gamma_1^{\ell_1} = \gamma_2^{-\ell_2}$ in contradiction to our hypothesis that γ_1 and γ_2 generate an abelian group of rank 2.

Now suppose Γ is an arbitrary irreducible lattice. We have the

<u>Proposition 1</u>: Any finitely generated subgroup of $GL(n,\mathbb{R})$ contains a net subgroup of finite index.

Proof: This [Ra 1] Theorem 6.11. □

Also recall that all lattices are finitely generated (cf. [Ra 1] Theorem 6.16 and remarks 6.18).

By the proposition, we find $\Gamma' \subset \Gamma$ net and of finite index. By the above, there is a singular closed geodesic α' in ${}_{\Gamma'}\backslash G/_K$ that is not contained in any compact higher dimensional flat. Project α' to α in ${}_{\Gamma}\backslash G/_K$. As $\Gamma' \subset \Gamma$ has finite index α also is not contained in a compact flat of dimension greater than 2.

We summarize this discussion in the

Proposition 2: Suppose G is semisimple of rank ≥ 2 and without compact factors. Let G contain a Cartan subgroup H whose split part $H_{\mathfrak{P}}$ is one dimensional. Then any locally symmetric space of finite volume contains a closed singular geodesic that is not contained in a compact flat F with $\dim F \geq 2$.

Proof: It remains to show that α as above is singular. Suppose not. Then any $\exp X \in H_{\mathfrak{P}}$ is polar regular. Hence, $H \subset \operatorname{centr} X$ is contained in the centraliser $M \cdot A$ of a maximal polar subalgebra $\mathfrak{a}$ (for notation cf.A 10..3) of $\mathfrak{g}$. As $M \cdot A$ is an abelian extension of a compact group all its Cartan subgroups are conjugate. This proves that H is a maximally split Cartan subgroup in contradiction to $\operatorname{rk} G \geq 2$ and $\dim H = 1$. $\square$

Note: If ${}_{\Gamma}\backslash G$ is uniform then α is singular by Section 2, Proposition 1.

We still have to exhibit a higher rank group that has a Cartan subalgebra with one dimensional split part.

Example 1 The algebra $s\ell(3,\mathbb{R})$ has a Cartan of the form

$$\begin{pmatrix} h_1 & -h_2 & 0 \\ h_2 & h_1 & 0 \\ 0 & 0 & -2h_1 \end{pmatrix} , \quad h_i \in \mathbb{R} .$$

Clearly, the split part has dimension 1.

Example 2: The algebra $s\ell(4,\mathbb{R})$ has a Cartan as above given by

$$\begin{pmatrix} h_1 & 0 & -h_3 & 0 \\ 0 & h_2 & 0 & -h_4 \\ h_3 & 0 & h_1 & 0 \\ 0 & h_4 & 0 & h_2 \end{pmatrix}, \quad h_i \in \mathbb{R}$$

It is easy to see that $s\ell(n,\mathbb{R})$ for $n > 4$ does not have a Cartan with one dimensional split part (cf. [Wa 1], 1.3.1 Example 1). For other real Lie groups see [Su 1].

One may also notice that our arguments prove the

Proposition 2': Let G have no compact factors and let rank $G = r > 1$. Suppose G has a Cartan H of split dimension $0 < k < r$. Then any locally symmetric space $_\Gamma\backslash G/_K$ contains a compact flat of dimension k that is not contained in a compact flat F with $\dim F > k$.

Proof: Theorem 2.8 of [Ra-Pr 1] asserts that Γ intersects a conjugate of H in a uniform lattice. Hence, there exists a compact flat of dimension k. The rest is proved as above. □

Finally, Let us refine Proposition 2 by

Proposition 2": Let Γ be an irreducible lattice in a semisimple connected group G of rank ≥ 2 and without compact factors. Assume $_\Gamma\backslash G/_K$ contains a singular closed geodesic α. Then there are infinitely many different singular closed geodesics.

Idea of Proof: Let $C(\Gamma)$ be the commensurability group of Γ in G, i.e., $C(\Gamma) = \{g \in G \mid g^{-1}\Gamma g \cap \Gamma$ has finite index in $\Gamma\}$. By [Ma 3] Γ is arithmetic and hence $C(\Gamma)$ is dense in G. Let $\gamma \in \Gamma$ translate α. For $c \in C(\Gamma)$, $c\gamma^n c^{-1}$ lies in Γ for some n and translates the geodesic $c(\alpha)$. By the above the geodesics $c(\alpha)$ for $c \in C(\Gamma)$ are dense in $G(\alpha)$. This clearly proves our claim. □

Section 4. Specification and Expansiveness

As before, we will study the geodesic flow ϕ_t on an ergodic component $E = {}_{\Gamma}\backslash G/_{M_H}$ where $H \in C$ is regular. We refine the tools of Section 2 and prove uniform specification and expansiveness. We mean weak specification in the sense of [Ru 2], i.e. we shadow orbit segments by the orbit of only a point rather than a periodic point. Strong specification fails to hold in higher rank. One may compare this with nonhyperbolic toral automorphisms (cf. [Mar 3], [L1]). Our expansiveness is slightly different from the usual notion as we allow "perturbations into all flat directions" rather than just along the flow. This section is really just understanding Bowen's ideas in [Bo 1] in our case. The proofs are similar, for the specification almost identical.

4.1 Expansiveness: Let γ be so small that we have canonical coordinates in a 2γ-ball about any point. By abuse of notation, for $a \in A$ we say that $||a|| \le u$ if $||\log a|| \le u$ (where $\log : A \to \mathfrak{a}$).

Proposition: For $u > 0$ small there is an $\alpha > 0$ such that: if $x,y \in E$ and $s_i : \mathbb{R} \to A$, $i = 1,2$, are continuous functions such that (i) $s_1(0) = s_2(0) = 1$

(ii) $\phi_t s_2(t)$ lies in the closure of the positive Weyl chamber C for $t > 0$ and $\phi_t s_2(t) \in -\bar{C}$ for $t < 0$.

(iii) $d(\phi_t s_1(t)x, \phi_t s_2(t)y) \le \alpha$ for all $|t| \le L$ some $L > 0$

then $||s_1(t)s_2(t)^{-1}|| \le 3u$ for all $|t| \le L$ and there is an $a \in A$, $||a|| \le u$ such that

$$d(\phi_t s_2(t)y, \phi_t a s_2(t)x) \le \gamma$$

for all $|t| \le L$.

Proof: We adopt Bowen's argument in [Bo 1], Proposition 1.6. Let

$\eta < \gamma/8$ such that diam $\{ux , u \epsilon A , ||u|| \leq 8\eta\} < \gamma/8$ for all $x \epsilon E$. Pick $\alpha \leq \eta$ so small that for $x,y \epsilon E$ with $d(x,y) \leq \alpha$ all canonical coordinates are at most η apart.

Let x,y be as in the proposition and let v,w be the strong unstable and stable coordinates. Let $0 \leq t_1$ be the smallest $t < L$ such that either $|| s_1(t_1) s_2(t_1)^{-1}|| \geq 3\eta$ or $d(\phi_{t_1} s_2(t_1)v, \phi_{t_1} s_2(t_1) w) \geq \gamma/2$. We will derive a contradiction. As $s_1(0) = s_2(0) = 1$, $t_1 > 0$ and $|| s_1(t_1)s_2(t_1)^{-1} || \leq 3\eta$. We claim that the unstable and stable coordinates of $\phi_{t_1} s_2(t_1)y$ with respect to $\phi_{t_1} s_1(t_1)x$ are $\phi_{t_1} s_2(t_1)v$ and $\phi_{t_1} s_2(t_1)w$:

By the definition of t_1, $d(\phi_{t_1} s_2(t_1)v, \phi_{t_1} s_2(t_1)w) \leq \gamma/2$. For $u \geq 0$, $d(\phi_{t_1-u} s_2(t_1) v, \phi_{t_1-u} s_2(t_1)w) \leq \gamma/2$ obviously. Hence $\phi_{t_1} s_2(t_1)w \,\epsilon\, W^{uu}_{\gamma}(\phi_{t_1} s_2(t_1)v)$.

On the other hand, for $u \geq 0$, $d(\phi_{t_1+u} s_2(t_1)w, \phi_{t_1+u} s_2(t_1)y) \leq \gamma$ as $w \,\epsilon\, W^{ss}_{\gamma}(y)$ and $\phi_{t_1} s_2(t_1)$ is in $\bar{C}$. This means that $\phi_{t_1} s_2(t_1)w$ $W^{ss}_{\gamma}(\phi_{t_1} s_2(t_1)y)$. Hence the stable coordinate of $\phi_{t_1} s_2(t_1)y$ with respect to $\phi_{t_1} s_2(t_1)v$ is $\phi_{t_1} s_2(t_1)w$. Note that $\phi_{t_1} s_1(t_1)v$ and $\phi_{t_1} s_2(t_1)v$ differ by $s_1(t_1)s_2(t_1)^{-1}$ which has norm $\leq 3\eta$. Hence $\phi_{t_1} s_1(t_1)$ lies on the 'same coordinate patch' and our claim is clear.

As $d(\phi_{t_1} s_1(t_1)x, \phi_{t_1} s_2(t_1)y) \leq \alpha$ the canonical coordinates are η close. In particular, $|| s_1(t_1)s_2(t_1)^{-1}|| \leq \eta$. Also $d(\phi_{t_1} s_2(t_1)v, \phi_{t_1} s_2(t_1)w) \leq \eta < \gamma/2$. This contradicts the choice of t_1.

A similar argument proves that $|| s_1(t)s_2(t)^{-1}|| \leq 3\eta$ for

$-L < t \le 0$ and $d(\phi_t s_2(t)a^{-1}y, \phi_t s_2(t)a^{-1}w) \le \frac{\gamma}{2}$ where $a \in A$, $||a|| \le \eta$ satisfies $v = a\cdot x$. Now the last statement is obvious. □

Note: If we let $s_2(t) = 1$ for all t we retrieve Bowen's proposition. Just observe in this case that $d(\phi_t y, \phi_t ax) < \gamma$ for $|t| \le L$ implies $d(\phi_t y, \phi_t ax) < \gamma\, e^{-\lambda(L-|t|)}$. Unfortunately, we need our version to prove a "uniqueness" for specification (4.2 Lemma 2).

4.2 Weak Specification: We call (T,Γ) an L-specification if $T = \{t_i\}_{i=-\infty}^{\infty}$, $t_i \in \mathbb{R}$ and $t_{i+1}-t_i \ge L$ for all $i \in \mathbb{Z}$ and $\Gamma = \{x_i\}_{i=-\infty}^{\infty}$, $x_i \in E$. We call (T,Γ) δ-possible if $d(\phi_{t_i}(x_i), \phi_{t_i}(x_{i-1})) \le \delta$ for all i.

Consider $s : \mathbb{R} \to A$. Let $U_\varepsilon(s,T,\Gamma) = \{y \in E : d(\phi_t s(t)y, \phi_t(x_i)) \le \varepsilon$ for $t_i \le t \le t_{i+1}, i \in \mathbb{Z}\}$ and $\mathrm{Step}_\varepsilon(T) = \{s : s$ is constant on (t_i,t_{i+1}), $s(t_i) = s(t_i+0)$ or $s(t_i-0)$, $||s(t_0)|| \le \varepsilon$ and $||s(t_i+0) - s(t_i-0)|| \le \varepsilon\}$.

Finally, let $U_\varepsilon^*(T,\Gamma) = \cup\{U_\varepsilon(s,T,\Gamma),\ s \in \mathrm{Step}_\varepsilon(T)\}$.

Now we can prove an analogue of Bowen's

Approximation Theorem: Given $\varepsilon > 0$ there are L and δ such that $U_\varepsilon^*(T,\Gamma) \neq \emptyset$ whenever (T,Γ) is a δ-possible L-specification.

Proof: We follow the argument in [Bo 1] with only minor changes.

Let $\delta_1 > 0$. Choose $\delta < \delta_1$ such that $W^{uu}_{\delta_1}(ax) \cap W^{ss}_{\delta_1}(y) \neq \emptyset$ for some $a \in A$, $||a|| \le \delta_1$ whenever $d(x,y) \le 2\delta$. Pick L so that $L^* = L - \delta_1$ satisfies $ce^{-\lambda L^*}\delta_1 < \delta$ and $\sum_{k=1}^{\infty} ce^{-\lambda L^* k} = \frac{ce^{-\lambda L^*}}{1-e^{-\lambda L^*}} < 1$.

Let (T,Γ) be δ-possible and suppose that $t_0 \le 0 < t_1$. Let $z_0 = x_0$ and define z_n inductively:

given z_n such that $d(\phi_{t_{n+1}-t_n}(z_n), \phi_{t_{n+1}}(x_{n+1}) \le 2\delta$

pick $z_{n+1} \in W^{uu}_{\delta_1}(\phi_{t_{n+1}-t_n} a_{n+1}(z_n)) \cap W^{ss}_{\delta_1}(\phi_{t_{n+1}}(x_{n+1}))$

with $a_{n+1} \in A, ||a_{n+1}|| < \delta_1$. Clearly,

$$d(\phi_{t_{n+2}-t_{n+1}}(z_{n+1}), \phi_{t_{n+2}}(x_{n+1})) \le \delta_1 ce^{-\lambda(t_{n+2}-t_{n+1})} < \delta .$$

Since (T,Γ) is δ-possible,

$$d(\phi_{t_{n+2}-t_{n+1}}(z_{n+1}), \phi_{t_{n+2}}(x_{n+2}) \le 2\delta_1$$

and we can continue with the next induction step.

Let $r_{n+1} = a_{n+1}\exp(t_{n+1}-t_n)H \in A$. Then $||r_{n+1}|| \ge L - \delta_1 \ge L^*$.

Clearly, $r^{-1}_{n+1}(z_{n+1}) \in W^{uu}_{\delta_1 ce^{-\lambda||r_{n+1}||}}(z_n)$.

Hence
$$r^{-1}_{n+1} r^{-1}_n (z_{n+1}) \in W^{uu}_{\delta_1 ce^{-\lambda||r_{n+1}+r_n||}}(r^{-1}_n z_n)$$
$$\subset W^{uu}_{\delta_1 ce^{-\lambda||r_{n+1}+r_n||} + \delta_1 ce^{-\lambda||r_n||}}(z_n)$$

Inductively, we let $u_{n,j} = (r_n \dots r_{j+1})^{-1}(z_n)$

Then $u_{n,j} \in W^{uu}_{\delta_1 c\sum_{k=1}^{\infty} e^{-\lambda L^* k}}(z_j) \subset W^{uu}_{\delta_1}(z_j)$.

Lemma 1. Fix j. Then $v_j = \lim_{n\to\infty} u_{n,j}$ exists, $v_j \in W^{uu}_{\delta_1}(z_j)$

and $v_{j+1} = r_{j+1}v_j$.

Proof: For $n \ge j+k$, we have $u_{n,j+k} \in W^{uu}_{\delta_1}(z_{j+k})$ and

$d(u_{n,j}\ \ u_{j+k,k}) =$

$d((r_{j+k}\dots r_{j+1})^{-1}(u_{n,j+k}), (r_{j+k}\dots r_{j+1})^{-1}(z_{j+k})) \le c\delta_1 e^{-\lambda L^* k}$.

Clearly, $\{u_{n,j}\}_{n=j}^{\infty}$ is Cauchy. Hence the limit v_j exists and lies in $W^{uu}_{\delta_1}(z_j)$. Since $u_{n,j} = (r_{j+1})^{-1}u_{n,j+1}$ we get

$v_j = (r_{j+1})^{-1}(v_{j+1})$ by continuity. □

We define $s:[t_0,\infty) \to A$ by $s|_{[t_0,t_1)} \equiv 1$ and

$s|_{[t_i,t_{i+1})} = a_1 \cdot \ldots \cdot a_i$. Let $y_1 = \phi_{-t}(v_1)$. For $t_i < t < t_{i+1}$,

$$\phi_t s(t) y_1 = \phi_{t-t_i}(\phi_{t_i - t_{i-1}} \cdot a_i) \cdot \ldots (\phi_{t_1 - t_0} \cdot a_1)\phi_{t_0}(y_1) =$$

$$\phi_{t-t_i} r_i \cdot \ldots \cdot r_1(v_0) = \phi_{t-t_i}(v_i) = \phi_{t-t_i} r_{i+1}^{-1}(v_{i+1}).$$

Therefore

$$d(\phi_t s(t)(y_1), \phi_t(x_i)) \le d(\phi_{t-t_i} r_{i+1}^{-1}(v_{i+1}), \phi_{t-t_i} r_{i+1}^{-1}(z_{i+1})) +$$

$$d(\phi_{t-t_i} r_{i+1}^{-1}(z_{i+1}), \phi_{t-t_i}(z_i)) + d(\phi_{t-t_i}(z_i), \phi_t(x_i)).$$

For a number δ_2 to be determined make $\delta_1 < \delta_2/3$ so small that

if $x,y \in E$, $d(x,y) < \delta$ and $||a|| < \delta_1$ then

$d(a(x), a(y)) \le \delta_2/3$.

As $v_{i+1} \in W^{uu}_{\delta_1}(z_{i+1})$ and $t < t_{i+1}$ we see that

$$\phi_{t-t_{i+1}}(v_{i+1}) \in W^{uu}_{\delta_1}(\phi_{t-t_{i+1}}(v_{i+1})).$$

Notice that $\phi_{t-t_i} r_{i+1}^{-1} = \phi_{t-t_{i+1}} a_{i+1}^{-1}$. By the above

$$d(\phi_{t-t_1} r_{i+1}^{-1}(v_{i+1}), \phi_{t-t_i} r_{i+1}^{-1}(z_{i+1})) \le \delta_2/3$$

Since $z_i \in W^{ss}_{\delta_1}(\phi_{t_i}(x_i))$, $d(\phi_{t-t_i}(z_i), \phi_t(x_i)) \le \delta_1 \le \delta_2/3$

Finally, $z_{i+1} \in W^{uu}_{\delta_1}(r_{i+1}(z_i))$. As for the first term we get

$$d(\phi_{t-t_i} r_{i+1}^{-1}(z_{i+1}), \phi_{t-t_i} r_{i+1}^{-1} r_{i+1}(z_i)) \le \delta_2/3.$$

Consequently, $d(\phi_t s(t) y_1, \phi_t(x_i)) \le \delta_2$ for $t_i < t < t_{i+1}, i \ge 0$.

Applying this argument to ϕ_{-t} we may extend s to a step function on all of $\mathbb{R}$ and find y_2 such that

$$d(\phi_{t+s(t)}(y_2), \phi_t(x_i)) \le \delta_2$$

for $t_i < t < t_{i+1}$, $i \le 0$. For $i = 0$ we find that

$$d(\phi_{t_0}(y_1), \phi_{t_0}(y_2)) = \lim_{t \to t_0^+} d(\phi_t(y_1), \phi_t(y_2)) \le 2\delta_2 .$$

Let $\eta < \varepsilon/8$ such that for all $x \in E$

$\operatorname{diam} \{ax, |a| < \eta, a \in A\} \le \varepsilon/10$

Assume $\delta_2 < \varepsilon/10$ is so small that

$$W^{uu}_{\eta}(a(x)) \cap W^{ss}_{\eta}(y) \neq \emptyset$$

for some $a \in A, ||a|| \le \eta$ whenever $d(x,y) \le 2\delta_2$.

Pick $y' \in W^{uu}_{\eta}(u\phi_{t_0}(y_2)) \cap W^{ss}_{\eta}(\phi_{t_0}(y_1))$ where $||u|| \le \eta$.

Let $y = \phi_{-t_0}(y')$ and let $s' \in STEP_{\delta_1+\eta}(T) \subset STEP_{\varepsilon}(T)$ be defined by

$$s'(t) = \begin{cases} s(t) & \text{for } t \ge t_0 \\ s(t)u^{-1} & \text{for } t < t_0 \end{cases}$$

Then $y \in U_{\varepsilon}(s',T,\Gamma)$. Indeed, if $t_i < t < t_{i+1}$, $i \ge 0$ then

$$d(\phi_t s'(t)(y), \phi_t(x_i)) \le d(\phi_t s(t)(y), \phi_t s(t)(y_1)) +$$
$$d(\phi_t s(t)(y_1), \phi_t(x_i)) \le \frac{\varepsilon}{2} + \delta_2 < \varepsilon$$

as $\phi_t s(t) = s(t)\phi_{t-t_0}\phi_{t_0}, ||s(t)|| \le \eta$ and

$$d(\phi_{t-t_0}\phi_{t_0}(y), \phi_{t-t_0}\phi_{t_0}(y_1)) < \eta .$$

For $t_i < t < t_{i+1}, i < 0$,

$$d(\phi_t s'(t)(y), \phi_t(x_i)) \le d(\phi_t s'(t)(y), \phi_t s(t)(y)) +$$
$$d(\phi_t s(t)(y), \phi_t s(t)(y_2)) +$$
$$d(\phi_t s(t)y_2, \phi_t(x_i)) \le \frac{\varepsilon}{10} + \eta + \frac{\varepsilon}{10} + \frac{\varepsilon}{5} + \delta_2 < \varepsilon$$

since $d(\phi_t(y), \phi_t(y_2) \le d(\phi_{t-t_0}(y'), \phi_{t-t_0}\, u\, \phi_{t_0}(y_2)) + \frac{\varepsilon}{10} \le \eta + \frac{\varepsilon}{10}$

and hence $d(\phi_t s(t)(y), \phi_t s(t)(y_2)) \le \eta + \frac{\varepsilon}{10} + 2\frac{\varepsilon}{10}$

by our assumption on η.

□

We proceed as in [Bo 1]. The next lemma says that two different shadows of a specification lie close on a leaf $F \in \mathcal{F}$:

<u>Lemma 2</u>: Given $\beta > 0$ there is $\varepsilon > 0$ such that for any L-specifi-

cation (T,Γ) with $\varepsilon/L << 1$ and $y_1, y_2 \in U^*_\varepsilon(T,\Gamma)$ there is a A, $||a|| < \beta$ such that $y_1 = ay_2$.

Proof: To apply expansiveness we need to make our step functions continuous. We follow Bowen. Suppose $y_k \in U_\varepsilon(s_k,T,\Gamma)$ for the step functions $s_k \in Step_\varepsilon(T)$. Define a map ls^*_k by

$$ls^*_k((t_i+t_{i+1})/2) = \log s_k((t_i+t_{i+1})/2)$$

and extending it linearly. Let $s^*_k = \exp(ls^*_k) : \mathbb{R} \to A$. W.l.o.g. we may assume that $t_o = 0$. As $\varepsilon/L << 1$ it is clear that $\phi_t \cdot s^*_k(t)$ lies in the positive Weyl chamber for $t > 0$ (respectively in $-\overline{C}$ for $t < 0$).

Let $\varepsilon' = \sup\{ d(ax,x),\ x\varepsilon E,\ a\varepsilon A,\ ||a|| < \varepsilon\}$. Then for $t_i < t < t_{i+1}$: $d(\phi_t s^*_1(t)y_1, \phi_t s^*_2(t)y_2) \leq d(\phi_t s^*_1(t)y_1, \phi_t s_1(t)y_1)$

$$+d(\phi_t s_1(t), \phi_t x_i) + d(\phi_t x_i, \phi_t s_2(t)y_2)$$

$$+d(\phi_t s_2(t)y_2, \phi_t s^*_2(t)y_2) \leq 2\varepsilon + 2\varepsilon^*.$$

As $s^*_k(0) = 1$, $k = 1,2$ we can apply 4.1 Proposition (with $\eta = \beta$ and $2\varepsilon + 2\varepsilon^* \leq \alpha$) to find an $a\varepsilon A$ such that $||a|| < \beta$ and

$$d(\phi_t s_2(t)y, \phi_t s_2(t)ax) \leq \beta \text{ for all } t.$$

As $\varepsilon/L << 1$, $||\phi_t s_2(t)|| \to \infty$ as $t \to \pm\infty$ and stays inside a cone in C (or $-C$). Clearly, $y = ax$. □

We need one more ingredient to prove the weak specification, namely the " C-density":

<u>Theorem</u>: The strong stable manifold $W^{ss}(x)$ is dense in E for any $x\varepsilon E$.

Proof: This is an obvious consequence of the minimality of the horospherical flow (cf. [Vel,2] or [Bo5]) as the strong stable foliation comes from the orbit foliation of the horospherical flow on $\Gamma\backslash G$. □

A simple consequence is the

<u>Lemma 3</u>: Let $\delta > 0$. Then there is a T such that $B_\delta(\phi_t W^{uu}_\delta(x)) = E$ for any $T \leq t$ and $x\varepsilon E$.

Proof: First fix x. By the theorem, some large but bounded piece W of $W^{uu}(x)$ (in the metric on $W^{uu}(x)$) is δ-dense. Pick t_k such that $\phi_{t_k} x \to x$ as $k \to \infty$. For some big k, $\phi_{t_k} W^{uu}(x)$ is $\delta/2$ - close to W. Hence the claim is clear for a single x.

For variable x, suppose the claim is false. Then there are x_i, y_i and T_i such that $d(y_i, \phi_{T_i} W^{uu}(x)) \geq \delta$, W.l.o.g. let $x_i \to x$, $y_i \to y$. By the above, there is a T such that $d(y, \phi_T W^{uu}(x)) < \delta$. As $T_i \geq T$ eventually and $\phi_T W^{uu}(x_i) \to \phi_T W^{uu}(x)$ this gives rise to a contradiction. □

We can deduce the

<u>Proposition:</u> Let $\varepsilon > 0$. There is an N such that, for any N-specification (T,Γ) one can find $y \in E$ and $s \in \mathrm{Step}_\varepsilon(T)$ such that

$$d(\phi_t s(t) y, \phi_t(x_i)) \leq \varepsilon \qquad \text{for } t_i \leq t \leq t_{i+1}-N.$$

Proof: This is exactly the proof of Proposition 3.7 in [Bo 1] . We include it for completeness.

Let δ and L be as in the approximation theorem, but for $\varepsilon/2$ instead of ε . Make sure $\delta \leq \varepsilon$. Let $N \geq L$ be the T of Lemma 3, for $\delta/2$ instead of δ. Pick $y_i \in \phi_N W^{uu}_{\delta/2}(\phi_{t_{i+1}-N}(x_i)) \cap B_{\delta/2}(\phi_{t_{i+1}}(x_{i+1}))$. Define $\Gamma' = \{x_i'\}$ by $x_i' = \phi_{-t_{i+1}} y_i$. Then $d(\phi_t x_i, \phi_t x_i') \leq \delta/2$ for $t_i \leq t \leq t_{i+1}-N$. As

$$d(\phi_{t_{i+1}} x_i', \phi_{t_{i+1}} x_{i+1}') \leq d(\phi_{t_{i+1}} x_i', \phi_{t_{i+1}} x_{i+1}) + d(\phi_{t_{i+1}} x_{i+1}, \phi_{t_{i+1}} x_{i+1}') \leq \delta \quad ,$$

(T,Γ') is δ-possible. By the approximation theorem there is $y \in E$ and $s \in \mathrm{Step}_{\varepsilon/2}(T)$ so that

$$d(\phi_t s(t) y, \phi_t x_i') \leq \varepsilon/2 \qquad \text{for } t_i \leq t \leq t_{i+1} .$$

Our claim follows from the triangle inequality. □

Finally we can prove the

<u>Weak Specification Theorem:</u> For any $\eta > 0$ and $n \geq 1$ there is an $N = N(\eta, n)$ such that:

if $z_o, \ldots, z_n$ are in E and $t_o, \ldots, t_n$ in $\mathbb{R}$ with $t_{k+1} - t_k \geq N$ then there is a point x such that for $0 \leq k \leq n$

$$d(\phi_{t_k+u} x, \phi_u z_k) \leq \eta \quad \text{for} \quad 0 \leq u \leq t_{k+1} - t_k - N .$$

Proof: This is even easier than in Bowen.

Pick $\varepsilon < \eta/2$ so small that $\operatorname{diam}\{ax, \|a\| < (n+2)\varepsilon\} < \eta/4$. Let N be as in the proposition. Extend the $\phi_{-t_i} z_i = x_i$ and the t_i to an N-specification. By the proposition there is a point y and an $s \in \mathrm{Step}_\varepsilon(T)$ such that $d(\phi_t s(t) y, \phi_t x_k) \leq \varepsilon \leq \eta/2$ for $t_k \leq t \leq t_{k+1} - N$. For such a t and $0 \leq k \leq n+1$ we have $\|s(t)\| \leq (n+2)\varepsilon$. By choice of ε and the triangle inequality we get $d(\phi_t y, \phi_t x_k) \leq \eta$ for $t_k \leq t \leq t_{k+1} - N$ and $0 \leq k \leq n+1$. $\square$

Finally, let us observe that strong specification doesn't hold in our case. In fact, there are only countably many regular ergodic components on which ϕ_t has periodic points. Recall that each regular periodic orbit lies on a maximal compact flat. There are only countably many such flats as each of them corresponds to an element of the free homotopy group of M. (cf. [Eb 4], Proposition 3.1). On each compact flat there are only countably many flow directions that have periodic orbits if $\dim F \geq 2$.

We do not know whether we have strong specification on some ergodic components. The technique of [L1] may be helpful. Also notice the discussion in Section 5.2 . There we see that strong specification holds in a weak sense on the whole unit tangent bundle.

Section 5. Growth of Maximal Flats

Bowen and Margulis studied the relationship between the number of closed geodesics $\psi(t)$ of length $\leq t$ and the topological entropy for a compact manifold of negative curvature. In fact, they obtained the asymptotics $\psi(t) \sim e^{ht}/ht$ as $t \to \infty$ where h is the topological entropy.

For a higher rank locally symmetric space the compact maximal flats replace the closed geodesics. There are at least two natural invariants for a compact flat F, its volume vol F and its systol sys F. Recall that the systol is the length of the shortest closed geodesic on F. While the systol is of an obvious dynamic nature this is not so clear for the volume. It comes in via the weak specification theorem 4.2 as we shadow a pseudoorbit only up to an ε-ball in a flat about the orbit points.

More precisely, we study the shortest regular closed geodesic on a compact flat F whose length we call the regular systol reg sys F. We will see that the function

$$VS(t) = \sum_{\text{regsysF} \leq t} \text{vol } F$$

is well defined. Then we calculate that the exponential growth of US is given by the topological entropy of the geodesic flow on the unit tangent bundle.

5.1 Volume and Systol: We discuss two characteristics of a compact flat on a compact locally symmetric manifold $M = \Gamma\backslash G/K$.

Definition: The systol sys F is the length of a shortest closed geodesic on F. We denote the volume of F by vol F and the length of a closed geodesic α by $\ell(\alpha)$. The regular systol regsys F is the length of a shortest regular closed geodesic.

Recall the connection between closed geodesics and free homotopy classes. We have the

Lemma 1: If two closed geodesics α and β belong to the same free homotopy class in M then α and β lie on a flat F and are translates of each other.

Proof: This is a reformulation of [Eb4], Proposition 3.1. □

Lemma 2: The closed geodesics of length less than t correspond to a finite number of free homotopy classes.

Proof: Consider a sequence of closed geodesics α_n of length $\leq t$. W.l.o.g. the α_n converge to a closed geodesic α of length $\leq t$. Pick lifts α', α_n' of α, α_n in the universal cover G/K such that $\alpha_n' \to \alpha'$ as $n \to \infty$. Let γ and γ_n translate α' and α_n' respectively. Let $x \in \alpha'$ and pick a fundamental domain D for Γ such that x lies in the interior of D. Clearly $\gamma^{-1}\gamma_n x \to x$ as $n \to \infty$. Hence $\gamma^{-1}\gamma_n x$ lies in the interior of D for all big n. As Γ is torsionfree $\gamma = \gamma_n$. □

Recall that a crystallographic group is a discrete uniform subgroup of the group of Euclidean motions E(n). We have the famous

Bieberbach Theorem: For each n, there are only finitely many crystallographic groups up to isomorphism. A crystallographic group Φ has a unique maximal normal abelian subgroup Φ^* of finite index.

Proof: This is [Wo 1], Theorem 3.2.2 and 3.2.9. □

Geometrically, Φ is the fundamental group of a compact flat manifold $F = \Phi\backslash R^n$ which is covered by the flat n-torus $T = \Phi^*\backslash R^n$. In particular, we see that for all n there is a constant c(n) such that $[\Phi^*:\Phi] \leq c(n)$.

<u>Lemma:</u> Let L be the space of flat tori. For any compact subset of L there is a constant $c > 0$ such that for $F \in k$

$$\text{vol } F \geq c \prod_{i=1}^{n} \ell(\alpha_i)$$

where $\alpha_1, \ldots, \alpha_n$ are the first n independent shortest geodesics on F.

<u>Proof:</u> Notice that for $F \in k$ all the angles between the sides of F are bounded away from 0. The volume of F is a product of the length of sides and sin's of these angles. Now the claim is obvious. □

<u>Corollary 2:</u> There are only finitely many compact r-flats F in M with vol F ≤ . , any > 0.

Proof: Let F_n be a sequence of compact r-flats such that volF_n ≤ . Then a subsequence converges to a compact flat F with vol F ≤ . Let α_n^i, i=1,...,r be the first r shortest geodesics on F_n. By Corollary 1 $l(\alpha_n^i) \leq l'$, some $l' > 0$ (use also Lemma 1).

We want to bound the length of a closed regular geodesic α_n in F_n. If no α_n^i, some i , is regular for all big n consider the closed geodesics $d_n\alpha_n^i$ where d_n is the index of the maximal abelian subgroup Φ_n^* of the fundamental group $\pi_1(F_n) = \Phi_n$. We may think of $d_n\alpha_n^i$ as a closed geodesic on a torus covering F_n. Then $l(\Pi\, a_i d_n \alpha_n^i) \leq \Sigma\, a_i d_n l(\alpha_n^i)$ for all positive integers a_i. Here Π denotes the closed geodesic wrapping around $d_n\alpha_n^i$ a_i times. Clearly, $a_i d_n \alpha_n^i$ is regular for some a_i C where C only depends on the shape of the Weyl chamber.

We have found a sequence of regular closed geodesics α_n in F_n with $l(\alpha_n) \leq C\, c(n)\, l'$. By Lemma 1 the α_n are all homotopic for a subsequence of n's. These α_n lie on a flat F'. By regularity of α_n, $F' = F_n$ for all big n. □

Finally, we have the

<u>Corollary 3</u>: For all $t > 0$, there are only finitely many r-flats F such that reg sys $F \leq t$.

Proof: Otherwise let F_n be a sequence of such flats. Let c_n in F_n be a shortest regular geodesic. Then $\ell(c_n) \leq t$. By Lemma 1 there is a subsequence of the n's such that these c_n lie on a unique flat F (by regularity). Hence $F_n = F$. □

If we drop the regularity in Corollary 3 the conclusion unfortunately doesn't hold. One counterexample is trivial: simply consider the product of two manifolds of negative curvature. The product of any two closed geodesics is a compact maximal flat. Making the second geodesic longer and longer we obtain infinitely many compact flats with the same systol.

Even if the manifold is irreducible the corollary is still false in general. We describe one example.

<u>Example</u>: There is a uniform lattice $\Gamma \subset G = SL(3,R)$ and a subgroup H of G conjugate to $GL(2,R)$ in G over R such that $\Gamma \cap H$ is uniform. One can either write down Γ explicitly as a group of units of a number field or follow the procedure in [Bor 1]. By [Eb 5] $H/\Gamma \cap H$ is finitely covered by $M' \times S^1$ where M' is a surface of negative curvature and S^1 corresponds to the center of H (cf. Corollary 2, loc.cit.). Any compact flat F is covered by $F' \times S^1$. In particular, F contains the closed geodesic α coming from the center of H. Since the center of H is conjugate to the group $\begin{pmatrix} a & & \\ & a & \\ & & a^{-2} \end{pmatrix}$ in G, α is singular. As there are infinitely many compact flats we are done.

5.2. A Growth Rate for Compact r-Flats: Corollary 3 of 5.1 shows that

$$VS(t) = \sum_{\text{regsys} F \le t} \text{vol } F$$

is well defined. We follow Bowen in [Bo 1] in order to determine the logarithmic growth rate of VS(t).

Call a subset E of T_1M (t,ε)-separated if $d(\phi_s x, \phi_s y) > \varepsilon$ for some $0 \le s \le t$. As ϕ_t expands and contracts monotonically a set E is (n,ε)-separated iff it is (n,ε)-separated for the time one map ϕ_1 of ϕ_t. Hence if $M(\varepsilon,t)$ denotes the maximal number of (t,ε)-separated points then the topological entropy h is given by

$$h = \lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} \log M(\varepsilon,t)/t \quad .$$

The same is true on any ergodic component.

We make a few observations leading to a multiplicative asymptotic law for $M(\varepsilon,t)$ on an ergodic component $\mathcal{E}$, $\mathcal{E}$ regular:

(1) Let E be a maximal (t,ε)-separated set. Then for any x in $\mathcal{E}$ there is a y in $\mathcal{E}$ such that $d(\phi_s x, \phi_s y) \le \varepsilon$ for all $0 \le s \le t$.

(2) For small $\eta, \varepsilon > 0$ there are constants C and D such that

$$M(\eta, t+C) \ge D\, M(\varepsilon,t) \quad \text{for all } t \ge 0 \; .$$

Proof: Let $\phi_{-t/2}E$ be (t,ε)-separated. Let F be maximal such that $\phi_{-(t+C)/2}F$ is $(t+C,\eta)$-separated. By (1) for all x in E there is gx in F such that $d(\phi_u x, \phi_u gx) \le \eta$ for all $|u| \le (t+C)/2$.

If $gx = gy$ then $d(\phi_u x, \phi_u y) \le 2\eta$ for $|u| \le (t+C)/2$. By 4.1 Note there is an $a = a(x,y)$ in A , $||a|| \le \delta \le 1$ (for η small enough) such that

$$d(\phi_p y, \phi_p ax) \le \gamma\, e^{-\lambda(C-2)/2} < \eta/3$$

for all $|p| \le t/2$ and C very big. Let $\beta > 0$ be so small that $\text{diam}\{ax, \; ||a|| < \beta\} < \eta/3$ for all x in $\mathcal{E}$. If $||a(x,y)|| < \beta$ then $d(\phi_p y, \phi_p x) < \eta$ for $|p| \le t/2$. Then $x = y$ by our assumption on E.

For η small the set $\{x_1,\ldots,x_m\} = g^{-1}gx$ lies in a chart for the canonical coordinates. Then $v(x,y) = a(x,y)x$ is the strong unstable coordinate of y with respect to x (cf. 4.1). If all the mutual strong unstable coordinates are at least ηdistant then it is clear that $v(x_i,x_1)$, $i = 1,\ldots,m$, are η'-distant for some η' that only depends on η (by the continuity of the canonical coordinates). Hence

$$m \le \mathrm{vol}\ (B_{2\eta}(x_1))/\eta' = \text{constant} .$$

□

The next five observations are exactly like in[Bo 1]. We include the proof for completeness.

(3) $M(\varepsilon,t_1+\ldots+t_n) \le M(\varepsilon/2,t_1)\cdot\ldots\cdot M(\varepsilon/2,t_n)$.

Proof: Let E be $(t_1+\ldots+t_n,\varepsilon)$-separated, E_k be maximal $(t_k,\varepsilon/2)$-separated. By (1) there is a map $g\colon E \to \prod_k E_k$ so that $g_k x$ satisfies

$$d(\phi_{u+t_1+\ldots+t_{k-1}}x,\phi_u g_k x) \le \varepsilon/2$$

for $0 \le u \le t_k$. Clearly g is injective. □

(4) For any L and small $\varepsilon > 0$ there is a constant C' so that for $t > 0$

$$M(\varepsilon,t+L) \le C'M(\varepsilon,t) .$$

Proof: It is sufficient to prove this for large t . By (2) there are constants C and D such that $M(\varepsilon,t+C) \ge DM(\varepsilon/2,t)$. By (3)

$$M(\varepsilon,t+L) \le M(\varepsilon/2,t-C)M(\varepsilon/2,C+L) \le M(\varepsilon,t)M(\varepsilon/2,C+L)/D .$$ □

(5) For all small η,ε there is a constant C" so that for all $t \ge 0$

$$C''\,M(\eta,t) \ge M(\varepsilon,t) .$$

Proof: For C and D as in (2) $M(\eta,t+C) \ge DM(\varepsilon,t)$. By (4) there is C' such that $M(\eta,t+C) \le C'M(\eta,t)$. □

(6) For ε small there is a constant C* such that for all $t,s \ge 0$

$$M(\varepsilon,t+s) \le C^*\,M(\varepsilon,t)\,M(\varepsilon,s) .$$

Proof: Obvious from (3) and (4). □

(7) For small η and $n \ge 1$ there is a constant $c > 0$ such that for all sufficiently large s_i

$$M(\eta,s_1+\ldots+s_n) \ge c\,M(\eta,s_1)\cdot\ldots\cdot M(\eta,s_n) .$$

Proof: Like in Bowen's case this follows from the specification theorem.

Let $N = N(\eta,n)$ be as for specification. Assume $s_i \geq N$. Let E_i be $(s_i-N,3\eta)$-separated. Assume E_i is maximal. Let $t_0 = 0$ and $t_k = s_0+\ldots+s_{k-1}$ for $1 \leq k \leq n+1$. Let $z_i \in E_i$. By weak specification there is a point $g(z_0,\ldots,z_n)$ such that

$$d(\phi_{t_k+u}g(z_0,\ldots,z_n),\phi_u z_k) < \eta$$

for $0 \leq u \leq s_k-N$. By the triangle inequality, the $g(z_0,\ldots,z_n)$ are (t_n,η)-separated. Hence

$$M(\eta,t_n) \geq M(3\eta,s_0-N)\cdots M(3\eta,s_n-N).$$

Now the claim follows from (5) and (4).□

We obtain the desired multiplicative asymptotic law by combining (3) and (7).

We start to examine the interplay between the dynamics on an ergodic component and on the whole unit tangent bundle. As we saw in 4.2, strong specification fails on an ergodic component. But given orbit segments on one component we can shadow them by a periodic orbit on a nearby ergodic component.

Let $X = T_1M$. We first get the

Closed Orbit Theorem: For $\beta > 0$ there are δ, $L > 0$ such that: if $r \geq L$ and $d(\phi_r x,x) \leq \delta$ then there are y in X and r' such that $\phi_{r'}y = y$, $|r'-r| \leq \beta$ and $d(\phi_t y,\phi_t x) \leq \beta$ for $0 \leq t \leq r$.

Proof: Let $\varepsilon \ll \beta$, $t_i = ir$ and $x_i = \phi_{-t_i}x$. This is a δ-possible L-specification. Apply the approximation theorem to the ergodic component of x to find y' such that $\phi_t y'$ ε-shadows $\varphi_t x_i$, $t_i \leq t \leq t_{i+1}$. As in Section 2, Lemma 6 we see that $\phi_t y$ is contained in a compact flat. As $d(\phi_r y,y) \leq 2\varepsilon$ we may vary the flow direction a little to get a closed orbit of length r' where $|r'-r| \leq 10\varepsilon$ and such that the new orbit 10ε shadows the orbit of y. □

This has an easy corollary:

A Weak Strong Specification Theorem: For any $\eta > 0$ and $n \geq 1$ there is an $N = N(\eta, n)$ such that:

if $z_o, \ldots, z_n$ lie on an ergodic component and $t_o, \ldots, t_{n+1} \in R$ with $t_{k+1} - t_k \geq N$ then there is a point x in X such that $d(\phi_{t_k+u} x, \phi_u z_k) \leq \eta$ for $0 \leq u \leq t_{k+1} - t_k - N$ and $0 \leq k \leq n$ and x is a periodic point with period $t_{n+1} - t_o \pm \eta$.*

Proof: Use the proof for weak specification where you extend $\phi_{-t_i} z_i = x_i$ to an N-specification such that $\phi_{t_{n+1}} x_{n+1} = z_o$. We get a point y that comes back close to itself after time $t_{n+1} - t_o$. Now we can apply the closed orbit theorem.

Notice that the N in principal depends on the ergodic component in question. Checking through 4.2 Lemma 3 and Proposition we see that one N works for an open set of ergodic components. As the space of ergodic components = WS is compact we are done. □

Recall from the discussion in Section 2 that the topological entropy h of the geodesic flow on X is achieved on the ergodic component $E = E(H)$ where H is dual to $\rho = 1/2\, \Sigma\alpha$, α a positive root. Let $M(\eta, t)$ count the maximal number of (t, η)-separated points on E.

Lemma 1: There is a constant d such that $VS(t) \geq d\, M(\eta, t)$.

Proof: Let N be as in the weak strong specification theorem. Let E be a $(t-N-\eta, \eta)$-separated set in E. For e in E we can find x(e) in X such that x(e) is periodic with period t and

$$d(\phi_s e, \phi_s x(e)) < \eta/3 \qquad \text{for} \quad 0 \leq s \leq t-N-\eta.$$

For $e \neq e'$ in E, $d(\phi_s x(e), \phi_s x(e')) > \eta/3$ for some $0 \leq s \leq t-N-\eta$.

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* By period we don't necessarily mean least period.

Let B_β be the β-ball about 1 in A. Pick $\beta > 0$ so small that the diam $B_{3\beta}x < \eta/3$ for all x in X. If $y \in B_{3\beta}x(e)$ then

$$d(\phi_s y, \phi_s x(e)) < \eta/3 \quad \text{for all } s .$$

Hence for $e' \neq e$ in E, $x(e') \notin B_{3\beta}x(e)$. Hence $B_\beta x(e) \cap B_\beta x(e') = \emptyset$. The x(e) determine flats F(e) with systol $\leq t$. As the x(e) lie close to E and E is regular (ρ lies deep in the Weyl chamber) the F(e) are compact. As the volume of $B_\beta x(e) = d' \beta^r$ for some constant d' we get $$VS(t) \geq d'\beta^r M(\eta,t). \quad \square$$

<u>Corollary 1:</u> $\underline{\lim}_{t\to\infty} \log VS(t) \;/t \geq h$.

Proof: By property (5) of $M(\eta,t)$

$$h = \lim_{\alpha\to o} \overline{\lim_{t\to\infty}} \log M(\eta,t)\;/t = \underline{\lim}_{t\to\infty} \log M(\eta,t)\;/t.$$

Now the claim is obvious from the lemma. $\square$

Let $N(\eta,t)$ count the maximal number of (t,η)-separated points in X. Let

$$VS_\varepsilon(t) = \sum_{F\in S(t,\varepsilon)} \text{vol } F$$

where we sum over the set $S(t,\varepsilon)$ of all compact r-flats with $t-\varepsilon$ sys $F \leq t+\varepsilon$. Fix ε very small.

<u>Lemma 2:</u> For some $c > 0$ and all small η, $VS_\varepsilon(t) \leq c\,\eta^r N(\eta,t)$.

Proof: Let $F,F' \in S = S(t,\varepsilon)$. Let γ,γ' be shortest regular geodesics of F,F' and suppose that γ and γ' are η-close. Let x,y be points in X, tangent to γ,γ' such that $d(\phi_s x,\phi_s y) \leq \eta$ for $0 \leq s \leq t$. Let $t-\varepsilon \leq \tau,\tau' \leq t+\varepsilon$ such that $\phi_\tau x = x$, $\phi_{\tau'} y = y$. Let $t_i = it$ and $x_i = \phi_{-t_i} x$ define a specification (T,Γ). Let $s_1 \in \text{Step}_\varepsilon(T)$ be given by $s_1(u) = i(\tau-t)$ for $t_i \leq u < t_{i+1}$. Then $x \in U^*_\varepsilon(\Gamma)$. Define s_2 the same way using τ' instead of τ . For $0 \leq u \leq t$ we see that

$$\phi_{t_i+u+s_2(t_i+u)} y = \phi_u y .$$

As $d(\phi_u x,\phi_u y) \leq \eta$, $y \in U^*(\Gamma)$. By 4.2 Lemma 2 and $\varepsilon << 1$, $x = ay$ for some $a \in A$, $||a|| < \beta$ (some universal β). In particular, $F = F'$ (as γ,γ' are regular).

For $F \in S$, $c > 0$ a universal constant, there are $\operatorname{vol} F/c\eta^r$ many points on F that are at least η-distant. They give rise to η-distant points x_i on the ergodic component corresponding to a shortest regular geodesic of F. For $F \neq F' \in S$ the argument above shows that the x_i and x_i' are (t,η)-separated. Hence $VS_\varepsilon(t) \leq c\eta^r N(\eta,t)$. □

<u>Proposition:</u> $\lim\limits_{t\to\infty} \log VS(t)/t = h$.

Proof: Clearly, $\lim\limits_{t\to\infty} \log VS_\varepsilon(t)/t \leq \lim\limits_{\eta\to 0} \overline{\lim\limits_{t\to\infty}} \log (c\eta^r N(\eta,t))/t = h.$

As $VS(t) \leq VS_\varepsilon(t) + VS_\varepsilon(t-2\varepsilon) + \ldots + VS_\varepsilon(0)$ our claim is clear. □

Appendix

In this appendix we will briefly review the basic structure theory of real semisimple Lie groups. Our main sources are [He 1] and [Wa 1]. We will assume the general theory of Lie groups.

1: Let G be a real connected Lie group , $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$.

1.1: Definition: $\mathfrak{g}$ is simple if $\mathfrak{g}$ has no ideals except $\{0\}$ and $\mathfrak{g}$.

$\mathfrak{g}$ is semisimple if $\mathfrak{g}$ is a direct sum of simple Lie algebras.

The Lie group G is called (semi)simple iff its Lie algebra $\mathfrak{g}$ is (semi)simple.

1.2: We denote the adjoint action of $\mathfrak{g}$ on itself by ad. We recall that

$$\mathrm{ad}(X)(Y) = [X,Y].$$

Exponentiating ad we get the adjoint action Ad of G on $\mathfrak{g}$. Recall that a finite dimensional representation is semisimple iff every invariant subspace has an invariant complementary subspace. In these terms we get a first important consequence of the semisimplicity that nearly characterizes it:

Proposition: If $\mathfrak{g}$ is semisimple then $\mathrm{ad}\,\mathfrak{g}$ is semisimple. Conversely, if $\mathrm{ad}\,\mathfrak{g}$ is semisimple then $\mathfrak{g}$ is reductive i.e. $\mathfrak{g}$ is the direct sum of an abelian and a semisimple Lie algebra.

Proof: This is Theorem 3.16.3 of [Va 1]. □

1.3: The next characterization is in terms of the Cartan-Killing form.

Definition: The bilinear form $B(X,Y) = \text{trace}(\text{ad } X \text{ ad } Y)$ is called the Cartan-Killing form of the Lie algebra $\mathfrak{g}$.

Proposition: A Lie algebra $\mathfrak{g}$ is semisimple iff its Cartan-Killing form is non-degenerate.

Proof: This is Proposition 6.1 and its corollaries in [He 1]. □

Notice that the Cartan-Killing form is invariant under all automorphisms of $\mathfrak{g}$.

2: An important element in the structure theory of semisimple Lie algebras is the Cartan subalgebra.

2.1: Definition 1: A subalgebra $\mathfrak{h} \in \mathfrak{g}$ is called a Cartan subalgebra of $\mathfrak{g}$ if $\mathfrak{h}$ is maximal abelian and $\text{ad}(H)$ is a semisimple endomorphism for each $H \in \mathfrak{g}$.

There is an intimate connection between Cartan subalgebras and regular elements. Let us first recall the

2.2: Definition 2: Expand the characteristic polynomial $\det(t-\text{ad}(X)) = \sum_0^n d_i(X)t^i$ for $X \in \mathfrak{g}$. Let ℓ be the least integer such that $d_\ell(X) \neq 0$ for some $X \in \mathfrak{g}$. We call $X \in \mathfrak{g}$ regular if $d_\ell(X) \neq 0$.

We first get an existence result:

Proposition 1: Let $\mathfrak{g}$ be semisimple. Then the centralizer of a regular element is a Cartan subalgebra of dimension ℓ. Conversely, every Cartan subalgebra arises in this manner.

Proof: This is [He 1], III, § 3, Theorem 3.1. To prove the converse we need:

Proposition 2: Over $\mathbb{C}$, Cartan subalgebras are unique up to conjugacy.

Proof of Proposition 2: This is the remark following Proposition 1.3.1.2 in [Wa 1]. □

Back in the proof of Proposition 1, suppose we are given a Cartan subalgebra $\mathfrak{h}$. Its complexification $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra in $\mathfrak{g}_{\mathbb{C}}$. By Proposition 2 and the first part of Proposition 1 $\mathfrak{h}_{\mathbb{C}}$ is the centralizer of a regular element. In particular, $\mathfrak{h}_{\mathbb{C}}$ contains a regular element. Since $\mathfrak{h}$ is Zariski dense (over $\mathbb{C}$) in $\mathfrak{h}_{\mathbb{C}}$ we also find a regular element in $\mathfrak{h}$. □

Corollary 1: Every semisimple Lie algebra $\mathfrak{g}$ has a Cartan subalgebra.

Proof: This is obvious from Proposition 1. □

Let us observe that the whole structure theory hinges on this result.

2.3: As a complement to Proposition 2 and Corollary 1 we have:

Proposition 3: Every real semisimple Lie algebra $\mathfrak{g}$ has a finite number of conjugacy classes of Cartan subalgebras. They may be described in terms of the root system.

Proof: This is Theorem 1.3.1.10 of [Wa 1]. □

Finally, let us observe that the set of regular elements $\mathfrak{g}'$ is open and dense in $\mathfrak{g}$. If $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$ are representative Cartan subalgebras of all the conjugacy classes then

$\mathfrak{g}' = \bigcup_{\substack{g \in G \\ 1 \le i \le k}} Ad(g)\mathfrak{h}_i'$ where $\mathfrak{h}_i'$ are the regular elements of $\mathfrak{h}_i$, (obvious from Proposition 1). In particular, $\mathfrak{g}'$ has a finite number of connected components.

3: Since the adjoint representation restricted to a Cartan subalgebra $\mathfrak{h}$ is a semisimple representation of an abelian Lie algebra we can diagonalize it. This leads to the

3.1: Definition: Let $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$, the dual of the complexification of $\mathfrak{h}$. Let $\mathfrak{g}^\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} | [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}_{\mathbb{C}}\}$. If $\mathfrak{g}^\alpha \neq 0$ we call α a root and $\mathfrak{g}^\alpha$ a root space. For a given Cartan subalgebra $\mathfrak{h}$, we denote the set of all non-zero roots by Φ.

Since the Cartan-Killing form B is non-degenerate even when restricted to $\mathfrak{h}_{\mathbb{C}}$ we get a canonical isomorphism $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{h}_{\mathbb{C}}^*$. In particular, we have a dual element H_α for α such that $B(H,H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}_{\mathbb{C}}$. Moreover, we can transport B to $\mathfrak{h}_{\mathbb{C}}^*$ and let $(\alpha,\beta) = B(H_\alpha,H_\beta)$.

3.2: The diagonalization of $ad|\mathfrak{h}_{\mathbb{C}}$ and its special properties is obtained in the root space decomposition.

Theorem: (i) $\mathfrak{g}_{\mathbb{C}}$ decomposes into a direct sum:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha .$$

(ii) $\dim_{\mathbb{C}} \mathfrak{g}^\alpha = 1$.

(iii) $\mathfrak{g}^\alpha \perp \mathfrak{g}^\beta$ with respect to B unless $\alpha = -\beta$.

(iv) If $\alpha \in \Phi$ and $c\alpha \in \Phi$ then $c = \pm 1$.

(v) For each $\alpha \in \Phi$ we can choose $X_\alpha \in \mathfrak{g}^\alpha$ such that

(1) $[X_\alpha, X_{-\alpha}] = H_\alpha$ and $[H, X_\alpha] = \alpha(H)X_\alpha$ for all $H \in \mathfrak{h}_{\mathbb{C}}$.

(2) There are real numbers $N_{\alpha,\beta}$ for all $\alpha,\beta \in \Phi$ such that

$$\begin{cases} [X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \\ N_{\alpha,\beta} = 0 & \text{if } \alpha+\beta \notin \Phi \text{ and } \alpha \neq -\beta. \end{cases}$$

Note that $N_{\alpha,-\alpha}$ is not defined.

We have the following relations whenever the terms are well-defined:

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta} = -N_{\beta,\alpha}$$

$$N_{-\alpha,\alpha+\beta} = N_{\alpha+\beta,-\beta} = N_{-\beta,-\alpha}.$$

(vi) Given $\alpha,\beta \in \Phi$, the roots of the form $\beta + n\alpha$ for $n \in \mathbb{Z}$ are an uninterrupted progression. If p and q denote the ends of this progression $p \leq n \leq q$ then

$$-2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = p + q$$

and $$N_{\alpha,\beta}^{2} = \frac{q(1-p)}{2}(\alpha,\alpha).$$

<u>Proof</u>: This is contained in [He 1], III, §4.5. □

We will call a set of X_α, H_α satisfying the properties of the theorem a Weyl basis of $\mathfrak{g}_{\mathbb{C}}$.

4: To advance further we have to use the unitary trick which presents us with a lot of compactness in $\mathfrak{g}$. We also discuss normal real forms. First recall the

4.1: Definition: A real Lie algebra is called compact if its Cartan-Killing form is negative definite.

Now we can formulate Weyl's unitary trick:

4.2: Theorem: Every complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ has a compact real form $\check{\mathfrak{u}}$, i.e. a compact real Lie algebra $\check{\mathfrak{u}}$ such that $\check{\mathfrak{u}}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$. Up to isomorphism $\check{\mathfrak{u}}$ is unique.

Proof: This is Theorem 6.3 of [He 1], III, §6. □

This result allows us to introduce Cartan decompositions:

4.3: Let σ be the complex conjugation of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ induced by $\mathfrak{g}$.

Definition: A direct sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for $\mathfrak{k}$ a Lie subalgebra and $\mathfrak{p}$ a vector subspace is called a Cartan decomposition for $\mathfrak{g}$ if there exists a compact real form $\check{\mathfrak{u}}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\sigma\check{\mathfrak{u}} = \check{\mathfrak{u}}$ and

$$\mathfrak{k} = \mathfrak{g}\cap\check{\mathfrak{u}} \quad \text{and} \quad \mathfrak{p} = \mathfrak{g}\cap i\check{\mathfrak{u}}.$$

Theorem: Every real semisimple Lie algebra $\mathfrak{g}$ has a Cartan decomposition which is unique up to conjugacy.

Proof: This is Theorem 7.1 and 7.2 of [He 1], III, §7. □

One characterizes Cartan decompositions in terms of the Killing form B:

Proposition: Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a direct decomposition into a Lie subalgebra $\mathfrak{k}$ and a vector subspace $\mathfrak{p}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition iff B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$ and if the involution θ defined by $\theta(K+P) = K - P$ is an automorphism of $\mathfrak{g}$.

Proof: This is Proposition 7.4 of [He 1], III, §7. □

We call θ the Cartan involution associated to the Cartan decomposition. This condition also implies that $\mathfrak{k}$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}$. We call $\mathfrak{k}$ the maximal compact and $\mathfrak{p}$ the vector part of a Cartan decomposition.

4.4: Normal real forms form the opposite of compact real forms. They provide real semisimple Lie algebras whose structure is about as simple as that of complex Lie algebras.

Definition: Let $\mathfrak{g}^{\mathbb{C}}$ be a semisimple complex Lie algebra. A real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ is called normal if for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{p}$ contains a Cartan subalgebra.

Theorem: Each semisimple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ has a normal real form $\mathfrak{g}$. In terms of the root system Φ of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g} = \sum_{\alpha\in\Phi} \mathbb{R} H_\alpha + \sum_{\alpha\in\Phi} \mathbb{R} X_\alpha .$$

Proof: This is [He 1], IX, Theorem 5.10. Note that our definition of a normal real form coincides with Helgason's because a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$ is a Cartan. This is clear from A5.1 Proposition 1(i). The description in terms of the roots is given in the proof of the theorem. □

5: We pursue the theme of A4 on the group level.

5.1: Theorem 1: Let K be any Lie subgroup of G with Lie algebra $\mathfrak{k}$, $\mathfrak{k}$ a maximal compact. Then

(i) K is connected, closed and contains the center

Z of G. Moreover, K is compact iff Z is finite.

(ii) There exists an involutive, analytic automorphism $\tilde{\theta}$ of G whose fixed point set is K and whose differential at 1 is θ.

(iii) The map $\varphi : (X,k) \mapsto (\exp X)\, k$ is a diffeomorphism of $\mathfrak{p} \times K$ onto G.

Proof: This is Theorem 1.1 of [He 1], VI, §1. □

In particular, the theorem shows that maximal compact subgroups of G exist provided the center of G is finite. Very important is the

Theorem 2: All maximal compact subgroups of G are connected and conjugate in G.

Proof: This is Theorem 2.2 of [He 1], VI, §2. □

5.2: From the fact that θ is an automorphism one easily concludes that $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$. In particular, Ad K leaves $\mathfrak{p}$ invariant as a set.

Proposition 1: Let $\mathfrak{a}$ be any maximal abelian subalgebra of $\mathfrak{p}$. Then

(i) $\mathfrak{a}$ can be extended to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Moreover $\mathfrak{a}$ is unique up to conjugacy.

(ii) $\mathfrak{p} = \mathrm{Ad}(K) \cdot \mathfrak{a}$ i.e. $\mathfrak{a}$ is a "cross-section" to the Ad K action.

Proof: (i) is part of the Iwasawa decomposition, [He 1], VI, §3. Uniqueness and (ii) are Lemma 6.3 of [He 1], V, §6. □

One can use this result to improve on 5.1 Theorem 2:

Proposition 2: Suppose that G has finite center. Then K is a maximal closed subgroup of G.

Proof: This is Theorem 1.3 in [Gl 1], VI. □

6: We want to tie up the structures in A4 and A5 with the theory of the roots of $\mathfrak{g}_{\mathbb{C}}$ in A3. Notice that we only used the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ in A3. So we have to study the effects of the real structure on the root system. We first have to recall abstract root systems and their properties in this section.

6.1: Definition 1: Let E be a finite dimensional real vector space. A reflection w_α with respect to $\alpha \in E$ is an automorphism of E such that

(i) $w_\alpha(\alpha) = -\alpha$.

(ii) The fixed points of w_α are a hyperplane in E.

Definition 2: A subset $\Phi \subset E$ is a root system in E if

(i) Φ is finite, generates E and $0 \notin \Phi$

(ii) For every $\alpha \in \Phi$ there exists a reflection w_α with respect to α which leaves Φ invariant.

(iii) For every $\alpha, \beta \in \Phi$, $w_\alpha(\beta) - \beta$ is an integral multiple of α.

Notice that w_α is unique.

Definition 3: The group $W = W(\Phi)$ generated by the reflections $w_\alpha, \alpha \in \Phi$ is called the Weyl group of Φ.

Fix a positive non-degenerate symmetric bilinear form

(,) on E invariant under W. (Since W is finite it exists). Then we may describe w_α explicitly by:

$$w_\alpha(e) = e - 2\,\frac{(e,\alpha)}{(\alpha,\alpha)}\,\alpha.$$

By property (iii) of a root system $A_{\alpha,\beta} = 2\,\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer, called the <u>Cartan</u> <u>integer</u> for all $\beta \in \Phi$.

<u>6.2</u>: <u>Definition</u>: Let Φ be a root system in E. A subset ψ of Φ is called a <u>fundamental system</u> of roots for Φ if:

(i) ψ is a vector space basis for E.

(ii) Every root can be written as a linear combination

$$\sum_{\alpha\in\psi} n_\alpha \alpha$$

where n_α are all integers of the same sign.

We call the elements of ψ <u>simple</u> <u>roots</u>.

Clearly, a fundamental system defines a unique vector space ordering on E such that the simple roots are positive. Conversely, one can prove that the positive roots with respect to some order on E contain a unique fundamental system.

<u>Proposition</u>: Let ψ be a fundamental system. Then the w_α for $\alpha \in \psi$ are a system of generators for W.

<u>Proof</u>: This is [Wa 1] Proposition 1.1.2.3. □

Using this proposition one can prove the

<u>Theorem</u>: Any two fundamental systems of Φ are conjugate under a unique element $w \in W$.

<u>Proof</u>: This is [Wa 1] Theorem 1.1.2.6. □

6.3: Every root $\alpha \in \Phi$ defines a hyperplane orthogonal to α with respect to $(\, , \,)$, called a singular hyperplane.

Definition: A connected component of the complement of the singular hyperplanes is called a Weyl chamber.

Notice that each Weyl chamber defines an order on E and conversely. Hence 6.2 Theorem says that W acts simply transitively on the Weyl chambers i.e. for any two Weyl chambers C, C' there is a unique $w \in W$ such that $wC = C'$. Even more is true:

Theorem 1: Let C be a Weyl chamber. The closure of C (in E) is a fundamental domain for the action of W on E, i.e. the closure of C meets each W-orbit exactly once.

Proof: This [Wa 1] Theorem 1.1.2.7. □

We also need a result of Chevalley.

Theorem 2: Let F be a subset of E, W_F the subgroup of W that fixes F pointwise. Then W_F is generated by the reflections $w_\alpha, \alpha \in \psi$ that fix F pointwise.

Proof: This is [Wa 1] Theorem 1.1.2.8. □

6.4: Definition 1: A root system Φ is called reduced if $+\alpha$ and $-\alpha$ are the only roots in Φ proportional to $\alpha \in \Phi$.

The most important example is the root system associated to a complex semisimple Lie algebra $\mathfrak{g}$: More precisely we take $E = \mathfrak{h}^*$ and we let $\Phi = \{\alpha \in \mathfrak{h}^* | \alpha$ a root as in A3.1$\}$. All the properties of a reduced root system are contained in A3.2 Theorem.

Given two root systems (E_1,Φ_1) and (E_2,Φ_2) we can form their <u>direct sum</u> $(E_1+E_2,\Phi_1\cup\Phi_2)$.

<u>Definition 2</u>: A root system Φ is <u>irreducible</u> if Φ is not the direct sum of two subsystems.

One can show easily that the root system of a complex semisimple Lie algebra $\mathfrak{g}$ is irreducible iff $\mathfrak{g}$ is simple. Also

<u>Proposition</u>: Φ is irreducible iff there are no two orthogonal subsets A, B of Φ iff W acts irreducibly on E.

<u>Proof</u>: If W does not act irreducibly on E let $E = E_1 + E_2$ a W-invariant decomposition. Take $\alpha \in \Phi$ and let $\alpha = \alpha_1 + \alpha_2$, $\alpha_i \in E_i$. Then $-\alpha = w_\alpha \alpha = w_\alpha \alpha_1 + w_\alpha \alpha_2 = -\alpha_1 - \alpha_2$. By W-invariance of the decomposition $w_\alpha \alpha_1 = -\alpha_1$, so $\alpha_1 - \frac{(\alpha_1,\alpha)}{(\alpha,\alpha)}\alpha = -\alpha_1$. As $(\alpha,\alpha_1) = (\alpha_1,\alpha_1) \neq 0$ unless $\alpha_1 = 0$ we get a linear dependence between α_1 and α_2. Therefore either α_1 or α_2 is 0 and every root lives either in E_1 or E_2. As $E_1 \perp E_2$ this defines two orthogonal subsets of Φ. The other claims are obvious. □

<u>7</u>: Now we study the effects of a real structure on the root system. For clarity, we first deal with an abstract root system.

<u>7.1: Definition</u>: Let Φ be a reduced root system. If σ is a linear involutive isometry of E such that $\sigma\Phi = \Phi$ and $\sigma \neq \pm 1$ then (Φ,σ) or Φ is called a <u>σ-system of roots</u>. We call Φ <u>normal</u> iff, for all $\alpha \in \Phi$, $\alpha^\sigma - \alpha \notin \Phi$.

In the following Φ will always be a normal σ-system.

We let $\Phi_+ = \{\alpha \in \Phi | \alpha \neq -\alpha^\sigma\}$ and $\Phi_- = \{\alpha \in \Phi | \alpha = -\alpha^\sigma\}$. Also let $E = E_+ + E_-$ be the decomposition into the ± 1-eigenspaces of σ. Finally, we can project $\alpha \in \Phi_+$ to $\tilde{\alpha} \in E_+$. The collection of these $\tilde{\alpha}$ is called Σ. Then Σ is finite and generates E_+. In fact, Araki proved:

Proposition: For Φ as above Σ is a root system in E_+.

Proof: This is [Wa 1] Proposition 1.1.3.1. □

Let us notice that Σ is not a reduced root system in general.

7.2: Notice that Φ_- is a root system. Let W_- be the Weyl group of Φ_- and $W_\sigma = \{w \in W | w\sigma = \sigma w\}$. Then W_σ is the subgroup of $W = W_\Phi$ consisting precisely of those elements which stabilize E_+. Also W_- is a normal subgroup of W_σ.

Proposition: Restriction of $w \in W_\sigma$ to E_+ defines a homomorphism from W_σ to W_Σ with kernel W_-.

Proof: This is [Wa 1] Proposition 1.1.3.3. □

7.3: Definition: The multiplicity $m(\lambda)$ for $\lambda \in \Sigma$ is the cardinality of $\alpha \in \Phi_+$ such that the orthogonal projection to E_+ is λ.

The multiplicities turn out to be an important invariant in classifying real semisimple Lie algebras.

7.4. We discuss the most important example of a σ-system, namely let $\mathfrak{g}$ be a real semisimple Lie algebra. Let Φ be the root system of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ as in A5.2 and

$\mathfrak{h}$ a Cartan subalgebra that contains $\mathfrak{a}$. Then $\mathfrak{h} = (\mathfrak{h}_\cap \mathfrak{k}) + \mathfrak{a}$. Let σ be the complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ induced by $\mathfrak{g}$. Let $\bar{\ }$ denote complex conjugation on $\mathbb{C}$. For $\alpha \in \mathfrak{h}^*_{\mathbb{C}}$ let $\alpha^\sigma(H) = \overline{\alpha(\sigma H)}$, $H \in \mathfrak{h}_{\mathbb{C}}$. Choose compatible orderings on $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathfrak{h}^*_{\mathbb{C}}$, i.e. the restriction of $\alpha \geq 0$ to $\mathfrak{a}_{\mathbb{C}}$ is also positive (≥ 0).

<u>Lemma</u>: For all $\alpha \in \Phi$, $\alpha^\sigma - \alpha \notin \Phi$. Therefore (Φ, σ) is a normal σ-system.

<u>Proof</u>: This is [Wa 1], Lemma 1.1.3.6. □

Now the preceding machinery applies and we get a root system Σ, called the root system of $(\mathfrak{g}, \mathfrak{a})$. One can prove that $\mathfrak{a}$, Σ and the multiplicities are a complete invariant of $\mathfrak{g}$ (follows from the classification). We may interpret Σ as a subset of $\mathfrak{a}^*_{\mathbb{C}}$. Choosing compatible orders gives us a correspondence between fundamental systems of Φ and Σ (after fixing an order on Φ_-).

<u>8</u>: We describe the Iwasawa decomposition of $\mathfrak{g}$ into a compact and a solvable respectively nilpotent subalgebra.

<u>8.1</u>: We assume the notations from 7.4. For each $\lambda \in \mathfrak{a}^*$ let $\mathfrak{g}^\lambda = \{X \in \mathfrak{g} : [H,X] = \lambda(H)X$ for each $H \in \mathfrak{a}\}$. Since $\mathrm{ad}\,\mathfrak{a}$ is a commuting family of self-adjoint operators we get a decomposition $\mathfrak{g} = \sum_{\lambda \in \mathfrak{a}^*} \mathfrak{g}^\lambda$. By comparison with the root space decomposition of $\mathfrak{g}_{\mathbb{C}}$ we see that $\mathfrak{g}^\lambda \neq 0$ iff $\lambda \in \Sigma$ or $\lambda = 0$. Hence $\mathfrak{g} = \sum_{\lambda \in \Sigma} \mathfrak{g}^\lambda + \mathfrak{g}^0$. Fix an order on Σ and let Σ^+ denote the positive roots. Let $\mathfrak{n}^+ = \sum_{\lambda \in \Sigma^+} \mathfrak{g}^\lambda$. Since $[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subset \mathfrak{g}^{\lambda+\mu}$ $\mathfrak{n}^+$ is a nilpotent subalgebra. Let $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}^+$. Then $\mathfrak{s}$ is a solvable subalgebra.

Theorem: We have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^+$.

Proof: This is [He 1], VI, §3, Theorem 3.4. □

8.2: Let N^+, A be the subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}^+$. We get the global decomposition

Theorem: The map $K \times A \times N^+ \to G$ given by $(k,a,n) \mapsto k \cdot a \cdot n$ is a diffeomeorphism. Moreover, A and N^+ are simply connected. In particular, G/K is diffeomorphic to an $\mathbb{R}^n$, some n.

We call $G = K \cdot A \cdot N^+$ an Iwasawa decomposition. Any two Iwasawa decompositions are conjugate.

Proof: The first two claims are [He 1], VI, §5, Theorem 5.1. Uniqueness of the decomposition follows easily from the uniqueness of the Cartan decomposition (A4.3) and the maximal subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ (A5.2). □

8.3: Proposition: The centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is $\mathfrak{a} + \mathfrak{m}$ where we let $\mathfrak{m} = (\sum_{\alpha \in \Phi_-} \mathfrak{g}^\alpha) + (\mathfrak{h} \cap \mathfrak{k})$. Also $\mathfrak{m} \subset \mathfrak{k}$.

Proof: The centralizer of $\mathfrak{a}$ clearly consists of $\mathfrak{h}$ and all those $\mathfrak{g}^\alpha$ whose root α is identically 0 on $\mathfrak{a}$ i.e. $\alpha \in \Phi_-$. Notice that $\mathfrak{m} \cap \mathfrak{p} = \{0\}$ since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Notice that $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ (from an eigenvalue consideration of θ). Hence if $K + P \in \mathfrak{m}$ then $K \in \mathfrak{m}$ and $P \in \mathfrak{m}$, so by the above $\mathfrak{m} \subset \mathfrak{k}$. □

8.4: Definition: The complex dimension of $\mathfrak{h}_{\mathbb{C}}$ is called the complex rank of $\mathfrak{g}$, while the rank, real rank or split rank is the real dimension of $\mathfrak{a}$.

Rank 1 and rank ≥ 2 groups have different fundamental properties.

8.5: With the real structure on the root system we can describe the terms of the Cartan decomposition: Let $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h} \cap \mathfrak{k}$. Then

Proposition: The following decompositions are direct:

$$\mathfrak{k}_{\mathbb{C}} = (\mathfrak{h}_{\mathfrak{k}})_{\mathbb{C}} + \sum_{\alpha \in \Phi_-} (\mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha}) + \sum_{\alpha \in \Phi_+} \mathbb{C}(X_\alpha + \theta X_\alpha)$$

and $$\mathfrak{p}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} + \sum_{\alpha \in \Phi_+} \mathbb{C}(X - \theta X_\alpha).$$

where θ is the Cartan involution.

Proof: This is [He 1], VI, §3, Lemma 3.6. □

8.6: We describe the Weyl group in terms of the semisimple group. Let M^* be the normalizer, M the centralizer of $\mathfrak{a}$ in K i.e.

$$M^* = \{k \in K | \mathrm{Ad}\, k\, \mathfrak{a} = \mathfrak{a}\}$$

and $$M = \{k \in K | \mathrm{Ad}\, K\, H = H \text{ for all } H \in \mathfrak{a}\}.$$

Then M^* and M are compact as they are closed in K, and they have the same Lie algebra $\mathfrak{m}$: this is clear for M and follows easily for M^*. If there is any root $\alpha \notin \Phi_-$ such that X_α enters into some term of the root space decomposition of an element $X \in \mathrm{Lie}\, M^*$ then $[X_\alpha, H] = \alpha(H) X_\alpha$ for some $H \in \mathfrak{a}$ with $\alpha(H) \neq 0$ and X clearly does not normalize $\mathfrak{a}$. In particular M^*/M is a finite group and it acts on $\mathfrak{a}$.

Proposition 1: By the embedding into $Gl(\mathfrak{a})$ by the adjoint action, M^*/M is the Weyl group of the root system of $(\mathfrak{g},\mathfrak{a})$.

Proof: This is a consequence of [He 1], VII, §2 (Theorem 2.12 in particular). □

As a corollary we see that the Weyl group of the complex root system is $\text{Norm}\ \mathfrak{h}_\cap U/\text{Centr}\ \mathfrak{h}_\cap U$ where U is a compact real form of $G_{\mathbb{C}}$.

We also need a kind of 'rigidity' of $\mathfrak{a}$:

Proposition 2: Let A be a subset of $\mathfrak{a}$ and suppose $k \in K$ such that $\text{Ad}k(A) \subset \mathfrak{a}$. Then there exists an element $w \in W = W(\mathfrak{g},\mathfrak{a})$ such that $w \cdot H = \text{Ad}k\ H$ for each $H \in A$.

Proof: This is [He 1, VII, §2, Proposition 2.2. □

Let us note that one can write down explicit formulas for representing elements in K of the Weyl group, cf. [Wa 1], Lemma 1.1.3.9 and Lemma 1.3.2.4.

8.7: Quite important is Cartan's polar coordinate decomposition:

Proposition: $G = K \cdot A \cdot K$.

Proof: Recall that $(k,P) \mapsto k \cdot \exp P$ is a diffeomorphism (A5.1 Theorem 1 (iii)). We only need to prove that $\exp \mathfrak{p} \subset K \cdot A \cdot K$ or that $\mathfrak{p} \subset \text{Ad}k \cdot \mathfrak{a}$. This is A5.2 Proposition 1(ii). □

9: The main point of this section is the Jordan decomposition of an element into a semisimple and unipotent part.

9.1: Definition: An element $X \in \mathfrak{g}$ is called semisimple if $\mathrm{ad}X$ is diagonalizable over $\mathbb{C}$. We call $X \in \mathfrak{g}$ nilpotent if $\mathrm{ad}X$ is nilpotent.

Proposition: Every element $X \in \mathfrak{g}$ can be written in a unique way as $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent and $[X_s, X_n] = 0$. Moreover, if Y commutes with X then Y commutes with X_s and X_n.

Proof: Cf. [Bl], p. 79. □

9.2: We also need the Jordan decomposition on the group level.

Definition: We call $x \in G$ semisimple if $\mathrm{Ad}x$ is semisimple. Moreover, the exponential of a nilpotent element is called unipotent.

Proposition: We can write $x \in G$ uniquely in the form $x = x_s x_u$ where x_s is semisimple, x_u is unipotent and x_s and x_u commute. Moreover, if y commutes with x then it also commutes with x_s and x_u.

Proof: This is [Wa 1], Proposition 1.4.3.3. □

9.3: One can characterize semisimple elements in terms of Cartan subalgebras.

Proposition: The centralizer $\mathfrak{g}_x$ in $\mathfrak{g}$ of a semisimple element x of $\mathfrak{g}$ or G is reductive in $\mathfrak{g}$ with $\mathrm{rank}\ \mathfrak{g}_x = \mathrm{rank}\ \mathfrak{g}$. In particular, the set of semisimple elements is the union of all Cartan subalgebras of $\mathfrak{g}$ respectively all Cartan subgroups of G.

Note: A Cartan subgroup is a centralizer in G of a Cartan subalgebra in $\mathfrak{g}$.

Proof: This is [Wa 1], Proposition 1.3.5.4 and 1.4.3.2. □

9.4: Definition: We call a semisimple element $x \in G$ elliptic if all the eigenvalues of Adx lie on the unit circle. We call x hyperbolic if the eigenvalues of Adx are positive real.

Proposition 1: Let $G = K \cdot A \cdot N$ be an Iwasawa decomposition. Then

(i) $G \ni g$ is elliptic iff g is conjugate to an element of K.

(ii) $G \ni g$ is hyperbolic iff g is conjugate to an element of A.

(iii) $G \ni g$ is unipotent iff g is conjugate to an element of N.

Proof: This is [He 1], IX, §7, Theorem 7.2. □

Proposition 2: Every semisimple element x has a unique decomposition $x = e \cdot h$ where e is elliptic and h hyperbolic.

Proof: This is in [Mo 2]. □

Definition: We call h in $x = e \cdot h$ the polar part of a semisimple element $x \in G$. We write $h = \text{pol } x$. For arbitrary $x \in G$ let $\text{pol } x = \text{pol } x_s$ where $x = x_s \cdot x_u$ is the Jordan decomposition.

10: We discuss the Bruhat decomposition and parabolic subgroups. We will always assume that G has finite center.

10.1: Definition: For an Iwasawa decomposition $G = K \cdot A \cdot N_+$ let M = centralizer A in K and set $P = M \cdot A \cdot N_+$. We call P a minimal parabolic in G.

Notice that P is a closed subgroup of G and that all minimal parabolics are conjugate. Recall from A8.6 Proposition 1 that each $w \in W$ (the Weyl group) has a representative m in $M^* =$ normalizer of A in K. Since two representatives of w differ only by an element in M the double coset PmP only depends on w. By abuse of notation we write PwP for PmP. We have the Bruhat decomposition

Theorem: We can decompose G into a disjoint union of double cosets: $G = \bigcup_{w \in W} PwP$.

Proof: This is [Wa 1] Theorem 1.2.3. □

A simple but quite important consequence is

Proposition: The minimal parabolic P is the normalizer of N_+ in G. Moreover, P is selfnormalizing. Also there exists a unique double coset of P whose dimension is equal to G. It is open and dense in G and has full measure.

Proof: This is [Wa 1] Propositions 1.2.3.4 and 1.2.3.5. □

10.2: Definition: A parabolic subgroup is any subgroup containing a minimal parabolic subgroup.

Quite surprisingly one can describe all parabolics containing a given minimal parabolic P quite easily:

Fix a fundamental system ψ for Σ. For any subset $\theta \subset \psi$ let W_θ be the subgroup of W generated by the w_i for $i \in \theta$. Notice that $W_\psi = W$ by A6.2 Proposition.

We let $P_\theta = PW_\theta P$. This is a parabolic subgroup. Clearly, $P_\phi = P$ and $P_\psi = G$ by the Bruhat decomposition. Tits proved

Theorem: The subgroups P_θ are all the parabolics containing P. No two of them are conjugate or equal. Hence there are 2^r of them where r is the split rank of G. All the parabolics are their own normalizers.

Proof: This is [Wa 1] Theorem 1.2.1.1. □

To describe the Lie algebra of P_θ let $\Delta \subset \Sigma$ be the set of all roots that are either positive or that are a $\mathbb{Z}$-linear combination of $i \in \theta$. Then

Proposition: The Lie algebra of P_θ is $\mathfrak{m} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$.

Proof: This is contained in [Wa 1] Theorem 1.2.4.8. □

10.3: Definition 1: The unipotent radical $R_u(P)$ is the greatest connected normal subgroup of P all of whose elements are unipotent.

Note: Cf. [Hu 1] 19.5 to see that this is well defined.

Definition 2: A Levi subgroup L of a parabolic P is a closed reductive subgroup L of P such that $P = L \cdot R_u(P)$ defines a unique decomposition $p = l \cdot r$ of any $p \in P$.

Proposition: Every parabolic possesses a Levi subgroup that is unique up to conjugation by $R_u(P)$.

Proof: This is [Wa 1] Proposition 1.2.4.14. □

For a parabolic P with a Levi subgroup L we let A be the unique maximal connected split abelian subgroup of the center of L. We let $M = \bigcap_{\chi \in X(L)} \ker \chi$ where $X(L)$ are all continuous homomorphisms of L into the multiplicative group of the reals. Clearly $L = M \cdot A$ and $M \cap A = \{1\}$. This gives us the Langlands decomposition of P:

$$P = M \cdot A \cdot R_u(P).$$

Clearly, any two Langlands decompositions of P are conjugate since any two Levi subgroups are conjugate. Notice that for a minimal parabolic P we have $R_u(P) = N_+$, $L = \text{centr}\ \mathfrak{a}$ is a Levi subgroup, $A = \exp \mathfrak{a}$ the split abelian component as above and $U = \text{centr}\ A \cap K$ (otherwise said, our notation is consistent with previous denominations in this case.)

10.4: Definition: A homogeneous space G/H is called a boundary of G if for every probability measure μ on G/H there exists a sequence $x_n \in G$ such that $x_n * \mu$ converges to a point measure.

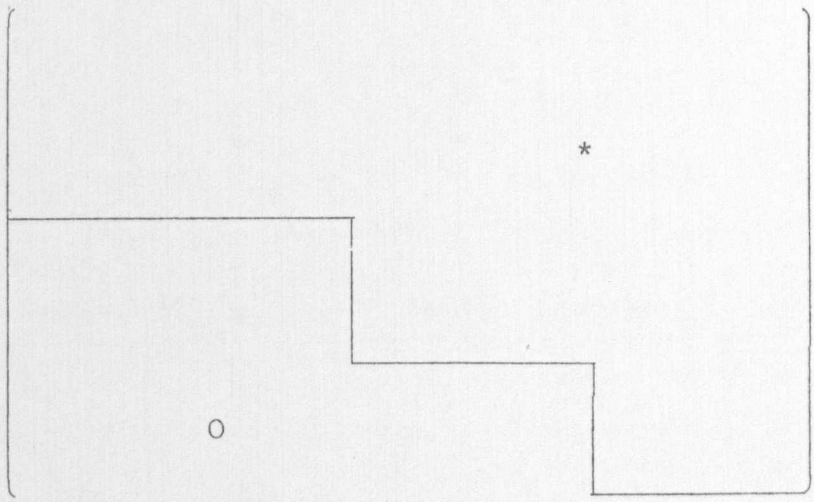
Fürstenberg proved that boundaries are very special:

Theorem: G/H is a boundary iff H is a parabolic.

Proof: This is [Wa 1] Proposition 1.2.3.11. □

Note: One can compactify the globally symmetric space G/K in such a way that G/P for P a minimal parabolic arises as the only compact G-orbit. Hence the name boundary (cf. I Section 2.3.)

10.5: We discuss $G = SL(n, \mathbb{R})$ as an example. For each sequence of integers $n_1 < n_2 < \ldots < n_k < n$ consider a flag $V_1 \subset V_2 \subset \ldots \subset V_k$ of linear subspaces of $\mathbb{R}^n$ of dimensions dim $V_i = n_i$. The space of all such flags for a given sequence (n_i) is called a <u>flag manifold</u> F. Clearly $SL(n,\mathbb{R})$ acts transitively on each F with an isotropy subgroup consisting of matrices of the form,



i.e. generalized upper triangular matrices. It is easy to see that these are all the parabolics of $SL(n,\mathbb{R})$. Accordingly, the flag manifolds are all the boundaries of $SL(n,\mathbb{R})$. The minimal parabolic is the group of upper triangular matrices.

<u>11</u>: Without going into any details we want to mention the notion of an algebraic group.

<u>Definition</u>: An algebraic variety G is called an algebraic group if G has a group structure such that $(x,y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are morphisms of algebraic varieties.

<u>Note</u>: Here $G \times G$ carries the Zariski product topology.

The basic fact for our purposes is

<u>Proposition</u>: Every connected semisimple Lie group G is locally isomorphic (i.e. has the same Lie algebra as) to a

real algebraic group G'.

Proof: The adjoint group of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is a complex algebraic group defined over $\mathbb{R}$. The real points form a real algebraic group whose connected component of the identity is the adjoint group of $\mathfrak{g}$. By semisimplicity this is locally isomorphic to G. □

This fact allows us to use the language of algebraic geometry. Also we would like to point out that the theory of semisimple algebraic groups is similar to the theory of real semisimple groups, cf. [Hu 1] and [B - T]. Let us note that the notion of a parabolic subgroup is very natural in this context.

Definition: A parabolic subgroup P of a real semisimple algebraic group G is an algebraic subgroup such that G/P is a complete projective variety.

Finally, we want to remark on Chevalley's theorem on the rationality of semisimple Lie algebras.

Theorem: Every real semisimple Lie algebra has a Weyl basis with rational structure constants.

Proof: This is [Ch 1], Theorem 1. □

12: We outline the theory of locally and globally symmetric spaces. We assume all the differential geometry used.

12.1: Let M be a Riemannian manifold. Let $p \in M$ and define the geodesic symmetry s_p to be the map defined locally by: for any geodesic γ with $\gamma(0) = p$ let $s(\gamma(t)) = \gamma(-t)$.

Definition 1: The manifold M is called locally symmetric if s_p is an isometry in a neighborhood of p for all $p \in M$. We call M globally symmetric if s_p extends to an isometry of all of M for all $p \in M$. The main point is that each complete locally symmetric manifold M has a globally symmetric universal cover. (Cf. [He 1], IV, §6, Corollary 5.7) and that one can classify globally symmetric spaces. The first step is the

Theorem: Let M be globally symmetric. Then the connected component G of the isometry group of M acts transitively on M with compact isotropy group K.

Proof: This [He 1], IV, §3, Theorem 3.3 □

One can further classify M into non-compact, compact and Euclidean type. Any M admits a decomposition (in the sense of de Rham) into these types. We are only interested in the non-compact type:

Definition 3: A globally symmetric space M is said to be of the non-compact type if G is a semisimple group with no compact factors and K a maximal compact of G.

This program reduces the study and classification of symmetric spaces to a group theoretic problem. Conversely, given a semisimple group G without compact factors and a maximal compact K we can give G/K a globally symmetric structure: Recall from A4.3 Proposition that the Cartan-Killing form B is positive definite on $\mathfrak{p}$. Clearly, we can identify the tangent space to G/K at $1 \cdot K$ with $\mathfrak{p}$. Hence B defines a Riemannian structure on G/K. The group-theoretic and the Riemannian exponential map are very similar: Let $X \in \mathfrak{p}$ then $\mathrm{Exp}_{Riem}\, X = \exp X \cdot K$. Moreover, A5.1 Theorem 1(ii) asserts that G/K is globally symmetric. One can also calculate the curvature in terms of the group structure.

Proposition 1: Let R denote the curvature tensor of G/K. Then at $0 = 1 \cdot K$ we find that

$$R_0(X,Y)Z = -[[X,Y],Z] \quad \text{for } X,Y,Z \in \mathfrak{p}.$$

Proof: This is [He 1], IV, §4, Theorem 4. □

This allows us to characterize the spaces of non-compact type:

Proposition 2: A globally symmetric space M is of non-compact type iff M has non-positive curvature and none of the de Rham components are flat.

Proof: This clear from [He 1], IV, §3, Theorem 3.1. □

Let us finally note that the fundamental group of a locally symmetric space of the non-compact type and of finite volume defines a lattice Γ in the semisimple group G, i.e. a discrete subgroup Γ of G such that G/Γ has finite volume. Conversely, any torsion-free lattice gives rise to a locally symmetric space.

We will need the following:

Definition: A lattice $\Gamma \subset G$ in a connected semisimple group without compact factors is reducible if G admits connected normal subgroups H, H' such that $G = H \cdot H'$, $H \cap H'$ is discrete and $\Gamma/(\Gamma \cap H)(\Gamma \cap H')$ is finite. We call Γ irreducible if Γ is not reducible.

One has the following decomposition theorem.

Proposition 3: Let Γ be a lattice in a connected semi-simple Lie group G without compact factors. Then there exist connected closed normal subgroups $G_1,\ldots,G_n$ such that $G_i \cap \Gamma$ is a lattice in G_i and $\prod_{i=1}^{n} (G_i \cap \Gamma)$ is a subgroup of finite index of Γ. Moreover, let π_i denote the projection $\pi_i\colon\ G \to G/G_1\ldots\hat{G}_i\ldots G_n$. Then $\Gamma' = \bigcap_{i=1}^{n} \pi_i^{-1}\pi_i\Gamma$ is a lattice in G and $\Gamma' \supset \Gamma$.

Proof: This is obvious from [Ra 1], Corollary 5.19. □

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