ON THE STRUCTURE OF THE YANG-MILLS-HIGGS EQUATIONS ON $\mathbb{R}^3$.

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Preface

The Yang-Mills-Higgs theory has its origins in Physics. It describes particles with masses via the Higgs mechanism and predicts magnetic monopoles.

We study here the mathematical aspects of the theory following an analytical and geometric approach. Our motivation comes from physics and we work all the time with the full Lagrangian of the theory. At the same time, we are interested in it from the variational point of view, as a functional on an infinite dimensional space and as a system of non-linear equations on a non-compact manifold with finite energy as the only constraint.

We are concerned mainly with the configuration space of the theory, the existence of solutions and their behaviour at infinity.

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INTRODUCTION.

1. Basic definitions.

The natural setting for the Yang-Mills-Higgs Theory is in terms of principal and associated bundles over $\mathbb{R}^3$. For that, let P be a principal G-bundle over $\mathbb{R}^3$. Since $\mathbb{R}^3$ is contractible, the bundle will always be isomorphic to the product one $\mathbb{R}^3 \times G$, but not necessarily equal to the product one. In particular, there exist global sections. In general, G can be any compact Lie Group with a non-degenerate inner product on its Lie Algebra **g**, invariant under the Adjoint action of the group. Let L be a finite dimensional vector space with an inner product, so that G acts on it unitarily, i.e. there is a homomorphism

$$T : G \longrightarrow Aut(L).$$

This differentiates to a representation of the Lie Algebra

$$t : \mathbf{g} \longrightarrow End(L).$$

Let E denote the associated vector bundle $P \times_G L$.

Example 1: Take G = SU(2), L = su(2), its Lie Algebra, and consider the adjoint action of SU(2) on su(2) and the Lie bracket action of su(2) on itself. An inner product is given by minus the Killing product. AdP denotes the corresponding associated bundle.

We shall consider connections ω on P as **g**-valued 1-forms. Given a connection ω, we denote by Ω its curvature[1]

$$\Omega = d\omega + 1/2\,[\,\omega \wedge \omega\,].$$

More often than not, we shall be working with the pull-back of ω by a global section s:

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[1] There are many natural operations one can perform on the sections of the various bundles, like taking inner products, Lie brackets, wedge products and, of course, all their combinations. We hope that the notation is self-explanatory.

$$A = s^*\omega = \sum_{i=1}^{3} A_i \, dx_i$$

and the curvature

$$F_A = \sum_{i<j} (\partial_j A_i - \partial_i A_j + [A_i, A_j]) \, dx_i \wedge dx_j .$$

We shall also consider sections Φ of E and of the other associated bundles. The covariant differentiation a connection ω induces on sections of E can be defined by viewing a section Φ as an equivariant function from P to L. Then

$$d_A : \Omega^0(E) \longrightarrow \Omega^1(E)$$

$$\Phi \longmapsto d_A \Phi = d\Phi + t(\omega)\Phi. \qquad (1)$$

Here Ω^p denotes the degree p forms with values in the bundle E. Notice that thanks to the inner products on the Lie Algebra and the representation space L, all bundles have natural inner products defined on them.

The operator d_A extends in two different ways to higher degree forms:

i) By the covariant exterior derivative

$$d_A : \Omega^p(E) \longrightarrow \Omega^{p+1}(E)$$

given by

$$d_A s(X_1, \ldots, X_{p+1}) = \sum_i (-1)^i d_A(s(X_1, \ldots, \widehat{X}_i, \ldots, X_{p+1}))(X_i)$$

$$+ \sum_{i,j} (-1)^{j+i} s([X_i, X_j], \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{p+1}).$$

ii) By the full covariant derivative:

$$\nabla_A : \Omega^p(E) \longrightarrow \Gamma(T^* \otimes \Lambda^p T^* \otimes E).$$

This is defined by formula (1) when we view $\Omega^p(E)$ as $\Omega^0(\Lambda^p T^* \otimes E)$.

Note that d_A is the antisymmetrization of ∇_A. The two operators clearly agree on Ω^0.

An important ingredient of the theory is the group of gauge transformations of the bundle P. They can be realized in many different ways:

i) as bundle isomorphisms from P to itself over the identity

ii) as sections of the bundle $\mathrm{Aut}P = P \times_G G$, where G acts on itself by conjugations.

iii) as mappings $g: \mathbb{R}^3 \longrightarrow G$, once a trivialization has been chosen.

We equip the gauge transformations by an inner product, too. This can be done either by viewing G as a group of matrices and therefore naturally sitting in some $\mathbb{R}^n$, or by choosing an invariant metric on G.

The automorphisms, viewed as G-valued functions on $\mathbb{R}^3$, act on A and Φ:

$$g \cdot A = \mathrm{Ad}_g A + g d g^{-1}$$

and

$$g \cdot \Phi = T(g)(\Phi).$$

On L consider a G-invariant function

$$V : L \longrightarrow \mathbb{R}$$

with the following properties:

i) It is smooth and G-invariant as a function defined on the representation space.

ii) It takes only nonnegative values.

iii) It gives symmetry breaking, that is it achieves the minimum value zero on a single non-trivial orbit, the vacuum. (This excludes 0 as a minimum.)

iv) The vacuum is a non-degenerate critical manifold: the kernel of the Hessian of V at a point on the vacuum orbit is exactly the tangent space to the orbit.

v) It is of degree at most 4. When V is not given by a polynomial in $|\Phi|$ we make sense of this condition by asking that $D^{(n)}_{\Phi} V = 0$ as an operator, for $n \geq 5$ and for any Φ in the representation space.

These properties are far from being arbitrarily chosen. We shall explain each of them in detail when we come to the physical part of the theory. We call such a V a **Higgs Potential.**

Example 2: On su(2), consider the function $V(\Phi) = \frac{\lambda}{2}(|\Phi|^2 - 1)^2$, $\lambda > 0$. The vacuum is the unit sphere orbit in su(2).

Among the aims of the Yang-Mills-Higgs Theory is to prove the existence, study the properties and perhaps make a use of the solutions of the Yang-Mills-Higgs equations:

$$d_A{}^*F_A = -\sum_i \langle t(\xi_i)\cdot\Phi, \nabla_A\Phi\rangle \xi_i \qquad \text{(YMH1)}$$

$$\nabla_A{}^*\nabla_A\Phi = \frac{\partial V}{\partial \Phi}\,. \qquad \text{(YMH2)}$$

Here $\{\xi_i\}$ denotes a basis of the Lie Algebra **g**. $\nabla_A{}^*$ is the formal L^2-adjoint of ∇_A. Equations (YMH1) and (YMH2) are (very formally) the variational equations of the **Yang-Mills Higgs Lagrangian** (or energy functional or action functional)

$$YMH(A,\Phi) = \int_{\mathbb{R}^3} \{\frac{1}{2}|F_A|^2 + \frac{1}{2}|\nabla_A\Phi|^2 + V(\Phi)\}\, d^3x$$

defined on pairs (A,Φ) of connections on P and sections of E. The precise definition of the domain of this functional will occupy Chapter B of this thesis.

Example 3. For the adjoint-SU(2) case with potential $V(\Phi) = \lambda/2(|\Phi|^2 - 1)^2$ the variational equations become

$$d_A{}^*F_A = [\Phi, \nabla_A\Phi]$$

$$\nabla_A{}^*\nabla_A\Phi = \frac{\lambda}{2}(|\Phi|^2 - 1)\Phi\,.$$

Recall here the Yang-Mills Lagrangian

$$\int_M |F_A|^2 dM$$

and its variational equation

$$\nabla_A {*F_A} = 0 .$$

The connection between Yang-Mills and Yang-Mills-Higgs theory will be the topic of Chapter D.

2. The Physics Point of View and the Mathematical Problems.

From a mathematical point of view, the first property one notices for the Yang-Mills-Higgs Lagrangian is that it is bounded below by 0. Second, it is gauge invariant. The term $V(\Phi)$ in the functional is clearly invariant since V was chosen to be invariant. An easy calculation shows that as a gauge transformation g acts on A and Φ, F_A transforms to $Ad_g F_A$ and $d_A \Phi$ transforms to $T(g)\cdot(d_A\Phi)$. By the invariance of the inner products the value of the functional remains unaltered.

However, the power of the theory lies in its physical interpretations, see for example [G-O], [o' R], [I] and [C]. We are to think of A as a (gauge) potential and its curvature as the field it creates. In this field we have another field Φ, usually called the **Higgs field**, created by some massive particle. Then $V(\Phi)$ measures the potential, or self-interaction, of the Φ-field. The "coupling term" $\nabla_A \Phi$, the only one involving both A and Φ, measures their interaction. YMH(A,Φ) is the energy of the system.

The model has the following properties:

1) **Symmetry breaking:** Apart from the gauge group G, the choice of the potential V introduces another group to the theory: the vacuum consists of a single orbit. Any point on it has (up to conjugacy) the same isotropy subgroup of G. We denote it by H and refer to it as the **small** or the **unbroken group** or the theory. By choosing the potential V carefully, we can incorporate in the same model two different groups, such as the electromagnetic U(1) and the SU(3) of the weak interactions. Unification

is served this way.

Chapter D shows the precise manner, from a mathematical point of view, in which starting from a G-bundle and a G-connection we end up with an H-bundle and a connection on it.

2) **Massive components (Higgs Effect):** It seems to be common in Physics to interpret as masses the positive coefficients in front of the squares of the fields. In this sense the Lagrangian offers masses to components of both A and Φ. The starting point is a fixed point on the vacuum orbit, Φ_0 say.

In an adjoint representation for example, write the coupling term as

$$\langle d_A\Phi, d_A\Phi\rangle = \langle d\Phi + [A,\Phi], d\Phi + [A,\Phi]\rangle = \langle [A,\Phi_0], [A,\Phi_0]\rangle + R$$
$$= \langle -\mathrm{ad}^2\Phi_0 A, A\rangle + R,$$

where R denotes the remaining terms. Since $-\mathrm{ad}^2\Phi_0$ is a positive operator, its kernel determines the zero mass components of the gauge potential. The rest of the components will have strictly positive masses. If we denote by **h** the Lie subalgebra of the isotropy subgroup H and its complement by $\mathbf{h}^\perp$, we see that the **h**-components of A have zero masses, while the $\mathbf{h}^\perp$-components acquire positive masses.

For the Φ field, expanding V around the vacuum Φ_0, we have

$$V(\Phi) = V(\Phi_0) + DV_{\Phi_0}(\Phi - \Phi_0) + D^2V_{\Phi_0}(\Phi - \Phi_0)^2 + R.$$

By property (iii) of the potential, $V(\Phi_0) = 0$ and since Φ_0 is a critical point $DV_{\Phi_0} = 0$. The Hessian D^2V at the minimum Φ_0 is again a positive operator. By property (iv) of V, we can write the Hessian as the diagonal matrix

$$\begin{bmatrix} 0 & & & & \\ & \cdot & & & \\ & & M_{d+1} & & \\ & & & \cdot & \\ & & & & M_n \end{bmatrix}$$

where d is the dimension of the orbit and n the dimension of the whole group. That is, only the components of Φ transversal to the vacuum orbit acquire masses.

Example 4: Consider the case of the gauge group SU(2) acting on its Lie Algebra via the adjoint representation and the potential V of Example 2. Since we shall come back to this example again and again, we shall name it the **adjoint-SU(2) case.** The invariance of the inner product gives for an infinitesimal gauge transformation ξ

$$< \exp t\xi\Phi_0, \exp t\xi\Phi_0 > = < \Phi_0, \Phi_0 >,$$

which on differentiation gives

$$< \frac{d}{dt}\Big|_0 \exp t\xi\Phi_0, \Phi_0 > = 0,$$

i.e. Φ_0 is orthogonal to the orbit of Φ_0. Since the isotropy group is U(1), Φ_0 generates the isotropy. One expects then the fields parallel to Φ_0 to be massless and the rest of the fields to be massive.

We shall see in Chapter D how much of these arguments actually survive mathematical scrutiny. It suffices to say here that one expects the fields generated by massive particles to decay very fast (exponentially) as one goes away from the particle. On the other hand, non-massive fields should decay much slower (power law decay).

3) **Monopole solutions:** One could argue that a "decently" behaving field Φ should have a limit Φ_∞ as the distance from the origin tends to infinity. Then for the energy YMH(A,Φ) to be finite, the only alternative is that Φ_∞ defines a mapping from S^2 to the vacuum orbit G/H. This in turn defines an element of $\pi_2(G/H)$. This homotopy class has the significance of a magnetic charge, see [S2], [H-R1] and [H-R2]. In this sense, the theory predicts **monopole solutions.**

By "decent" field we mean a field with appropriate decay. We show in Chapter C why finite energy is enough to define the limit Φ_∞. However, there is no guaranty that Φ_∞ is continuous for all finite energy pairs. One must assume further decay conditions, as for example in [J-T], section II.3.

Example 5: The fibration

$$H \longrightarrow G \longrightarrow G/H$$

gives the exact homotopy sequence

$$\cdots \longrightarrow \pi_2(G) \longrightarrow \pi_2(G/H) \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \cdots .$$

For G simply connected, $\pi_2(G/H) \approx \pi_1(H)$. For the adjoint-SU(2) case we have that the magnetic charge is defined as an element of $\pi_1(U(1)) \approx \mathbb{Z}$. The corresponding integer can be calculated in terms of explicit integrals, see [S2], [J-T], [H-R1] and [H-R2].

From a mathematical point of view, we are faced with a variational problem which leads to a system of non-linear, non-elliptic partial differential equations. The non-linearity results from the quadratic term of the curvature in (YMH1). The non-ellipticity is a symptom of gauge invariance in (YMH2) and can be cured by fixing the gauge. It is however incurable in (YMH1): $d_A{*}F_A$ will give a top term $d_A{*}d_A A$ which becomes elliptic only when working in a gauge with $d{*}A = 0$. It is not known whether such gauges exist globally. We shall have the opportunity to comment on this again.
We are also faced with the non-compact domain $\mathbb{R}^3$. This gives the rich structure to the theory from the physics point of view, as the discussion above shows. Mathematically, it gives non-trivial decay properties to the fields but also creates some very subtle analytic difficulties. To make things more interesting, the theory is not conformally invariant in three dimensions. Easy compactifications are thus excluded.
Why should we insist on three dimensions? For one thing, it is the natural thing to do when we study a static physical theory. Further, a scaling argument, see [J-T], page 32, shows that in the SU(2)-adjoint case, for example, there are non-trivial solutions only in dimensions 2, 3 and 4. In dimension 4 any solution is gauge equivalent to a

pure Yang-Mills field. The dimension 2 has been studied extensively, too, see [J-T].

3. The Prasad-Sommerfield Limit.

Both mathematicians and physicists have studied the SU(2)-adjoint case. Much is known for the case when the parameter λ in the potential

$$V(\Phi) = \frac{\lambda}{2}(|\Phi|^2 - 1)^2$$

is zero. This is known as the **Prasad-Sommerfield** limit. Prasad and Sommerfield gave exact solutions and their name to this part of the theory. The idea here has been that results obtained for the Lagrangian

$$PS(A,\Phi) = \int_{\mathbb{R}^3} \{\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\Phi|^2\} d^3x$$

should in some way carry over to the full YMH Lagrangian.

The PS Lagrangian is exceptional in that its minima correspond exactly to the solutions of the first order **Bogomol'nyi** equations

$$\pm d_A\Phi = *F_A.$$

If the functional PS is viewed as the dimensional reduction of the Yang-Mills functional in four dimensions, then the Bogomol'nyi equations correspond to the (anti)self-duality equations

$$\pm F_A = * F_A.$$

In the full Lagrangian case there are no corresponding equations for the minima of the theory.

The Prasad-Sommerfield limit has been extensively studied both by twistorial and by analytical means. There is no reason now for Φ to tend to any vacuum orbit at infinity, so this condition is added ad hoc.

For the Bogomol'nyi case the twistorial approach has been extremely successful, see [A-H], [H1] and [H2], for example. The complete set of solutions for the

Bogomol'nyi case is known, as well as the way these solutions form the moduli space, see [Don].

The analytical approach is mainly due to Taubes, see [T1-8]. The topology of the configuration space and the way it separates into path connected components according to magnetic charge is well understood. Taubes has also proved that infinitely many non-stable solutions exist with energy arbitrarily large, see [T4]. In the Prasad-Sommerfield limit the YMH functional " behaves like a good Morse function".

It is conceivable that the technics created for the Prasad-Sommerfield limit will carry over to the full Lagrangian case. This has not being carried out yet, see [G1] for an attempt.

4. Sobolev Spaces.

By picking a connection ∇, for example the canonical one, we can describe the space of all connections as the affine space $\nabla + \Omega^1(\mathrm{AdP})$. On this space and on the obviously linear space of sections of an associated bundle we define natural generalizations of the standard Sobolev norms. Here covariant differentiation replaces the usual derivatives. For a section s and for a connection A define

$$\| s \|_{L^p_k(A)} = \Big(\int_{\mathbb{R}^3} \sum_{j=0}^{k} | \nabla_A^{(j)} s |^p \Big)^{1/p}.$$

One would usually proceed by considering the completion of the compactly supported, smooth sections with respect to this norm. However, it is here that we have one of the complications of working over a non-compact domain: such a completion depends on the choice of the connection A.

For that, we need to define the local Sobolev spaces $L^p_{k,loc}$ as the set of measurable sections which lie in $L^p_k(B)$ for any compact domain B in $\mathbb{R}^3$. This definition is independent of the connection used on B. For all the theorems we shall be using

(Sobolev, Rellich, Sobolev Inequality), see [Pa1] and the last chapter of [J- T].

Since the Yang-Mills-Higgs Lagrangian involves first order differentiations on A (through the curvature) and on Φ (through the covariant derivative) it is reasonable to ask for A and Φ to be in the space $L^2_{1,loc}$. By the way they act on A and Φ, it is reasonable to ask for the gauge transformations to have one more derivative, so they must be in $L^2_{2,loc}$. Standard Multiplication theorems, see [Pa1], show that the integral of the Lagrangian is locally finite. Therefore, one more condition is needed: YMH(A,Φ) must be finite. We shall study the set of all such configurations much more carefully in Chapter B.

5. About this thesis.

The purpose of this thesis has been to study Yang-Mills-Higgs Theory when the Lagrangian includes a Higgs potential term (not the Prasad-Sommerfield limit), this being the case closer to the physical theory. We have tried throughout this thesis not make any other assumptions on the asymptotics of the fields apart from the finite energy one.

Inevitably, we learned the techniques we have used from what has already been studied. Hence most of the results we obtain concern the case G = SU(2), H = U(1) with G acting on its Lie Algebra via the adjoint representation. We understand that this is still far from being of any physical consequence. One has to deal with larger groups, the first case of some importance being H = SU(3)×U(1), see [H-R2]. The SU(2)-adjoint case then is only a model case, something to learn the rules of the game from.

Between the two mathematical trends of the theory, the twistorial and the analytical one, we have found ourselves following the latter. It seems to us that away from the self-duality Bogomol'nyi case a more variational point of view is needed. In that we benefited from K. Uhlenbeck's and C. Taubes' work. In fact, it is tempting to think of the Yang-Mills-Higgs functional as yet another example to be understood before an infinite-dimensional analogue of Morse Theory is developed (even when Palais-Smale type conditions fail).

6. Summary.

There are four main chapters in this thesis. In the first one we examine the existence of solutions. First we review the known ones and then we give a rigorous proof of the existence of spherically symmetric solutions along the lines of an argument by Romanov, Frolov and Schwarz. We find that the proof easily applies to any compact Lie Group as gauge group. We also argue why this method gives solutions of arbitrary charge for $H = U(1)$. We also take the opportunity to clarify some of the less obvious points of spherical symmetry as developed by the Russian school.

In the second chapter we prove that the configuration space of the theory, as sketched above, has a Banach (in fact Hilbert) Manifold Structure with respect to which the Lagrangian is at least C^1. The original motivation for this comes from the proof in the first chapter. One has to appeal to the Principle of Symmetric Criticality, a crucial ingredient of which is the Manifold Structure. If this seems overcautious, another justification for such a structure on the configuration space is given by the generalized Morse Theory plan, see [T6]. The proof is based on Floer's proof of the analogous fact for the Prasad-Sommerfield limit. It applies to the SU(2)-adjoint case.

The third chapter examines the asymptotics of finite energy configurations and the fourth the asymptotics of finite energy solutions. The mathematical meaning of symmetry breaking is presented in terms of reductions of bundles defined by the limit of the Higgs field. The main result is the proof of the fact that the limit of a non-abelian monopole is a Dirac monopole: we find the appropriate mathematical formulation and prove that the connection part of the solution has a limit that reduces to the bundle defined by the Higgs field. The reduced connection is a pure Yang-Mills field.

In the final chapter we summarize some of the problems we think should be tackled next.

Throughout this thesis C is a generic name for constants appearing in estimates. We emphasize their dependence only, not their precise value.

[B-L], [F-U] and [La] are some of the general references for gauge theories. The standard reference for monopole theory is [J-T].

A. SOME REMARKS ON THE EXISTENCE OF SOLUTIONS.

A0. Introduction.

We give a rigorous proof of the existence of spherically symmetric solutions. Spherical symmetry here is understood in terms of fixed points of a given lifting of the SO(3) action on $\mathbb{R}^3$ to the bundle P. The basic ideas can be found in the papers by the Russian authors included in the bibliography. The main point is to minimize directly the functional YMH over all spherically symmetric configurations. This we carry out in section A4, up to a Banach Manifold structure on the space of all configurations. Such a structure will be the topic of the next chapter.

A1. The known solutions.

Much is known about solutions in the Prasad-Sommerfield limit of the theory. There, we have explicit formulas and detailed knowledge of the moduli space in the Bogomol'nyi case. In the general case though, with self-interaction term $V(\Phi)$, there are only some existence results. We list here the ones we know:

1) Ansatz solutions: Starting from the original SU(2) solutions suggested by 't Hooft and Polyakov in ['t H] and [P] respectively, the first rigorous proof of existence of solutions was given by Tyupkin, Fateev and Schwarz in [T-F-S]. There they use an Ansatz describing configurations with specified angular dependence and they prove that the Yang-Mills-Higgs functional attains its minimum over all the configurations described by the Ansatz. Their solutions are for gauge group SU(2) and SU(3).
Their techniques have been generalized by Rawnsley in [R] to any gauge group and quite general potential V. The fact that the minimizers are indeed critical points has been checked in [Pl]. Using different techniques but more or less the same Ansatz, another proof of the existence of "dyon" solutions was given in [S-W].

When G = SU(2) all these solutions have magnetic charge 1, see [Pl], page 62.

2) Non-minimal solutions: Following Taubes' methods in [T2] for the Prasad-Sommerfield limit, Groisser has proved in [G1] the existence of non-spherically symmetric, non-minimal solutions for gauge group SU(2) in the adjoint representation and for the potential $V(\Phi) = \lambda/2(|\Phi|^2 - 1)^2$, $\lambda \le \lambda_0$. Notice the restriction on λ: it is the price one pays for using the methods used for $\lambda = 0$, the Prasad-Sommerfield limit. Groisser's solutions have zero magnetic charge.[1]

3) General spherically symmetric solutions: these we study in this chapter, were we give a rigorous proof of their existence along the lines of [R-F-S]. They are more interesting than those of the first category in that they are minimizers among all spherically symmetric configurations. Therefore they stand a better chance of being stable solutions, although this we have not been able to prove yet. Furthermore, they are not necessarily of magnetic charge 1, see the last section of this chapter.

A 2. The spherically symmetric case: Basic definitions and relations.

Let $\pi: P \longrightarrow \mathbb{R}^3$ be a principal G bundle over $\mathbb{R}^3$ with G a compact and non-abelian Lie group. For a representation $T: G \longrightarrow Aut(L)$ with L a finite dimensional vector space with G-invariant inner product, let $E \longrightarrow \mathbb{R}^3$ be the associated vector bundle. Let also t denote the action of **g**, t: **g** $\longrightarrow$ End(L).

The Lie group SO(3) acts on $\mathbb{R}^3$ as the usual orientation-preserving rotations. As in [S1], we consider liftings of this SO(3) action to bundle automorphisms of P, a differentiable homomorphism

$$\sigma: SO(3) \longrightarrow Aut(P)$$
$$R \longmapsto \sigma(R) = \sigma_R$$

[1]The solutions in Taubes' paper also have zero magnetic charge. However, Taubes claims in his paper that the same method should work in any monopole sector. He also thinks, [T8], that the "small λ" condition in Groisser should be removable: the starting point for the construction of a non-contractible loop in the space of all configurations should be two widely separated exact solutions as in (1), for example. Groisser has used two widely separated Prasad-Sommerfield solutions.

such that for any R in SO(3) the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\sigma_R} & P \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{R}^3 & \xrightarrow[R]{} & \mathbb{R}^3 \end{array}$$

Such liftings exist. For example, with respect to a trivialisation $P \approx \mathbb{R}^3 \times G$ of the bundle, consider the lifting $\sigma_R(x,g) = (Rx,g)$.

Given such a lifting we have an action of SO(3) on the space of connections C by:

$$SO(3) \times L^2_{1,loc}(AdP) \longrightarrow L^2_{1,loc}(AdP)$$

$$(R\,, A) \longmapsto (\sigma_{R^{-1}})^* A$$

The connection ω is **spherically symmetric with respect to the lifting σ** if

$$(\sigma_{R^{-1}})^* \omega = \omega \qquad \text{(A2.1)}$$

for any R in SO(3), that is if ω is a fixed point for the action of SO(3) on the connection forms.

At the same time, SO(3) acts on the Higgs sections of the associated vector bundle E as follows: For each automorphism σ_R of the bundle P there is a corresponding automorphism σ_R^E of the bundle E defined by

$$\sigma_R^E : E \longrightarrow E$$

$$[\,(p,l)\,] \longmapsto [\,(\sigma_R(p),l)\,]$$

for p in P and l in L. The action of SO(3) on the sections Φ is now given by:

$$SO(3) \times L^2_{1,loc}(E) \longrightarrow L^2_{1,loc}(E)$$

$$(R,\Phi) \longmapsto \sigma_R^E \circ \Phi \circ R^{-1}.$$

Again, Φ is spherically symmetric if it is a fixed point for this action:

$$\sigma_R^E \circ \Phi \circ R^{-1} = \Phi, \qquad \text{(A2.2)}$$

for all R in SO(3).

From now on we are assuming that a lifting σ of the SO(3) action on $\mathbb{R}^3$ has been chosen and we shall be studying spherical symmetry with respect to this lifting.

As in [H-S-V], we want to describe spherical symmetry in terms of trivialisations. One chooses a trivialisation

$$\varphi : P \longrightarrow \mathbb{R}^3 \times G$$

of the principal bundle P with special section $s(x) = \varphi^{-1}(x,e)$. For ω a spherically symmetric connection form on P consider the Lie Algebra-valued 1-form $s^*\omega = A$ on $\mathbb{R}^3$. For this pulled-back form the spherical symmetry condition can be described as follows:

For any R in SO(3) the section $\sigma_R \circ s \circ R^{-1}$ of P is related to s in terms of the trivialisation φ by

$$(\sigma_R \circ s \circ R^{-1})(x) = s(x)\tau_R(x) \qquad \text{(A2.3)}$$

where $\tau_R : \mathbb{R}^3 \longrightarrow G$ is the description of a gauge transformation in terms of the trivialisation. If the gauge transformations have smoothness properties or belong to Sobolev spaces then so does τ_R for all R. The usual law of transforming the pull-backs of connections by (local) sections (see [K-N]) becomes over the trivial bundle P

$$(\sigma_R \circ s \circ R^{-1})^*\omega = \tau_R^{-1}(s^*\omega)\,\tau_R + \tau_R^{-1} d\tau_R$$

or

$$(R^{-1})^* s^*(\sigma_R{}^*\omega) = \tau_R^{-1}(s^*\omega)\,\tau_R + \tau_R^{-1} d\tau_R\,.$$

Using that $\sigma_R{}^*\omega = \omega$ and that $A = s^*\omega$ we have

$$(R^{-1})^*A = \tau_R^{-1} A\, \tau_R + \tau_R^{-1} d\tau_R$$

which we finally write as

$$A = \tau_R\,(R^{-1})^*A\,\tau_R^{-1} + \tau_R\, d\tau_R^{-1}\,. \qquad \text{(A2.4)}$$

This is the condition used in [R-S-T], [S1] and [R-F-S].

We can easily repeat the same for the spherically symmetric sections of the adjoint bundle: having fixed the section s of the bundle P, we can describe a section

$$\Phi : \mathbb{R}^3 \longrightarrow E$$

in terms of a function $\Phi' : \mathbb{R}^3 \longrightarrow L$:

$$\Phi(x) = [\,(s(x),\Phi'(x))\,]\,.$$

The spherical symmetry condition $\sigma^E_R \circ\Phi\circ R^{-1} = \Phi$ then gives:

$$\sigma^E_R\,(s(R^{-1}x),\,\Phi'(R^{-1}x)) = (s(x),\Phi'(x))$$

$$(\sigma_R \circ s \circ R^{-1}(x),\,\Phi'(R^{-1}x)) = (s(x),\Phi'(x))$$

$$(s(x)\tau_R(x),\,\Phi'(R^{-1}x)) = (s(x),\Phi'(x))$$

$$(s(x),\,T(\tau_R(x))\Phi'(R^{-1}x)) = (s(x),\Phi'(x))$$

which finally gives

$$T(\tau_R(x))\Phi'(R^{-1}x) = \Phi'(x)\,. \tag{A2.5}$$

Again, this is the relation used in [R-S-T] and [S1]. We refer to τ_R as the compensating function for the rotation R^{-1} for the lifting σ of the action with respect to the trivialisation φ.

For the special section s′ of some other trivialisation (i.e. for some other section of the bundle P) with $s'(x) = s(x)k(x)$ we have that the compensating functions for this new gauge are given by

$$\tau'_R(x) = k(x)\,\tau_R(x)\,k^{-1}(R^{-1}x)\,. \tag{A2.6}$$

Furthermore, for R_1 and R_2 rotations in SO(3) we have that in any gauge

$$\tau_{R_1R_2}(x) = \tau_{R_1}(x)\,\tau_{R_2}(R_1^{-1}x)\,. \tag{A2.7}$$

These last two assertions are simply a matter of calculations.

Conversely, if a family of compensating functions $\{\tau_R\,;\ R \in SO(3)\}$ satisfying condition (A2.7) is given for each trivialisation and if the various families are related by the compatibility condition (A2.6) then there exists a unique lifting of the SO(3) action on $\mathbb{R}^3$ having these families as compensating functions. The lifting is defined by (A2.3). It is well defined thanks to (A2.6).

Finally, a piece of terminology that is going to be important in what follows: we say that a gauge is a **rigid gauge** if τ_R is independent of $x \in \mathbb{R}^3$ for each R in SO(3). We shall see that rigid gauges exist as a result of the existence of finite energy configurations. It is immediately clear that for G = SU(2) such a gauge cannot exist: if it did, (A2.6) would define a homomorphism from SO(3) to SU(2). It is well known no such nontrivial homomorphism exists.

A3. Some particular gauges.

We want to calculate the functional on a spherically symmetric configuration (A,Φ). First observe that in any gauge the Lagrangian reduces to a one-dimensional integral:

$$\int_{\mathbb{R}^3} \{ |F_A|^2(x) + |D_A\Phi|^2(x) + V(\Phi)(x) \} \, d^3x =$$

$$4\pi \int_0^\infty r^2 \{ |F_A|^2(0,0,r) + |D_A\Phi|^2(0,0,r) + V(\Phi)(0,0,r) \} \, dr$$

since by the Ad-invariance of the inner product in **g**, the G-invariance of V and the fact that SO(3) acts by isometries on $\mathbb{R}^3$, the integrand is constant on S^2:

$$|F_A|(R\omega_o) = |F_A|(\omega_o),$$

$$|D_A\Phi|(R\omega_o) = |D_A\Phi|(\omega_o),$$

$$V(\Phi)(R\omega_o) = V(\Phi)(\omega_o).$$

This dimensional reduction is the main reason for studying spherical symmetry. How desirable it is from a technical point of view to work on a 1-dimensional space rather than 3-dimensional one will become clear in section A4.

Remark: We shall prove that $L^2_{1,loc}$ functions are continuous in almost any radial direction, see Lemma C1. Here and in what follows we are implicitly assuming that the positive z-axis is one of the directions on which both A and Φ are continuous. When working with sequences of configurations we shall be assuming that on the z-axis all the members of the sequence are continuous.

In general, the compensating functions $\tau_R(x)$ are not homomorphisms of SO(3) for fixed x. However, for $x_o = (0,0,r)$, $r > 0$, a point on the z-axis[2] the isotropy group of the SO(3) action is an SO(2) subgroup and (A2.7) gives that for h_1, h_2 in SO(2)

$$\tau_{h_1 h_2}(x) = \tau_{h_1}(x)\,\tau_{h_2}(x)\,. \qquad (A3.1)$$

That is, for each fixed x_o on the z-axis τ: SO(2) $\longrightarrow$ G is a homomorphism. Furthermore, using the fact that $\mathbb{R}^3 \setminus \{0\}$ consists of orbits of points on the z-axis for any R in SO(3) the same relation gives :

$$\tau_R(x) = \tau_R((R_x)^{-1}x_o) = (\tau_{R_x})^{-1}(x_o)\tau_{R_x R}(x_o)$$

where R_x is such that $x = (R_x)^{-1}x_o$. Therefore, it is enough to know the compensating functions on the z-axis. We shall avoid any mention of the singularity at the origin until the last section.

a) A **preliminary gauge:** Let so(2) and so(3) denote the Lie Algebras of SO(2) and SO(3) respectively. We then choose a basis $\{\omega_1, \omega_2, \omega_3\}$ of so(3) so that ω_3 generates so(2) as it sits in so(3). Then for R in a sufficiently small neighborhood of the identity in SO(3) we have that R can be decomposed in a unique way into an SO(2)-part h(R) and a non-SO(2) part v(R):

$$R = h(R)\,v(R) = \exp(\varphi\omega_3)\exp(a_1\omega_1 + a_2\omega_2).$$

Using (A2.7) again:

$$\tau_R(x_o) = \tau_{h(R)}(x_o)\,\tau_{v(R)}(x_o)\,. \qquad (A3.2)$$

Define

$$\lambda_{x_o}(h) = \tau_h(x_o)$$

for h in the SO(2) isotropy subgroup of any element on the z-axis. We now choose gauges where λ has a particularly simple form.

First we need to establish that for A in the configuration space there exists a gauge such that $\sum x_i A_i = 0$, or equivalently $A_r = 0$. The existence of such a gauge is not hard to see if A is C^1: simply solve the equation

[2]From now on by z-axis we mean positive z-axis.

$$g^{-1}A_r\, g + g^{-1}\partial_r\, g = 0$$

or

$$A_r\, g + \partial_r\, g = 0$$

with some initial conditions. For A in $L^2_{1,loc}$ we can again solve the equation for any initial conditions by approximating A with C^1 functions. We show how to do this in Lemma II of the Appendix to this chapter.

LEMMA A3.1: There exists a gauge such that for all h in SO(2) τ_h is independent of x_o on the z-axis.

Proof: The spherical symmetry condition (A2.4) for the gauge potential $A=\sum A_\mu dx_\mu$ can be written as

$$A_\mu(x) = \tau_R(x)\,(R^{-1})_{\mu\nu}\,A_\nu(R^{-1}x)\,\tau_R^{-1}(x) + \tau_R(x)\,\partial_\mu\tau_R^{-1}(x)$$

where $(R^{-1})_{\nu\mu} = R_{\mu\nu}$ is the matrix (of the differential of) the rotation R^{-1}. In particular, for h in SO(2), x_o on the z-axis and $\mu = 3$,

$$A_3(x_o) = \tau_h(x_o)\,A_3(x_o)\tau_h^{-1}(x) + \tau_h(x_o)\partial_3\tau_h^{-1}(x_o).$$

In a radial gauge as above and for $x_o = (0,0,r)$, $r > 0$, the condition $rA_3(x_o) = 0$ gives

$$\tau_h(x_o)\partial_3\tau_h^{-1}(x_o) = 0$$

and hence $\tau_h(x_0)$ is constant.

LEMMA A3.2: There exists a gauge with the same τ_h on the z-axis as in Lemma A3.1 and such that $d\tau_h = 0$ on the z-axis.

Proof: Pick a gauge as in Lemma A3.1. Define the gauge transformation k in a neighborhood of the z-axis as follows: for $x = R^{-1}x_o$ with x_o on the z-axis and R small enough

$$k(R^{-1}x) = \tau_{v(R)}(x_o).$$

Then in the new gauge we have:

$$\tau'_h(x_o) = k(x_o)\,\tau_h(x_o)\,k^{-1}(x_o) = \tau_h(x_o)$$

since the only rotation with no SO(2)-part that takes x_o to itself is the identity. Furthermore,

$$\tau_R'(x_o) = k(x_o)\,\tau_R(x_o)\,k^{-1}(R^{-1}x_o)$$

or

$$\tau'_{v(R)}(x_o) = k(x_o)\,\tau_{v(R)}(x_o)\,k^{-1}(R^{-1}x_o) = e\,\tau_{v(R)}(x_o)\,(\tau_{v(R)})^{-1}(x_o) = e.$$

Then for $x = R^{-1}x_o$ with R having no SO(2)-part

$$\begin{aligned}\tau_h'(R^{-1}x_o) &= (\tau'_R)^{-1}(x_o)\,\tau'_{Rh}(x_o) = \tau'_{Rh}(x_o)\\ &= \tau'_{hh^{-1}Rh}(x_o) = \tau'_h(x_o)\,\tau'_{h^{-1}Rh}(x_o)\\ &= \tau'_h(x_o)\end{aligned}$$

since h^{-1} R h has no SO(2) part and is still close to the identity. Hence for any $x = R^{-1}x_o$ close to the z-axis τ'_h is constant.

Remark: The final calculation in Lemma 2 is exactly the one in [R-S-T]. Their claim that τ_h is constant is thus true only on SO(3)-orbits and becomes globally true thanks to Lemma A3.1.

In such a gauge define λ to be λ_{x_o} for any x_o on the z-axis and

$$K_3 = \frac{d}{dt}\Big|_0 \lambda(\exp t\omega_3).$$

b) The energy expression: In such a gauge (as in Lemma A3.2) the expression of the energy has been calculated in [R-S-T] and [R-F-S] to be

$$YMH(A,\Phi) = 4\pi \int_0^\infty r^2 \{ \frac{1}{r^2} | \partial_r (r A_1(r)) |^2 + \frac{1}{r^2} | \partial_r (r A_2(r)) |^2$$

$$+ \frac{1}{r^4} | K_3 + [r A_1(r), r A_2(r)] |^2 + | \partial_r \Phi(r) |^2$$

$$+ | t(A_1(r)\Phi(r)) |^2 + | t(A_2(r)\Phi(r)) |^2 + V(\Phi(r)) \} \, dr . \tag{A3.1}$$

We have checked and completed the calculation of the energy expression in [D]. Since the main ideas are contained in [R-S-T] we are not going to repeat the calculation here.

c) The construction of a rigid gauge: We combine here [H-S-V] and [R-S-T] to give a full proof of the existence of a rigid gauges. As in the construction of the preliminary gauges of the preceding paragraph, we shall use a finite energy configuration to construct such gauges. The authors in [R-F-S] err in assuming the existence of a rigid gauge before writing an expression for the energy.

In a gauge with $A_3(0, 0, r) = 0$ and the homomorphism λ independent of the point on the z-axis, we write the spherical symmetry condition as

$$A_\mu(x) = \tau_h(x_o)\, h_{\mu\nu}\, A_\nu(x_o) \tau_h^{-1}(x_o) + \tau_h(x_o)\, \partial_\mu \tau_h^{-1}(x_o) =$$

$$= \tau_{\exp(\varphi\omega_3)} (\exp(\varphi\omega_3))_{\mu\nu}\, A_\nu(x_o) \tau_{\exp(\varphi\omega_3)}^{-1} .$$

Differentiating with respect to φ at $\varphi = 0$ we have:

$$0 = \frac{\partial}{\partial\varphi}\Big|_0 \tau_{\exp(\varphi\omega_3)} A_\mu(x_o) - A_\mu(x_o) \frac{\partial}{\partial\varphi}\Big|_0 \tau_{\exp(\varphi\omega_3)} + (\omega_3)_{\mu\nu} A_\nu(x_o)$$

or

$$[K_3, A_\mu(x_o)] = -(\omega_3)_{\mu\nu} A_\nu(x_o).$$

Since

$$\omega_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

setting $\mu = 1$ and $\mu = 2$ gives

$$[K_3, A_1(x_o)] = A_2(x_o) \tag{A3.2}$$

and

$$[K_3, A_2(x_o)] = -A_1(x_o) \tag{A3.3}$$

respectively.

For a finite energy configuration (A,Φ) we have:

$$|r' A_1(r') - r A_1(r)| \le \int_r^{r'} \left| \frac{d(rA_1)}{dr}(r) \right| dr \le \left(\int_r^{r'} dr \int_r^{r'} \left| \frac{d(rA_1)}{dr}(r) \right|^2 dr \right)^{1/2}$$

$$\le C (r' - r)^{1/2} \, \mathrm{YMH}(A,\Phi)^{1/2}. \tag{A3.4}$$

Therefore $\lim_{r\to 0} r A_1(r)$ exists. We call it $K_1(A)$. Similarly, set $K_2(A) = \lim_{r\to 0} rA_2(r)$.

Then taking the limits of (A3.2) and (A3.3) as r tends to 0 we have

$$[K_3, K_1] = K_2$$

and

$$[K_3, K_2] = -K_1.$$

Furthermore, using once again that the energy is finite we have that the integral

$$\int_0^\infty \frac{1}{r^2} |K_3 + [r A_1(r), r A_2(r)]|^2 dr$$

is finite, which gives that

$$-K_3 = [K_1, K_2].$$

That is, the elements $L_1 = K_1$, $L_2 = K_2$ and $L_3 = -K_3$ satisfy the relations

$$[L_i, L_j] = \varepsilon_{ijk} L_k.$$

Since the generators ω_1, ω_2 and ω_3 satisfy the same relations, we have an isomorphism of the Lie algebra of SO(3) with the subalgebra of the Lie Algebra of the gauge group G generated by K_1, K_2 and $-K_3$ which we want to view as a homomorphism

$$d\Lambda: su(2) \longrightarrow g.$$

Since SU(2) is simply connected this gives a homomorphism

$$\Lambda: SU(2) \longrightarrow G$$

satisfying

$$\Lambda(-I) = \exp\,(d\Lambda)(2\pi\omega_3) = \exp 2\pi K_3 =$$

$$= \exp 2\pi \frac{d}{dt}\Big|_0 \tau_{\exp t\omega_3}(x_o) = \exp \frac{d}{dt}\Big|_0 \tau_{\exp 2\pi t\omega_3}(x_o)$$

$$= \tau_{\exp 2\pi\omega_3}(x_o)$$

$$= \tau_e(x_o) = e\,.$$

Therefore we have a homomorphism $d\Lambda$: $SO(3) \longrightarrow G$ which clearly extends the homomorphism λ: $SO(2) \longrightarrow G$.

To show that such a homomorphism determines a rigid gauge we argue as follows: in any gauge the compensating function τ_R is characterized by

$$(\sigma_R \circ s \circ R^{-1})(x) = s(x)\,\tau_R(x)$$

where s is the special section of the given trivialisation.

Define a new section s′ by

$$s'(Rx_o) = \sigma_R(s(x_o))\,\Lambda(R^{-1})$$

for any $x = R_x x_o$ in $\mathbb{R}^3\backslash\{0\}$. It is clearly well defined. In the gauge which has s′ as its special section we follow the same recipe for finding compensating functions as for (A2.4). For any rotation R:

$$(\sigma_R \circ s' \circ R^{-1})(x) = (\sigma_R \circ s' \circ R^{-1})(R_x x_o) = (\sigma_R \circ s')(R^{-1}R_x x_o)$$

$$= \sigma_R \circ \sigma_{R^{-1}R_x}(s(x_o))\,\Lambda((R_x)^{-1}R)$$

$$= \sigma_R \circ s(x_o)\Lambda((R_x)^{-1})\Lambda(R) = s'(x)\,\Lambda(R).$$

Therefore in this gauge $\tau'_R(x) = \Lambda(R)$, depending only on R.

We see then that the existence of a rigid gauge is equivalent to the extension of the homomorphism λ to a homomorphism from SO(3) to the gauge group. The existence of such an extension follows from the existence of finite energy nontrivial configurations. Therefore, no finite energy configurations enjoying spherical symmetry exist when G = SU(2). However, we can replace in this case SO(3) by SU(2), still acting on $\mathbb{R}^3$ by rotations in the obvious way. We can then repeat the definition of spherical symmetry and all the results up to now word by word, substituting SO(3) by SU(2). Finite energy configurations with this kind of symmetry now exist, as the formulas in [R-S-T] show.

A4. Minimizing over all spherically symmetric configurations.

The method for minimizing the YMH functional we present here is the well-known Direct Method in the Calculus of Variations. It has been used before in the context of configurations described by particular Ansatze, see [R-F-S], [T-F-S] and [R]. Here we wish to apply it for a minimizing sequence of spherically symmetric configurations as above. The fact that the technique works in this case has been indicated in [R-F-S] where it has also been carried out for the Skyrmion problem. We carry it out here for the YMH case since some not entirely obvious technicalities are involved. We also find that the method applies for any gauge group.

First notice that YMH ≥ 0 on the spherically symmetric configurations. We can then choose a sequence (A^n, Φ^n) of spherically symmetric configurations such that as $n \to \infty$

$$\mathrm{YMH}(A^n, \Phi^n) \to \inf \{\, \mathrm{YMH}(A, \Phi) : (A, \Phi) \text{ is spherically symmetric} \,\}.$$

We want to comment on the choice of gauge since it is an important ingredient of the argument and one of the obscure points in [R-F-S]. For each n use the gauge transformation g^n solving the equation

$$\partial_r g^n(r) + A_3{}^n(r) = 0$$

with initial condition

$$g^n(0,0,1) = e \in G.$$

Then we are in the situation where $A_3{}^n = 0$ on the z-axis and, as proved above, $\lambda^n(\varphi)$ is constant on the z-axis. Notice that, although the type of symmetry is fixed once and for all, the description of the particular functions (like λ) depends on the choice of gauge. At the same time, the initial condition shows that the gauge transformation from the g^n-gauge to the g^m-gauge at the point (0,0,r) is the identity. The way the compensating functions change from gauge to gauge gives

$$\tau^n{}_h(0,0,r) = \tau^m{}_h(0,0,r)$$

and hence

$$\tau^n{}_h = \tau^m{}_h,$$
$$K^n{}_3 = K^m{}_3.$$

Further modify the gauges as in Lemma A3.2 without changing the K_3-parts. Then the energy of the elements of the minimizing sequence is given by (A3.1) for a uniform K_3. Thus the elements of the minimizing sequence are described in different gauges. Recall that the value of YMH is gauge invariant.

The next step is to show that A^n and Φ^n as functions on $(0, \infty)$ are bounded in Hilbert spaces that are naturally defined using the derivative terms in the energy expression. They turn out to be enough to control the remaining terms in the expression.

Define H_1 to be the space of all measurable functions f that have finite H_1-norm:

$$\| f \|_{H_1} = \left(\int_0^\infty \left| \frac{d}{dr} f(r) \right|^2 dr + | f(1) |^2 \right)^{1/2}$$

and H_2 to be the space of all measurable functions f that have finite H_2-norm:

$$\| f \|_{H_2} = \left(\int_0^\infty r^2 \left| \frac{d}{dr} f(r) \right|^2 dr + | f(1) |^2 \right)^{1/2}.$$

Note that it is not clear that these spaces are the completion spaces of smooth functions with compact support and therefore the constant terms in the norm are of importance. We now estimate $\| A^n \|_{H_1}$ and $\| \Phi^n \|_{H_2}$:

Recall that for a finite energy configuration (A,Φ) we have by (A3.4) defined

$$K_1(A) = \lim_{r\to 0} rA_1(r),$$
$$K_2(A) = \lim_{r\to 0} rA_2(r).$$

Using (A3.4) again, for the elements of the minimizing sequence

$$|A_1^n(1)| \leq C\cdot YMH(A,\Phi)^{1/2} + |K_1^n|$$

and

$$|A_2^n(1)| \leq C\cdot YMH(A,\Phi)^{1/2} + |K_2^n|.$$

However, because of the relations among K_1, K_2 and K_3 and the uniformity of K_3, the invariance of the inner product under the adjoint action gives

$$|K_3|^2 = \langle K_3, K_3 \rangle = \langle Ad_{\exp(-tK_2^n)} K_3 , Ad_{\exp(-tK_2^n)} K_3 \rangle$$

for all t.

Further,

$$Ad_{\exp(-tK_2^n)} K_3 = e^{t\, ad(-K_2)} K_3 = \sin t K_1^n + \cos t K_3^n .$$

Setting $t = \pi/2$ gives

$$|K_1^n| = |K_3|,$$

while for the same value of t

$$Ad_{\exp(-tK_1^n)} K_3 = e^{t\, ad(-K_1^n)} K_3 = \sin t K_2^n + \cos t K_3^n .$$

gives

$$|K_2^n| = |K_3|.$$

Since the numerical sequence $YMH(A_n, \Phi_n)$ is bounded, say by a constant M, we have

$$|A_1^n(1)| \leq C\cdot M^{1/2} + |K_3|$$

and

$$|A_2^n(1)| \leq C\cdot M^{1/2} + |K_3|.$$

Since also

$$\int_0^\infty |\frac{d}{dr} rA_1{}^n(r)|^2 dr + \int_0^\infty |\frac{d}{dr} r A_2{}^n(r)|^2 dr \le M$$

we see that the components of $r A^n$ are bounded in H_1.

Similarly for the Φ-part of the configurations we estimate that:

$$|\Phi(r') - \Phi(r)| \le \int_r^{r'} |\frac{d}{dr}\Phi(r)| dr \le \left(\int_r^{r'} \frac{1}{r^2} dr \int_r^{r'} r^2 |\frac{d}{dr}\Phi(r)|^2 dr\right)^{1/2}$$
$$\le C(r^{-1} - (r')^{-1})^{1/2} \, YMH(A^n,\Phi^n)^{1/2}.$$

which gives that the limit of $\Phi(r)$ exists as r tends to infinity. By the symmetry breaking assumption this limit has to lie on a unique orbit in the representation space. Since G is compact so is the orbit and therefore bounded in the norm of the representation space by a constant N. We then have the uniform bound

$$|\Phi^n(1)| \le C \cdot YMH(A^n,\Phi^n)^{1/2} + N.$$

Since by finite energy

$$\int_0^\infty r^2 |\frac{d}{dr}\Phi^n(r)|^2 dr \le M$$

we see that the sequence Φ_n is bounded in H_2.

By the weak compactness of the Hilbert spaces H_1 and H_2 we obtain weak limits A^o and Φ^o in H_1 and H_2 respectively . By rotating A^o and Φ^o we can construct a spherically symmetric configuration, described in a K_3-gauge. To prove that $YMH(A^o,\Phi^o) = \inf \{ YMH(A,\Phi): (A,\Phi) \text{ spherically symmetric} \}$, we argue as in [R]:

First assume that $\| rA^n \|_{H_1}$ and $\| \Phi^n \|_{H_2}$ are convergent, by choosing a subsequence if necessary.

For each closed subinterval [a,b] of $(0,\infty)$ with $a < 1 < b$ define $H_1(a,b)$ and $H_2(a,b)$ to be the spaces of measurable functions with finite norm

$$\| f \|_{H_1} = \left(\int_a^b | \frac{d}{dr} f(r) |^2 dr + | f(a) |^2 \right)^{1/2}$$

and

$$\| f \|_{H_2} = \left(\int_a^b r^2 | \frac{d}{dr} f(r) |^2 dr + | f(a) |^2 \right)^{1/2}$$

respectively.

Using the obvious continuity of the restriction mappings $H_1 \longrightarrow H_1(1,b)$ and $H_2 \longrightarrow H_2(1,b)$ we conclude that $r A^n$ still converges weakly to $r A^o$ in $H_1(1,b)$ and Φ^n weakly to Φ^o in $H_2(1,b)$. However, over compact domains finite H-norms imply finite $L^2{}_1(a,b)$ norms. For a proof of this see Lemma I of the Appendix. We can therefore consider our sequences as weakly convergent sequences in $L^2{}_1$. By Sobolev's Embedding Theorem in dimension 1, they are strongly convergent sequences in $C^o(a,b)$. This implies (uniform and hence) pointwise convergence and that A^o and Φ^o are continuous.

In particular we have

$$\lim_{n\to\infty} A^n(1) = A(1)$$

and (A4.1)

$$\lim_{n\to\infty} \Phi^n(1) = \Phi(1) .$$

Since we are assuming $\| rA^n \|_{H_1}$ and $\| \Phi^n \|_{H_2}$ convergent we have that

$$\int_0^\infty \alpha_n \, dr = \int_0^\infty | \frac{d}{dr} r A^n(r) |^2 dr$$

and

$$\int_0^\infty \beta_n \, dr = \int_0^\infty r^2 | \frac{d}{dr} \Phi^n(r) |^2 dr$$

converge as n tends to ∞.

Notice that

$$YMH(A_n, \Phi_n) = \int_0^\infty \alpha_n \, dr + \int_0^\infty \beta_n \, dr + \int_0^\infty \gamma_n \, dr,$$

where γ_n denotes all the terms in the energy integral that do not involve derivatives.

Since $YMH(A_n, \Phi_n)$ converges as $n \to \infty$, so does $\int_0^\infty \gamma_n \, dr$. Then

$$\int_1^b \gamma(r) \, dr = \int_1^b \{ \gamma(r) - \gamma_n(r) \} \, dr + \int_1^b \gamma_n(r) \, dr$$

gives

$$\int_1^b \gamma(r) \, dr \le \int_1^b \{ \gamma(r) - \gamma_n(r) \} \, dr + \int_0^\infty \gamma_n \, dr.$$

Since the convergence on [1,b] is uniform, taking limits as $n \to \infty$:

$$\int_1^b \gamma(r) \, dr \le \lim_{n\to\infty} \int_0^\infty \gamma_n(r) \, dr$$

as $n \to \infty$. Now repeat the proof up to this point substituting f(1) in the norms with any value f(a) and take $a < b$. The last inequality becomes

$$\int_a^b \gamma(r) \, dr \le \lim_{n\to\infty} \int_0^\infty \gamma_n(r) \, dr$$

for all a and b, which finally gives:

$$\int_0^\infty \gamma(r) \, dr \le \lim_{n\to\infty} \int_0^\infty \gamma_n(r) \, dr . \qquad \text{(A4.2)}$$

Using the standard property of weak limits we have:

$$\|r A_1^o\|_{H_1} \le \lim_{n\to\infty} \|r A_1^n\|_{H_1},$$
$$\|r A_2^o\|_{H_1} \le \lim_{n\to\infty} \|r A_2^n\|_{H_1}, \qquad (A4.3)$$
$$\|\Phi^o\|_{H_2} \le \lim_{n\to\infty} \|\Phi^n\|_{H_2}.$$

Since

$$\mathrm{YMH}(A,\Phi) = \|r A_1\|_{H_1} - |A_1(1)| + \|r A_2\|_{H_1} - |A_2(1)|$$
$$+ \|\Phi\|_{H_2} - |\Phi(1)| + \int_0^\infty \gamma(r)\, dr\ ,$$

adding up (A4.1), (A4.2) and (A4.3) gives

$$\mathrm{YMH}(A^o,\Phi^o) \le \lim_{n\to\infty} \mathrm{YMH}(A^n,\Phi^n).$$

To see that $\mathrm{YMH}(A^o,\Phi^o)$ is indeed the energy of a spherically symmetric configuration of the same type as the configurations (A^n, Φ^n) we argue as follows: choose any gauge as described in Lemma A3.1 and A3.2 and characterized by K_3, for example any of the g_n-gauges. By the Rigid Gauge Construction, we can assume that this gauge is rigid without changing K_3. In such a gauge then we define the spherically symmetric configuration (A,Φ) at $x = Rx_o$ for x_o on the z-axis by

$$A_\mu(Rx_o) = \tau_R R_{\mu\nu} A_\nu^o(x_o) \tau_R^{-1}$$

and

$$\Phi(Rx_o) = T(\tau_R)\, \Phi^o(x_o).$$

Such a configuration is well defined: for R' such that $R'x_o = x$ we have $R' = Rh$ for h in the isotropy of x_o. Then

$$\begin{aligned} A_\mu(R' x_o) &= \tau_{Rh} (Rh)_{\mu\nu} A_\nu^o(x_o) \tau_{Rh}^{-1} \\ &= \tau_R \tau_h (Rh)_{\mu\nu} A_\nu^o(x_o) \tau_h^{-1} \tau_R^{-1} = \\ &= \tau_R \tau_h (R_{\mu j} h_{j\nu}) A_\nu^o(x_o) \tau_h^{-1} \tau_R^{-1} \\ &= \tau_R R_{\mu j} (\tau_h h_{j\nu} A_\nu^o(x_o) \tau_h^{-1}) \tau_R^{-1}. \end{aligned}$$

Since $A_j^n(x_o) = \tau_h h_{j\nu} A_\nu^n(x_o) \tau_h^{-1}$ by the uniformity of K_3, taking pointwise limits we have

$$A_j^o(x_o) = \tau_h h_{j\nu} A_\nu^o(x_o) \tau_h^{-1}.$$

Hence

$$A(R'x_o) = A(Rx_o).$$

Summarizing, we have proved:

THEOREM A4.1: For any compact, semi-simple gauge group G and any symmetry breaking potential V the functional YMH attains its minimum over all configurations that are spherically symmetric with respect to a given type of spherical symmetry.

A5. The existence of spherically symmetric solutions.

How far is this from proving the existence of critical points among ALL configurations? As mentioned in [R-S-T] and as widely accepted by physicists, see [C1], imposing a symmetry and finding a critical point among all symmetric fields should yield a genuine critical point. However, it has long been recognized by mathematicians that this is not always true and that the correct statement of this principle is as follows. We quote from [Pa2]:

THE PRINCIPLE OF SYMMETRIC CRITICALITY: Let K be a compact Lie Group acting on a smooth manifold X and on a fibre-bundle Y over X. If B is a Banach Manifold of sections of Y, consider the natural action of K on B. Then the set Σ of K-equivariant sections is a smooth submanifold of B. Furthermore, if $F: B \longrightarrow \mathbb{R}$ is a K-invariant smooth function the critical points of $F|_{\Sigma}$ are critical points of $F|_B$.

By natural action on the space of sections we mean

$$(k\cdot s)(x) = k\cdot(s(k^{-1}x))$$

for k in K and s in B.

As it becomes clear to anyone who studies Palais' paper, the technical condition that the space of sections forms a Banach Manifold is very important. It is the "linearity of the enveloping space" implied in [L], where another attempt for symmetric solutions for the Skyrme problem can be found[3]. The smoothness of the functional can be relaxed to F being C^1. The proof of the fact that the configuration space admits a Banach Manifold structure such that YMH is smooth for the case G = SU(2) and for the adjoint representation occupies the whole of the next chapter. Only then will the existence of spherically symmetric solutions be completed.

In our case, of course, K = SO(3) and acts by rotations on $X = \mathbb{R}^3$ and by the prescribed lifting on $Y = (T^* \otimes AdP) \oplus E$. Σ then is the submanifold of spherically symmetric configurations and the minimizers of Theorem A4.1 are critical points of YMH on the full configuration space.

However, there is no reason why these particular critical points should be stable with respect to variations among all configurations. So far as we know this problem remains open and we intend to address it in the future. Suffices to mention here that the Harmonic Mappings paradigm indicates that stability in Variational Problems can be quite unpredictable, see [E-L], section 6.6.

Of course, the solutions we obtained are useless unless they are non-trivial. Trivial solutions define trivial homotopy classes and therefore have magnetic charge 0. However, the solutions described in this chapter have non-zero charge, they are genuine monopoles. To see this, consider a fixed type of spherical symmetry described in a rigid gauge, the corresponding λ-homomorphism and its derivative K_3. Going back to the defining relation (A2.5) for symmetric Higgs fields, for x_o on the z-axis

$$T(\lambda(\varphi))\,\Phi(x_o) = \Phi(x_o)$$

which gives by differentiating with respect to φ:

$$t(K_3)\Phi(x_o) = 0$$

and taking the limit as $| x_o |$ tends to infinity

[3]Ladynzeskaya refers to Coleman's paper [C] for the Principle. According to Palais, it is not clear what Coleman proves there.

$$t(K_3)\Phi_\infty = 0.$$

That is, K_3 lies in the isotropy subalgebra of the vacuum Φ_∞. Another way of saying the same thing is that the type of symmetry determines the asymptotic vacuum value on the z-axis. For the SU(2)-adjoint case it is determined up to a one-dimensional subspace and the isotropy subgroup is the same for all the points on the z-axis. If we denote it by H we have a homomorphism

$$\lambda: U(1) \longrightarrow H.$$

As proved in [R-S-T], the magnetic charge of any spherically symmetric configuration as an element of $\pi_1(H)$ is the homotopy class of λ: $U(1) \longrightarrow H$.[4] Notice that the homotopy class does not change from gauge to gauge: the transformation rule for compensating functions gives

$$\lambda'(\varphi) = k(x)\lambda(\varphi)k^{-1}(x)$$

for all x on the z-axis and all φ. Therefore, each k(x) describes a homotopy between λ and λ'.

There is no obstruction for such a homomorphism to be the characterizing homomorphism of some type of spherical symmetry when the gauge group is simply connected, see [S1]. In fact, [S1] refers to single orbits but the bundles he obtains are trivial. We can therefore take λ to be the same for all points on the z-axis and then the result carries over to $\mathbb{R}^3\setminus\{0\}$.

For H = U(1) the homotopy class is characterized by the degree of λ. The point then is that by choosing the U(1) subgroup and the homomorphism λ (of arbitrary degree) we can determine the asymptotic values of the Higgs fields and the magnetic charge of the symmetric configurations. This establishes the existence of solutions of arbitrary magnetic charge in the case of semi-simple gauge group and small group U(1).

---

[4]This is proved by considering the bundle homomorphism (τ,Φ_∞) from the bundle $(SU(2),U(1),S^2)$ to the bundle (G,H,G/H) and the corresponding homotopy sequences. The induced map on the fibres is given by λ.

We need not be concerned with the regularity of the solutions. As we have argued, they are continuous on the z-axis and since we have used $L^2_{2,loc}$ transformations to define them on the rest of the space they are definitely in $L^2_{1,loc}$. By a standard theorem any solution is gauge equivalent to a smooth one, see [J-T], Chapter V. This also settles the problem of the singularity at the origin.

Appendix: Two technical Lemmas.

LEMMA I: On any interval [a,b],

$$\| f \|_{2;1} \leq C \| f \|_{H_1}.$$

Proof: The proof consists of modifying appropriately the standard proof for the Poincare Inequality for functions with compact support. Since such functions are not dense in L^2_1[a,b] a bit more care is needed:

$$\int_a^b | f(r) |^2 dr = \int_a^b \left| \int_a^r f'(t)\, dt + f(a) \right|^2 dr$$

$$\leq \int_a^b \left(\left\{ \int_a^r | f'(t) | dt \right\}^2 + | f(a)|^2 + 2 \int_a^r | f(a) f'(t)| dt \right) dr$$

$$\leq C \int_a^b | f'(t) |^2 dt + C | f(a) |^2 + 2\, C | f(a) | \| f' \|_2$$

$$= C(| f(a) | + \| f' \|_2)^2 .$$

LEMMA II: For each A in $L^2_{1,loc}$ there exists an $L^2_{2,loc}$ gauge transformation such that $(g\cdot A)_r = 0$.

Proof: We construct the gauge on closed intervals of the positive z-axis, as it can be extended on the whole space in the obvious way. Any A in L^2_1[a,b] can be approximated in L^2_1[a,b] by a sequence A_n of C^1 functions. Let g_n be the unique solution of the ordinary differential equation

$$\partial_r g_n + A_n g_n = 0,$$
$$g_n(a) = e.$$

Then $\| \partial_r g_n \|_2 = \| A_n g_n \|_2 \leq \| A_n \|_2$. Since $\| A_n \|_2$ is bounded using Lemma 1 and the uniform initial condition we have that g_n is bounded in L^2_1 and hence weakly convergent to g, say. Using Sobolev's Embedding again, g_n converges in $C[a,b]$ to g and therefore pointwise to g. Taking limits then we have

$$\partial_r g + Ag = 0$$

which also shows that $\partial_r g$ is in L^2_1 and hence g is in L^2_2. It is clear that we can choose the uniform initial condition in an arbitrary way. By pointwise convergence g will satisfy the same initial condition.

B. A BANACH MANIFOLD STRUCTURE ON THE CONFIGURATION SPACE OF THE YANG-MILLS-HIGGS FUNCTIONAL.

B0. Introduction.

In this chapter we show that the effective configuration space of the theory, the equivalence classes under the action of the gauge group, is a Banach manifold. It is clear from the previous chapter that such a structure is desirable in proving the existence of spherically symmetric solutions. For further evidence on the importance in general of such a structure on the configuration space of a variational problem, see [T 6].

The method we are following is the more or less standard method of finding a local slice. The analytic difficulties were first realized by C. Taubes who tackled them in the case of smooth configurations, see [T3], and suggested ways of overcoming them in the case of the general configuration space. This construction was carried out for the case of the Prasad-Sommerfield limit by A. Floer in [F]. Here we are interested in showing that there is a manifold structure so that the full Yang-Mills-Higgs functional is differentiable, something that Floer has not addressed, not even for the Prasad-Sommerfield case. It turns out that a small modification of the norm of the tangent space at each configuration is enough. Precisely, where Floer finds that the norm of the tangent vector (a,φ) should be

$$(\| \nabla_A a \|_2^2 + \| \nabla_A \varphi \|_2^2 + \| [\Phi, a] \|_2^2 + \| [\Phi, \varphi] \|_2^2)^{1/2} + \| \Phi \cdot \nabla_A^* a \|_{6/5}$$

we find that it should be

$$(\| \nabla_A a \|_2^2 + \| \nabla_A \varphi \|_2^2 + \| [\Phi, a] \|_2^2 + \| \varphi \|_2^2)^{1/2} + \| \Phi \cdot \nabla_A^* a \|_{6/5} .$$

However, in order to do this, a non-trivial step is involved: since the construction essentially takes place in a "regularized" configuration space C_R we must find a way of going from the configuration space C to C_R by adding an L^2 field to the original Higgs field. We show how to do this in section B5.

Although the main aim of this section is to solve the problems for the case where the Lagrangian has a potential term, we hope that at the same time we have managed to present a readable account of the original proof.

Another attempt at a Banach manifold structure can be found in [P], for a similar functional on a four-dimensional compact manifold. There the compactness makes the analytic difficulties less severe. The question of smoothness of the functional has not been addressed there either.

B1. The local slice method: The Yang-Mills analogue.

The prototype for a Banach manifold structure for a configuration space of the kind we are studying here is the structure on the configuration space of the Yang-Mills functional on a compact 4-dimensional manifold M, as it appears for example in [F-U]. We choose to describe everything by analogy with this situation, for reasons of clarity. There, the the configuration space is the space of irreducible connections in the Sobolev space $L^2_3(T^*\otimes Ad(P))$ acted upon by the gauge transformations in **G** = L^2_4 (Aut(P)). Notice that both these spaces are themselves manifolds: L^2_3 $(T^*\otimes Ad(P))$ is affine and L^2_4 (Aut(P)) has tangent spaces described by the L^2_4 sections of the bundle of Lie algebras. By contrast, in our situation we have only the Frechet spaces $L^2_{1,loc}(\mathbb{R}^3;T^*\otimes Ad(P))$ and $L^2_{2,loc}(\mathbb{R}^3;Aut(P))$ to begin with. This will cause the first complications.

In the case of the Yang-Mills functional, one proceeds by constructing a **slice** at each point D in the manifold of connections. A slice **S** consists of an open submanifold containing D such that:

(i) the restriction on **S** of the projection to the quotient is one to one and

(ii) the tangent space at the point splits into the space tangent to the orbit and the space tangent to **S**.

This is a standard procedure on spaces with a group action on them. In any case, the slice is easy to imagine: we need only exploit the natural inner product on each tangent space and go orthogonally to the orbit directions. If ξ is a section of the bundle of Lie algebras, an element of the tangent space at the identity gauge transformation, it exponentiates to the path of transformations $\exp(t\xi)$. In the space of connections the corresponding tangent vector to the connection D is

$$\frac{d}{dt}\Big|_0 \exp(t\xi)D = D\xi.$$

A tangent vector orthogonal to all such vectors is given by

$$< a , D\xi > = 0$$

or

$$< D^* a , \xi > = 0.$$

The slice we are after then has tangent space at D given by

$$T_D = \{ a \text{ in } L^2{}_3 \text{ with } D^* a = 0 \}.$$

One wants to argue that locally the configuration space is like $G \times T_D$. Dividing by the group action only T_D will survive. This will be the tangent space to the quotient space at the equivalence class [D]. To prove that locally we have a cross product we only need to solve for g the equation

$$D^*(gAg^{-1} + gDg^{-1}) = 0\,.$$

To do this, we need to know that the mapping

$$(g,A) \longmapsto D^*(gAg^{-1} + gDg^{-1})$$

has nonsingular derivative with respect to g at (id,0). The Implicit Function Theorem will then take over. This derivative is luckily given by D^*D which for irreducible connections has no kernel. Being also self-adjoint and elliptic it is non-singular by the Fredholm Alternative.

It then becomes clear that to follow the same strategy for the Yang-Mills-Higgs theory we must overcome the following problems:

Problem 1: Try to describe the configuration space at least around a given configuration in terms of Banach spaces, so that we can use the Implicit Function Theorem. We settle this in sections B5 to B7.

Problem 2: The operator corresponding to D^*D in the above description is $D^*D + \mathrm{ad}^2\Phi$, see below. Inverting this operator is a bit more tricky since $\mathrm{ad}\Phi$ has non-trivial kernel. This is done in section B7.

Sections B3 and B4 provide some motivation and discuss the differences of the case with a potential term from the Prasad-Sommerfield limit. The heuristic approach of section B4 comes from [T8].

B2. The configuration space: Definition and Topology.

The configuration space is of the Yang-Mills-Higgs theory is

$$C = \{ (A,\Phi): A \in L^2_{1,loc}(T^*(\mathbb{R}^3) \otimes AdP), \Phi \in L^2_{1,loc}(AdP) \text{ with } YMH(A,\Phi) < \infty \},$$

where the functional YMH is the full Lagrangian of the physical theory:

$$YMH(A,\Phi) = \int_{\mathbb{R}^3} \{\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\Phi|^2 + V(\Phi)\}\, d^3x .$$

Here V is a Higgs interaction potential, as explained in the introduction.

This is the configuration space as a set. As a topological space it has the intersection of the $L^2_{1,loc}$ topologies of the corresponding Frechet spaces with the topology that renders continuous the following functions:

$$C \longrightarrow L^2(\Omega^2(AdP))$$
$$c = (A,\Phi) \longmapsto F_A ,$$

$$C \longrightarrow L^2(\Omega^1(AdP))$$

$$c = (A,\Phi) \longmapsto D_A\Phi,$$

and

$$C \longrightarrow L^1(\mathbb{R}^3)$$

$$c = (A,\Phi) \longmapsto V(\Phi).$$

This is a natural adaptation of the standard configuration space used so far, see [T3], [F] and [G2], to the case of the full Lagrangian. Special care has to be taken so that the original topology used by Taubes is compatible with our presentation: it is the topology needed to prove one of the main results of the theory at the level of the Prasad- Sommerfield limit (with all the care one has to take for generalizations): the configuration space of smooth objects is homotopically equivalent to Maps $(S^2; S^2)$ and there are infinitely many unstable solutions of arbitrarily large energy. Further, in this topology the configuration space has countable many path components, counted by magnetic charge, see [G2].

The gauge transformations we are allowing in the theory are naturally in

$$G = L^2_{2,loc}\mathrm{Aut}\,(P).$$

Equivalently, once a trivialization has been chosen, they are the $L^2_{2,loc}$ mappings from $\mathbb{R}^3$ to G. They act continuously on the configuration space C in the usual way:

$$g{\cdot}A = Ad_gA + gdg^{-1}$$

and

$$g{\cdot}\Phi = T(g)(\Phi).$$

In this chapter we consider G = SU(2) in the adjoint representation only. We sum up the reasons for this restriction at the end of the chapter.

We shall exploit the group action to achieve the manifold structure. Naively speaking, given a configuration (A,Φ), we cannot expect that every configuration in a neighborhood of (A,Φ) has the same "decay" so that their difference is in some normed space. What one can prove is that any configuration sufficiently close to

(A,Φ) can be gauge transformed by an $L^2_{2,loc}$ transformation so that this is true. The manifold structure then is on the quotient C/G. It follows from the definition of the action that the constant gauge transformations 1 and -1 applied on a configuration give the same result. It is then clear that we need to work not with G but rather with $G/\{-1,+1\}$. Further, in order to achieve a Hausdorff quotient, we shall need to exclude the reducible parts of the theory:

$$C_{red} = \{ c=(A,\Phi) \in C \text{ such that there is } g \neq 1 \text{ with } g\cdot c = c \} .$$

We lose nothing from the physical point of view: a configuration is reducible if and only if its magnetic charge is zero, [F] Lemma 2.3. We are always interested in magnetic monopoles, objects with magnetic charge.

It is then possible to prove that the quotient is in fact Hausdorff, [F] Lemma 3.1. This is done by proving that the action of $G/\{\pm 1\}$ on $C\backslash C_{red}$ has a closed graph. It depends entirely on the connection part of the configuration and follows in exactly the same way as in the Yang-Mills theory, see [F-U]. The non-compactness difficulties are minor here.

We end this section with some standard technical facts.

The linearization of the action of the gauge group at the point (A,Φ) is given by

$$\xi \longmapsto \frac{d}{dt}\Big|_0 \exp(t\xi)(A\oplus\Phi)$$

$$= \frac{d}{dt}\Big|_0 \{\exp(t\xi)A\exp(t\xi)^{-1} + \exp(t\xi)d\exp(t\xi)^{-1}\} \oplus \frac{d}{dt}\Big|_0 \{\exp(t\xi)\Phi\exp(t\xi)^{-1}\}$$

$$= (\xi A - A\xi - d\xi) \oplus (\xi\Phi - \Phi\xi)$$

$$= -\{\nabla_A \xi \oplus \mathrm{ad}\Phi(\xi)\}. \tag{B3.1}$$

Here and throughout this chapter we view the tangent space at (A,Φ) as the direct sum of the tangent space at A and the tangent space at Φ.

The formal L^2-adjoint is given by the relation

$$-\langle a\oplus\varphi, \nabla_A\xi \oplus \mathrm{ad}\Phi(\xi) \rangle = -\{\langle a, \nabla_A\xi \rangle + \langle \varphi, \mathrm{ad}\Phi(\xi) \rangle\}$$

$$= -\langle \nabla_A{}^* a, \xi \rangle + \langle \mathrm{ad}\Phi(\varphi), \xi \rangle$$

$$= - \langle \nabla_A{}^* a + \mathrm{ad}\Phi(\varphi), \xi \rangle.$$

The corresponding operator then is

$$(a,\varphi) \longmapsto -\nabla_A{}^* a + \mathrm{ad}\Phi(\varphi)\,. \tag{B3.2}$$

The composition of the mappings above gives the Laplacian of the theory;

$$\xi \longmapsto \nabla_A{}^* \nabla_A \xi - \mathrm{ad}^2\Phi(\xi) = -(\nabla_A{}^2 \xi + \mathrm{ad}^2\Phi(\xi))\,.$$

Here $\nabla_A{}^2$ denotes the covariant Laplacian on sections. Of great use will be the following technical Lemmas:

LEMMA B2.1: (Kato's inequality) The following is true pointwise almost everywhere:

$$|\, d|s|\, |(x) \leq |\nabla_A s|(x).$$

For a proof, see [J-T], page 268.

LEMMA B2.2: Let V_A denote the completion of compactly supported smooth sections with respect to the norm

$$\| \varphi \|_{V_A} = \| \nabla_A \varphi \|_2.$$

Then if φ lies in V_A it lies in L^6.

Proof: Assume φ of compact support. By Kato's inequality, $d|\varphi| \in L^2$. Since it has compact support, by Sobolev's Inequality in three dimensions $\varphi \in L^6$.

LEMMA B2.3: Let φ have finite $\mathbf{V}_A$-norm, without necessarily belonging to $\mathbf{V}_A$. Then there exists a constant $M(\varphi) \in \mathbb{R}$ such that $|\varphi| - M(\varphi) \in L^6$.

Proof: See [G2], Lemma 1.1.

For example, let (A,Φ) be a finite energy configuration. Then by Lemma B2.3 there exists a constant $M(\Phi)$ such that $\Phi - M(\Phi) \in L^6$. By finite energy again, $\Phi - 1 \in L^2$. Hence $M(\Phi) = 1$.

B3. Differentiating the potential term.

There is no physics in the particular form of the potential V. It suffices to be chosen so that it satisfies properties (i) to (v) of the introduction, see page 3.

It is property (iii) that is the crucial one here and we should comment a bit more on it. From the Physics point of view the Lagrangian must describe a breaking of the original G-invariance to an H-invariance, for H a subgroup of G. This is made more precise in the chapters to follow. Here we shall only say that H is the (uniquely determined up to conjugation) isotropy group of the vacuum. Given G and H any potential satisfying the conditions above and having H as isotropy group will do and in fact it will be of the same value as any other having the same properties.

Having said all this let us concentrate on the SU(2) case and its adjoint representation on its Lie algebra. For this representation there are two sorts of orbits we can hope for: a) the orbit of 0, excluded by the non-triviality condition and b) the orbits of all other elements, all equivalent from our point of view since they all give H = U(1). Such orbits are spheres in the Lie algebra with respect to the standard inner product. Then everything depends on $|\Phi|$ and therefore we can choose V to be a fourth degree polynomial in $|\Phi|$:

$$V(\Phi) = a_4 |\Phi|^4 + a_3 |\Phi|^3 + a_2 |\Phi|^2 + a_1 |\Phi| + a_0 .$$

Also choose the vacuum orbit to be the sphere of radius 1. Then $|\Phi|=1$ must be a root of V and V must be of the form:

$$V(\Phi) = (|\Phi| - 1)(b_3 |\Phi|^3 + b_2 |\Phi|^2 + b_1 |\Phi| + b_0).$$

The positivity condition shows that for $|\Phi|$ smaller than 1

$$b_3 |\Phi|^3 + b_2 |\Phi|^2 + b_1 |\Phi| + b_0$$

has to be negative and for $|\Phi|$ greater than 1 it has to be positive. Being continuous it has a zero at 1 and V has the form:

$$V(\Phi) = (|\Phi| - 1)^2(c_2 |\Phi|^2 + c_1 |\Phi| + c_0).$$

We can then dispose of the $c_2 |\Phi|^2 + c_1 |\Phi| + c_0$ part since it does not contribute anything. We only need remind ourselves that physicists insist on an expression polynomial in Φ and then the potential takes the form:

$$V(\Phi) = (|\Phi|^2 - 1)^2.$$

Now let us try to complement the norm of the tangent space so that the full functional is at least once continuously differentiable in the Gateaux (directional) sense. Formally differentiating[1],

$$\frac{d}{dt}\Big|_0 \int V(\Phi + t\varphi)\, d^3x = \int DV_\Phi(\varphi)\, d^3x.$$

In the case that the Higgs potential V is a polynomial of $|\Phi|^2$, this gives

$$\int V'(|\Phi|^2) <\Phi,\varphi>\, d^3x.$$

In the SU(2) case this is

$$\int (|\Phi|^2 - 1) <\Phi,\varphi>\, d^3x.$$

Now since $(|\Phi|^2 - 1)$ is in L^2, using Holder's inequality

$$\int (|\Phi|^2 - 1) <\Phi,\varphi>\, d^3x \leq \| |\Phi|^2 - 1 \|_2 \| <\Phi,\varphi> \|_2.$$

Hence, we need to estimate the L^2 norm of $<\Phi,\varphi>$. For this notice that:

1) Φ does not belong to any L^p space since $|\Phi| - 1$ is in L^6.

2) It is easy to construct a Φ in $L^2_{1,loc}$ which is not bounded at all and with good enough behavior at large distances to guarantee finite energy. Therefore we cannot factor Φ out of the integral and try to control some norm of φ only.

3) Although we can prove that the limits at infinity of Φ exist for almost any radial direction, see chapter C, they are not achieved in a uniform way. We therefore cannot attempt to integrate $<\Phi,\varphi>$ separately over a compact and a non-compact region using Sobolev inequalities for the compact and an essential bound for the non-compact region.

We are then forced to try and control the L^2 norm of $<\Phi,\varphi>$ directly. This we do as follows:

[1]Notice that even in this simple instance the "naturality conditions", as in [Gi] for example, are not satisfied.

$$\int_{\mathbb{R}^3} |<\Phi,\varphi>|^2 \le \int_{\mathbb{R}^3} |\Phi|^2 |\varphi|^2 \le \int_{\mathbb{R}^3} (|\Phi|^2 - 1)|\varphi|^2 + \int_{\mathbb{R}^3} |\varphi|^2 \le$$

$$\le \| |\Phi|^2 - 1 \|_2 \| \varphi \|_4^2 + \| \varphi \|_2^2 \qquad \text{(B3.1)}$$

The fact that $\| |\Phi|^2 - 1 \|_2$ is finite follows from the finite energy condition. It is the third term in the Lagrangian. We then need to know that the L^2 and the L^4 norm of φ are finite and controlled by the norm in the tangent space at c.

Suppose that we choose the tangent space (so far as the Higgs part is concerned) to be the completion of compactly supported, smooth sections with respect to the following norm:

$$\| \nabla_A \varphi \|_2 + \| \varphi \|_2 \qquad \text{(B3.2)}$$

Then, according to Lemma B2.2, φ lies in L^6. By Holder's inequality and since φ is in L^2, it is in L^4, too:

$$\int_{\mathbb{R}^3} |\varphi|^4 = \int_{\mathbb{R}^3} |\varphi||\varphi|^3 \le \| \varphi \|_2 \| \varphi \|_6^3.$$

B4. A heuristic approach: the mysterious norm 6/5.

The norm must also be chosen so that the following is also true: given (a,φ) in some Banach space yet to be specified, try to solve for g the equation

$$D_A^*(g(A + a) - A) + [\Phi, [\Phi, g(\Phi + \varphi) - \Phi] = 0.$$

Linearize by considering g of the form $g = \exp\xi$ to get the approximating equation

$$D_A^* D_A \xi + [\Phi,[\Phi,\xi]] = D_A^* a + [\Phi,\varphi] \qquad \text{(B4.1)}$$

which we should solve for ξ.

The first step is to ask for the right-hand-side of the equation to be in L^2. This is a natural thing to do and leads to considering the Banach spaces of the completions of compactly supported, smooth objects with respect to the norm:

$$\| (a,\varphi) \|_c = (\| \nabla_A a \|_2^2 + \| \nabla_A \varphi \|_2^2 + \| [\Phi,a] \|_2^2 + \| [\Phi,\varphi] \|_2^2)^{1/2}.$$

Notice that the first and the last term in the norm gives the L^2 condition. In fact, this is the norm considered by Taubes since his early papers, as the natural norm associated to the configuration (A,Φ), see in particular [T5]. It measures, in an L^2 way, the action of a finite energy pair (A,Φ) on the sections of the associated bundles. However, our considerations in the previous section for controlling the interaction term in the Lagrangian, lead us to the norm

$$\| (a,\varphi) \|_c = (\| \nabla_A a \|_2^2 + \| \nabla_A \varphi \|_2^2 + \| [\Phi,a] \|_2^2 + \| \varphi \|_2^2)^{1/2}.$$

Notice that we can bound the norm $\| [\Phi,\varphi] \|_2$ in exactly the same way as we did for the $\| \langle\Phi,\varphi\rangle \|_2$ term in (B3.1). For that use that

$$| \langle\Phi,\varphi\rangle |^2 + | [\Phi,\varphi] |^2 = | \Phi |^2 | \varphi |^2,$$

hence

$$| [\Phi,\varphi] |^2 \leq | \Phi |^2 | \varphi |^2.$$

It is natural to assume that ξ lies in a similar space, for example the completion H_A of compactly supported, smooth objects with respect to the norm[2]

$$\| \xi \|_{H_A} = (\| \nabla_A \xi \|_2^2 + \| [\Phi,\xi] \|_2^2)^{1/2}.$$

According to Lemma B2.2, $\| \xi \|_6$ is finite when ξ is in H_A.

A way to solve the linearized equation then is to view it as variational equation and try to minimize the functional

[2] It turns out that this is a naive choice. This norm will be improved on the way.

$$\int|D_A\xi|^2+\int|[\Phi,\xi]|^2-\int<\xi,q>$$

where q is in L^2 and ξ in H_A. To minimize the functional we should at least ask that it is coercive on H_A. One is then lead to ask that q is in $L^{6/5}$, since we have by Holder's inequality that

$$-\int<\xi,q>\leq \|q\|_{6/5}\|\xi\|_6.$$

We are then forced to ask that the pairs (a,φ) are such that $q = D_A{}^*a + [\Phi,\varphi]$ is in $L^{6/5}$. It is a matter of an easy calculation to see that it is enough to ask that $\Phi\cdot D_A{}^*a$ is in $L^{6/5}$.

Putting together the discussion of this section and the estimates of the previous one, the problem now is:

Find for each configuration (A',Φ') close to (A,Φ) a gauge transformation g such that $(a,\varphi) = g\cdot(A',\Phi') - (A,\Phi)$ lies in the completion of compactly supported smooth sections with respect to the norm:

$$\|(a,\varphi)\|_{Y_C} = (\|\nabla_A a\|_2{}^2+\|\nabla_A\varphi\|_2{}^2+\|[\Phi,a]\|_2{}^2+\|\varphi\|_2{}^2)^{1/2}+\|\Phi\cdot\nabla_A{}^*a\|_{6/5}.$$

It is clearly a norm on compactly supported objects: By Kato's inequality and because of the first two terms in the norm the derivatives of the lengths have L^2 norm zero, hence they are zero, hence the lengths are constant. Being compactly supported, they are identically zero. We denote the corresponding completion by Y_C.

B5. The mapping from C to C_R.

As we shall see in Chapter C, given a finite energy pair (A,Φ) the Higgs field Φ approaches the vacuum orbit at large distances from the origin. One of the main difficulties in handling a finite energy configuration is that Φ does not approach its asymptotic value uniformly. However, if the extra assumption

$$\nabla_A \nabla_A \Phi \in L^2(\mathbb{R}^3)$$

is satisfied it implies that $|\Phi| - 1$ is in $L^6_1(\mathbb{R}^3)$.

Indeed, since $|\nabla_A \Phi| \in L^2$ using Lemma B2.3 there exists a constant M such that $|\Phi| - M \in L^6$. For the SU(2) case and the standard potential of section B3 the constant M cannot be anything else but 1, as we have already argued. On the other hand, $\nabla_A \nabla_A \Phi \in L^2$ gives that there exists a constant M' such that $|\nabla_A \Phi| - M'$ is in L^6. Since $\nabla_A \Phi \in L^2$ the constant M' has to be 0 and $|\nabla_A \Phi| \in L^6$. By the pointwise Kato's inequality $d|\Phi| = d(|\Phi| - 1) \in L^6$. Hence $|\Phi| - 1 \in L^6_1$. Now it is well known that if a function is in $L^p_1(\mathbb{R}^n)$ for $p > n$ then the function decays uniformly to zero. Therefore, in C_R we have that $|\Phi|$ tends uniformly to 1.

This leads us to consider, as in [T3] and [F], the "regularized space"

$$C_R = \{(A,\Phi) \text{ in } C \text{ with } \nabla_A \nabla_A \Phi \in L^2(\mathbb{R}^3)\}.$$

On it we consider the topology inherited by the topology on C intersected with the topology that renders continuous the mapping

$$C_R \longrightarrow L^2; (A,\Phi) \longmapsto \nabla_A \nabla_A \Phi .$$

The main point of this section is to show how to go from C to C_R by adding an L_2 term to the Higgs field Φ. This L^2 condition is not only natural, since a $\int |\varphi|^2$ term appears in the linearization of the potential term, but also necessary for the construction we are proposing, see sections B6 and B7. More specifically, we are proving the following:

PROPOSITION B5.1: There exists a canonically defined continuous mapping

$$C \longrightarrow C_R ; (A,\Phi) \longmapsto (A,\Phi_R)$$

with the property: $\Phi_R - \Phi \in L^2$.

The proof of the proposition occupies the rest of this section. Notice that the mapping does not influence the connection part of the configuration at all. Several equally natural mappings from C to C_R have been described in [T3] but none of them serves us here: the L^2 condition has to be satisfied so that $\int V(\Phi_R)$ is finite.

To achieve this, first consider the completion of the smooth, compactly supported sections of the adjoint bundle with respect to the norm:

$$\|\varphi\| = (\|\nabla_A\varphi\|_2^2 + \|\varphi\|_2^2)^{1/2}.$$

On this space, which it is natural to call $L^2_1(A)$, consider the functional

$$\int_{\mathbb{R}^3} <\nabla_A\Phi,\nabla_A\varphi> + \frac{1}{2}\int_{\mathbb{R}^3} <\nabla_A\varphi,\nabla_A\varphi> + \frac{1}{2}\int_{\mathbb{R}^3} <\varphi,\varphi>.$$

On $L^2_1(A)$ the functional has the following properties:

1) It is coercive:

$$\frac{1}{4}\|\nabla_A\varphi\|_2^2 + \|\nabla_A\Phi\|_2^2 + \int_{\mathbb{R}^3} <\nabla_A\Phi,\nabla_A\varphi> = \|\frac{1}{2}\nabla_A\varphi + \nabla_A\Phi\|_2^2 \geq 0$$

gives by adding $1/4\|\nabla_A\varphi\|_2^2$ on both sides

$$\frac{1}{2}\|\nabla_A\varphi\|_2^2 + \int_{\mathbb{R}^3} <\nabla_A\Phi,\nabla_A\varphi> \geq \frac{1}{4}\|\nabla_A\varphi\|_2^2 - \|\nabla_A\Phi\|_2^2$$

and

$$\frac{1}{2}\|\nabla_A\varphi\|_2^2 + \int_{\mathbb{R}^3} <\nabla_A\Phi,\nabla_A\varphi> + \int_{\mathbb{R}^3} |\varphi|^2 \geq \frac{1}{4}(\|\varphi\|_{L^2_1(A)})^2 - \|\nabla_A\Phi\|_2^2,$$

which is a coercivity relation.

2) The functional is continuous:

$$\frac{1}{2}\|\nabla_A\varphi\|_2^2 + \int_{\mathbb{R}^3} <\nabla_A\Phi,\nabla_A\varphi> + \int_{\mathbb{R}^3} |\varphi|^2 \leq$$

$$\leq \| \nabla_A \Phi \|_2 \| \nabla_A \varphi \|_2 + \frac{1}{4} | \nabla_A \varphi \|_2^2 + \| \varphi \|_2 .$$

3) Consisting purely of linear and quadratic terms, the functional is strictly convex.

4) Being convex and continuous, it is lower semicontinuous.

The above four properties are enough, see [J-T], to guarantee that the functional attains a unique minimum on $L^2_1(A)$ which we call φ_c. Then φ_c satisfies the corresponding Euler Lagrange equation:

$$-\nabla_A^2(\Phi + \varphi_c) + \varphi_c = 0. \qquad \text{(B5.1)}$$

This follows from an easy differentiation. Our conventions are such that

$$-\nabla_A^2 = \nabla_A^* \nabla_A .$$

We want to prove that equation (B5.1) is enough to deduce that $\nabla_A \nabla_A(\Phi + \varphi_c)$ lies in L^2 and hence that $(A, \Phi + \varphi_c)$ lies in C_R. Notice that $\Phi + \varphi_c$ itself does not lie in any L^2 space since Φ does not, recall that $|\Phi| - 1$ is in L^2. However, we can go round this problem using some of the standard techniques of the theory as developed by Taubes, c.f. [G2]:

Use the completion V_A of smooth, compactly supported sections with respect to the norm

$$\| \varphi \|_{V_A} = \| \nabla_A \varphi \|_2 .$$

LEMMA B5.2: There exists a canonically defined Φ' with $\Phi - \Phi'$ in $\mathbf{V}_A$ and such that $-\nabla_A^2 \Phi' = 0$, $\| \Phi' \|_\infty < \infty$ and $\nabla_A \nabla_A \Phi'$ in L^2.

Proof: With Φ in $L^2_{1,loc}$ and $\nabla_A \Phi$ in L^2, consider on H_A the functional

$$\int_{\mathbb{R}^3} | \nabla_A(\Phi + \varphi) |^2 .$$

Once again, it is easy to see that it is continuous, strictly convex (and as such weakly lower semicontinuous) and coercive on $\mathbf{V}_A$. Therefore there exists a unique minimizer φ_0 in H_A. It solves the corresponding Euler-Lagrange equation which is easily calculated to be:

$$\nabla_A{}^*\nabla_A(\Phi+\varphi_0) = 0 \ .$$

Set $\Phi' = \Phi+\varphi$. Then it is easy to see that Φ' satisfies the rest of the conditions, see [G2]. The point is that one can use the maximum principle here, whereas one cannot for (B5.1).

Note that by Lemma B2.2 and since $\Phi - \Phi' = \varphi_0$ in V_A, we have that $|\Phi - \Phi'|$ is in L^6. Adding a trivial term to equation (B5.1) gives:

$$-\nabla_A{}^2(\Phi - \Phi' + \varphi_c) + \varphi_c = 0$$

with $\Phi - \Phi'$ in L^6 and φ_c in $L^2{}_1(A)$. For the rest of the argument we shall be studying the local behavior of this equation.

Choose a ball of fixed radius R around a point x_o. We wish to apply the following standard result, see [G-T]:

THEOREM B5.3: Let u be a positive function over a domain Ω in $\mathbb{R}^3$, subsolution of

$$\Delta u \geq g$$

with g in $L^{q/2}(\Omega)$, $q > 3$ and u in $L^2{}_1(\Omega)$. Then for any ball $B_{2R}(x_o)$ contained in Ω and any $p > 1$ we have that:

$$\sup \{u(x): x \text{ in } B_{2R}(x_o)\} \leq C\,(R^{-3/p} \| u \|_{L^p(B_{2R}(x_o))} + R^{2\delta} \| g \|_{L^{q/2}(B_{2R}(x_o))})$$

where the constant C depends only on the radius R, the values of p and q and $\delta = 1 -$ $- 3/q$.

Notice that we cannot apply Theorem B5.3 directly to $|\Phi+\varphi|^2 - 1$ since it does not have a sign. We can however apply it for $u = |\Phi - \Phi' + \varphi_c|^2$. Indeed, we have:

1) $|\Phi - \Phi' + \varphi_c|^2$ is in $L^2(\Omega)$ for any bounded domain Ω since on such an Ω:

$$\| |\Phi - \Phi' + \varphi_c|^2 \|_2 = \|\Phi - \Phi' + \varphi_c\|_4^2 \leq C \|\Phi - \Phi' + \varphi_c\|_6^2$$

and both $\Phi - \Phi'$ and φ_c are in L^6.

2) $\nabla|\Phi - \Phi' + \varphi_c|^2$ is in L^2: For that, use that

$$\nabla|\Phi - \Phi' + \varphi_c|^2 = 2 < \nabla_A(\Phi - \Phi' + \varphi_c), \Phi - \Phi' + \varphi_c > .$$

Setting $\Psi = \Phi - \Phi' + \varphi_c$, we have by Holder's inequality ($1 = 1/3 + 2/3$)

$$\int |<\nabla_A\Psi, \Psi>|^2 \leq (\int |\nabla_A\Psi|^2 |\Psi|^2) \leq \|\nabla_A\Psi\|_3 \|\Psi\|_6 .$$

As remarked above , Ψ is in L^6. To see that $\nabla_A\Psi$ is in L^3 we need the following lemma, which is mentioned in [G2]:

LEMMA B5.4: Let Z be in L^2_{loc}, A and Ψ in $L^2_{1,loc}$ and such that $\nabla_A^2\Psi = Z$. Then Ψ is in $L^p_{1,loc}$ with $2 \leq p \leq 6$.

We prove this at the end of this section.

Now (2) follows: according to the Lemma and since Ψ is in $L^2_{1,loc}$, $|\nabla\Psi|$ is in L^3. To see that $|\nabla_A\Psi|$ is in L^3 use that A and Ψ are in L^6_{loc} and hence

$$\|[A,\Psi]\|_3 \leq \|A\|_6 \|\Psi\|_6 .$$

We can now apply Theorem B5.3 to the following relation:

$$\begin{aligned}\Delta|\Phi - \Phi' + \varphi_c|^2 &= 2 < \nabla_A^2(\Phi - \Phi' + \varphi_c), \Phi - \Phi' + \varphi_c > + 2|\nabla_A(\Phi - \Phi' + \varphi_c)|^2 \\ &\geq 2 < \nabla_A^2(\Phi - \Phi' + \varphi_c), \Phi - \Phi' + \varphi_c > \\ &= 2 < \varphi_c, \Phi - \Phi' + \varphi_c >.\end{aligned}$$

As for the right -hand-side of the inequality, we estimate:

$$\begin{aligned}\| < \varphi_c, \Phi - \Phi' + \varphi_c > \|_{2,loc} &\leq \|\varphi_c\|_{4,loc} \|\Phi - \Phi' + \varphi_c\|_{4,loc} \\ &\leq \|\varphi_c\|_{4,loc} \|\Phi - \Phi' + \varphi_c\|_{6,loc} \\ &\leq \|\varphi_c\|_4 \|\Phi - \Phi' + \varphi_c\|_6 .\end{aligned}$$

We have seen above why $\Phi - \Phi' + \varphi_c$ is in L^6. Concerning φ_c, recall that it is in $L^2_1(A)$, hence in L^6 and in L^2. By Holder's inequality it is in L^4. Applying Theorem

B5.3 then, we get:

$$\sup|\Psi(x)|^2 \leq C(\| |\Psi|^2 \|_{2,loc} + \|\varphi\|_4 \|\Psi\|_6)$$
$$\leq C(\|\Psi\|_6 + \|\varphi\|_4 \|\Psi\|_6).$$

Now repeat this for any point in $\mathbb{R}^3$ and for the fixed ball of radius R around it. Since the constant C in Theorem B5.3 depends only on the radius and not on the particular point, we have the above estimate for any point in $\mathbb{R}^3$. Since Φ' is bounded by construction, we have that $\Phi + \varphi_c$ is bounded.

Finally we can apply the inequality

$$\|\nabla_A \nabla_A(\Phi+\varphi_c)\|_2 \leq \|\nabla_A{}^* \nabla_A(\Phi+\varphi_c)\|_2 + \|\Phi+\varphi_c\|_\infty \|F_A\|_2 + \|\nabla_A(\Phi+\varphi_c)\|_2 \|F_A\|_2{}^2$$

(see [J-T] and [G1] for a complete proof) to deduce that $\nabla_A \nabla_A(\Phi+\varphi)$ is in L^2.

The argument will be complete once we have proved Lemma B5.4. For this we need the following:

THEOREM B5.5: ([Mo], 5.5.3): Let Ω be a domain and D with closure in Ω. Let f be in $L^{p'}(D)$ with $p' = 3q'/(3+q')$ and u in $L^q{}_1(D)$. Further assume that u is a weak solution of

$$\Delta u + \partial_\alpha(b^\alpha u) + c^\alpha \partial_\alpha u + f = 0$$

with coefficients satisfying the $H^1{}_q$ and $H^1{}_{q'}$ conditions (see below). Then u is in $L^{q'}{}_1(D)$ where $p' = 3q'/(3+q')$.

According to Morrey (see parts (i) and (iii) of definition 5.5.2 in his book), the coefficients b^α and c^α satisfy condition $H^1{}_q$ in a domain Γ

(i) for $3/2 < q < 3$: if they are measurable and they lie in $L^3(\Gamma)$

(iii) for $q > 3$: if b^α lies in L^q and c^α lies in L^3.

In our case, the diagonal system

$$-\nabla_A^{\,2}(\Phi - \Phi' + \varphi_c) + \varphi_c = 0$$

gives rise to three equations with coefficients coming from A and therefore in $L^2_{1,loc}(\mathbb{R}^3)$:

$$\Delta u + \partial_i(A)\, u + (A)\partial_i u + (A)(A)u = v\,,$$

where (A) denotes various combinations of components of A and u now is a component of $\Phi - \Phi' + \varphi_c$.

Then in a bounded domain Γ the coefficients are certainly in $L^3(\Gamma)$, in fact in all $L^p(\Gamma)$ for $2 < p < 6$. Then conditions H^1_q and $H^1_{q'}$ are satisfied for $q = 2$ and $q' = 6$. In fact the rest of the conditions of the theorem are satisfied since the components u lie in L^2_1. We can then conclude that $\Phi - \Phi' + \varphi_c$ lies in $L^6_{1,loc}$ and therefore in $L^p_{1,loc}$ for $2 \le p \le 6$, as claimed.

This competes the construction of the mapping from C to C_R. It is clear that (A,Φ_R) still has finite energy. To prove the continuity of the mapping one follows [T3], proposition B3.2, Lemma B3.7 and Corollary B3.5. The L^2 norms of the Lie brackets there should be replaced by the L^2 norms of φ_c. It is here that the topology on C and C_R becomes important.

It should be emphasized that the continuity of all the mappings involved in the construction is crucial. Its role becomes clear when we try to prove the global effectiveness of the slice, see below.

B6. The correct gauge in C_R.

It is here that all the work we did to bring our configurations to C_R is justified. In the space C_R we have the following:

LEMMA B 6.1. (A. Floer): For each Φ_R in C_R there exists a neighborhood U_R of Φ_R such that for all Φ'_R in U_R there exists a gauge transformation $g(\Phi'_R)$ such that at large distances Φ and $g(\Phi')\Phi'$ are parallel:

$$\frac{g(\Phi'_R)\Phi'_R}{|\Phi'_R|} = \frac{\Phi_R}{|\Phi_R|}. \tag{B6.1}$$

Furthermore, the mapping

$$U_R \longrightarrow G;$$
$$\Phi'_R \longmapsto g(\Phi'_R)$$

is continuous with $g(\Phi_R) = id$.

Notice that without uniform convergence to a non-zero constant at infinity we would be in trouble trying to explain what we mean be parallel directions. The fact that we are working in C_R is crucial in one more way. The defining condition $\nabla_A \nabla_A \Phi_R \in L^2$ together with the condition $\Phi_R \in L^2_{1,loc}$ give that Φ_R is in $L^2_{2,loc}$ and therefore continuous, by the Sobolev embedding

$$L^p_k \hookrightarrow C^j \ , k > j + n/p.$$

This is used to construct an $L^2_{2,loc}$ gauge transformation for each Φ'_R with the properties of the lemma.[3] It is also the reason why we put so much effort in proving that in our case $\Phi_R = \Phi + \varphi_c$ is in C_R.

Recall that our main aim is to find a gauge so that everything lies in Y_c. To that end, we need to observe that $g\Phi_R' - \Phi_R$ follows the rate of decay of $|\Phi'| - |\Phi|$:

LEMMA B6.2: For g as above and for Φ' in the same neighborhood as above, $g(\Phi')\Phi' - \Phi$ is in L^2.

Proof: Rewrite the defining relation (B6.1) of $g(\Phi'_R)$ as

$$g(\Phi'_R)\Phi'_R - |\Phi'_R| \ (\widehat{\Phi_R}) = 0.$$

Here $\widehat{\ }$ indicates the field divided by its length, a new field of unit length.By adding and subtracting $|\Phi| \ (\widehat{\Phi_R})$:

[3]In fact the gauge transformations of the lemma are constructed in two steps. The first is a local geometric construction where the continuity of the fields is used. This step heavily relies on the fact that we work in the Lie Algebra of SU(2). The second one, a correction to make the gauge transformations globally defined, does not influence the property described in the lemma.

$$g(\Phi'_R)\Phi'_R - |\Phi'_R|\,(\widehat{\Phi_R}) + |\Phi_R|\,(\widehat{\Phi_R}) - |\Phi_R|\,(\widehat{\Phi_R}) = 0$$

or

$$g(\Phi'_R)\Phi'_R - |\Phi_R|\,(\widehat{\Phi_R}) = (|\Phi'_R| - |\Phi_R|)\,(\widehat{\Phi_R})\,. \qquad \text{(B6.2)}$$

Now $(\widehat{\Phi_R})$ is bounded by the way it is defined (it is identically 1 far away and it can be bumped out near the origin). Whereas

$$|\Phi'_R| - |\Phi_R| = |\Phi' + \varphi_{c'}| - |\Phi + \varphi_c|$$

where φ_c and $\varphi_{c'}$ come from the minimization recipe we described in the previous section. Since

$$|\Phi' + \varphi_{c'}| - |\Phi + \varphi_c| \leq |\Phi'| + |\varphi_{c'}| - |\Phi| + |\varphi_c| = (|\Phi'| - |\Phi|) + |\varphi'| + |\varphi|.$$

Now notice that by definition φ_c and $\varphi_{c'}$ are in L^2. Φ and Φ' are in the configuration space C and hence have finite energy:

$$\int (|\Phi|^2 - 1)^2 = \int (|\Phi| - 1)^2 (|\Phi| + 1)^2 \geq \int (|\Phi| - 1)^2,$$

which proves that $|\Phi| - 1$ and similarly $|\Phi'| - 1$ are in L^2. Therefore $|\Phi'| - |\Phi|$ is in L^2 and hence $g(\Phi')\Phi' - \Phi$ is in L^2, by (B6.2).

We have then achieved the first step towards bringing everything in Y_c: the one but last term of the norm works in this gauge.

As for the rest of the terms in the norm, one checks that $\nabla_A(g(\Phi')\Phi' - \Phi) \in L^2$ directly, as in Floer. Finally we need an extra gauge transformation to bring the remaining terms in the appropriate norm. This transformation is of the form $\exp(f(\Phi_R)')$ and therefore does not influence at all the Higgs part of the construction up to now.

B7: Back to the configuration space C.

What we have proved so far is that in C_R we can transform any Φ'_R in a neighborhood of Φ_R so that $g(\Phi'_R)\cdot\Phi'_R - \Phi_R$ lies in Y_c. We can now go back to the configuration space C and check that this is still true if we use the gauge transformations described by the mapping

$$U_c \longrightarrow G$$

$$c \longmapsto c_R \longmapsto g(c_R).$$

Once again, we check this for the $\| \varphi \|_2$ part of the Y_c norm and refer to Floer for the remaining parts. We have

$$g(\Phi'_R)\,\Phi'_R - \Phi_R = g(\Phi_R')\,(\Phi' + \varphi_{c'}) - (\Phi + \varphi_c) \in L^2$$

or

$$g(\Phi')\Phi' - \Phi + g(\Phi')\varphi_{c'} - \varphi_c \in L^2$$

and since $\varphi_c \in L^2$

$$g(\Phi')\Phi' - \Phi + g(\Phi')\varphi_{c'} \in L^2.$$

Finally notice that since $g(\Phi')$ is unitary $| g(\Phi')\varphi_{c'} | = | \varphi_{c'} |$ and since $\varphi_{c'}$ is in L^2 we get that $g(\Phi')\Phi' - \Phi$ is in L^2, as desired.

Remark: It looks as if we simply used the φ's to go from C to C_R and back again. The point is that we had to go to C_R to construct the gauge transformation there in the class of the gauge transformations we have used in the theory.

B8: Answering Problem 2: Solving the slice equation.

What we have achieved is a Banach model in the configuration space, up to gauge transformations. That was Problem 1. In this model, we must solve the slice equation up to a gauge, that is we must find for each (A',Φ') a gauge transformation g with

$$\nabla_A^*(gA' - A,\Phi) + [\,\Phi, g\cdot\Phi' - \Phi\,] = 0\,.$$

As we have already argued in the Yang-Mills case, this is done by considering the mapping

$$(g, (A',\Phi')) \longmapsto \nabla_A^*(gA' - A,\Phi) + [\,\Phi, g\cdot\Phi' - \Phi\,] \qquad \text{(B7.1)}$$

for g in a neighborhood of the identity and (A',Φ') close to (A,Φ) in Y_c. To apply the Implicit Function Theorem we need to make a good choice of gauge transformations. At this point recall the heuristic discussion of section B4 and introduce the following spaces:[4]

Define Z_c to be the completion of the compactly supported, smooth sections of Ad(P) with respect to the norm:

$$\|\xi\|_{Z_c} = \|\xi\|_2 + \|<\Phi,\xi>\|_{6/5}.$$

Define X_c to be the completion of the compactly supported, smooth sections of Ad(P) bundle with respect to the norm:

$$\|\xi\|_{X_c} = \|\nabla_A\nabla_A\xi\|_2 + \|[\Phi,\xi]\|_2 + \|\nabla_A\xi\|_2 + \|\Phi\cdot\nabla_A{}^2\xi\|_{6/5}.$$

There is no mystery in choosing these norms. One has to understand only the choice of the Z_c-norm, and that we have explained in section B4. The rest are of the norms are chosen so that the following mappings are continuous. They naturally appear when one tries to apply the Implicit Function Theorem:

LEMMA B7.1: (i) The linearization of the gauge action mapping:

$$\xi \longmapsto \nabla_A\xi + [\Phi,\xi]$$

is continuous as a mapping from X_c to Y_c.

(ii) The adjoint of the linearization mapping, the slice operator

$$(\alpha,\varphi) \longmapsto \nabla_A{}^*\alpha + [\Phi,\varphi]$$

is continuous as a mapping from Y_c to Z_c.

(iii) Finally, the composition of the two mappings, the Laplacian of the system

$$\xi \longmapsto \nabla_A{}^2\xi + [\Phi,[\Phi,\xi]]$$

is continuous as a mapping from X_c to Z_c.

Proof:

(i) $\|(\nabla_A\xi,[\Phi,\xi])\|_{Y_c} = \|\nabla_A\nabla_A\xi\|_2 + \|[\Phi,\nabla_A\xi]\|_2$

$$+\|[\Phi,\xi]\|_2 + \|\nabla_A[\Phi,\xi]\|_2 + \|\nabla_A{}^*\nabla_A\xi\|_{6/5}$$

The first, third and last term are included in the X_c norm and hence are naturally

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[4]These norm spaces where first introduced in Floer's preprints. Here we have modified some of the terms and the proofs to suit the case of the full Lagrangian.

bounded by it. For the second term notice that as in (B3.1),

$$\|[\Phi, \nabla_A\xi]\|_2^2 \le \|(|\Phi|-1)^2|\nabla_A\xi|^2\|_1 + \|\nabla_A\xi\|_2^2$$
$$\le \|(|\Phi|-1)\|_2 \|\nabla_A\xi\|_4^2 + \|\nabla_A\xi\|_2^2 .$$

Again, the fact that (A,Φ) is a pair with finite energy gives $\|(|\Phi|-1)\|_2$ finite. At the same time, by Sobolev's inequality

$$\|\nabla_A\xi\|_6 \le \|\nabla_A\nabla_A\xi\|_2$$

since ξ is in the completion of compactly supported objects. This, together with the fact that $\|\nabla_A\xi\|_2$ is finite bounds $\|\nabla_A\xi\|_4$.

As for the fourth term, use that

$$\|\nabla_A[\Phi, \xi]\|_2 \le \|[\nabla_A\Phi, \xi]\|_2 + \|[\Phi, \nabla_A\xi]\|_2.$$

But

$$\|[\nabla_A\Phi, \xi]\|_2 \le \|\xi\|_\infty \|\nabla_A\Phi\|_2.$$

Again, recall that $\|\nabla_A\Phi\|_2$ is finite since the energy of the configuration is finite. That ξ is bounded follows from the fact that both $\nabla_A\xi$ and $\nabla_A\nabla_A\xi$ are in L^2 and hence ξ is continuous and in L^6_1.

(ii) Obvious, by the definitions of the norms.

(iii) Since

$$\|\nabla_A{}^2\xi + [\Phi,[\Phi,\xi]]\|_{Z_c} = \|\nabla_A{}^2\xi + [\Phi,[\Phi,\xi]]\|_2 + \|\Phi\cdot\nabla_A{}^2\xi\|_{6/5},$$

use that

$$\|\nabla_A{}^2\xi\|_2 \le \|\nabla_A\nabla_A\xi\|_2$$

and that

$$\|[\Phi,[\Phi,\varphi]]\|_2^2 = \|\,|\Phi|^2|[\Phi,\xi]]|^2\|_1$$
$$\le \|(|\Phi|-1)|^2[\Phi,\xi]]|^2\|_1 + \|[\Phi,\xi]]|\,\|_2^2.$$

Once again, we need to know that $\|[\Phi, \xi]]\|_4$ is finite. But we have just shown above how to control the $\|\nabla_A[\Phi,\xi]\|_2$ by the X_c norm. This gives $[\Phi,\xi]$ in L^6 and by the familiar argument in L^4.

Of course the most important property of the above spaces is the behaviour of the derivative of the mapping (B7.1) with respect to g , which is given by the operator

$$\nabla_A{}^2 + \mathrm{ad}^2\Phi.$$

For this we have the following all-important

THEOREM B7.2: (A. Floer). For any connection ∇ with square integrable curvature, the operator

$$\nabla_A{}^2 + \mathrm{ad}^2\Phi$$

defines a linear homeomorphism from the space X_c to the space Z_c.

Proof: The proof is essentially the heuristic argument of section B4. Technically, although we have slightly changed the norm of $X_c{}'$ we only have to repeat (and complete!) the arguments of the proof in [F].

Therefore, the mapping

$$\exp(X_c) \times Y_c \longrightarrow Z_c$$

$$(\exp(\xi), (a,\varphi)) \longmapsto \nabla_A{}^* \exp(\xi).a + [\Phi, \exp(\xi)\varphi]$$

has non-singular derivative with respect to ξ at the point (e, 0). By the Implicit function Theorem, the slice equation can be solved.

Recall however that a slice has to be globally effective, that is it has to intersect orbits only once (the restriction of the projection has to be one-to-one). To be able to prove this Floer finds that one has to use more gauge transformations than the exponentials of the space X_c.

With hindsight, this was to be expected: First, we chose the slice using a formal L^2 adjoint but had to use a completely different inner product on the tangent space in the sequel. Second, the gauge transformations coming from X_c decay to zero at infinity. This is not true for all gauge transformations in $L^2_{2,loc}$ and indicates that we have somehow ignored too many of the original gauge transformations.

The space X_c has to be complemented to $X_c' = X_c \oplus \mathbb{R}\Phi_o$. Here Φ_o is a canonically defined element[5] and X_c' is exactly the space of $L^2_{2,loc}$ gauge transformations with finite X_c norm. This naturally leads to complementing Z_c to $Z_c' = Z_c \oplus \mathbb{R}$ and to considering the mapping

$$Y_c \longrightarrow Z_c'$$
$$(a,\varphi) \longmapsto (\nabla_A{}^* a + [\Phi,\varphi], <(a,\varphi), \nabla_A\Phi_o + [\Phi,\Phi_o]>_c).$$

Therefore the Laplacian must also change into

$$X_c' \longrightarrow Z_c'$$
$$(\xi,t) \longmapsto (\nabla_A{}^2\xi + ad^2\Phi(\xi), <(\nabla_A\xi, [\Phi,\xi]), \nabla_A\Phi_o + [\Phi,\Phi_o]>_c).$$

Theorem B7.2 is still true for X_c and Z_c replaced by X_c' and Z_c' respectively. This gives a genuine slice and, by the obvious way, coordinates for the quotient space. One can check that the change of coordinates is differentiable.

To summarize, the tangent space at a given equivalence class $[(A,\Phi)]$ in the quotient space is given by:

$$T_{[(A,\Phi)]}C^*/G^* = \{(a,\varphi) \text{ in } Y_c \text{ with } \nabla_A{}^* a + [\Phi,\varphi] = 0 \text{ and } <(a,\varphi), \nabla_A\Phi_o>_c = 0\}.$$

As a reward to this rather technical construction, we can differentiate the Yang-Mills-Higgs functional and find that it has a continuous derivative at each point. Since we have already argued for the potential term, we check here the remaining terms:

For (a,φ) in the tangent space at (A,Φ),

---

[5]In Floer's construction Φ_o can be used as Φ_R. In fact, it is crucial to realize that in Floer's construction Φ_o serves two purposes: Provides the passing to the regularized space and completes the tangent space to the orbit. In our case we have to use Φ_R to reach the regularized space in an L^2 manner.

$$\frac{d}{dt}\Big|_0 \int_{\mathbb{R}^3} \{ |F_{A+ta}|^2 + |\nabla_{A+ta}(\Phi + t\varphi)|^2 \} d^3x =$$

$$= \langle F_A, \nabla_A a \rangle + \langle \nabla_A \Phi, \nabla_A \varphi \rangle + \langle \nabla_A \Phi, [a, \Phi] \rangle,$$

which can obviously be controlled by the Y_c-norm with the help of Holder's inequality.

As a final remark, we would like to underline the two points of this construction that are particular to the SU(2)-adjoint case (or SO(3)-adjoint): First, the gauge transformation of Lemma B6.1 uses the geometry of the Lie Algebra su(2), viewed as the Lie Algebra $\mathbb{R}^3$ with the standard skew product on it. Second, we have used more than once gauge transformations of the form exp(fΦ), which automatically assumes that we are in the adjoint representation case.

C. FINITE ENERGY CONFIGURATIONS: THE GENERAL CASE.

We present here some preliminary results concerning the asymptotics of finite energy fields. In particular, we are not assuming that the fields solve any equations. We are assuming the structure group G to be any compact Lie group, the small group H to be any subgroup of G and V to be any symmetry breaking Higgs potential. It turns out that much more can be said about the Higgs field Φ than the gauge potential A. In the next chapter, where we specialize to solutions for G = SU(2) and H = U(1), we can deal with the asymptotics of A using known estimates for F_A.

By configuration we mean a pair (A,Φ) with both members of the pair in the corresponding $L^2_{1,loc}$ spaces and such that the energy is finite. We use spherical coordinates (r,θ,φ) with $0 \le r$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$ on $\mathbb{R}^3$ so that

$$x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi,\ x_3 = r\cos\theta.$$

Then the volume element on $\mathbb{R}^3$ is

$$r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

and the metric is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin\theta\, d\varphi^2.$$

Therefore an orthonormal basis for the cotangent space at a point is given by

$$\{ dr, r\, d\theta, r\sin\theta\, d\varphi \}.$$

We write $d\Omega$ for the volume element of the unit sphere, $d\Omega = \sin\theta\, d\theta\, d\varphi$.

LEMMA C1: If Φ is in $L^2_{1,loc}$ then Φ is continuous in almost any radial direction.[1]

Proof: Since Φ is in $L^2_{1,loc}$, Φ is in L^2_1 on the annulus $\{ x: 1 \le |x| \le R_n \}$ for $R_n > 1$. Therefore, in spherical coordinates we have that

$$\int_{S^2}\int_1^{R_n} r^2(|\Phi|^2 + |d\Phi|^2)\, dr\, d\Omega$$

[1] We acknowledge inspiration from [S-Y].

is finite. This means that in almost any radial direction the integral

$$\int_1^{R_n} r^2(|\Phi|^2 + |d\Phi|^2)\, dr$$

and since $r \geq 1$ the integral

$$\int_1^{R_n} (|\Phi|^2 + |d\Phi|^2)\, dr$$

is finite, too. Hence Φ is in L^2_1 in almost any radial direction within the annulus. By Sobolev's Embedding Theorem for dimension 1, Φ is continuous in each such direction. Taking an increasing sequence of R_n's so as to cover the whole of $\mathbb{R}^3$ and forgetting each time a set of measure zero, we end up with almost all radial directions on each of which Φ is continuous.

One of the major technical problems when dealing with the coupling term $d_A\Phi$ of the Lagrangian is that it involves both the Φ and the A field and therefore in general gives information for none of them unless something is known about one of them. This difficulty can be avoided for the radial components when working in the radial gauge, which is characterized by the condition $\Sigma x_i A_i = 0$ or, in terms of the spherical coordinates of the connection form, $A_r = 0$. For the existence of such gauges see the next section. We use such a gauge in the following:

PROPOSITION C2: Let (A,Φ) be a finite energy configuration (not necessarily a solution). Then in a radial gauge Φ achieves a limit in almost any radial direction.

Proof: The finite energy condition means that $\| d_A\Phi \|_2$ is finite. Written out in a radial gauge this gives

$$\int_{S^2}\int_0^{\infty} \{r^2|\partial_r\Phi|^2 + |\partial_\theta\Phi + [A_\theta,\Phi]|^2 + \sin\theta^{-2}|\partial_\varphi\Phi + [A_\varphi,\Phi]|^2\}\, dr\, d\Omega < \infty .$$

Then for almost any radial direction

$$\int_0^{\infty} r^2 \, | \partial_r \Phi |^2 \, dr$$

is finite.

Pick a generic radial direction $(.,\, \omega_o)$ in $\mathbb{R}^3$ for which this integral is finite and the previous lemma is true and two points (R_1, ω_o) and (R_2, ω_o) with $R_1 < R_2$. On such a direction, using Holder's inequality and the continuity of Φ, we have

$$| \Phi(R_1, \omega_o) - \Phi(R_2, \omega_o) | \leq \int_{R_1}^{R_2} | \partial_r \Phi(r, \omega_o) | \, dr$$

$$\leq \left(\int_{R_1}^{R_2} \frac{1}{r^2} dr \right)^{1/2} \left(\int_{R_1}^{R_2} r^2 | \partial_r \Phi(r, \omega_o) |^2 \, dr \right)^{1/2}$$

$$\leq M \left(\frac{1}{R_1} - \frac{1}{R_2} \right)^{1/2} .$$

Therefore, for each such radial direction the Higgs field has a limit as the distance from the origin tends to infinity.

Notice that the constant M in the proof of the Proposition depends on the direction and hence the estimate is not uniform.

Let $\Phi_{\infty}(\omega)$ denote the limit on the radial direction (r, ω) of the Higgs field Φ as r tends to infinity whenever this limit exists. Exploiting the finite energy condition through the third term in the Lagrangian gives that

$$\int_{S^2} \int_0^{\infty} r^2 \sin\theta \, V (\Phi (r,\omega)) \, dr \, d\omega < \infty .$$

Therefore

$$\int_0^\infty r^2 V(\Phi(r,\omega))\,dr$$

is finite for almost any radial direction ω.

Since $\Phi(r,\omega)$ has a limit as r tends to infinity and V is at least continuous, $V(\Phi(r,\omega))$ must go to zero as r tends to infinity, for the last integral to be finite. But V achieves the value 0 only on the vacuum orbit, therefore Φ_∞ defines a map

$$\Phi_\infty : S^2 \longrightarrow G/H.$$

Such a map defines a reduction of any trivial G-bundle over S^2 to an H-subbundle in the following way:

It is well known that reductions of a G-bundle P to H-subbundles are in one to one correspondence with sections of the associated bundle

$$Q = P \times_G G/H \approx P/H.$$

Here G acts on the quotient space by left multiplication, see [K-N]. In our case P is trivial and hence isomorphic to $S^2 \times G$. Using this identification, the bundle Q is isomorphic to $S^2 \times G/H$ via the following isomorphism:

$$[(\omega, g), g'H] \longmapsto (\omega, gg'H).$$

It is then clear that a map like Φ_∞ defines a section of $S^2 \times G/H$, hence a section of Q and therefore a reduction of P to an H-bundle.

We have deliberately avoided any adjectives like smooth, continuous and the similar. As we are going to prove in the next section, Φ_∞ is continuous when dealing with solutions and therefore the reduction will be within the known framework. We just mention here that measurable reductions of bundles have been studied, see [Z]. This kind of analysis together with the methods of [U 2] for Sobolev connections should give a way of defining a magnetic charge in the general setting as a generalized characteristic class of some measurable reduction.

Notice that by lemma B2.2, there exists a constant $M(\Phi)$ such that $|\Phi| - M(\Phi) \in L^6$. Since we prove that Φ tends to the vacuum as $|x|$ tends to infinity we have that

$$M(\Phi) = |\Phi_0|,$$

for Φ_0 any point on the vacuum orbit. That is, the asymptotics for the case with a potential term in the Lagrangian have more rigidity than the Prasad-Sommerfield limit.

D. G=SU(2), ADJOINT REPRESENTATION: THE ASYMPTOTIC BEHAVIOUR OF A FINITE ENERGY SOLUTION.

D0. Introduction.

We give a detailed proof of a fact long conjectured by physicists, see [G-N-O], and often referred to, see [H-R 1] and [H-R 3]:

Given any finite energy solution (A,Φ) of the Yang-Mills-Higgs equations on $\mathbb{R}^3$, A becomes a pure U(1)-Yang-Mills connection at the "sphere at infinity ".

For a physicist this means that at large scales compared to the atom all that is left from a non-abelian t'Hooft-Polyakov monopole is an abelian Dirac monopole. This was one of the main reasons for introducing the theory, after all.

Now it is well known that the holonomy of a Yang-Mills field on the sphere is either $\mathbb{R}$ or U(1), see [A-B] and [F-H]. Starting with a compact gauge group immediately excludes $\mathbb{R}$. On the other hand a connection always reduces to its holonomy bundle. The conjecture then is that any finite energy monopole becomes a pure Yang-Mills field at infinity. (In the case we study here, G=SU(2), H=(1). The above complication is not apparent since we prove directly that the limit is a U(1)-Yang-Mills field.)

Notice that although Φ does not appear in the final statement, it influences A through the coupling term $d_A\Phi$ in the Lagrangian.

A number of points should be emphasized:

Of course, the first thing one has to make sense of is what exactly is meant by "sphere at infinity". Since $\mathbb{R}^3\backslash\{0\}$ is $S^2\times(0,\infty)$ only topologically but not metrically, some care has to be taken. In fact, we have found this point to be a major step in the proof.

The idea is that the "sphere at infinity" should be interpreted as a family of configurations on the fixed unit sphere S^2 in $\mathbb{R}^3$ with its standard Riemannian metric. This family is parametrized by r, the distance from the origin in $\mathbb{R}^3$. The limits at infinity are nothing but the limits of the family as the parameter tends to infinity. This does not influence the calculations for the Higgs field at all, since we deal with limits of functions. It does however clarify the form part of the configuration, where the Riemannian structure comes to the fore.

Recall that in the adjoint-SU(2) case a monopole solves the equations

$$d_A{}^*F_A = [\nabla_A\Phi, \Phi]$$

$$\nabla_A{}^*\nabla_A\Phi = \frac{\lambda}{2}(|\Phi|^2 - 1)\Phi .$$

Morally speaking then, if $\nabla_A\Phi$ decays to zero the first equation should give the Yang-Mills equation:

$$\nabla_A{}^*F_A = 0.$$

It is this observation one has to make sense of. For this, we have found that the Sobolev spaces of fields over S^2 are the appropriate setting: although one starts with solutions, therefore smooth objects, some differentiability is lost by passing to the limit. Such a limit can be realized only in a Sobolev space.

Having realized these two points, the rest of the proof relies on Taubes' estimates in [T 1], Uhlenbeck's Weak Compactness theorem, see [U1], and a formula by Taubes as it appears here[2]. The rest consists of analytic pleasantries.

Finally, viewing the problem as the behaviour of solutions to a system of partial differential equations, with the finite energy condition replacing boundary values, we see that monopoles behave quite differently than harmonic functions, see [A-S]. However, for the Bogomol'nyi case with hyperbolic metric the conjecture is no longer true, see [B-A].

Throughout this chapter, G=SU(2), $V(\Phi) = \lambda/2(|\Phi|^2 - 1)^2$ and the representation is the adjoint one. The method we present here also applies to any solution in the Prasad-Somerfield limit $\lambda=0$, see below. (A,Φ) will always be a solution configuration.

D1. Taubes' estimates.

In [T 1] and [J-T] Taubes proves the following for the adjoint-SU(2) case:

[2]Taubes' formula proves the conjecture in an asymptotic manner. We learned of it towards the end of our study on the problem.

THEOREM D1.1: Let (A,Φ) be a smooth finite energy solution of the Yang-Mills-Higgs equations. Then we have the following a priori estimates:

Coupling term estimate: there exists a positive constant m and for any positive ε there exists a positive real number $M(\varepsilon)$ such that:

$$|d_A\Phi|(x) \leq M(\varepsilon)\, e^{-(1-\varepsilon)m|x|}$$

and

Higgs field estimate:

$$0 \leq 1 - |\Phi| \leq M(\varepsilon)\, e^{-(1-\varepsilon)m|x|}.$$

Curvature estimate: there exists a constant M such that for x with $|x|$ sufficiently large

$$|F_A|(x) \leq M(1+|x|^2)^{-1}.$$

In particular, we have for the transverse to Φ components:

$$|[F_A,\Phi]| \leq M(\varepsilon)e^{-(1-\varepsilon)|x|}.$$

Comments on the proof: The proof occupies almost the whole of chapter 4 in [J-T] and the estimates for the coupling term and the curvature term appear as Theorem 10.2. In fact, this Theorem is the answer to the problem of the masses as we discussed it in the introduction. Here the relevant decomposition to massless and massive components is given by the projections of the fields onto Φ (longitudal components) and the components orthogonal to Φ (transverse components) respectively. This is best described by the following decomposition in su(2) with respect to any unit vector η of any vector ξ:

$$\xi = \langle \xi,\eta \rangle \eta + [\eta,[\eta,\xi]].$$

In our case, η is the Higgs field Φ divided by its length. Notice that this makes sense in large distances thanks to the Higgs field estimate.

Observe that in a radial gauge the exponential convergence of the lengths implies the exponential convergence of the fields themselves:

LEMMA D1.2: In a radial gauge Φ tends to its limit exponentially in any radial direction where it does have a limit.

Proof: Using the coupling term estimate we can write the inequality in the proof of Proposition C2 in a stronger way:

$$|\Phi(R_1,\omega_o)-\Phi(R_2,\omega_o)| \le \int_{R_1}^{R_2} |\partial_r\Phi(r,\omega_o)|\,dr \le \int_{R_1}^{R_2} e^{-Mr}\,dr .$$

Letting R_2 tend to infinity,

$$|\Phi(R_1,\omega_o)-\Phi_\infty(\omega_o)| \le M^{-1}e^{-MR_1} .$$

<u>D2. The limit of A at infinity.</u>

We start in a gauge where the configuration (A,Φ) is smooth on $\mathbb{R}^3$. Such a gauge exists, see [J.-T.], section V. Gauge transform to a radial gauge using a smooth gauge transformation, which we can obtain by solving the following ordinary differential equation for $g(.,\varphi,\theta)$:

$$g^{-1}(r,\varphi,\theta)\,A_r(r,\varphi,\theta)\,g(r,\varphi,\theta) + g^{-1}(r,\varphi,\theta)\,\partial_r g(r,\varphi,\theta) = 0,$$

with some initial conditions. We are then in a gauge where A and Φ are smooth and $A_r = \sum x_i A_i = 0$.

We shall now see how the connection part of the configuration behaves in this gauge. Let

$$i_R : S^2 \to \mathbb{R}^3$$

be the family of embeddings that send the point (φ,θ) of the sphere to (r,φ,θ) in $\mathbb{R}^3$. Using them to pull back the bundle P and the connection A we have the one parameter family $i_R^*(P)$ of bundles over S^2, all equivalent to the trivial one, each supplied with the connection $i_R^*(A)$. Since we are in a radial gauge and we can write A over $\mathbb{R}^3$ as

$$A(r,\varphi,\theta) = A_\varphi(r,\varphi,\theta)d\varphi + A_\theta(r,\varphi,\theta)d\theta ,$$

on the sphere we have that

$$i_R^*(A)(\varphi,\theta) = A_\varphi(R,\varphi,\theta)d\varphi + A_\theta(R,\varphi,\theta)d\theta.$$

From now on we write A_R for $i_R^*(A)$ when there is no confusion and

$$A_R = (A_R)_\varphi(\varphi,\theta)d\varphi + (A_R)_\theta(\varphi,\theta)d\theta.$$

That is, we want to view the r-variable in $\mathbb{R}^3$ as a parameter for S^2. We can start now seeing why there is a limit for the A_R's as R tends to infinity:

The respective curvatures F_{A_R} on S^2 are

$$F_{A_R}(\varphi,\theta) = F_{\varphi\theta}(R,\varphi,\theta)\, d\varphi\wedge d\theta,$$

where

$$F_A(r,\varphi,\theta) = F_{\varphi\theta}(r,\varphi,\theta)\, d\varphi\wedge d\theta + F_{r\theta}(r,\varphi,\theta)\, dr\wedge d\theta + F_{r\varphi}(r,\varphi,\theta)\, dr\wedge d\varphi$$

on $\mathbb{R}^3$.

The Curvature estimate of Theorem D1 then gives that

$$| F_{\varphi\theta}(r,\varphi,\theta)\, d\varphi\wedge d\theta | \leq M\, (\, 1 + r^2\,)^{-1}.$$

An orthonormal basis for the cotangent space of $\mathbb{R}^3$ at the point (r,φ,θ) is given by

$$\{dr, r\sin\theta d\varphi, r d\theta\}.$$

Therefore,

$$| r^{-2}F_{A_R}(\varphi,\theta)| = | r^{-2} \sin\theta^{-1} F_{\varphi\theta}(r,\varphi,\theta)\, (r\sin\theta)\, d\varphi\wedge r\, d\theta\, |$$
$$\leq M\, (\, 1 + r^2\,)^{-1},$$

which gives that

$$| F_{A_R}(\varphi,\theta)| \leq M,$$

for all R.

That is, the A_R's are connections with uniform bounds on the curvature in the sense of Uhlenbeck, see [U 1]. This provides us with an elegant, if somewhat sophisticated, way of finding the limit of $\{A_R\}$. We know of no other way.

The main result in [U 1] is:

THEOREM D2.1: Let M be a compact manifold of dimension M and $\{A_n\}$ a sequence of connections on a bundle P over M, in $L^p{}_1(M)$ with $2p > n$. If there exists a constant B such that

$$\| F_{A_n} \|_{L^p} \leq B$$

then there exists a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ and a sequence $\{g_{n_i}\}$ of gauge transformations in $L^p{}_2(M)$ with the property: $g_{n_i} \cdot A_{n_i}$ converges weakly to a connection A in $L^p{}_1(M)$.

It is part of the proof of the Theorem that A defines a connection on a bundle isomorphic to the original P. For p = 2n this is no longer the case, see [Sed].
In our case, we have that $M = S^2$ and then n = 2. The family of connections is smooth and therefore each of them is in the $L^p{}_1(S^2)$ Sobolev space required by the theorem, for any p. To avoid any ambiguity concerning the limit connection we take the sequence on which to apply the theorem to be $\{A_R\}$ for all positive integers R. Then the A_R's live on bundles that are equivalent to the trivial one and the theorem applies with B = M. We call the weak limit connection A_∞. It lives on the trivial bundle over S^2 and is in $L^p{}_1(S^2)$, for p > 1. Of course, we rename the subsequences to $\{A_R\}$ and $\{g_R\}$.
To make sure that we are still within the configuration space we have chosen, we want to realize the corresponding gauge in $\mathbb{R}^3$. Define g: $\mathbb{R}^3 \longrightarrow G$ by:

$$g(r,\varphi,\theta) = g_R(\varphi,\theta)$$

when r is in the strip

$$\frac{(R-1)+R}{2} < r < \frac{(R+1)+R}{2}.$$

If we take p = 2, each g_R is $L^2{}_2$ on the sphere and g is $L^2{}_2$ on the strips. Using a bump function identically 1 on the narrower strips

$$\frac{4R-1}{4} < r < \frac{4R+1}{4},$$

it is clear that we can join things together so that g is $L^2{}_{2,loc}$. The resulting configuration then on $\mathbb{R}^3$ is gauge equivalent to the original one via one of the gauge transformations of the theory. This is the gauge we wish to work in.

D3. The Finite Energy Condition: A_∞ Reduces.

In this section we prove that the limit of the Higgs field in the final gauge is continuous and therefore defines a reduction to a U(1)-subbundle as explained in Chapter C. We also prove that the limit connection reduces to this subbundle, or, to use a piece of terminology from Physics, the Finite Energy Condition is satisfied.
(It is well known that in a radial gauge Φ has a continuous limit at infinity, see [J-T] page 38. The problem here is that since the gauge transformations g_R in $L^p{}_2$ do not necessarily have a limit we cannot conclude immediately that the limit of Φ in the final gauge exists.)

We claim that Φ_R has a pointwise limit Φ_∞ in the gauge where A_∞ exists. To prove this, first notice that since $|\Phi| \leq 1$, $\{\Phi_R\}$ is bounded in any $L^p(S^2)$, for any p:

$$\| \Phi_R \|_p \leq (\mathrm{vol}(S^2))^{1/p}.$$

This is true for any gauge, since $|\Phi|$ is a gauge invariant quantity. Now it is a standard fact that in a reflexive space bounded sets are weakly compact. Therefore, in any gauge Φ_R has a subsequence that converges weakly in L^p, for any $p \geq 2$.

We also have that A_R converge weakly to A_∞ in $L^p{}_1$ for all p. By the Rellich-Kondrachov Theorem, they converge strongly in L^q for $q \geq 1$, and therefore (up to subsequences) pointwise. In particular, A_R is bounded in L^q, $q \geq 1$.
Then $[A_R , \Phi_R]$ is bounded in L^q, too: using the elementary inequality

$$|[A_R , \Phi_R]|^2 + |\langle A_R , \Phi_R \rangle|^2 = |A|^2 |\Phi_R|^2$$

we see that

$$|[A_R , \Phi_R]| \leq |A_R| |\Phi_R| \leq |A_R|.$$

Applying this for p = 2 we have that $[A_R , \Phi_R]$ converges weakly to a limit B_∞ in L^p. (We shall prove in a while that this limit is independent of p.)
Now use the coupling term estimate of Theorem 1: the exponential decay of $|d_A \Phi|$ on

$\mathbb{R}^3$ means that $| d_{A_R}\Phi_R | \to 0$ on S^2, much faster [3] than R^{-1}. Hence $d_{A_R}\Phi_R \to 0$ in any L^p strongly.

Then

$$d(\Phi_R) + [A_R, \Phi_R] \to 0$$

and

$$[A_R, \Phi_R] \to B_\infty \text{ weakly}$$

give that

$$\partial_{\varphi,\theta}\Phi_r \to -(B_\infty)_{\varphi,\theta} \qquad \text{(D3.1)}$$

in L^p, weakly. Notice that Φ_R are differentiable since we started from a smooth gauge and transformed by $L^p{}_2$, that is C^1 if $p > 2$, transformations. This means that Φ_R has a weak limit in $L^p{}_1(S^2)$. Let Φ_∞ denote this limit.

(Naive proof: For a smooth function f on the sphere we have that

$$\int < \partial_{\varphi,\theta}\Phi_R, f > = -\int < \Phi_R, \partial_{\varphi,\theta} f > \to -\int < \Phi_\infty, \partial_{\varphi,\theta} f >$$

while

$$\int < \partial_{\varphi,\theta}\Phi_R, f > \to \int < (B_\infty)_{\varphi,\theta}, f >$$

which gives that

$$\int < \Phi_\infty, \partial_{\varphi,\theta} f > = -\int < (B_\infty)_{\varphi,\theta}, f > .)$$

Since Φ_R converges weakly in $L^p{}_1$ for $p \geq 2$, it converges strongly in L^q for $q \geq 1$. In particular, its weak limits in L^p for $p \geq 2$ are its pointwise limit and the weak limit B_∞ of $[A_R, \Phi_R]$ is nothing but the pointwise limit $[A_\infty, \Phi_\infty]$.

This has the following two consequences:

First, the limit of the Higgs field in the final gauge is continuous: Taking $p = 3$ in equation (D3.1), for example, we have that Φ_∞ lies in $L^3{}_1$ and hence is continuous.

Second, equation (D3.1) shows that $d\Phi_\infty$ is $-[A_\infty, \Phi_\infty]$. That is, we have the

[3] This is similar to the way the global estimates on the 3-space give estimates for F_R on the sphere. The only difference is that when dealing with 1-forms we lose only one power of r. Therefore the argument is still valid for the Prasad-Sommerfield limit where we have that the coupling term on the 3-space decays like r^{-2}, see [J-T].

Reduction (Finite Energy) Condition

$$d_{A_\infty}\Phi_\infty = 0.$$

As an elementary instance of bootstrapping, notice that by Embedding Theorems again A_∞ is continuous and since we just proved that Φ_∞ is continuous we have that the derivatives of Φ_∞ are continuous, therefore Φ_∞ is C^1. Therefore the finite energy condition holds in a strong sense. Summarizing,we have the following:

THEOREM D3.2: Every finite energy solution is gauge equivalent to a smooth solution (A,Φ) with the following properties:

a) The connections A_R on the trivial bundle over S^2 converge to a connection A_∞ on the same bundle. The convergence is strong in $L^p(S^2)$ and weak in $L^p{}_1(S^2)$. In any case A_∞ is continuous.

b) The Higgs fields converge pointwise to Φ_∞ and weakly in $L^p{}_1(S^2)$ and Φ_∞ is at least C^1.

c) A_∞ and Φ_∞ satisfy the Finite Energy Condition $d_{A_\infty}\Phi_\infty = 0$.

Recall now the discussion on the reduction of the previous section. Since in the case we are studying the Higgs potential is given by $V(\Phi) = (|\Phi|^2 - 1)^2$, the small group of the theory is U(1). Therefore, Φ_∞ defines a reduction of the trivial bundle over S^2 on which A_∞ is defined, to a U(1) subbundle. The meaning of the finite energy condition is that A_∞ reduces on this subbundle. That is, its restriction on the subbundle is a U(1) connection. (Recall from [K-N] that, given a section s of the associated bundle $P\times_G G/H$ defining a reduction of the G-bundle P to an H-bundle S, a given connection A on P reduces to S if and only if s is parallel with respect to A.)

We would like to remark here that the finite energy condition is a geometrical way of proving something that ought to be provable using analysis: Since A_∞ reduces to a U(1) connection only the corresponding U(1) components of A on $\mathbb{R}^3$ survive and the rest fade away. Referring back to our discussion on massive and massless components, one should be able to form appropriate equations that would give exponential decay to all the components but the ones corresponding to the U(1) subgroup. A considerable amount of effort has been made to this direction without any success until now. The major technical problem we have is that we do not know of any global gauge on $\mathbb{R}^3$ in which the Yang-Mills-Higgs equations are elliptic for A.[4] Only local gauges are known to exist in which the extra condition d^*A is satisfied. In fact, these are the gauges used by Uhlenbeck in her Weak Compactness Theorem.

D4. A_∞ is Yang-Mills.

We shall now show that the reduced connection is Yang-Mills.

First recall that the curvature form for the connection induced by A_∞ on the subbundle defined by the Φ_∞ section is given (up to a multiple of $\sqrt{(-1)}$) by

$$< F_{A_\infty}, \Phi_\infty > + < [d_{A_\infty} \Phi_\infty , d_{A_\infty} \Phi_\infty], \Phi_\infty > ,$$

see for example [M]. The same formula appears also in [S 𝕽]. By the finite energy condition we are left with

$$< F_{A_\infty}, \Phi_\infty >.$$

This is the curvature of the reduced connection since by definition a connection that reduces equals its induced connection. To prove that on the abelian U(1) bundle this is the curvature of a Yang-Mills field we only need to know that

$$d^* < F_{A_\infty}, \Phi_\infty > = 0$$

where now $*$ denotes the Hodge star operator on the 2-sphere.

To see why this is true, one might suppose for the moment that $d^* < F_{A_\infty}, \Phi_\infty >$ can be approximated by $d^* < F_{A_R}, \Phi_R >$ and calculate:

4 The well-known result for the impossibility of an existence of a global gauge, Gribov's ambiguity, refers to fields with asymptotic conditions that guaranty compactification. It is basically a result for the 4-dimensional sphere, see [Si].

$$d^*\langle F_{A_R},\Phi_R\rangle = *d\langle *F_{A_R},\Phi_R\rangle$$
$$= *\langle d_{A_R}*F_{A_R},\Phi_R\rangle + *\langle *F_{A_R}, d_{A_R}\Phi_R\rangle$$
$$= \langle *d_{A_R}*F_{A_R},\Phi_R\rangle + \langle *F_{A_R}, *d_{A_R}\Phi_R\rangle.$$

The exponential decay of $| d_A\Phi |$ on $\mathbb{R}^3$ means that we have only the term $\langle *d_{A_R}*F_{A_R},\Phi_R\rangle$ to worry about. Using $*_E$ to denote the Hodge star operator on $\mathbb{R}^3$, we calculate

$$*d_{A_R}*F_{A_R} = \{ R^2(*_E d_{A_R}*_E F_{A_R})_\varphi - R^2\partial_r F_{r\varphi}(R,.)\} d\varphi +$$
$$+ \{ R^2(*_E d_{A_R}*_E F_{A_R})_\theta - R^2\partial_r F_{r\theta}(R,.)\} d\theta .$$

Now from the first Yang-Mills-Higgs equation we have that

$$\langle *_E d_{A_R}*_E F_{A_R}, \Phi_R\rangle = 0$$

i.e. we only need to know that $\partial_r\langle F_{r\varphi}(R,.),\Phi_R\rangle$ decays at least like r^{-3}.

At this point we learned of the following formula by Taubes, which provides us with the desirable decay:

Taubes' formula: On $\mathbb{R}^3$,

$$\langle F_A, \Phi\rangle = CdS^2 + \omega,$$

where dS^2 is the area element of the unit sphere in $\mathbb{R}^3$ and ω is a real valued 2-form on $\mathbb{R}^3$ with $|(\partial_r)^k \omega| \le r^{-3-k}$.

We give a proof of this in the Appendix. Notice that the formula proves much more than the decay we were asking: every constant multiple of dS^2 is a Yang-Mills curvature. Since ω decays to zero, at large distances we are left only with a Yang-Mills field. However, the formula does not explain why only the $\langle F_A, \Phi\rangle$ part is relevant, or why this limit is actually realized on a bundle "at infinity".

Using the formula, we finally prove:

Theorem D4.1: $< F_{A_\infty}, \Phi_\infty >$ is a pure Yang-Mills solution ~~on the~~ sphere.

Proof: From Taubes' formula we see that $< F_{A_R}, \Phi_R >$ converges to CdS^2 strongly in any $L^p(S^2)$:

$$< F_A, \Phi > = CdS^2 + \omega$$

on $\mathbb{R}^3$, gives that

$$< F_{A_R}, \Phi_R > = CdS^2 + \omega_{\varphi\theta}(R,.)\, d\varphi \wedge d\theta.$$

Since $|\omega| \leq |x|^{-3}$ on $\mathbb{R}^3$, $|\omega_{\varphi\theta}(R,.)| \leq R^{-1}$ on S^2. Hence $< F_{A_R}, \Phi_R > - CdS^2$ tends to zero in any L^p norm. Notice that this is a gauge invariant statement.

We want to argue that in our gauge the limit of $< F_{A_R}, \Phi_R >$ is actually $< F_{A_\infty}, \Phi_\infty >$. Since A_R converges weakly to A_∞ in $L^p{}_1$ it follows that F_{A_R} converges weakly to F_{A_∞} in L^p, but this does not seem to be enough to prove that $< F_{A_R}, \Phi_R >$ converges in any sense to $< F_{A_\infty}, \Phi_\infty >$. We present here a somewhat indirect argument:

As argued above, $< F_{A_R}, \Phi_R >$ has a pointwise limit and, in the gauge we are working in, so does Φ_R, see above. Therefore, $|\Phi_R|^{-2} < F_{A_R}, \Phi_R > \Phi_R$ has a pointwise limit. (We also use the fact that $|\Phi_R|$ tends to 1, another gauge invariant argument.) Similarly, from the estimate on the transverse components of Theorem 1, $|\Phi_R|^{-2} [\Phi_R, [\Phi_R, F_{A_R}]]$ has pointwise limit zero. Since this accounts for the whole of the curvature, F_{A_R} has a pointwise limit which of course has to be equal to its weak L^p limit, F_{A_∞}, by the uniqueness of a weak limit. Here we use the standard fact that a bounded sequence in L^p with pointwise limit converges weakly to this limit for $p \geq 2$, see [A]. Then $< F_{A_R}, \Phi_R >$ converges pointwise to $< F_{A_\infty}, \Phi_\infty >$ (therefore also weakly and strongly) and hence $< F_{A_\infty}, \Phi_\infty > = C\, dS^2$.

Remark: Had we chosen some other sequence of A_R's they would still have the same curvature on the reduced bundle, as the Theorem shows. Then their limits on the reduced bundle would be gauge equivalent: for any two connections A_1 and A_2 on the sphere with $dA_1 = dA_2$ we have $A_1 = A_2 + gdg^{-1}$, $g = \exp f$ with $df = A_1 - A_2$.

Appendix : The formula.

We describe how one proves the formula as we learned it from [T8]. Basic ideas of the estimates for a slightly more complicated situation in the Prasad-Sommerfield limit can be found also in [T4].

One starts with the real valued 1-form $\alpha = \langle\Phi, *_E F_A\rangle$ on $\mathbb{R}^3$. The Bianchi identity and the first Yang-Mills-Higgs equation give

$$d*_E\alpha = d*_E\langle\Phi,*_EF_A\rangle = d\langle\Phi,F_A\rangle = \langle d_A\Phi\wedge F_A\rangle + \langle \Phi,d_AF_A\rangle$$
$$= \langle d_A\Phi\wedge F_A\rangle =: p$$

and

$$*_Ed\alpha = *_Ed\langle\Phi,*_EF_A\rangle = *_E\langle d_A\Phi\wedge *_EF_A\rangle + *_E\langle\Phi,d_A*_EF_A\rangle$$
$$= *_E\langle d_A\Phi\wedge *_EF_A\rangle + \langle\Phi,*_Ed_A*_EF_A\rangle$$
$$= *_E\langle d_A\Phi\wedge *_EF_A\rangle =: q$$

respectively. Note that once again the coupling term estimate and the curvature estimate give that both p and q have exponentially decaying lengths.

We now define the operator

$$L: \Omega^0(\mathbb{R}^3)\oplus\Omega^1(\mathbb{R}^3) \to \Omega^0(\mathbb{R}^3)\oplus\Omega^1(\mathbb{R}^3)$$

by

$$L(f,\beta) = (d^*\beta\,,\, df + *_Ed\beta)$$

In Taubes' quaternionic notation, if

$$\Psi = (f,\beta) = \Psi_o + \sum \Psi_i\,\tau_i$$

with $\Psi_o = f$ and $\Psi_i = \beta_i$, the formula for L becomes

$$L(\Psi) = \sum (\partial_i\Psi)\,\tau_i\,,$$

where quaternionic multiplication is meant.

We can then write the equations above in a compact form as

$$L(0,\alpha) = (*_E p, q) .$$

The main point now is that L is in a sense the square root of the Laplacian on $\Omega^0(\mathbb{R}^3)\oplus\Omega^1(\mathbb{R}^3)$:

$$L^2(f,\beta) = L(d^*\beta , df + *_E d\beta) = (d^*df , dd^*\beta + *_E d *_E d\beta)$$
$$= (d^*df , dd^*\beta + d^*d\beta)$$
$$= (- \Delta f , -\Delta\beta) .$$

Here we have used that on 2-forms over $\mathbb{R}^3$

$$*_E d *_E = d^*$$

and Δ denotes the Laplacian both on functions and forms.

One uses this observation to write a Green's function for L and therefore a formula for α. Following the quaternionic notation, since $L(0,\alpha) = (*p , q)$ and since Green's function for the Laplacian on $\mathbb{R}^3$ is $|x - y|^{-1}$,

$$(0,\alpha)(x) = \sum_i a_i(x)\tau_i = - \int_{\mathbb{R}^3} L(|x - y|^{-1},0) (*p + \sum_i q_i\tau_i)$$

$$= - \int_{\mathbb{R}^3} (0 , \sum_i \frac{x_i - y_i}{|x - y|^3}\tau_i) (*p + \sum_i q_i\tau_i)$$

$$= \int_{\mathbb{R}^3} (\sum_i \frac{x_i - y_i}{|x - y|^3} q_i , - \sum_i \frac{x_i - y_i}{|x - y|^3} *p\, \tau_i - \sum_{i,j} \frac{x_i - y_i}{|x - y|^3} \tau_i q_j \tau_j) \quad \text{(D5.1)}$$

where quaternionic multiplication is implied.

The way to prove this is similar to the way one proves that the unique solution that vanishes at infinity for the equation

$$\Delta u = \nabla f$$

is given by the formula:

$$u (x) = \int_{\mathbb{R}^3} d\,|x - y|^{-1} f(y)\, dy,$$

see the last chapter of [J-T]. The decay of the fields guarantees that the integrals are finite.

The first thing that equation (D5.1) implies is that

$$\int_{\mathbb{R}^3} \sum_i \frac{x_i - y_i}{|x - y|^3} q_i(y)\, dy = 0 .$$

Now use the multipole expansion

$$\frac{x_i - y_i}{|x - y|^3} = \frac{x_i}{|x|^3} - \frac{y_i}{|x|^3} + \dots = \frac{x_i}{|x|^3} + O(|x|^{-3}) .$$

We then have that for all x in $\mathbb{R}^3$

$$\sum_i \int_{\mathbb{R}^3} \frac{x_i}{|x|^3} q_i(y)\, dy - O(|x|^{-3}) \sum_i \int_{\mathbb{R}^3} F(y)\, q_i(y)\, dy = 0 . \quad \text{(D5.2)}$$

Notice that we have enough decay on q so that the last integral is finite no matter what power of y appears in the integrand. Now choose x = (t,0,0), t > 0. Then (D5.2) becomes

$$\int_{\mathbb{R}^3} |t|^{-2} q_1(y)\, dy + O(|t|^{-3}) \sum_i \int_{\mathbb{R}^3} F(y)\, q_i(y)\, dy = 0 .$$

Multiplying by $|t|^2$ and letting t tend to infinity we have

$$\int_{\mathbb{R}^3} q_1(y)\, dy = 0.$$

Treat q_2 and q_3 similarly.

The second thing that equation (D5.1) implies is that

$$\alpha = - \int_{\mathbb{R}^3} \sum \frac{x_i - y_i}{|x - y|^3} * p\, \tau_i - \int_{\mathbb{R}^3} \sum_{i,j} \frac{x_i - y_i}{|x - y|^3} \tau_i\, q_j \tau_j .$$

Using again the multipole expansion

$$\frac{x_i - y_i}{|x-y|^3} = \frac{x_i}{|x|^3} - \frac{y_i}{|x|^3} + \ldots = \frac{x_i}{|x|^3} + O(|x|^{-3})$$

and the fact that the decay conditions on p and q give bounded integrals,

$$\alpha = -\sum_i \frac{x_i}{|x|^3}\int_{\mathbb{R}^3} *p\,\tau_i + \frac{1}{|x|^3}\sum_i \int_{\mathbb{R}^3} y_i *p\,\tau_i + \ldots$$

$$\ldots - \sum_{i\neq j} \frac{x_i}{|x|^3}\int_{\mathbb{R}^3} \tau_i\, q_j \tau_j + \frac{1}{|x|^3}\sum_{i\neq j}\int_{\mathbb{R}^3} y_i \tau_i\, q_j \tau_j + O(|x|^{-3}).$$

Finally, using that

$$\int_{\mathbb{R}^3} q_i(y)\,dy = 0$$

we can write, going back to the differential forms notation:

$$\alpha = -\left(\int_{\mathbb{R}^3} *p(y)dy\right)\sum_i \frac{x_i}{|x|^3}dx_i + O(|x|^{-3})$$

Now notice that in polar coordinates

$$\sum_i \frac{x_i}{|x|^3}dx_i = r\,dr$$

hence

$$\alpha = C\,r^{-2}dr + O(|x|^{-3}).$$

Since we had set $\alpha = \langle\Phi, *_E F_A\rangle$, we have on $\mathbb{R}^3$:

$$\langle \Phi, F_A\rangle = C\sin\theta\, d\varphi \wedge d\theta + O(|x|^{-3}).$$

Since we have not presented any formulas for the magnetic charge of a monopole solution, we do it here. Notice that by definition

$$C = \int_{\mathbb{R}^3} *p(y)dy = \int_{\mathbb{R}^3} < d_A\Phi \wedge F_A > .$$

Now the first Chern class of the reduced bundle over S^2 is given by

$$c_1 = \frac{1}{4\pi} \int_{S^2} < F_{A_\infty}, \Phi_\infty > = \frac{1}{4\pi} C \operatorname{vol}(S^2) = C.$$

Since the reduced bundle is nothing but the pull-back bundle via Φ_∞ of the Hopf fibration $U(1) \longrightarrow SU(2) \longrightarrow S^2$ we have that $c_1 = \deg(\Phi_\infty)$. That is, we recover the well-known formula for magnetic charge

$$\text{Magnetic charge} \equiv \deg(\Phi_\infty) = \int_{\mathbb{R}^3} < d_A\Phi \wedge F_A >.$$

Or, as physicists argue, see [S2], the magnetic field is the projection of the electromagnetic field on the Higgs direction and the magnetic charge is obtained by integrating the magnetic flux.

E. THE NEXT STEPS.

We summarize here some questions arising from this thesis and some of the problems we have not addressed at all.

1) By now it is standard in variational problems to ask the following: To what extent do the critical points of the theory capture the topology of the relative configuration space? In trying to answer this question for the Yang-Mills-Higgs functional one has the help of the work already done by Taubes for the Prasad-Sommerfield limit.

2) Are the spherically symmetric solutions described in Chapter A stable with respect to any variation in the configuration space as described in Chapter B?

3) Compare the solutions of Chapter A to the Ansatz solutions as in [R]. Use this as a first step for understanding the moduli space of the full Lagrangian.

4) Prove that the configuration space is a Banach manifold for groups other than SU(2) breaking to U(1).

5) Repeat Chapter D for other groups. Notice that the proof for the existence of the limits carries over to any group given the estimates of Theorem D1.1. The problem then is to obtain similar estimates in general. In fact, Taubes can obtain exactly the same estimates for the case of "maximal symmetry breaking", i.e. for a group G breaking to an abelian H (unpublished). However, the proof does not apply to the case of a non-abelian small group. The conjecture here is that at large distances only abelian components survive in any case. For supporting topological evidence see [H-R1]

6) If the situation of Chapter D carries over to any gauge group and any small group, we are faced with two theories: the Yang-Mills-Higgs theory over $\mathbb{R}^3$ and the Yang-Mills theory over S^2. Is there a deep relation between them? In particular, is the stability of the Yang-Mills-Higgs pair (A,Φ) reflected in the stability of A_∞? See [H-o'R-R] for arguments about this.

7) We hope that the reader has been convinced that the natural setting for a Yang-Mils-Higgs theory is a non-compact three dimensional manifold, possibly with many

ends. Apart from $\mathbb{R}^3$, the parts the theory that have been developed on such manifolds seem to indicate that the Analysis easily carries over, see [F]. The question then is whether one can use the moduli space of monopoles to study such manifolds the way the four-dimensional Yang-Mills theory has been used by S. Donaldson. The direct analogue of his work would be to use the Bogomol'nyi solutions, see [B] for Riemannian manifolds with nice compactifications.

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