Nonlinear Controllability via the Initial State: with Application to the Spread of Rabies

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Thesis submitted for the degree of Doctor of Philosophy
to the University of Warwick

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September 1999

In memory of Grandad Gibson and Grandma Evans, who both died during the preparation of this thesis.

And for Grandad Evans who died a short time after.

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Acknowledgements

I wish to express my deep gratitude to Prof. Tony Pritchard who has been a source of guidance and encouragement to throughout this work. The work in this thesis was funded by an EPSRC studentship and so a big thank you to them.

Thank you to my colleagues in the School of Engineering who have had to put up with their Research Assistant spending a lot of his time writing this thesis. I would particularly like to thank Prof. Keith Godfrey for giving me so much time off and Dr. Mike Chappell for some helpful discussions.

Thanks also to Stephen Williams who wrote the Makefile used to compile my computer programs and for providing help for all of my programming questions.

Finally, thanks to God, I owe everything to Him.

Declaration

The material contained in this thesis is the work of the author under the supervision of Prof. A. J. Pritchard unless otherwise stated. The following material is based on that in a paper entitled "Controlling the spread of rabies" by Evans and Pritchard provisionally accepted by the IMA

Journal of Mathematics Applied in Medicine and Biology subject to corrections:

The Summary is an expanded form of the abstract. Chapter 1 provides additional background information from the ecology and biology literature and includes an analysis of the steady states of the extended model. Other material from Chapter 1 is based on that in the paper.

The paper summarises the use of the fixed-point theorem to solve the control problem when the nonlinearity maps the state-space into itself. Furthermore, the time-varying perturbation $P(\cdot)$ is assumed to be bounded.

The numerical results of Chapter 5 are based on those from the paper.

Summary

There are many problems in medicine and biology involving some kind of spatial spread. Often the aim in such problems is to control the spread. A large proportion of medical and biological systems distinguish themselves from the types of system found in engineering by the way the control acts. This is illustrated by considering the specific example of the spread of rabies among foxes

A brief description of a model for the spatial spread of rabies among foxes, developed by Murray et al. (1986), is given. This model is then extended to include the control mechanism. The problem is to prevent the spread of the rabies virus by vaccinating or culling foxes via the distribution of bait in a region around an observed outbreak.

The extended model can be formulated as a nonlinear time-varying control system described by partial differential equations. In contrast to most engineering type control problems the control does not continuously affect the system but only acts through the initial distributions. A general theory is developed for dealing with such nonlinear systems by the use of a fixed point theorem.

In a similar way to Pritchard and Salamon (1987) and Hinrichsen and Pritchard (1994) the dynamics are considered on a triple of Banach spaces $\underline{Z} \subset Z \subset \overline{Z}$ to allow for the possible unboundedness of the nonlinearity. Thus the nonlinearity is considered as a map from $\underline{Z}$ into $\overline{Z}$. A mild form of the time-varying system is introduced to allow for a wider class of nonlinearities. Assumptions are introduced so that the mild form of system equation is well-defined and has a fixed point that, at least partially, solves the control problem.

An adaptive scheme is introduced that constructs the control that gives rise to the fixed-point but is easier to implement computationally. This scheme is less intuitive than that provided by the fixed point theorem. However the method exploits the existence of the fixed point while only requiring the final states (and not the states on the whole time interval) to be stored at each step.

By assuming that the linear part of the system is a time-varying perturbation of a time-invariant operator it is shown how a mild form for the system equation can be derived from the original dynamics. Moreover suppose that the time-invariant operator is the generator of a strongly continuous semigroup. Then the conditions for the mild form of the system to be well-defined and have a fixed point can be reduced to conditions on the semigroup and perturbation.

Existence theorems are provided for solutions of semilinear systems with unbounded nonlinearities.

The theory is applied to the rabies model. The problem and the theory are illustrated by some numerical simulations.

Introduction

This thesis is concerned with the controllability of time-varying, infinite-dimensional systems where the control acts only via the initial state. The motivation for such systems are some of the models being proposed for medical and biological problems involving some form of spatial spread.

In many biological systems the control acts only via the initial state (Roberts, 1992; Tracqui et al., 1995; Allen et al., 1996,for example), though it is sometimes repeated. This control, in systems involving the spread of an epidemic, typically consists of a cull or vaccination program that removes a certain proportion of the susceptible population.

For some systems spatial heterogeneity is an important part of the model (Lewis et al., 1996; Cruywagen et al., 1996) and quite often spatial spread is modelled by a simple diffusion term (Okubo et al., 1989; Louie et al., 1993). This leads us to adopt an infinite-dimensional setting and, since these models are invariably nonlinear, in this thesis we will be considering the controllability of such nonlinear systems. A theory is developed that allows for the possible unboundedness of the nonlinearity.

A good example of the problems associated with the controllability of biological systems is provided by the spread of rabies (Murray et al., 1986). This example will be used to motivate and illustrate the mathematical approach used.

Mathematical modelling of the spread of rabies

Many mathematical models have been proposed for the spread of rabies (see Smith and Harris, 1991,for a review). These models have been used to better understand the epidemiological patterns observed in an epidemic, the mechanism and rate of spread of the disease, and the important question of the possibility of controlling rabies. The principal reservoir of rabies in Europe is the Red Fox, *Vulpes vulpes* (Anderson, 1986) and many of the models study the spread of rabies within a fox population.

These models can help biologists to better understand the disease by highlighting key parameters within the model. For example, in the models of Anderson et al. (1981) and Murray et al. (1986) it is seen that the ability of the environment to support a fox population—the environmental carrying capacity—is an important parameter. In these models it is seen, roughly speaking, that if this value is above a critical one then there will be an epidemic.

The controls methods considered are usually that of vaccination, culling or a combination of both. These methods are seen as a way of reducing the environmental carrying capacity in a certain region where one wishes to prevent the spread of the disease. Källen et al. (1985) and Murray et al. (1986) consider when travelling wave solutions of rabid foxes exist and the possibility of creating a break ahead of the wave to prevent further spread.

There is no consensus of opinion over the best method of reducing the environmental carrying capacity. Some authors have suggested that vaccination is the best method (Anderson et al., 1981; Murray et al., 1986,for example) while others in recent years have been proposing culling as the only effective strategy (Harris and Smith, 1990).

At present there has been no attempt made to apply the techniques of mathematical control theory to this problem. We treat this problem by extending a well-known model to include a control term and then applying a fixed point approach to nonlinear control.

Time-varying systems

Consider the time-varying abstract differential equation given by

$$\dot{z}(t) = A(t)z(t), \qquad t \geq 0 \tag{1}$$

where, for all $t \geq 0$, $A(t)$ is an unbounded linear operator on some Banach space Z. Kato (1953, 1956) was the first to construct the fundamental solution of (1) by approximating it by fundamental solutions corresponding to piecewise constant generators. Hence (Yosida, 1980) Kato's method is an abstraction of the classical polygon method of Cauchy for the ordinary differential equation given by

$$\frac{dz(t)}{dt} = a(t)z(t).$$

Tanabe (1961) constructed a fundamental solution of (1) by representing the system generator as a time-invariant generator with a time-varying perturbation using the theory of holomorphic semigroups. Essential to both approaches is the assumption that $A(t)$ is the generator of a strongly continuous semigroup for all $t \geq 0$. A different approach is provided by Lions (1971) who assumes that $A(t)$ is defined via a time-varying bilinear form.

For time-invariant linear differential equations ($A(t) \equiv A$) the Hille-Yosida Theorem provides a necessary and sufficient condition for the existence of solutions. However, for time-varying differential equations of the form (1) the existence theory is not so well developed.

Suppose that a fundamental solution $U(t,s)$ of (1) exists and consider the inhomogeneous differential equation given by

$$\dot{z}(t) = A(t)z(t) + f(t), \qquad z(0) \in D(A(0)). \tag{2}$$

Then if $f(\cdot)$ is suitably smooth the solution of (2) is given by

$$z(t) = U(t,0)z(0) + \int_0^t U(t,s)f(s)\,ds. \tag{3}$$

Fundamental solutions of (1) are strong evolution operators and $A(t)$ is said to be the generator of $U(t,s)$. If, in (3), $U(t,s)$ is a mild evolution operator and $z(0) \in Z$ is arbitrary then (3) defines a continuous function that is independent of $A(t)$. Hence by weakening the assumptions on $U(t, s)$ and studying this system equation directly Hinrichsen and Pritchard (1994) were able to allow for a wider class of perturbed dynamical system. This will be the approach followed in this thesis where $f(\cdot)$ is replaced by a possibly unbounded nonlinearity.

Hence systems described by equations of the form

$$z(t) = U(t,s)Bu + \int_0^t U(t,s)D(s)N(s, E(s)z(s))\, ds$$

where u is the control, B is a bounded input operator, $D(\cdot)$ and $E(\cdot)$ characterise the unboundedness of the nonlinearity, will be considered in this thesis. The unboundedness is represented by the triple of Banach spaces $\underline{Z} \subset Z \subset \overline{Z}$, where the canonical injections are continuous with dense range. With respect to these spaces it will be assumed that $E(t)$ is a bounded linear operator from $\underline{Z}$, and $D(t)$ is a bounded linear operator into $\overline{Z}$. Hence assumptions will be introduced so that this mild form of system equation is well-defined.

This approach has been used by Pritchard and Salamon (1987) to consider the linear quadratic control problem with unbounded input and output operators. For time-varying systems Hinrichsen and Pritchard (1994), who were the first to work with this style of system equation and setting, have used this approach to study the stability of (1) for unbounded unknown perturbations.

If $A(t) = A + P(t)$, where A is the generator of a strongly continuous semigroup $S(t)$, and $P(t) \in \mathcal{L}(Z)$ is piecewise continuous, then it is known (Curtain and Zwart, 1995) that $A(\cdot)$ is the generator of a mild evolution operator $U(t, s)$ in the sense that the unique solution of

$$U(t,s)z = S(t-s)z + \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\, ds$$

is $U(t, s)$. This provides the means by which a mild form of system equation can be associated with the original form of the system when $A(t) = A + P(t)$. This is particularly useful when the original system arises from a semilinear one by performing a local approximation about a solution trajectory. A framework is developed for this association even when the perturbation is allowed to exhibit the same unboundedness as the nonlinearity. This is represented by assuming that $P(t) \in \mathcal{L}(\underline{Z}, \overline{Z})$ for each t.

Application of fixed point theorems in nonlinear control

While a well developed theory exists for linear control systems, even in infinite-dimensional spaces, for nonlinear systems this is not the case. Any success in this area is dependent upon particular classes of nonlinearity, and advances have been limited. Some progress is possible using the well-known fixed point methods of nonlinear analysis.

The earliest use of fixed point methods in a control text was by Hermes (1965) for finite-dimensional systems. A description of the application of such methods to finite-dimensional time-varying systems that is used as a basis for other authors' work is given by Davison and Kunze (1970).

The methods for finite-dimensional systems have been extended to infinite-dimensional systems by Magnusson and Pritchard (1981) and Magnusson et al. (1985). For a review of the use of fixed point methods in nonlinear control and observation see Carmichael and Quinn (1988). In this thesis the fixed point approach will be extended to nonlinear time-varying infinite-dimensional systems where the control acts only via the initial state.

Suppose that $U(t,s)$ is a mild evolution operator on some Banach space Z and consider the system described by

$$z(t) = U(t,0)Bu$$
$$y = Cz(T)$$

where the output $y \in Y$ a Hilbert space, $u \in U$ the Hilbert space of controls, $B \in \mathcal{L}(U,Z)$ and $C \in \mathcal{L}(Z,Y)$. The control problem then becomes that of finding, if possible, a control u that solves the following equation

$$y_d = CU(T,0)Bu$$

where y_d is the target output. If the operator $\phi = CU(T,0)B$ is invertible then there is a unique solution given by $u = \phi^{-1}y_d$. Now suppose that there is a nonlinear term in the system equation, then this approach suggests that the control is given by

$$u = \phi^{-1}\left(y_d - C\int_0^T U(T,s)\mathcal{N}(z(s))\,ds\right).$$

Note, however, that this is an implicit expression since the control depends on the trajectory $z(\cdot)$. If this trajectory exists and is known then it is given by

$$z(t) = U(t,0)B\phi^{-1}\left(y_d - C\int_0^T U(T,s)\mathcal{N}(z(s))\,ds\right) + \int_0^t U(t,s)\mathcal{N}(z(s))\,ds. \qquad (4)$$

Hence the control problem is reduced to finding a fixed point of the operator defined by the right-hand side of (4). The fixed point theorem used in this thesis is by Collatz (1966) and guarantees the uniqueness of the fixed point.

Normally the fixed point problem is considered on a subspace, with a suitable topology, such that the linear part of the system is exactly controllable to this region. Adopting this approach would require restricting attention to the range of ϕ and this is too restrictive. Therefore, for the linear part of the system, the least squares problem of minimising

$$\|y_d - \phi u\|_Y$$

over all choices of $u \in U$ is posed. The least squares solution with minimum norm, provided $y_d \in \operatorname{ran}\phi + (\operatorname{ran}\phi)^{\perp}$, is given by $u = \phi^{\dagger}y_d$. Hence the fixed point approach will be used for the nonlinear system with control given by

$$u = \phi^{\dagger}\left(y_d - C\int_0^T U(T,s)\mathcal{N}(z(s))\,ds\right).$$

The operator $\phi^{\dagger}$ is called the generalised inverse of ϕ. For a treatment of generalised inverses of linear operators on infinite-dimensional spaces see Nashed (1971). Generalised inverses in the fixed point approach have been used by Pritchard (1981) to obtain observers and minimum-energy controls for nonlinear finite-dimensional systems.

Organisation of thesis

This thesis deals with a particular control problem and can be roughly divided into three parts. The first part is concerned with defining the control problem; the second develops a mathematical theory to solve it; and in the third part the theory is applied.

The primary concern of Chapter 1 is the control problem. Background material is provided for the rabies virus and control methods currently in use. Then the control problem is defined from an ecological stand-point. A mathematical model for the spread of rabies (Murray et al., 1986) is presented in Section 1.2 and is then extended to include a control term.

In terms of this extended model the control problem is then to choose, if possible, a suitable initial density of vaccinated foxes (or level of cull) such that, in a region where rabies is not endemic, the total population density of infected foxes is driven to a specified target in a certain time. This poses a novel control problem since the control is allowed to influence the system only via the initial state.

The system of nonlinear partial differential equations comprising the extended model is then formulated as an abstract differential equation in a Banach space setting.

Chapters 2 and 3 comprise the main theory of this thesis. A theory is developed for solving the mathematical control problem while allowing for the possible unboundedness of the nonlinearity.

The mathematical control problem is dealt with in Chapter 2 in a time-varying system framework similar to that of Hinrichsen and Pritchard (1994) used for the study of unbounded perturbations of linear evolution equations. This involves considering a mild form for the system equation and constructing assumptions that imply that this equation is well-defined.

Once this framework has been developed, in Section 2.2 a fixed point theorem of Collatz (1966) is applied to construct an input that gives rise to a mild solution with the desired properties. An equivalent, but less intuitive, method for constructing the control is presented in Section 2.3. This is an adaptive scheme that proves to be easier to implement computationally by making use of the original dynamics of the system.

It is shown that it is possible to drive the output of the system to the target only on some subspace of the space of outputs Y. For the rabies model it is the actual output that is of primary concern and so in Section 2.4 the important question of how to choose the target to minimise the actual output is considered.

For the rabies model the time-varying nature of the system arises because of a local approximation that is made about some initial control and corresponding trajectory. In this case

$A(t) = A + P(t)$ describes the linear part of the system. In Chapter 3 systems of this form are considered.

The concept of $A(\cdot)$ being the generator of a mild evolution operator is introduced for the case where $P(t)$ exhibits the same unboundedness as the nonlinearity for each t. It is assumed that A is the generator of a strongly continuous semigroup and the conditions of Chapter 2 are reduced to corresponding ones for the semigroup and perturbation.

Chapters 4 and 5 apply the theory of Chapters 2 and 3. First to a general example in a Hilbert space setting in Chapter 4 and then to the rabies model itself in Chapter 5. The results of Chapter 5 can be considered as corollaries to those in Chapter 4 for which some of the Hilbert space structure is lost. This is because in the rabies model the diffusion term that gives rise to a strongly continuous semigroup with smoothing properties appears only in the last equation. Therefore the natural space to consider each of the other parts is the Banach space of continuous functions.

Numerical results are provided for a specific example of the rabies control problem. The three control strategies—vaccination, culling and a combination of both—are compared. The adaptive scheme of Chapter 2 provides an easily implemented method for constructing the desired control.

Chapter 1

The Control Problem

The question of rabies spread and control has been widely studied by ecologists and mathematical biologists (see Smith and Harris, 1991,for a review of some of the principal models suggested), but so far the techniques of mathematical control theory have not been applied. In the chapters that follow we will apply some of these techniques.

In this chapter we define the control problem, first as an ecological one, and then as a mathematical one. The mathematical control problem will be treated in the chapters that follow.

1.1 Rabies and its control

In this section background information on rabies is provided. The topics covered are the virus and the resulting disease itself (MacDonald, 1980; Anderson, 1986; MacKenzie, 1990); the history of rabies in Europe (Harris and Smith, 1990; MacKenzie, 1990, 1997); and the controls being employed to stop its spread (Anderson, 1986; Harris and Smith, 1990; MacKenzie, 1990, 1997). This section is concluded by defining the control problem from an ecological and biological point of view.

1.1.1 What is rabies?

Rabies, one of the oldest recognised diseases, is an acute viral infection of the central nervous system. In the Nineteenth century Louis Pasteur developed a vaccine that, if used immediately, can be used to treat the disease. Unfortunately once rabies has reached the clinical phase it is nearly always fatal.

Multiplication of the virus in the brain results in the well-known 'furious' symptoms, although if it is the spinal cord that is predominately affected then paralytic or dumb rabies results. One of the more infamous and distressing symptoms in humans is the fear of water (hydrophobia). Together with the fatality of the clinical phase of infection and the other distressing symptoms, this helps to make rabies one of the most feared of all diseases.

An example of the fear that rabies produces is recorded by MacDonald (1980): A blacksmith demanded treatment for rabies after shoeing a pony that later became rabid. His wife also demanded treatment because she had brushed down the clothes he had been wearing at the time. Even in Britain, isolated from the rest of Europe by the English Channel, strict and harsh regulations exist to keep rabies out.

The actual threat to human life (in Europe) is quite small; in the 1960s and 1970s there were only 1–4 deaths per year (Anderson, 1986). Between 1945 and 1997, 250 human deaths were reported in Europe (MacKenzie, 1997).

An outbreak of the disease is costly to treat: all domestic animals in an infected area are vaccinated; any animal thought to be rabid is destroyed and the owners given post-exposure vaccinations. In the United States, where the cost per person for the vaccine is 1000 to 1500 dollars, over 1800 people in Texas were given post-exposure treatment for rabies in 1995.

It is widely believed that the principal reservoir of infection in the wild is the Red Fox *Vulpes vulpes* (Anderson, 1986). Therefore in this thesis we will be considering the spread of rabies in foxes.

1.1.2 A history of rabies in Europe

An epizootic of rabies in wild dogs was eradicated in 1928 by the mandatory vaccination of domestic animals and by destroying packs of wild dogs. In the 1930s rabies reemerged on the Polish-Russian border and the outbreak of the Second World War helped to spread the disease, primarily through dogs and foxes.

By the time dogs were brought under control rabies was sufficiently maintained in foxes for an epizootic to begin. The current epizootic, which was a result, started in Poland in 1945 and spread west at a rate of 30–40 kilometres per year reaching (West) Germany in 1950; Belgium in 1966; and France in 1989 (MacKenzie, 1990).

In Victorian Britain the muzzling of all dogs and the shooting on sight of any that were not muzzled led to the eradication of rabies. Since then, except for a brief spell just after the First World War, Britain has remained free of rabies. The building of the Channel Tunnel provoked fears that rabies might once again be introduced onto the British mainland.

1.1.3 Control methods for rabies

The problem of controlling the spread of rabies is normally tackled either through the culling or vaccination of a proportion of the fox population. The idea of this is to block the chain of transmission by reducing the probability that an infected fox will pass on the disease to a healthy one. If an infected fox is unable to pass on the infection before it dies, then the virus dies with it. To maintain a high level of immunity against rabies repeated vaccinations are required because of the short life expectancy of the fox—approximately 1.5 to 2.5 years (Anderson, 1986).

Oral rabies vaccine, contained in pellets of fishmeal or lard, is distributed throughout the countryside in some European countries. This distribution, performed by helicopters or hunters, has occurred twice a year—in Spring and Autumn—since the 1980s (MacKenzie, 1997).

MacKenzie (1997) reported that this distribution of vaccinated bait had been largely successful: There had been no cases in France for one year; Switzerland and Belgium for almost a year; the Netherlands, Luxembourg and (East) Germany for two years and so these countries were officially considered 'rabies-free'.

The two principal vaccines being used are the classical vaccine, based on live weakened rabies virus; and a live recombinant vaccinia virus (Blancou et al., 1986). In order for the former vaccine to be effective it must be given live. If not, the virus can not multiply and an ineffective immune response results. This creates major problems in the distribution of this vaccine as it dies out after only a few weeks at field temperatures. To maximise the life-span and hence the usefulness of this vaccine it is distributed in Spring and Autumn. The recombinant virus is more stable and can be used at different times of the year, and in hotter climates.

The recombinant vaccinia virus is based on the pox virus and harbours a gene that codes for a surface antigen specific to rabies. The pox virus replicates exposing the immune system to rabies protein and thereby inducing an immunity to the disease. This vaccine gives good protection against rabies infection when administered orally and is easily distributed.

Neither vaccine is without fears over its use. There are fears that the weakened rabies virus might revert to a stronger strain and so produce infection itself. At the time of MacKenzie (1990) a major fear was that too little was known about the vaccinia virus to know whether it is a human pathogen. Since then it has become the vaccine used by the Texas Department of Health in their Oral Rabies Vaccination Programs.

The first successful distribution of the live vaccine stopped the spread of rabies across the Swiss Alps. Between 1983 and 1990, 5.2 million baits had been distributed in an effort to eradicate rabies from Europe.

The vaccinia virus was first tested in 1987 on 6 square kilometres of a closed military base in Belgium. In the following year 435 square kilometres of Southern Belgium was covered, and by 1990, 2200 square kilometres had been vaccinated.

The other main method of tackling the problem of rabies spread is culling. The widespread public concern over the slaughter of foxes and the development of oral vaccines has called into question its use. It was estimated in Anderson (1986) that 1.25 million foxes were being killed annually in rabies control. This wholesale slaughter had had a limited success though, only slowing, not stopping, the rate of spread.

More recently, Harris and Smith (1990) have pointed out the advantages of culling over vaccination. An immediate effect of using poison in bait instead of vaccine is that foxes would not return after taking a bait to take more. This would hopefully increase the proportion of the fox population taking bait. Culling makes it easier to pin-point the source of infection and observe the effect of the control strategy; bodies are provided that can be tested for the rabies virus. A psychological advantage of culling is that any fox seen has the possibility of being infected. With a vaccination program it would be unknown whether a fox sighted in the wild was vaccinated or not.

Anderson (1986) suggests that in high fox population densities a program of vaccination could be supplemented by culling.

1.1.4 The control problem

Harris and Smith (1990) pointed out that the situation in Britain, should rabies be introduced back onto the mainland, would be very different from that being faced by the rest of Europe. Whereas in Europe, where rabies is endemic and an epizootic wave 2000 kilometres long has to be dealt with, in Britain only a local point-source outbreak would need to be treated. If this outbreak were caught early enough the infection could be confined to this small area while being eradicated. Britain has a high population of foxes in urban areas; numbers in cities are five-times those in rural ones, and (in 1990) much higher than those in continental Europe.

The success of the vaccination procedures in Europe over the past two decades has meant that fox populations are soaring. MacKenzie (1997) reported that in Switzerland and Germany a three-fold increase had been observed; while in parts of France a five-fold increase had occurred. Some scientists fear that with governments thinking the fight is over that vaccinations will be stopped prematurely. An example of this is in Slovenia were the number of cases of rabies jumped five-fold between 1992 and 1995. By 1997 the outbreak had only just been brought under control (MacKenzie, 1997).

The fear is that with fox populations so high, if rabies were to return the current vaccination procedures would be unable to deal with it. Therefore it seems that, with the success of past vaccination programs, the situation in Europe is becoming like that in Britain. The problem we consider in this thesis is the following: Suppose that in some area were rabies is not endemic, an outbreak occurs. This outbreak must be contained in some neighbourhood of the site of infection and eradicated.

Associated with the control problem is the question of what method should be used to reduce the population. A baiting trial performed by Trewhella et al. (1991) in Bristol showed that bait uptake was generally lower than in Europe and North America. The abundance of other sources of food in British cities can help to explain the low uptake in Bristol. Another possible reason is that in high densities foxes tend to live in family units and so the distribution of bait will only reach certain family members.

We consider what form the control will take, whether it is culling, vaccination or a combination of both.

1.2 A mathematical model for controlling the spread of rabies

In this section we derive a mathematical model for controlling the spread of rabies among foxes. The starting point is the model proposed by Murray et al. (1986). This model is itself an extension of that given by Anderson et al. (1981) by including the spatial spread of the disease. The paper by Anderson et al. (1981) was concerned with the overall population dynamics of rabies in foxes and neglected spatial effects.

It was proposed by Källen et al. (1985) that the spatial spread of the disease is due primarily to the random dispersal of rabid foxes. Their model was concerned mainly with the wavefront of an epizootic. This is the mechanism assumed by Murray et al. (1986).

A key feature of rabies that was omitted in the Källen et al. (1985) model is the variable and often lengthy incubation period. This was taken into account and studied by Anderson et al. (1981).

We start this section with a description and a summary of the analysis of the Murray et al. (1986) model.

1.2.1 The basic model

To model the incubation period of rabies, the fox population is divided into three classes : the susceptible, with population density S; the infected, but not infectious, with density I; and the rabid class, with density R. There is no class of recovered and immune foxes because once rabies has reached the clinical phase it is almost certainly fatal. The total population density is $N = S + I + R$ and varies in time. In this the formulation of Anderson et al. (1981) differed from other conventional epidemiological models.

In the absence of rabies it is assumed that the population dynamics can be approximated by the logistic equation

$$\frac{dN}{dt} = (a - b)\left(1 - \frac{N}{K}\right) N,$$

where a, b are the intrinsic birth and death rates respectively; K is the environmental carrying capacity and models the ability of the environment to support the fox population. Typical values for K are 2 foxes per square kilometre for continental Europe and 4.6 foxes per square kilometre for the U.K.

It is assumed that foxes of both infected classes continue to use environmental resources and die through natural means, but they produce negligible healthy offspring. Therefore the rate of change of fox density due to the population dynamics of both infected classes of fox omits any term for births.

The principal methods of rabies transmission are biting or licking which require direct contact between foxes. Therefore it is assumed that foxes become infected at an average rate per head of βR, where β is the disease transmission coefficient and measures the rate of contact between rabid and susceptible foxes.

The rate of change of the susceptible population density is then the rate due to population dynamics minus the rate of loss due to rabies infection:

$$\frac{\partial S}{\partial t} = (a - b)\left(1 - \frac{N}{K}\right) S - \beta RS \tag{1.1}$$

where $N = S + I + R$ is the total fox population density.

Newly infected foxes remain in the infected but not infectious class for an average incubation period of $1/\sigma$. Therefore the rate at which incubating foxes become rabid and hence infectious is σI.

The rate of change of the incubating fox density satisfies

$$\frac{\partial I}{\partial t} = \beta RS - \sigma I - \left(b + (a-b)\frac{N}{K}\right) I. \tag{1.2}$$

If $1/\alpha$ is the average duration of the clinical disease, since rabies is usually fatal once it has reached the clinical phase, αR is the average rate at which foxes die from the disease.

The mechanism for the spatial spread of the disease is assumed to be the random dispersal of rabid foxes. Non–rabid foxes are generally territorial and hence the absence of any migration terms in the equations governing the dynamics of S or I. Behavioural changes in rabid foxes are caused by the rabies virus attacking the central nervous system; while about half of foxes gradually become paralysed, the other half exhibit the more infamous 'furious' symptoms. It is the latter that lose their territorial behaviour and disperse randomly. This is modelled by a simple diffusion term (as was the case in Källen et al., 1985).

The population dynamics of the rabid class is then modelled by the following equation

$$\frac{\partial R}{\partial t} = \sigma I - \alpha R - \left(b + (a-b)\frac{N}{K}\right) R + D\nabla^2 R \tag{1.3}$$

where D is the diffusion coefficient. Some typical values (given in Anderson et al., 1981) for the parameters in this model are given in Table 1.1.

The mechanism for the spatial spread of the disease is the random dispersal or migration of infected foxes. Infection is through the uniform mixing of susceptibles and infectives. In this model the diffusion coefficient is constant and hence independent of the spatial domain of interest. In contrast, Cruywagen et al. (1996) argue that an important feature of the natural world is spatial heterogeneity. This can be modelled by incorporating spatial dependence of the various parameters in the model, including the diffusion coefficient.

The spatially uniform case ($D = 0$) of the model (1.1)–(1.3) was studied by Anderson et al. (1981). If an infectious fox is introduced into a population of K susceptible ones, then the expected number of secondary cases produced as a result in its lifetime is the 'basic reproductive rate'. It

Parameter	*Symbol*	*Value*
average birth rate	a	1 per year
average death rate	b	0.5 per year
average duration of clinical disease	$1/\alpha$	1/73 year (5 days)
average incubation period	$1/\sigma$	1/13 year (28 days)
carrying capacity	K	0.25 to 4.0 foxes km^{-2}
disease transmission coefficient	β	80 km^2 per year

Table 1.1: Values for the model parameters as given in Anderson et al. (1981).

was found, by determining when the basic reproductive rate is one, that a critical condition for the carrying capacity is

$$K_c = \frac{(\alpha + a)(\sigma + a)}{\beta\sigma}. \tag{1.4}$$

$K > K_c$ corresponds to the basic reproductive rate being greater than one resulting in rabies being maintained in the population.

Three types of behaviour are possible: For $K < K_c$ rabies dies out and the susceptible population density tends to K, the disease–free steady state value; if $K > K_c$ rabies becomes endemic and the population densities oscillate about a steady state $S = S_0$, $I = I_0$, $R = R_0$. These oscillations are damped if K is not much bigger than K_c in which case the densities tend to the steady state.

This critical value is in keeping with epidemiological evidence and is between 0.2 and 1.0 foxes per square kilometre (MacDonald, 1980; Anderson et al., 1981; Murray et al., 1986). Given this value it is possible, from (1.4), to obtain an estimate for β, a parameter which cannot be estimated directly because of the difficulty in observing fox contacts. This method was employed by Anderson et al. (1981).

A numerical analysis performed by Murray et al. (1986) shows that for the full system ($D \neq 0$), if $K < K_c$ rabies dies out. However, if $K > K_c$, then an epidemic wave forms travelling with near–constant velocity.

The method of control employed by Murray et al. (1986) was to consider a barrier ahead of the epidemic wave where the susceptible population has been reduced below the critical density. This reduction can either be achieved through vaccination or culling, or a combination of the two. Estimates are made from the model (1.1)—(1.3) for the width of this break region and the level of reduction necessary. This method is being employed successfully both in Europe (MacKenzie, 1990, 1997) and Texas (Zoonosis Control Division, 1998).

In studying the control of the spread of rabies, it is implicit in the analysis carried out by Murray et al. (1986) that once the population reduction has occurred it is maintained at this level. Hence a control zone of vaccinated foxes ahead of an epizootic wave is treated as being equivalent to a region where the carrying capacity K has been reduced. In practice we would expect that, after the population reduction has been completed, the susceptible population would begin to recover and increase towards the environmental carrying capacity.

It has been argued (Harris and Smith, 1990) that the situation would be very different if rabies reached mainland Britain. Unlike in continental Europe where rabies is endemic and control strategies have to deal with an epizootic wave 2000 kilometres long, in Britain only a local, 'point-source' outbreak, would be experienced initially. A control strategy in Britain should, therefore, have the goals of both containing the outbreak and eliminating it in the original local area.

To study the problem of containing and eliminating point–source outbreaks; and to consider the recovery of the susceptible population we extend the model given by (1.1)–(1.3).

1.2.2 The extended model

Present vaccination programs are carried out by distributing bait with oral rabies vaccine contained inside. An example of such a vaccination program is being carried out in Texas (Zoonosis Control Division, 1998). This program began in February 1995 in response to two rabies epizootics (one in grey foxes) which started in 1988. The aim of the program is to produce a zone of vaccinated animals in front of each wave front to prevent the further spread of the disease and to bring about its subsequent elimination. In the last year this has involved the aerial distribution of, on average, 10 000 baits per flight over the course of 254 flights.

With this in mind we can divide the control of rabies into three phases:

1. An input phase where the bait containing the vaccine is delivered. The control parameters are the trajectory of the deliverer and the amount deposited.

2. A vaccination phase where the bait is taken by the foxes and an immune response is produced in some.

3. An observation phase where we observe the effects of our control strategy.

We remark that culling could also be performed by the distribution of bait, substituting poison for vaccine (see, for example, Harris and Smith, 1990).

The first phase can either be performed by the manual placement or aerial distribution of bait. The speed of distribution depends on the delivery method and the area over which bait is distributed. For example, the 1998 vaccination program carried out in Texas lasted for 34 days over a region of 108 780 square kilometres and involved the distribution of 2.6 million baits.

If the delivery of the bait is fast compared with the time frame of the rest of the dynamics the control or input to the system is the initial distribution (in space) of the bait. Suppose that the density of bait is B then a possible form for the dynamics of the bait is

$$\frac{\partial B}{\partial t} = -\left(\gamma_S S + \gamma_I I + \gamma_V V\right) B - HB, \tag{1.5}$$

where V is the density of vaccinated foxes; γ_S, γ_I and γ_V measure the rate at which susceptible (respectively incubating, vaccinated) foxes take the bait; H is an increasing function in t and models the decomposition of the bait. To allow for environmental heterogeneity this function could be made spatially dependent.

After the initial distribution, $B(x, 0)$, of bait is made the bait is taken by foxes or decomposes until all the bait has been used up. Therefore the length of this second phase is limited by the period of effectiveness of the vaccine. At present, in vaccines whose active component is weakened rabies virus, the virus dies out after a few weeks in the field. To maximise the useful life expectancy of the vaccinated baits they are currently distributed in Spring and Autumn (MacKenzie, 1990).

In the second phase susceptible foxes take the bait and an immune response is produced in some. Suppose that the susceptible foxes take the bait and become immune at an average rate per head of γB. If we suppose that vaccinated foxes give birth to susceptible ones so that immunity is not inherited, and that the vaccine does not produce any behavioural changes the dynamics of the susceptible class of fox becomes

$$\frac{\partial S}{\partial t} = (a-b)\left(1-\frac{N}{K}\right)S - \beta RS - \gamma BS + aV. \tag{1.6}$$

The bait used in the Texas program also contains a biomark agent. This has been used to indicate bait uptake. Following the 1996 program this marker was found in 35% of a representative sample of grey foxes. Of this sample 32% showed a positive response. In Europe typically 70% of foxes in the target areas take the bait (MacKenzie, 1990).

If we assume that incubating foxes do not become immune by taking the bait then the dynamics of I and R remain unchanged; foxes in the clinical phase cannot be vaccinated against rabies. Therefore the remaining equations of the extended model are

$$\frac{\partial V}{\partial t} = \gamma BS - \left(b + (a-b)\frac{N}{K}\right)V \tag{1.7}$$

$$\frac{\partial I}{\partial t} = \beta RS - \sigma I - \left(b + (a-b)\frac{N}{K}\right)I \tag{1.8}$$

$$\frac{\partial R}{\partial t} = \sigma I - \alpha R - \left(b + (a-b)\frac{N}{K}\right)R + D\nabla^2 R, \tag{1.9}$$

where $N = S + V + I + R$ is the total fox population density.

A similar extension in the spatially uniform case can be found in Anderson et al. (1981), though we do not consider a rate of vaccination. We are dealing with the situation where bait is laid down and produces a level of vaccination which is allowed to have an effect. Possibly this program is repeated at a later stage, for example each year.

To simplify the model and make use of existing estimates for the parameters in the original system (1.1)–(1.3), we consider the situation after the bait has been delivered and taken (or decomposed). Hence we consider the problem of determining what distribution of vaccinated foxes we need to produce in order to contain and eliminate an outbreak of rabies. Thus the simplified version of the system of equations (1.5)–(1.9) that we consider in the following is

$$\frac{\partial S}{\partial t} = (a-b)\left(1-\frac{N}{K}\right)S - \beta RS + aV \tag{1.10}$$

$$\frac{\partial V}{\partial t} = -\left(b+(a-b)\frac{N}{K}\right)V \tag{1.11}$$

$$\frac{\partial I}{\partial t} = \beta RS - \sigma I - \left(b+(a-b)\frac{N}{K}\right)I \tag{1.12}$$

$$\frac{\partial R}{\partial t} = \sigma I - \alpha R - \left(b+(a-b)\frac{N}{K}\right)R + D\nabla^2 R, \tag{1.13}$$

Our control or input to this system is the level of vaccination (or cull). This is the initial distribution (in space) of the vaccinated class produced by some baiting strategy. This poses an interesting and novel control problem. Usually in engineering control systems the control acts continuously in time throughout the period considered. Therefore in tackling this problem we must deal with a new style of control system.

1.3 Analysis of extended model

In this section we analyse the steady states of (1.10)–(1.13) for the spatially uniform case and compare the results with those obtained by Murray et al. (1986). In a similar fashion we first introduce non-dimensionalised quantities. We conclude the section with an interesting analysis of the time-varying nature of a vaccination procedure.

1.3.1 Analysis of steady states

It is helpful for the analysis of the steady states to introduce non-dimensionalised quantities for the model. We make the following substitutions:

$$s = S/K, v = V/K, q = I/K, r = R/K, n = N/K,$$
$$\epsilon = (a-b)/\beta K, \delta = b/\beta K, \mu = \sigma/\beta K,$$
$$d = (\alpha+b)/\beta K, \overline{x} = (\beta K)^{1/2}x, \overline{t} = \beta Kt.$$

These quantities are along the lines of those introduced in Murray et al. (1986). With these substitutions the model becomes, on dropping the bar notation for simplicity,

$$\frac{\partial s}{\partial t} = \epsilon(1-n)s + (\epsilon+\delta)v - rs \tag{1.14}$$

$$\frac{\partial v}{\partial t} = -(\delta+\epsilon n)v \tag{1.15}$$

$$\frac{\partial q}{\partial t} = rs - (\mu+\delta+\epsilon n)q \tag{1.16}$$

$$\frac{\partial r}{\partial t} = \mu q - (d+\epsilon n)r + \frac{\partial^2 r}{\partial x^2}. \tag{1.17}$$

For the spatially uniform case we set $\frac{\partial^2 r}{\partial x^2} = 0$. The steady states satisfy $\dot{s} = \dot{v} = \dot{q} = \dot{r} = 0$. We see immediately from equation (1.15) that $v = 0$ for a steady state. Therefore equations (1.14)–(1.17) reduce to

$$\dot{s} = \epsilon(1-n)s - rs = 0 \tag{1.18}$$

$$\dot{q} = rs - (\mu+\delta+\epsilon n)q = 0 \tag{1.19}$$

$$\dot{r} = \mu q - (d+\epsilon n)r = 0. \tag{1.20}$$

at a steady state.

Two possible solutions are $s = q = r = 0$ and $s = 1$, $q = r = 0$. These correspond to the population free and disease free steady states respectively. To find the disease persistent steady state we assume that $q \neq 0$, and $r \neq 0$. Dividing (1.18) through by s gives

$$r = \epsilon\,(1-n)\,,$$

and in particular $n \neq 1$ since $r \neq 0$. Now (1.20) yields

$$q = \frac{\epsilon}{\mu}\,(d+\epsilon n)\,(1-n)$$

and, upon substitution, (1.19) becomes

$$s = \frac{1}{\mu}\,(\mu+\delta+\epsilon n)\,(d+\epsilon n)\,.$$

Therefore adding these equations together gives the following formula for n_*, the total population density at the disease persistent steady state:

$$n_* = \frac{d\,(\mu+\delta+\epsilon)+\mu\epsilon}{\mu-\epsilon\,(\delta+\epsilon)}. \tag{1.21}$$

Thus the disease persistent steady state is given by $(s_*, 0, q_*, r_*)$ where

$$s_* = \frac{1}{\mu}\left(\mu+\delta+\epsilon\left(\frac{d(\mu+\delta+\epsilon)+\mu\epsilon}{\mu-\epsilon(\delta+\epsilon)}\right)\right)\left(d+\epsilon\left(\frac{d(\mu+\delta+\epsilon)+\mu\epsilon}{\mu-\epsilon(\delta+\epsilon)}\right)\right) \quad (1.22)$$

$$q_* = \frac{\epsilon}{\mu}\left(d+\epsilon\left(\frac{d(\mu+\delta+\epsilon)+\mu\epsilon}{\mu-\epsilon(\delta+\epsilon)}\right)\right)\left(1-\left(\frac{d(\mu+\delta+\epsilon)+\mu\epsilon}{\mu-\epsilon(\delta+\epsilon)}\right)\right) \quad (1.23)$$

$$r_* = \epsilon\left(1-\left(\frac{d(\mu+\delta+\epsilon)+\mu\epsilon}{\mu-\epsilon(\delta+\epsilon)}\right)\right). \quad (1.24)$$

Murray et al. (1986) made the observation that $\epsilon, \delta \ll 1$ and so performed an asymptotic analysis. In agreement with this analysis, to first order in ϵ and δ, s_*, q_*, r_* are given by

$$\begin{aligned} s_* &= d + \frac{d}{\mu}\delta + d\left(1+\frac{d}{\mu}\right)\epsilon \\ q_* &= \frac{d}{\mu}(1-d)\epsilon \\ r_* &= (1-d)\epsilon. \end{aligned}$$

We now determine when this steady state is realistic—that is, when each of s_*, q_*, r_* is nonnegative. Since $r_* \neq 0$ we see that r_* is realistic when

$$\left(1-\left(\frac{d(\mu+\delta+\epsilon)+\mu\epsilon}{\mu-\epsilon(\delta+\epsilon)}\right)\right) > 0.$$

We expect $\mu > \epsilon(\delta+\epsilon)$ since ϵ and δ are small and so this inequality simplifies to

$$\mu-\epsilon(\delta+\epsilon)-d(\mu+\delta+\epsilon)-\mu\epsilon > 0,$$

which can be rewritten in the form

$$d < \left[1+\frac{\delta+\epsilon}{\mu}\right]^{-1} - \epsilon. \quad (1.25)$$

Similarly for q_* to be realistic when (1.25) is satisfied the following condition must hold:

$$\left(d+\epsilon\left(\frac{d(\mu+\delta+\epsilon)+\mu\epsilon}{\mu-\epsilon(\delta+\epsilon)}\right)\right) > 0.$$

Again this simplifies to, assuming that $\mu > \epsilon(\delta+\epsilon)$,

$$\mu d(\epsilon+1)+\mu\epsilon^2 > 0,$$

which is clearly satisfied. Finally, for s_* to be realistic the following condition must be satisfied:

$$\left(\mu + \delta + \epsilon \left(\frac{d\,(\mu + \delta + \epsilon) + \mu\epsilon}{\mu - \epsilon\,(\delta + \epsilon)}\right)\right) > 0.$$

This will be the case, provided $\mu > \epsilon\,(\delta + \epsilon)$, if

$$d > \delta - \frac{\mu(\mu + \delta)}{\epsilon(\mu + \delta + \epsilon)}.$$

Since δ is small we expect that the right-hand side of this equation is negative; this occurs when $-\mu^2 - \mu\delta(1-\epsilon) + \delta^2 + \epsilon^2 < 0$. Now since $d > 0$ we see that the steady state is realistic if condition (1.25) holds. It now remains to check the stability of the system at each of the steady states.

Consider the general dynamical system

$$\dot{\overline{z}} = f(\overline{z}).$$

Suppose that the function f is differentiable with respect to $\overline{z}$. Then making the substitution $\overline{z} = z + z_0$ we have

$$\dot{z} = f(z_0) + \frac{df}{d\overline{z}}(z_0)z + \overline{f}(z) - \dot{z}_0.$$

Therefore, if z_0 is a steady state and $\|z\|$ small,

$$\dot{z} = \frac{df}{d\overline{z}}(z_0)z + \overline{f}(z) = Az + \overline{f}(z).$$

The following well-known result gives a sufficient criterion for the stability of a steady state.

Theorem 1.3.1. *Consider the general differential equation given by*

$$\dot{\overline{z}} = f(\overline{z}).$$

Suppose that f is differentiable and $\sigma(A) \subset \mathbb{C}_-$. Then the steady state is asymptotically stable.

This means that, if the eigenvalues of A have negative real part, then the steady state is stable; small perturbations of the steady state return to the steady state.

Suppose that f is given by the equations (1.18)–(1.20), with $\overline{z} = (s,v,q,r)^{\mathrm{T}}$ and that $z_0 = (s_0, 0, q_0, r_0)^{\mathrm{T}}$ is the steady state under consideration. Then

$$\frac{df}{d\overline{z}}(z_0) = \begin{pmatrix} \epsilon(1-n_0) - s_0\epsilon - r_0 & -\epsilon s_0 + (\epsilon+\delta) & -\epsilon s_0 & -s_0(\epsilon+1) \\ 0 & -(\delta+\epsilon n_0) & 0 & 0 \\ r_0 - \epsilon q_0 & -\epsilon q_0 & -(\mu+\delta+\epsilon n_0) - \epsilon q_0 & s_0 - \epsilon q_0 \\ -\epsilon r_0 & -\epsilon r_0 & \mu - \epsilon r_0 & -(d+\epsilon n_0) - \epsilon r_0 \end{pmatrix}$$

where $n_0 = s_0 + q_0 + r_0$. We will consider each of the three steady states in turn.

$\mathbf{z_0 = (0,0,0,0)}$

For this steady state the matrix A is given by

$$\begin{pmatrix} \epsilon & \epsilon+\delta & 0 & 0 \\ 0 & -\delta & 0 & 0 \\ 0 & 0 & -(\mu+\delta) & 0 \\ 0 & 0 & \mu & -d \end{pmatrix}.$$

We see immediately that one of the eigenvalues is real and positive. Therefore this steady state is unstable.

$\mathbf{z_0 = (1,0,0,0)}$

For this steady state the matrix A is given by

$$\begin{pmatrix} -\epsilon & \delta & -\epsilon & -(\epsilon+1) \\ 0 & -(\delta+\epsilon) & 0 & 0 \\ 0 & 0 & -(\mu+\delta+\epsilon) & 1 \\ 0 & 0 & \mu & -(d+\epsilon) \end{pmatrix}.$$

By inspection we see that two of the eigenvalues are real and negative. The other two eigenvalues are roots of

$$\lambda^2 + \lambda(\mu + 2\epsilon + \delta + d) + (d\delta + d\epsilon + \delta\epsilon + \epsilon^2 - \mu + d\mu + \epsilon\mu). \tag{1.26}$$

Necessary and sufficient conditions for the roots to have negative real part are provided by the *Routh-Hurwitz conditions* (Murray, 1993). These conditions are, for equation (1.26),

$$\mu + 2\epsilon + \delta + d > 0, \qquad \text{and} \qquad d(\delta + \epsilon + \mu) + \delta\,\epsilon + \epsilon^2 - \mu + \epsilon\,\mu > 0.$$

The first equation is automatically satisfied since each of the parameters is positive. The second equation yields the following necessary and sufficient condition for the steady state to be stable

$$d > \frac{\mu}{\delta + \epsilon + \mu} - \epsilon = \left[1 + \frac{\delta + \epsilon}{\mu}\right]^{-1} - \epsilon. \tag{1.27}$$

$\mathbf{z}_0 = (\mathbf{s}_*, \mathbf{0}, \mathbf{q}_*, \mathbf{r}_*)$

For this steady state the matrix A is given by

$$\begin{pmatrix} \epsilon(1-n_*) - s_*\epsilon - r_* & -\epsilon s_* + (\epsilon + \delta) & -\epsilon s_* & -s_*(\epsilon + 1) \\ 0 & -(\delta + \epsilon n_*) & 0 & 0 \\ r_* - \epsilon q_* & -\epsilon q_* & -(\mu + \delta + \epsilon n_*) - \epsilon q_* & s_* - \epsilon q_* \\ -\epsilon r_* & -\epsilon r_* & \mu - \epsilon r_* & -(d + \epsilon n_*) - \epsilon r_* \end{pmatrix}$$

where n_*, s_*, q_*, r_* are given by equations (1.21) and (1.22)–(1.24). Two of the eigenvalues of this matrix are

$$\begin{aligned} \lambda_1 &= \epsilon(1 - n_*) - \epsilon s_* - r_* \\ \lambda_2 &= -\delta - \epsilon n_* \end{aligned}$$

with the remaining eigenvalues being the roots of the following polynomial:

$$\lambda^2 + \lambda\,(d + \epsilon\,(2n_* + r_* + q_*) + \delta + \mu) \\ + (\mu + \delta + \epsilon\,(n_* + q_*))\,(d + \epsilon\,(n_* + r_*)) - (\epsilon r_* - \mu)\,(\epsilon q_* - s_*)\,.$$

Clearly $\lambda_2 < 0$ and λ_1 is given by

$$\lambda_1 = \epsilon(1 - n_*) - \epsilon s_* - r_* = -\epsilon s_* < 0,$$

(by (1.18) and the fact that $s_* \neq 0$). Therefore it only remains to check whether the two roots of the above quadratic have negative real part. Again we apply the Routh-Hurwitz conditions: the roots have negative real part if and only if

$$d + \epsilon(2n_* + r_* + q_*) + \delta + \mu > 0$$

and

$$(\mu + \delta + \epsilon(n_* + q_*))\,(d + \epsilon(n_* + r_*)) - (\epsilon r_* - \mu)\,(\epsilon q_* - s_*) > 0.$$

The first of these conditions is clearly satisfied. For the second condition we expand out the terms to get

$$d(\mu + \delta) + \epsilon n_*(\mu + \delta) + \epsilon r_*(\mu + \delta) + \epsilon d n_* + \epsilon d q_* \\ + \epsilon^2 n_*^2 + \epsilon^2 n_* r_* + \epsilon^2 n_* q_* + \epsilon r_* s_* + \epsilon \mu q_* - \mu s_*.$$

Now $s_* = \frac{1}{\mu}(\mu + \delta + \epsilon n)(d + \epsilon n)$ and substituting for the last term gives

$$\epsilon r_*(\mu + \delta) + \epsilon d q_* + \epsilon^2 n_* r_* + \epsilon^2 n_* q_* + \epsilon r_* s_* + \epsilon \mu_* q_*$$

which is clearly positive. Hence this steady state is stable.

Remark 1.3.2. Recall that this steady state is realistic if condition (1.25) is satisfied which means that (1.27) is violated. Hence this steady state is realistic (and stable) if the disease free steady state is unstable. This is the requirement for an epidemic.

The condition (1.27) is the non-dimensionalised version of (1.4) and determines whether an epidemic will be maintained.

1.3.2 Some remarks

The analysis of the previous subsection is not unexpected. By including the vaccinated class of fox, with its time-dependence, rather than reducing the environmental carrying capacity, we do not change the stability of the system. Intuitively, if $K > K_c$ (or in non-dimensionalised form $d < \left[1 + \frac{\delta+\epsilon}{\mu}\right]^{-1} - \epsilon$), once the population density of the vaccinated class of fox has been reduced to zero we are back in the unstable situation. Therefore if the infected fox population density is positive anywhere when this happens an epidemic will begin.

Roughly speaking, while the population density of the susceptible class is less than K_c, the system is stable in the sense that the infected population density will decrease. Once the susceptible population density has risen above K_c, the system becomes unstable in the sense that an epidemic will begin and rabies will be maintained in the fox population.

We must remark at this stage a problem associated with using classical diffusion to model dispersal and continuous densities for populations of individuals. Both Källen et al. (1985) and Murray et al. (1986) point out that mathematically the infected population density will always be positive everywhere and for all later times. This means that we must be careful in our analysis of the above extended model since mathematically if $K > K_c$ whatever our vaccination program is, an epidemic will occur at some later point. The problem of continuous densities can be overcome by setting $R = 0$ (or $I = 0$) if R (I respectively) is small enough.

1.3.3 Time dependence of vaccination procedures

Once a baiting trial has been completed and all of the bait has either been taken by foxes or decomposed the vaccinated population density will decrease. The susceptible population density will increase as vaccinated foxes give birth to susceptible ones. A critical question therefore for us to ask is: When will the density of the susceptible population be greater than or equal to the critical carrying capacity for an epidemic? If the outbreak of rabies has not been eliminated before this density has risen to this critical value an epidemic will occur.

To illustrate the problem of the recovery of the susceptible population density after a vaccination has been carried suppose that a fox population has reached the environmental carrying capacity K, which is a steady state for the fox population in the absence of rabies.

After a baiting trial suppose that a proportion, λ, of the susceptible foxes have been vaccinated. In the absence of any infected foxes the model equations (1.10)–(1.13) become

$$\frac{dS}{dt} = (a-b)\left(1 - \frac{S+V}{K}\right)S + aV, \qquad S(0) = (1-\lambda)K$$
$$\frac{dV}{dt} = -\left(b + (a-b)\frac{S+V}{K}\right)V, \qquad V(0) = \lambda K.$$

Since the total fox population, $S+V$, is at the steady state value K, we have $S(t)+V(t) = K$ for

λ	T_c (Days)
0.9	215
0.8	172
0.7	123
0.6	67

Table 1.2: Recovery of susceptible population density for a carrying capacity of 2 foxes km^{-1} and a critical carrying capacity of 1 fox km^{-1}.

λ	T_c (Days)
0.9	51
0.8	8

Table 1.3: Recovery of susceptible population density for a carrying capacity of 4.6 foxes km^{-1} and a critical carrying capacity of 1 fox km^{-1}.

all times t. Therefore

$$\frac{dS}{dt} = a(K - S)$$

which has the solution

$$S(t) = K(1 - \lambda e^{-at}).$$

We are interested in the time T_c when $S(T_c) = K_c$, the critical carrying capacity for an epidemic. This occurs at

$$T_c = \frac{-1}{a} \ln\left[\frac{1}{\lambda}\left(1 - \frac{K_c}{K}\right)\right].$$

We give some numerical results in Table 1.2 and Table 1.3.

We note from the results in Table 1.3 that in the U.K. a minimum reduction of 80% is required and then the system remains stable for only eight days. This means that the rabies has to be eliminated in eight days or an epidemic will occur. This has serious repercussions if an outbreak of rabies occurs in Britain.

These results seem to suggest that the time-dependence of the vaccination procedure is important when considering the efficacy of a control strategy.

1.4 Mathematical formulation of the control problem

Suppose that we wish to formulate the system of partial differential equations (1.10)–(1.13) as an abstract differential equation. To do this let $\Omega \subset \mathbb{R}^2$ be closed and bounded where Ω is the spatial domain. We set $\overline{s}(t) = S(t,\cdot)$, $\overline{v}(t) = V(t,\cdot)$, $\overline{q}(t) = I(t,\cdot)$, $\overline{r}(t) = R(t,\cdot)$, where, for example, $S(t,\cdot) = \{S(t,x) : x \in \Omega\}$. Now form the vector

$$\overline{z}(\cdot) = \begin{pmatrix} \overline{s}(\cdot) \\ \overline{v}(\cdot) \\ \overline{q}(\cdot) \\ \overline{r}(\cdot) \end{pmatrix},$$

and consider the dynamics

$$\dot{\overline{z}} = f(\overline{z}) \tag{1.28}$$

where $f(\cdot)$ is given by (1.10)–(1.13).

The mechanism for controlling the spread of rabies considered in this thesis is population reduction. We allow for this reduction to be carried out in three ways: through a vaccination program; a cull; or a combined scheme of vaccination and culling.

For the first method, suppose that an initial distribution, $\overline{u}_1(\cdot) \in PC(\Omega;\mathbb{R})$, of foxes has been vaccinated. If the initial distributions of susceptible, incubating, and rabid foxes are, respectively,

$\overline{s}_0$, $\overline{q}_0$, and $\overline{r}_0$, then the initial condition for (1.28) is

$$\overline{z}(0) = \begin{pmatrix} \overline{s}_0 \\ 0 \\ \overline{q}_0 \\ \overline{r}_0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \overline{u}_1 = \overline{z}_0 + B_1 \overline{u}_1. \tag{1.29}$$

For the second method we suppose that an initial distribution, $\overline{u}_2(\cdot) \in PC(\Omega; \mathbb{R})$, of foxes has been culled. Then the initial condition for (1.28) is

$$\overline{z}(0) = \begin{pmatrix} \overline{s}_0 \\ 0 \\ \overline{q}_0 \\ \overline{r}_0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \overline{u}_2 = \overline{z}_0 + B_2 \overline{u}_2. \tag{1.30}$$

Clearly these two methods of population reduction can be combined to give a third scheme for controlling the spread of rabies. In this case we suppose that initial distributions $\overline{u}_1$, and $\overline{u}_2$ of foxes have been vaccinated and culled respectively. Then the initial condition for (1.28) is

$$\overline{z}(0) = \begin{pmatrix} \overline{s}_0 \\ 0 \\ \overline{q}_0 \\ \overline{r}_0 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{u}_1 \\ \overline{u}_2 \end{pmatrix} = \overline{z}_0 + B\overline{u}. \tag{1.31}$$

For the purposes of reducing the total infected population density we consider the following observation

$$\overline{y}(t) = \overline{q}(t) + \overline{r}(t) = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \overline{z}(t) = C\overline{z}(t). \tag{1.32}$$

The problem then is as follows: The spatial domain Ω represents the 'target zone'—the local area in which we wish to contain the outbreak of rabies. Suppose that the distribution, $\overline{r}_0$, of rabid foxes represents the outbreak. The problem is to choose (if possible) an input u—level of vaccination or cull or both—so that the output—the total infected fox population density—at a particular time, T say, is some desired distribution y_d.

Chapter 2

Controllability via the Initial State

The control problem which this thesis aims to solve is to drive some part, or all, of a particular biological or medical process, to some desired state in a specified time. The main application considered throughout this thesis is controlling the spread of rabies in a fox population. In this case the parts of the system that one wishes to control are the population densities of the incubating and rabid foxes. The novelty of these systems is that the control acts only through the initial state.

In Chapter 1 the control problem was formulated mathematically as an abstract differential equation. The part of the state that is to be controlled is formulated as an output so that the problem becomes that of driving the output of the system to a desired value in a specified time. In this chapter the important question of whether a solution to the mathematical control problem exists is considered. A weakening of the aim of the control problem leads to a positive answer. More satisfactorily it is shown that, under certain circumstances, if a solution exists then the method used in this chapter will determine it.

The mathematical formulation of the control problem is as follows. Consider the abstract differential equation given by

$$\dot{\overline{z}}(t) = f(t, \overline{z}(t))$$

where f is nonlinear and a solution is sought with values in a Banach space Z. The equation is uncontrolled except for an input, denoted by $\overline{u}$, via part of the initial state. Supposing that z_0' is the known, given initial state of the equation without input, and B determines how the control acts, the initial state is given by

$$\overline{z}(0) = z'_0 + B\overline{u}.$$

The controls are assumed to belong to a Hilbert space U such that $B \in \mathcal{L}(U, Z)$. Therefore the treatment of this thesis is confined to the situation involving bounded inputs. If, for the underlying system, it is not possible to affect the state at every point of the spatial domain so that the controls are restricted to only a few points or parts of the boundary the resulting model will involve an *unbounded* input operator. Pritchard and Salamon (1987) considered systems with unbounded inputs and outputs. In this case it is assumed that there exists a Banach space $Z \subset Z_1$ with continuous injection and dense range. The input operator is then assumed to be bounded from U to Z_1.

The output associated with the differential equation is given by

$$y = C\overline{z}(T),$$

where T is the specified time and the output takes values in a Hilbert space Y. Mathematically the control problem is to choose u such that the resulting output is $y = y_d$, the desired target output. Again, if the part of the system that is to be controlled, that is the output, is the state at only certain points in the spatial domain or parts of the boundary the resulting model will have an unbounded output operator. If, in the rabies model, it was desired only to contain the spread of infection by keeping the density of infected foxes low on the boundary of the spatial domain, this would lead to an unbounded output operator. However in this thesis the aim is to reduce the infected population throughout the spatial domain.

Suppose that an initial guess is made for the control, u' say, with associated differential equation

$$\dot{z}'(t) = f(t, z'(t)), \qquad z'(0) = z'_0 + Bu', \tag{2.1}$$

and that this equation has a continuously differentiable solution $z'(\cdot)$. While this control might be a good initial guess there is no reason to assume that the output of this system, $Cz'(T)$, is the desired final state. Therefore a local approximation is made by setting $\overline{z} = z' + z$ and $\overline{u} = u' + u$

to get

$$\dot{z}(t) + \dot{z}'(t) = f(t, z(t) + z'(t)), \qquad z(0) = z_0' + Bu + Bu' - z'(0). \tag{2.2}$$

Now suppose that f is differentiable around the trajectory $\{(t, z'(t)) : t \in [0, T]\}$ in the sense that

$$f(t, \overline{z}) = f(t, z'(t)) + A(t)\,(\overline{z} - z'(t)) + D(t)N(t, E(t)(\overline{z} - z'(t))) \tag{2.3}$$

for some piecewise continuous $A(\cdot)$ such that $A(t)$ is an unbounded linear operator on Z for each $t \in [0, T]$; $D(\cdot)$ and $E(\cdot)$ characterise and describe the unboundedness of the nonlinearity. Therefore in the following it will be assumed that there are Banach spaces $\underline{Z}$ and $\overline{Z}$ such that $\underline{Z} \subset Z \subset \overline{Z}$ where the injections are continuous with dense ranges. With respect to these spaces we suppose that $D(t)$ and $E(t)$ are bounded (and so can be considered as unbounded operators with respect to Z). To give greater flexibility in the treatment of the nonlinearity we suppose that $N : [0, T] \times \underline{W} \longrightarrow \overline{W}$ where $\underline{W} \subset \overline{W}$ are Banach spaces. More precise assumptions will be introduced in the following.

Equation (2.2) can be rewritten as

$$\dot{z}(t) = A(t)z(t) + D(t)N(t, E(t)z(t)), \qquad z(0) = Bu. \tag{2.4}$$

The following assumption is made for the general differential equation.

Assumption 1 The initial guess for the control u' gives rise to a continuously differentiable (with respect to $\overline{Z}$) solution $z'(\cdot)$ of (2.1). The nonlinear function f satisfies (2.3) with respect to this solution.

In Section 2.1 it is determined how equation (2.4) will be interpreted and in what form solutions will be sought. To seek solutions of this equation is too restrictive and so a mild form of the system equation is introduced (in a similar fashion to that of Hinrichsen and Pritchard, 1994,for perturbations of linear evolution equations). This leads naturally to the definition of a *mild solution.*

In Section 2.2 the first stage of the control problem is considered, namely the construction of an input that gives rise to a mild solution with the desired properties. A fixed point theorem is applied to give the solution. A review of the use of fixed point theorems in nonlinear control problems is given in Carmichael and Quinn (1988).

An equivalent, but less intuitive, method for constructing the control is given in Section 2.3. This method exploits the original system dynamics and uses an adaptive scheme to give the solution. This method readily lends itself to numerical simulation and constructs the control directly rather than via a mild solution (as is the case in the fixed point approach).

Throughout the chapter it is seen that, in general, the target output cannot be hit everywhere, but only only on some smaller subspace. In Section 2.4 the important question of how to minimise the actual output over all choices of target is considered.

2.1 Time-Varying Systems

In this section the system given by

$$\dot{z}(t) = A(t)z(t) + D(t)N(t, E(t)z(t)), \qquad z(0) = Bu \tag{2.5}$$

is considered. The meaning of equation (2.5) and a solution of it must be made precise. One of the aims of this thesis is to allow for the possible unboundedness of the nonlinearity and this must be taken into consideration.

In the following subsection the abstract Cauchy problem on $[0, T]$ defined by

$$\dot{z}(t) = A(t)z(t), \qquad z(s) = z_s$$

where $s \in [0, T]$, is considered. A definition is given for what is meant for this problem to be well-posed, and a (unique) family of operators governing the evolution of the system is associated with solutions of the problem. This family of *evolution* operators takes the place of the strongly continuous semigroup in the time-invariant situation.

After considering the linear time-varying Cauchy problem the full system of (2.5) is considered. In general, systems of this form will not be well-posed in the sense of the linear case. As a result, in a similar fashion to Hinrichsen and Pritchard (1994), a mild version of the system equation is introduced.

The section is concluded by allowing for the possible unboundedness of the nonlinearity in the mild version of the system equation. Solutions of the abstract Cauchy problem will then be interpreted in terms of *mild solutions* of this version of the problem.

2.1.1 The abstract Cauchy problem

Consider the linear part of equation (2.5) with arbitrary initial state given by

$$\dot{z}(t) = A(t)z(t), \qquad z(s) = z_s, \tag{2.6}$$

with $s \in [0, T]$. It is well-known (see Pazy, 1983,for example) that in the time-invariant case, where $A(t) = A$ for all t, if A is a densely defined linear operator on Z with non-empty resolvent set $\rho(A)$, then the Cauchy problem (2.6) is well-posed if and only if A is the generator of a strongly continuous semigroup. Well-posed is meant in the sense of the following definition:

Definition 2.1.1. The Cauchy problem given by (2.6) where $A(t) = A$ is independent of t and s is fixed, is said to be *well-posed* if there exists a unique function $z(\cdot, s) \in C(s, T; Z)$, that is continuously differentiable on $[s, T]$, $z(t, s) \in D(A)$ and satisfies (2.6) for all $t \in [s, T]$.

If A is the generator of the strongly continuous semigroup $S(t)$ then the unique solution of (2.6) (provided $z_s \in D(A)$) is given by $z(t, s) = S(t - s)z_s$.

In the time-varying case Tanabe (1979) considered the situation where $A(t)$ is the generator of a strongly continuous semigroup for each $t \in [0, T]$ and Krein (1971) that when $A(t)$ is strongly continuous with domain independent of t. Hinrichsen and Pritchard (1994) weakened the latter assumption by allowing $A(t)$ to be piecewise continuous. This will be the setting for the rest of the chapter. More formally, it is assumed that the following is satisfied.

Assumption 2 $A(t)$ is a linear operator on Z for all $t \in [0, T]$ with domain $D(A)$, independent of t and dense in Z. For all $z \in D(A)$ the map $t \mapsto A(t)z$ is continuous except on a finite set of discrete points J. For each $\tau \in J$ and $z \in D(A)$ the one-sided limits $\lim_{t\downarrow\tau} A(t)z$, $\lim_{t\uparrow\tau} A(t)z$ exist.

In parallel with the time-invariant case the precise meaning of the time-varying Cauchy problem given by (2.6) being well-posed must be defined. Let $\Delta(T) = \{(t, s) : 0 \leq s \leq t \leq T\}$. By allowing $A(t)$ to be only piecewise continuous Hinrichsen and Pritchard (1994) were able to slightly weaken the definition appearing in the literature (see for example Krein, 1971).

Definition 2.1.2. The Cauchy problem (2.6) is said to be *well-posed on* $\Delta(T)$ if the following conditions are satisfied:

(i) For all $s \in [0,T]$ and $z_s \in D(A)$ there exists a unique continuous function $z(\cdot,s) : [s,T] \longrightarrow Z$ such that $z(t,s) \in D(A)$ for all $t \in [s,T]$, is strongly differentiable on $[s,T]\setminus J$ and satisfies (2.6) in the following sense:

(a) For all $t \in [s,T] \setminus J$ (2.6) is satisfied;

(b) For all $\tau \in J$ we have the following one-sided limits

$$\lim_{h\downarrow 0} \frac{z(\tau+h)-z(\tau)}{h} = \lim_{t\downarrow\tau} A(t)z(t),$$
$$\lim_{h\downarrow 0} \frac{z(\tau)-z(\tau-h)}{h} = \lim_{t\uparrow\tau} A(t)z(t);$$

(c) $z(s) = z_s$ and

$$\lim_{h\downarrow 0} \frac{z(s+h)-z(s)}{h} = \lim_{t\downarrow s} A(t)z(t).$$

(ii) The function $z(t,s)$ is continuous and its derivative $\frac{\partial z}{\partial t}(t,s)$ is piecewise continuous (with discontinuities in J only) in both t and s with $(t,s) \in \Delta(T)$.

(iii) The function depends continuously on the initial data in the following sense:

Let $z_s^n \in D(A)$ be a sequence of initial states such that $z_s^n \to 0$ as $n \to \infty$. Then the corresponding functions $z^n(t,s)$ converge to zero uniformly in $(t,s) \in \Delta(T)$.

For each (fixed) $s \in [0,T]$ a function $z(\cdot,s)$ satisfying condition (i) of Definition 2.1.2 is said to be a *strong solution* of (2.6).

If the time-invariant Cauchy problem is well-posed then A is the generator of a strongly continuous semigroup. For the time-varying problem Hinrichsen and Pritchard proved a similar result. In this case the family of evolution operators associated with solutions of the Cauchy problem are functions of $(t,s) \in \Delta(T)$.

Proposition 2.1.3. *Suppose that Assumption 2 holds and the Cauchy problem* (2.6) *is well-posed. Then there exists a unique family of bounded linear operators on* Z, $(U(t,s))_{(t,s)\in\Delta(T)}$, *such that the solutions of* (2.6) *are given by* $z(t,s) = U(t,s)z_s$ *for each* $z_s \in D(A)$ *and* $s \in [0,T]$ *where the following conditions are satisfied.*

(i) *For all* $t \in [0,T]$, $U(t,t) = I$;

(ii) *For all* $(t,s) \in \Delta(T)$, *and* $r \in [s,t]$, $U(t,r)U(r,s) = U(t,s)$;

(iii) *For all* $(t,s) \in \Delta(T)$, $D(A)$ *is* $U(t,s)$*-invariant;*

(iv) *For each* $s \in [0,T]$ *and* $z \in D(A)$ *the map* $t \mapsto A(t)U(t,s)z$ *is piecewise continuous (with discontinuities in* J *only) on* $[s,T]$ *and*

$$U(t,s)z - z = \int_s^t A(\sigma)U(\sigma,s)z\,d\sigma, \quad \textit{for } (t,s) \in \Delta(T);$$

(v) *For all* $t \in [0,T]$ *and* $z \in D(A)$ *the map* $s \mapsto U(t,s)A(s)z$ *is piecewise continuous (with discontinuities in* J *only) on* $[0,t]$ *and*

$$U(t,s)z - z = \int_s^t U(t,\sigma)A(\sigma)z\,d\sigma, \quad \textit{for } (t,s) \in \Delta(T);$$

(vi) $U(\cdot,s)$ *is strongly continuous on* $[s,T]$ *and* $U(t,\cdot)$ *is strongly continuous on* $[0,t]$ *for all* $(t,s) \in \Delta(T)$.

Remark 2.1.4. For $s \geq 0$, $z \in D(A)$ and $t \in [s,T] \setminus J$,

$$\frac{\partial}{\partial t}U(t,s)z = A(t)U(t,s)z.$$

Now let $z \in D(A)$, $s \in [0,t] \setminus J$, and $h > 0$ such that $s + h \in [s,t]$. Then

$$\begin{aligned}\frac{\partial}{\partial s}U(t,s)z &= \lim_{h\downarrow 0}\frac{U(t,s+h)z - U(t,s)z}{h}\\ &= \lim_{h\downarrow 0}\frac{1}{h}U(t,s+h)\,(z - U(s+h,s)z)\\ &= -U(t,s)A(s)z,\end{aligned}$$

from the previous statement and the strong continuity of $U(t,\cdot)$ on $[0,t]$.

The family of operators in Proposition 2.1.3 serves in the place of the semigroup in the time-dependent Cauchy problem.

Definition 2.1.5. Let $A(t)$ be a family of operators satisfying Assumption 2 and $(U(t,s))_{(t,s)\in\Delta(T)}$ be a family of bounded linear operators on Z that satisfy conditions (i)–(vi) of Proposition 2.1.3. We say that $(U(t,s))_{(t,s)\in\Delta(T)}$ (or $U(t,s)$ for short) is a *strong evolution operator* with generator $A(\cdot)$.

In fact Hinrichsen and Pritchard (1994) proved, as with the time-invariant case, that the converse is also true:

Proposition 2.1.6. *Suppose that* $(U(t,s))_{(t,s)\in\Delta(T)}$ *is a strong evolution operator with generator* $A(\cdot)$. *Then the Cauchy problem* (2.6) *is well-posed.*

Now for the full problem (2.5) one might expect to have to assume that $A(\cdot)$ is the generator of a strong evolution operator and impose suitable conditions on $D(t)N(t,E(t)z(t))$ for the resulting Cauchy problem to be well-posed in the sense of Definition 2.1.2. Even in the case where $A(t)=A$ is independent of time and the nonlinearity is bounded it is well-known that one cannot guarantee that this problem is well-posed. In this case a mild form of solution is introduced and the system dynamics are interpreted in terms of it. To motivate a similar idea for the time-dependent case suppose that $D(t)N(t,E(t)z(t))$ is continuous with values in Z, and the Cauchy problem is well-posed with solution $z(\cdot)$ and strong evolution operator $U(t,s)$. Then (Hinrichsen and Pritchard, 1994)

$$\frac{\partial}{\partial\sigma}\left[U(t,\sigma)z(\sigma)\right]=\left.\frac{\partial U(t,\rho)z(\sigma)}{\partial\rho}\right|_{\rho=\sigma}+U(t,\sigma)\frac{\partial z(\sigma)}{\partial\sigma}$$

and so, using Remark 2.1.4,

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left[U(t,\sigma)z(\sigma)\right]&=U(t,\sigma)\left(-A(\sigma)z(\sigma)+A(\sigma)z(\sigma)+D(\sigma)N(\sigma,E(\sigma)z(\sigma))\right)\\&=U(t,\sigma)D(\sigma)N(\sigma,E(\sigma)z(\sigma)).\end{aligned}$$

Integrating both sides with respect to σ and rearranging we have

$$z(t)=U(t,s)z_s+\int_s^t U(t,\sigma)D(\sigma)N(\sigma,E(\sigma)z(\sigma))\,d\sigma.$$

This will be the mild form of solution to (2.5) and how 'solutions' of the system will be interpreted. Weaker conditions will be imposed on the evolution operator $U(t,s)$ than those needed for a strong operator.

Definition 2.1.7. A family of linear operators $(U(t,s))_{(t,s)\in\Delta(T)}$ such that $U(t,s) \in \mathcal{L}(Z)$ for each $(t,s) \in \Delta(T)$, is a *mild evolution operator* on Z if it satisfies conditions (i), (ii) and (vi) of Proposition 2.1.3.

The definition of a mild evolution operator is independent of any generator whereas for a strong evolution operator this is not the case. Since the mild form of solution to (2.5) does not require an $A(t)$ Hinrichsen and Pritchard (1994) studied this object directly and so allowed for a wider class of perturbed dynamical systems.

In a similar way the analysis that follows will be confined to this mild form for the system equation. In the next chapter, motivated by the rabies model, a certain structure for $A(t)$ will be considered that admits, in some sense to be made precise, a mild evolution operator on Z.

2.1.2 Unboundedness of nonlinearity

Suppose that $U(t,s)$ is a mild evolution operator on Z and consider the following system equation

$$z(t) = U(t,0)Bu + \int_0^t U(t,s)D(s)N(s, E(s)z(s))\,ds, \tag{2.7}$$

where $u \in U$ a Hilbert space such that $B \in \mathcal{L}(U,Z)$. To allow for the possible unboundedness of the nonlinearity, which will be characterised by the operators $D(\cdot)$ and $E(\cdot)$, the following spaces are introduced.

TV I. *$\underline{Z}, Z, \overline{Z}$ are Banach spaces such that $\underline{Z} \subset Z \subset \overline{Z}$ and the canonical injections $\underline{Z} \hookrightarrow Z$, $Z \hookrightarrow \overline{Z}$ are continuous with dense ranges. Moreover, $E(\cdot) \in PC(0,T;\mathcal{L}(\underline{Z},\underline{W}))$ and $D(\cdot) \in PC(0,T;\mathcal{L}(\overline{W},\overline{Z}))$.*

In view of this condition, for all $t \in [0,T]$, $E(t)$ can be viewed as an unbounded linear operator from Z to $\underline{W}$ and $D(t)$ as an unbounded operator from $\overline{W}$ to Z. It is assumed that the nonlinearity $N : [0,T] \times \underline{W} \longrightarrow \overline{W}$ maps functions in $L^r(0,T;\underline{W})$ to functions in $L^s(0,T;\overline{W})$ for $r, s \geq 1$ real numbers in the sense that

$$\mathcal{N}(w)(\cdot) = N(\cdot, w(\cdot)) \in L^s(0,T;\overline{W}) \quad \text{whenever} \quad w(\cdot) \in L^r(0,T;\underline{W}).$$

Clearly to allow for this unboundedness further assumptions must be made so that the system equation (2.7) is well–defined. The next assumption ensures that the integrand makes sense.

TV II. *For every* $0 \leq s \leq t \leq T$ $U(t,s)$ *can be extended to a bounded linear operator on* $\overline{Z}$*, which we again denote* $U(t,s)$.

Now integrability in the upper space must be guaranteed.

TV III. *For every* $h(\cdot) \in L^s(0,T;\overline{W})$*, and* $t \in (0,T]$, $U(t,\cdot)D(\cdot)h(\cdot)$ *from* $[0,t]$ *to* $\overline{Z}$ *is integrable in* $\overline{Z}$.

With these assumptions the following operator is well-defined

$$(\mathrm{M}_U h)(t) = \int_0^t U(t,s)D(s)h(s)\,ds \tag{2.8}$$

for all $h(\cdot) \in L^s(0,T;\overline{W})$. A continuous solution with respect to Z is sought and so some further assumptions are required. In the following the system equation will be considered as a map from the space of all $w(\cdot) = E(\cdot)z(\cdot)$ to itself. Therefore more is required than $(\mathrm{M}_U h)(t) \in Z$:

TV IV. *For every* $h(\cdot) \in L^s(0,T;\overline{W})$, $(\mathrm{M}_U h)(t) \in \underline{Z}$ *for almost every* $t \in [0,T]$*, and the map* $t \mapsto (\mathrm{M}_U h)(t)$ *is continuous with respect to* $\|\cdot\|_Z$.

Definition 2.1.8. A *mild solution* of (2.7) is any continuous function $z(\cdot) : [0,T] \longrightarrow Z$ such that $E(\cdot)z(\cdot)$ is L^r-integrable in $\underline{W}$ and (2.7) is satisfied for all $t \in [0,T]$.

It should be noted that these conditions do not guarantee that a mild solution actually exists. The existence of a mild solution that solves the control problem is the topic of the next section.

2.2 Existence of a mild solution

In this section the question of the existence of a mild solution of (2.7) that solves a specific control problem is addressed. The output associated with the system equation is given by

$$y = Cz'(T) + Cz(T) \tag{2.9}$$

where $Cz'(T)$ is some given output. In this thesis C is bounded in the sense that there exists a Hilbert space, $\overline{Y}$ say, such that $C \in \mathcal{L}(Z, \overline{Y})$. Other considerations, to be discussed in greater detail in Section 2.2.2, might require the restriction of this space so that output values are considered in a Hilbert space $Y \subseteq \overline{Y}$. In this case one might not be able to consider C as a bounded linear operator from Z to Y. To allow for this it will be assumed that there exists a Banach space V ($\subset Z$) with continuous injection such that $C \in \mathcal{L}(V, Y)$.

The control problem considered is to choose an input u such that the output is some specified value y_d. The idea of this section is to construct a control that gives rise to a mild solution which has the desired output. The approach is to first reduce the problem to one of finding a fixed point of a certain map. Next it is ensured that the map is well-defined when allowing for the unboundedness of the nonlinearity. Finally a version of the Contraction Mapping Theorem is applied to construct the fixed point.

The approach followed necessitates a weakening of the objective of the control problem. It will no longer be required that the output of the system be equal to the desired one everywhere, but only on a subspace. The section is concluded with some interesting results that relate the achieved output to the target one. In Section 2.4 the problem of minimising the achieved output by a suitable choice of target is analysed.

2.2.1 The fixed point problem

In infinite dimensional systems the ability to control the state (or the output in this thesis) to any other state is a strong and restrictive condition. In applying a fixed point theorem to the question of controllability one usually considers the states to which the linearised system can be steered, obtaining a subregion to which the nonlinear system can be controlled (see, for example, Carmichael and Quinn, 1988).

To motivate the use of the fixed point theorem consider the linear part of the system equation

$$z(t) = U(t,0)Bu,$$

where $U(t,s)$ is a mild evolution operator on some Banach space Z that the problem is to be considered on. The output is

$$y = Cz(T) + Cz'(T),$$

where $y \in Y$, a Hilbert space, and $Cz'(T) \in Y$ is some given output. Therefore for the linear system a control $\hat{u}$ is sought such that the output

$$y = CU(T,0)B\hat{u} + Cz'(T),$$

is the desired value y_d. If $y_d - Cz'(T) \in \operatorname{ran}\phi$, the range of the operator $\phi = CU(T,0)B$, then a solution exists. In particular, if ϕ is invertible, then there is a unique solution given by

$$\hat{u} = \phi^{-1}\left(y_d - Cz'(T)\right).$$

Therefore the region the linearised system can be steered to is $\{v \in Y : v - Cz'(T) \in \operatorname{ran}\phi\}$.

In the literature (Magnusson et al., 1985; Carmichael and Quinn, 1988) it is usual to consider the nonlinear problem on this subspace, with some suitable topology defined on it. Following this approach we would therefore have to assume, for the full nonlinear system, that

$$y_d - Cz'(T) - C\int_0^T U(T,s)D(s)N(s,E(s)z(s))\,ds \in \operatorname{ran}\phi.$$

This assumption is too restrictive and so the problem is considered on the whole space Y. In fact in the next subsection we will require that the range of ϕ is a closed subspace of Y. Therefore in some sense we expand the space on which we consider the problem from the range of ϕ to some larger space for which the range remains a closed subspace. Ideally this space can be taken to be $\overline{Y}$ but in general this might not be the case and so we restrict ourselves to a smaller space Y. To do this we must reconsider what we mean by a solution to the control problem.

Suppose that $\phi : U \longrightarrow Y$ is a bounded linear operator. If $y_d - Cz'(T) \notin \operatorname{ran}\phi$ then a solution of the equation

$$y_d - Cz'(T) - \phi u = 0 \tag{2.10}$$

does not exist. Instead the least squares problem of minimising

$$\|\left[y_d - Cz'(T)\right] - \phi u\|_Y \tag{2.11}$$

over all choices of $u \in U$ is posed. In the rabies model the control represents the level of population reduction—either by vaccination or culling—and therefore it seems appropriate to choose the solution of the least squares problem having the smallest norm in U. This control, provided $y_d - Cz'(T) \in \operatorname{ran}\phi + (\operatorname{ran}\phi)^{\perp}$ is given by (Nashed, 1971),

$$\hat{u} = \phi^{\dagger}\left(y_d - Cz'(T)\right)$$

where $\phi^{\dagger}$ is the pseudo, or generalised inverse of ϕ.

Definition 2.2.1. Let $\phi : U \longrightarrow Y$ be a bounded linear operator between Hilbert spaces. The *generalised inverse* $\phi^{\dagger}$ is the linear extension of $\left(\phi|_{(\ker\phi)^{\perp}}\right)^{-1}$ so that its domain of definition is $\operatorname{ran}\phi + (\operatorname{ran}\phi)^{\perp}$ and its null-space is $(\operatorname{ran}\phi)^{\perp}$.

If the range of ϕ is closed in Y then the generalised inverse is defined on the whole of $Y = \operatorname{ran}\phi \oplus (\operatorname{ran}\phi)^{\perp}$.

Remark 2.2.2. The property of the generalised inverse in solving the least squares problem with minimal norm is closely connected with the Hilbert space inner product. While generalised inverses of linear operators between Banach spaces can be defined (see Nashed, 1971,for example) and least squares problems posed in a Banach space setting, the connection between solutions and inverses is lost.

Example 2.2.3. Consider the matrix equation $Ax = b$ where

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

The matrix A is singular with range given by

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Since b is in the range of A there exists an $x \in \mathbb{R}^3$ satisfying the matrix equation. Therefore

$$A^\dagger = \frac{1}{63}\begin{pmatrix} -11 & 19 & 8 \\ 1 & 4 & 5 \\ 26 & -22 & 4 \end{pmatrix} \quad \text{and} \quad A^\dagger b = \frac{1}{21}\begin{pmatrix} 8 \\ 5 \\ 4 \end{pmatrix}.$$

We confirm that $AA^\dagger b = b$ as expected.

Suppose that $y = (1,1,3)^\top$. Then $y \notin \operatorname{ran} A$ and so a solution of the matrix equation $Ax = y$ does not exist. In this case

$$A^\dagger y = \frac{1}{63}\begin{pmatrix} 32 \\ 20 \\ 16 \end{pmatrix} \quad \text{and} \quad AA^\dagger y = \frac{1}{3}\begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix}.$$

The latter is $b + \frac{1}{3}b$ and is, in the sense of least squares, the closest element in the range of A to y.

We make the following assumption that ϕ is bounded.

TV V. *There exists a constant K_1 such that*

$$\|CU(T,0)Bu\|_Y \leq K_1\|u\|_U$$

for all $u \in U$.

The following result from Nashed (1971) characterises when the generalised inverse of a bounded linear operator is itself bounded.

Lemma 2.2.4. *Let $\phi : U \longrightarrow Y$ be a bounded linear operator between two Hilbert spaces. Then the generalised inverse $\phi^\dagger$ is bounded if and only if the range of ϕ is a closed subspace of Y.*

Therefore the following condition ensures that the generalised inverse of ϕ is bounded on Y.

TV VI. *The range of ϕ is a closed subspace of Y.*

The choice of the Hilbert space Y is crucial for the boundedness of the generalised inverse of ϕ. Thus it cannot, in general, be assumed that $C \in \mathcal{L}(Z, Y)$. A method will now be outlined for constructing a suitable space of outputs.

Suppose that there exists some Hilbert space $\overline{Y}$ such that $C \in \mathcal{L}(Z, \overline{Y})$. Then $\phi : U \longrightarrow \overline{Y}$ is a bounded linear operator. If the range of ϕ is a closed subspace of $\overline{Y}$ then set $Y = \overline{Y}$. If the range is not closed a new Hilbert space is sought such that $Y \subset \overline{Y}$ and the range is closed in Y. A topology can be defined on the range of ϕ so that it is a Banach space. We use the construction of Magnusson et al. (1985):

Lemma 2.2.5. *The range of ϕ is a Banach space $R(\phi)$, with a suitably defined norm.*

Proof. We know that ϕ is a bounded linear operator. Define the space X by

$$X := U / \ker \phi.$$

Then $\ker \phi$ is closed, and X is a Banach space under the norm

$$\|[u]\|_X = \inf_{u \in [u]} \|u\|_U = \inf_{\phi \tilde{u} = 0} \|u + \tilde{u}\|_U.$$

We now define $\tilde{\phi} : X \longrightarrow \overline{Y}$ by

$$\tilde{\phi}[u] = \phi \tilde{u} \qquad \tilde{u} \in [u].$$

Then $\tilde{\phi}$ is injective and

$$\|\tilde{\phi}[u]\|_{\overline{Y}} = \|\phi \tilde{u}\|_{\overline{Y}} \leq \|\phi\| \, \|\tilde{u}\|_U, \qquad \tilde{u} \in [u].$$

This is true for every $\tilde{u} \in [u]$ so

$$\|\tilde{\phi}[u]\|_{\overline{Y}} \leq \|\phi\| \, \|[u]\|_X.$$

We now define a norm on the range of $\tilde{\phi}$ by

$$\|v\|_{R(\phi)} := \|\tilde{\phi}^{-1} v\|_X.$$

Let $\mathcal{G}(\tilde{\phi}^{-1})$ denote the graph of $\tilde{\phi}^{-1}$ given by

$$\mathcal{G}(\tilde{\phi}^{-1}) = \left\{ (y, \tilde{\phi}^{-1} y) : y \in D(\tilde{\phi}^{-1}) \right\}$$

and $\| \cdot \|_{\mathcal{G}}$ the corresponding graph norm:

$$\|(y, \tilde{\phi}^{-1} y)\|_{\mathcal{G}} = \|y\|_{\overline{Y}} + \|\tilde{\phi}^{-1} y\|_X.$$

Note that the two norms $\|\cdot\|_{R(\phi)}$ and $\|\cdot\|_{\mathcal{G}}$ are equivalent:

$$\|v\|_{R(\phi)} \leq \|(v, \tilde{\phi}^{-1}v)\|_{\mathcal{G}} \leq (\|\phi\| + 1)\, \|v\|_{R(\phi)}, \qquad v \in R(\phi).$$

Since $\tilde{\phi}$ is bounded and $D(\tilde{\phi})(= X)$ is closed we have that $\tilde{\phi}$ is a closed linear operator. Therefore (Yosida, 1980) $\tilde{\phi}^{-1}$ is also closed.

Let (v_n) be a Cauchy sequence into $R(\phi)$. Since $\|\cdot\|_{R(\phi)}$ and $\|\cdot\|_{\mathcal{G}}$ are equivalent $\left((v_n, \tilde{\phi}^{-1}v_n)\right)$ is a Cauchy sequence into $\overline{Y} \times X$. This is a Banach space since X and $\overline{Y}$ are, and so the latter sequence converges to some $(v, x) \in \overline{Y} \times X$. Since $\tilde{\phi}^{-1}$ is closed we have $v \in D(\tilde{\phi}^{-1}) = R(\phi)$ and $\tilde{\phi}^{-1}v = x$. The equivalence of the norms means that $v_n \longrightarrow v$ in $R(\phi)$ with respect to $\|\cdot\|_{R(\phi)}$. Hence $R(\phi)$ is complete and so a Banach space. □

Note that we have the following Corollary.

Corollary 2.2.6. $\phi \in \mathcal{L}(U, R(\phi))$.

Proof. Let $u \in U$, then

$$\begin{aligned} \|\phi u\|_{R(\phi)} &= \|\tilde{\phi}^{-1}\phi u\|_X = \|\tilde{\phi}^{-1}\tilde{\phi}[u]\|_X \\ &= \|[u]\|_X = \inf_{\tilde{u}\in[u]} \|\tilde{u}\|_U \leq \|u\|_U. \end{aligned}$$

□

The following result gives an important property of any Hilbert space containing ran ϕ into which ϕ is bounded.

Lemma 2.2.7. *Let Y be a Banach space such that $Y \supset \operatorname{ran}\phi$. Then $\phi \in \mathcal{L}(U,Y)$ if and only if the inclusion map $\imath : R(\phi) \longrightarrow Y$ is continuous.*

Proof. Suppose that $\imath$ is continuous. Then there exists a constant M such that

$$\|y\|_Y \leq M\|y\|_{R(\phi)}$$

for all $y \in R(\phi)$. Now from Corollary 2.2.6 we have

$$\|\phi u\|_Y \leq M\|\phi u\|_{R(\phi)} \leq M\|u\|_U,$$

and so $\phi \in \mathcal{L}(U, Y)$.

Conversely, suppose that $\phi \in \mathcal{L}(U, Y)$. Then there exists a constant M such that

$$\|\phi u\|_Y \leq M\|u\|_U,$$

for all $u \in U$. Let $y \in R(\phi)$. Then there exists a $u \in U$ such that $\phi u = y$. Therefore

$$\|y\|_Y = \|\phi\tilde{u}\|_Y \leq M\|\tilde{u}\|_U$$

for all $\tilde{u} \in [u]$ (recall notation from Lemma 2.2.5). Hence

$$\|y\|_Y \leq M\| \, [u] \, \|_X.$$

Now

$$\|y\|_{R(\phi)} = \|\tilde{\phi}^{-1}\phi u\|_X = \| \, [u] \, \|_X$$

and so

$$\|y\|_Y \leq M\|y\|_{R(\phi)}$$

as required. □

Therefore a Hilbert space Y is constructed such that the inclusion $R(\phi) \subset Y$ is continuous and the range of ϕ is closed in Y. Since it might be necessary to restrict the output space to $Y \subseteq \overline{Y}$ it cannot, in general, be assumed that $C \in \mathcal{L}(Z, Y)$.

For the nonlinear problem this suggests that the control given by

$$\hat{u} = \phi^\dagger \left(y_d - Cz'(T) - C\int_0^T U(T,s)D(s)N(s, E(s)z(s))\, ds \right) \tag{2.12}$$

be applied. However this is an implicit expression since the state $z(\cdot)$ is dependent on the control. Suppose, for the moment, that a solution exists, then substituting into (2.7) gives

$$\begin{aligned} z(t) = {} & U(t,0)B\phi^\dagger \left(y_d - Cz'(T) - C\int_0^T U(T,s)D(s)N(s, E(s)z(s))\, ds \right) \\ & + \int_0^t U(t,s)D(s)N(s, E(s)z(s))\, ds \end{aligned}$$

with

$$y = \phi\phi^\dagger y_d + (I - \phi\phi^\dagger) \left[Cz'(T) + C\int_0^T U(T,s)D(s)N(s, E(s)z(s))\, ds \right]. \tag{2.13}$$

Remark 2.2.8. The first term of (2.13), in the case where the range of ϕ is closed, is the orthogonal projection of y_d onto ran ϕ and the second is the orthogonal projection onto ran $\phi^{\perp}$. Therefore, on the range of ϕ, the control drives the system to the desired final state.

In the case where ran ϕ is not closed then the orthogonal projection onto the closure of ran ϕ is the unique continuous extension of $\phi\phi^{\dagger}$ to the whole of Y.

If ϕ is invertible then $\phi^{\dagger} = \phi^{-1}$ and (2.13) reduces to $y = y_d$.

Therefore the controllability problem is reduced to that of finding a fixed point of

$$(\psi w)(t) = E(t)U(t,0)B\phi^{\dagger}\left(y_d - Cz'(T) - C\left(\mathrm{M}_U\mathcal{N}w\right)(T)\right) + E(t)\left(\mathrm{M}_U\mathcal{N}w\right)(t), \qquad (2.14)$$

where

$$w(\cdot) = E(\cdot)z(\cdot) \quad \text{and} \quad \mathcal{N}w(\cdot) = N(\cdot, w(\cdot)).$$

Once a fixed point, $w(\cdot)$, has been found this can be readily substituted into the right hand side of (2.12) to find the control.

To allow for the unboundedness of the nonlinearity further assumptions are required to ensure that ψ is a well-defined map $L^r(0,T;\underline{W}) \longrightarrow L^r(0,T;\underline{W})$.

2.2.2 Unboundedness of nonlinearity

In addition to the existing conditions (TV I–VI) the following are required.

TV VII. *$U(t,s)z \in \underline{Z}$ for all $z \in \underline{Z}$.*

This condition ensures that $E(t)$ can be applied to $U(t,0)z$ for all $t \in [0,T]$ and $z \in \underline{Z}$. Recall though that it cannot be guaranteed that the control will be mapped into $\underline{Z}$ by B (since $B \in \mathcal{L}(U, Z)$). The next condition ensures that $E(\cdot)U(\cdot,0)$ can be extended to a bounded linear operator on Z.

TV VIII. *There exists a positive constant K_2 such that*

$$\|E(\cdot)U(\cdot,0)z\|_{L^r(0,T;\underline{W})} \leq K_2\|z\|_Z,$$

for each $z \in \underline{Z}$.

Remark 2.2.9. For each $z \in Z$ we can take a sequence $(z_n)_{n=1}^{\infty}$ into $\underline{Z}$ such that $\|z_n - z\|_Z \to 0$. This assumption means that a bounded extension of $E(\cdot)U(\cdot, 0)$ can be defined on Z by setting

$$E(\cdot)U(\cdot,0)z = \lim_{n\to\infty} E(\cdot)U(\cdot,0)z_n$$

for $z \in Z$.

For $h \in L^s(0,T;\overline{W})$ define

$$(\mathbb{L}_U h)(t) = E(t)\int_0^t U(t,s)D(s)h(s)\,ds. \tag{2.15}$$

We assume that the operator $\mathbb{L}_U : L^s(0,T;\overline{W}) \longrightarrow L^r(0,T;\underline{W})$ is bounded:

TV IX. *There exists a positive constant K_3 such that*

$$\|\mathbb{L}_U h\|_{L^r(0,T;\underline{W})} \leq K_3 \|h\|_{L^s(0,T;\overline{W})},$$

for $h \in L^s(0,T;\overline{W})$.

Recall that, in general, it cannot be assumed that $C \in \mathcal{L}(Z,Y)$. Instead it will be assumed that by restricting C to a subset of Z, with some suitable topology, it is bounded. To ensure that the map $h \mapsto C\,(\mathbb{M}_U h)\,(T)$ is bounded from $L^s(0,T;\overline{W})$ to Y the following assumption is made.

TV X. *$V \subset Z$ is a Banach space such that $C \in \mathcal{L}(V,Y)$. There exists a constant K_4 such that*

$$\|\,(\mathbb{M}_U h)\,(T)\|_V \leq K_4 \|h\|_{L^s(0,T;\overline{W})}$$

for all $h \in L^s(0,T;\overline{W})$.

Now all that remains is to construct a fixed point of ψ. In the next subsection we apply a version of the Contraction Mapping Theorem to obtain a unique fixed point.

2.2.3 Application of a fixed point theorem

We are now in a position to state and prove the main result of this chapter. The idea of the proof is to show that ψ is a contraction on a certain ball about the origin. Then the Contraction Mapping Theorem is applied to show the existence of a unique solution to our control problem.

Theorem 2.2.10. *Consider the nonlinear system governed by*

$$z(t) = U(t,0)Bu + \int_0^t U(t,s)D(s)N(s, E(s)z(s))\, ds \tag{2.16}$$

with output

$$y = Cz(T) + Cz'(T) \tag{2.17}$$

where $Cz'(T)$ is a given output. Suppose that the following conditions are satisfied:

(i) Assumptions TV I–X hold.

(ii) For all $w(\cdot) \in L^r(0,T;\underline{W})$ we have $N(\cdot, w(\cdot)) = \mathcal{N}w(\cdot) \in L^s(0,T;\overline{W})$. Furthermore, $N : [0,T] \times \underline{W} \longrightarrow \overline{W}$ satisfies the following Lipschitz condition on the ball of radius a' about the origin, $B_{a'}$:

$$\|\mathcal{N}w_1 - \mathcal{N}w_2\|_{L^s(0,T;\overline{W})} \le k(\|w_1\|, \|w_2\|)\|w_1 - w_2\|_{L^r(0,T;\underline{W})} \tag{2.18}$$

for each $w_1, w_2 \in B_{a'}$ and some continuous symmetric function $k(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $k(0,0) = 0$.

(iii) Choose $a \le a'$ such that

$$\left(K_2\|B\|\,\|\phi^\dagger\|\,\|C\|K_4 + K_3\right)\tilde{K} = K < 1 \tag{2.19}$$

where

$$\tilde{K} = \sup_{0 \le \theta_1, \theta_2 \le a} k(\theta_1, \theta_2).$$

Suppose that $y_d \in Y$ satisfies

$$\|y_d - Cz'(T)\|_Y \le \frac{a\,(1-K)}{K_2\|B\|\,\|\phi^\dagger\|}. \tag{2.20}$$

Then there exists a control $\tilde{u}$ (2.12) which drives the output (2.17) to y_d on the range of ϕ.

Proof. We first show that ψ is a contraction on B_a :

$$\begin{aligned}\|\psi w_1 - \psi w_2\|_{L^r(0,T;\underline{W})} &\leq \|E(\cdot)U(\cdot,0)B\phi^\dagger C\left(\mathbb{M}_U(\mathcal{N}w_2 - \mathcal{N}w_1)\right)(T)\|_{L^r(0,T;\underline{W})} \\ &\quad + \|\mathbb{L}_U(\mathcal{N}w_1 - \mathcal{N}w_2)\|_{L^r(0,T;\underline{W})} \\ &\leq \left(K_2\|B\|\,\|\phi^\dagger\|\,\|C\|K_4 + K_3\right)\tilde{K}\,\|w_1 - w_2\|_{L^r(0,T;\underline{W})}.\end{aligned}$$

Therefore by (2.19) ψ is a contraction on B_a.

Since we have included our initial guess for the control in the initial state for z', it seems natural to consider the iterative process given by $w_n = \psi w_{n-1}$ with $w_0 = 0$. Then we have

$$w_1 = E(\cdot)U(\cdot,0)B\phi^\dagger\left(y_d - Cz'(T)\right).$$

Let

$$\begin{aligned}S = \Big\{w \in L^r(0,T;\underline{W}) : \|w(\cdot) - E(\cdot)U(\cdot,0)B\phi^\dagger\left(y_d - Cz'(T)\right)\|_{L^r(0,T;\underline{W})} \\ \leq \frac{K}{1-K}\|E(\cdot)U(\cdot,0)B\phi^\dagger\left(y_d - Cz'(T)\right)\|_{L^r(0,T;\underline{W})}\Big\}.\end{aligned}$$

This will be contained in the ball of radius a if

$$\left(1 + \frac{K}{1-K}\right)\|E(\cdot)U(\cdot,0)B\phi^\dagger\left(y_d - Cz'(T)\right)\|_{L^r(0,T;\underline{W})} \leq a.$$

This will be the case if

$$\frac{1}{1-K}K_2\|B\|\,\|\phi^\dagger\|\,\|y_d - Cz'(T)\|_Y \leq a.$$

Rearranging we have

$$\|y_d - Cz'(T)\|_Y \leq \frac{a(1-K)}{K_2\|B\|\,\|\phi^\dagger\|}.$$

Now applying the Contraction Mapping Theorem (Theorem A.1) proves the existence of a unique fixed point for ψ. Substituting this fixed point into (2.12) gives the control $\tilde{u}$. □

Note that the output resulting from applying the control given by the last theorem is only guaranteed to coincide with the desired state on the range of ϕ. If there exists a control $\tilde{u}$ such that the output, when applying this control, is the desired state, then it remains an open question whether the previous theorem gives the same control.

The proof of Theorem 2.2.10 provides an iterative scheme for obtaining the fixed point (and hence the control). In the next section a method of finding this fixed point is sought that is easier to implement computationally. The proof also provides an estimate (2.20) for a region of possible targets.

2.3 Computing control by iteration

The constructive method of the proof of Theorem 2.2.10 can be used to find the control which solves the control problem. The iterative scheme used involves integral equations and not the differential equations of the original model. In this section an alternative method for obtaining the control which gives rise to the solution of the fixed point problem is constructed. The method exploits the original dynamics and so is easier to solve numerically. A further advantage of the scheme is that it produces the desired control that gives rise to the fixed point without the need for further substitution.

The method used in this section is as follows: Consider the dynamical equations

$$z_n(t) = U(t,0)Bu_n + \left(\mathbb{M}_U \mathcal{N} w_n\right)(t)$$

where (as before) $w_n(\cdot) = E(\cdot)z_n(\cdot)$ and $\mathcal{N}w_n(\cdot) = N(\cdot, w_n(\cdot))$. Suppose that the control u_{n+1} is defined in terms of the previous control as follows

$$u_{n+1} = u_n + v(n)$$

and for each $n \in \mathbb{N}$ there exists a solution of the dynamical equation, ignoring for the moment the possible unboundedness of the nonlinearity. Note that

$$y_{n+1} - y_d + y_d - y_n = y_{n+1} - y_n = \phi v(n) + C\mathbb{M}_U \left(\mathcal{N}w_{n+1} - \mathcal{N}w_n\right).$$

Assume that the scheme converges and so $y_{n+1} - y_d \to 0$ and $w_{n+1} - w_n \to 0$ as $n \to \infty$. This suggests that $v(\cdot)$ is chosen to be

$$v(n) = \phi^{\dagger}\left(y_d - y_n\right).$$

Again, to allow for the possible unboundedness of the nonlinearity, suitable conditions will be imposed so that the system equation is well-defined. We have not shown that each of the system equations has a solution for any n and further conditions are required for this. Finally it is shown that the iterative scheme converges under certain conditions.

2.3.1 Existence of solutions

Let $U(t,s)$ be a mild evolution operator on Z and consider the following sequence of system equations

$$z_n(t) = U(t,0)Bu_n + (\mathbb{M}_U\mathcal{N}w_n)(t) \tag{2.21}$$

where $u_n \in U$, a Hilbert space, with output

$$y_n = Cz_n(T) + Cz'(T) \tag{2.22}$$

in some Hilbert space Y for $n \in \mathbb{N}$. The purpose of this subsection is to show that, under suitable conditions, (2.21) has a solution for each $n \in \mathbb{N}$. To allow for the possible unboundedness of the nonlinearity the setting for this section will be that of the previous one, namely that provided by TV I.

Assuming that TV II is satisfied ensures that the integrand in (2.21) now makes sense in $\overline{Z}$ for $w_n(\cdot) \in L^r(0,T;\underline{W})$ and integrability follows if TV III holds. Recall that these conditions ensure that the operator $(\mathbb{M}_U\cdot)$ defined in Section 2.1 is well-defined. The system equations are required to be continuous with respect to the norm in Z and so it is assumed that TV IV is also satisfied.

To show the existence of each solution the Contraction Mapping Theorem will be applied to the operator Ψ given by

$$(\Psi w_n)(t) = E(t)U(t,0)Bu_n + E(t)(\mathbb{M}_U\mathcal{N}w_n)(t). \tag{2.23}$$

Suppose that TV VII–IX are satisfied. Then Ψ can be considered as a bounded operator from $L^r(0,T;\underline{W})$ to itself.

Theorem 2.3.1. *For each $n \in \mathbb{N}$ consider the nonlinear system governed by*

$$z_n(t) = U(t,0)Bu_n + \int_0^t U(t,s)D(s)N(s,E(s)z_n(s))(s)\,ds. \tag{2.24}$$

Suppose that the following conditions are satisfied:

(i) Assumptions TV I–IV and TV VII–IX hold.

(ii) $N : [0,T] \times \underline{W} \longrightarrow \overline{W}$ where $\underline{W} \subset \overline{W}$ are Banach spaces satisfies the following:

(a) For every $w(\cdot) \in L^r(0,T;\underline{W})$ we have $N(\cdot, w(\cdot)) \in L^s(0,T;\overline{W})$, where $r, s \geq 1$ are real numbers;

(b) On the ball of radius a' about the origin, $B_{a'}$,

$$\|\mathcal{N}w_1 - \mathcal{N}w_2\|_{L^s(0,T;\overline{W})} \leq k(\|w_1\|, \|w_2\|)\|w_1 - w_2\|_{L^r(0,T;\underline{W})} \tag{2.25}$$

for each $w_1, w_2 \in B_{a'}$ and some continuous symmetric function $k(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $k(0,0) = 0$.

(iii) Choose $a \leq a'$ such that

$$K_3\tilde{K} = K < 1 \tag{2.26}$$

where

$$\tilde{K} = \sup_{0 \leq \theta_1, \theta_2 \leq a} k(\theta_1, \theta_2).$$

Suppose that $u_n \in U$ satisfies

$$\|u_n\|_U \leq \frac{a\left(1 - K_3\tilde{K}\right)}{K_2\|B\|}. \tag{2.27}$$

Then there exists a solution z_n of (2.24) such that $E(\cdot)z_n(\cdot) \in L^r(0,T;\underline{W})$.

Proof. Let $n \in \mathbb{N}$ be fixed. We first show that Ψ is a contraction on the ball B_a. Let $w^1, w^2 \in L^r(0,T;\underline{W})$, then

$$\begin{aligned}\|\Psi w^1 - \Psi w^2\|_{L^r(0,T;\underline{W})} &= \|E(\cdot)\left(\mathrm{M}_U\mathcal{N}w^1\right)(\cdot) - E(\cdot)\left(\mathrm{M}_U\mathcal{N}w^2\right)(\cdot)\|_{L^r(0,T;\underline{W})} \\ &\leq K_3\tilde{K}\|w^1 - w^2\|_{L^r(0,T;\underline{W})}.\end{aligned}$$

Therefore by (2.26) Ψ is a contraction.

Now we define the ball S needed for the Contraction Mapping Theorem and show that it is contained in B_a. Let $w_n^0 = 0$. Then $w_n^1(\cdot) = E(\cdot)U(\cdot,0)Bu_n$ and so the ball is given by

$$S = \left\{ w : \|w(\cdot) - E(\cdot)U(\cdot,0)Bu_n\|_{L^r(0,T;\underline{W})} \leq \frac{K_3\tilde{K}}{1 - K_3\tilde{K}}\|E(\cdot)U(\cdot,0)Bu_n\|_{L^r(0,T;\underline{W})} \right\}.$$

The ball will be contained in B_a if

$$\left(1 + \frac{K_3\tilde{K}}{1 - K_3\tilde{K}}\right) \|E(\cdot)U(\cdot,0)Bu_n\|_{L^r(0,T;\underline{W})} \leq a$$

which will certainly be the case if

$$\frac{1}{1 - K_3\tilde{K}} K_2\|B\| \, \|u_n\|_U \leq a$$

that is, if

$$\|u_n\|_U \leq \frac{a\left(1 - K_3\tilde{K}\right)}{K_2\|B\|}.$$

This is condition (2.27) and so applying the Contraction Mapping Theorem we have the existence of a unique fixed point w_n of Ψ. Substituting this fixed point into the right hand side of (2.24) gives a solution z_n as required. □

Remark 2.3.2. Whilst condition (2.26) is weaker than that required in the previous section for a contraction (2.19), further conditions are required for the iterative scheme to converge.

The theorem guarantees the existence of a solution for each n provided the input u_n satisfies (2.27). Each input is defined in terms of the previous one as follows: $u_0 = 0$ and for all $n > 0$

$$u_n = u_{n-1} + \phi^\dagger \left(y_d - y_{n-1}\right). \tag{2.28}$$

To ensure that this iterative formula is well-defined, it will be assumed that TV V and TV VI hold. Thus $\phi : U \longrightarrow Y$ is a bounded linear operator with a well-defined and bounded generalised inverse $\phi^\dagger$.

The next result ensures that there is a solution to (2.21) for each n.

Corollary 2.3.3. *Suppose that the conditions of Theorem 2.3.1 hold. In addition suppose that the following are satisfied.*

(i) *Conditions TV V, TV VI, and TV X hold.*

(ii) *For each $n \geq 1$ let*

$$u_n = u_{n-1} + \phi^\dagger (y_d - y_{n-1})$$

where $u_0 = 0$.

(iii) *y_d satisfies*

$$\|y_d - Cz'(T)\|_Y \leq \frac{a(1-K)}{K_2\|B\|\,\|\phi^\dagger\|}, \tag{2.29}$$

where $K = \tilde{K}\left(K_2\|B\|\,\|\phi^\dagger\|\,\|C\|K_4 + K_3\right)$.

Then for each $n \in \mathbb{N}$ there exists a solution z_n of (2.16).

Proof. From Theorem 2.3.1 it is only necessary to show that u_n satisfies (2.27) for each $n \in \mathbb{N}$. First note that

$$\begin{aligned}
u_n &= u_{n-1} + \phi^\dagger (y_d - y_{n-1}(T)) \\
&= u_{n-1} + \phi^\dagger (y_d - Cz'(T) - C(\mathbb{M}_U \mathcal{N} w_{n-1})(T) - \phi u_{n-1}) \\
&= \left(1 - \phi^\dagger\phi\right) u_{n-1} + \phi^\dagger (y_d - Cz'(T) - C(\mathbb{M}_U \mathcal{N} w_{n-1})(T)) \\
&= \left(1 - \phi^\dagger\phi\right) u_{n-2} + \phi^\dagger (y_d - Cz'(T) - C(\mathbb{M}_U \mathcal{N} w_{n-1})(T)) \\
&= \ldots = \phi^\dagger (y_d - Cz'(T) - C(\mathbb{M}_U \mathcal{N} w_{n-1})(T)).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|u_n\| &\leq \|\phi^\dagger\| \left(\|y_d - Cz'(T)\|_Y + \|C(\mathbb{M}_U \mathcal{N} w_{n-1})(T)\|_Y\right) \\
&\leq \|\phi^\dagger\| \left(\frac{a\left(1 - K + \tilde{K}K_2\|B\|\,\|\phi^\dagger\|\,\|C\|K_4\right)}{K_2\|B\|\,\|\phi^\dagger\|}\right) \\
&= \frac{a\left(1 - K_3\tilde{K}\right)}{K_2\|B\|}
\end{aligned}$$

and since $u_0 = 0$ condition (2.27) is satisfied for all $n \geq 0$ as required. □

Remark 2.3.4. For $n = 0$ applying the iterative method of Theorem 2.3.1 we start with $w^0 = 0$. But then $\Psi(0) = E(\cdot)U(\cdot, 0)Bu_0 = 0$ and so the fixed point $w_0 = 0$.

The conditions required for the existence of solutions to (2.21) are the same as those required in the last two sections for the fixed point theorem approach. The only difference is (2.26); so far the iterative scheme of this section can be considered on a larger ball than the fixed point approach. Now it only remains to show that the iterative scheme converges.

2.3.2 Convergence of iterative scheme

In this subsection we consider the important questions of whether the sequence of controls used by the iterative scheme converge; and if they do, whether they converge to the same control as given by the fixed point approach.

To ensure that the sequence of controls converge we must restrict the ball considered in Theorem 2.3.1 to that considered in the previous section for the fixed point approach. The answer to the second question question is yes. Thus the iterative scheme approach provides a good method for implementation while still yielding the same results as the fixed point approach.

Theorem 2.3.5. *Suppose that the conditions of Corollary 2.3.3 are satisfied. If in addition we restrict the ball B_a so that*

$$\left(K_2\|B\|\,\|\phi^\dagger\|\,\|C\|K_4 + K_3\right)\tilde{K} < 1,$$

where

$$\tilde{K} = \sup_{0\leq\theta_1,\theta_2\leq a} k(\theta_1, \theta_2),$$

then the sequence of controls given by the iterative scheme considered in this section converges. Moreover the limit is the same as that given by the fixed point approach of Section 2.2.

Proof. By restricting ourselves to the ball on which

$$\left(K_2\|B\|\,\|\phi^\dagger\|\,\|C\|K_4 + K_1\right)\tilde{K} < 1$$

we can apply the results of the last subsection to show that the iterative scheme is well-defined on B_a in the sense that for each $n \geq 0$ there exists a solution of (2.21). In particular for each solution $E(\cdot)z_n(\cdot)$ is unique.

From the proof of Corollary 2.3.3 we have

$$\begin{aligned}||u_m - u_n||_U &= ||\phi^\dagger [C (\mathbb{M}_U \mathcal{N} w_{n-1}) (T)] - \phi^\dagger [C (\mathbb{M}_U \mathcal{N} w_{m-1}) (T)] ||_U \\ &\leq ||\phi^\dagger|| \, ||C|| K_4 \tilde{K} \, ||w_{m-1} - w_{n-1}||_{L^r(0,T;\underline{W})}\end{aligned}$$

and so if the sequence of solutions $(w_n)_{n \in \mathbb{N}}$ converges on the ball B_a then the sequence of controls also converges. Now we have

$$\begin{aligned}||w_m - w_n||_{L^r(0,T;\underline{W})} &= ||E(\cdot)U(\cdot,0)B(u_m - u_n) + E(\cdot)(\mathbb{M}_U(\mathcal{N}w_m - \mathcal{N}w_n))(\cdot)||_{L^r(0,T;\underline{W})} \\ &\leq K_2||B|| \, ||u_m - u_n||_U + K_3\tilde{K}||w_m - w_n||_{L^r(0,T;\underline{W})} \\ &\leq K_2||B|| \, ||\phi^\dagger|| \, ||C||K_4\tilde{K}||w_{m-1} - w_{n-1}||_{L^r(\underline{W})} + K_3\tilde{K}||w_m - w_n||_{L^r(\underline{W})} \\ &\leq \frac{K_2||B|| \, ||\phi^\dagger|| \, ||C||K_4\tilde{K}}{1 - K_3\tilde{K}}||w_{m-1} - w_{n-1}||_{L^r(0,T;\underline{W})}.\end{aligned}$$

On the ball B_a we have $\left(K_2||B|| \, ||\phi^\dagger|| \, ||C||K_4 + K_3\right)\tilde{K} < 1$ which implies that

$$\frac{K_2||B|| \, ||\phi^\dagger|| \, ||C||K_4\tilde{K}}{1 - K_3\tilde{K}} < 1$$

and so the sequence of solutions converges as $n \to \infty$.

Now

$$u_n = \phi^\dagger (y_d - Cz'(T) - C(\mathbb{M}_U \mathcal{N} w_{n-1})(T))$$

and so letting $n \to \infty$, since $\phi^\dagger$ and $C(\mathbb{M}_U \cdot)(T)$ are bounded, the limit is given by

$$u = \phi^\dagger (y_d - Cz'(T) - C(\mathbb{M}_U \mathcal{N} w)(T)).$$

But then w (the limit of the sequence $(w_n)_{n \in \mathbb{N}}$) satisfies

$$w(\cdot) = E(\cdot)U(\cdot,0)B\phi^\dagger (y_d - Cz'(T) - C(\mathbb{M}_U \mathcal{N} w)(T)) + E(\cdot)(\mathbb{M}_U \mathcal{N} w)(\cdot)$$

and so is a fixed point of ψ. Hence the iterative and fixed point schemes converge to the same w and yield the same control. □

An advantage of the iterative scheme over the fixed point one is that only y_n is needed after each iteration for the next one. This makes the iterative scheme more desirable for computation as only one value of the output needs to be stored at each step rather than the whole output. The iterative scheme also makes use of the original dynamics

$$\dot{\overline{z}}(t) = f(t, \overline{z}(t)), \qquad \overline{z}(0) = z_0 + B\overline{u}. \tag{2.30}$$

Suppose that an initial guess, u', is made for the control. A further control u is then sought such that, if $\overline{z}(\cdot)$ is a solution of (2.30) with initial state

$$\overline{z}(0) = z_0 + Bu' + Bu,$$

then

$$C\overline{z}(T) = y_d$$

for some fixed T and y_d. The differential equation (2.30) is repeatedly solved numerically (via a computer) for a sequence of controls $(u_n)_{n\in\mathbb{N}}$. The controls are related via the iterative scheme so that the n^{th} control, u_n is given by

$$u_n = u_{n-1} + \phi^{\dagger}(y_d - y_{n-1})$$

and the dynamics are solved for the n^{th} time with initial state

$$\overline{z}_n(0) = z_0 + Bu' + Bu_{n-1} + B\phi^{\dagger}\,(y_d - y_{n-1})$$

with $u_0 = 0$. The sequence of outputs $y_n = C\overline{z}(T)$ will then converge, at least on the range of ϕ, to the desired target y_d.

2.4 Choosing the target output

The previous sections have shown how it is possible to drive certain systems to a specified output in a desired time. Unfortunately it can only be guaranteed that the actual output of the system is equal to the desired one on the range of $\phi = CU(T, 0)B$.

In the case of the rabies model it is the actual output achieved that is crucial. The aim of a control strategy is to reduce the infected fox population density sufficiently for rabies to be eradicated. Given an initial guess and the resulting trajectory with output $Cz'(T)$, condition (2.20) determines a region of possible target states y_d. Therefore one is now faced with the problem of how to choose the target output y_d so that the actual, achieved output, y is minimised.

Before performing an analysis of this problem some preliminary results that help in determining the best choice for y_d are presented. Throughout this section the system defined by

$$\begin{aligned} z(t) &= U(t,0)Bu + \int_0^t U(t,s)D(s)N(s,E(s)z(s))\,ds \\ y &= Cz'(T) + Cz(T) \end{aligned} \tag{2.31}$$

where $Cz'(T)$ is given, is considered. It will be assumed that the conditions of Theorem 2.2.10 (or equivalently Theorem 2.3.5) are satisfied.

2.4.1 Some preliminary results

The first result demonstrates that it is the orthogonal projection of y_d onto the range of ϕ that is important when choosing the target.

Proposition 2.4.1. *Suppose that the target outputs y_d^1 and y_d^2 satisfy (2.20) and give rise to the actual outputs y_1 and y_2. If $y_d^1 = y_d^2$ on the range of ϕ then $y_1 = y_2$.*

Proof. The actual outputs are given by ($i \in \{1,2\}$)

$$y_i = Cz'(T) + \phi\phi^\dagger\left(y_d^i - Cz'(T) - C\left(\mathbb{M}_U\mathcal{N}w_i\right)(T)\right) + C\left(\mathbb{M}_U\mathcal{N}w_i\right)(T)$$

where w_i is the unique fixed point of

$$\psi_i(w) = E(\cdot)U(\cdot,0)B\phi^\dagger\left(y_d^i - Cz'(T) - C\left(\mathbb{M}_U\mathcal{N}w\right)(T)\right) + E(\cdot)\left(\mathbb{M}_U\mathcal{N}w\right)(\cdot).$$

Now

$$\begin{aligned} \phi^\dagger y_d^1 &= \phi^\dagger\left(\phi\phi^\dagger y_d^1 + \left(1 - \phi\phi^\dagger\right)y_d^1\right) \\ &= \phi^\dagger\left(\phi\phi^\dagger y_d^2 + \left(1 - \phi\phi^\dagger\right)y_d^1\right) \\ &= \phi^\dagger y_d^2. \end{aligned}$$

Therefore

$$E(\cdot)U(\cdot,0)B\phi^{\dagger}y_d^1 = E(\cdot)U(\cdot,0)B\phi^{\dagger}y_d^2$$

and so $\psi_1(w) = \psi_2(w)$ for all $w \in L^r(0,T;\underline{W})$. In particular a fixed point of ψ_1 is also a fixed point of ψ_2 and so $w_1 = w_2$. Hence we see from the above that $y_1 = y_2$. □

The orthogonal projection of the target output onto ran $\phi^{\perp}$ does not affect the actual output. Therefore one may choose this part of the target to simplify our analysis. As a consequence of this set $\left(1 - \phi\phi^{\dagger}\right) y_d = \left(1 - \phi\phi^{\dagger}\right) Cz'(T)$. This does not affect the actual output but does imply that $y_d - Cz'(T) \in \operatorname{ran} \phi$.

Remark 2.4.2. An important consequence of $y_d - Cz'(T) \in \operatorname{ran} \phi$, given in Pritchard (1981), is

$$\phi^{\dagger}\left(y_d - Cz'(T)\right) = \phi^{*}\lambda,$$

where ϕ^{*} is the adjoint operator of ϕ and

$$\phi\phi^{*}\lambda = \left(y_d - Cz'(T)\right).$$

In particular if $\phi\phi^{*}$ is invertible $\phi^{\dagger}\left(y_d - Cz'(T)\right) = \phi^{*}\left(\phi\phi^{*}\right)^{-1}\left(y_d - Cz'(T)\right)$.

An easy consequence of choosing y_d such that $y_d - Cz'(T) \in \operatorname{ran} \phi$ is given in the next result.

Lemma 2.4.3. *Suppose that y_d is chosen so that $\left(1 - \phi\phi^{\dagger}\right) y_d = \left(1 - \phi\phi^{\dagger}\right) Cz'(T)$. Then the actual output y satisfies*

$$\|y - y_d\| = \|\left(1 - \phi\phi^{\dagger}\right) C\left(\mathrm{M}_U \mathcal{N} w\right)(T)\|.$$

Proof. We know that

$$\begin{aligned} y - y_d &= Cz'(T) + \phi\phi^{\dagger}\left(y_d - Cz'(T) - C\left(\mathrm{M}_U \mathcal{N} w\right)(T)\right) + C\left(\mathrm{M}_U \mathcal{N} w\right)(T) - y_d \\ &= -\left(1 - \phi\phi^{\dagger}\right)\left(y_d - Cz'(T) - C\left(\mathrm{M}_U \mathcal{N} w\right)(T)\right) \\ &= \left(1 - \phi\phi^{\dagger}\right) C\left(\mathrm{M}_U \mathcal{N} w\right)(T) \end{aligned}$$

as required. □

As a result of this lemma note that

$$\|y(T)\| \leq \|y_d\| + \|\left(1 - \phi\phi^\dagger\right) C\left(\mathbb{M}_U \mathcal{N} w\right)(T)\|.$$

The next result gives an estimate for the second term on the right hand side in terms of the difference $y_d - Cz'(T)$. Coupled with the previous lemma this result is crucial in demonstrating the trade-off that is required in the minimisation.

Lemma 2.4.4. *Let y_d be the target output (satisfying (2.20)) that gives rise to the solution w of the fixed point problem. Then there exists a constant M_1 such that*

$$\|C\left(\mathbb{M}_U \mathcal{N} w\right)(T)\| \leq M_1 \|y_d - Cz'(T)\|. \tag{2.32}$$

Proof. The fixed point is the limit of the iterative sequence $w_n = \psi(w_{n-1})$ with $w_0 = 0$. Therefore

$$\begin{aligned}
\|w_n\| &= \|E(\cdot)U(\cdot,0)B\phi^\dagger\left(y_d - Cz'(T) - C\left(\mathbb{M}_U \mathcal{N} w_{n-1}\right)(T)\right) + E(\cdot)\left(\mathbb{M}_U \mathcal{N} w_{n-1}\right)(\cdot)\| \\
&\leq K_2\|B\|\,\|\phi^\dagger\|\,\|y_d - Cz'(T)\| + \left(K_2\|B\|\,\|\phi^\dagger\|\,\|C\|K_4 + K_3\right)\tilde{K}\|w_{n-1}\| \\
&= K_2\|B\|\,\|\phi^\dagger\|\,\|y_d - Cz'(T)\| + K\|w_{n-1}\| \\
&\leq (1+K)\,K_2\|B\|\,\|\phi^\dagger\|\,\|y_d - Cz'(T)\| + K\|w_{n-2}\|
\end{aligned}$$

Thus by induction (since $w_0 = 0$) we have

$$\|w_n\| \leq \left(\sum_{i=0}^{n-1} K^i\right) K_2\|B\|\,\|\phi^\dagger\|\,\|y_d - Cz'(T)\|.$$

Therefore

$$\|C\left(\mathbb{M}_U \mathcal{N} w_n\right)(T)\| \leq \left(\sum_{i=0}^{n-1} K^i\right) \|C\|K_4\tilde{K}K_2\|B\|\,\|\phi^\dagger\|\,\|y_d - Cz'(T)\|$$

and letting $n \to \infty$ this converges to

$$\|C\left(\mathbb{M}_U \mathcal{N} w\right)(T)\| \leq \frac{\tilde{K}\|C\|K_4K_2\|B\|\,\|\phi^\dagger\|}{1-K}\|y_d - Cz'(T)\|.$$

This completes the proof. □

From this result and Lemma 2.4.3 it is seen that by choosing y_d close to $Cz'(T)$ on the range of ϕ reduces the actual output on ran $\phi^\perp$. But then on ran ϕ it is known that $y = y_d$ and so one should choose y_d so that $\|\phi\phi^\dagger y_d\|$ is much less than $\|\phi\phi^\dagger Cz'(T)\|$.

The question of how to choose y_d to achieve a balance of these two effects is the topic of the next subsection.

2.4.2 Minimising the actual output

Using the estimates in Lemma 2.4.3 and Lemma 2.4.4 it is seen that

$$||y|| \leq ||y_d|| + ||\left(1-\phi\phi^\dagger\right)||\left(\frac{\tilde{K}K_2||B||\,||\phi^\dagger||\,||C||K_4}{1-K}\right)||y_d - Cz'(T)||.$$

The problem is to minimise the right-hand side, subject to the constraint

$$||y_d - Cz'(T)|| \leq \frac{a\,(1-K)}{K_2||B||\,||\phi^\dagger||}.$$

Squaring the first of these equations gives

$$\begin{aligned} ||y||^2 \leq\ & 2||\phi\phi^\dagger y_d||^2 + 2||\left(1-\phi\phi^\dagger\right)y_d||^2 \\ & + 2||\left(1-\phi\phi^\dagger\right)||^2\left(\frac{\tilde{K}K_2||B||\,||\phi^\dagger||\,||C||K_4}{1-K}\right)^2 ||y_d - Cz'(T)||^2. \end{aligned}$$

The only choice open to us for the purposes of the optimisation is $\phi\phi^\dagger y_d$.

Proposition 2.4.5. *Let $\alpha, \beta \in \operatorname{ran}\phi$. Consider the optimisation problem of minimising, over all choices of α,*

$$||\alpha||^2 + \frac{m^2}{c^2}||\alpha - \beta||^2 \tag{2.33}$$

subject to the constraint

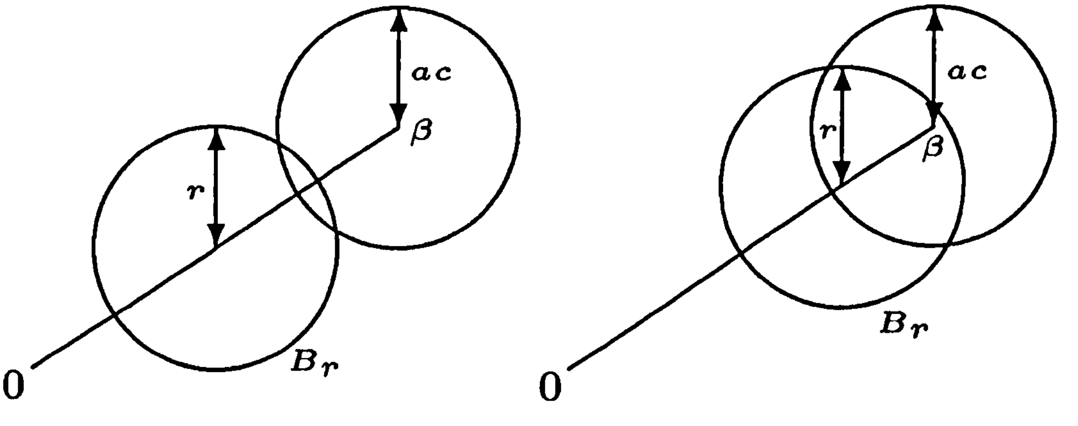
$$||\alpha - \beta|| \leq ac. \tag{2.34}$$

(i) *If $||\beta|| \leq \frac{a}{c}\left(m^2+c^2\right)$ then (2.33) is minimised by $\alpha = \frac{m^2}{m^2+c^2}\beta$. The minimum of (2.33) is*

$$\left(\frac{m^2}{m^2+c^2}\right)||\beta||^2.$$

(ii) *If $||\beta|| > \frac{a}{c}\left(m^2+c^2\right)$ then (2.33) is minimised by $\alpha = \frac{||\beta||-ac}{||\beta||}\beta$. The minimum of (2.33) is*

$$||\beta||^2 - 2ac||\beta|| + a^2(m^2+c^2).$$



(a) $\left(\frac{m^2}{m^2+c^2}\right)\beta \notin B(\beta)$ (b) $\left(\frac{m^2}{m^2+c^2}\right)\beta \in B(\beta)$

Figure 2.1: Geometric interpretation of minimisation problem: The problem is to minimise the radius r such that the intersection of the balls is non-empty.

Proof. We start by completing the square for α in (2.33)

$$\frac{m^2+c^2}{c^2}\|\alpha - \left(\frac{m^2}{m^2+c^2}\right)\beta\|^2 + \left(\frac{m^2}{m^2+c^2}\right)\|\beta\|^2.$$

Therefore to minimise this we need to minimise

$$\|\alpha - \left(\frac{m^2}{m^2+c^2}\right)\beta\|^2,$$

subject to the constraint (2.34). We can think of this geometrically as follows (see Figure 2.1): Let $B(\beta)$ be the (closed) ball of radius ac. Let B_r be the (closed) ball of radius r about the point $\left(\frac{m^2}{m^2+c^2}\right)\beta$. Then the minimisation problem is to find the smallest r such that $B_r \cap B(\beta) \neq \emptyset$. Any α in the intersection is a solution of the minimisation problem.

There are two cases to consider here: Whether the origin of B_r lies in the ball $B(\beta)$ or outside of it. Whether this point is in the ball or not is characterised by

$$\|\left(\frac{m^2}{m^2+c^2} - 1\right)\beta\| = \frac{c^2}{m^2+c^2}\|\beta\|$$

being less than ac. This is the case when

$$\|\beta\| \leq \frac{a\,(m^2+c^2)}{c}. \tag{2.35}$$

(i) If (2.35) holds then the origin of B_r lies in the ball $B(\beta)$. In particular the smallest value of r is zero and the solution is

$$\alpha = \left(\frac{m^2}{m^2+c^2}\right)\beta.$$

Therefore the minimum of (2.33) is

$$\left(\frac{m^2}{m^2+c^2}\right)\|\beta\|^2.$$

(ii) If (2.35) does not hold then the origin does not lie in $B(\beta)$. In this case we can shrink B_r until the balls are just touching. The point of contact, which lies on the line joining the two centres, is the solution. Now since $\left(\frac{m^2}{m^2+c^2}\right)$ is a scalar, the line joining the two centres also goes through the origin. Hence the point of intersection will be of the form $\lambda\beta$ for some scalar $0 < \lambda < 1$. Then

$$\|\alpha - \left(\frac{m^2}{m^2+c^2}\right)\beta\|^2 = \left(\frac{(\lambda-1)m^2+\lambda c^2}{m^2+c^2}\right)^2\|\beta\|^2$$

and

$$(1-\lambda)\|\beta\| = ac \quad \Longrightarrow \quad \lambda = \frac{\|\beta\| - ac}{\|\beta\|}.$$

Substituting for this gives the minimum of (2.33) to be

$$\begin{aligned}
\frac{m^2+c^2}{c^2}&\left(\frac{(\lambda-1)m^2+\lambda c^2}{m^2+c^2}\right)^2\|\beta\|^2 + \left(\frac{m^2}{m^2+c^2}\right)\|\beta\|^2 \\
&= \|\beta\|^2\left(\frac{\left(-am^2+(\|\beta\|-ac)c\right)^2}{\|\beta\|^2(m^2+c^2)} + \frac{m^2}{m^2+c^2}\right) \\
&= \frac{1}{m^2+c^2}\left(a^2m^4 - 2am^2(\|\beta\|-ac)c + (\|\beta\|-ac)^2c^2 + \|\beta\|^2m^2\right) \\
&= \frac{1}{m^2+c^2}\left(\|\beta\|^2(m^2+c^2) - 2ac\|\beta\|(m^2+c^2) + a^2(m^2+c^2)^2\right) \\
&= \|\beta\|^2 - 2ac\|\beta\| + a^2(m^2+c^2)
\end{aligned}$$

and we are done. □

This proposition now gives some advice on how to choose the target output y_d. Recall that y_d is chosen such that $(1-\phi\phi^\dagger)y_d = (1-\phi\phi^\dagger)Cz'(T)$. Hence, in the original problem considered in this subsection, the constraint is given by

$$\|y_d - Cz'(T)\| = \|\phi\phi^\dagger(y_d - Cz'(T))\| \leq \frac{a(1-K)}{K_2\|B\|\,\|\phi^\dagger\|}.$$

Since $||(1-\phi\phi^\dagger)y_d||$ is fixed an equivalent problem to the original one is to minimise

$$||\alpha||^2 + ||(1-\phi\phi^\dagger)||^2 \left(\frac{\tilde{K}K_2||B||\,||\phi^\dagger||\,||C||K_4}{1-K}\right)^2 ||\alpha-\beta||^2$$

where $\alpha = (\phi\phi^\dagger)y_d$ and $\beta = (\phi\phi^\dagger)Cz'(T)$. Therefore the proposition is applied with $m = ||1-\phi\phi^\dagger||\tilde{K}$, $c = \frac{1-K}{K_2||B||\,||\phi^\dagger||\,||C||K_4}$, β and α as already defined. This suggests that the best choice for y_d is

$$y_1 = \frac{||\phi^\dagger||^2||1-\phi\phi^\dagger||^2||B||^2K_2^2||C||^2K_4^2\tilde{K}^2}{||\phi^\dagger||^2||1-\phi\phi^\dagger||^2||B||^2K_2^2||C||^2K_4^2\tilde{K}^2+(1-K)^2}\left[(\phi\phi^\dagger)Cz'(T)\right]$$

if

$$||\phi\phi^\dagger Cz'(T)|| \le \frac{aK_2||B||\,||\phi^\dagger||\,||C||K_4}{(1-K)}\left(||1-\phi\phi^\dagger||^2\tilde{K}^2+\frac{(1-K)^2}{K_2^2||B||^2||\phi^\dagger||^2||C||^2K_4^2}\right);$$

or

$$y_2 = \frac{||\phi\phi^\dagger Cz'(T)||\,K_2||B||\,||\phi^\dagger||\,||C||K_4 - a(1-K)}{||\phi\phi^\dagger Cz'(T)||\,K_2||B||\,||\phi^\dagger||\,||C||K_4}\left[\phi\phi^\dagger Cz'(T)\right]$$

otherwise.

With these values the following estimates are obtained:

$$||y||^2 \le 2||(1-\phi\phi^\dagger)Cz'(T)||^2 + 2\left(\frac{m^2}{m^2+c^2}\right)||\phi\phi^\dagger Cz'(T)||^2 \tag{2.36}$$

setting $y_d = y_1$ and

$$||y||^2 \le 2||(1-\phi\phi^\dagger)Cz'(T)||^2 + 2\left(||\phi\phi^\dagger Cz'(T)||^2 - 2ac||\phi\phi^\dagger Cz'(T)|| + a^2(m^2+c^2)\right) \tag{2.37}$$

for $y_d = y_2$. An important question is whether these estimates are less than $||Cz'(T)||^2$? If this is the case then it is known that the initial guess can be improved upon. In the Hilbert space setting it is known that

$$||Cz'(T)||^2 = ||\phi\phi^\dagger Cz'(T)||^2 + ||\left(1-\phi\phi^\dagger\right)Cz'(T)||^2.$$

The right-hand side of (2.36) is strictly less than $||Cz'(T)||^2$ if

$$\begin{aligned} 0 &> ||(1-\phi\phi^\dagger)Cz'(T)||^2 + \left(\frac{2m^2}{m^2+c^2}-1\right)||\phi\phi^\dagger Cz'(T)||^2 \\ &= ||(1-\phi\phi^\dagger)Cz'(T)||^2 + \left(\frac{m^2-c^2}{m^2+c^2}\right)||\phi\phi^\dagger Cz'(T)||^2. \end{aligned}$$

Therefore at the very least $m^2 < c^2$ is required. Similarly the right-hand side of (2.37) is strictly less than $||Cz'(T)||^2$ if

$$0 > ||(1 - \phi\phi^\dagger)Cz'(T)||^2 + ||\phi\phi^\dagger Cz'(T)||^2 - 4ac||\phi\phi^\dagger Cz'(T)|| + 2a^2(m^2 + c^2).$$

Consider this last equation as a quadratic in $||\phi\phi^\dagger Cz'(T)||$. The inequality holds if there are two distinct real roots of the quadratic and $||\phi\phi^\dagger Cz'(T)||$ lies in the open interval between them. The roots of the quadratic satisfy

$$\lambda_{1,2} = \frac{1}{2}\left(4ac \pm \left[8a^2(c^2 + m^2) - 4||(1 - \phi\phi^\dagger)Cz'(T)||^2\right]^{1/2}\right).$$

The condition required for two distinct real roots is, therefore,

$$2a^2(c^2 + m^2) > ||(1 - \phi\phi^\dagger)Cz'(T)||^2.$$

Both these cases show the importance of the initial guess. Ideally $||(1 - \phi\phi^\dagger)Cz'(T)||$ should be small.

Chapter 3

Perturbed Systems

The previous chapter considered a control problem associated with the system

$$\dot{z}(t) = A(t)z(t) + D(t)N(t, E(t)z(t))$$

on the Banach space Z, where Assumption 2 of Section 2.1.1 and condition TV I of Section 2.1.2 are satisfied. The problem was to choose a control u that acts only via the initial state:

$$z(0) = Bu$$

such that the output

$$y = Cz'(T) + Cz(T)$$

is some desired value, y_d say, where $Cz'(T)$ is a given output. The method involved generating conditions that ensured that the problem was well-defined and that a solution could be found. The conditions involved a mild evolution operator associated with the system. In this chapter a particular type of the previous system, namely those for which $A(t) = A + P(t)$ where $P(\cdot)$ is piecewise continuous and may exhibit the same unboundedness as the nonlinearity, will be considered. Thus perturbations $P(\cdot) \in PC(0, T; \mathcal{L}(\underline{Z}, \overline{Z}))$ of closed unbounded linear operators A are considered. Suppose that A is the infinitesimal generator of a strongly continuous semigroup. The main aim of this chapter is to show that suitable conditions can be imposed on the semigroup and perturbation such that a mild evolution operator satisfying the conditions of the previous chapter can be associated with $A(t)$, in a way to be made precise in Section 3.1. Therefore only conditions on the semigroup and perturbation need to be checked in applications involving this type of system.

The first section deals with the problem of associating a mild evolution operator with $A(t)$ on Z. This association leads naturally to the concept of $A + P(\cdot)$ being the generator of a mild evolution operator (Curtain and Zwart, 1995). It is firstly shown that $A + P(\cdot)$ is the generator of a mild evolution operator on $\underline{Z}$ that can be extended to one on Z. This same method is then used in the second section to extend it to a mild evolution operator on $\overline{Z}$.

The problem of deriving sufficient conditions for the mild evolution operator generated by $A(\cdot)$ to satisfy those of the previous chapter is the topic of the second section. These conditions, together with TV VI, allow us to apply Theorem 2.2.10 to the control problem.

The rabies model of Chapter 1 that motivated the work of this thesis is semilinear in form. The linearisation performed in the introduction to Chapter 2, when performed on a semilinear system gives rise to one of the form considered in this chapter and this is the motivation for the work here. The chapter is concluded by obtaining existence results for solutions of general semilinear systems. The method follows that used by Pazy (1983) while allowing for the unboundedness of the nonlinear part.

3.1 The perturbation

Suppose that $A(\cdot) = A + P(\cdot)$ is the generator of a strong evolution operator $U(t, s)$ on Z, a Banach space, and consider the following Cauchy problem:

$$\dot{z}(t) = A(t)z(t), \qquad z(s) = z_s,$$

where $s \in [0, T)$ is fixed, $t \in (s, T]$ and $z_s \in D(A)$. Recall from the last chapter that this means that the Cauchy problem is well-posed with solution given by

$$z(t) = U(t, s)z_s.$$

Now if A is the generator of a strongly continuous semigroup $S(t)$ and $P(\cdot) \in PC(s,T;\mathcal{L}(Z))$ then (Curtain and Zwart, 1995)

$$U(t,s)z_s = S(t-s)z_s + \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z_s\,d\sigma. \tag{3.1}$$

A natural question arising from this is whether the properties that were required for the evolution operator in Chapter 2 can be guaranteed by imposing suitable assumptions on the perturbation and semigroup. If $A(t)$ is not the generator of a strong evolution operator one might ask under what circumstances does there exist a *mild* evolution operator equal to the right-hand side of (3.1). Associated with these questions is the problem of the possible unboundedness of the nonlinearity and this must be taken into consideration.

The setting for this chapter is very much the same as that for the previous one.

PS I. *$\underline{Z}, Z, \overline{Z}$ are Banach spaces such that $\underline{Z} \subset Z \subset \overline{Z}$ and the canonical injections $\underline{Z} \hookrightarrow Z$, $Z \hookrightarrow \overline{Z}$ are continuous with dense ranges.*

The nonlinearity is a map from $[0,T] \times \underline{Z}$ to $\overline{Z}$ and so can be considered as unbounded on Z. The perturbation is at least piecewise continuous, with the same degree of possible unboundedness as the nonlinearity.

PS II. *$P(\cdot) \in PC(0,T;\mathcal{L}(\underline{Z},\overline{Z}))$ and the intersection of the domain of A and $\underline{Z}$ is a dense subspace of Z.*

Thus $A(t)$ is an unbounded linear operator on Z whose domain contains $D(A) \cap \underline{Z}$ for each $t \in [0,T]$.

The subject of this section is to associate with $A + P(t)$ a mild evolution operator via equation (3.1). In the next section conditions are determined for the perturbation and semigroup that imply that the associated mild evolution operator satisfies the hypotheses of the previous chapter.

3.1.1 Associated mild evolution operator

Equation (3.1) provides the means by which we can define the concept of $A(\cdot) = A + P(\cdot)$ being the generator of a mild evolution operator.

Definition 3.1.1. On the Banach space Z, suppose that A is the generator of a strongly continuous semigroup $S(t)$ and $P(\cdot) \in PC(0,T;\mathcal{L}(Z))$. $A + P(\cdot)$ is the *generator of the mild evolution operator* $U(t,s)$ on $[0,T]$ if, for all $s \in [0,T]$, $U(t,s)$ is the unique solution of (3.1).

Now consider (3.1) with $P(\cdot)$ possibly unbounded (so that $P(t) \in \mathcal{L}(\underline{Z},\overline{Z})$ for each $t \in [0,T]$)

$$U(t,s)z_s = S(t-s)z_s + \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z_s\,ds.$$

The first step is to construct a mild evolution operator on $\underline{Z}$ satisfying this equation. This operator will then be extended to Z. The semigroup is required to act on all three spaces, namely

PS III. *$S(t)$ is a strongly continuous semigroup on all three spaces $\underline{Z}$, Z and $\overline{Z}$.*

From the integral term in (3.1) it is seen that a smoothing condition for the semigroup is required. To allow for trajectories in all three spaces the following assumption is made.

PS IV. *There exists a continuous function $\overline{K}_1(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that $\overline{K}_1(0) = 0$, and for all $s \in [0,T)$, $z(\cdot) \in C(s,T;\underline{Z})$, $t \in [s,T]$, the map $t \mapsto \int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma$ is continuous from $[s,T]$ to $\underline{Z}$ and*

$$\|\int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma\|_{\underline{Z}} \le \overline{K}_1(t-s)\|z\|_{C(s,t;\overline{Z})}. \tag{3.2}$$

Remark 3.1.2. The estimate (3.2) is sufficient for the map $t \mapsto \int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma$ to be continuous from the right. To see this let $h > 0$ and note that

$$\begin{aligned}
&\|\int_s^{t+h} S(t+h-\sigma)P(\sigma)z(\sigma)\,d\sigma - \int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma\|_{\underline{Z}} \\
&\quad = \|\int_t^{t+h} S(t+h-\sigma)P(\sigma)z(\sigma)\,d\sigma + \int_s^t \left(S(t+h-\sigma) - S(t-\sigma)\right)P(\sigma)z(\sigma)\,d\sigma\|_{\underline{Z}} \\
&\quad \le \overline{K}_1(h)\|z\|_{C(t,t+h;\overline{Z})} + \|\left(S(h)-I\right)\int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma\|_{\underline{Z}}.
\end{aligned}$$

These last two terms converge to zero as $h \downarrow 0$ by the continuity of $\overline{K}_1(\cdot)$ and the strong continuity of the semigroup on $\underline{Z}$.

Proposition 3.1.3. *Let A be the infinitesimal generator of the strongly continuous semigroup $S(t)$ and suppose that PS I–IV hold. Then $A(t) = A + P(t)$ is the generator of a mild evolution operator*

$U(t,s)$ on $\underline{Z}$ in the sense that $U(t,s)$ is the unique, in the class of strongly continuous operators on $\underline{Z}$, solution of

$$U(t,s)z = S(t-s)z + \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma \tag{3.3}$$

for $z \in \underline{Z}$.

Proof. Suppose that $\|S(t)\|_{\mathcal{L}(\underline{Z})} \le M_0 e^{\omega_0 t}$. We let $U_0(t,s) = S(t-s)$ and define the following iterative scheme

$$U_n(t,s)z = \int_s^t S(t-\sigma)P(\sigma)U_{n-1}(\sigma,s)z\,d\sigma. \tag{3.4}$$

Let

$$U(t,s) = \sum_{n=0}^{\infty} U_n(t,s). \tag{3.5}$$

Note that $U_0(\cdot,s)$ is strongly continuous on $[s,T]$ for each $s \in [0,T]$ and

$$\|U_0(t,s)\|_{\mathcal{L}(\underline{Z})} \le M_0 e^{\omega_0(t-s)}.$$

For each $s \in [0,T]$ if $U_n(\cdot,s)$ is strongly continuous then PS IV and (3.4) imply that $U_{n+1}(\cdot,s)$ is as well. Therefore $U_n(\cdot,s)$ is strongly continuous for each $n \in \mathrm{N}$ by induction. Furthermore we have the following estimate for each $z \in \underline{Z}$:

$$\begin{aligned}\|U_n(t,s)z\|_{\underline{Z}} &= \left\| \int_s^t S(t-\sigma)P(\sigma)U_{n-1}(\sigma,s)z\,d\sigma \right\|_{\underline{Z}} \\ &\le \overline{K}_1(t-s)\|U_{n-1}(\cdot,s)z\|_{C(s,t;\overline{Z})}.\end{aligned}$$

Since there exist constants R_1, R_2 such that $\|z\|_Z \le R_1\|z\|_{\underline{Z}}$ for each $z \in \underline{Z}$, and $\|z\|_{\overline{Z}} \le R_2\|z\|_Z$ for all $z \in Z$ we can rewrite this as:

$$\begin{aligned}\|U_n(t,s)z\|_{\underline{Z}} &\le R_1R_2\overline{K}_1(t-s)\|U_{n-1}(\cdot,s)z\|_{C(s,t;\underline{Z})} \\ &\le R_1R_2\overline{K}_1(t-s)\sup_{\sigma\in[s,t]}\left(R_1R_2\overline{K}_1(\sigma-s)\|U_{n-2}(\cdot,s)\|_{C(s,\sigma;\underline{Z})}\right) \\ &\le \left(R_1R_2\overline{K}_1(\tau)\right)^2\|U_{n-2}(\cdot,s)\|_{C(s,t;\underline{Z})}\end{aligned}$$

where $\overline{K}_1(\tau) = \sup_{\sigma\in[0,t-s]} \overline{K}_1(\sigma)$. That this supremum exists and is achieved, is a result of the continuity of $\overline{K}_1(\cdot)$ on $[s,t]$. Therefore by induction it is easy to see that

$$\|U_n(t,s)\| \leq \left(R_1 R_2 \overline{K}_1(\tau)\right)^n \gamma_0$$

where $\gamma_0 = \max\{M_0, M_0 e^{\omega_0 T}\}$. We restrict attention, for the moment, to the interval $[s_i, t_i]$ where for $\overline{K}_1(\tau_i) = \sup_{\sigma\in[0,t_i-s_i]} \overline{K}_1(\sigma)$ we have $\overline{K}_1(\tau_i) R_1 R_2 < 1$. Now for all s and t such that $s_i \leq s \leq t \leq t_i$ the series (3.5) is majorised by

$$\gamma_0 \sum_{n=0}^{\infty} \left(R_1 R_2 \overline{K}_1(\tau_i)\right)^n$$

and so converges absolutely in the uniform topology of $\mathcal{L}(\underline{Z})$. Therefore $U(\cdot,\cdot)$ is uniformly bounded by

$$\underline{M_U}(i) = \frac{\gamma_0}{1 - \left(R_1 R_2 \overline{K}_1(\tau_i)\right)}$$

on $\Delta(t_i, s_i) = \{(t,s) : s_i \leq s \leq t \leq t_i\}$. Since, for each $z \in \underline{Z}$ and $s \in [s_i, t_i]$, $U_n(\cdot, s)z \in C(s, t_i; \underline{Z})$ for each $n \in \mathbb{N}$ we have $\sum_{n=0}^{N} U_n(\cdot, s)z \in C(s, t_i; \underline{Z})$ for each $N \in \mathbb{N}\cup\{0\}$. Therefore $U(\cdot, s)z$ is the uniform limit of a sequence of continuous functions and so is continuous; this is the strong continuity of $U(\cdot, s)$.

Next we see that, for all $s_i \leq s \leq t \leq t_i$,

$$\begin{aligned} U(t,s)z &= \sum_{n=0}^{\infty} U_n(t,s) \\ &= S(t-s)z + \sum_{n=1}^{\infty} \int_s^t S(t-\sigma)P(\sigma)U_{n-1}(\sigma,s)z\,d\sigma \\ &= S(t-s)z + \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma. \end{aligned}$$

Therefore $U(t,s)$ satisfies (3.3) on $\Delta(t_i, s_i)$.

We now show that $U(t,s)$ for $s_i \leq s \leq t \leq t_i$ satisfies the conditions for a mild evolution operator. It is clear that $U(t,t) = I$ for every $t \in [s_i, t_i]$. Let $(t,s) \in \Delta(t_i, s_i)$ and $p \in [s,t]$. Then for any $z \in \underline{Z}$,

$$\begin{aligned}
U(t,p)U(p,s)z - U(t,s)z &= S(t-p)U(p,s)z + \int_p^t S(t-\sigma)P(\sigma)U(\sigma,p)U(p,s)z\,d\sigma \\
&\quad - S(t-s)z - \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma \\
&= S(t-p)S(p-s)z + S(t-p)\int_s^p S(p-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma \\
&\quad + \int_p^t S(t-\sigma)P(\sigma)U(\sigma,p)U(p,s)z\,d\sigma \\
&\quad - S(t-s)z - \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma \\
&= \int_p^t S(t-\sigma)P(\sigma)\left(U(\sigma,p)U(p,s)z - U(\sigma,s)z\right)\,d\sigma
\end{aligned}$$

and so

$$\begin{aligned}
\|U(t,p)U(p,s)z - U(t,s)z\|_{\underline{Z}} &\leq \overline{K}_1(\tau_i)\sup_{\sigma\in[p,t]}\|U(\sigma,p)U(p,s)z - U(\sigma,s)z\|_{\overline{Z}} \\
&\leq \overline{K}_1(\tau_i)\sup_{\sigma\in[p,t]}\int_p^\sigma \gamma\|P\|_\infty\|U(\alpha,p)U(p,s)z - U(\alpha,s)z\|_{\underline{Z}}\,d\alpha \\
&\leq \int_p^t (\text{const})\|U(\sigma,p)U(p,s)z - U(\sigma,s)z\|_{\underline{Z}}\,d\sigma,
\end{aligned}$$

where $\gamma = \sup_{t\in[0,T]}\|S(t)\|_{\mathcal{L}(\overline{Z})}$ and $\|P\|_\infty = \sup_{t\in[0,T]}\|P(t)\|_{\mathcal{L}(\underline{Z},\overline{Z})}$. Let $t' = t-p$, $\sigma' = \sigma - p$ and $g(t') = \|U(t'+p,p)U(p,s)z - U(t'+p,s)z\|_{\underline{Z}}$ to see that

$$0 \leq g(t') \leq \int_0^{t'} (\text{const})g(\sigma')\,d\sigma'.$$

Applying Gronwall's lemma we see that $U(t,s) = U(t,p)U(p,s)$.

The strong continuity of $U(\cdot,s)$ for each $s \in [s_i,t_i]$ has already been established and so it only remains to show that $U(t,\cdot)$ is strongly continuous for each $t \in [s_i,t_i]$. Fix $(t,s) \in \Delta(t_i,s_i)$ and let $h > 0$ be such that $h \leq t - s$. Then for all $z \in \underline{Z}$ we have

$$\begin{aligned}
\|U(t, s+h)z - U(t,s)z\|_{\underline{Z}} &= \| \int_{s+h}^{t} S(t-\sigma)P(\sigma)\left(U(\sigma, s+h)z - U(\sigma, s)z\right) d\sigma \\
&\quad + \int_{s}^{s+h} S(t-\sigma)P(\sigma)U(\sigma,s)z\, d\sigma\|_{\underline{Z}} \\
&\leq \overline{K}_1(\tau_i)\|U(\cdot, s+h)z - U(\cdot, s)z\|_{C(s+h,t;\overline{Z})} \\
&\quad + \|S(t-(s+h)) \int_{s}^{s+h} S((s+h)-\sigma)P(\sigma)U(\sigma,s)z\, d\sigma\|_{\underline{Z}} \\
&\leq \int_{s+h}^{t} \gamma\overline{K}_1(\tau_i)\|P\|_\infty \|U(\sigma, s+h)z - U(\sigma,s)z\|_{\underline{Z}}\, d\sigma \\
&\quad + \gamma\overline{K}_1(h)\|U(\cdot, s)z\|_{C(s,s+h;\overline{Z})}.
\end{aligned}$$

Therefore applying Gronwall's Lemma we have

$$\|U(t, s+h)z - U(t,s)z\|_{\underline{Z}} \leq \gamma\overline{K}_1(h)\|U(\cdot,s)z\|_{C(s,s+h;\overline{Z})}\, e^{\gamma\overline{K}_1(\tau_i)\|P\|_\infty (t-s-h)}.$$

Now letting $h \downarrow 0$ we see that the right-hand side of this equation converges to zero. Similarly for $h > 0$ such that $h \leq s - s_i$ and $z \in \underline{Z}$ we have

$$\begin{aligned}
\|U(t, s-h)z - U(t,s)z\|_{\underline{Z}} &= \| \int_{s}^{t} S(t-\sigma)P(\sigma)\left(U(\sigma, s-h)z - U(\sigma, s)z\right) d\sigma \\
&\quad + \int_{s-h}^{s} S(t-\sigma)P(\sigma)U(\sigma,s-h)z\, d\sigma\|_{\underline{Z}} \\
&\leq \int_{s}^{t} \gamma\overline{K}_1(\tau_i)\|P\|_\infty \|U(\sigma, s-h)z - U(\sigma,s)z\|_{\underline{Z}}\, d\sigma \\
&\quad + \gamma\overline{K}_1(h)\|U(\cdot, s-h)z\|_{C(s-h,s;\overline{Z})}.
\end{aligned}$$

Applying Gronwall's Lemma and using the uniform boundedness of $U(\cdot,\cdot)$ on $\Delta(t_i, s_i)$ gives

$$\|U(t, s-h)z - U(t,s)z\|_{\underline{Z}} \leq \gamma\overline{K}_1(h)R_1R_2\underline{M}_U(i)\|z\|_{\underline{Z}}\, e^{\gamma\overline{K}_1(\tau_i)\|P\|_\infty (t-s)}.$$

The right-hand side of this equation converges to zero as $h \downarrow 0$ by the continuity of $\overline{K}_1$. Hence $U(t,\cdot)$ is strongly continuous for every $t \in [s_i, t_i]$.

All that now remains to complete the proof is to extend $U(t,s)$ from $\Delta(t_i, s_i)$ to $\Delta(T)$ and to prove uniqueness. We do this by using the above arguments to construct $U(t,s)$ on finite intervals covering the whole of $[0,T]$ and so defining $U(t,s)$ on $\Delta(T)$. Note that the continuity of $\overline{K}_1(\cdot)$ and the fact that $\overline{K}_1(0) = 0$ implies that there exists a constant $\delta > 0$ such that $\overline{K}_1(\tau) < 1/(R_1R_2)$ for all $\tau \in [0, \delta)$. Now cover the interval $[0, T]$ with the finite union $\bigcup_{i=0}^{N}[s_i, t_i]$ such that $t_i - s_i < \delta$, $s_0 = 0, t_N = T$ and $s_i = t_{i-1}$ for each $1 \leq i \leq N$. For each interval we have $R_1R_2\overline{K}_1(\tau_i) < 1$ and so we can apply the above to construct a mild evolution operator $U^i(t, s)$ on $\Delta(t_i, s_i)$ satisfying

$$U^i(t,s)z = S(t-s)z + \int_s^t S(t-\sigma)P(\sigma)U^i(\sigma,s)z\,d\sigma$$

for each $z \in \underline{Z}$ and $(t, s) \in \Delta(t_i, s_i)$. We piece these operators together to give $U(t, s)$ on $\Delta(T)$: For all $(t, s) \in \Delta(t_0)$ we define $U(t, s) = U^0(t, s)$ and so $U(t, s)$ is a mild evolution operator on $\Delta(t_0)$. To extend $U(t, s)$ to $\Delta(t_1)$ we define

$$U(t,s)z = \begin{cases} U^1(t,s)z & (t,s) \in \Delta(t_1, s_1) \\ U^1(t,s_1)U(s_1,s)z & s \in [0,s_1), t \in (s_1,t_1] \end{cases}$$

for all $z \in \underline{Z}$. Then $U(t, s)$ is a mild evolution operator on $\Delta(t_1, s_1) \cup \Delta(t_0)$ by construction. Clearly $U(t, t) = I$ for all $t \in [0, t_1]$. Let $(t, s) \in \Delta(t_1)$ and $p \in [s, t]$. It only remains to consider the case where $s \in [0, s_1)$ and $t \in (s_1, t_1]$. Then if $p = s_1$ we are done so let $p \in [s, s_1)$:

$$U(t,s)z = U^1(t,s_1)U(s_1,s)z = U^1(t,s_1)U(s_1,p)U(p,s)z = U(t,p)U(p,s)z$$

for every $z \in \underline{Z}$. Similarly for $p \in (s_1, t]$.

Clearly $U(\cdot, s)$ and $U(t, \cdot)$ are strongly continuous for every $s \in [s_1, t_1]$, $t \in [0, s_1]$ respectively. Let $s \in [0, s_1)$. $U(\cdot, s)$ is strongly continuous at every $t \in [s, s_1]$ so suppose that $t \in (s_1, t_1]$. Now for $h > 0$ such that $h \leq t_1 - t$ we have, for every $z \in \underline{Z}$,

$$U(t+h,s)z - U(t,s)z = U^1(t+h,s_1)U(s_1,s)z - U^1(t,s_1)U(s_1,s)z$$

and for $h \leq t - s_1$

$$U(t,s)z - U(t-h,s)z = U^1(t,s_1)U(s_1,s)z - U^1(t-h,s_1)U(s_1,s)z.$$

Therefore the strong continuity of $U(\cdot, s)$ follows from the strong continuity of $U^1(\cdot, s_1)$. Similarly the strong continuity of $U(t,\cdot)$ follows from the strong continuity of $U(s_1,\cdot)$. Finally we see that

$$\begin{aligned} U(t,s)z &= U^1(t,s_1)U(s_1,s)z \\ &= S(t-s_1)U(s_1,s)z + \int_{s_1}^{t} S(t-\sigma)P(\sigma)U^1(\sigma,s_1)U(s_1,s)z\,d\sigma \\ &= S(t-s)z + \int_{s}^{s_1} S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma + \int_{s_1}^{t} S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma \\ &= S(t-s)z + \int_{s}^{t} S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma \end{aligned}$$

for all $z \in \underline{Z}$, $s \in [0,s_1)$ and $t \in (s_1,t_1]$. Therefore we have extended $U(t,s)$ to a mild evolution operator on $\Delta(t_1)$ that satisfies (3.3). Furthermore, $U(\cdot,\cdot)$ is uniformly bounded by

$$\max\left\{\underline{M}_U(\tau_0), \underline{M}_U(\tau_1), \underline{M}_U(\tau_0)\underline{M}_U(\tau_1)\right\}.$$

Continuing this process we construct a mild evolution operator $U(t,s)$ on $\Delta(T)$ that satisfies (3.3) and is uniformly bounded, by $\underline{M}_U$ say.

To prove uniqueness: Suppose that $\tilde{U}(t,s)$ is another strongly continuous solution of (3.3). Then

$$U(t,s)z - \tilde{U}(t,s)z = \int_s^t S(t-\sigma)P(\sigma)\left(U(\sigma,s)z - \tilde{U}(\sigma,s)z\right)\,d\sigma,$$

and so

$$\begin{aligned} \|U(t,s)z - \tilde{U}(t,s)z\|_{\underline{Z}} &\le \overline{K}_1(\tau) \sup_{\sigma\in[s,t]} \|U(\sigma,s)z - \tilde{U}(\sigma,s)z\|_{\overline{Z}} \\ &\le \int_s^t (\text{const})\|U(\sigma,s)z - \tilde{U}(\sigma,s)z\|_{\underline{Z}}\,d\sigma \end{aligned}$$

where $\overline{K}_1(\tau) = \sup_{\sigma\in[0,T]} \overline{K}_1(\sigma)$. Applying Gronwall's Lemma proves the uniqueness on $\Delta(T)$.

□

A mild evolution operator is required on Z and in the next subsection the operator given in the last result is extended.

3.1.2 Extension of the mild evolution operator

Our assumption about the smoothing property of the semigroup on the perturbation is sufficient for the mild evolution operator of the last subsection to be extended to a bounded linear operator on Z. The uniform boundedness is sufficient for this extension to be a mild evolution operator.

Corollary 3.1.4. *Let $U(t,s)$ be the mild evolution operator generated by $A(t)$ given in Proposition 3.1.3 and suppose that the hypothesis of the proposition holds. Then we can define an extension of $U(t,s)$ to a bounded linear operator on Z by*

$$\tilde{U}(t,s)z = \lim_{n\to\infty} U(t,s)z_n \tag{3.6}$$

for each $z \in Z$, where $(z_n)_{n=1}^{\infty}$ is a sequence into $\underline{Z}$ such that $\|z_n - z\|_Z \to 0$ as $n \to \infty$.

Furthermore, this extension, which will be denoted by $U(t,s)$, is a mild evolution operator on Z.

Proof. Let $z \in \underline{Z}$ and $(t,s) \in \Delta(t_{i+1}, t_i)$ where t_i, t_{i+1} are chosen such that

$$\overline{K}_1(\tau_i) = \sup_{\sigma\in[0,t_{i+1}-t_i]} K_1(\sigma)$$

satisfies $R_1R_2\overline{K}_1(\tau_i) < 1$. Then

$$\begin{aligned}\|U(t,s)z\|_Z &\le \|S(t-s)z\|_Z + R_1\|\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma\|_{\underline{Z}}\\ &\le M_1e^{\omega_1(t-s)}\|z\|_Z + R_1R_2\overline{K}_1(t-s)\|U(\cdot,s)z\|_{C(s,t;Z)}\end{aligned}$$

where we assume that $\|S(t-s)\|_{\mathcal{L}(Z)} \le M_1e^{\omega_1(t-s)}$. Therefore

$$\|U(\cdot,s)z\|_{C(s,t_1;Z)} \le \gamma_1\|z\|_Z + R_1R_2\overline{K}_1(\tau_i)\|U(\cdot,s)z\|_{C(s,t_1;Z)}$$

where $\gamma_1 = \max\{M_1, M_1e^{\omega_1 T}\}$ and so

$$\|U(t,s)z\|_Z \le \|U(\cdot,s)z\|_{C(s,t_1;Z)} \le \frac{\gamma_1}{1-R_1R_2\overline{K}_1(\tau_i)}\|z\|_Z.$$

Hence $U(\cdot,\cdot)$ is uniformly bounded, by $M_U(i)$ say, on $\Delta(t_{i+1},t_i)$ with respect to the norm in Z. By partitioning the interval $[0,T]$ into the union of N intervals:

$$[0,T] = \bigcup_{i=0}^{N-1}[t_i, t_{i+1}]$$

such that $t_0 = 0$, $t_N = T$ and $R_1 R_2 \overline{K}_1(\tau_i) < 1$, we see that $U(\cdot,\cdot)$ is uniformly bounded, with respect to the norm in Z, on $\Delta(T)$ by

$$M_U = \max\left\{\prod_{i=0}^{N-1} \alpha_i M_U(i) : (\alpha_0, \dots, \alpha_{N-1}) \in \mathbb{R}^N, \alpha_i \in \{1, 1/M_U(i)\}\right\}.$$

Now for $z \in Z$ let $(z_n)_{n=1}^{\infty}$ be a sequence into $\underline{Z}$ such that $\|z_n - z\|_Z \to 0$ as $n \to \infty$. Then

$$\begin{aligned}\|\tilde{U}(t,s)z\|_Z &= \lim_{n\to\infty} \|U(t,s)z_n\|_Z \\ &\le \lim_{n\to\infty} M_U\|z_n\|_Z = M_U\|z\|_Z.\end{aligned}$$

Therefore the extension of $U(t,s)$ to a linear operator on Z is (uniformly) bounded.

We now show that $\tilde{U}(t,s)$ is a mild evolution operator on Z. First note that $\tilde{U}(s,s)z = z$ for all $s \in [0,T]$ and $z \in \underline{Z}$. Now for $z \in Z$, let $(z_n)_{n=1}^{\infty}$ be a sequence into $\underline{Z}$ such that $\|z_n - z\|_Z \to 0$ as $n \to \infty$. Then

$$\|U(s,s)z_n - z\|_Z = \|z_n - z\|_Z \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Therefore, by the definition of the extension, $\tilde{U}(s,s) = I$ for all $s \in [0,T]$ on Z.

Let $(t,s) \in \Delta(T)$ and $p \in [s,t]$. Clearly $\tilde{U}(t,p)\tilde{U}(p,s)z = \tilde{U}(t,s)z$ for each $z \in \underline{Z}$. Now for $z \in Z$, let $(z_n)_{n=1}^{\infty}$ be a sequence into $\underline{Z}$ such that $\|z_n - z\|_Z \to 0$ as $n \to \infty$, and consider

$$\begin{aligned}\|U(t,p)U(p,s)z_n - \tilde{U}(t,s)z\|_Z &= \|\tilde{U}(t,s)z_n - \tilde{U}(t,s)z\|_Z \\ &\le M_U\|z_n - z\|_Z \\ &\longrightarrow 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Therefore, by definition,

$$\tilde{U}(t,p)\tilde{U}(p,s)z = \lim_{n\to\infty} U(t,p)U(p,s)z_n = \tilde{U}(t,s)z$$

for each $z \in Z$.

Now it only remains to show that $\tilde{U}(\cdot,\cdot)$ is strongly continuous. For each $s \in [0,T]$ and $z \in Z$ we have, where $(z_n)_{n=1}^{\infty}$ is a sequence into $\underline{Z}$ such that $\|z_n - z\|_Z \to 0$ as $n \to \infty$,

$$\|\tilde{U}(t,s)z - U(t,s)z_n\|_Z = \|\tilde{U}(t,s)z - \tilde{U}(t,s)z_n\|_Z \le M_U\|z - z_n\|_Z,$$

for $t \in [s, T]$. Therefore $\tilde{U}(\cdot, s)z$ is the uniform limit of the sequence of continuous functions $(U(\cdot, s)z_n)_{n=1}^{\infty}$ and so is continuous. Similarly $U(t, \cdot)z$ is continuous for all $z \in Z$ and $t \in [0, T]$. Hence the extension $\tilde{U}(t, s)$ is a mild evolution operator on Z which we will again denote by $U(t, s)$. □

Remark 3.1.5. Applying Hinrichsen and Pritchard (1994) we remark that the linear extension $U(t, s)$ to Z, of the mild evolution operator is itself a mild evolution (on Z) if and only if the sets $\{U(t, s) : s \in [0, t]\}$ and $\{U(t, s) : t \in [s, T]\}$ are bounded in $\mathcal{L}(Z)$ for, respectively, every $t \geq 0$ and $s \geq 0$.

In the following section certain conditions are imposed on the semigroup and perturbation such that the perturbed system

$$z(t) = U(t, 0)Bu + \int_0^t U(t, s)D(s)N(s, E(s)z(s))\, ds$$

satisfies the hypotheses of Chapter 2. In doing so it is seen that the mild evolution operator given by Corollary 3.1.4 can be extended to a bounded linear operator on $\overline{Z}$. Moreover, this extension is a mild evolution operator.

3.2 Mild form for perturbed systems

Consider the abstract differential equation

$$\dot{z}(t) = A(t)z(t) + D(t)N(t, E(t)z(t)), \qquad z(0) = Bu, \tag{3.7}$$

with output

$$y = Cz'(T) + Cz(T),$$

where $Cz'(T)$ is some given output. The system is considered with the state taking values in a Banach space Z and the output taking values in a Hilbert space Y. Let $u \in U$, a Hilbert space, $B \in \mathcal{L}(U, Z)$, and suppose that there exists a Banach space $V \subseteq Z$, with continuous injection, such that $C \in \mathcal{L}(V, Y)$.

The nonlinearity $N : [0,T] \times \underline{W} \longrightarrow \overline{W}$, with $\underline{W} \subset \overline{W}$ Banach spaces, satisfies

$$N(\cdot, w(\cdot)) \in L^s(0,T;\overline{W}) \quad \text{for all} \quad w(\cdot) \in L^r(0,T;\underline{W}),$$

where $r, s \geq 1$ are real numbers. The operators $D(\cdot)$ and $E(\cdot)$ characterise the unboundedness of the nonlinearity. In addition to PS I suppose that

$$E(\cdot) \in PC(0,T;\mathcal{L}(\underline{Z},\underline{W})) \quad \text{and} \quad D(\cdot) \in PC(0,T;\mathcal{L}(\overline{W},\overline{Z}))$$

so that TV I is satisfied.

In this section (3.7) will be considered for $A(\cdot) = A + P(\cdot)$, where A is the infinitesimal generator of a strongly continuous semigroup on Z that satisfies PS III and $P(\cdot)$ satisfies PS II. Recall from the last section that if, in addition, PS IV is satisfied then $A(\cdot)$ is the generator of a mild evolution operator $U(t,s)$ on the Banach space Z. The system (3.7) is interpreted in terms of the corresponding mild form given by

$$z(t) = U(t,0)Bu + \int_0^t U(t,s)D(s)N(s,E(s)z(s))\,ds. \tag{3.8}$$

Conditions are derived for the semigroup generated by $S(t)$ and the perturbation such that the mild evolution operator $U(t,s)$ satisfies the hypotheses of the previous chapter. First the conditions for the system equation to be well-defined are considered and then those that are required for the existence of a mild solution.

3.2.1 System equation is well-defined

In this subsection conditions are derived that are sufficient for the conditions TV I–IV to be satisfied for the perturbed system (3.8). It has already been assumed that TV I holds. For TV II the mild evolution operator $U(t,s)$ must be extended to a bounded linear operator on $\overline{Z}$ for all $(t,s) \in \Delta(T)$. Condition PS IV guarantees this and more as can be seen from the following corollary to Proposition 3.1.3.

Corollary 3.2.1. *The mild evolution operator $U(t,s)$ on $\underline{Z}$ defined in Proposition 3.1.3 can be extended to a bounded linear operator on $\overline{Z}$. Moreover, the extension is a mild evolution operator on $\overline{Z}$.*

Proof. Note that for any $z \in \underline{Z}$ we have

$$\begin{aligned}\|U(t,s)z\|_{\overline{Z}} &\le \|S(t-s)z\|_{\overline{Z}} + R_1R_2\|\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma\|_{\underline{Z}} \\ &\le M_2e^{\omega_2(t-s)}\|z\|_{\overline{Z}} + R_1R_2\overline{K}_1(t-s)\|U(\cdot,s)z\|_{C(s,t;\overline{Z})}\end{aligned}$$

where we assume that $\|S(t-s)\|_{\mathcal{L}(\overline{Z})} \le M_2e^{\omega_2(t-s)}$. Therefore we can proceed as in Corollary 3.1.4 where we restrict attention to $(t,s) \in \Delta(t_{i+1},t_i)$, where $(t_{i+1},t_i) \in \Delta(T)$ are chosen such that

$$R_1R_2\overline{K}_1(\tau_i) = R_1R_2 \sup_{\sigma\in[0,t_{i+1}-t_i]} \overline{K}_1(\sigma) < 1,$$

and $U(\cdot,\cdot)$ is uniformly bounded with respect to $\overline{Z}$ on $\Delta(t_{i+1},t_i)$ by

$$\overline{M}_U(i) = \frac{\gamma_2}{1-R_1R_2\overline{K}_1(\tau_i)}.$$

Hence, as in Corollary 3.1.4, the extension to a bounded linear operator on $\overline{Z}$, again denoted $U(t,s)$, is uniformly bounded, by $\overline{M}_U$ say, on $\Delta(T)$ and is a mild evolution operator on $\overline{Z}$. □

Therefore TV II is satisfied and the following remark shows that TV III is also satisfied.

Remark 3.2.2. Let $h(\cdot) \in L^s(0,T;\overline{W})$. Then as a result of the previous corollary we see that $U(t,\cdot)D(\cdot)h(\cdot) : [0,t] \longrightarrow \overline{Z}$ is measurable. Now, setting $z(\cdot) = D(\cdot)h(\cdot) \in L^s(0,T;\overline{Z})$,

$$\begin{aligned}\int_0^t \|U(t,s)z(s)\|_{\overline{Z}}\,ds &\le \int_0^t \overline{M}_U\|z(s)\|_{\overline{Z}}\,ds \\ &\le \overline{M}_U t^{1/s'}\|z\|_{L^s(0,T;\overline{Z})} < \infty\end{aligned}$$

where $\frac{1}{s}+\frac{1}{s'}=1$ and so $U(t,\cdot)D(\cdot)h(\cdot) : [0,t] \longrightarrow \overline{Z}$ is integrable.

Before considering TV IV a further remark is made on the proof of Corollary 3.2.1 by looking at the extension of $U(t,s)$ to $\overline{Z}$ more closely.

Remark 3.2.3. Let $z \in \overline{Z}$ and $(z_n)_{n=1}^{\infty}$ be a sequence into $\underline{Z}$ such that $\|z_n - z\|_{\overline{Z}} \to 0$ as $n \to \infty$. Then

$$\begin{aligned}U(t,s)z &= \lim_{n\to\infty} U(t,s)z_n \\ &= \lim_{n\to\infty}(S(t-s)z_n) + \lim_{n\to\infty}\left(\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z_n\,d\sigma\right) \\ &= S(t-s)z + \lim_{n\to\infty}\left(\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z_n\,d\sigma\right).\end{aligned}$$

Furthermore, taking the norm of the last term in $\underline{Z}$

$$\begin{aligned}\| \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z_n\,d\sigma\|_{\underline{Z}} &\leq \overline{K}_1(t-s)\|U(\cdot,s)z_n\|_{C(s,t;\overline{Z})}\\ &\leq \overline{K}_1(\tau)\overline{M}_U\|z_n\|_{\overline{Z}},\end{aligned}$$

where $\overline{K}_1(\tau) = \sup_{\sigma\in[0,T]} \overline{K}_1(\sigma)$. Therefore the integral term converges in $\underline{Z}$ and the limit is the same as with respect to $\|\cdot\|_{\overline{Z}}$. Therefore we can extend the map

$$z \longmapsto \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma$$

for $z \in \underline{Z}$ to a bounded linear map from $\overline{Z}$ to $\underline{Z}$ by

$$\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma := \lim_{n\to\infty} \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z_n\,d\sigma, \tag{3.9}$$

for $z \in \overline{Z}$.

Since, for all $z \in \underline{Z}$,

$$\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma = U(t,s)z - S(t-s)z$$

it is seen that the left-hand side is continuous with respect to the norm in $\underline{Z}$ in both s and t. In particular for each $z \in \overline{Z}$

$$\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma$$

is continuous with respect to the norm in $\underline{Z}$ in both t and s since it is the uniform limit of a sequence of continuous functions.

Let

$$(\mathrm{M}_S h)(t) = \int_0^t S(t-s)D(s)h(s)\,ds$$

for $h(\cdot) \in L^s(0,T;\overline{W})$ and suppose that the semigroup satisfies TV IV.

PS V. *For every $h(\cdot) \in L^s(0,T;\overline{W})$, $(\mathrm{M}_S h)(t) \in \underline{Z}$ for almost every $t \in [0,T]$, and $t \mapsto (\mathrm{M}_S h)(t)$ is continuous with respect to $\|\cdot\|_Z$.*

The following proposition shows that this assumption, together with the previous ones, is sufficient for TV IV to be satisfied for the perturbed system.

Proposition 3.2.4. *Suppose that PS I–V hold. Then the mild evolution operator $U(t,s)$ defined in Corollary 3.2.1 satisfies TV IV.*

Proof. We will treat each part of TV IV separately. For the first part notice that

$$\int_0^t S(t-s)D(s)h(s)\,ds = (\mathbb{M}_S h)(t) \in \underline{Z}$$

for all $h(\cdot) \in L^s(0,T;\overline{W})$ by assumption. We see from Remark 3.2.3 that

$$s \mapsto \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma$$

is measurable. Furthermore we see that this map is integrable in $\underline{Z}$:

$$\begin{aligned}\int_0^t \left\| \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma \right\|_{\underline{Z}} ds &\le \int_0^t \overline{K}_1(\tau)\overline{M}_U \|D(s)h(s)\|_{\overline{Z}}\,ds \\ &\le \overline{K}_1(\tau)\overline{M}_U t^{1/s'} \|D(\cdot)h(\cdot)\|_{L^s(0,t;\overline{Z})}\end{aligned}$$

where $1/s' + 1/s = 1$. Therefore

$$(\mathbb{M}_U h)(t) = (\mathbb{M}_S h)(t) + \int_0^t \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds \in \underline{Z}$$

almost everywhere as required.

For the second part let $h > 0$ and $h(\cdot) \in L^s(0,T;\overline{W})$. We have

$$\begin{aligned}&(\mathbb{M}_U h)(t+h) - (\mathbb{M}_U h)(t) \\ &= (\mathbb{M}_S h)(t+h) - (\mathbb{M}_S h)(t) + \int_0^{t+h}\int_s^{t+h} S(t+h-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds \\ &\quad - \int_0^t \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds \\ &= (\mathbb{M}_S h)(t+h) - (\mathbb{M}_S h)(t) + \int_t^{t+h}\int_s^{t+h} S(t+h-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds \\ &\quad + \int_0^t \left(\int_s^{t+h} S(t+h-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma \right. \\ &\quad \left. - \int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma \right) ds.\end{aligned}$$

Condition PS V ensures that

$$\|\left(\mathbb{M}_S h\right)(t+h)-\left(\mathbb{M}_S h\right)(t)\|_Z \longrightarrow 0 \quad \text{as} \quad h \downarrow 0.$$

Using Remark 3.2.3 note that

$$\begin{aligned}&\|\int_t^{t+h}\int_s^{t+h} S(t+h-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds\|_Z \\ &\qquad\qquad \leq R_1\int_t^{t+h}\overline{K}_1(\tau)\overline{M}_U\|D(s)h(s)\|_{\overline{Z}}\,ds \to 0 \quad \text{as} \quad h\downarrow 0.\end{aligned}$$

Furthermore

$$\begin{aligned}&\|\int_0^t\int_s^{t+h} S(t+h-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma-\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma ds\|_Z \\ &\qquad \leq R_1\int_0^t\|\int_s^{t+h} S(t+h-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma \\ &\qquad\qquad -\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\|_{\underline{Z}}\,ds\end{aligned}$$

which converges to zero by the continuity in t of the extension

$$D(s)h(s)\mapsto\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma$$

given in Remark 3.2.3 and the Lebesgue Dominated Convergence Theorem. Therefore

$$\|\left(\mathbb{M}_U h\right)(t+h)-\left(\mathbb{M}_U h\right)(t)\|_Z \longrightarrow 0 \quad \text{as} \quad h \downarrow 0.$$

Similarly

$$\begin{aligned}&\left(\mathbb{M}_U h\right)(t-h)-\left(\mathbb{M}_U h\right)(t) \\ &\quad=\left(\mathbb{M}_S h\right)(t-h)-\left(\mathbb{M}_S h\right)(t)+\int_{t-h}^t\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds \\ &\quad+\int_0^{t-h}\left(\int_s^{t-h} S(t-h-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\right. \\ &\qquad\qquad\left.-\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\right)ds.\end{aligned}$$

Note that

$$\|\int_{t-h}^t\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds\|_Z\leq R_1\int_{t-h}^t\overline{K}_1(\tau)\overline{M}_U\|D(s)h(s)\|_{\overline{Z}}\,ds$$

which converges to zero as $h \downarrow 0$. Proceeding as before shows that

$$\| (\mathbb{M}_U h)(t-h) - (\mathbb{M}_U h)(t) \|_Z \longrightarrow 0 \quad \text{as} \quad h \downarrow 0.$$

Hence the map $t \mapsto (\mathbb{M}_U h)(t)$ is continuous with respect to the norm in Z. □

Therefore if conditions PS I–V hold together with the basic assumptions introduced at the beginning of this section then the perturbed system satisfies TV I–IV.

In the next subsection the remaining conditions that are sufficient to be able to apply Theorem 2.2.10 to the perturbed system are considered.

3.2.2 Existence of mild solution

To show the existence of a mild solution for (3.8) conditions on the semigroup which imply TV V, TV VII–X hold for the evolution operator generated by $A + P(\cdot)$ are determined.

Since $U(t,s)$ is a mild evolution operator on all three spaces $\underline{Z}$, Z, and $\overline{Z}$ condition TV VII is satisfied automatically. Suppose that the semigroup satisfies TV V:

PS VI. *There exists a constant $\overline{K}_2$ such that*

$$\|CS(T)Bu\|_Y \leq \overline{K}_2 \|u\|_U$$

for every $u \in U$.

Recall that, for each $z \in \overline{Z}$,

$$U(T,0)z = S(T)z + \int_0^T S(T-\sigma)P(\sigma)U(\sigma,0)z \, d\sigma$$

where the last term is the extension given in Remark 3.2.3. If this extension is a bounded map into V for $t = T$, then replacing z by Bu shows that $\phi \in \mathcal{L}(U,Y)$. The next condition ensures that this is the case, though it is slightly stronger than is necessary.

PS VII. *There exists a continuous function $\overline{K}_3(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that $\overline{K}_3(T) = 0$ and for every $s \in [0,T]$, $z(\cdot) \in C(s,T;\underline{Z})$,*

$$\int_s^T S(T-\sigma)P(\sigma)z(\sigma)\, d\sigma \in V$$

with

$$\|\int_s^T S(T-\sigma)P(\sigma)z(\sigma)\,d\sigma\|_V \leq \overline{K}_3(s)\|z\|_{C(s,T;\overline{Z})}.$$

This condition ensures that the extension given in Remark 3.2.3 maps into V for each $s \in [0,T]$ when $t = T$.

Remark 3.2.5. For every $z \in \underline{Z}$ we have

$$\begin{aligned}\|\int_s^T S(T-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma\|_V &\leq \overline{K}_3(s)\|U(\cdot,s)z\|_{C(s,T;\overline{Z})}\\ &\leq \overline{K}_3(\tau)\overline{M}_U\|z\|_{\overline{Z}}\end{aligned}$$

where $\overline{K}_3(\tau) = \sup_{\sigma\in[0,T]}\overline{K}_3(\sigma)$. By the comments in Remark 3.2.3 and the continuous injectivity of V into Z (and hence $\overline{Z}$), we see that the extension of

$$z \longmapsto \int_s^T S(T-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma$$

to a bounded linear map from $\overline{Z}$ to $\underline{Z}$ also maps into V. Furthermore this extension is strongly continuous in s with respect to $\mathcal{L}(\overline{Z},V)$; that is, for every $z \in \overline{Z}$, the map from $[0,T]$ to V given by

$$s \longmapsto \int_s^T S(T-\sigma)P(\sigma)U(\sigma,s)z\,d\sigma$$

is continuous.

Proposition 3.2.6. *Suppose that PS I–VII are satisfied by the perturbed system. Then there exists a constant K_1 such that*

$$\|CU(T,0)Bu\|_Y \leq K_1\|u\|_U$$

for every $u \in U$.

Proof. Let $u \in U$. Then

$$\begin{aligned}\|CU(T,0)Bu\|_Y &= \|CS(T)Bu + C\int_0^T S(T-\sigma)P(\sigma)U(\sigma,0)Bu\,d\sigma\|_Y\\ &\leq \overline{K}_2\|u\|_U + \|C\|\overline{K}_3(0)\overline{M}_U\|Bu\|_{\overline{Z}}\\ &\leq \left(\overline{K}_2 + \|C\|\overline{K}_3(0)\overline{M}_U R_2\|B\|\right)\|u\|_U\end{aligned}$$

as required. □

This is TV V and in fact, together with the following condition PS VII is sufficient for TV X to be satisfied.

PS VIII. *There exists a constant* $\overline{K}_4$ *such that* $(\mathbb{M}_S h)(T) \in V$ *with*

$$\| (\mathbb{M}_S h)(T)\|_V \le \overline{K}_4 \|h\|_{L^s(0,T;\overline{W})}$$

for all $h(\cdot) \in L^s(0,T;\overline{W})$.

Proposition 3.2.7. *Suppose that PS I–VIII are satisfied. Then* $(\mathbb{M}_U h)(T) \in V$ *and there exists a constant* K_4 *such that*

$$\| (\mathbb{M}_U h)(T)\|_V \le K_4 \|h\|_{L^s(0,T;\overline{W})}$$

for all $h(\cdot) \in L^s(0,T;\overline{W})$.

Proof. Let $h(\cdot) \in L^s(0,T;\overline{W})$. Note that

$$\int_0^T U(T,s)D(s)h(s)\,ds = \int_0^T S(T-s)D(s)h(s)\,ds + \int_0^T \int_s^T S(T-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds.$$

Condition PS VIII ensures that $(\mathbb{M}_S h)(T) \in V$ and

$$\| \int_0^T S(T-s)D(s)h(s)\,ds\|_V \le \overline{K}_4 \|h\|_{L^s(0,T;\overline{W})}.$$

We see from Remark 3.2.5 that

$$s \longmapsto \int_s^T S(T-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma$$

is measurable as a map from $[0,T]$ to V. Furthermore

$$\int_0^T \| \int_s^T S(T-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\|_V\,ds \le \int_0^T \overline{K}_3(\tau)\overline{M}_U \|D(s)h(s)\|_{\overline{Z}}\,ds$$
$$\le \overline{K}_3(\tau)\overline{M}_U T^{1/s'} \|D(\cdot)h(\cdot)\|_{L^s(0,T;\overline{Z})}$$

where $1/s' + 1/s = 1$ and so the map is integrable in V. Putting these estimates together yields

$$\| \int_0^T U(T,s)D(s)h(s)\,ds\|_V \le \overline{K}_4 \|h\|_{L^s(0,T;\overline{W})} + \overline{K}_3(\tau)\overline{M}_U T^{1/s'} \|D(\cdot)h(\cdot)\|_{L^s(0,T;\overline{Z})}$$
$$\le \left(\overline{K}_4 + \overline{K}_3(\tau)\overline{M}_U T^{1/s'} \|D\|_\infty\right) \|h\|_{L^s(0,T;\overline{W})}$$

as required. □

Now suppose that the semigroup satisfies condition TV VIII.

PS IX. *There exists a positive constant* $\overline{K}_5$ *such that*

$$\|E(\cdot)S(\cdot)z\|_{L^r(0,T;\underline{W})} \le \overline{K}_5\|z\|_Z$$

for each $z \in \underline{Z}$.

Proposition 3.2.8. *Suppose that PS I–IX hold. Then there exists a positive constant* K_2 *such that*

$$\|E(\cdot)U(\cdot,0)z\|_{L^r(0,T;\underline{W})} \le K_2\|z\|_Z,$$

for each $z \in \underline{Z}$.

Proof. Let $z \in \underline{Z}$. Then

$$\begin{aligned}
\|E(\cdot)U(\cdot,0)z\|_{L^r(0,T;\underline{W})} &\le \|E(\cdot)S(\cdot)z\|_{L^r(0,T;\underline{W})} + \|E(\cdot)\int_0^{\cdot} S(\cdot-\sigma)P(\sigma)U(\sigma,0)z\,d\sigma\|_{L^r(0,T;\underline{W})} \\
&\le \overline{K}_5\|z\|_Z + \left(\int_0^T \|E(t)\int_0^t S(t-\sigma)P(\sigma)U(\sigma,0)z\,d\sigma\|_{\underline{W}}^r\,dt\right)^{1/r} \\
&\le \overline{K}_5\|z\|_Z + \|E\|_\infty \left(\int_0^T \overline{K}_1(\tau)^r\|U(\cdot,0)z\|_{C(0,T;\overline{Z})}^r\,dt\right)^{1/r} \\
&\le \overline{K}_5\|z\|_Z + T^{1/r}\|E\|_\infty\overline{K}_1(\tau)R_2M_U\|z\|_Z,
\end{aligned}$$

where $\|E\|_\infty = \sup_{t\in[0,T]}\|E(t)\|_{\mathcal{L}(\underline{Z},\underline{W})}$. Therefore we set $K_2 = \overline{K}_5 + T^{1/r}\|E\|_\infty\overline{K}_1(\tau)R_2M_U$. □

Again, assuming TV IX holds for the semigroup is sufficient, together with the preceding assumptions, for it to hold for the evolution operator. Define

$$(\mathbb{L}_S h)(t) = E(t)\int_0^t S(t-s)D(s)h(s)\,ds. \tag{3.10}$$

for $h \in L^s(0,T;\overline{W})$.

PS X. *There exists a positive constant* $\overline{K}_6$ *such that*

$$\|\mathbb{L}_S h\|_{L^r(0,T;\underline{W})} \le \overline{K}_6\|h\|_{L^s(0,T;\overline{W})},$$

for $h \in L^s(0,T;\overline{W})$.

Proposition 3.2.9. *Suppose that PS I–X hold. Then there exists a positive constant K_3 such that*

$$\|\mathbb{L}_U h\|_{L^r(0,T;\underline{W})} \leq K_3 \|h\|_{L^s(0,T;\overline{W})},$$

for $h \in L^s(0,T;\overline{W})$.

Proof. We have

$$\begin{aligned}
&\|E(\cdot)\int_0^{\cdot} U(\cdot,s)D(s)h(s)\,ds\|_{L^r(0,T;\underline{W})} \\
&\quad \leq \|E(\cdot)\int_0^{\cdot} S(\cdot - s)D(s)h(s)\,ds\|_{L^r(0,T;\underline{W})} \\
&\quad + \left(\int_0^T \|E(t)\int_0^t\int_s^t S(t-\sigma)P(\sigma)U(\sigma,s)D(s)h(s)\,d\sigma\,ds\|_{\underline{W}}^r dt\right)^{1/r} \\
&\quad \leq \overline{K}_6\|h\|_{L^s(0,T;\overline{W})} + \|E\|_\infty \left(\int_0^T \overline{K}_1(\tau)^r \overline{M}_U^r t^{r/s'} \|D(\cdot)h(\cdot)\|_{L^s(0,t;\overline{Z})}^r\,dt\right)^{1/r}
\end{aligned}$$

where $1/s' + 1/s = 1$ using an estimate from the proof of Proposition 3.2.4. Hence

$$\begin{aligned}
&\|E(\cdot)\int_0^{\cdot} U(\cdot,s)D(s)h(s)\,ds\|_{L^r(0,T;\underline{W})} \\
&\qquad \leq \overline{K}_6\|h\|_{L^s(0,T;\overline{W})} + \|E\|_\infty \overline{K}_1(\tau)\overline{M}_U T^{\frac{r+s'}{rs'}} \|D(\cdot)h(\cdot)\|_{L^s(0,T;\overline{Z})} \\
&\qquad \leq \left(\overline{K}_6 + \|E\|_\infty \overline{K}_1(\tau)\overline{M}_U T^{\frac{r+s'}{rs'}} \|D\|_\infty\right) \|h(\cdot)\|_{L^s(0,T;\overline{W})}
\end{aligned}$$

where $\|D\|_\infty = \sup_{t\in[0,T]} \|D(t)\|_{\mathcal{L}(\overline{W},\overline{Z})}$ and the proof is completed. □

Therefore if the conditions PS I–X are satisfied together with TV VI, then we can apply Theorem 2.2.10 to the problem of finding a control u such that the output $y = y_d$, the desired value.

3.3 A semilinear system

Suppose that the original system dynamics are semilinear

$$\dot{\overline{z}}(t) = A\overline{z}(t) + g(t,\overline{z}(t)), \qquad \overline{z}(0) = \overline{z}_0 \tag{3.11}$$

where PS I is satisfied, $g : [0,T] \times \underline{Z} \longrightarrow \overline{Z}$ and the state is considered with values in Z. Recall that the first step in applying the theory of Chapter 2 is to linearise the system about a solution trajectory. In this section it is shown that, with certain conditions on g, there exists a solution of (3.11) and the system can be linearised about the resulting trajectory. The solution in this case will be based on the following definition.

Definition 3.3.1. A function $\overline{z}(\cdot) \in C(0,T;Z)$ is a *classical solution* of (3.11) on $[0,T]$ if it is continuously differentiable (with values in Z), $\overline{z}(t) \in D(A)$, the domain of the operator A, and $\overline{z}(t)$ satisfies (3.11) for all $t \in [0,T]$.

Throughout this section it will be assumed that A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on Z that satisfies PS III.

Under certain conditions there exists a mild solution of (3.11) and the proof of this is the subject of the next subsection. In the following subsection it is shown that this mild solution is a classical one, except that the continuous differentiability is considered with respect to $\overline{Z}$. The approach taken follows that of Pazy (1983) who considered (3.11) with $g : [0,T] \times Z \longrightarrow Z$.

3.3.1 Existence of a mild solution

Definition 3.3.2. A *mild solution* of (3.11) is any function $\overline{z}(\cdot) \in C(0,T;\underline{Z})$ such that $\overline{z}(0) = \overline{z}_0 \in \underline{Z}$ and $\overline{z}$ satisfies the following equation

$$\overline{z}(t) = S(t)\overline{z}_0 + \int_0^t S(t-s)g(s,\overline{z}(s))\,ds \tag{3.12}$$

for all $t \in [0,T]$.

PS XI. *There exists a continuous function $k_1(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that $k_1(0) = 0$ and for all $s \in [0,T]$, $\overline{z}(\cdot) \in C(s,T;\overline{Z})$, $t \in [s,T]$ we have $\int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \in \underline{Z}$ with*

$$\left\| \int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \right\|_{\underline{Z}} \leq k_1(t-s)\|\overline{z}\|_{C(s,t;\overline{Z})}. \tag{3.13}$$

The following remark shows that the previous assumption is sufficient for the mild solution (if it exists) to be well-defined.

Remark 3.3.3. If PS XI holds then for each $s \in [0,T]$ the map

$$t \longmapsto \int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma$$

is continuous (with respect to the norm in $\underline{Z}$) for every $\overline{z}(\cdot) \in C(s,T;\overline{Z})$. To see this let $h > 0$, $s \in [0,T]$, $\overline{z}(\cdot) \in C(s,T;\overline{Z})$ and consider

$$\begin{aligned}
&\| \int_s^{t+h} S(t+h-\sigma)\overline{z}(\sigma)\,d\sigma - \int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \|_{\underline{Z}} \\
&\quad \le \| \int_t^{t+h} S(t+h-\sigma)\overline{z}(\sigma)\,d\sigma \|_{\underline{Z}} + \| \int_s^t \left(S(t+h-\sigma) - S(t-\sigma)\right)\overline{z}(\sigma)\,d\sigma \|_{\underline{Z}} \\
&\quad \le k_1(h)\|\overline{z}\|_{C(t,t+h;\overline{Z})} + \| (S(h) - I) \int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \|_{\underline{Z}}
\end{aligned}$$

which converge to zero as $h \downarrow 0$. Similarly we have

$$\begin{aligned}
&\| \int_s^{t-h} S(t-h-\sigma)\overline{z}(\sigma)\,d\sigma - \int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \|_{\underline{Z}} \\
&\quad = \| \int_{s+h}^t S(t-\sigma)\overline{z}(\sigma-h)\,d\sigma - \int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \|_{\underline{Z}} \\
&\quad \le \| \int_s^{s+h} S(t-\sigma)\overline{z}(\sigma)\,d\sigma \|_{\underline{Z}} + \| \int_{s+h}^t S(t-\sigma)\left(\overline{z}(\sigma-h) - \overline{z}(\sigma)\right)\,d\sigma \|_{\underline{Z}} \\
&\quad \le \|S(t-s-h)\| k_1(h) \|\overline{z}\|_{C(s,s+h;\overline{Z})} + k_1(t-h-s)\|\overline{z}(\cdot - h) - \overline{z}(\cdot)\|_{C(s,t;\overline{Z})}
\end{aligned}$$

which converge to zero as $h \downarrow 0$ by the continuity of $k_1(\cdot)$, $\overline{z}(\cdot)$ and the norm in $C(s,t;\overline{Z})$.

The following results show that the hypotheses introduced so far are sufficient for there to exist a mild solution of (3.11) if the nonlinearity satisfies a uniform Lipschitz condition. The proof follows the method of Pazy (1983) for Lipschitz perturbations of linear evolution equations.

Proposition 3.3.4. *Suppose that for the system given by* (3.11) *PS I holds, A is the generator of a strongly continuous semigroup $S(t)$ that satisfies PS III and PS XI. Furthermore suppose that $g : [t_0,T] \times \underline{Z} \longrightarrow \overline{Z}$ is continuous in t and there exists a constant k_2 such that for each t $\in [t_0,T]$*

$$\|g(t,z) - g(t,y)\|_{\overline{Z}} \le k_2 \|z - y\|_{\underline{Z}}$$

for all $z, y \in \underline{Z}$. Then for all $w(\cdot) \in C(t_0,T;\underline{Z})$ there exists a unique solution of the integral equation

$$z(t) = w(t) + \int_{t_0}^t S(t-s)g(s,z(s))\,ds$$

in $C(t_0,T;\underline{Z})$.

Proof. We start by defining the map $\mathcal{F} : C(t_0, T; \underline{Z}) \longrightarrow C(t_0, T; \underline{Z})$ by

$$(\mathcal{F}z)(t) = w(t) + \int_{t_0}^{t} S(t-s)g(s, z(s))\, ds.$$

We show that $\mathcal{F}$ is a contraction when restricted to the interval $[t_0, t_1]$, and by applying the Contraction Mapping Theorem we obtain a unique solution of the integral equation on this interval.

We have

$$(\mathcal{F}z_1)(t) - (\mathcal{F}z_2)(t) = \int_{t_0}^{t} S(t-s)\left(g(s, z_1(s)) - g(s, z_2(s))\right) ds$$

for every $z_1, z_2 \in C(t_0, T; \underline{Z})$, and so

$$\begin{aligned} \|(\mathcal{F}z_1)(t) - (\mathcal{F}z_2)(t)\|_{\underline{Z}} &\leq k_1(t-t_0) \sup_{\sigma\in[t_0,t]} \|g(\sigma, z_1(\sigma)) - g(\sigma, z_2(\sigma))\|_{\underline{Z}} \\ &\leq k_1(t-t_0)k_2\, \|z_1 - z_2\|_{C(t_0,t;\underline{Z})}. \end{aligned}$$

Since $k_1(\cdot)$ is continuous on the closed and bounded interval $[0, t_1 - t_0]$ it is bounded and attains its bounds. Therefore suppose $\tau \in [0, t_1 - t_0]$ is such that

$$k_1(\tau) = \sup_{t\in[0,t_1-t_0]} k_1(t).$$

Then

$$\|\mathcal{F}z_1 - \mathcal{F}z_2\|_{C(t_0,t_1;\underline{Z})} \leq k_1(\tau)k_2\|z_1 - z_2\|_{C(t_0,t_1;\underline{Z})}.$$

Since $k_1(\cdot)$ is continuous and $k_1(0) = 0$ we can choose $t_1 > t_0$ such that $k_1(\tau)k_2 < 1$. Therefore $\mathcal{F}$ is a contraction on $C(t_0, t_1; \underline{Z})$ and applying the Contraction Mapping Theorem gives the unique fixed point $\overline{z}(\cdot) \in C(t_0, t_1; \underline{Z})$. Let $\delta = t_1 - t_0$, and partition the interval $[0, T]$ into N intervals such that

$$[0, T] = \bigcup_{i=0}^{N-1} [t_i, t_{i+1}],$$

$t_N = T$ and $t_{i+1} - t_i < \delta$ for $0 \leq i \leq N-1$. Therefore for $0 \leq i \leq N-1$

$$\sup_{\sigma\in[0,t_{i+1}-t_i]} k_1(\sigma) < k_1(\tau)$$

and so, applying the above method, we see that $\mathcal{F}$ is a contraction on each subspace $C(t_i, t_{i+1}; \underline{Z})$. Let z_i be the unique fixed point in $C(t_i, t_{i+1}; \underline{Z})$ for $0 \leq i \leq N-1$ after applying the Contraction Mapping Theorem. By defining $\overline{z}(t) = z_i(t)$ for $t \in [t_i, t_{i+1}]$ we see that $\overline{z}$ is the unique solution of the integral equation

$$z(t) = w(t) + \int_{t_0}^{t} S(t-s)g(s, z(s))\, ds$$

as required. □

Clearly, if $\overline{z}_0 \in \underline{Z}$ then $S(\cdot)\overline{z}_0 \in C(0, T; \underline{Z})$ and so applying the proposition yields a unique mild solution (3.12) of (3.11). A similar result holds for the slightly weaker hypothesis that g is only locally Lipschitz continuous in z, uniformly in t.

Proposition 3.3.5. *Suppose that for the system given by* (3.11) *PS I holds, A is the generator of a strongly continuous semigroup $S(t)$ that satisfies PS III and PS XI. Furthermore suppose that $g : [0, T] \times \underline{Z} \longrightarrow \overline{Z}$ is continuous in t and there exists a constant $k_3(c)$ such that for each $t \in [0, T]$*

$$\|g(t, z) - g(t, y)\|_{\overline{Z}} \leq k_3(c)\|z - y\|_{\underline{Z}}$$

for all $z, y \in \underline{Z}$ such that $\|y\|_{\underline{Z}}, \|z\|_{\underline{Z}} \leq c$. Then there exists a $\tilde{T} \leq T$ and a unique mild solution of (3.11) *in $C(0, \tilde{T}; \underline{Z})$.*

Proof. Let $t_0 \in [0, T]$ and define $\mathcal{F}$ as in the proof of Proposition 3.3.4. We will again restrict attention, for the moment, to the interval $[t_0, t_1]$ where $t_1 = t_0 + \delta(t_0)$ for some $\delta(t_0) > 0$. On this interval we will show that there exists a unique solution of (3.12) provided δ is suitably chosen. Let $K(t_0) = 2\gamma_0\|\overline{z}_0\|_{\underline{Z}}$ where $\gamma_0 = \sup_{0 \leq t \leq T} \|S(t)\|$, and choose $\delta(t_0)$ such that

$$\sup_{t \in [0, \delta(t_0)]} k_1(t) = k_1(\tau_0) \leq \frac{\gamma_0\|\overline{z}_0\|}{K(t_0)k_3(K(t_0)) + N_0} \tag{3.14}$$

where $N_0 = \sup\{\|g(t, 0)\|_{\overline{Z}} : 0 \leq t \leq T\}$. The continuity of k_1 and the fact that $k_1(0) = 0$ ensure that $\delta(t_0)$ can be chosen in this way. We firstly show that $\mathcal{F}$ maps the ball of radius $K(t_0)$ in

$C(t_0, t_1; \underline{Z})$ into itself.

$$
\begin{aligned}
\|(\mathcal{F}\overline{z})(t)\|_{\underline{Z}} &\le \gamma_0\|\overline{z}_0\|_{\underline{Z}} + \|\int_{t_0}^{t} S(t-s)g(s,\overline{z}(s))\,ds\|_{\underline{Z}} \\
&\le \gamma_0\|\overline{z}_0\|_{\underline{Z}} + k_1(t-t_0)\sup_{s\in[t_0,t_1]}\left(\|g(s,\overline{z}(s)) - g(s,0)\|_{\overline{Z}} + \|g(s,0)\|_{\overline{Z}}\right) \\
&\le \gamma_0\|\overline{z}_0\|_{\underline{Z}} + k_1(\tau_0)\left(k_3(K(t_0))K(t_0) + N_0\right) \\
&\le 2\gamma_0\|\overline{z}_0\|_{\underline{Z}}.
\end{aligned}
$$

On this ball $\mathcal{F}$ satisfies a uniform Lipschitz condition and so applying Proposition 3.3.4 shows that there exists a unique solution of (3.12) on $[t_0, t_1]$.

Suppose that $\overline{z}$ is the unique mild solution on $[0, \tau]$ then the above shows that it can be extended to the interval $[0, \tau+\delta]$ with $\delta > 0$ by defining $\overline{z}$ on $[\tau, \tau+\delta]$ by $\overline{z}(t) = w(t)$ where $w(t)$ is the unique solution of the integral equation

$$
w(t) = S(t-\tau)\overline{z}(\tau) + \int_{\tau}^{t} S(t-s)g(s,w(s))\,ds.
$$

Moreover, δ depends only on $\|\overline{z}(\tau)\|$. Therefore we can extend the mild solution to the whole interval $[0, \tilde{T}]$ for some suitable $\tilde{T} \le T$. □

Provided the initial guess for the control is chosen such that $z'(0) = z_0' + Bu' \in \underline{Z}$ the proposition shows that there exists a mild solution $z'(\cdot) \in C^0(0, \tilde{T}; \underline{Z})$. For the purposes of the control problem T is chosen to be $\tilde{T}$.

3.3.2 Existence of a classical solution

The inhomogeneous differential equation

$$
\dot{\overline{z}}(t) = A\overline{z}(t) + h(t), \qquad \overline{z}(0) = \overline{z}_0 \tag{3.15}
$$

where $h(\cdot) \in C^1(0, T; Z)$ has been shown by many authors to have a classical solution satisfying Definition 3.3.1 (Curtain and Pritchard, 1977, 1978; Pazy, 1983; Curtain and Zwart, 1995). To show that there exists a classical solution of (3.11) it will be shown that, for a mild solution $\overline{z}$, the map from $[0, T]$ to $\overline{Z}$ defined by $h(t) = g(t, \overline{z}(t))$ is continuously differentiable in some suitably chosen space, to be made precise later, containing $\overline{Z}$. This suggests that a classical solution is sought that is continuously differentiable with respect to $\overline{Z}$. Now considering the homogeneous part of (3.15) note that any solution is of the form

$$\overline{z}(t) = S(t)\overline{z}_0.$$

Since $S(t)$ is a strongly continuous semigroup on $\overline{Z}$ if $\overline{z}_0 \in \overline{D}(A)$, the domain of A when considered as the generator of $S(t)$ on $\overline{Z}$, then this solution is continuously differentiable. These considerations motivate:

Definition 3.3.6. A *solution* of (3.11) on $[0,T]$ is any function $\overline{z}(\cdot) \in C(0,T;\underline{Z})$ such that $\overline{z}(\cdot) \in C^1(0,T;\overline{Z})$, $\overline{z}(t) \in \overline{D}(A)$, the domain of the operator A with respect to $\overline{Z}$, and $\overline{z}(t)$ satisfies (3.11) for all $t \in [0,T]$.

To ensure that $S(\cdot)\overline{z}_0 \in C(0,T;\underline{Z})$ for all $\overline{z}_0 \in \overline{D}(A)$ the following assumption is introduced.

PS XII. $\overline{D}(A) \subset \underline{Z}$.

Suppose that there exists a solution $\overline{z}$ of (3.11). Then the map $s \mapsto S(t-s)\overline{z}(s)$ on $[0,t]$ is differentiable for each $t \in (0,T)$. Therefore, for $t \in (0,T)$,

$$\begin{aligned}\frac{d}{ds}S(t-s)\overline{z}(s) &= -AS(t-s)\overline{z}(s) + S(t-s)\dot{\overline{z}}(s)\\ &= S(t-s)\left(\dot{\overline{z}}(s) - A\overline{z}(s)\right)\\ &= S(t-s)g(s,\overline{z}(s))\end{aligned}$$

and so, if $g(\cdot,\overline{z}(\cdot))$ is continuous, integrating (in $\overline{Z}$) yields

$$\overline{z}(t) = S(t)\overline{z}_0 + \int_0^t S(t-s)g(s,\overline{z}(s))\,ds.$$

Therefore a solution of (3.11) is a mild solution. Hence in this subsection the mild solution of the previous subsection is shown to be, under suitable conditions, a solution. Clearly, for a mild solution, it is only the integral term that must be shown to be continuously differentiable and satisfy (3.11). The following modified version of a result from Pazy (1983,Theorem 4.2.4, page 107) gives some general criteria for the existence of a solution of (3.15).

Theorem 3.3.7. *Suppose that PS I holds; A is the generator of a strongly continuous semigroup $S(t)$ on Z that satisfies PS III, PS XI and PS XII; and $h(\cdot) \in C(0,T;\overline{Z})$. Let*

$$v(t) = \int_0^t S(t-s)h(s)\,ds, \qquad 0 \le t \le T.$$

The initial value problem (3.15) *has a solution on* $[0,T]$ *for every* $\overline{z}_0 \in \overline{D}(A)$ *if one of the following conditions is satisfied:*

(i) $v(\cdot) \in C(0,T;\underline{Z}) \cap C^1(0,T;\overline{Z})$.

(ii) $v(t) \in \overline{D}(A)$ *for every* $t \in [0,T]$ *and* $Av(\cdot)$ *is continuous on* $[0,T]$ *with respect to the norm in* $\overline{Z}$.

If (3.15) *has a solution for some* $\overline{z}_0 \in \overline{D}(A)$ *then* v *satisfies both (i) and (ii).*

Proof. Suppose that (3.15) has a solution for some $\overline{z}_0 \in \overline{D}(A)$. Then this solution is given by

$$\overline{z}(t) = S(t)\overline{z}_0 + \int_0^t S(t-s)h(s)\,ds$$

which gives, on rearranging,

$$v(t) = \overline{z}(t) - S(t)\overline{z}_0 \in \overline{D}(A) \subset \underline{Z},$$

and is continuous with respect to $\|\cdot\|_{\underline{Z}}$. Furthermore, since the right-hand side is differentiable in $\overline{Z}$, we have

$$\dot{v}(t) = \dot{\overline{z}}(t) - S(t)A\overline{z}_0$$

which is continuous with respect to $\|\cdot\|_{\overline{Z}}$. Therefore the solution satisfies *(i)*. To see that the solution satisfies *(ii)* we note that

$$Av(t) = A\overline{z}(t) - AS(t)\overline{z}_0 = \dot{\overline{z}}(t) - h(t) - S(t)A\overline{z}_0$$

which is again continuous with respect to the norm in $\overline{Z}$.

For the first part of the theorem we notice that, for any $h > 0$, we have

$$\frac{S(h)-I}{h}v(t) = \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} S(t+h-s)h(s)\,ds. \tag{3.16}$$

Since $h(\cdot) \in C(0,T;\overline{Z})$ we see that the last term on the right-hand side of this equation converges to $h(t)$ with respect to the norm in $\overline{Z}$ as $h \downarrow 0$. If *(i)* is satisfied then from (3.16) we see that $v(t) \in \overline{D}(A)$ for all $t \in [0,T]$ and

$$Av(t) = \dot{v}(t) - h(t),$$

so $Av(\cdot)$ is continuous in $\overline{Z}$. Thus *(i)* implies *(ii)*.

If *(ii)* is satisfied then (3.16) implies that v is differentiable from the right in t and that this derivative satisfies

$$D^+v(t) = Av(t) + h(t).$$

The continuity of the right-hand side of this equation implies the continuity of the left-hand side and consequently that v is continuously differentiable with respect to the norm in $\overline{Z}$. This, together with PS XI and Remark 3.3.3, shows that *(i)* is satisfied. In both cases we have

$$\dot{v}(t) = Av(t) + h(t)$$

where $v(\cdot) \in C(0,T;\underline{Z}) \cap C^1(0,T;\overline{Z})$ and $v(t) \in \overline{D}(A)$ for each $t \in [0,T]$. Since $v(0) = 0$ we see that

$$\overline{z}(t) = S(t)\overline{z}_0 + v(t)$$

is a solution of (3.15). □

Remark 3.3.8. Suppose that in the last theorem $h(\cdot) \in C^1(0,T;\overline{Z})$. Then by definition we have

$$v(t) = \int_0^t S(t-s)h(s)\,ds = \int_0^t S(s)h(t-s)\,ds$$

and so PS XI shows that $v(\cdot) \in C(0,T;\underline{Z})$, while the last equality shows that v is differentiable. Now

$$\begin{aligned}\dot{v}(t) &= S(t)h(0) + \int_0^t S(s)\dot{h}(t-s)\,ds \\ &= S(t)h(0) + \int_0^t S(t-s)\dot{h}(s)\,ds\end{aligned}$$

and so $v(\cdot) \in C^1(0,T;\overline{Z})$. Hence condition *(i)* of Theorem 3.3.7 is satisfied and so there exists a solution of the inhomogeneous equation.

This remark demonstrates the main method used in the literature to show that there exists a classical solution of the inhomogeneous equation (for example Corollary 4.2.5 in Pazy, 1983; Curtain and Zwart, 1995,Theorem 3.1.3). Following the approach of Pazy (1983) it will not be possible (in general) to show that, for a mild solution $\overline{z}$, the map $t \mapsto g(t, \overline{z}(t))$ is continuously differentiable with respect to $\overline{Z}$. Therefore we introduce the following Banach space containing $\overline{Z}$.

PS XIII. $\overline{Z} \subset \overline{\overline{Z}}$ *where the canonical injection is continuous with dense range.*

The differentiability of $t \mapsto g(t, \overline{z}(t))$ will now be considered in $\overline{\overline{Z}}$. A further property for the semigroup is required.

PS XIV. $S(t)$ *is a strongly continuous semigroup on* $\overline{\overline{Z}}$. *Moreover, there exists a continuous function* $k_4(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ *such that* $k_4(0) = 0$ *and for all* $s \in [0,T]$, $\overline{z}(\cdot) \in C(s,T;\overline{\overline{Z}})$, $t \in [s,T]$ *we have* $\int_s^t S(t-\sigma)\overline{z}(\sigma)\, d\sigma \in \overline{Z}$ *with*

$$\| \int_s^t S(t-\sigma)\overline{z}(\sigma)\, d\sigma \|_{\overline{Z}} \leq k_4(t-s) \|\overline{z}\|_{C(s,t;\overline{\overline{Z}})}.$$

Corollary 3.3.9. *Suppose that PS I and PS XIII hold; A is the generator of a strongly continuous semigroup* $S(t)$ *on* Z *that satisfies PS III, PS XI–XII and PS XIV; and* $h(\cdot) \in C(0,T;\overline{Z}) \cap C^1(0,T;\overline{\overline{Z}})$. *Then* (3.15) *has a solution for each* $\overline{z}_0 \in \overline{D}(A)$.

Proof. We show that $v(t)$ (as defined in Theorem 3.3.7) is continuously differentiable with respect to the norm in $\overline{Z}$. We see that

$$\begin{aligned}
&\|\frac{1}{h}(v(t+h) - v(t)) - S(t)h(0) - \int_0^t S(t-s)\dot{h}(s)\, ds\|_{\overline{Z}} \\
&\quad \leq \|\frac{1}{h}\int_0^h S(t+h-s)h(s)\, ds - S(t)h(0)\|_{\overline{Z}} \\
&\qquad + \|\frac{1}{h}\int_h^{t+h} S(t+h-s)h(s)\, ds - \frac{1}{h}\int_0^t S(t-s)h(s)\, ds - \int_0^t S(t-s)\dot{h}(s)\, ds\|_{\overline{Z}} \\
&\quad \leq \epsilon(h) + \| \int_0^t S(t-s)\left(\frac{1}{h}(h(s+h) - h(s)) - \dot{h}(s)\right) ds\|_{\overline{Z}} \\
&\quad \leq \epsilon(h) + k_4(t) \sup_{\sigma\in[0,t]} \| \left(\frac{1}{h}(h(\sigma+h) - h(\sigma)) - \dot{h}(\sigma)\right) \|_{\overline{\overline{Z}}}
\end{aligned}$$

where $\epsilon(h) \to 0$ as $h \to 0$. Hence the right-hand side tends to zero as $h \to 0$ and v is differentiable from the right. The right-sided derivative is

$$S(t)h(0) + \int_0^t S(t-s)\dot{h}(s)\,ds$$

which is continuous in $\overline{Z}$ and so v is continuously differentiable in $\overline{Z}$. Now applying Theorem 3.3.7 yields a solution $\overline{z}$. □

To show that (3.11) has a solution it now only remains to show that for a mild solution, $\overline{z}$, $t \mapsto g(t, \overline{z}(t))$ is continuously differentiable (in $\overline{\overline{Z}}$). Unfortunately this requires that the mild solution is continuously differentiable and so further work is required. We follow the approach of Pazy (1983) and start with the following preliminary result based on Proposition 3.3.4.

Lemma 3.3.10. *Suppose that for the system given by* (3.11) *PS I and PS XIII hold; and A is the generator of a strongly continuous semigroup $S(t)$ that satisfies PS III, and PS XIV. Furthermore suppose that $h : [t_0, T] \times \overline{Z} \longrightarrow \overline{\overline{Z}}$ is continuous in t and there exists a constant k such that for each $t \in [0, T]$*

$$\|h(t,z) - h(t,y)\|_{\overline{\overline{Z}}} \leq k\|z - y\|_{\overline{Z}}$$

for all $z, y \in \overline{Z}$. Then for all $w(\cdot) \in C(t_0, T; \overline{Z})$ there exists a unique solution of the integral equation

$$u(t) = w(t) + \int_{t_0}^t S(t-s)h(s, u(s))\,ds$$

in $C(t_0, T; \overline{Z})$.

Proof. This follows from the method of the proof of Proposition 3.3.4. □

The next result, based on Pazy (1983,Theorem 6.1.5), guarantees the existence of a solution of (3.11) provided certain conditions are satisfied.

Theorem 3.3.11. *Suppose that for the system given by* (3.11) *PS I and PS XIII hold; A is the generator of a strongly continuous semigroup $S(t)$ that satisfies PS III, PS XI, PS XII and PS XIV; $g(\cdot,\cdot)$*

is continuously differentiable from $[0,T] \times \underline{Z} \longrightarrow \overline{Z}$*; and, if we let* $B(s,z) = (\partial/\partial z)g(s,z(s))$ *for* $z \in C(0,T;\underline{Z})$*, there exists a constant* $k_5(z)$ *(dependent on* z*) such that*

$$\|B(s,z)w\|_{\overline{\overline{Z}}} \leq k_5(z)\|w\|_{\overline{Z}} \tag{3.17}$$

for all $w \in \underline{Z}$*. Then for every* $\overline{z}_0 \in \overline{D}(A)$ *there exists a solution of* (3.11).

Proof. The continuous differentiability of $g(t,z)$ implies that g is continuous in t and locally Lipschitz continuous in z, uniformly in t. Therefore Proposition 3.3.5 shows that there exists a unique mild solution $\overline{z}$ of (3.11), provided T is suitably chosen. We will show that this mild solution is continuously differentiable in $\overline{Z}$.

Choose $\delta > 0$ such that $k_4(\tau) = \sup_{t\in[0,\delta]} k_4(t) < 1/k_5(\overline{z})$. Now partition the interval $[0,T]$ into the union

$$[0,T] = \bigcup_{i=0}^{N-1} [t_i, t_{i+1}]$$

such that $t_0 = 0$, $t_N = T$, $t_i < t_{i+1}$ and $t_{i+1} - t_i < \delta$. We will restrict attention for the moment to the interval $[t_i, t_{i+1}]$ for some $i \in \{0, \ldots, N-1\}$. Suppose that $\overline{z}(t_i) = \overline{z}_i \in \overline{D}(A)$.

From the assumptions of the theorem there exists an extension $\tilde{B}(t,\overline{z})$ of $B(t,\overline{z})$ to a bounded linear operator from $\overline{Z}$ into $\overline{\overline{Z}}$ for each $t \in [t_i, t_{i+1}]$. Furthermore, for each $w \in \overline{Z}$ there exists a sequence (w_n) into $\underline{Z}$ such that $w_n \to w$ in $\overline{Z}$. Hence

$$\|\tilde{B}(t,\overline{z})w - B(t,\overline{z})w_n\|_{\overline{\overline{Z}}} \leq k_5(\overline{z})\|w - w_n\|_{\overline{Z}} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

uniformly in t. Hence $\tilde{B}(\cdot,\overline{z})w$ is the uniform limit of the sequence of functions $(B(\cdot,\overline{z})w_n)$. By our assumptions the latter are continuous and so the function $h : [t_i, t_{i+1}] \times \overline{Z} \longrightarrow \overline{\overline{Z}}$ defined by $h(t,w) = \tilde{B}(t,\overline{z})w$ is continuous in t. Moreover, this function is uniformly Lipschitz continuous in w. Set

$$w(t) = S(t)g(t_i,\overline{z}(t_i)) + AS(t)\overline{z}_i + \int_{t_i}^{t} S(t-s)\frac{\partial}{\partial s}g(s,\overline{z}(s))\,ds.$$

From our assumptions we have $w(\cdot) \in C(t_i, t_{i+1}; \overline{Z})$. Therefore Lemma 3.3.10 guarantees the existence of a unique solution (in $C(t_i, t_{i+1}; \overline{Z})$) of the integral equation

$$u(t) = w(t) + \int_{t_i}^{t} S(t-s)\tilde{B}(s,\overline{z})u(s)\,ds.$$

Our assumptions on g yield the following estimates:

$$g(s,\overline{z}(s+h)) - g(s,\overline{z}(s)) = B(s,\overline{z})\left(\overline{z}(s+h) - \overline{z}(s)\right) + w_1(s,h)$$

and

$$g(s+h,\overline{z}(s+h)) - g(s,\overline{z}(s+h)) = \frac{\partial}{\partial s} g(s,\overline{z}(s+h)).h + w_2(s,h)$$

where $h^{-1}\|w_j(s,h)\|_{\overline{Z}} \to 0$ uniformly on $[t_i, t_{i+1}]$ for $j = 1,2$ as $h \downarrow 0$. Let

$$u_h(t) = h^{-1}\left(\overline{z}(t+h) - \overline{z}(t)\right) - u(t).$$

Then

$$\begin{aligned}
u_h(t) &= \frac{1}{h}\left(S(t+h)\overline{z}_i - S(t)\overline{z}_i\right) - AS(t)\overline{z}_i \\
&\quad + \frac{1}{h}\int_{t_i}^{t_i+h} S(t+h-s)g(s,\overline{z}(s))\,ds - S(t)g(t_i,\overline{z}(t_i)) \\
&\quad + \frac{1}{h}\left(\int_{t_i}^{t} S(t-s)g(s+h,\overline{z}(s+h))\,ds - \int_{t_i}^{t} S(t-s)g(s,\overline{z}(s))\,ds\right) \\
&\quad - \int_{t_i}^{t} S(t-s)\frac{\partial}{\partial s}g(s,\overline{z}(s))\,ds - \int_{t_i}^{t} S(t-s)\tilde{B}(s,\overline{z})u(s)\,ds \\
&= \left[\frac{1}{h}\left(S(h)-I\right)S(t)\overline{z}_i - AS(t)\overline{z}_i\right] + \left[\frac{1}{h}\int_{t_i}^{t} S(t-s)\left(w_1(s,h)+w_2(s,h)\right)\,ds\right] \\
&\quad + \left[\frac{1}{h}\int_{t_i}^{t_i+h} S(t+h-s)g(s,\overline{z}(s))\,ds - S(t)g(t_i,\overline{z}(t_i))\right] \\
&\quad + \left[\int_{t_i}^{t} S(t-s)\frac{\partial}{\partial s}g(s,\overline{z}(s+h))\,ds - \int_{t_i}^{t} S(t-s)\frac{\partial}{\partial s}g(s,\overline{z}(s))\,ds\right] \\
&\quad + \left[\frac{1}{h}\int_{t_i}^{t} S(t-s)B(s,\overline{z})\left(\overline{z}(s+h)-\overline{z}(s)\right)\,ds - \int_{t_i}^{t} S(t-s)\tilde{B}(s,\overline{z})u(s)\,ds\right].
\end{aligned}$$

Since the norm (in $\overline{Z}$) of each of the first four terms converges to zero as $h \downarrow 0$ we can estimate the sum of them by a function $\epsilon(h)$ that tends to zero as $h \downarrow 0$. Therefore

$$\begin{aligned}
\|u_h(t)\|_{\overline{Z}} &\leq \epsilon(h) + \left\|\frac{1}{h}\int_{t_i}^{t} S(t-s)\tilde{B}(s,\overline{z})\left(\overline{z}(s+h)-\overline{z}(s)\right)\,ds - \int_{t_i}^{t} S(t-s)\tilde{B}(s,\overline{z})u(s)\,ds\right\|_{\overline{Z}} \\
&\leq \epsilon(h) + k_4(t-t_i)k_5(\overline{z}) \sup_{s\in[t_i,t]} \|u_h(s)\|_{\overline{Z}}.
\end{aligned}$$

Hence

$$\|u_h(\cdot)\|_{C(t_i,t_{i+1};\overline{Z})} \leq \frac{\epsilon(h)}{1 - k_4(\tau)k_5(\overline{z})}$$

and the right-hand side converges to zero as $h \downarrow 0$. This means that, on $[t_i, t_{i+1}]$, $\overline{z}$ is differentiable from the right with derivative u. The continuity of u implies that $\overline{z}$ is continuously differentiable in $\overline{Z}$. Furthermore we see that

$$\frac{1}{h}\left(g(s+h,\overline{z}(s+h)) - g(s,\overline{z}(s))\right) - \frac{\partial}{\partial t}g(t,\overline{z}(t)) - \tilde{B}(t,\overline{z})u(t)$$
$$= \frac{1}{h}\left(w_1(t,h) + w_2(t,h)\right) + \frac{\partial}{\partial t}g(t,\overline{z}(t+h)) - \frac{\partial}{\partial t}g(t,\overline{z}(t))$$
$$+ \frac{1}{h}\tilde{B}(t,\overline{z})\left(\overline{z}(t+h) - \overline{z}(t)\right) - \tilde{B}(t,\overline{z})u(t)$$

and so $t \mapsto g(t,\overline{z}(t))$ is differentiable from the right in $\overline{\overline{Z}}$. The derivative $\frac{\partial}{\partial t}g(t,\overline{z}(t)) + \tilde{B}(t,\overline{z})u(t)$ is continuous and so we see that this map is continuously differentiable from $[t_i, t_{i+1}]$ to $\overline{\overline{Z}}$.

From Corollary 3.3.9 it follows that

$$v(t) = S(t)\overline{z}_i + \int_{t_i}^{t} S(t-s)g(s,\overline{z}(s))\,ds$$

is a solution of (3.11). But $\overline{z}$ is the unique mild solution and so $\overline{z} = v$ on $[t_i, t_{i+1}]$.

To complete the proof we note that at $t = t_0 = 0$ we have $\overline{z}(0) = \overline{z}_0 \in \overline{D}(A)$. Therefore we can apply the above to show that $\overline{z}$ is a solution on $[0, t_1]$. By the definition of the solution $\overline{z}(t_1) = \overline{z}_1 \in \overline{D}(A)$ and so we extend the solution to $[0, t_2]$. Proceeding in this way shows that $\overline{z}$ is the unique solution of (3.11) on $[0, T]$. □

In this section we have introduced some general criteria for the semilinear system given by (3.11) to have a solution in the classical sense. If the hypotheses of Theorem 3.3.11 are satisfied then for the controlled system

$$\dot{\overline{z}}(t) = A\overline{z}(t) + g(t,\overline{z}(t)), \qquad \overline{z}(0) = z_0' + B\overline{u}$$

if the initial guess for the control, u' say, is chosen such that $z'(0) = z_0' + Bu' \in \overline{D}(A)$ then there exists a solution z' when $\overline{z}(0) = z'(0)$. Let $P(t) = g'(t, z'(t))$ where g' denotes the derivative of g with respect to z. Since g is continuously differentiable from $[0,T] \times \underline{Z} \longrightarrow \overline{Z}$ and $z'(\cdot)$ is a classical solution, $P(\cdot) \in C(0,T;\mathcal{L}(\underline{Z},\overline{Z}))$. Define the following function from $[0,T] \times \underline{Z} \to \overline{Z}$:

$$h(t,z) = D(t)N(t,E(t)z) = g(t,z+z'(t)) - g(t,z'(t)) - P(t)z,$$

where D and E are linear operators chosen such that $N : [0,T] \times \underline{W} \longrightarrow \overline{W}$, where $\underline{W} \subset \overline{W}$ are Banach spaces, $D(\cdot) \in PC(0,T;\mathcal{L}(\overline{W},\overline{Z}))$ and $E(\cdot) \in PC(0,T;\mathcal{L}(\underline{Z},\underline{W}))$. Therefore, setting $\overline{z} = z' + z$ and $\overline{u} = u' + u$,

$$\begin{aligned}\dot{\overline{z}}(t) &= A\overline{z}(t) + g(t,\overline{z}(t)) \\ &= Az(t) + Az'(t) + g(t, z(t) + z'(t)) \\ &= (A + P(t))z(t) + D(t)N(t, E(t)z(t)) + Az'(t) + g(t, z'(t))\end{aligned}$$

and so

$$\dot{z}(t) = (A + P(t))z(t) + D(t)N(t, E(t)z(t)), \qquad z(0) = \overline{z}(0) - z'(0) = Bu.$$

Let $z_1, z_2 \in \underline{Z}$ and consider

$$\begin{aligned}\|h(t,z_1) - h(t,z_2)\|_{\overline{Z}} &= \|g(t, z_1 + z'(t)) - g(t, z'(t)) - P(t)z_1 \\ &\quad - g(t, z_2 + z'(t)) + g(t, z'(t)) + P(t)z_2\|_{\overline{Z}} \\ &\leq \|g(t, z_1 + z'(t)) - g(t, z_2 + z'(t))\|_{\overline{Z}} + \|P(t)\,(z_1 - z_2)\,\|_{\overline{Z}} \\ &\leq \left(k_2(c+\eta) + \sup_{t\in[0,T]} \|P(t)\|_{\mathcal{L}(\underline{Z},\overline{Z})} \right) \|z_1 - z_2\|_{\underline{Z}}\end{aligned}$$

where $k_2(c+\eta)$ is the Lipschitz constant for g when $(t, z_i) \in [0,T] \times B(z'(t), c)$ for $i = 1, 2$ and $\eta = \|z'(\cdot)\|_{C(0,T;\underline{Z})}$.

For the semilinear system we have shown how it is possible to perform the linearisation about a solution trajectory resulting from a suitably chosen initial guess u' given in the introduction to Chapter 2. The resulting system is of the form considered in this chapter.

Chapter 4

General Example

In Chapters 2 and 3 the nonlinear control problem has been considered. It has been shown that a solution to the control problem exists provided certain basic assumptions for the system hold. Chapter 2 dealt with the general mathematical system with the assumptions formulated in terms of an associated mild evolution operator, while Chapter 3 considered a specific type of system. In this more specific case the assumptions for the mild evolution operator were reformulated for the strongly continuous semigroup generated by the time-invariant operator and the time-varying perturbation describing the linear dynamics of the system.

In this chapter a general example will be considered. Conditions will be introduced that are sufficient for the example to satisfy the assumptions of Chapter 3. The dynamics of the rabies example can be split into two distinct parts: the first corresponds to the spatially uniform dynamics and the second to the mechanism for the spread of the disease—the diffusion term. While the first part can be treated within the Banach space of continuous functions, the second requires the introduction of products of spaces based on $L^2(\Omega;\mathbb{R})$. Each space will correspond to a particular sequence of weights $(\alpha_n)_{n=1}^{\infty}$ with respect to the coefficients of a chosen orthonormal basis of $L^2(\Omega;\mathbb{R})$, $\{\phi_n : n \in \mathbb{N}\}$ say (Pritchard and Salamon, 1987; Curtain and Zwart, 1995). These spaces are defined in Section 4.1.1.

The general form of the dynamics for the example will be that of Chapter 3. Each of the operators will be assumed to have a particular form and properties with respect to the spaces introduced. This is the subject of Section 4.1.2.

The conditions imposed on the example that are sufficient for the assumptions of Chapter 3 to be satisfied are stated in terms of the weights and the particular forms of the operators. In the next chapter the rabies model will be formulated in a similar fashion to that of the general example considered here. For the rabies model the spatially uniform part of the system will be considered in the Banach space of continuous functions. The results in the next chapter that show that the theory of Chapters 2 and 3 can be applied to the rabies model will be based on those of this chapter.

Recall that the dynamics of the rabies model are semilinear in form and that the initial value problem corresponding to these general types of semilinear systems was considered in Section 3.3. Therefore a semilinear general example will be introduced in the final section of this chapter and conditions will be imposed that are sufficient for the assumptions of Section 3.3 to be satisfied.

4.1 The basic model

4.1.1 The spaces

Let $\Omega \subset \mathbb{R}^2$ be closed and bounded. The general example will be considered on Z which will be decomposed into the finite product of Hilbert spaces. The construction of these spaces is the topic of this subsection and performed with respect to $L^2(\Omega;\mathbb{R})$.

Let $\{\phi_n\}$ be an orthonormal basis for $L^2(\Omega;\mathbb{R})$. Given a sequence $\alpha_i = (\alpha_{i,n})_{n\in\mathbb{N}}$ such that $\alpha_{i,n} > 0$ for all n, and $h \in C^\infty(\Omega;\mathbb{R})$, define the following inner product:

$$\langle h_1, h_2\rangle_{\alpha_i} = \sum_{n=1}^{\infty} \alpha_{i,n}\langle h_1,\phi_n\rangle_{L^2(\Omega)}\langle h_2,\phi_n\rangle_{L^2(\Omega)} \tag{4.1}$$

with corresponding norm:

$$\|h\|_{\alpha_i} := \left(\sum_{n=1}^{\infty} \alpha_{i,n}\langle h,\phi_n\rangle^2_{L^2(\Omega)}\right)^{1/2}. \tag{4.2}$$

Definition 4.1.1. $H^{\alpha_i}(\Omega;\mathbb{R})$ is defined to be the completion of

$$\{h \in C^\infty(\Omega;\mathbb{R}) : \|h\|_{\alpha_i} < \infty\}$$

with respect to $\|\cdot\|_{\alpha_i}$.

With the inner product $\langle \cdot, \cdot \rangle_{\alpha_i}$ defined above we see that $H^{\alpha_i}(\Omega; \mathbb{R})$ is a Hilbert space.

Definition 4.1.2. Let $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ be any two given sequences of real numbers such that $\alpha_n, \beta_n > 0$ for all $n \in \mathbb{N}$. Then we write $\alpha \prec \beta$ if and only if the sequence $(\alpha_n/\beta_n)_{n \in \mathbb{N}}$ is bounded; that is, there exists a constant R such that $\alpha_n \leq R\beta_n$ for all $n \in \mathbb{N}$.

This order can be used to characterise when a space $H^{\alpha_i}(\Omega; \mathbb{R})$ is contained in another.

Remark 4.1.3. $H^{\alpha_1}(\Omega; \mathbb{R}) \subset H^{\alpha_2}(\Omega; \mathbb{R})$ if and only if $\alpha_1 \succ \alpha_2$.

With respect to these spaces Z is decomposed into the product of n spaces given by

$$Z = H^{\alpha_1^1}(\Omega; \mathbb{R}) \times \ldots \times H^{\alpha_1^1}(\Omega; \mathbb{R}) \times H^{\alpha_1^2}(\Omega; \mathbb{R}). \tag{4.3}$$

The Hilbert space Z is considered with the following inner product

$$\langle z^1, z^2 \rangle = \sum_{k=1}^{n-1} \langle z_k^1, z_k^2 \rangle_{\alpha_1^1} + \langle z_n^1, z_n^2 \rangle_{\alpha_1^2},$$

where $z^i = \begin{pmatrix} z_1^i & \ldots & z_n^i \end{pmatrix}^{\mathsf{T}}$, and corresponding norm. Similarly we suppose that

$$\underline{Z} = H^{\alpha_1^1}(\Omega; \mathbb{R}) \times \ldots \times H^{\alpha_1^1}(\Omega; \mathbb{R}) \times H^{\alpha_0^2}(\Omega; \mathbb{R}), \tag{4.4}$$

and

$$\overline{Z} = H^{\alpha_1^1}(\Omega; \mathbb{R}) \times \ldots \times H^{\alpha_1^1}(\Omega; \mathbb{R}) \times H^{\alpha_2^2}(\Omega; \mathbb{R}), \tag{4.5}$$

with corresponding inner products and norms. The state for the general example will be considered taking values in Z and the spaces $\underline{Z}$, $\overline{Z}$ are introduced to allow for the unboundedness of the nonlinearity. These spaces will be related according to the following condition:

Condition 1. $\alpha_1^1 \prec \alpha_2^2 \prec \alpha_1^2 \prec \alpha_0^2$.

This condition ensures that $\underline{Z} \subset Z \subset \overline{Z}$ with continuous injections and dense ranges.

Remark 4.1.4. There exist constants R_1, R_2, and R_3 such that

$$\begin{aligned} \|h\|_{\alpha_1^2} &\leq R_1 \|h\|_{\alpha_0^2} && \forall h \in H^{\alpha_0^2}(\Omega; \mathbb{R}) \\ \|h\|_{\alpha_2^2} &\leq R_2 \|h\|_{\alpha_1^2} && \forall h \in H^{\alpha_1^2}(\Omega; \mathbb{R}) \\ \|h\|_{\alpha_1^1} &\leq R_3 \|h\|_{\alpha_2^2} && \forall h \in H^{\alpha_2^2}(\Omega; \mathbb{R}). \end{aligned}$$

Moreover we have the following inequalities

$$\|z\|_Z \leq \overline{R}_1 \|z\|_{\underline{Z}} \qquad z \in \underline{Z}$$

$$\|z\|_{\overline{Z}} \leq \overline{R}_2 \|z\|_Z \qquad z \in Z$$

where $\overline{R}_i = \max\{1, R_i\}$.

The nonlinearity is assumed to be a map from $[0,T] \times \underline{W}$ into $\overline{W}$ where these spaces are decomposed as follows:

$$\overline{W} = H^{\beta_2^1}(\Omega;\mathbb{R}) \times \ldots \times H^{\beta_2^m}(\Omega;\mathbb{R}) \tag{4.6}$$

$$\underline{W} = H^{\beta_0^1}(\Omega;\mathbb{R}) \times \ldots \times H^{\beta_0^m}(\Omega;\mathbb{R}) \tag{4.7}$$

where $H^{\beta_0^i}(\Omega;\mathbb{R}) \subset H^{\beta_2^i}(\Omega;\mathbb{R})$ for every $1 \leq i \leq m$. This is ensured by the following condition.

Condition 2. $\beta_2^i \prec \beta_0^i$ for all $1 \leq i \leq m$.

The output operator, that is not necessarily bounded as a linear map from Z to the space of output values, will be considered on the Hilbert space V given by

$$V = H^{\alpha_3^1}(\Omega;\mathbb{R}) \times \ldots \times H^{\alpha_3^n}(\Omega;\mathbb{R}).$$

To ensure that $V \subset Z$ with continuous injection—in fact continuous injection with dense range—we suppose that the following condition holds.

Condition 3. $\alpha_1^1 \prec \alpha_3^i$ for all $1 \leq i \leq n-1$ and $\alpha_1^2 \prec \alpha_3^n$.

Remark 4.1.5. Condition 3 implies that there exist constants r_i such that

$$\|h\|_{\alpha_1^1} \leq r_i \|h\|_{\alpha_3^i} \qquad h \in H^{\alpha_3^i}(\Omega;\mathbb{R}),\ 1 \leq i \leq n-1$$

$$\|h\|_{\alpha_1^2} \leq r_n \|h\|_{\alpha_3^n} \qquad h \in H^{\alpha_3^n}(\Omega;\mathbb{R})$$

and so, for $z \in V$,

$$\begin{aligned}
\|z\|_Z^2 &= \sum_{k=1}^{n-1} \|z_k\|_{\alpha_1^1}^2 + \|z_n\|_{\alpha_1^2}^2 \\
&\leq \sum_{k=1}^{n-1} r_k^2 \|z_k\|_{\alpha_3^k}^2 + r_n^2 \|z_n\|_{\alpha_3^n}^2 \\
&\leq \overline{R}_3^2 \|z\|_V^2
\end{aligned}$$

where $\overline{R}_3 = \max\{r_i : 1 \leq i \leq n\}$.

It will be assumed that the operators of the general example have particular forms with respect to the spaces that have been introduced. This is the subject of the next subsection.

4.1.2 The operators

Consider

$$\begin{aligned} \dot{z}(t) &= (A + P(t))\, z(t) + D(t)N(t, E(t)z(t)), \\ z(0) &= Bu \\ y &= Cz'(T) + Cz(T), \end{aligned}$$

where $u \in U$, $y \in Y$ are Hilbert spaces with the state taking values in Z, as defined in the last subsection. Assumptions will be introduced in this subsection to ensure that the general example has the form of the system considered in Chapter 3.

Suppose that A is (formally) of the form

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{(n-1)\times(n-1)}$, $A_2 \in \mathbb{R}^{1\times(n-1)}$ and A_3 is a self-adjoint (unbounded) linear operator on $L^2(\Omega;\mathbb{R})$. Suppose that $A_1 = (a_1(i,j))_{(n-1)\times(n-1)}$ and $A_2 = (a_2(j))_{1\times(n-1)}$. Then for $z^1(\cdot) = \begin{pmatrix} z_1^1(\cdot) & \dots & z_{n-1}^1(\cdot) \end{pmatrix}^{\mathsf{T}}$ define the action of A_1 and A_2 as follows:

$$\left(A_1 z^1\right)(x) = A_1 z^1(x) \quad \text{and} \quad \left(A_2 z^1\right)(x) = A_2 z^1(x), \qquad (x \in \Omega).$$

Remark 4.1.6. $A_1 \in \mathcal{L}(Z_1)$ and $A_2 \in \mathcal{L}(Z_2, H^{\alpha_1^2}(\Omega;\mathbb{R}))$ where

$$\begin{aligned} Z_1 &= H^{\alpha_1^1}(\Omega;\mathbb{R}) \times \ldots \times H^{\alpha_1^1}(\Omega;\mathbb{R}) \qquad ((n-1) \text{ times}) \\ Z_2 &= H^{\alpha_1^2}(\Omega;\mathbb{R}) \times \ldots \times H^{\alpha_1^2}(\Omega;\mathbb{R}) \qquad ((n-1) \text{ times}). \end{aligned}$$

Let $z \in Z_1$. Then

$$\|A_1 z\|_{Z_1}^2 = \sum_{k=1}^{n-1} \| (A_1 z)_k \|_{\alpha_1^1}^2$$

where $(A_1 z)_k$ is the k-th component of the vector $A_1 z$. Now

$$\begin{aligned}
\|A_1 z\|_{Z_1}^2 &= \sum_{k=1}^{n-1}\left(\sum_{l=1}^{\infty}\alpha_{1,l}^1 \langle (A_1 z)_k, \phi_l\rangle_{L^2(\Omega)}^2\right) \\
&= \sum_{k=1}^{n-1}\left(\sum_{l=1}^{\infty}\alpha_{1,l}^1\left[\sum_{j=1}^{n-1} a_1(k,j)\langle z_j, \phi_l\rangle_{L^2(\Omega)}\right]^2\right) \\
&\leq (n-1)\sum_{k=1}^{n-1}\left(\sum_{l=1}^{\infty}\alpha_{1,l}^1\sum_{j=1}^{n-1} a_1(k,j)^2\langle z_j, \phi_l\rangle_{L^2(\Omega)}^2\right) \\
&\leq (n-1)^2\overline{a}_1^2\sum_{j=1}^{n-1}\|z_j\|_{\alpha_1^1}^2,
\end{aligned}$$

where $\overline{a}_1 = \max_{k,j}|a_1(k,j)|$. Therefore

$$\|A_1 z\|_{Z_1} \leq (n-1)\overline{a}_1\|z\|_{Z_1}.$$

Let $z \in Z_2$, then

$$\begin{aligned}
\|A_2 z\|_{\alpha_1^2}^2 &= \sum_{l=1}^{\infty}\alpha_{1,l}^2\langle A_2 z, \phi_l\rangle_{L^2(\Omega)}^2 \\
&\leq (n-1)\overline{a}_2^2\sum_{j=1}^{n-1}\|z_j\|_{\alpha_1^2}^2 \\
&= (n-1)\overline{a}_2^2\|z\|_{Z_2}^2
\end{aligned}$$

where $\overline{a}_2 = \max_j|a_2(j)|$. In the same way

$$\|A_2 z\|_{\alpha_1^1} \leq (n-1)^{1/2}\overline{a}_2\|z\|_{Z_1},$$

for $z \in Z_1$.

Clearly A_1 is the generator of the strongly continuous semigroup $S_1(t) = e^{A_1 t}$ on Z_1 where, again,

$$\left(e^{A_1 t}z_1\right)(x) = e^{A_1 t}z_1(x) \qquad \forall t \in [0,T].$$

Suppose that A_3 has eigenvalues λ_k satisfying $|\lambda_k| \to \infty$ as $k \to \infty$ and

$$\ldots < \lambda_k < \ldots < \lambda_1 \leq \text{const},$$

with corresponding eigenvectors $\phi_k \in L^2(\Omega;\mathbb{R})$ such that $||\phi_k||_{L^2(\Omega;\mathbb{R})} = 1$. Then

$$A_3 z_2 = \sum_{k=1}^{\infty} \lambda_k \langle z_2, \phi_k \rangle_{L^2(\Omega)} \phi_k,$$

for

$$z_2 \in D(A_3) = \left\{ z_2 \in H^{\alpha_1^2}(\Omega;\mathbb{R}) : \sum_{k=1}^{\infty} \alpha_{1,k}^2 \lambda_k^2 \langle z_2, \phi_k \rangle_{L^2}^2 < \infty \right\}.$$

Furthermore, A_3 generates the strongly continuous semigroup defined by

$$S_3(t) z_2 = \sum_{k=1}^{\infty} e^{\lambda_k t} \langle z_2, \phi_k \rangle_{L^2} \phi_k.$$

One of the assumptions required for A is that it is the generator of a strongly continuous semigroup on Z. The following condition is sufficient for this.

Condition 4. There exists a constant R_4 such that

$$\frac{\alpha_{1,k}^2}{|\lambda_k| \alpha_{1,k}^1} \leq R_4 \tag{4.8}$$

for all $k \in \mathbb{N}$.

This condition is needed because of the unboundedness of A_2 as a linear operator from Z_1 to $H^{\alpha_1^2}(\Omega;\mathbb{R})$. In the case where $A_2 \in \mathcal{L}(Z_1, H^{\alpha_1^2}(\Omega;\mathbb{R}))$ it is well-known (Curtain and Zwart, 1995,for example) that A is the generator of a strongly continuous semigroup. The following result is based on this and uses the smoothing effect of $S_3(t)$ provided by Condition 4 to allow for the unboundedness of A_2.

Proposition 4.1.7. *With A defined as above, A generates a strongly continuous semigroup $S(t)$ on Z where*

$$S(t) = \begin{pmatrix} e^{A_1 t} & 0 \\ S_2(t) & S_3(t) \end{pmatrix} \tag{4.9}$$

and

$$S_2(t) z = \int_0^t S_3(t-s) A_2 e^{A_1 s} z \, ds, \tag{4.10}$$

provided that Condition 4 is satisfied.

Proof. Clearly $\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}$ generates the strongly continuous semigroup

$$T(t) = \begin{pmatrix} e^{A_1 t} & 0 \\ 0 & S_3(t) \end{pmatrix}.$$

Let $\mathcal{A} = \begin{pmatrix} 0 & 0 \\ A_2 & 0 \end{pmatrix}$. Then for $z \in Z$,

$$\begin{aligned}
&T(t)z + \int_0^t T(t-s)\mathcal{A}S(s)z\,ds \\
&= \begin{pmatrix} e^{A_1 t}z^1 \\ S_3(t)z^2 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{A_1(t-s)} & 0 \\ 0 & S_3(t-s) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A_2 & 0 \end{pmatrix} \begin{pmatrix} e^{A_1 s}z^1 \\ S_2(s)z^1 + S_3(s)z^2 \end{pmatrix} ds \\
&= \begin{pmatrix} e^{A_1 t}z^1 \\ S_2(t)z^1 + S_3(t)z^2 \end{pmatrix} \\
&= S(t)z.
\end{aligned}$$

Actually $S(t)$ is the unique solution of this equation for, if $\tilde{S}(t)$ is another solution, then

$$\begin{aligned}
S(t)z - \tilde{S}(t)z &= \int_0^t T(t-s)\mathcal{A}\left(S(s)z - \tilde{S}(s)z\right) ds \\
&= \int_0^t S_3(t-s)A_2\left(e^{A_1 s}z^1 - e^{A_1 s}z^1\right) ds \\
&= 0.
\end{aligned}$$

We now show that $S(t) \in \mathcal{L}(Z)$. Since $e^{A_1 t}$ and $S_3(t)$ are strongly continuous semigroups on Z_1 and $H^{\alpha_1^2}(\Omega;\mathbb{R})$ respectively there exist M_1, M_2 and ω_1, ω_2 such that

$$\|e^{A_1 t}z^1\|_{Z_1} \leq M_1 e^{\omega_1 t}\|z^1\|_{Z_1},$$

and

$$\|S_3(t)z^2\|_{\alpha_1^2} \leq M_2 e^{\omega_2 t}\|z^2\|_{\alpha_1^2}.$$

Let $n_0 = \min\{n \in \mathbb{N} : \lambda_n < 0\}$. We see that

$$\begin{aligned}
\|S_2(t)z^1\|^2_{\alpha_1^2} &= \| \int_0^t S_3(t-s)A_2e^{A_1s}z^1\, ds\|^2_{\alpha_1^2} \\
&= \sum_{k=1}^{\infty} \alpha_{1,k}^2 \left(\int_0^t e^{\lambda_k(t-s)} \langle A_2 e^{A_1s}z^1, \phi_k\rangle_{L^2(\Omega)}\, ds \right)^2 \\
&\le \sum_{k=1}^{n_0} \alpha_{1,k}^2 \left(\int_0^t e^{2\lambda_k s}\, ds \right) \left(\int_0^t \langle A_2 e^{A_1s}z^1, \phi_k\rangle^2_{L^2(\Omega)}\, ds \right) \\
&\quad + \sum_{k=n_0+1}^{\infty} \frac{\alpha_{1,k}^2}{2|\lambda_k|} \left(\int_0^t \langle A_2 e^{A_1s}z^1, \phi_k\rangle^2_{L^2(\Omega)}\, ds \right) \\
&\le R_4' \sum_{k=1}^{\infty} \alpha_{1,k}^1 \int_0^t \sum_{j=1}^{n-1} a_2(j)^2 \langle (e^{A_1s}z^1)_j, \phi_k\rangle^2_{L^2(\Omega)}\, ds \\
&\le R_4'(n-1)\bar{a}_2^2 \sum_{j=1}^{n-1} \max_i \left(\int_0^t s_1(s;j,i)^2\, ds \right) \|z^1\|^2_{Z_1},
\end{aligned}$$

where $e^{A_1s} = (s_1(s;i,j))_{(n-1)\times(n-1)}$ and

$$R_4' = \frac{1}{2} \max\left\{ |e^{2\lambda_1 t} - 1|, \dots, |e^{2\lambda_{n_0} t} - 1|, 1 \right\} R_4(n-1).$$

Therefore

$$\begin{aligned}
&\|S(t)z\|^2_Z \\
&= \|e^{A_1t}z^1\|^2_{Z_1} + \|S_2(t)z^1 + S_3(t)z^2\|^2_{\alpha_1^2} \\
&\le M_1^2 e^{2\omega_1 t}\|z^1\|^2_{Z_1} + 2R_4'(n-1)\bar{a}_2^2 \sum_{j=1}^{n-1} \max_i \left(\int_0^t s_1(s,j,i)^2\, ds \right) \|z^1\|^2_{Z_1} + 2M_2^2 e^{2\omega_2 t}\|z^2\|^2_{\alpha_1^2} \\
&\le \Gamma(t)\|z\|^2_Z.
\end{aligned}$$

We now show that $S(t)$ is strongly continuous. Let $h > 0$. Then

$$\begin{aligned}
\|S(h)z - z\|^2_Z &\le \|e^{A_1h}z^1 - z^1\|^2_{Z_1} + \|S_2(h)z^1 + S_3(h)z^2 - z^2\|^2_{\alpha_1^2} \\
&\le \|e^{A_1h}z^1 - z^1\|^2_{Z_1} + \|S_3(h)z^2 - z^2\|^2_{\alpha_1^2} \\
&\quad + \| \int_0^h S_3(h-s)A_2e^{A_1s}z^1\, ds\|^2_{\alpha_1^2}.
\end{aligned}$$

We know from the above that the last term is bounded, with the bound continuous in h. Furthermore, as $h \to 0$, the bound tends to zero. Therefore since e^{A_1t}, and $S_3(t)$ are strongly continuous semigroups on Z_1 and $H^{\alpha_1^2}(\Omega;\mathbb{R})$ respectively we see that $S(t)$ is strongly continuous.

Now

$$S(0) = \begin{pmatrix} I & 0 \\ S_2(0) & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Therefore it only remains to show that $S(t+s) = S(t)S(s)$ to see that $S(t)$ is a strongly continuous semigroup. We have, for $z \in Z$,

$$S(t)S(s)z - S(t+s)z = \begin{pmatrix} e^{A_1 t}\left(S(s)z\right)^1 \\ S_2(t)\left(S(s)z\right)^1 + S_3(t)\left(S(s)z\right)^2 \end{pmatrix} - \begin{pmatrix} \left(S(t+s)z\right)^1 \\ \left(S(t+s)z\right)^2 \end{pmatrix}.$$

The first component is

$$e^{A_1 t}e^{A_1 s}z^1 - e^{A_1(t+s)}z^1 = 0.$$

The second component is

$$\begin{aligned}
&\left(S_2(t)e^{A_1 s} + S_3(t)S_2(s) - S_2(t+s)\right)z^1 + \left(S_3(t)S_3(s) - S_3(t+s)\right)z^2 \\
&\quad = \left(S_2(t)e^{A_1 s} + S_3(t)S_2(s) - S_2(t+s)\right)z^1 \\
&\quad = \int_0^t S_3(t-\sigma)A_2 e^{A_1(\sigma+s)}z^1\,d\sigma + S_3(t)\int_0^s S_3(s-\sigma)A_2 e^{A_1\sigma}z^1\,d\sigma \\
&\qquad - \int_0^{t+s} S_3(t+s-\sigma)A_2 e^{A_1\sigma}z^1\,d\sigma \\
&\quad = \int_s^{t+s} S_3(t+s-\sigma)A_2 e^{A_1\sigma}z^1\,d\sigma + \int_0^s S_3(t+s-\sigma)A_2 e^{A_1\sigma}z^1\,d\sigma \\
&\qquad - \int_0^{t+s} S_3(t+s-\sigma)A_2 e^{A_1\sigma}z^1\,d\sigma \\
&\quad = 0
\end{aligned}$$

as required.

We have shown that $S(t)$ is a strongly continuous semigroup on Z. We now show that it is generated by A. We have

$$\begin{aligned}
\|\frac{1}{h}\left(S(h)z - z\right) - Az\|_Z^2 &= \|\frac{1}{h}\left(e^{A_1 h}z^1 - z^1\right) - A_1 z\|_{Z_1}^2 \\
&\quad + \|\frac{1}{h}\left(S_2(h)z^1 + S_3(h)z^2 - z^2\right) - A_2 z^1 - A_3 z^2\|_Z^2 \\
&\le \|\frac{1}{h}\left(e^{A_1 h}z^1 - z^1\right) - A_1 z\|_{Z_1}^2 + 2\|\frac{1}{h}\left(S_2(h)z^1\right) - A_2 z^1\|_{\alpha_1^2}^2 \\
&\quad + 2\|\frac{1}{h}\left(S_3(h)z^2 - z^2\right) - A_3 z^2\|_{\alpha_1^2}^2.
\end{aligned}$$

The first and third terms converge to zero as $h \to 0$ since $e^{A_1 t}$ and $S_3(t)$ are generated by A_1 and A_3 respectively. Consider the second term:

$$\|\frac{1}{h}\left(S_2(h)z^1\right) - A_2 z^1\|_{\alpha_1^2} = \|\frac{1}{h}\int_0^h S_3(h-s)A_2 e^{A_1 s} z^1\, ds - A_2 z^1\|_{\alpha_1^2}.$$

Therefore we must restrict A_2 to $Z_2 \subset Z_1$. We have, for $z^1 \in Z_2$,

$$\|\frac{1}{h}\int_0^h S_3(h-s)A_2 e^{A_1 s} z^1\, ds - A_2 z^1\|_{\alpha_1^2}$$
$$\leq \|\frac{1}{h}\int_0^h S_3(s)A_2\left(e^{A_1(h-s)}z^1 - z^1\right)\, ds\|_{\alpha_1^2} + \|\frac{1}{h}\int_0^h S_3(s)A_2 z^1\, ds - A_2 z^1\|_{\alpha_1^2}.$$

The last term converges to zero since $S_3(s)$ is a semigroup on $H^{\alpha_1^2}(\Omega;\mathbb{R})$. We can estimate the first by

$$\frac{1}{h}\int_0^h \|S_3(h-s)\|\,\|A_2\|\,\|\left(e^{A_1 s}z^1 - z^1\right)\|\, ds.$$

Since $e^{A_1 s}$ is strongly continuous on Z_2 there exists a h such that

$$\|e^{A_1 s}z^1 - z^1\| \leq \epsilon \qquad \text{for } s \in [0,h].$$

Therefore

$$\frac{1}{h}\int_0^h \|S_3(h-s)\|\,\|A_2\|\,\|\left(e^{A_1 s}z^1 - z^1\right)\|\, ds \leq \frac{1}{h}\int_0^h \|S_3(h-s)\|\,\|A_2\|\,\epsilon\, ds$$
$$\leq (\text{const})\|A_2\|\,\epsilon.$$

Thus $S(t)$ is generated by A with the domain of A equal to $Z_2 \times D(A_3)$. □

Let $Bu = \begin{pmatrix} B_1 u & \dots & B_n u \end{pmatrix}^{\mathsf{T}}$ and for each $1 \leq i \leq n$, $(b_k(i))_{k \in \mathbb{N}}$ a sequence into U such that

$$B_i u = \sum_{k=1}^{\infty} \langle b_k(i), u\rangle_U \phi_k.$$

To ensure that the input operator is bounded a further condition is required.

Condition 5. For each $1 \leq i \leq n-1$ we have $B_i \in \mathcal{L}(U, H^{\alpha_1^i}(\Omega;\mathbb{R}))$ and $B_n \in \mathcal{L}(U, H^{\alpha_1^2}(\Omega;\mathbb{R}))$.

In the scalar case, that is when $B_i u = b_i u$ for some $b_i \in \mathbb{R}$, the following is necessary and sufficient for Condition 5 to be satisfied.

Condition 5.a. If $b_n \neq 0$ then $U \subseteq H^{\alpha_1^2}(\Omega;\mathbb{R}) \left(\subset H^{\alpha_1^1}(\Omega;\mathbb{R})\right)$ otherwise $U \subseteq H^{\alpha_1^1}(\Omega;\mathbb{R})$.

When B is not scalar the following is sufficient for Condition 5 to be satisfied.

Condition 5.b. For each $1 \le i \le n-1$, the sequence $(b_k(i))_{k\in\mathrm{N}}$ satisfies

$$\sum_{k=1}^{\infty} \alpha_{1,k}^1 \|b_k(i)\|_U^2 < \infty \tag{4.11}$$

and the sequence $(b_k(n))_{k\in\mathrm{N}}$ satisfies

$$\sum_{k=1}^{\infty} \alpha_{1,k}^2 \|b_k(n)\|_U^2 < \infty. \tag{4.12}$$

Lemma 4.1.8. *Suppose that Condition 5.b is satisfied. Then Condition 5 holds for the general example.*

Proof. For $u \in U$ and $1 \le i \le n-1$ we have

$$\begin{aligned}\|B_i u\|_{\alpha_1^1}^2 &= \sum_{k=1}^{\infty} \alpha_{1,k}^1 \langle B_i u, \phi_k\rangle_{L^2(\Omega)}^2 \\ &= \sum_{k=1}^{\infty} \alpha_{1,k}^1 \langle b_k(i), u\rangle_U^2 \\ &\le \left[\sum_{k=1}^{\infty} \alpha_{1,k}^1 \|b_k(i)\|_U^2\right] \|u\|_U^2\end{aligned}$$

and a similar result result holds for $i = n$. □

The scalar case can be treated in the same way but this treatment leads to a condition that is not as precise. This is illustrated in the following example.

Example 4.1.9. Suppose that $\alpha_1^1 = (n^{2\alpha})_{n\in\mathrm{N}}$, $\alpha_3 = (n^{2\gamma})_{n\in\mathrm{N}}$, where α, γ are real numbers and $U = H^{\alpha_3}(\Omega;\mathbb{R})$. If $Bu = \begin{pmatrix} bu & 0 & 0\end{pmatrix}^{\mathrm{T}}$ for some $b \in \mathbb{R}$ and all $u \in U$, then $B \in \mathcal{L}(U, Z)$ if and only if $H^{\alpha_3}(\Omega;\mathbb{R}) \subset H^{\alpha_1^1}(\Omega;\mathbb{R})$. This is the case if and only if $\gamma \ge \alpha$.

Alternatively, if we use Condition 5.b then we require

$$\sum_{k=1}^{\infty} \alpha_{1,k}^1 \|b_k(1)\|_U^2 < \infty \tag{4.13}$$

where, for any $u \in U$,

$$bu = \sum_{k=1}^{\infty} \langle b_k(1), u\rangle_U \phi_k.$$

The latter implies that

$$\langle bu, \phi_k\rangle = \langle b_k(1), u\rangle_U = \sum_{n=1}^{\infty} n^{2\gamma}\langle b_k(1), \phi_n\rangle_{L^2(\Omega)}\langle u, \phi_n\rangle_{L^2(\Omega)},$$

and so, setting $u = \phi_m \in H^{\alpha_3}(\Omega;\mathbb{R})$, we have

$$m^{2\gamma}\langle b_k(1), \phi_m\rangle_{L^2(\Omega)} = \begin{cases} b & m = k \\ 0 & m \neq k \end{cases}.$$

Substituting this into equation (4.13) gives

$$\sum_{n=1}^{\infty} n^{2\alpha}\|b_n(1)\|_U^2 = \sum_{n=1}^{\infty} n^{2\alpha}\sum_{k=1}^{\infty} k^{2\gamma}\langle b_n(1), \phi_k\rangle_{L^2(\Omega)}^2$$
$$\left(\sum_{n=1}^{\infty} n^{2(\alpha-\gamma)}\right) b$$

and this series converges if and only if $2(\alpha - \gamma) < -1$. This condition is much stronger than is necessary (namely $\alpha - \gamma \leq 0$).

In the same way suppose that, for $v \in V$, $Cv = \sum_{i=1}^{n} C_i v_i$ and for each $1 \leq i \leq n$, $(c_k(i))_{k\in\mathbb{N}}$ is a sequence into Y such that

$$C_i v_i = \sum_{k=1}^{\infty} c_k(i)\langle v_i, \phi_k\rangle_{L^2(\Omega)}.$$

The following condition ensures that C is a bounded linear operator from V to Y.

Condition 6. For each $1 \leq i \leq n$ we have $C_i \in \mathcal{L}(H^{\alpha_3^i}(\Omega;\mathbb{R}), Y)$.

As in the case of the input operator B if C is scalar the following is necessary and sufficient for Condition 6.

Condition 6.a. For each $1 \leq i \leq n$, if $c_i \neq 0$ then $H^{\alpha_3^i}(\Omega;\mathbb{R}) \subset Y$.

If C is not scalar the following is a sufficient (but not necessary) condition.

Condition 6.b. The sequence $(c_k(i))_{k\in\mathbb{N}}$ satisfies

$$\sum_{k=1}^{\infty} \left(\alpha_{3,k}^i\right)^{-1} \|c_k(i)\|_Y^2 < \infty \tag{4.14}$$

for each $1 \leq i \leq n$.

Lemma 4.1.10. *Suppose that Condition 6.b is satisfied. Then $C \in \mathcal{L}(V,Y)$.*

Proof. For all $v \in V$ we have

$$\begin{aligned} \|Cv\|_Y^2 &= \|\sum_{i=1}^{n} C_i v_i\|_Y^2 \\ &\leq n \sum_{i=1}^{n} \|C_i v_i\|_Y^2 \\ &\leq n \sum_{i=1}^{n} \left(\sum_{k=1}^{\infty} (\alpha_{3,k}^i)^{-1} \|c_k(i)\|_Y^2 \right) \|v\|_V^2 \end{aligned}$$

and so the result follows from Condition 6.b. □

Remark 4.1.11. Clearly the scalar cases for either B or C can be generalised to the situation where, for example, $b_i \in \mathbb{R}^{1\times r}$ for some $r \in \mathbb{N}$ and $u \in U$ can be decomposed as

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}.$$

For the linear part of the system the time-variation is given by $P(\cdot)$ where, with respect to the decomposition of Z, we have

$$P(\cdot) = \begin{pmatrix} & & \\ & P_{ij}(\cdot) & \\ & & \end{pmatrix}_{n\times n}.$$

It will be assumed that the P_{ij} satisfy the following condition.

Condition 7. $P_{ij}(\cdot) \in PC(0,T;\mathcal{L}(H^{\alpha_0^j}(\Omega;\mathbb{R}), H^{\alpha_2^i}(\Omega;\mathbb{R})))$ where

$$\alpha_i^j := \begin{cases} \alpha_i^1 & 1 \leq j \leq n-1 \\ \alpha_i^2 & j = n \end{cases}.$$

Furthermore

$$P_{ij}(s)\phi_l = \sum_{k=1}^{\infty} p_l^k(s;i,j)\phi_k$$

is well-defined.

Remark 4.1.12. We can calculate the adjoint operator of $P_{ij}(s)$, with respect to the inner product on $L^2(\Omega;\mathbb{R})$. Let $w \in L^2(\Omega;\mathbb{R})$. Then

$$\begin{aligned}\langle P_{ij}(s)w, \phi_k\rangle_{L^2} &= \langle \sum_{l=1}^{\infty}\langle w,\phi_l\rangle P_{ij}(s)\phi_l, \phi_k\rangle_{L^2}\\ &= \sum_{l=1}^{\infty}\langle\langle w,\phi_l\rangle \sum_{m=1}^{\infty} p_l^m(s;i,j)\phi_m, \phi_k\rangle_{L^2}\\ &= \sum_{l=1}^{\infty} p_l^k(s;i,j)\langle w,\phi_l\rangle,\end{aligned}$$

and so

$$P_{ij}^*(s)\phi_k = \sum_{l=1}^{\infty} p_l^k(s;i,j)\phi_l$$

for all $1 \le i,j \le n$ and $s \in [0,T]$.

The following is sufficient for $P_{ij}(\cdot)$ to be piecewise continuous and bounded operator valued.

Condition 7.a. For each $1 \le i,j \le n$, $p_l^k(\cdot;i,j)$ is piecewise continuous for all $l,k \in \mathbb{N}$, and

$$\sum_{k=1}^{\infty} \alpha_{2,k}^i \|\underline{p}_k(s;i,j)\|_{\ell^2}^2 < \infty \qquad \forall s \in [0,T]$$

where $\underline{p}_k(s;i,j)$ is the sequence $\left((\alpha_{0,l}^j)^{-1/2} p_l^k(s;i,j)\right)_{l\in\mathbb{N}}$.

Lemma 4.1.13. *Condition 7.a is sufficient for $P(\cdot)$ to satisfy Condition 7.*

Proof. Let $z \in H^{\alpha_0^j}(\Omega;\mathbb{R})$. We have

$$\begin{aligned}\|P_{ij}(s)z\|_{\alpha_2^i}^2 &= \sum_{k=1}^{\infty} \alpha_{2,k}^i \langle P_{ij}(s)z, \phi_k\rangle_{L^2(\Omega)}^2\\ &= \sum_{k=1}^{\infty} \alpha_{2,k}^i \langle z, P_{ij}^*(s)\phi_k\rangle_{L^2(\Omega)}^2\\ &= \sum_{k=1}^{\infty} \alpha_{2,k}^i \left(\sum_{l=1}^{\infty} p_l^k(s;i,j)\langle z,\phi_l\rangle_{L^2(\Omega)}\right)^2\\ &\le \left[\sum_{k=1}^{\infty} \alpha_{2,k}^i \|\underline{p}_k(s;i,j)\|_{\ell^2}^2\right] \sum_{l=1}^{\infty} \alpha_{0,l}^j \langle z,\phi_l\rangle_{L^2(\Omega)}^2\end{aligned}$$

and so Condition 7.a implies that $P_{ij}(s) \in \mathcal{L}(H^{\alpha_0^j}(\Omega;\mathbb{R}), H^{\alpha_2^i}(\Omega;\mathbb{R}))$ as required. □

Similarly, we suppose that

$$D(\cdot) = \left(\quad D_{ij}(\cdot) \quad \right)_{n \times m}, \qquad E(\cdot) = \left(\quad E_{ij}(\cdot) \quad \right)_{m \times n},$$

with respect to the decomposition of Z.

Condition 8. For $1 \leq i \leq n-1$

$$D_{ij}(\cdot) \in PC(0,T; \mathcal{L}(H^{\beta_2^j}(\Omega;\mathbb{R}), H^{\alpha_1^1}(\Omega;\mathbb{R})))$$
$$E_{ji}(\cdot) \in PC(0,T; \mathcal{L}(H^{\alpha_1^1}(\Omega;\mathbb{R}), H^{\beta_0^j}(\Omega;\mathbb{R})))$$

and

$$D_{nj}(\cdot) \in PC(0,T; \mathcal{L}(H^{\beta_2^j}(\Omega;\mathbb{R}), H^{\alpha_2^2}(\Omega;\mathbb{R})))$$
$$E_{jn}(\cdot) \in PC(0,T; \mathcal{L}(H^{\alpha_0^2}(\Omega;\mathbb{R}), H^{\beta_0^j}(\Omega;\mathbb{R}))).$$

The following are well-defined:

$$D_{ij}(s)\phi_k = \sum_{l=1}^{\infty} d_k^l(s;i,j)\phi_l$$
$$E_{ji}(s)\phi_k = \sum_{l=1}^{\infty} e_k^l(s;j,i)\phi_l$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Remark 4.1.14. Proceeding as in Remark 4.1.12 we can similarly calculate the adjoint operators of $D_{ij}(s)$ and $E_{ij}(s)$ with respect to the inner product in $L^2(\Omega;\mathbb{R})$:

$$D_{ij}^*(s)\phi_l = \sum_{k=1}^{\infty} d_k^l(s;i,j)\phi_k,$$
$$E_{ji}^*(s)\phi_l = \sum_{k=1}^{\infty} e_k^l(s;j,i)\phi_k,$$

for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $s \in [0,T]$.

Condition 8.a. Let $\underline{d}^k(s;i,j)$ denote the sequence $\left((\beta_{2,l}^j)^{-1/2} d_l^k(s;i,j)\right)_{l \in \mathbf{N}}$. Then

$$\sum_{k=1}^{\infty} \alpha_{2,k}^i \|\underline{d}^k(s;i,j)\|_{\ell^2}^2 < \infty \tag{4.15}$$

for all $1 \leq i \leq n$, $1 \leq j \leq m$, and $s \in [0,T]$.

Condition 8.b. Let $\underline{e}^k(s;i,j)$ denote the sequence $\left((\alpha_{0,l}^j)^{-1/2}e_l^k(s;i,j)\right)_{l\in\mathbb{N}}$. Then

$$\sum_{k=1}^{\infty}\beta_{0,k}^i\|\underline{e}^k(s;i,j)\|_{\ell^2}^2<\infty \tag{4.16}$$

for all $1\le i\le m$, $1\le j\le n$, and $s\in[0,T]$.

Lemma 4.1.15. *Conditions 8.a and 8.b are sufficient for*

$$D_{ij}(\cdot)\in PC(0,T;\mathcal{L}(H^{\beta_2^j}(\Omega;\mathbb{R}),H^{\alpha_2^i}(\Omega;\mathbb{R})))$$
$$E_{ij}(\cdot)\in PC(0,T;\mathcal{L}(H^{\alpha_0^j}(\Omega;\mathbb{R}),H^{\beta_0^i}(\Omega;\mathbb{R})))$$

respectively, where

$$\alpha_i^j=\begin{cases}\alpha_i^1 & 1\le j\le n-1\\ \alpha_i^2 & j=n\end{cases}$$

for $i=0,2$.

Proof. The results follow from the method used in the proof of Lemma 4.1.13. □

Throughout the remainder of this chapter we will assume that for any $w(\cdot)\in L^4(0,T;\underline{W})$ we have $N(\cdot,w(\cdot))\in L^2(0,T;\overline{W})$.

4.2 Model considered as a perturbed system

In this section we will consider the general dynamical model introduced so far as a perturbed system studied in Chapter 3. Conditions will be derived for the general example to satisfy the assumptions introduced in Chapter 3.

In the following, by the term *general example* we will mean the system with the operators and spaces defined in the previous subsection; in addition we will assume that any conditions previously derived for the system also hold.

The assumptions PS I and PS II are satisfied by the construction of the previous section.

4.2.1 Assumptions for evolution operator

Proposition 4.1.7 and the following condition imply that the general example satisfies PS III.

Condition 9. There exists a constant R_5 such that for the general example

$$\frac{\alpha_{0,k}^2}{|\lambda_k|\alpha_{1,k}^1} \leq R_5, \tag{4.17}$$

for all $k \in \mathbb{N}$.

Corollary 4.2.1. *Suppose that the general example satisfies Condition 9. Then the semigroup $S(t)$ generated by A in Proposition 4.1.7 is a strongly continuous semigroup on $\underline{Z}$, and $\overline{Z}$.*

Proof. This follows for $\underline{Z}$ immediately from the proof of Proposition 4.1.7 with (4.17) instead of (4.8). Furthermore we see that

$$\frac{\alpha_{2,k}^2}{|\lambda_k|\alpha_{1,k}^1} \leq R_2\frac{\alpha_{1,k}^2}{|\lambda_k|\alpha_{1,k}^1} \leq R_1R_2\frac{\alpha_{0,k}^2}{|\lambda_k|\alpha_{1,k}^1}$$

and so the result holds for $\overline{Z}$ in the same way.

In particular following the approach used in the proof of Proposition 4.1.7 shows that there exist $\underline{M}_2(\cdot), \overline{M}_2(\cdot) \in C(0,T;\mathbb{R})$ such that

$$||S_2(t)z||_{\alpha_0^2} \leq \underline{M}_2(t)||z||_{Z_1}$$

and

$$||S_2(t)z||_{\alpha_2^2} \leq \overline{M}_2(t)||z||_{Z_1}$$

for all $z \in Z_1$. □

We now introduce conditions on $P(t)$ and $S(t)$ for the general example to satisfy the smoothing property PS IV.

Condition 10. For all $1 \leq i \leq n-1$ and $z \in H^{\alpha_0^2}(\Omega;\mathbb{R})$, there exists a constant $k_1(i)$ (dependent on i) such that

$$||P_{in}(\sigma)z||_{\alpha_1^1} \leq k_1(i)||z||_{\alpha_2^2}, \tag{4.18}$$

for $\sigma \in [0,T]$.

Let $\underline{p}^l_2(\sigma; i,j)$ be the sequence $\left(\left(\alpha^j_{2,k}\right)^{-1/2} p^l_k(\sigma; i,j)\right)_{k=1}^{\infty}$. For all $1 \le i,j \le n$,

$$\operatorname*{ess\,sup}_{\sigma\in[0,T]} \|\underline{p}^l_2(\sigma; i,j)\|_{\ell^2} < \infty \tag{4.19}$$

and if $p^l_2(i,j) = \operatorname{ess\,sup}_{\sigma\in[0,T]} \|\underline{p}^l_2(\sigma; i,j)\|_{\ell^2}$, then

$$\sum_{l=n_0+1}^{\infty} \frac{\alpha^2_{0,l} p^l_2(n,i)^2}{|\lambda_l|} < \infty \tag{4.20}$$

(for all $1 \le i \le n$) where $n_0 = \min\{n \in \mathbb{N} : \lambda_n < 0\}$.

Proposition 4.2.2. *For the general example, suppose that Condition 10 is satisfied. Then there exists a continuous function $\overline{K}_1(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that $\overline{K}_1(0) = 0$, and for all $s \in [0,T)$, $z(\cdot) \in C(s,T;\underline{Z})$, $t \in (s,T]$, the map $t \mapsto \int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma$ is continuous from $[s,T]$ to $\underline{Z}$ with*

$$\| \int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma\|_{\underline{Z}} \le \overline{K}_1(t-s)\|z\|_{C(s,t;\underline{Z})}.$$

Proof. Note that for $z(\cdot) \in C(0,T;\underline{Z})$

$$\sup_{\sigma\in[0,T]} \|z(\sigma)\|^2_{\underline{Z}} = \sup_{\sigma\in[0,T]} \left[\sum_{k=1}^{n-1} \|z_k(\sigma)\|^2_{\alpha^1_1} + \|z_n(\sigma)\|^2_{\alpha^2_0}\right] < \infty.$$

In particular

$$\sup_{\sigma\in[0,T]} \|z_k(\sigma)\|^2_{\alpha^k_0} < \infty$$

where $\alpha^k_0 = \begin{cases} \alpha^1_1 & 1 \le k \le n-1 \\ \alpha^2_0 & k = n \end{cases}$.

Consider

$$\begin{aligned}
\int_s^t \| \left(P(\sigma)z(\sigma)\right)^1 \|^2_{Z_1}\,d\sigma &= \sum_{k=1}^{n-1} \int_s^t \| \left(P(\sigma)z(\sigma)\right)_k \|^2_{\alpha^1_1}\,d\sigma \\
&= \sum_{k=1}^{n-1} \int_s^t \| \sum_{i=1}^{n} P_{ki}(\sigma)z_i(\sigma)\|^2_{\alpha^1_1}\,d\sigma \\
&\le n\sum_{k=1}^{n-1}\left(\sum_{i=1}^{n-1} \int_s^t \|P_{ki}(\sigma)\|^2\|z_i(\sigma)\|^2_{\alpha^1_1}\,d\sigma + \int_s^t k_1(k)^2\|z_n(\sigma)\|^2_{\alpha^2_2}\,d\sigma\right) \\
&\le n\sum_{k=1}^{n-1} \max\left\{\|P_{k1}(\cdot)\|^2_{\infty}, \dots, \|P_{k(n-1)}(\cdot)\|^2_{\infty}, k_1(k)^2\right\} \int_s^t \|z(\sigma)\|^2_{\underline{Z}}\,d\sigma.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\| \int_s^t e^{A_1(t-\sigma)} \left(P(\sigma)z(\sigma)\right)^1 \, d\sigma\|_{Z_1}^2 \\
&\le \left(\int_s^t M_1 e^{\omega_1(t-\sigma)} \| \left(P(\sigma)z(\sigma)\right)^1 \|_{Z_1} \, d\sigma\right)^2 \\
&\le \int_0^{t-s} M_1^2 e^{2\omega_1\sigma} \, d\sigma \left[n \sum_{k=1}^{n-1} \max\left\{ \|P_{k1}(\cdot)\|_\infty^2, \dots, \|P_{k(n-1)}(\cdot)\|_\infty^2, k_1(k)^2 \right\} \|z(\cdot)\|^2_{L^2(s,t;\overline{Z})} \right].
\end{aligned}$$

Hence

$$\| \int_s^t e^{A_1(t-\sigma)} \left(P(\sigma)z(\sigma)\right)^1 \, d\sigma\|_{Z_1} \le (\text{const}) \left((t-s) \int_0^{t-s} e^{2\omega_1\sigma} \, d\sigma \right)^{1/2} \|z\|_{C(s,t;\overline{Z})}.$$

Now consider

$$\begin{aligned}
&\| \int_s^t S_2(t-\sigma) \left(P(\sigma)z(\sigma)\right)^1 \, d\sigma\|_{\alpha_0^2}^2 \\
&\le \left(\int_s^t \underline{M}_2(t-\sigma) \| \left(P(\sigma)z(\sigma)\right)^1 \|_{Z_1} \, d\sigma\right)^2 \\
&\le \int_0^{t-s} \underline{M}_2(\sigma)^2 \, d\sigma \left[n \sum_{k=1}^{n-1} \max\left\{ \|P_{k1}(\cdot)\|_\infty^2, \dots, \|P_{k(n-1)}(\cdot)\|_\infty^2, k_1(k)^2 \right\} \|z(\cdot)\|^2_{L^2(s,t;\overline{Z})} \right].
\end{aligned}$$

Therefore

$$\| \int_s^t S_2(t-\sigma) \left(P(\sigma)z(\sigma)\right)^1 \, d\sigma\|_{\underline{Z}_2} \le (\text{const}) \left((t-s) \int_0^{t-s} \underline{M}_2(\sigma)^2 \, d\sigma \right)^{1/2} \|z\|_{C(s,t;\overline{Z})}.$$

Finally consider

$$\begin{aligned}
&\| \int_s^t S_3(t-\sigma) \left(P(\sigma)z(\sigma)\right)_n \, d\sigma\|_{\alpha_0^2}^2 \\
&= \sum_{l=1}^\infty \alpha_{0,l}^2 \langle \int_s^t S_3(t-\sigma) \left(P(\sigma)z(\sigma)\right)_n \, d\sigma, \phi_l \rangle_{L^2}^2 \\
&= \sum_{l=1}^\infty \alpha_{0,l}^2 \left(\int_s^t e^{\lambda_l(t-\sigma)} \langle \sum_{i=1}^n P_{ni}(\sigma) z_i(\sigma), \phi_l \rangle_{L^2} \, d\sigma \right)^2 \\
&\le n \sum_{l=1}^\infty \alpha_{0,l}^2 \sum_{i=1}^n \left(\int_0^{t-s} e^{2\lambda_l\sigma} \, d\sigma \right) \left(\int_s^t \|\underline{p}_2^l(\sigma; n, i)\|_{\ell^2}^2 \, \|\underline{z}_i(\sigma; \alpha_2)\|_{\ell^2}^2 \, d\sigma \right) \\
&\le n(t-s) \max_i \left[\sum_{l=1}^{n_0} \alpha_{0,l}^2 \left(\int_0^{t-s} e^{2\lambda_l\sigma} \, d\sigma \right) p_2^l(n,i)^2 + \sum_{l=n_0+1}^{\infty} \frac{\alpha_{0,l}^2 p_2^l(n,i)^2}{2|\lambda_l|} \right] \|z\|^2_{C(s,t;\overline{Z})}
\end{aligned}$$

which is finite by condition (4.20), where $\underline{z}_i(\sigma;\alpha_2)$ is the sequence $\left(\left(\alpha_{2,l}^i\right)^{1/2}\langle z_i(\sigma),\phi_l\rangle_{L^2}\right)_{l\in\mathbf{N}}$.

Therefore

$$\begin{aligned}
&\|\int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma\|_{\underline{Z}}^2\\
&\qquad=\sum_{k=1}^{n-1}\|\int_s^t \left(e^{A_1(t-\sigma)}\left(P(\sigma)z(\sigma)\right)^1\right)_k d\sigma\|_{\alpha_1^1}\\
&\qquad\quad+\|\int_s^t \left[S_2(t-\sigma)\left(P(\sigma)z(\sigma)\right)^1+S_3(t-\sigma)\left(P(\sigma)z(\sigma)\right)_n\right] d\sigma\|_{\alpha_0^2}^2\\
&\qquad\leq \overline{K}_1(t-s)^2\|z\|_{C(s,t;\overline{Z})}^2
\end{aligned}$$

as required. □

To show that the general example satisfies PS IV it now only remains to show that the map $t\mapsto\int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma$ is continuous. The following corollary proves that this is the case. The method of the proof will also be used in the following.

Corollary 4.2.3. *Suppose that the condition of Proposition 4.2.2 is satisfied. Then the map $t\mapsto\int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma$ is continuous from $[s,T]$ to $\underline{Z}$.*

Proof. By Remark 3.1.2 we know that this map is continuous from the right. Therefore let $s\in[0,T]$, $h>0$ and consider the difference

$$\begin{aligned}
&\int_s^{t-h} S(t-h-\sigma)P(\sigma)z(\sigma)\,d\sigma-\int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma\\
&\qquad=\int_s^{t-h}\left(S(t-h-\sigma)-S(t-\sigma)\right)P(\sigma)z(\sigma)\,d\sigma+\int_{t-h}^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma.
\end{aligned}$$

For the second term Proposition 4.2.2 proves that

$$\|\int_{t-h}^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma\|_{\underline{Z}}\leq\overline{K}_1(h)\|z(\cdot)\|_{C(t-h,t;\overline{Z})}$$

which converges to zero as $h \to 0$. For the first term note that

$$\begin{aligned}
\| \int_s^{t-h} (S(t-\sigma) - S(t-h-\sigma))\, P(\sigma)z(\sigma)\, d\sigma \|_{\underline{Z}}^2 \\
\leq \| \int_s^{t-h} \left(e^{A_1(t-\sigma)} - e^{A_1(t-h-\sigma)}\right) (P(\sigma)z(\sigma))^1 \, d\sigma \|_{Z_1}^2 \\
+ 2\| \int_s^{t-h} (S_2(t-\sigma) - S_2(t-h-\sigma))\, (P(\sigma)z(\sigma))^1 \, d\sigma \|_{\alpha_0^2}^2 \\
+ 2\| \int_s^{t-h} (S_3(t-\sigma) - S_3(t-h-\sigma))\, (P(\sigma)z(\sigma))^2 \, d\sigma \|_{\alpha_0^2}^2. \quad (4.21)
\end{aligned}$$

Each of the terms on the right-hand side will be considered separately: Since $e^{A_1 t}$ is a strongly continuous semigroup on Z_1 and $(P(\sigma)z(\sigma))^1 \in Z_1$ for each $\sigma \in [0, T]$, there exists a positive constant m_1 such that

$$\| \left(e^{A_1(t-\sigma)} - e^{A_1(t-h-\sigma)}\right) (P(\sigma)z(\sigma))^1 \|_{Z_1} \leq m_1 \| (P(\sigma)z(\sigma))^1 \|_{Z_1},$$

and for all $\sigma \in [0, t-h]$

$$\left(e^{A_1(t-\sigma)} - e^{A_1(t-h-\sigma)}\right) (P(\sigma)z(\sigma))^1 \to 0$$

as $h \to 0$. Therefore the first term on the right-hand side of equation (4.21) converges to zero by the Lebesgue Dominated Convergence Theorem.

For the second term, the proofs of Corollary 4.2.1 and Proposition 4.1.7 imply that there exists a constant m_2 such that

$$\| (S_2(t-\sigma) - S_2(t-h-\sigma))\, (P(\sigma)z(\sigma))^1 \|_{\alpha_0^2} \leq m_2 \| (P(\sigma)z(\sigma))^1 \|_{Z_1}.$$

Furthermore, for any $z^1 \in Z_1$ and $t' \in [0, T]$,

$$\begin{aligned}
\| (S_2(t') - S_2(t'-h))\, z^1 \|_{\alpha_0^2} \leq \| \int_{t'-h}^{t'} S_3(t'-\sigma) A_2 e^{A_1\sigma} z^1 \, d\sigma \|_{\alpha_0^2} \\
+ \| \int_0^{t'-h} (S_3(t'-\sigma) - S_3(t'-h-\sigma))\, A_2 e^{A_1\sigma} z^1 \, d\sigma \|_{\alpha_0^2}.
\end{aligned}$$

Using the approach used in the proof of Proposition 4.1.7 with (4.17), the first term is

$$\begin{aligned}
\| \int_{t'-h}^{t'} S_3(t'-h) A_2 e^{A_1\sigma} z^1 \, d\sigma \|_{\alpha_0^2} = \sum_{k=1}^{\infty} \alpha_{0,k}^2 \left(\int_{t'-h}^{t'} e^{\lambda_k(t'-\sigma)} \langle A_2 e^{A_1\sigma} (P(s)z(s))^1, \phi_k \rangle \, d\sigma \right)^2 \\
\leq R'(h)(n-1)\bar{\alpha}_2^2 \sum_{j=1}^{n-1} \max_i \left(\int_{t'-h}^{t'} s_1(\sigma; j, i)^2 \, d\sigma \right) \|z^1\|_{Z_1}^2
\end{aligned}$$

and this converges to zero as $h \to 0$ since

$$R'(h) = \frac{1}{2}\max\left\{|e^{2\lambda_1 h} - 1|, \dots, |e^{2\lambda_{n_0} h} - 1|, 1\right\} R_5 \longrightarrow \frac{1}{2}R_5.$$

Note that for the second term

$$\begin{aligned}
&\| \int_0^{t'-h} \left(S_3(t'-\sigma) - S_3(t'-h-\sigma)\right) A_2 e^{A_1\sigma} z^1 \, d\sigma \|^2_{\alpha_0^2} \\
&\quad = \sum_{k=1}^{\infty} \alpha_{0,k}^2 \left(\int_h^{t'} \left(e^{\lambda_k \sigma} - e^{\lambda_k(\sigma-h)}\right) \langle A_2 e^{A_1(t'-\sigma)} z^1, \phi_k \rangle_{L^2(\Omega)} \, d\sigma \right)^2 \\
&\quad \le \sum_{k=1}^{\infty} \alpha_{0,k}^2 \left[\left(e^{\lambda_k h} - 1\right)^2 \int_0^{t'-h} e^{2\lambda_k \sigma} \, d\sigma \right] \int_h^{t'} \langle A_2 e^{A_1(t'-\sigma)} z^1, \phi_k \rangle^2_{L^2(\Omega)} \, d\sigma \\
&\quad \le \sum_{k=1}^{n_0} \alpha_{0,k}^2 \left(e^{\lambda_k h} - 1\right)^2 \int_0^{t'-h} e^{2\lambda_k \sigma} \, d\sigma \int_h^{t'} \langle A_2 e^{A_1(t'-\sigma)} z^1, \phi_k \rangle^2_{L^2(\Omega)} \, d\sigma \\
&\qquad + \sum_{k=n_0+1}^{\infty} \frac{\alpha_{0,k}^2}{2|\lambda_k|} \left(e^{\lambda_k h} - 1\right)^2 \int_h^{t'} \langle A_2 e^{A_1(t'-\sigma)} z^1, \phi_k \rangle^2_{L^2(\Omega)} \, d\sigma.
\end{aligned}$$

Since, for $k > n_0$, $\left(e^{\lambda_k h} - 1\right)^2 < 1$ the series converges uniformly in h. Moreover, $e^{\lambda_k h} - 1 \to 0$ for each k as $h \to 0$, and so the series converges to zero as $h \to 0$.

Finally,

$$\begin{aligned}
&\| \int_0^{t-h} \left(S_3(t-\sigma) - S_3(t-h-\sigma)\right) \left(P(\sigma) z(\sigma)\right)^2 \, d\sigma \|^2_{\alpha_0^2} \\
&\quad \le n \sum_{k=1}^{\infty} \alpha_{0,k}^2 \left[\left(e^{\lambda_k h} - 1\right)^2 \int_0^{t-h} e^{2\lambda_k \sigma} \, d\sigma \right] \max_i p_2^k(n,i)^2 \int_0^{t-h} \|z(\sigma)\|^2_{\mathbb{Z}} \, d\sigma.
\end{aligned}$$

This series converges to zero as $h \to 0$ in the same way as before. □

4.2.2 Assumptions for mild form

In this subsection it is shown that the general example, with suitable conditions imposed, satisfies assumption PS V. This is achieved in the following two stages.

Condition 11. Let $\underline{d}_2^k(s;n,i)$ be the sequence $\left(\left(\beta_{2,l}^i\right)^{-1/2} d_l^k(s;n,i)\right)_{l \in \mathbb{N}}$. Then

$$d_2^k(n,i) := \operatorname*{ess\,sup}_{s \in [0,T]} \|\underline{d}_2^k(s;n,i)\|_{\ell^2} < \infty \tag{4.22}$$

for all $1 \le i \le m$. Let $n_0 = \min\{n \in \mathbb{N} : \lambda_n < 0\}$, then

$$\sum_{l=n_0+1}^{\infty} \frac{\alpha_{0,l}^2 d_2^l(n,i)^2}{|\lambda_l|} < \infty \tag{4.23}$$

for all $1 \le i \le m$.

Proposition 4.2.4. *Suppose that the general example satisfies Condition 11. Then, for every* $h(\cdot) \in L^2(0,T;\overline{W})$, $(\mathrm{M}_S h)(t) \in \underline{Z}$ *for all* $t \in [0,T]$.

Proof. Let $\underline{h}_i(s;\beta_2) = \left(\left(\beta_{2,l}^i\right)^{1/2} \langle h_i(s), \phi_l\rangle_{L^2}\right)_{l\in\mathbb{N}}$ and observe that

$$\begin{aligned}
&\|\int_0^t \sum_{k=1}^{\infty} e^{\lambda_k(t-s)} \langle D_{ni}(s)h_i(s), \phi_k\rangle_{L^2}\phi_k\, ds\|_{\alpha_0^2}^2 \\
&\quad = \sum_{l=1}^{\infty} \alpha_{0,l}^2 \left(\int_0^t e^{\lambda_l(t-s)} \langle D_{ni}(s)h_i(s), \phi_l\rangle_{L^2}\, ds\right)^2 \\
&\quad \le \sum_{l=1}^{\infty} \alpha_{0,l}^2 \left(\int_0^t e^{2\lambda_l s}\, ds\right) \int_0^t \|\underline{d}_2^l(s;n,i)\|_{\ell^2}^2\, \|\underline{h}_i(s;\beta_2)\|_{\ell^2}^2\, ds \\
&\quad \le \left[\sum_{l=1}^{n_0} \alpha_{0,l}^2 \left(\int_0^t e^{2\lambda_l s}\, ds\right) d_2^l(n,i)^2 + \sum_{l=n_0+1}^{\infty} \frac{\alpha_{0,k}^2 d_2^l(n,i)^2}{2|\lambda_l|}\right] \int_0^t \|h_i(s)\|_{\beta_2^i}^2\, ds
\end{aligned}$$

which is finite, for all $1 \le i \le m$, by conditions (4.22) and (4.23).

Since $(D(\cdot)h(\cdot))^1 \in L^2(0,T;Z_1)$,

$$\|\int_0^t e^{A_1(t-s)}\,(D(s)h(s))^1\, ds\|_{Z_1} \le \int_0^t M_1 e^{w_1(t-s)}\,\|\,(D(s)h(s))^1\,\|_{Z_1}\, ds$$

and

$$\|\int_0^t S_2(t-s)\,(D(s)h(s))^1\, ds\|_{\alpha_0^2} \le \int_0^t \underline{M}_2(t-s)\|\,(D(s)h(s))^1\,\|_{Z_1}\, ds$$

are finite. Therefore

$$\begin{aligned}
&\|\int_0^t S(t-s)D(s)h(s)\, ds\|_{\underline{Z}}^2 \\
&\quad = \|\int_0^t e^{A_1(t-s)}\,(D(s)h(s))^1\, ds\|_{Z_1}^2 \\
&\qquad + \|\int_0^t S_2(t-s)\,(D(s)h(s))^1\, ds + \int_0^t S_3(t-s)\,(D(s)h(s))_n\, ds\|_{\alpha_0^2}^2 \\
&\quad \le \left(\int_0^t M_1 e^{w_1(t-s)}\,\|\,(D(s)h(s))^1\,\|_{Z_1}\, ds\right)^2 + 2\left(\int_0^t \underline{M}_2(t-s)\|\,(D(s)h(s))^1\,\|_{Z_1}\, ds\right)^2 \\
&\qquad + 2m\sum_{i=1}^{m} \|\int_0^t \sum_{k=1}^{\infty} e^{\lambda_k(t-s)} \langle D_{ni}(s)h_i(s), \phi_k\rangle_{L^2}\phi_k\, ds\|_{\alpha_0^2}^2
\end{aligned}$$

is finite as required. □

Remark 4.2.5. Proposition 4.2.4 gives a stronger condition than is required for PS V; the proposition shows that $(\mathbb{M}_S h)(t) \in \underline{Z}$ for all $t \in [0,T]$, not just almost everywhere.

All that remains to show that the general example satisfies condition PS V is to show the continuity of $t \mapsto (\mathbb{M}_S h)(t)$ with respect to $\|\cdot\|_Z$. Before proceeding to the proof of this, note that the following corollary applies.

Corollary 4.2.6. *Under the conditions of Proposition 4.2.4 the general example satisfies PS X; namely, there exists a positive constant* $\overline{K}_6$ *such that*

$$\|\mathbb{L}_S h\|_{L^4(0,T;\underline{W})} \leq \overline{K}_6 \|h\|_{L^2(0,T;\overline{W})}$$

for all $h(\cdot) \in L^2(0,T;\overline{W})$.

Proof. From the proof of Proposition 4.2.4 we have

$$\begin{aligned}
&\|\int_0^t S(t-s)D(s)h(s)\,ds\|_{\underline{Z}}^2 \\
&\quad\leq \int_0^t (M_1^2 e^{2\omega_1 s} + 2\underline{M}_2(s)^2)\,ds \int_0^t \sum_{k=1}^{n-1} \|\sum_{i=1}^{m} D_{ki}(s)h_i(s)\|_{\alpha_1^1}^2\,ds \\
&\quad\quad + 2m \max_i \left(\sum_{k=1}^{\infty} \alpha_{0,k}^2 \int_0^t e^{2\lambda_k s}\,ds\, d_2^k(n,i)^2\right) \int_0^t \|h(s)\|_{\overline{W}}^2\,ds \\
&\quad\leq m\left[(\text{const}) \sum_{k=1}^{n-1} \max_i \|D_{ki}(\cdot)\|_\infty^2 + 2\max_i \left(\sum_{k=1}^{n_0} \alpha_{0,k}^2 \left(\int_0^t e^{2\lambda_k s}\,ds\right) d_2^k(n,i)^2\right.\right. \\
&\quad\quad \left.\left. + \sum_{k=n_0+1}^{\infty} \frac{\alpha_{0,k}^2 d_2^k(n,i)^2}{2|\lambda_k|}\right)\right] \|h\|_{L^2(0,t;\overline{W})}^2 \\
&\quad= m(t)\|h\|_{L^2(0,t;\overline{W})}^2
\end{aligned}$$

for some $m(\cdot) \in L^\infty(0,T;\mathbb{R})$. Therefore

$$\begin{aligned}
\|E(\cdot)\int_0^{\cdot} S(\cdot - s)D(s)h(s)\,ds\|_{L^4(0,T;\underline{W})}^4 &\leq \int_0^T \|E(\cdot)\|_\infty^4 \|\int_0^t S(t-s)D(s)h(s)\,ds\|_{\underline{Z}}^4\,ds \\
&\leq T\|E(\cdot)\|_\infty^4 \|m(\cdot)\|_\infty^2 \|h\|_{L^2(0,T;\overline{W})}^4
\end{aligned}$$

as required. □

To conclude the proof that, under suitable conditions, the general example satisfies PS V, the following lemmas are required.

Lemma 4.2.7. *Let $\delta > 0$. For the general example*

$$\|S(\delta)\int_t^{t+\delta} S(t-s)D(s)h(s)\,ds\|_Z \longrightarrow 0$$

as $\delta \to 0$.

Proof. Applying the same method used in the proof of Corollary 4.2.6 yields

$$\begin{aligned}\|\int_t^{t+\delta} S(t+\delta-s)D(s)h(s)\,ds\|_Z &\leq \overline{R}_1\|\int_t^{t+\delta} S(t+\delta-s)D(s)h(s)\,ds\|_{\underline{Z}}\\ &\leq \overline{R}_1(m(\delta))^{1/2}\|h\|_{L^2(t,t+\delta;\overline{W})}\end{aligned}$$

which converges to zero as $\delta \to 0$. □

Lemma 4.2.8. *Let $\delta > 0$. For the general example*

$$\|\int_{t-\delta}^{t} S(t-s)D(s)h(s)\,ds\|_Z \longrightarrow 0$$

as $\delta \to 0$.

Proof. Again applying the method used in the proof of Corollary 4.2.6 yields

$$\begin{aligned}\|\int_{t-\delta}^{t} S(t-s)D(s)h(s)\,ds\|_Z &\leq \overline{R}_1\|\int_{t-\delta}^{t} S(t-s)D(s)h(s)\,ds\|_{\underline{Z}}\\ &\leq \overline{R}_1(m(\delta))^{1/2}\|h\|_{L^2(t-\delta,t;\overline{W})}\end{aligned}$$

which converges to zero as $\delta \to 0$. □

Lemma 4.2.9. *Let $\delta > 0$. Then, for the general example, and for all $h \in L^2(0,T;\overline{W})$*

$$\|\int_0^{t-\delta} \left(S(t-s) - S(t-\delta-s)\right) D(s)h(s)\,ds\|_Z \longrightarrow 0$$

as $\delta \to 0$.

Proof. Proceeding in the same fashion as used in the proof of Corollary 4.2.3 with $s = 0$, $h = \delta$, replacing $P(\cdot)$ by $D(\cdot)$ and $z(\cdot)$ by $h(\cdot)$ proves the lemma. □

Remark 4.2.10. The proof of Corollary 4.2.3 shows that the lemma is also true when the norm is taken is that for $\underline{Z}$.

Proposition 4.2.11. *For the general example the map* $t \mapsto (\mathrm{M}_S h)(t)$ *for any* $h \in L^2(0,T;\overline{W})$ *is continuous with respect to* $\|\cdot\|_Z$.

Proof. First consider the question of continuity from the right : Let $\delta > 0$. Then

$$\begin{aligned}
&\|(\mathrm{M}_S h)(t+\delta) - (\mathrm{M}_S h)(t)\|_Z \\
&\quad = \|\int_0^{t+\delta} S(t+\delta-s)D(s)h(s)\,ds - \int_0^t S(t-s)D(s)h(s)\,ds\|_Z \\
&\quad \le \|(S(\delta)-I)\int_0^t S(t-s)D(s)h(s)\,ds\|_Z + \|S(\delta)\int_t^{t+\delta} S(t-s)D(s)h(s)\,ds\|_Z.
\end{aligned}$$

Since $(\mathrm{M}_S h)(t) \in \underline{Z}$ and $S(t)$ is a strongly continuous semigroup on Z, we see that the first term tends to zero as $\delta \to 0$. The second term converges to zero by Lemma 4.2.7. Therefore $t \mapsto (\mathrm{M}_S h)(t)$ is continuous from the right with respect to $\|\cdot\|_Z$.

Now consider continuity from the left. Again let $\delta > 0$. Then

$$\begin{aligned}
&\|(\mathrm{M}_S h)(t) - (\mathrm{M}_S h)(t-\delta)\|_Z \\
&\quad \le \|\int_{t-\delta}^{t} S(t-s)D(s)h(s)\,ds\|_Z + \|\int_0^{t-\delta} (S(t-s) - S(t-\delta-s))\, D(s)h(s)\,ds\|_Z.
\end{aligned}$$

We see that the first term converges to zero as $\delta \to 0$ by Lemma 4.2.8. The second term converges to zero by Lemma 4.2.9 which concludes the proof. □

In imposing suitable conditions on the general example for condition PS VI to be satisfied, the special case when both B and C are scalar will be taken into consideration.

Condition 12. For all $1 \le i, j \le n-1$

$$\sum_{k=1}^{\infty} \|c_k(i)\|_Y \|b_k(j)\|_U < \infty \tag{4.24a}$$

or, in the scalar situation,

$$U \subseteq Y \quad \text{with continuous injection.} \tag{4.24b}$$

In addition

$$\sum_{k=n_0+1}^{\infty} \frac{\|c_k(n)\|_Y \|b_k(j)\|_U}{|\lambda_k|} < \infty. \tag{4.25}$$

Furthermore

$$\sum_{k=1}^{\infty} e^{\lambda_k T} \|c_k(n)\|_Y \|b_k(n)\|_U < \infty. \tag{4.26}$$

Proposition 4.2.12. *Condition 12 is sufficient for there to exist a constant $\overline{K}_2$, for the general example, such that*

$$\|CS(T)Bu\|_Y \leq \overline{K}_2 \|u\|_U$$

for all $u \in U$.

Proof. For any $1 \leq i, j \leq n-1$

$$\|C_i \left(s_1(T;i,j)B_j u\right)\|_Y = |s_1(T;i,j)| \, \|C_i B_j u\|_Y.$$

In the scalar case (4.24b) is sufficient for the right-hand side to be a bounded function of u. Otherwise

$$\begin{aligned}\|C_i \left(s_1(T;i,j)B_j u\right)\|_Y &= |s_1(T;i,j)| \, \| \sum_{k=1}^{\infty} c_k(i) \langle b_k(j), u \rangle_U \|_Y \\ &\leq |s_1(T;i,j)| \left(\sum_{k=1}^{\infty} \|c_k(i)\|_Y \|b_k(j)\|_U \right) \|u\|_U\end{aligned}$$

where the term in the bracket is finite by (4.24a). Furthermore,

$$\begin{aligned}\|C_n S_2(T) \left(Bu\right)^1\|_Y &= \| \sum_{k=1}^{\infty} c_k(n) \langle \int_0^T S_3(T-s) A_2 e^{A_1 s} \left(Bu\right)^1 ds, \phi_k \rangle_{L^2} \|_Y \\ &= \| \sum_{k=1}^{\infty} c_k(n) \int_0^T e^{\lambda_k (T-s)} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_2(i) s_1(s;i,j) \langle B_j u, \phi_k \rangle_{L^2} ds \|_Y \\ &\leq \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(|a_2(i)| \, \overline{s}_1(i,j) \sum_{k=1}^{\infty} \|c_k(n)\| \int_0^T e^{\lambda_k s} ds \, |\langle B_j u, \phi_k \rangle_{L^2}| \right) \\ &\leq \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} |a_2(i)| \, \overline{s}_1(i,j) \left(\sum_{k=1}^{n_0} \|c_k(n)\|_Y \|b_k(j)\|_U \int_0^T e^{\lambda_k s} ds \right. \\ &\qquad \left. + \sum_{k=n_0+1}^{\infty} \frac{\|c_k(n)\|_Y \|b_k(j)\|_U}{|\lambda_k|} \right) \|u\|_U\end{aligned}$$

and the term in the bracket is finite by (4.25).

Note that

$$\begin{aligned}\|C_nS_3(T)B_nu\|_Y &= \|C_n\left(\sum_{k=1}^{\infty} e^{\lambda_k T}\langle B_nu,\phi_k\rangle_{L^2}\phi_k\right)\|_Y \\ &\leq \sum_{k=1}^{\infty}\|c_k(n)\|_Y e^{\lambda_k T}|\langle B_nu,\phi_k\rangle_{L^2}| \\ &\leq \left(\sum_{k=1}^{\infty}\|c_k(n)\|_Y\|b_k(n)\|_U e^{\lambda_k T}\right)\|u\|_U.\end{aligned}$$

The term in the bracket is finite by assumption (4.26).

Finally,

$$\begin{aligned}\|CS(T)Bu\|_Y &= \|\sum_{i=1}^{n-1}C_i\left(\sum_{j=1}^{n-1}s_1(T;i,j)B_ju\right)+C_nS_2(T)\,(Bu)^1+C_nS_3(T)B_nu\|_Y \\ &\leq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\|C_i\,(s_1(T;i,j)B_ju)\,\|_Y+\|C_nS_2(T)\,(Bu)^1\,\|_Y+\|C_nS_3(T)B_nu\|_Y\end{aligned}$$

which is bounded above by (const) $\|u\|_U$ using the above estimates. □

Assumption PS VII of the previous chapter is similar to PS IV and the proof of the following result demonstrates this. Note though, that there is no requirement of continuity.

Condition 13. For $1 \leq i, j \leq n-1$ and all $z \in H^{\alpha_1^i}(\Omega;\mathbb{R})$ there exists a constant $k_2(i,j)$ such that

$$\|P_{ij}(\sigma)z\|_{\alpha_3^i} \leq k_2(i,j)\|z\|_{\alpha_1^j} \tag{4.27}$$

for all $\sigma \in [0,T]$. Similarly, for all $z \in H^{\alpha_0^2}(\Omega;\mathbb{R})$, there exists a constant $k_2(i,n)$ such that

$$\|P_{in}(\sigma)z\|_{\alpha_3^i} \leq k_2(i,n)\|z\|_{\alpha_2^2} \tag{4.28}$$

for all $\sigma \in [0,T]$. There exists a constant R_7 such that

$$\frac{\alpha_{3,k}^n}{|\lambda_k|\alpha_{1,k}^1} \leq R_7 \tag{4.29}$$

for all $k \in \mathbb{N}$. Furthermore,

$$\sum_{k=n_0+1}^{\infty}\frac{\alpha_{3,k}^n p_2^k(n,j)^2}{|\lambda_k|} < \infty \tag{4.30}$$

for all $1 \leq j \leq n$.

Proposition 4.2.13. *Suppose that the general example satisfies Condition 13. Then there exists a continuous function* $\overline{K}_3(\cdot) : [0,T] \to \mathbb{R}^+$ *such that* $\overline{K}_3(T) = 0$*, and for all* $s \in [0,T]$*,*

$$\int_s^T S(T-\sigma)P(\sigma)z(\sigma)\,d\sigma \in V$$

with

$$\| \int_s^T S(T-\sigma)P(\sigma)z(\sigma)\,d\sigma \|_V \le \overline{K}_3(s)\|z\|_{C(s,T;\underline{Z})}$$

for all $z(\cdot) \in C(s,T;\underline{Z})$.

Proof. Note that, for $1 \le i \le n-1$,

$$\begin{aligned}
&\| \int_s^T \sum_{j=1}^{n-1} s_1(T-\sigma;i,j)\left(P(\sigma)z(\sigma)\right)_j d\sigma\|^2_{\alpha_3^i} \\
&\qquad \le (n-1)\sum_{j=1}^{n-1} \| \sum_{l=1}^{n} \int_s^T s_1(T-\sigma;i,j)P_{jl}(\sigma)z_l(\sigma)d\sigma\|^2_{\alpha_3^i} \\
&\qquad \le (n-1)\sum_{j=1}^{n-1} \overline{s}_1(i,j)^2 \| \sum_{l=1}^{n} \int_s^T P_{jl}(\sigma)z_l(\sigma)d\sigma\|^2_{\alpha_3^i} \\
&\qquad \le n(n-1)(T-s)^2 \sum_{j=1}^{n-1} \overline{s}_1(i,j)^2 \max_{1\le l\le n} k_2(j,l)^2 \sup_{\sigma\in[s,T]} \sum_{l=1}^{n} \|z_l(\sigma)\|^2_{\alpha_2^l}
\end{aligned}$$

where this last term is equal to (const)$(T-s)^2\|z\|^2_{C(s,T;\overline{Z})}$. Here we have made use of the estimates (4.27) and (4.28) from Condition 13.

Using (4.29) we obtain the following estimates:

$$\begin{aligned}
&\| \int_s^T S_2(T-\sigma)\left(P(\sigma)z(\sigma)\right)^1 d\sigma\|^2_{\alpha_3^n} \\
&\qquad \le (T-s)\int_s^T \| \int_0^{T-\sigma} S_3(T-\sigma-t)A_2e^{A_1t}\left(P(\sigma)z(\sigma)\right)^1 dt\|^2_{\alpha_3^n} d\sigma \\
&\qquad \le (T-s)\int_s^T \Bigg(\sum_{k=1}^{n_0} \alpha_{3,k}^n \int_0^{T-\sigma} e^{2\lambda_k t}dt \int_0^{T-\sigma} \langle A_2e^{A_1t}\left(P(\sigma)z(\sigma)\right)^1, \phi_k\rangle^2_{L^2}dt \\
&\qquad\qquad + \sum_{k=n_0+1}^{\infty} \frac{\alpha_{3,k}^n}{2|\lambda_k|} \int_0^{T-\sigma} \langle A_2e^{A_1t}\left(P(\sigma)z(\sigma)\right)^1, \phi_k\rangle^2_{L^2}dt \Bigg) d\sigma \\
&\qquad \le (T-s)\int_s^T R(\sigma) \left(\sum_{k=1}^{\infty} \alpha_{1,k}^1 \int_0^{T-\sigma} \langle A_2e^{A_1t}\left(P(\sigma)z(\sigma)\right)^1, \phi_k\rangle^2_{L^2}dt \right) d\sigma
\end{aligned}$$

where

$$R(\sigma) = \frac{1}{2}\max\left\{|e^{2\lambda_1(T-\sigma)} - 1|, \ldots, |e^{2\lambda_{n_0}(T-\sigma)} - 1|, 1\right\} R_7.$$

The proof of Proposition 4.1.7 shows that the last term is bounded by

$$(n-1)\overline{a}_2^2 \sum_{j=1}^{n-1} \max_i \overline{s}_1(j,i)^2 (T-s) \int_s^T R(\sigma)(T-\sigma) \| (P(\sigma)z(\sigma))^1 \|_{Z_1}^2 d\sigma$$

while the proof of Proposition 4.2.2 shows that

$$\int_s^T (T-\sigma)R(\sigma) \| (P(\sigma)z(\sigma))^1 \|_{Z_1}^2 d\sigma \le T \sup_\sigma R(\sigma) \int_s^T \| (P(\sigma)z(\sigma))^1 \|_{Z_1}^2 d\sigma$$
$$\le (\text{const}) T(T-s) \sup_\sigma R(\sigma) \|z\|_{C(s,T;\overline{Z})}^2$$

and so

$$\| \int_s^T S_2(T-\sigma) (P(\sigma)z(\sigma))^1 d\sigma \|_{\alpha_3^n} \le (\text{const})(T-s)\|z\|_{C(s,T;\overline{Z})}^2.$$

The proof of Proposition 4.2.2 also shows (replacing $\underline{Z}$ by V) that

$$\| \int_s^T S_3(T-\sigma) (P(\sigma)z(\sigma))_n d\sigma \|_{\alpha_3^n}^2$$
$$\le n(T-s) \max_i \left[\sum_{k=1}^{n_0} \alpha_{3,k}^n \left(\int_0^{T-s} e^{2\lambda_k \sigma} d\sigma \right) + \sum_{k=n_0+1}^{\infty} \frac{\alpha_{3,k}^n p_2^k(n,i)^2}{2|\lambda_k|} \right] \|z\|_{C(s,T;\overline{Z})}^2.$$

Therefore to complete the proof we note that

$$\| \int_s^T S(T-\sigma) P(\sigma) z(\sigma) d\sigma \|_V^2$$
$$= \sum_{i=1}^{n-1} \| \int_s^T \sum_{j=1}^{n-1} s_1(T-\sigma; i,j) (P(\sigma)z(\sigma))_j d\sigma \|_{\alpha_3^i}^2$$
$$+ \| \int_s^T \left(S_2(T-\sigma) (P(\sigma)z(\sigma))^1 + S_3(T-\sigma) (P(\sigma)z(\sigma))_n \right) d\sigma \|_{\alpha_3^n}^2$$

and use the above estimates. □

Condition 14. For each $1 \le i \le n-1$, $1 \le j \le m$, there exists a $k_3(\cdot; i,j) \in PC(0,T;\mathbb{R}^+)$ such that

$$\|D_{ij}(s)h\|_{\alpha_3^i} \le k_3(s;i,j)\|h\|_{\beta_2^j} \tag{4.31}$$

for all $h \in H^{\beta_2^j}(\Omega;\mathbb{R})$. Furthermore,

$$\sum_{k=n_0+1}^{\infty} \frac{\alpha_{3,k}^n d_2^k(n,i)^2}{|\lambda_k|} < \infty \tag{4.32}$$

for all $1 \le i \le m$.

The estimate (4.31) takes account of the fact that the semigroup $S(t)$ has no smoothing properties on the first $n-1$ components. If V is such that $\alpha_3^i = \alpha_1^i$ for $i = 1,\dots,n-1$ then (4.31) is automatic.

Proposition 4.2.14. *Suppose that the general example satisfies Condition 14. Then* $(\mathbb{M}_S h)(T) \in V$ *and there exists a constant* $\overline{K}_4$ *such that*

$$\|(\mathbb{M}_S h)(T)\|_V \le \overline{K}_4 \|h\|_{L^2(0,T;\overline{W})}$$

for all $h \in L^2(0,T;\overline{W})$.

Proof. From the proof of Proposition 4.2.4 the following estimate is obtained by substituting α_3^n for α_0^2:

$$\begin{aligned}&\| \int_0^T S_3(T-s)\,(D(s)h(s))_n\, ds\|_{\alpha_3^n}^2 \\ &\qquad \le m \max_{1\le i\le m}\left[\sum_{k=1}^{n_0} \alpha_{3,k}^n \left(\int_0^T e^{2\lambda_k s} ds\right) d_2^k(n,i)^2 + \sum_{k=n_0+1}^{\infty} \frac{\alpha_{3,k}^n d_2^k(n,i)^2}{2|\lambda_k|}\right] \int_0^T \|h(s)\|_{\overline{W}}^2 ds\end{aligned}$$

which is finite by (4.32). Similarly the proof of Proposition 4.2.13 yields the following estimate:

$$\begin{aligned}&\| \int_0^T S_2(T-s)\,(D(s)h(s))^1\, ds\|_{\alpha_3^n}^2 \\ &\qquad \le (n-1)\overline{a}_2^2 \sum_{j=1}^{n-1} \max_i \overline{s}_1(j,i)^2 T^2 \sup_{\sigma\in[0,T]} R(\sigma) \int_0^T \|(D(s)h(s))^1\|_{Z_1}^2 d\sigma \\ &\qquad \le \overline{a}_2^2 \sum_{j=1}^{n-1} \max_i \overline{s}_1(j,i)^2 T^2 \sup_{\sigma\in[0,T]} R(\sigma) \sum_{k=1}^{n-1} \max_i \|D_{ki}(\cdot)\|_\infty^2 \int_0^T \|h(s)\|_{\overline{W}}^2 ds\end{aligned}$$

where the proof of Corollary 4.2.6 has been used to obtain an estimate for the integrand in the previous line.

Again using the proof of Proposition 4.2.13, for all $1 \leq i \leq n-1$, we obtain

$$\begin{aligned}
&\| \int_0^T \sum_{j=1}^{n-1} s_1(T-s;i,j) \left(D(s)h(s)\right)_j ds\|^2_{\alpha_3^i} \\
&\qquad \leq n(n-1) \sum_{j=1}^{n-1} \bar{s}_1(i,j)^2 \sum_{k=1}^{n} \left(\int_0^T \|D_{jk}(s)h_k(s)\|_{\alpha_3^i} ds \right)^2 \\
&\qquad \leq n(n-1) \sum_{j=1}^{n-1} \bar{s}_1(i,j)^2 \max_k \left[\int_0^T k_3(s;j,k)^2 ds \right] \int_0^T \|h(s)\|^2_W ds
\end{aligned}$$

where (4.31) has been used to obtain the last inequality.

Finally the following estimate is obtained

$$\begin{aligned}
&\| \int_0^T S(T-s)D(s)h(s)ds\|^2_V \\
&\quad = \sum_{i=1}^{n-1} \| \int_s^T \sum_{j=1}^{n-1} s_1(T-s;i,j) \left(D(s)h(s)\right)_j ds\|^2_{\alpha_3^i} \\
&\quad\quad + \| \int_s^T \left(S_2(T-s) \left(D(s)h(s)\right)^1 + S_3(T-s) \left(D(s)h(s)\right)_n \right) ds\|^2_{\alpha_3^n} \\
&\quad \leq \left(\sum_{i=1}^{n-1} n(n-1) \sum_{j=1}^{n-1} \bar{s}_1(i,j)^2 \max_k \left[\int_0^T k_3(s;j,k)^2 ds \right] \right. \\
&\quad\quad + 2m(n-1)\bar{a}_2^2 \sum_{j=1}^{n-1} \max_i \bar{s}_1(j,i)^2 T^2 \sup_{\sigma\in[0,T]} R(\sigma) \sum_{k=1}^{n-1} \max_i \|D_{ki}(\cdot)\|^2_\infty \\
&\quad\quad \left. + 2m \max_{1\leq i\leq m} \left[\sum_{k=1}^{\infty} \alpha_{3,k}^n \left(\int_0^T e^{2\lambda_k s} ds \right) d_2^k(n,i)^2 \right] \right) \int_0^T \|h(s)\|^2_W ds
\end{aligned}$$

which proves the proposition. □

Now it only remains to show that the general example satisfies PS IX. This is the subject of the following lemma and proposition.

Condition 15. There exists a constant R_6 such that

$$\frac{\alpha_{0,k}^2 \alpha_{0,l}^2}{\alpha_{1,k}^2 \alpha_{1,l}^2 |\lambda_k + \lambda_l|} \leq R_6 \tag{4.33}$$

for all $k, l \in \mathbb{N}$.

Lemma 4.2.15. *Suppose that the general example satisfies Condition 15. Then there exists a constant $g_3(k)$ (dependent on k) such that*

$$\int_0^T \|E_{kn}(s)S_3(s)z\|^4_{\beta_0^k}\, ds \leq g_3(k)\|z\|^4_{\alpha_1^2}$$

for each $1 \leq k \leq m$ and $z \in H^{\alpha_0^2}(\Omega;\mathbb{R})$.

Proof. We see that

$$\|E_{kn}(t)S_3(t)z\|^4_{\beta_0^k} \leq \|E_{kn}(t)\|^4\|S_3(t)z\|^4_{\alpha_0^2}$$

for all $t \in [0,T]$. For $k \in \mathbb{N}$ we set $n(k) = \min\{l \in \mathbb{N} : \lambda_k + \lambda_l < 0\}$. Now let $n_1 = \max_{k\in\mathbb{N}} n(k)$.

We start by fixing k and considering

$$\begin{aligned}
&\sum_{l=1}^{\infty}\int_0^T \alpha_{0,k}^2\alpha_{0,l}^2 e^{2t(\lambda_l+\lambda_j)}\langle z,\phi_l\rangle^2_{L^2(\Omega)}\langle z,\phi_k\rangle^2_{L^2(\Omega)}\, dt\\
&\quad= \sum_{l=1}^{n_1}\alpha_{0,k}^2\alpha_{0,l}^2\int_0^T e^{2t(\lambda_k+\lambda_l)}\, dt\langle z,\phi_l\rangle^2_{L^2}\langle z,\phi_k\rangle^2_{L^2} + \sum_{l=n_1+1}^{\infty}\frac{\alpha_{0,k}^2\alpha_{0,l}^2}{|\lambda_k+\lambda_l|}\langle z,\phi_l\rangle^2_{L^2}\langle z,\phi_k\rangle^2_{L^2}\\
&\quad\leq \overline{R}_6\sum_{l=1}^{\infty}\alpha_{1,k}^2\alpha_{1,l}^2\langle z,\phi_l\rangle^2_{L^2(\Omega)}\langle z,\phi_k\rangle^2_{L^2(\Omega)}\\
&\quad= \overline{R}_6\alpha_{1,k}^2\langle z,\phi_k\rangle^2_{L^2(\Omega)}\|z\|^2_{\alpha_1^2} < \infty
\end{aligned}$$

where $\overline{R}_6 = \max\left\{R_6|e^{4\lambda_1 T}-1|, R_6\right\}$. Therefore

$$\int_0^T\sum_{l=1}^{\infty}\alpha_{0,k}^2\alpha_{0,l}^2 e^{2t(\lambda_l+\lambda_j)}\langle z,\phi_l\rangle^2_{L^2}\langle z,\phi_k\rangle^2_{L^2}dt = \sum_{l=1}^{\infty}\int_0^T \alpha_{0,k}^2\alpha_{0,l}^2 e^{2t(\lambda_l+\lambda_j)}\langle z,\phi_l\rangle^2_{L^2}\langle z,\phi_k\rangle^2_{L^2}dt$$

is finite and so

$$\sum_{l=1}^{\infty}\alpha_{0,k}^2\alpha_{0,l}^2 e^{2t(\lambda_l+\lambda_j)}\langle z,\phi_l\rangle^2_{L^2(\Omega)}\langle z,\phi_k\rangle^2_{L^2(\Omega)}$$

converges for almost all $t \in [0,T]$. Hence

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_0^T\sum_{l=1}^{\infty}\alpha_{0,k}^2\alpha_{0,l}^2 e^{2t(\lambda_l+\lambda_j)}\langle z,\phi_l\rangle^2_{L^2(\Omega}\langle z,\phi_k\rangle^2_{L^2(\Omega)}\, dt\\
&\qquad\leq \sum_{k=1}^{\infty}\overline{R}_6\sum_{l=1}^{\infty}\alpha_{1,k}^2\alpha_{1,l}^2\langle z,\phi_l\rangle^2_{L^2(\Omega)}\langle z,\phi_k\rangle^2_{L^2(\Omega)}\\
&\qquad= \overline{R}_6\left[\sum_{k=1}^{\infty}\alpha_{1,k}^2\langle z,\phi_k\rangle^2_{L^2(\Omega)}\right]^2 = \overline{R}_6\|z\|^4_{\alpha_1^2}
\end{aligned}$$

and so

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\alpha_{0,k}^2\alpha_{0,l}^2 e^{2t(\lambda_l+\lambda_j)}\langle z,\phi_l\rangle_{L^2(\Omega)}^2\langle z,\phi_k\rangle_{L^2(\Omega)}^2$$

converges for almost all $t \in [0,T]$.

Finally we see that

$$\begin{aligned}\int_0^T \|E_{kn}(t)S_3(t)z\|_{\beta_0^k}^4\,dt &\le \|E_{kn}(\cdot)\|_\infty^4\int_0^T \|S_3(t)z\|_{\alpha_0^2}^4\,dt\\ &= \|E_{kn}(\cdot)\|_\infty^4\int_0^T\left[\sum_{j=1}^{\infty}\alpha_{0,j}^2 e^{2\lambda_j t}\langle z,\phi_j\rangle_{L^2(\Omega)}^2\right]^2 dt\\ &= \|E_{kn}(\cdot)\|_\infty^4\int_0^T\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\alpha_{0,j}^2\alpha_{0,l}^2 e^{2t(\lambda_l+\lambda_j)}\langle z,\phi_l\rangle_{L^2}^2\langle z,\phi_j\rangle_{L^2}^2\,dt\\ &\le \|E_{kn}(\cdot)\|_\infty^4\overline{R}_6\|z\|_{\alpha_1^2}^4\end{aligned}$$

as required. □

Proposition 4.2.16. *Suppose that the general example satisfies the conditions of Lemma 4.2.15. Then there exists a constant $\overline{K}_5$ such that*

$$\|E(\cdot)S(\cdot)z\|_{L^4(0,T;\underline{W})} \le \overline{K}_5\|z\|_Z$$

for all $z \in \underline{Z}$.

Proof. First note that

$$[E(t)S(t)z]_k = \sum_{i=1}^{n-1} E_{ki}(t)\left(e^{A_1 t}z^1\right)_i + E_{kn}(t)S_2(t)z^1 + E_{kn}(t)S_3(t)z_n.$$

We have the following estimate for the first term, for $1 \le k \le m$ fixed,

$$\begin{aligned}\|\sum_{i=1}^{n-1} E_{ki}(t)\left(e^{A_1 t}z^1\right)_i\|_{\beta_0^k}^4 &\le (n-1)^3\sum_{i=1}^{n-1}\|E_{ki}(t)\|^4\,\|\left(e^{A_1 t}z^1\right)_i\|_{\alpha_1^i}^4\\ &\le (n-1)^6\sum_{i=1}^{n-1}\|E_{ki}(t)\|^4\sum_{j=1}^{n-1}|s_1(t;i,j)|^4\,\|z_j\|_{\alpha_1^i}^4\\ &\le (n-1)^6\sum_{i=1}^{n-1}\|E_{ki}(\cdot)\|_\infty^4\max_j \overline{s}_1(i,j)^4\|z^1\|_{Z_1}^4\end{aligned}$$

where $\bar{s}_1(i,j) = \sup_{t\in[0,T]} |s_1(t;i,j)|$. Therefore

$$\|\sum_{i=1}^{n-1} E_{ki}(t)\left(e^{A_1 t}z^1\right)_i\|_{\beta_0^k}^4 \le g_1(k)\|z^1\|_{Z_1}^4.$$

For the second term note that, from Corollary 4.2.1 (of Proposition 4.1.7), we have $S_2(t) \in \mathcal{L}(Z_1, H^{\alpha_0^2}(\Omega;\mathbb{R}))$ and

$$\|S_2(t)z^1\|_{\alpha_0^2} \le \underline{M}_2(t)\|z^1\|_{Z_1}.$$

Therefore

$$\|E_{kn}(t)S_2(t)z^1\|_{\beta_0^k}^4 \le \|E_{kn}(\cdot)\|_\infty^4 \sup_{t\in[0,T]} \underline{M}_2(t)^4\|z^1\|_{Z_1}^4 = g_2(k)\|z^1\|_{Z_1}^4$$

for every $1 \le k \le m$. Therefore

$$\begin{aligned}
\int_0^T \|E(t)S(t)z\|_{\underline{W}}^4\,dt &= \int_0^T \left(\sum_{k=1}^m \|\left[E(t)S(t)z\right]_k\|_{\beta_0^k}^2\right)^2 dt \\
&\le m\sum_{k=1}^m \int_0^T \|\left[E(t)S(t)z\right]_k\|_{\beta_0^k}^4\,dt \\
&\le 9m\sum_{k=1}^m \int_0^T \Bigg(\|\sum_{i=1}^{n-1} E_{ki}(t)\left(e^{A_1 t}z^1\right)_i\|_{\beta_0^k}^4 + \|E_{kn}(t)S_2(t)z^1\|_{\beta_0^k}^4 \\
&\qquad + \|E_{kn}(t)S_3(t)z_n\|_{\beta_0^k}^4\Bigg)\,dt \\
&\le 9m\sum_{k=1}^m \int_0^T \left(g_1(k)\|z^1\|_{Z_1}^4 + g_2(k)\|z^1\|_{Z_1}^4 + \|E_{kn}(t)S_3(t)z_n\|_{\beta_0^k}^4\right)dt
\end{aligned}$$

(using the above estimates)

$$\le 9m\sum_{k=1}^m \left[T\left(g_1(k)+g_2(k)\right)\|z^1\|_{Z_1}^4 + g_3(k)\|z_n\|_{\alpha_1^2}^4\right]$$

(by Lemma 4.2.15)

$$\begin{aligned}
&\le 9m^2\left(T(G_1+G_2)\|z^1\|_{Z_1}^4 + G_3\|z_n\|_{\alpha_1^2}^4\right) \\
&\le \overline{K}_5^4\|z\|_Z^4
\end{aligned}$$

where $G_i = \max_k g_i(k)$. □

4.3 A semilinear general example

The topic of this section is the study of a semilinear general example. The dynamics of the model for this example will be considered in the triple $\underline{Z} \subset Z \subset \overline{Z}$ defined in Section 4.1.1. The model is assumed to be of the form

$$\dot{\overline{z}}(t) = A\overline{z}(t) + g(t, \overline{z}(t)), \qquad \overline{z}(0) = \overline{z}_0 \tag{4.34}$$

where the state $\overline{z}$ takes values in Z; $g : [0,T] \times \underline{Z} \longrightarrow \overline{Z}$ is a nonlinear operator; and A is of the form considered in Section 4.1.2, namely

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{(n-1)\times(n-1)}$, $A_2 \in \mathbb{R}^{1\times(n-1)}$ and A_3 is a self-adjoint (unbounded) linear operator on $L^2(\Omega;\mathbb{R})$. The basic assumption of this section is that A is the generator of a strongly continuous semigroup $S(t)$ on all three spaces. The assumption is PS III and to ensure that it holds it will be assumed that Condition 9 is satisfied.

In Section 3.3 it was shown that under suitable conditions, namely PS I, PS III, PS XI–XIV, the semilinear model of (4.34) has a mild or classical solution—as defined in Section 3.3—depending on which assumptions about the nonlinearity are satisfied. The general example satisfies PS I by the construction of Section 4.1.1. It was shown in the last section that Condition 9 is sufficient for the general example to also satisfy PS III. Further conditions are imposed on the semilinear general example such that PS XI–XIV are also satisfied.

4.3.1 A mild solution exists

In this subsection it is shown that the semilinear general example satisfies PS XI which is required for the existence results (Propositions 3.3.3 and 3.3.4) which guarantee a mild solution. Suppose that the following condition is satisfied.

Condition A. There exists a constant R_8 such that

$$\frac{\alpha_{0,k}^2}{\alpha_{2,k}^2|\lambda_k|} \le R_8, \tag{4.35}$$

for all $k \in \mathbb{N}$.

Proposition 4.3.1. *Suppose that the general example satisfies Condition A. Then there exists a continuous function* $k_1(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ *such that* $k_1(0) = 0$ *and for all* $s \in [0,T]$, $\overline{z} \in C(s,T;\overline{Z})$, $t \in [s,T]$ *we have* $\int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \in \underline{Z}$ *with*

$$\|\int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma\|_{\underline{Z}} \le k_1(t-s)\|\overline{z}\|_{C(s,t;\overline{Z})}.$$

Proof. Given $s \in [0,T]$ and $\overline{z} \in C(s,T;\overline{Z})$ let $\overline{z}(t) = \begin{pmatrix}\overline{z}^1(t) & \overline{z}^2(t)\end{pmatrix}^\top \in Z_1 \times H^{\alpha_2^2}(\Omega;\mathbb{R})$. Note that, since $e^{A_1 t}$ is a strongly continuous semigroup on Z_1,

$$\|\int_s^t e^{A_1(t-\sigma)}\overline{z}^1(\sigma)\,d\sigma\|_{Z_1} \le \frac{M_1}{\omega_1}\left(e^{\omega_1(t-s)} - 1\right) \sup_{\sigma\in[s,t]} \|z^1(\sigma)\|_{Z_1},$$

for some suitable constants M_1, ω_1. Now consider

$$\begin{aligned}\|\int_s^t S_2(t-\sigma)\overline{z}^1(\sigma)\,d\sigma\|_{\alpha_0^2} &\le \int_s^t \underline{M}_2(t-\sigma)\|\overline{z}^1(\sigma)\|_{Z_1} d\sigma \\ &\le \int_0^{t-s} \underline{M}_2(\sigma)\,d\sigma \sup_{\sigma\in[s,t]} \|\overline{z}^1(\sigma)\|_{Z_1}\end{aligned}$$

which follows from Condition 9 and $\underline{M}_2(\cdot) \in C(0,T;\mathbb{R})$ (Corollary 4.2.1). Finally the following estimate follows from Condition A.

$$\begin{aligned}\|\int_s^t S_3(t-\sigma)\overline{z}^2(\sigma)\,d\sigma\|_{\alpha_0^2}^2 &= \sum_{k=1}^{\infty} \alpha_{0,k}^2 \left(\int_s^t e^{\lambda_k(t-\sigma)}\langle \overline{z}^2(\sigma), \phi_k\rangle_{L^2(\Omega)} d\sigma\right)^2 \\ &\le \sum_{k=1}^{n_0} \alpha_{0,k}^2 \int_0^{t-s} e^{2\lambda_k \sigma} d\sigma \int_s^t \langle \overline{z}^2(\sigma), \phi_k\rangle_{L^2(\Omega)}^2 d\sigma \\ &\quad + \sum_{k=n_0+1}^{\infty} \frac{\alpha_{0,k}^2}{2|\lambda_k|} \int_s^t \langle \overline{z}^2(\sigma), \phi_k\rangle_{L^2(\Omega)}^2 d\sigma \\ &\le (t-s)R_2'(t-s) \sup_{\sigma\in[s,t]} \|\overline{z}^2(\sigma)\|_{\alpha_2^2}^2\end{aligned}$$

where

$$R_2'(t-s) = \frac{1}{2}R_8 \max\left\{|e^{2\lambda_1(t-s)} - 1|, \ldots, |e^{2\lambda_{n_0}(t-s)} - 1|, 1\right\}.$$

Therefore

$$\begin{aligned}
&\| \int_s^t S(t-\sigma)\bar{z}(\sigma)\,d\sigma\|_{\underline{Z}}^2 \\
&\quad = \| \int_s^t e^{A_1(t-\sigma)}\bar{z}^1(\sigma)\,d\sigma\|_{Z_1}^2 + \| \int_s^t S_2(t-\sigma)\bar{z}^1(\sigma)\,d\sigma + \int_s^t S_3(t-\sigma)\bar{z}^2(\sigma)\,d\sigma\|_{\alpha_0^2}^2 \\
&\quad \le \frac{M_1^2}{\omega_1^2}\left(e^{\omega_1(t-s)}-1\right)^2 \sup_{\sigma\in[s,t]} \|z^1(\sigma)\|_{Z_1}^2 + 2\left(\int_0^{t-s} \underline{M}_2(\sigma)\,d\sigma\right)^2 \sup_{\sigma\in[s,t]} \|\bar{z}^1(\sigma)\|_{Z_1}^2 \\
&\qquad + 2(t-s)R_2'(t-s) \sup_{\sigma\in[s,t]} \|\bar{z}^2(\sigma)\|_{\alpha_2^2}^2 \\
&\quad \le k_1(t-s)^2\|\bar{z}\|_{C(s,t;\overline{Z})}^2
\end{aligned}$$

for some continuous function $k_1(\cdot)$ as required. □

Therefore the semilinear general example satisfies PS XI, and if g satisfies a uniform Lipschitz condition in $\bar{z}$ then Proposition 3.3.4 can be applied. However if g satisfies a local Lipschitz condition in $\bar{z}$, uniformly in t, then Proposition 3.3.5 can be applied to show the existence of a (unique) mild solution for the semilinear example.

4.3.2 A classical solution exists

The existence results (Theorem 3.3.7, Corollary 3.3.9, and Theorem 3.3.11) of Section 3.3.2 require that the semilinear example satisfy PS XII–XIV. In this subsection suitable conditions sufficient for this to be the case are derived.

Theorem 3.3.7 gives general criteria for the existence of a solution to a inhomogeneous differential equation. In addition to PS XI the general example must also satisfy PS XII. To see that this is the case is the subject of the next result.

Proposition 4.3.2. *For the semilinear general example $\overline{D}(A) \subset \underline{Z}$, where $\overline{D}(A)$ is the domain of A when considered as the generator of the strongly continuous operator $S(t)$ on $\overline{Z}$, and the canonical injection is continuous with dense range.*

Proof. The domain of A is

$$\overline{D}(A) = H^{\alpha_2^2}(\Omega;\mathbb{R}) \overset{(n-1)\text{ times}}{\times\cdots\times} H^{\alpha_2^2}(\Omega;\mathbb{R}) \times \overline{D}(A_3)$$

where $\overline{D}(A_3)$ is the domain of A_3 when considered as the generator of $S_3(t)$ on $H^{\alpha_2^2}(\Omega;\mathbb{R})$. Condition 1 implies that $H^{\alpha_2^2}(\Omega;\mathbb{R}) \subset H^{\alpha_2^1}(\Omega;\mathbb{R})$ such that the canonical injection is continuous with dense range.

Note that

$$\overline{D}(A_3) = \left\{ z \in H^{\alpha_2^2}(\Omega;\mathbb{R}) : \sum_{k=1}^{\infty} \alpha_{2,k}^2 \lambda_k^2 \langle z, \phi_k \rangle_{L^2(\Omega)}^2 < \infty \right\}.$$

To see that $\overline{D}(A_3)$ is a dense subspace of $H^{\alpha_2^2}(\Omega;\mathbb{R})$ observe that

$$\frac{\alpha_{0,k}^2}{\alpha_{2,k}^2 \lambda_k^2} \leq R_3 \frac{\alpha_{0,k}^2}{\alpha_{1,k}^1 \lambda_k^2} \leq R_3 R_5 \frac{1}{|\lambda_k|}$$

for all $k \in \mathbb{N}$. The last term is bounded for all $k \in \mathbb{N}$ and so the result follows. □

To be able to apply the remaining existence results of Section 3.3.2 a further space $\overline{\overline{Z}}$ must be constructed such that $\overline{Z} \subset \overline{\overline{Z}}$ and the canonical injection is continuous with dense range. This is hypothesis PS XIII. This construction follows that of Section 4.1.1. Let

$$\overline{\overline{Z}} = Z_1 \times H^{\alpha_4^2}(\Omega;\mathbb{R})$$

where Z_1 is defined in Remark 4.1.6. The following condition ensures that PS XIII is satisfied.

Condition B. $\alpha_4^2 \prec \alpha_2^2$ (with bound R_9).

The final hypothesis, PS XIV, is similar to PS XI. The condition imposed on the general example reflects this.

Condition C. There exists a constant R_{10} such that

$$\frac{\alpha_{2,k}^2}{\alpha_{4,k}^2 |\lambda_k|} \leq R_{10} \tag{4.36}$$

for all $k \in \mathbb{N}$.

Proposition 4.3.3. *Suppose that the general example satisfies Condition C. Then $S(t)$ is a strongly continuous semigroup on $\overline{\overline{Z}}$. Furthermore there exists a continuous function $k_3(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that $k_3(0) = 0$ and for all $s \in [0,T]$, $\bar{z}(\cdot) \in C(0,T;\overline{\overline{Z}})$, $t \in [s,T]$ we have $\int_s^t S(t-\sigma)\bar{z}(\sigma)\, d\sigma \in \overline{Z}$ with*

$$\left\| \int_s^t S(t-\sigma)\bar{z}(\sigma)\, d\sigma \right\|_{\overline{Z}} \leq k_3(t-s) \|\bar{z}\|_{C(0,T;\overline{\overline{Z}})}.$$

Proof. Note that, for all $k \in \mathbb{N}$,

$$\frac{\alpha_{4,k}^2}{\alpha_{1,k}^1|\lambda_k|} \leq R_9 \frac{\alpha_{2,k}^2}{\alpha_{1,k}^1|\lambda_k|} \leq R_1 R_2 R_5 R_9.$$

Therefore, as in Corollary 4.2.1, that $S(t)$ is a strongly continuous semigroup follows immediately from the proof of Proposition 4.1.7 and this bound. The remainder of the proof of the proposition follows the method of the proof of Proposition 4.3.1 and Condition C. □

Therefore the general example satisfies PS XIV and the existence results of Section 3.3.2 can be applied.

Chapter 5

Controlling the Spread of Rabies

In this chapter the problem of controlling the spread of rabies will again be considered. In Section 1.1.4 the control problem was defined as containing and eradicating an outbreak of rabies in a region where it is not endemic. By introducing and extending an existing mathematical model (Sections 1.2 and 1.3) for the spread of rabies in foxes—the principle reservoir of infection—this problem was reformulated as a mathematical one (Section 1.4). The mathematical control problem associated with the rabies model is to choose, if possible, a control u which corresponds to a level of vaccination or cull, such that the total infected population density is driven to a specified density y_d. The specific choice of y_d will be considered later in Section 5.3.

The main difference between this problem and many others appearing in engineering is that the control acts only via the initial state. Therefore a new approach to this problem has been developed in this thesis. Since the original dynamics involve coupled partial differential equations that are nonlinear the approach followed has been based on time-varying infinite-dimensional state-space systems with a possibly unbounded nonlinearity (Chapter 2).

Observing that the original dynamics, when formulated as an abstract differential equation on a Banach space Z, are semilinear in form motivated the development of a theory for time-varying systems where the linear operator is a time-varying perturbation of a time-invariant (unbounded) linear operator (Chapter 3). It was shown in Section 3.3 that, under suitable conditions, semilinear systems with an unbounded nonlinearity possess a solution in a classical sense.

The primary concern of this chapter is to show that the theory developed in Chapters 2 and 3 can be applied to the problem of controlling the spread of rabies. The model is firstly considered as a semilinear system and the existence results of Section 3.3 applied. This then motivates the application of the general theory of Chapter 3 to the rabies model.

The methods of this chapter follow those of the previous one that dealt with a general example of a similar form to the rabies model. For the rabies model it was not possible to deal solely with the Hilbert space structure used for the general example. However, by replacing certain $H^\alpha(\Omega;\mathbb{R})$ spaces with the Banach space of continuous functions the results of the previous chapter can be extended to the rabies model. Thus many of the results in this chapter are corollaries of the corresponding ones for the general example.

The chapter concludes with a specific example of the rabies problem. A target density y_d is chosen that is a test of whether the rabies has been eradicated or not. As a result of the discussion in Section 1.3.2 this density is nonzero and care must be taken over the particular choice made. Numerical results are given for this example for the three basic strategies of population reduction: vaccination only, culling only, and a combination of both.

5.1 Rabies model as a semilinear system

In this section the model proposed in Chapter 1 for the dynamics of the spread of rabies in a fox population will be considered as an abstract semilinear system. For simplicity the spatial domain Ω will be a closed and bounded interval in $\mathbb{R}$; let $L \in \mathbb{R}$ and set $\Omega = [0, L]$. Recall from Section 1.4, that to formulate the rabies model as an abstract differential equation we set $\overline{s}(t) = S(t,\cdot)$, $\overline{v}(t) = V(t,\cdot)$, $\overline{q}(t) = I(t,\cdot)$, and $\overline{r}(t) = R(t,\cdot)$, where, for example, $S(t,\cdot) = \{S(t,x) : x \in [0, L]\}$. To apply the results of Section 3.3 the dynamics (1.28) will be written as the semilinear system given by

$$\dot{\overline{z}}(t) = A\overline{z}(t) + g(\overline{z}(t)), \qquad \overline{z}(0) = \overline{z}_0 + B\overline{u}, \tag{5.1}$$

where $\overline{z}(\cdot) = \begin{pmatrix}\overline{s}(\cdot) & \overline{v}(\cdot) & \overline{q}(\cdot) & \overline{r}(\cdot)\end{pmatrix}^{\mathrm{T}}$.

It will shown that, under suitable conditions, (5.1) has a well-defined mild solution. This mild solution will be then shown to be classical if further conditions hold.

5.1.1 Model formulation

To write the abstract rabies model (1.28) as the semilinear one (5.1) let

$$A = \left(\begin{array}{ccc|c} (a-b) & a & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & -(b+\sigma) & 0 \\ \hline 0 & 0 & \sigma & -(b+\alpha)+DA_0 \end{array}\right) \tag{5.2}$$

where

$$A_0 h = \frac{d^2 h}{dx^2} \tag{5.3}$$

and

$$g(\overline{z}) = \begin{pmatrix} -\mu\left((\overline{z}_1)^2 + \overline{z}_1\overline{z}_2 + \overline{z}_1\overline{z}_3 + \overline{z}_1\overline{z}_4\right) - \beta\overline{z}_1\overline{z}_4 \\ -\mu\left(\overline{z}_1\overline{z}_2 + (\overline{z}_2)^2 + \overline{z}_2\overline{z}_3 + \overline{z}_2\overline{z}_4\right) \\ -\mu\left(\overline{z}_1\overline{z}_3 + \overline{z}_2\overline{z}_3 + (\overline{z}_3)^2 + \overline{z}_3\overline{z}_4\right) + \beta\overline{z}_1\overline{z}_4 \\ -\mu\left(\overline{z}_1\overline{z}_4 + \overline{z}_2\overline{z}_4 + \overline{z}_3\overline{z}_4 + (\overline{z}_4)^2\right) \end{pmatrix} \tag{5.4}$$

where $\mu = (a-b)/K$. The input operator B will take any one of the following three forms:

$$B_v = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_c = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad B_{vc} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.5}$$

corresponding to control via vaccination (B_v), culling (B_c) or a combination of both (B_{vc}). The semilinear equation will be considered on a triple of Banach spaces $\underline{Z} \subset Z \subset \overline{Z}$, such that

$$\underline{Z} = Z_1 \times \underline{Z}_2, \quad Z = Z_1 \times Z_2, \quad \overline{Z} = Z_1 \times C(0,L)$$

where $Z_1 = C(0,L) \times C(0,L) \times C(0,L)$ and $\underline{Z}_2 \subset Z_2 (\subset C(0,L))$ will be suitably chosen Hilbert spaces $H^{\alpha_i}(0,L)$ defined in Section 4.1.1. The natural space in which to study the rabies model is the continuous functions, namely $\overline{Z}$, but the other spaces are introduced to facilitate the estimation of the norms appearing in the assumptions of Chapter 3.

The orthonormal basis chosen for $L^2(0, L; \mathbb{R})$ is

$$\phi_0(x) = \frac{1}{\sqrt{L}}, \qquad \phi_n(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) \quad \text{for all } n \in \mathbb{N}$$

and the spaces $H^\alpha(0, L)$ defined in Section 4.1.1 are constructed in terms of $\{\phi_n\}$. Note that, since the orthonormal basis is defined for $n \geq 0$, the definitions of the previous chapter must be extended. Thus if $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ then we extend the definition of $||\cdot||_\alpha$ to the following

$$||h||_\alpha^2 = \langle z, \phi_0\rangle_{L^2(0,L)}^2 + \sum_{n=1}^{\infty} \alpha_n \langle z, \phi_n\rangle_{L^2(0,L)}^2 = \sum_{n=0}^{\infty} \alpha_n \langle z, \phi_n\rangle_{L^2(0,L)}^2$$

where α_0 is set equal to 1. For any $\alpha \in \mathbb{R}$ we define (α) to be the sequence $(n^{2\alpha})_{n\in\mathbb{N}}$. With a slight abuse of notation the corresponding Hilbert space $H^{(\alpha)}(\Omega; \mathbb{R})$ will be denoted by $H^\alpha(0, L)$ with norm $||\cdot||_\alpha$.

Remark 5.1.1. Recall that $(\alpha_n)_{n\in\mathbb{N}} \prec (\beta_n)_{n\in\mathbb{N}}$ if and only if the sequence $(\alpha_n/\beta_n)_{n\in\mathbb{N}}$ is bounded. Therefore

$$(\alpha) \prec (\beta) \iff \left(n^{2(\alpha-\beta)}\right)_{n\in N} \quad \text{bounded}$$
$$\iff \alpha \leq \beta$$

for $\alpha, \beta \in \mathbb{R}$.

The following result shows that, with a suitable condition imposed, the rabies model satisfies assumption PS I.

Proposition 5.1.2. *Suppose that*

$$\boxed{\tfrac{1}{2} < \alpha_1 \leq \alpha_0} \tag{5.6}$$

$\underline{Z}_2 = H^{\alpha_0}(0, L)$, *and* $Z_2 = H^{\alpha_1}(0, L)$. *Then* $\underline{Z}_2 \subset Z_2 \subset C(0, L)$, *where the canonical injections are continuous with dense ranges.*

Proof. As a result of the previous remark it is clear that (5.6) implies that the result holds for $\underline{Z}_2 \subset Z_2$. We now show that $Z_2 \subset C(0, L)$, and in particular that there exists a constant m_1 such that

$$\|h\|_{C(0,L)} \leq m_1 \|h\|_{\alpha_1}.$$

Let $h \in H^{\alpha_1}(0, L)$. Then

$$\begin{aligned}\left(\sum_{n=0}^{\infty} |\langle h, \phi_n \rangle|\right)^2 &= \left(\sum_{n=0}^{\infty} \alpha_{1,n}^{-1/2} \alpha_{1,n}^{1/2} |\langle h, \phi_n \rangle|\right)^2 \\ &\leq \left(\sum_{n=0}^{\infty} \alpha_{1,n}^{-1}\right)\left(\sum_{n=0}^{\infty} \alpha_{1,n} \langle h, \phi_n \rangle^2\right) \\ &= \left(1 + \sum_{n=1}^{\infty} n^{-2\alpha_1}\right) \|h\|_{\alpha_1}^2\end{aligned}$$

which is finite by (5.6). Therefore

$$\begin{aligned}\|h\|_{C(0,L)} &= \sup_{x \in [0,L]} |h(x)| \\ &= \sup_{x \in [0,L]} \left|\sum_{n=0}^{\infty} \langle h, \phi_n \rangle \phi_n(x)\right| \\ &\leq \sqrt{\frac{2}{L}} \left(1 + \sum_{n=1}^{\infty} n^{-2\alpha_1}\right)^{1/2} \|h\|_{\alpha_1}\end{aligned}$$

and so the result follows with $m_1 = \sqrt{\frac{2}{L}} \left(1 + \sum_{n=1}^{\infty} n^{-2\alpha_1}\right)^{1/2}$. □

Remark 5.1.3. We have the following estimates

$$\|h\|_{\alpha_1} \leq \|h\|_{\alpha_0} \qquad \forall h \in H^{\alpha_0}(0, L)$$

$$\|h\|_{C(0,L)} \leq m_1 \|h\|_{\alpha_1} \qquad \forall h \in H^{\alpha_1}(0, L)$$

and inequalities

$$\|z\|_Z \leq \|z\|_{\underline{Z}} \qquad z \in \underline{Z}$$

$$\|z\|_{\overline{Z}} \leq \max\{1, m_1\} \|z\|_Z \qquad z \in Z.$$

Hence, in the notation of Chapter 4, $\overline{R}_1 = 1$ and $\overline{R}_2 = \max\{1, m_1\}$.

For the nonlinearity, g_i maps continuous functions into $C(0,L)$ and so it is clear that $g : \underline{Z} \longrightarrow \overline{Z}$. The conditions of the previous proposition provide estimates that will be helpful later in showing that $g : \underline{Z} \longrightarrow \overline{Z}$ satisfies the Lipschitz condition of Theorem 2.2.10.

Remark 5.1.4. Let $z_1, z_2 \in C(0,L)$ and $z_3, z_4 \in H^{\alpha_0}(0,L)$, then

$$\begin{aligned}
||z_1 z_2||_{C(0,L)} &\leq ||z_1||_{C(0,L)} ||z_2||_{C(0,L)} \\
||z_1 z_3||_{C(0,L)} &\leq ||z_1||_{C(0,L)} ||z_3||_{C(0,L)} \leq m_1 ||z_1||_{C(0,L)} ||z_3||_{\alpha_0} \\
||z_3 z_4||_{C(0,L)} &\leq m_1^2 ||z_3||_{\alpha_0} ||z_4||_{\alpha_0}.
\end{aligned}$$

Recall that, writing

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix},$$

A_1 is the generator of a strongly continuous semigroup $e^{A_1 t}$ on Z_1. For the rabies model:

$$e^{A_1 t} = \begin{pmatrix} e^{(a-b)t} & e^{(a-b)t}(1 - e^{-at}) & 0 \\ 0 & e^{-bt} & 0 \\ 0 & 0 & e^{-(\sigma+b)t} \end{pmatrix}. \tag{5.7}$$

Note that, for $n \geq 1$, we have

$$\begin{aligned}
A_3 \phi_n &= -(\alpha + b)\phi_n + D\frac{d^2\phi_n}{dx^2} \\
&= -(\alpha + b)\phi_n - D\frac{n^2\pi^2}{L^2}\phi_n
\end{aligned}$$

and so the eigenvalues λ_n are given by

$$\lambda_n = -(\alpha + b + D\frac{n^2\pi^2}{L^2})$$

with $\lambda_0 = -(\alpha + b)$. Hence (see Section 4.1.2) A_3, with domain

$$D(A_3) = \left\{ h \in H^{\alpha_1}(0,L) : \sum_{n=1}^{\infty} n^{2\alpha_1} \lambda_n^2 \langle h, \phi_n \rangle_{L^2(0,L)}^2 < \infty \right\}, \tag{5.8}$$

is the infinitesimal generator of the strongly continuous semigroup

$$S_3(t)h = \sum_{n=0}^{\infty} e^{\lambda_n t} \langle h, \phi_n \rangle_{L^2(0,L)}^2 \phi_n \tag{5.9}$$

for $h \in H^{\alpha_1}(0,L)$.

Proposition 5.1.5. *Suppose that*

$$\boxed{\alpha_1 \leq 1} \tag{5.10}$$

with A as defined above. Then A is the generator of a strongly continuous semigroup $S(t)$ on Z where

$$S(t) = \begin{pmatrix} e^{A_1 t} & 0 \\ S_2(t) & S_3(t) \end{pmatrix}$$

and

$$S_2(t)z = \int_0^t S_3(t-s)A_2 e^{A_1 s} z \, ds.$$

Proof. A modified version of the proof of Proposition 4.1.7 (on page 112) is used here. In particular it is seen directly from this proof that $S(t)$ is the unique solution of the equation

$$S(t)z = T(t)z + \int_0^t T(t-s)\mathcal{A}S(s)z \, ds$$

where $T(t)$ and $\mathcal{A}$ are as defined in the proof of Proposition 4.1.7.

For the rabies model, if $z \in Z_1$,

$$\begin{aligned}
\|S_2(t)z\|_{\alpha_1}^2 &= \|\sigma \int_0^t S_3(t-s)e^{-(\sigma+b)s} z_3 ds\|_{\alpha_1}^2 \\
&\leq \sigma^2 \sum_{n=0}^{\infty} \alpha_{1,n} \left(\int_0^t e^{2\lambda_n s} ds\right) \left(\int_0^t e^{-(\sigma+b)s} \langle z_3, \phi_n \rangle_{L^2}^2 ds\right) \\
&\leq \frac{-\sigma^2}{2(\sigma+b)} \left(e^{-2(\sigma+b)t} - 1\right) \sum_{n=0}^{\infty} \frac{\alpha_{1,n}}{2|\lambda_n|} \langle z_3, \phi_n \rangle_{L^2}^2 .
\end{aligned}$$

Note that, for $n \geq 1$,

$$\frac{\alpha_{1,n}}{2|\lambda_n|} = \frac{n^{2\alpha_1}}{2|\lambda_n|} = \frac{L^2 n^{2\alpha_1}}{2(\alpha+b)L^2 + 2Dn^2\pi^2} \leq \frac{L^2}{2D\pi^2} n^{2(\alpha_1 - 1)} \leq \frac{L^2}{2D\pi^2},$$

and so

$$\begin{aligned}
\|S_2(t)z\|_{\alpha_1}^2 &\leq m_2 \left(1 - e^{-2(\sigma+b)t}\right) \|z_3\|_{L^2(0,L)}^2 \\
&\leq L m_2 \left(1 - e^{-2(\sigma+b)t}\right) \|z_3\|_{C(0,L)}^2
\end{aligned}$$

where

$$m_2 = \frac{\sigma^2}{4(\sigma+b)} \max\left\{\frac{1}{(\alpha+b)}, \frac{L^2}{D\pi^2}\right\}.$$

Therefore

$$\begin{aligned}
\|S(t)z\|_Z^2 &= \|e^{A_1 t}z^1\|_{Z_1}^2 + \|S_2(t)z^1 + S_3(t)z_4\|_{\alpha_1}^2 \\
&\le \|e^{A_1 t}\|^2\|z^1\|_{Z_1}^2 + 2L\,m_2\left(1-e^{-2(\sigma+b)t}\right)\|z_3\|_{C(0,L)}^2 + 2\|S_3(t)\|^2\|z_4\|_{\alpha_1}^2 \\
&\le \max\left\{\|e^{A_1 t}\|^2 + 2L\,m_2\left(1-e^{-2(\sigma+b)t}\right), 2\|S_3(t)\|^2\right\}\|z\|_Z^2.
\end{aligned}$$

This shows that $S(t) \in \mathcal{L}(Z)$. To see that $S(t)$ is strongly continuous let $h > 0$ and consider

$$\begin{aligned}
\|S(h)z - z\|_Z^2 &\le \|e^{A_1 h}z^1 - z^1\|_{Z_1}^2 + 2\|S_3(h)z_4 - z_4\|_{\alpha_1}^2 + 2\|S_2(h)z^1\|_{\alpha_1}^2 \\
&\le \|e^{A_1 h}z^1 - z^1\|_{Z_1}^2 + 2\|S_3(h)z_4 - z_4\|_{\alpha_1}^2 + 2Lm_2\left(1-e^{-2(\sigma+b)h}\right)\|z_3\|_{C(0,L)}^2 \\
&\longrightarrow 0
\end{aligned}$$

as $h \to 0$. That $S(t+s) = S(t)S(s)$ follows directly from the proof of Proposition 4.1.7.

Finally we show that A is the infinitesimal generator of $S(t)$. First recall that (5.6) implies $H^{\alpha_1}(0,L) \subset C(0,L)$. Applying the method of Proposition 4.1.7 it only remains to show that, for $h > 0$,

$$\|\frac{1}{h}S_2(t)z^1 - A_2 z^1\|_{\alpha_1} \longrightarrow 0$$

as $h \to 0$. For the rabies model this expression is

$$\|\frac{1}{h}\int_0^h S_3(h-s)A_2 e^{A_1 s}z^1 ds - A_2 z^1\|_{\alpha_1} = \|\frac{\sigma}{h}\int_0^h S_3(h-s)e^{-(\sigma+b)s}z_3 ds - \sigma z_3\|_{\alpha_1}.$$

Clearly $\sigma z_3 \in H^{\alpha_1}(0,L)$ if only if $z_3 \in H^{\alpha_1}(0,L)$, in which case

$$\begin{aligned}
\|\frac{\sigma}{h}\int_0^h S_3(h-s)e^{-(\sigma+b)s}z_3 ds - \sigma z_3\|_{\alpha_1} &\le \sigma\|\frac{1}{h}\int_0^h S_3(h-s)\left(e^{-(\sigma+b)s}z_3 - z_3\right) ds\|_{\alpha_1} \\
&\quad + \sigma\|\frac{1}{h}\int_0^h S_3(s)z_3 ds - z_3\|_{\alpha_1} \\
&\le \frac{\sigma}{h}\int_0^h \|S_3(h-s)\|\,\|e^{-(\sigma+b)s}z_3 - z_3\|\,ds \\
&\quad + \sigma\|\frac{1}{h}\int_0^h S_3(s)z_3 ds - z_3\|_{\alpha_1}.
\end{aligned}$$

The last term converges to zero since $S_3(t)$ is a strongly continuous semigroup on $H^{\alpha_1}(0,L)$. For the first term, note that for any $\epsilon > 0$ there exists a $h > 0$ such that

$$\|e^{-(\sigma+b)s} z_3 - z_3\| < \epsilon \qquad \text{for } s \in [0,h].$$

Thus

$$\begin{aligned}\frac{\sigma}{h}\int_0^h \|S_3(h-s)\|\,\|e^{-(\sigma+b)s} z_3 - z_3\|\,ds &\leq \frac{\sigma}{h}\int_0^h \|S_3(h-s)\|\epsilon\,ds \\ &\leq (\text{const})\epsilon.\end{aligned}$$

Therefore A is the generator of $S(t)$ and has domain

$$D(A) = C(0,L) \times C(0,L) \times H^{\alpha_1}(0,L) \times D(A_3).$$

□

We can explicitly calculate $S_2(t)$ as follows :

$$\begin{aligned}S_2(t)h &= \int_0^t S_3(t-s)A_2 e^{A_1 s} h\,ds \\ &= \sigma \sum_{n=0}^{\infty} \frac{L^2}{Dn^2\pi^2 + L^2(\alpha-\sigma)}\left(e^{-(\sigma+b)t} - e^{-(\frac{D\pi^2 n^2}{L^2}+\alpha+b)t}\right)\langle h_3, \phi_n\rangle_{L^2(0,L)}\phi_n \qquad (5.11)\end{aligned}$$

for $h \in Z_1$.

Corollary 5.1.6. *Suppose that the following condition is satisfied*

$$\boxed{\alpha_0 \leq 1} \qquad (5.12)$$

Then the semigroup $S(t)$, with generator A, is a strongly continuous semigroup on all three spaces $\underline{Z} \subset Z \subset \overline{Z}$.

Proof. Note that $\frac{1}{2} < \alpha_1 \leq \alpha_0$ and use the proof of Proposition 5.1.5. In particular the following estimate is obtained:

$$\|S_2(t)z\|_{\alpha_0} \leq M(t)\|z_3\|_{C(0,L)} \qquad (5.13)$$

for $z \in Z_1$, where $M(t)^2 = Lm_2\left(1 - e^{-2(\sigma+b)t}\right)$. The domain of A when considered as the generator of $S(t)$ on $\underline{Z}$ is given by

$$\underline{D}(A) = C(0,L) \times C(0,L) \times H^{\alpha_0}(0,L) \times \underline{D}(A_3)$$

where $\underline{D}(A_3)$ is the domain of A_3 considered as the generator of $S_3(t)$ on $H^{\alpha_0}(0,L)$.

Now it only remains to show that $S(t)$ is a strongly continuous semigroup on $\overline{Z}$. Note that $A_2 \in \mathcal{L}(Z_1, C(0,L))$ and so it is well-known (Curtain and Zwart, 1995,for example) that A is the generator of $S(t)$. Alternatively, the method of the proof of Proposition 5.1.5 can be used to show that $S(t)$ is a strongly continuous semigroup on $\overline{Z}$ with

$$||S_2(t)z||_{C(0,L)} \leq m_1 M(t) ||z_3||_{C(0,L)} \tag{5.14}$$

and the domain of A with respect to $\overline{Z}$ is

$$\overline{D}(A) = Z_1 \times \overline{D}(A_3)$$

where $\overline{D}(A_3) = \{h \in C(0,L) : h \in C^2(0,L)\}$. □

In the next subsection it will be shown that the rabies model, for a suitable initial guess for the control u' and suitable conditions on the spaces, has a solution in the sense of Definition 3.3.6.

5.1.2 Existence of a solution

To prove that there exists a solution to the rabies model (5.1) Theorem 3.3.11 will be applied. It is necessary to show that the rabies model satisfies conditions PS XI, PS XII, PS XIII, and PS XIV.

Proposition 5.1.7. *Under the conditions imposed so far for the rabies model, there exists a continuous function $k_1'(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that for all $s \in [0,T]$, $h(\cdot) \in L^2(0,T;\overline{Z})$, and $t \in [s,T]$, we have $\int_s^t S(t-\sigma)h(\sigma)\,d\sigma \in \underline{Z}$, with*

$$||\int_s^t S(t-\sigma)h(\sigma)\,d\sigma||_{\underline{Z}} \leq k_1'(t-s)||h||_{L^2(s,t;\overline{Z})}.$$

Proof. Note that

$$\begin{aligned}
\| \int_s^t S_3(t-\sigma)h_4(\sigma)d\sigma\|_{\alpha_0}^2 &= \sum_{n=0}^{\infty} \alpha_{0,n} \left(\int_s^t e^{\lambda_n(t-\sigma)} \langle h_4(\sigma), \phi_n \rangle_{L^2} d\sigma \right)^2 \\
&\leq \sum_{n=0}^{\infty} \alpha_{0,n} \left(\int_0^{t-s} e^{2\lambda_n \sigma} d\sigma \right) \left(\int_s^t \langle h_4(\sigma), \phi_n \rangle_{L^2}^2 d\sigma \right) \\
&\leq \frac{1}{2(\alpha+b)} \left(\int_s^t \langle h_4(\sigma), \phi_0 \rangle_{L^2}^2 d\sigma \right) \\
&\quad + \frac{L^2}{2D\pi^2} \sum_{n=1}^{\infty} n^{2(\alpha_0-1)} \left(\int_s^t \langle h_4(\sigma), \phi_n \rangle_{L^2}^2 d\sigma \right).
\end{aligned}$$

Condition (5.10) implies that

$$\sum_{n=1}^{\infty} n^{2(\alpha_0-1)} \langle h_4(\sigma), \phi_n \rangle_{L^2}^2 \leq \sum_{n=1}^{\infty} \langle h_4(\sigma), \phi_n \rangle_{L^2}^2.$$

Thus

$$\| \int_s^t S_3(t-\sigma)h_4(\sigma)d\sigma\|_{\alpha_0}^2 \leq \frac{L}{2} \max\left\{ \frac{1}{(\alpha+b)}, \frac{L^2}{D\pi^2} \right\} \int_s^t \|h_4(\sigma)\|_{C(0,L)}^2 d\sigma.$$

Therefore

$$\begin{aligned}
&\| \int_s^t S(t-\sigma)h(\sigma)\, d\sigma\|_{\underline{Z}}^2 \\
&\leq \| \int_s^t e^{A_1(t-\sigma)} h^1(\sigma)\, d\sigma\|_{Z_1}^2 + 2\| \int_s^t S_2(t-\sigma)h^1(\sigma)d\sigma\|_{\alpha_0}^2 + 2\| \int_s^t S_3(t-\sigma)h_4(\sigma)d\sigma\|_{\alpha_0}^2 \\
&\leq \int_0^{t-s} \|e^{A_1\sigma}\|^2 d\sigma \int_s^t \|h^1(\sigma)\|_{Z_1}^2 d\sigma + 2\int_0^{t-s} M(\sigma)^2 d\sigma \int_s^t \|h_3(\sigma)\|_{C(0,L)}^2 d\sigma \\
&\quad + L \max\left\{ \frac{1}{(\alpha+b)}, \frac{L^2}{D\pi^2} \right\} \int_s^t \|h_4(\sigma)\|_{C(0,L)}^2 d\sigma \\
&\leq \left(\int_0^{t-s} \|e^{A_1\sigma}\|^2 d\sigma + 2\int_0^{t-s} M(\sigma)^2 d\sigma \right) \int_s^t \|h^1(\sigma)\|_{Z_1}^2 d\sigma \\
&\quad + L \max\left\{ \frac{1}{(\alpha+b)}, \frac{L^2}{D\pi^2} \right\} \int_s^t \|h_4(\sigma)\|_{C(0,L)}^2 d\sigma.
\end{aligned}$$

Setting

$$k_1'(t)^2 = \max\left\{ \int_0^t \|e^{A_1\sigma}\|^2 d\sigma + 2\int_0^t M(\sigma)^2 d\sigma, L \max\left\{ \frac{1}{(\alpha+b)}, \frac{L^2}{D\pi^2} \right\} \right\}$$

completes the proof. □

A particular case of this proposition shows that the rabies model satisfies PS XI.

Corollary 5.1.8. *Under the conditions of the proposition there exists a continuous function $k_1(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that $k_1(0) = 0$, and for all $s \in [0,T]$, $\overline{z}(\cdot) \in C(s,T;\overline{Z})$, $t \in [s,T]$, we have $\int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \in \underline{Z}$, with*

$$\| \int_s^t S(t-\sigma)\overline{z}(\sigma)\,d\sigma \|_{\underline{Z}} \leq k_1(t-s)\|\overline{z}\|_{C(s,t;\overline{Z})}.$$

Proof. This follows directly from the previous proposition setting $k_1(t) = \sqrt{t}\,k_1'(t)$. □

Remark 5.1.9. PS XII follows as a natural consequence of the construction of the spaces $\underline{Z}$, Z, $\overline{Z}$ and Corollary 5.1.6. Recall that $\overline{D}(A) = Z_1 \times \overline{D}(A_3)$ and $\underline{Z} = Z_1 \times H^{\alpha_0}(0,L)$. To see that $\overline{D}(A_3) \subset H^{\alpha_0}(0,L)$ consider $h \in \overline{D}(A_3)$. We show that $h \in H^{\alpha_0}(0,L)$. Since $h \in \overline{D}(A_3)$ we know that $A_3 h \in C(0,L) \subset L^2(0,L)$ and so

$$\|A_3 h\|^2_{L^2(0,L)} = \sum_{n=0}^{\infty} \langle A_3 h, \phi_n\rangle^2_{L^2(0,L)} = \sum_{n=0}^{\infty} \lambda_n^2 \langle h, \phi_n\rangle^2_{L^2(0,L)} < \infty.$$

Now

$$\|h\|^2_{\alpha_0} = \sum_{n=0}^{\infty} \alpha_{0,n} \langle h, \phi_n\rangle^2_{L^2(0,L)} \leq \sum_{n=0}^{\infty} \lambda_n^2 \langle h, \phi_n\rangle^2_{L^2(0,L)} < \infty$$

since the sequence $(n^{2\alpha_0}/\lambda_n^2)_{n\in\mathbb{N}}$ is bounded by (5.10).

Let $\overline{\overline{Z}} = \overline{Z}$. Then assumption PS XIV is automatically satisfied. Hence the rabies model, with the relations introduced in this section, satisfies PS XI, PS XII, PS XIII, and PS XIV. It must now be shown that the remainder of the hypotheses of Theorem 3.3.11 hold for the rabies model.

First note that $g : \underline{Z} \longrightarrow \overline{Z}$ is continuously differentiable with

$$Dg(w) = \begin{pmatrix} -\mu(w_1 + \sum w_i) - \beta w_4 & -\mu w_1 & -\mu w_1 & -(\mu+\beta) w_1 \\ -\mu w_2 & -\mu(w_2 + \sum w_i) & -\mu w_2 & -\mu w_2 \\ -\mu w_3 + \beta w_1 & -\mu w_3 & -\mu(w_3 + \sum w_i) & -\mu w_3 + \beta w_1 \\ -\mu w_4 & -\mu w_4 & -\mu w_4 & -\mu(w_4 + \sum w_i) \end{pmatrix}$$

where $\sum w_i = \sum_{i=1}^4 w_i$. In particular g is locally Lipschitz continuous in $\overline{z}$. For $z(\cdot) \in C(0,T;\underline{Z})$ define the bounded linear operator $B(t,z) : \underline{Z} \longrightarrow \overline{Z}$ by $B(t,z)w = Dg(z(t))w$ for $w \in \underline{Z}$.

Proposition 5.1.10. *For each $z(\cdot) \in C(0,T;\underline{Z})$ there exists a constant $k_5(z)$ such that*

$$\|B(t,z)w\|_{\underline{Z}} \leq k_5(z)\|w\|_{\underline{Z}}$$

for all $w \in \underline{Z}$.

Proof. Let $z(\cdot) \in C(0,T;\underline{Z})$. For simplicity we write

$$B(t,z) = \begin{pmatrix} p_{11}(t) & \dots & p_{14}(t) \\ \vdots & & \vdots \\ p_{41}(t) & \dots & p_{44}(t) \end{pmatrix}.$$

Since α_0 is chosen such that $H^{\alpha_0}(0,L) \subset C(0,L)$ it is clear that $p_{ij}(\cdot) \in C(0,T;C(0,L))$. Consider

$$\begin{aligned}
\|B(t,z)w\|_{\underline{Z}}^2 &= \sum_{i=1}^4 \| (B(t,z)w)_i \|_{C(0,L)}^2 \\
&= \sum_{i=1}^4 \| \sum_{j=1}^4 p_{ij}(t)w_j\|_{C(0,L)}^2 \\
&\leq 4\sum_{i=1}^4 \sum_{j=1}^4 \|p_{ij}(t)\|_{C(0,L)}^2 \|w_j\|_{C(0,L)}^2 \\
&\leq \left(4\sum_{i=1}^4 \max_j \|p_{ij}(t)\|_{C(0,L)}^2\right) \|w\|_{\underline{Z}}^2.
\end{aligned}$$

Note that p_{ij} depends on z and by the continuity of the former we can take the supremum—with respect to t—to obtain the constant $k_5(z)$. □

So far in this section it has been shown that the rabies model, subject to certain constraints, satisfies the hypotheses of Theorem 3.3.11. Therefore for every initial state $\overline{z}(0) = \overline{z}_0 + B\overline{u} \in \overline{D}(A)$ there exists a solution of (5.1).

For the rabies model it will be assumed that the fox population is at equilibrium at the steady state value K when an outbreak of rabies occurs. Therefore the given initial state $\overline{z}_0$ will be of the form

$$\overline{z}_0 = \begin{pmatrix} \overline{s}(0) \\ 0 \\ 0 \\ \overline{r}(0) \end{pmatrix}$$

where $\overline{s}(0)(x) = K$ for all $x \in [0, L]$. The initial distribution of rabid foxes is $\overline{r}(0)$ and this must lie in the domain, $\overline{D}(A_3)$, of A_3 considered as an operator on $\overline{Z}$. Note that, if $\overline{u}$ is chosen to be a uniform distribution ($\overline{u}(x) = u$), then for $n \geq 1$,

$$\langle \overline{u}, \phi_n \rangle_{L^2(0,L)} = \int_0^L u\sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) dx = \frac{u\sqrt{2L}}{n\pi} \left[\sin\left(\frac{n\pi x}{L}\right)\right]_0^L = 0.$$

Therefore

$$||\overline{u}||_{\beta_1} = u\sqrt{L}$$

for some β_1. Moreover, if $H^{\beta_1}(0, L)$ is chosen such that $H^{\beta_1}(0, L) \subset C(0, L)$, then $B\overline{u} \in \overline{D}(A) = Z_1 \times \overline{D}(A_3)$, and so, provided $\overline{r}(0) \in C^2(0, L)$, there exists a solution corresponding to any uniform distribution for the initial control u'.

In the next section it will be shown that the rabies model, under suitable constraints, satisfies the hypotheses of Chapters 2 and 3. Hence the control problem for the rabies model will have a solution as defined in Chapter 2.

5.2 Applying general theory to rabies model

In this section it is shown that the rabies model can be considered as an example of a perturbed system as studied in Chapter 3. The first stage is to perform a local approximation about some initial control u' and resulting solution $z'(\cdot)$ to obtain a perturbed system of the form:

$$\dot{z}(t) = (A + P(t))\, z(t) + N(z(t)), \qquad z(0) = Bu, \tag{5.15}$$

where $\overline{z} = z + z'$ and $\overline{u} = u' + u$. Performing this approximation for the rabies model one obtains a system of the form (5.15) with A and B (B_v, B_c or B_{vc}) as given in the previous section;

$P(t) = Dg(z'(t))$, that is $P(t)$ is given by

$$\begin{pmatrix} -\mu(s'(t)+n'(t))-\beta r'(t) & -\mu s'(t) & -\mu s'(t) & -(\mu+\beta)s'(t) \\ -\mu v'(t) & -\mu(v'(t)+n'(t)) & -\mu v'(t) & -\mu v'(t) \\ -\mu q'(t)+\beta s'(t) & -\mu q'(t) & -\mu(q'(t)+n'(t)) & -\mu q'(t)+\beta s'(t) \\ -\mu r'(t) & -\mu r'(t) & -\mu r'(t) & -\mu(r'(t)+n'(t)) \end{pmatrix} \tag{5.16}$$

where $\mu = (a-b)/K$, $n'(t) = s'(t)+v'(t)+q'(t)+r'(t)$, and $z'(t) = \begin{pmatrix} s'(t) & v'(t) & q'(t) & r'(t) \end{pmatrix}^T$; and $N(z(t)) = g(z(t))$.

Since the aim of the control problem is to reduce the total population density of all infected foxes throughout the spatial domain, (5.15) is considered with the following output

$$y = Cz(T) + Cz'(T) \tag{5.17}$$

for some chosen time $T > 0$, and where

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}. \tag{5.18}$$

In the following subsection the rabies model is formulated as a perturbed system. The results of Chapters 2 and 3 will then be applied.

5.2.1 Model formulation

For the rabies model the triple of spaces $\underline{Z} \subset Z \subset \overline{Z}$ was defined in the last section. It was also shown that these spaces satisfy PS I. The output of the rabies model will be considered in the Hilbert space $Y = H^{\beta_2}(0, L)$. Suppose that the following condition is satisfied:

$$\boxed{\beta_2 \leq 0} \tag{5.19}$$

This condition ensures that $C(0, L) \subset H^{\beta_2}(0, L)(= Y)$ and so $C \in \mathcal{L}(Z, Y)$. Recall from the last section that A is the generator of a strongly continuous semigroup that satisfies PS III.

Each of the forms that will be considered for B are scalar in the terminology of Chapter 4. In particular

$$B = B_v : \qquad B_1u = -u,\ B_2u = u,\ B_3u = B_4u = 0;$$

$$B = B_c : \qquad B_1u = -u,\ B_2u = B_3u = B_4u = 0;$$

$$B = B_{vc} : \qquad B_1u = -(u_1 + u_2),\ B_2u = u_1,\ B_3u = B_4u = 0.$$

Note that the case $B = B_{vc}$ corresponding to a combined vaccination and culling strategy is an example of the generalised situation of Remark 4.1.11. When $B = B_v$ or B_c let $U = H^{\beta_1}(0, L)$, and for $B = B_{vc}$ let $U = H^{\beta_1}(0, L) \times H^{\beta_1}(0, L)$. The following condition ensures that $B \in \mathcal{L}(U, Z)$ for each of the cases that will be considered later.

$$\boxed{\beta_1 > \tfrac{1}{2}} \tag{5.20}$$

Recall that this condition ensures that $H^{\beta_1}(0, L) \subset C(0, L)$.

For simplicity $P(t)$ will be expressed in the following form

$$P(t) = \begin{pmatrix} p_{11}(t) & \dots & p_{14}(t) \\ \vdots & & \vdots \\ p_{41}(t) & \dots & p_{44}(t) \end{pmatrix}$$

(recall from the previous section the description of $B(t, z)$). Note, from the last section and the fact that $P(t) = Dg(z'(t))$, that $p_{ij}(\cdot) \in C(0, T; C(0, L))$. The model formulation is completed by noting, in the notation and terminology of Chapter 3, that $D(t) = I_{\overline{Z}}$, $E(t) = I_{\underline{Z}}$ (the identity operators on $\overline{Z}$ and $\underline{Z}$ respectively). In the next subsection it will be shown that the rabies model, under suitable constraints, satisfies the assumptions of Sections 3.1 and 3.2.

5.2.2 Rabies model as a perturbed system

By the construction at the beginning of this Section $P(t) = Dg(z'(t))$, where $z'(\cdot) \in C(0, T; \underline{Z})$, and so it is clear that $P(\cdot) \in C(0, T; \mathcal{L}(\underline{Z}, \overline{Z}))$. The following result finishes the proof that the rabies model satisfies PS II.

Proposition 5.2.1. *Suppose that*

$$\boxed{\alpha_0 - \alpha_1 \leq 2} \tag{5.21}$$

Then the intersection of the domain of A (with respect to Z) and $\underline{Z}$ is a dense subspace of Z.

Proof. Recall that $D(A) = C(0,L) \times C(0,L) \times H^{\alpha_1}(0,L) \times D(A_3)$. Now

$$\frac{n^{2\alpha_0}}{n^{2\alpha_1}\lambda_n^2} \leq (\text{const})n^{2(\alpha_0-\alpha_1-2)}$$

which is bounded by (5.21). Hence $D(A) \cap \underline{Z} = C(0,L) \times C(0,L) \times H^{\alpha_1}(0,L) \times D(A_3)$. This proves the result. □

The next two results deal with the assumption PS IV.

Proposition 5.2.2. *Suppose that the conditions introduced so far for the rabies model are satisfied. Then there exists a continuous function $\overline{K}_1(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that $\overline{K}_1(0) = 0$, and for all $s \in [0,T)$, $z(\cdot) \in C(s,T;\underline{Z})$, $t \in [s,T]$,*

$$\| \int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma \|_{\underline{Z}} \leq \overline{K}_1(t-s)\|z\|_{C(s,t;\overline{Z})}.$$

Proof. Let $s \in [0,T)$ and $z(\cdot) \in C(s,T;\underline{Z})$. If we define $h(\sigma) = P(\sigma)z(\sigma)$ then $h(\cdot) \in C(s,T;\overline{Z})$. Corollary 5.1.8 then implies that there exists a continuous function $k_1(\cdot) : [0,T] \longrightarrow \mathbb{R}^+$ such that $k_1(0) = 0$ and

$$\| \int_s^t S(t-\sigma)h(\sigma)\,d\sigma \|_{\underline{Z}} \leq k_1(t-s)\|h\|_{C(s,t;\overline{Z})}.$$

Moreover, Proposition 5.1.10 implies that $P(t) \in \mathcal{L}(\overline{Z})$ and so

$$\| \int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma \|_{\underline{Z}} \leq k_1(t-s)k_3(z')\|z\|_{C(s,t;\overline{Z})}.$$

This completes the proof. □

The second result proves the continuity required in assumption PS IV.

Corollary 5.2.3. *For all $s \in [0,T)$ and $z(\cdot) \in C(s,T;\underline{Z})$ the map*

$$t \longmapsto \int_s^t S(t-\sigma)P(\sigma)z(\sigma)\,d\sigma$$

is continuous from $[s,T]$ to $\underline{Z}$.

Proof. This follows from the proof of Corollary 4.2.3 with the following modifications: Using the notation of Corollary 4.2.3

$$\begin{aligned}\left\|\int_{t'-h}^{t'} S_3(t'-h)A_2e^{A_1\sigma}z^1 d\sigma\right\|_{\alpha_0}^2 &\le \sigma^2\sum_{n=0}^{\infty}\alpha_{0,n}\left(\int_0^h e^{2\lambda_n\sigma}d\sigma\right)\left(\int_{t'-h}^{t'} e^{-2(\sigma+b)\sigma}d\sigma\right)\langle z_3,\phi_n\rangle_{L^2}^2\\ &\le \frac{\sigma^2}{2}\max\left\{\frac{1}{(\alpha+b)},\frac{L^2}{D\pi^2}\right\}\|z_3\|_{L^2(0,L)}^2\left(\int_{t'-h}^{t'} e^{-2(\sigma+b)\sigma}d\sigma\right)\end{aligned}$$

which converges to zero as $h\to 0$. Similarly

$$\begin{aligned}&\left\|\int_0^{t'-h}\left(S_3(t'-\sigma)-S_3(t'-h-\sigma)\right)A_2e^{A_1\sigma}z^1 d\sigma\right\|_{\alpha_0}^2\\ &\quad\le \sigma^2\frac{1}{2(\alpha+b)}\left(e^{\lambda_0 h}-1\right)^2\left(\int_h^{t'} e^{-2(\sigma+b)(t'-\sigma)}\langle z_3,\phi_0\rangle_{L^2(0,L)}^2 d\sigma\right)\\ &\qquad+\sigma^2\frac{L^2}{2D\pi^2}\sum_{n=1}^{\infty}n^{2(\alpha_0-1)}\left(e^{\lambda_n h}-1\right)^2\left(\int_h^{t'} e^{-2(\sigma+b)(t'-\sigma)}\langle z_3,\phi_n\rangle_{L^2(0,L)}^2 d\sigma\right).\end{aligned}$$

As in Corollary 4.2.3 since $\left(e^{\lambda_n h}-1\right)^2<1$ this sum converges uniformly. In particular $e^{\lambda_n h}-1\to 0$ and so the right-hand side converges to zero as $h\to 0$. Finally

$$\begin{aligned}&\left\|\int_0^{t-h}\left(S_3(t-\sigma)-S_3(t-h-\sigma)\right)\left(P(\sigma)z(\sigma)\right)_4 d\sigma\right\|_{\alpha_0}^2\\ &\quad\le \frac{1}{2(\alpha+b)}\left(e^{\lambda_0 h}-1\right)^2\sum_{i=1}^{4}\int_0^{t-h}\langle P_{4i}(\sigma)z_i(\sigma),\phi_0\rangle_{L^2(0,L)}^2 d\sigma\\ &\qquad+\frac{L^2}{2D\pi^2}\sum_{n=1}^{\infty}\left(e^{\lambda_n h}-1\right)^2\sum_{i=1}^{4}\int_0^{t-h}\langle P_{4i}(\sigma)z_i(\sigma),\phi_n\rangle_{L^2(0,L)}^2 d\sigma\end{aligned}$$

and this also converges to zero as $h\to 0$. □

The following two Corollaries to Proposition 5.1.7 show that the rabies model satisfies assumptions PS V and PS X.

Corollary 5.2.4. *For any* $h\in L^2(0,T;\overline{Z})$, $(\mathrm{M}_S h)(t)\in \underline{Z}$ *for almost every* $t\in[0,T]$, *and* $t\mapsto(\mathrm{M}_S h)(t)$ *is continuous with respect to* $\|\cdot\|_Z$.

Proof. That $(\mathrm{M}_S h)(t)\in\underline{Z}$ for almost every $t\in[0,T]$ follows immediately from Proposition 5.1.7. Therefore it only remains to show continuity.

Let $\delta > 0$. Recall from Proposition 4.2.11 that

$$\| (\mathbb{M}_S h)(t+\delta) - (\mathbb{M}_S h)(t) \|_Z \\ \leq \| (S(\delta) - I) \int_0^t S(t-s)h(s)\,ds \|_Z + \|S(\delta) \int_t^{t+\delta} S(t-s)h(s)\,ds\|_Z.$$

The first term converges to zero as $h \to 0$ since $(\mathbb{M}_S h)(t) \in \underline{Z}$ and $S(t)$ is a strongly continuous semigroup on $\underline{Z}$. Using Proposition 5.1.7 we see that

$$\|S(\delta) \int_t^{t+\delta} S(t-s)h(s)\,ds\|_Z \leq \|S(\delta)\| \left(k_1'(\delta)\, \|h\|_{L^2(t,t+\delta;\overline{Z})} \right)$$

which also converges to zero as $\delta \to 0$. Therefore we have shown continuity from the right.

Similarly from Proposition 4.2.11 it is seen that

$$\| (\mathbb{M}_S h)(t) - (\mathbb{M}_S h)(t-\delta) \|_Z \\ \leq \| \int_{t-\delta}^t S(t-s)h(s)\,ds\|_Z + \| \int_0^{t-\delta} (S(t-s) - S(t-\delta-s))\, h(s)\,ds\|_Z.$$

Again applying Proposition 5.1.7 it is clear that the first term vanishes as $\delta \to 0$. For the second term note that

$$\| \int_0^{t-\delta} (S(t-s) - S(t-\delta-s))\, h(s)\,ds\|_Z^2 \\ \leq \| \int_0^{t-\delta} \left(e^{A_1(t-s)} - e^{A_1(t-\delta-s)}\right) h^1(s)\,ds\|_{Z_1}^2 \\ + 2\| \int_0^{t-\delta} (S_2(t-s) - S_2(t-\delta-s))\, h^1(s)\,ds\|_{\alpha_1}^2 \\ + 2\| \int_0^{t-\delta} (S_3(t-s) - S_3(t-\delta-s))\, h_4(s)\,ds\|_{\alpha_1}^2.$$

The first of these terms converges to zero since $e^{A_1 t}$ is a strongly continuous semigroup on Z_1. The second and third terms converge to zero applying the method of the proof of Corollary 5.2.3. □

Corollary 5.2.5. *There exists a positive constant $\overline{K}_6$ such that*

$$\|\mathbb{L}_S h\|_{L^4(0,T;\underline{Z})} \leq \overline{K}_6 \|h\|_{L^2(0,T;\overline{Z})}$$

for all $h \in L^2(0,T;\overline{Z})$.

Proof. First note that for the rabies model $\mathbb{L}_S h = (\mathbb{M}_S h)$. Hence Proposition 5.1.7 implies that

$$\begin{aligned}\|\mathbb{L}_S h\|^4_{L^4(0,T;\underline{Z})} &= \|\mathbb{M}_S h\|^4_{L^4(0,T;\underline{Z})} \\ &\leq \int_0^T k_1'(t)^4 \|h\|^4_{L^2(0,t;\overline{Z})} dt \\ &\leq \left(\int_0^T k_1'(t)^4 dt\right) \|h\|^4_{L^2(0,T;\overline{Z})}\end{aligned}$$

and taking the fourth root of both sides yields the result. □

Note that for the rabies model $CS(T)Bu = 0$ for all $u \in U$. Hence the assumption PS VI is automatically satisfied. Since $V = Z$ assumption PS VII follows immediately from Proposition 5.2.2. Similarly assumption PS VIII follows immediately from Proposition 5.1.7. It only remains in this subsection to prove that the rabies model satisfies the assumption PS IX.

Proposition 5.2.6. *Suppose that the following condition is satisfied for the rabies model.*

$$\boxed{\alpha_0 - \alpha_1 \leq \tfrac{1}{2}} \tag{5.22}$$

There exists a positive constant $\overline{K}_5$ such that

$$\|S(\cdot)z\|_{L^4(0,T;\underline{Z})} \leq \overline{K}_5 \|z\|_Z$$

for all $z \in \underline{Z}$.

Proof. Lemma 4.2.15 implies that there exists a constant g_3 such that

$$\int_0^T \|S_3(t)z\|^4_{\alpha_0} ds \leq g_3 \|z\|^4_{\alpha_1}$$

for all $z \in H^{\alpha_0}(0, L)$ provided

$$\frac{1}{|\lambda_n + \lambda_m|} n^{2(\alpha_0-\alpha_1)} m^{2(\alpha_0-\alpha_1)} \tag{5.23}$$

is bounded for all $m, n \in \mathbb{N}$. Note that, fixing $m \in \mathbb{N}$, we have

$$\frac{n^{2(\alpha_0-\alpha_1)} m^{2(\alpha_0-\alpha_1)}}{n^2 + m^2} = \frac{n^{2(\alpha_0-\alpha_1-1)} m^{2(\alpha_0-\alpha_1)}}{1 + m^2/n^2}$$

which is bounded by condition (5.22). Similarly, if $m = n$, we have

$$\frac{n^{2(\alpha_0-\alpha_1)}m^{2(\alpha_0-\alpha_1)}}{n^2+m^2} = \frac{n^{4(\alpha_0-\alpha_1)}}{2n^2} = \frac{1}{2}n^{4(\alpha_0-\alpha_1)-2}$$

which is also bounded by condition (5.22). Hence (5.23) is bounded. Therefore

$$\begin{aligned}\int_0^T \|S(t)z\|_{\underline{Z}}^4 dt &\leq \int_0^T \left(\|e^{A_1 t}z^1\|_{Z_1}^2 + 2\|S_2(t)z^1\|_{\alpha_0}^2 + 2\|S_3(t)z_4\|_{\alpha_0}^2\right)^2 dt \\ &\leq 3\int_0^T \|e^{A_1 t}z^1\|_{Z_1}^4 dt + 12L^2 m_2^2 \int_0^T \left(e^{-2(\sigma+b)t} - 1\right)^2 \|z_3\|_{C(0,L)}^4 dt \\ &\quad + 12g_3\|z_4\|_{\alpha_1}^4 \\ &\leq \max\left\{3\int_0^T \|e^{A_1 t}\|^4 dt + 12L^2 m_2^2 \int_0^T \left(e^{-2(\sigma+b)t} - 1\right)^2 dt, 12g_3\right\} \|z\|_Z^4\end{aligned}$$

as required. □

Hence the rabies model satisfies all of the assumptions required in Chapter 3 for perturbed systems. In the next subsection it will be shown that the rabies model satisfies the hypotheses of Theorem 2.2.10.

5.2.3 Solving the control problem

It remains to show that the rabies model satisfies assumption TV VI, and that the nonlinearity satisfies a Lipschitz condition.

Remark 5.2.7. First note that, if $z \in \underline{Z}$, then

$$\begin{aligned}\|Nz\|_{\overline{Z}}^2 &\leq 5\mu \sum_{k=1}^{4}\sum_{j=1}^{4} \|z_k\|_{C(0,L)}^2 \|z_j\|_{C(0,L)}^2 + 10\beta\|z_1\|_{C(0,L)}^2 \|z_4\|_{C(0,L)}^2 \\ &\leq 5\mu \overline{R}_1^2 \overline{R}_2^2 \|z\|_{\underline{Z}}^2 \sum_{k=1}^{4} \|z_k\|_{C(0,L)}^2 + 10\beta \overline{R}_1^2 \overline{R}_2^2 \|z_1\|_{C(0,L)}^2 \|z\|_{\underline{Z}}^2 \\ &\leq 5\overline{R}_1^4 \overline{R}_2^4 (\mu + 2\beta) \|z\|_{\underline{Z}}^4.\end{aligned}$$

Thus, if $z(\cdot) \in L^4(0,T;\underline{Z})$, then

$$\int_0^T \|N(z(t))\|_{\overline{Z}}^2 dt \leq 5\overline{R}_1^4 \overline{R}_2^4 (\mu + 2\beta) \int_0^T \|z(t)\|_{\underline{Z}}^4.$$

Hence $N(z(\cdot)) \in L^2(0,T;\overline{Z})$ for all $z(\cdot) \in L^4(0,T;\underline{Z})$.

The next result shows that, for the rabies model, the nonlinearity $N : \underline{Z} \longrightarrow \overline{Z}$ satisfies a Lipschitz condition.

Proposition 5.2.8. *For the rabies model there exists a constant c and a continuous symmetric function $k(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $k(0,0) = 0$ with*

$$\|N(z_1(\cdot)) - N(z_2(\cdot))\|_{L^2(0,T;\overline{Z})} \leq k(\|z_1\|, \|z_2\|)\|z_1 - z_2\|_{L^4(0,T;\underline{Z})}$$

for all $\|z_1\|, \|z_2\| \leq c$.

Proof. Let $z, w \in \underline{Z}$. Then for $k = 2, 4$ we have

$$\begin{aligned}
\|N_k z - N_k w\|_{C(0,L)} &= \mu \| \sum_{j=1}^{4} (z_k z_j - w_k w_j) \|_{C(0,L)} \\
&\leq \frac{\mu}{2} \sum_{j=1}^{4} \left(\|(z_k + z_j)^2 - (w_k + w_j)^2\| + \|z_k^2 - w_k^2\| + \|z_j^2 - w_j^2\| \right) \\
&\leq \frac{\mu}{2} \sum_{j=1}^{4} \Big[(\|z_k\| + \|z_j\| + \|w_k\| + \|w_j\|)(\|z_k - w_k\| + \|z_j - w_j\|) \\
&\qquad + (\|z_k\| + \|w_k\|)\|z_k - w_k\| + (\|z_j\| + \|w_j\|)\|z_j - w_j\| \Big] \\
&\leq \mu \sum_{j=1}^{4} (\|z_k\| + \|z_j\| + \|w_k\| + \|w_j\|)(\|z_k - w_k\| + \|z_j - w_j\|).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|N_k z - N_k w\|^2_{C(0,L)} &\leq 32\mu^2 \sum_{j=1}^{4} \left(\|z_k\|^2 + \|z_j\|^2 + \|w_k\|^2 + \|w_j\|^2 \right) \left(\|z_k - w_k\|^2 + \|z_j - w_j\|^2 \right) \\
&\leq 32\mu^2 \left(\|z\|^2_{\underline{Z}} + \|w\|^2_{\underline{Z}} + \|z_k\|^2 + \|w_k\|^2 \right) \left(4\|z_k - w_k\|^2 + \|z - w\|^2_{\underline{Z}} \right) \\
&\leq m_3 \left(\|z\|^2_{\underline{Z}} + \|w\|^2_{\underline{Z}} \right) \left(\|z - w\|^2_{\underline{Z}} \right)
\end{aligned}$$

where $m_3 = 320\mu^2$. Similarly, for $k = 1, 3$, we have

$$\begin{aligned}
\|N_k z - N_k w\|_{C(0,L)} &\leq \mu \sum_{j=1}^{4} (\|z_k\| + \|z_j\| + \|w_k\| + \|w_j\|)(\|z_k - w_k\| + \|z_j - w_j\|) \\
&\qquad + \beta (\|z_1\| + \|z_4\| + \|w_1\| + \|w_4\|)(\|z_1 - w_1\| + \|z_4 - w_4\|)
\end{aligned}$$

and so

$$\begin{aligned}\|N_k z - N_k w\|^2_{C(0,L)} &\le 2m_3 \left(\|z\|^2_{\underline{Z}} + \|w\|^2_{\underline{Z}}\right)\left(\|z-w\|^2_{\underline{Z}}\right)\\ &\quad + 16\beta^2\left(\|z_1\|^2 + \|z_4\|^2 + \|w_1\|^2 + \|w_4\|^2\right)\left(\|z_1-w_1\|^2 + \|z_4-w_4\|^2\right)\\ &\le \left(2m_3 + 16\beta^2\right)\left(\|z\|^2_{\underline{Z}} + \|w\|^2_{\underline{Z}}\right)\left(\|z-w\|^2_{\underline{Z}}\right).\end{aligned}$$

Now let $z(\cdot), w(\cdot) \in L^4(0,T;\underline{Z})$. We have

$$\begin{aligned}&\int_0^T \|N(z(t)) - N(w(t))\|^2_{\underline{Z}}\, dt\\ &\quad = \int_0^T \sum_{k=1}^{4} \|N_k(z(t)) - N_k(w(t))\|^2_{C(0,L)}\, dt\\ &\quad \le \left(6m_3 + 32\beta^2\right)\left[\int_0^T \left(\|z(t)\|^2_{\underline{Z}} + \|w(t)\|^2_{\underline{Z}}\right)^2 dt\right]^{1/2}\left[\int_0^T \|z(t)-w(t)\|^4_{\underline{Z}}\, dt\right]^{1/2}\\ &\quad \le 2\overline{R}_1^4\overline{R}_2^4\left(6m_3 + 32\beta^2\right)\left(\|z\|^4_{L^4(0,T;\underline{Z})} + \|w\|^4_{L^4(0,T;\underline{Z})}\right)^{1/2}\|z-w\|^2_{L^4(0,T;\underline{Z})}\end{aligned}$$

and taking square roots of both sides completes the proof. □

To summarise the analysis so far, α_0, α_1, β_1 and β_2 must be chosen such that

$$\frac{1}{2} < \alpha_1 \le \alpha_0 \le 1, \qquad \beta_1 > \frac{1}{2} \quad \text{and} \quad \beta_2 \le 0$$

to ensure that the assumptions of Chapter 3 are satisfied.

Finally it remains to be shown that the rabies model satisfies assumption TV VI. This assumption is similar to the conditions given in Curtain and Zwart (1995) for an infinite-dimensional system to be (exactly) controllable or observable. Unfortunately this assumption can prove to be difficult to establish for a given system. Let $\mathcal{L} = CU(T,0)B$ which has already been shown to be a bounded linear map from U to Y. For the rabies model we recall the equivalence between the boundedness of the generalised inverse of $\mathcal{L}$ and the range of $\mathcal{L}$ being closed. The output space $Y = H^{\beta_2}(0,L)$ should be chosen such that, numerically, the former property is satisfied. In fact it is this property that is necessary for the theory of Chapter 2 to be applied.

Note that, for all $y \in H^{\beta_2}(0,L)$ we have

$$y = \sum_{n=0}^{\infty} y_n \phi_n$$

where $y_n = \langle y, \phi_n \rangle_{L^2}$, provided $||y||_{\beta_2} < \infty$. Since

$$||\phi_n||_{\beta_2} = n^{\beta_2}$$

we can define an orthonormal basis $\{\psi_n\}$ for $H^{\beta_2}(0, L)$ where $\psi_0 = \phi_0$, and $\psi_n = n^{-\beta_2}\phi_n$.

Suppose that functions in $H^{\beta_1}(0, L)$ and $H^{\beta_2}(0, L)$ are approximated by taking only finitely many terms in their expansions. Hence each $y \in H^{\beta_2}(0, L)$ is approximated by the $N + 1$ dimensional (real) vector y^N defined by

$$y^N = \begin{pmatrix} \langle y, \phi_0 \rangle_{L^2(0,L)} \\ \vdots \\ \langle y, \phi_N \rangle_{L^2(0,L)} \end{pmatrix} \qquad \text{where} \qquad y = \sum_{n=0}^{\infty} \langle y, \phi_n \rangle_{L^2(0,L)} \phi_n.$$

Now consider the following abstract Cauchy problem:

$$\dot{z}(t) = (A + P(t))\, z(t), \qquad z(0) = z_0 \tag{5.24}$$

where A and $P(t)$ are defined as for the rabies problem. The action of $\mathcal{L}$ on any basis function ϕ_n can then be obtained by setting $z_0 = B\phi_n$ to get

$$Cz(T) = CU(T, 0)z_0 = CU(T, 0)B\phi_n = \mathcal{L}\phi_n.$$

Note that ϕ_n is approximated by the $n + 1$ vector of the standard basis of $\mathbb{R}^{N+1}$ $(n \leq N)$ given by

$$e_{n+1} = \begin{pmatrix} 0 & \ldots & 1 & 0 & \ldots \end{pmatrix}^{\mathrm{T}}.$$

$\uparrow$ $n + 1$ position

Thus if the dynamics of (5.24) are solved via a numerical scheme with the output $Cz(T)$ approximated by y^N, and $\mathcal{L}$ is approximated by the $(N + 1) \times (N + 1)$ matrix $\mathcal{L}^N$,

$$y^N = \mathcal{L}^N e_{n+1} \qquad 0 \leq n \leq N.$$

This output then gives the $n + 1$ column of $\mathcal{L}^N$. An approximation of the generalised inverse can then be calculated as $(\mathcal{L}^N)^\dagger$. Hence for increasing N the finite approximation of $||\mathcal{L}^\dagger y||^2_{\beta_1}$ given by

$$||(\mathcal{L}^N)^\dagger y^N||^2_{\beta_1} = (z_1)^2 + \sum_{n=1}^{N} n^{2\beta_1}(z_{n+1})^2$$

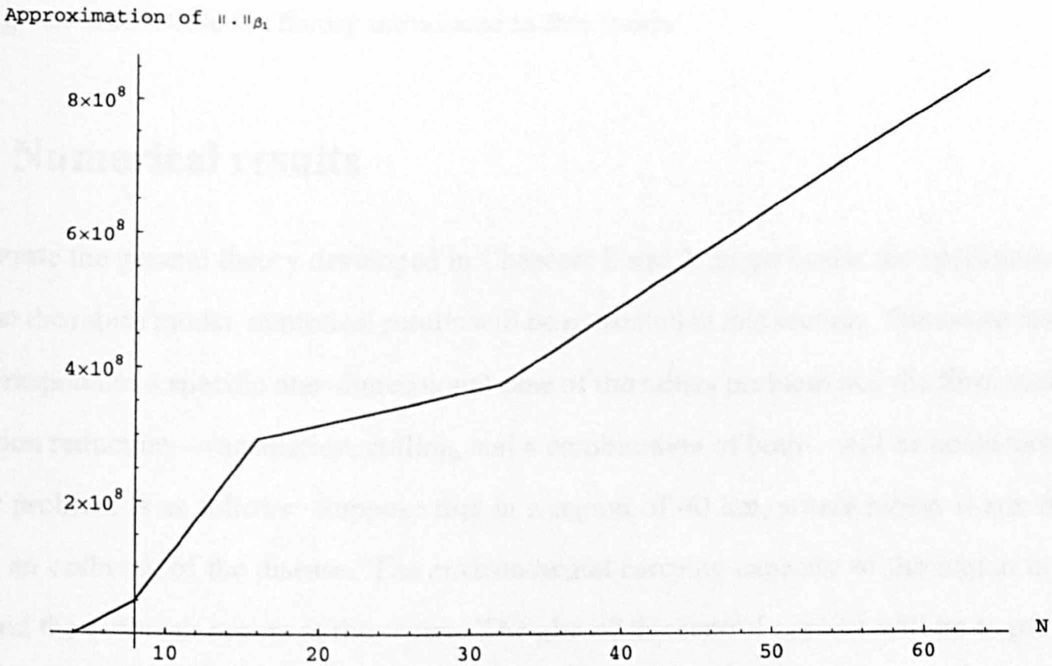


Figure 5.1: The approximation of $||\mathcal{L}^\dagger y||^2_{\beta_1}$ for $N = 4, 8, 16, 32$ and 64. The limitation to an integer power of 2 results from the use of the Fast Fourier Transform algorithm from Press et al. (1992). The data points are connected by straight lines.

is calculated, where

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_{N+1} \end{pmatrix} = \begin{pmatrix} (\mathcal{L}^N)^\dagger \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ N^{-\beta_2} \end{pmatrix}$$

where this last vector is ψ_N. The value β_2 should then be chosen such that the resulting sequence is bounded.

The numerical procedure used to select β_2 utilises the Fast Fourier Transform (introduced by Cooley and Tukey, 1965) and the implementation used (Press et al., 1992) requires N to be an integer power of 2. Thus only five values ($N = 4, 8, 16, 32, 64$) are plotted in Figure 5.1 for $\beta_2 = 0$ and $\beta_1 = 0.501$. The program was unable to obtain an estimate for $\mathcal{L}^N$ for $N = 128$ or higher. This is severely limiting in regard to the data that can be obtained. The few data points that are plotted in Figure 5.1 cast into doubt whether the generalised inverse is bounded or not.

In the next section a specific example of the rabies problem will be considered and numerical results given to illustrate the theory introduced in this thesis.

5.3 Numerical results

To illustrate the general theory developed in Chapters 2 and 3, in particular the application of this theory to the rabies model, numerical results will be presented in this section. The numerical results will correspond to a specific one–dimensional case of the rabies problem and the three methods of population reduction—vaccination, culling, and a combination of both—will be considered.

The problem is as follows: Suppose that in a region of 40 km, where rabies is not endemic, there is an outbreak of the disease. The environmental carrying capacity of the region is 2 foxes km^{-2} and the outbreak occurs at the centre. The aim of the control scheme will be to prevent the spread of the disease and to eradicate rabies in the region of interest.

It will be assumed that the initial fox population density is at the steady state value equal to the environmental carrying capacity. The outbreak of rabies will be modelled by assuming that the initial density of the rabid class is normally distributed with total area approximately equal to one (see Figure 5.2).

The environmental carrying capacity for this example is above the critical threshold of 1 fox km^{-2} (Anderson et al., 1981; Murray et al., 1986) and so, in the absence of any control measures, an epidemic begins. Figure 5.3 shows the evolution of the infected and susceptible population densities over an extended region of 60 km for a time–span of 0.5 year. In Figure 5.3(b) the distinctive travelling waves can be seen to be just forming and travelling away from the centre.

In applying the techniques of the previous chapters we must be careful in choosing the desired final state. Choosing a final state of zero total infected fox population density is not possible as discussed earlier in Chapter 1. Instead a y_d is chosen such that

$$\int_{-20}^{20} (y_d)(x)\,dx < m_1$$

with m_1 some small value. An example of such a distribution is

$$(y_d)(x) = \frac{\mu_1}{\sqrt{2\pi}} e^{-\mu_2 x^2}, \tag{5.25}$$

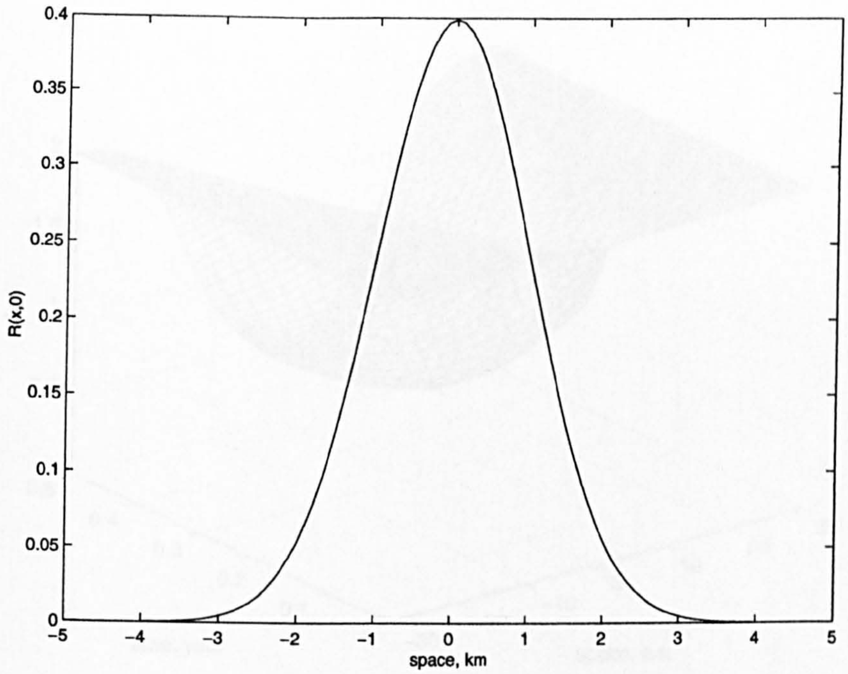


Figure 5.2: Initial distribution of rabid foxes.

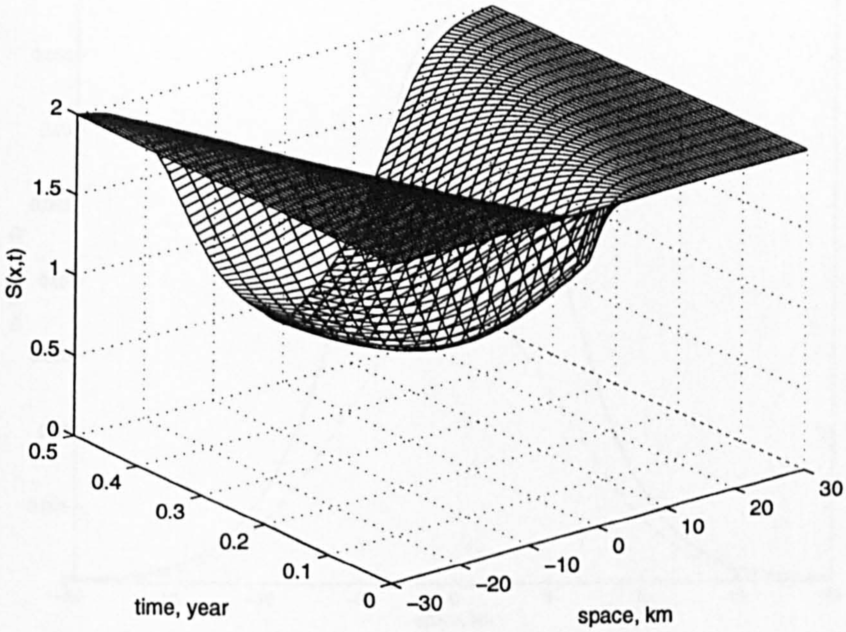
with μ_1 and μ_2 suitably chosen. To ensure containment of the disease it will also be required that the total density of infected foxes remains negligible at the boundary of the spatial domain for all time (ie on $[0, T]$). That is

$$R(-20, t) + I(-20, t) = R(20, t) + I(20, t) < m_2$$

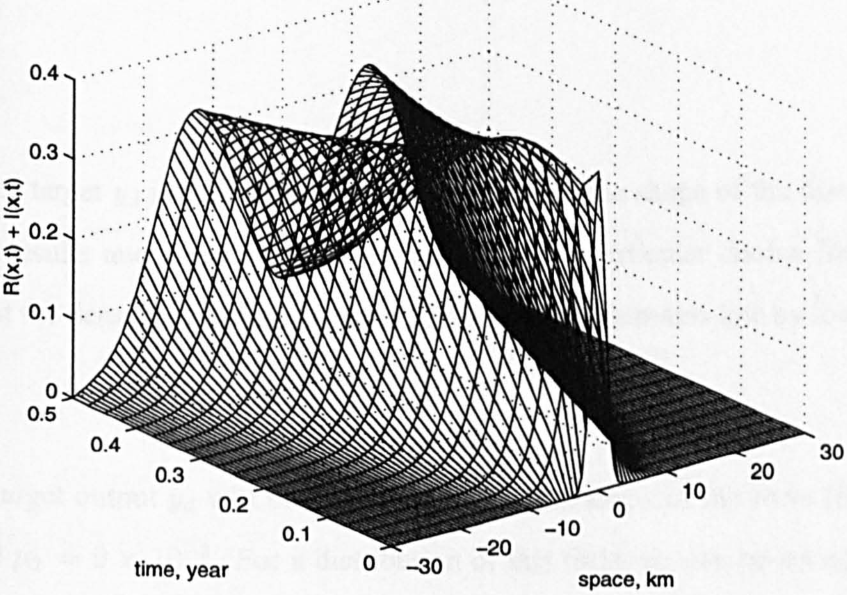
for all $t \in [0, T]$, where again, m_2 is some small number.

Note that a 50% reduction (at least) is required for the susceptible population density to be less that the critical threshold for an epidemic. Therefore the initial control u' that is applied corresponds to a uniform vaccination (or cull) of 60% of the susceptible foxes. When this population reduction is achieved through vaccination, Table 2 shows that the situation remains non-epidemic supporting for 67 days (approximately 0.18 year). For a uniform reduction of 80% the region does not support an epidemic for 172 days (approximately 0.47 year). Thus the final time T is chosen to be 0.3 year.

Figure 5.4 shows the output resulting from a initial control of vaccinating 60% of all susceptible foxes. Also shown is a distribution of the form (5.25) chosen to roughly approximate $Cz'(T)$, but



(a) Susceptible population



(b) Infected population

Figure 5.3: The progress of an epidemic over a time–span of 0.5 year. The situation is unstable with a carrying capacity of 2 foxes km^{-2} with no control applied.

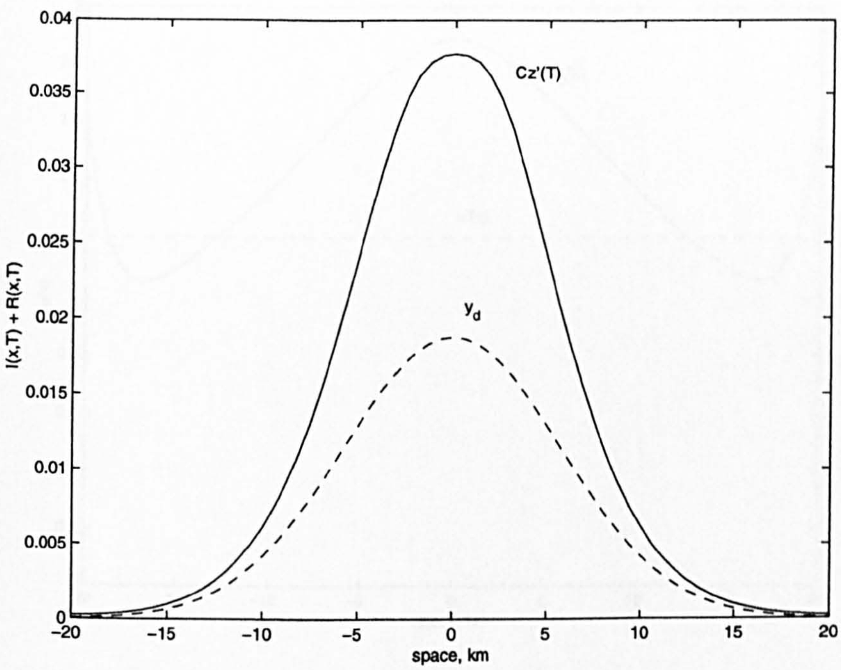


Figure 5.4: The output resulting from vaccinating 60% of all susceptible foxes. Also shown is a distribution of the form (5.25) with $\mu_1 = 0.0472$ and $\mu_2 = 0.015$.

scaled by $\frac{1}{2}$. The target y_d is chosen to roughly approximate the shape of the distribution $Cz'(T)$ because of the results and discussion of Section 2.4. The particular choice for y_d reflects the requirement that the density of infected foxes at the boundaries remains low by lowering the target density there.

Hence the target output y_d will be chosen to be a distribution of the form (5.25) with $\mu_1 = 5.3 \times 10^{-3}$ and $\mu_2 = 9 \times 10^{-3}$. For a distribution of this form m_1 can be set equal to 0.04. To ensure containment m_2 is chosen to be 6×10^{-5}.

In the subsections that follow the theory developed in the previous chapters will be applied to this control problem. The first mechanism for the control to be considered is vaccination. This is followed by a culling control strategy and then a combined one.

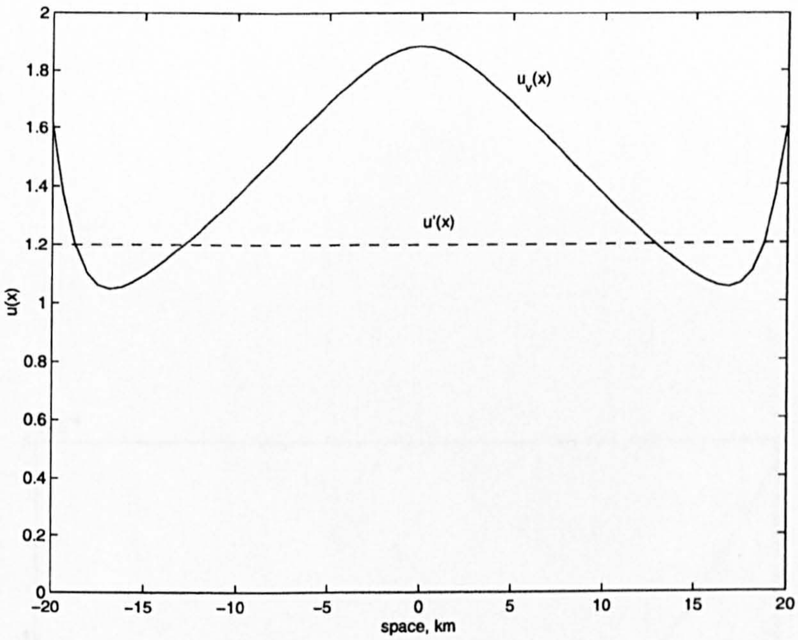


Figure 5.5: The control resulting from the application of the iterative scheme, which drives the system to the distribution y_d defined by (5.25) with $\mu_1 = 5.3 \times 10^{-3}$ and $\mu_2 = 9 \times 10^{-3}$. The control strategy is vaccination only.

5.3.1 Vaccination only strategy

The iterative scheme presented in earlier chapters gives rise to the control u_v shown in Figure 5.5, together with the initial control u'. As expected the greatest level of reduction occurs at the centre. To counter the spread of rabies from the region there are also greater levels of vaccination at the boundary of the spatial domain. This keeps the infected fox densities low at the boundary ensuring containment of the outbreak. This is illustrated in Figure 5.6 where it is shown that the density of infected foxes at each boundary remains below the desired level.

The evolution of the susceptible and rabid fox population densities when the control u_v is applied are plotted in Figure 5.7. Figure 5.7(b) shows that rabies dies away quickly; a magnified version is given in Figure 5.8. This plot shows more clearly the containment of the outbreak. Note that the population density of the susceptible class of fox rises above the critical threshold for an epidemic towards the end of the simulation. Hence the situation is becoming unstable in the sense

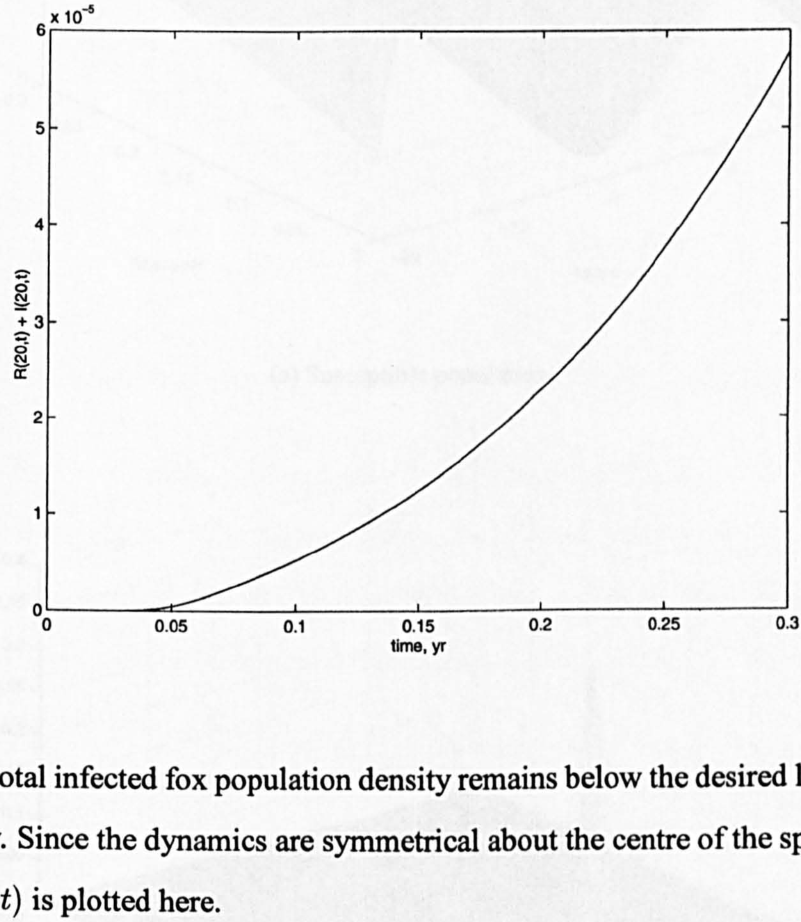
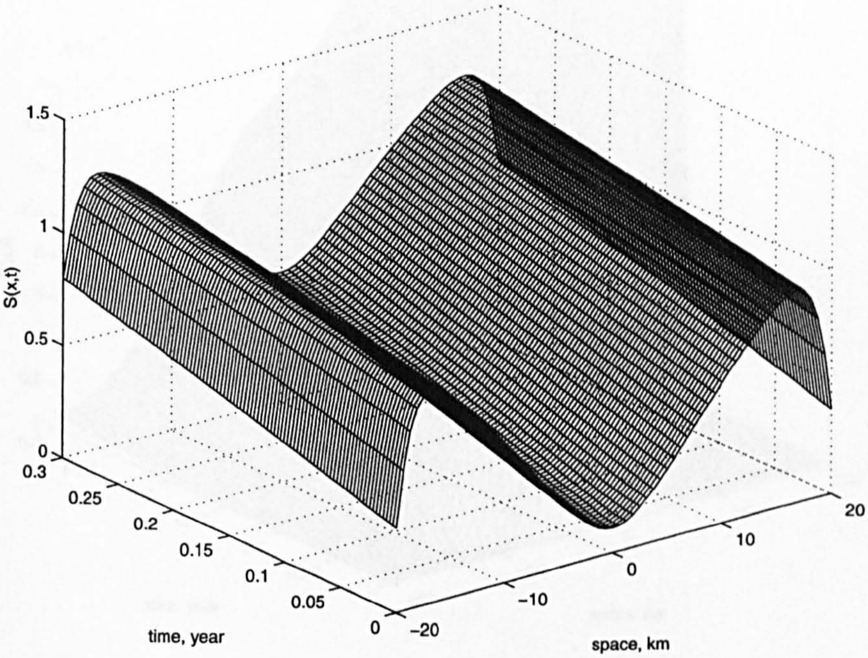
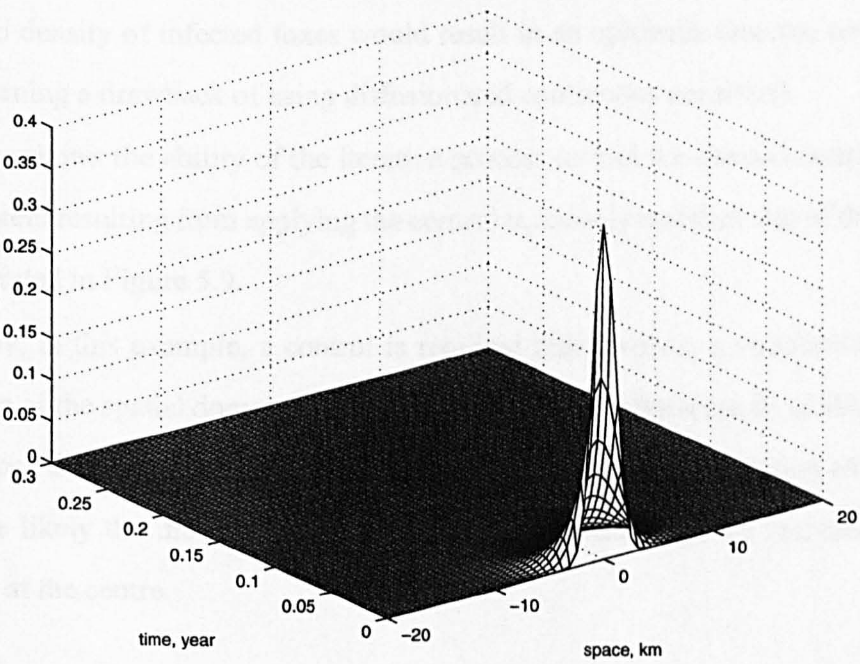


Figure 5.6: The total infected fox population density remains below the desired level set in the text at each boundary. Since the dynamics are symmetrical about the centre of the spatial domain only $I(20, t) + R(20, t)$ is plotted here.



(a) Susceptible population



(b) Rabid population

Figure 5.7: The effect of the control u_v during a time-span of 0.3 year.

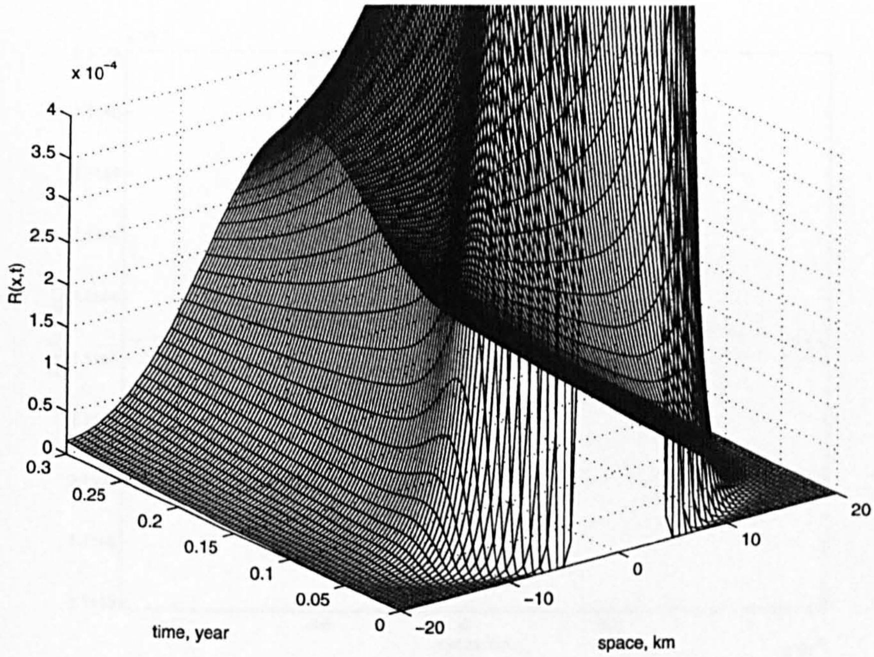


Figure 5.8: A magnified view of the evolution of the rabid fox population density.

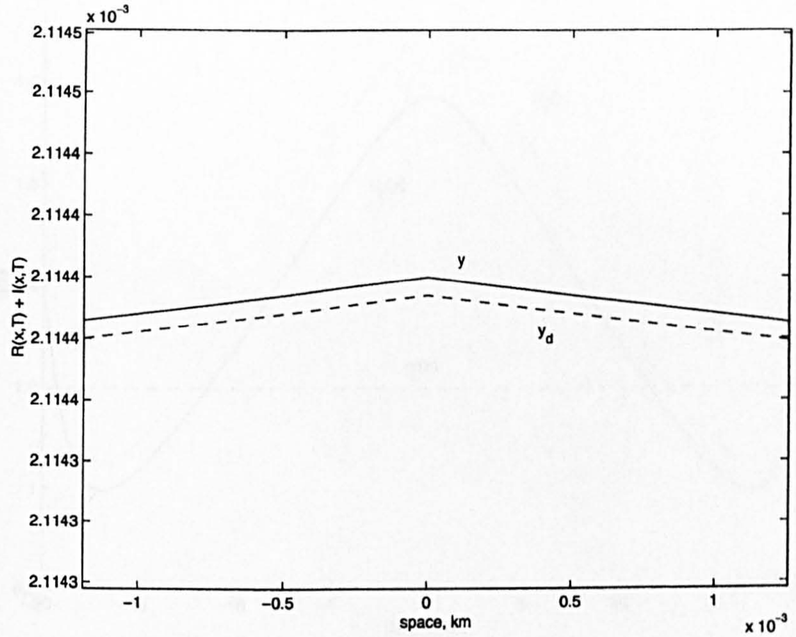
that any nonzero density of infected foxes would result in an epidemic (see the remarks made in Chapter 1 concerning a drawback of using diffusion and continuous densities).

This example shows the ability of the iterative process to find the desired control. In fact the output of the system resulting from applying the control u_v closely matches that of the target output y_d. This is illustrated in Figure 5.9.

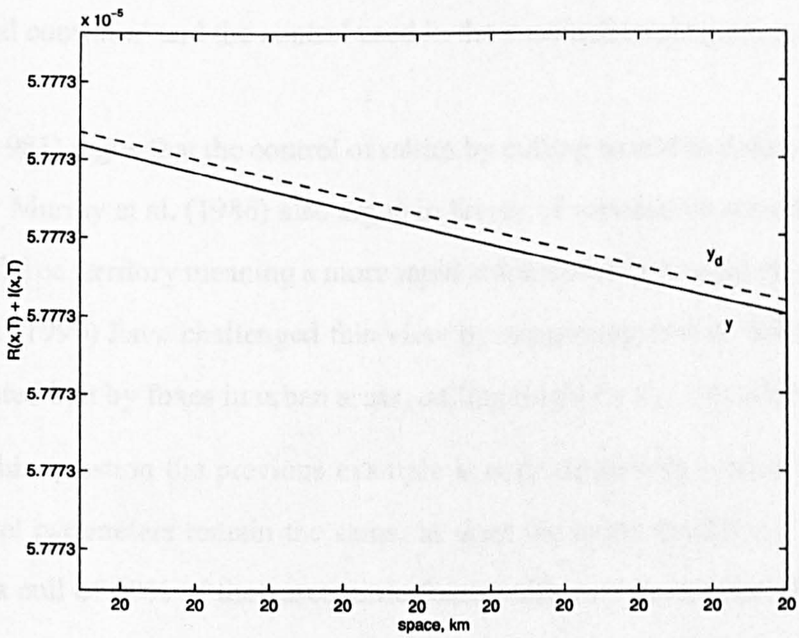
Unfortunately, in this example, a control is required that involves a vaccination rate of over 90% at the centre of the spatial domain. The function (5.25) provides some flexibility in the choice of y_d. The steeper the profile of the function chosen, with more concentration of density at the centre, the more likely the disease will be contained. Unfortunately this increases the level of control required at the centre.

5.3.2 Culling only strategy

Previous models for the spread of rabies have also attempted to answer the important question of whether the population reduction should be carried out through a vaccination or culling program.



(a) Comparison at the centre



(b) Comparison at the boundary

Figure 5.9: A comparison between the desired target output y_d and the actual output when applying the control u_v.

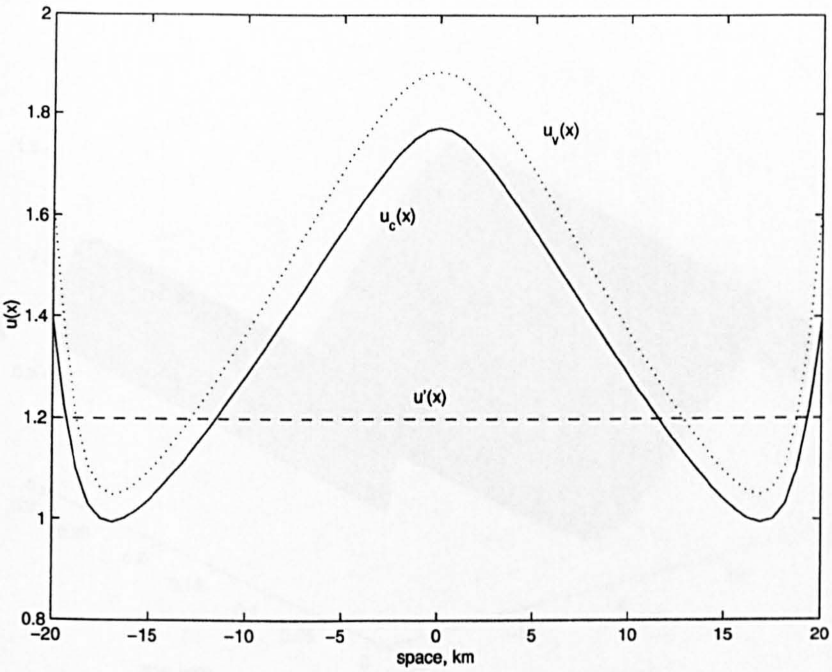
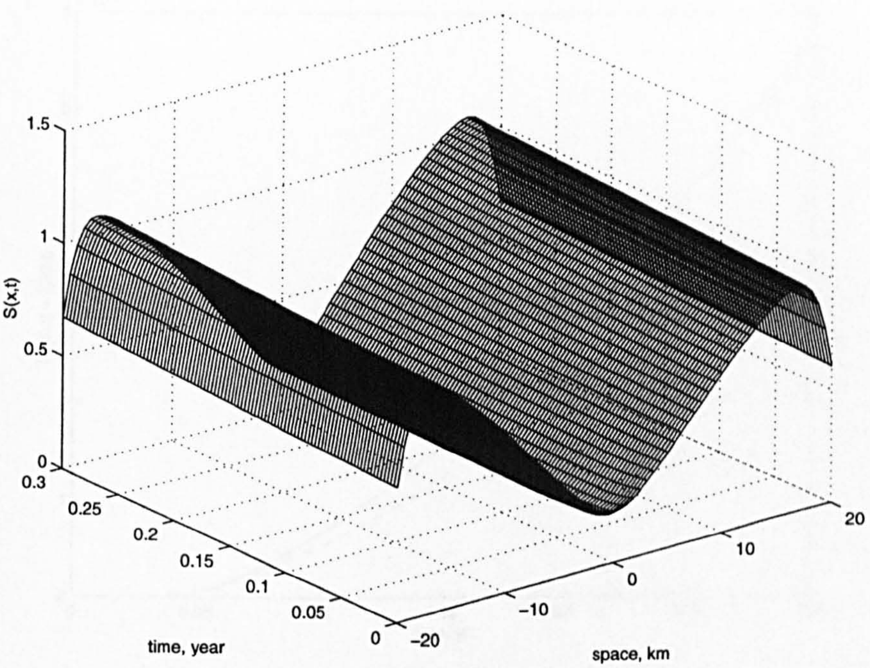


Figure 5.10: The control u_c that is returned by the iterative process in the case of a program of culling. The initial control u' and the control used in the vaccination program u_v are also shown.

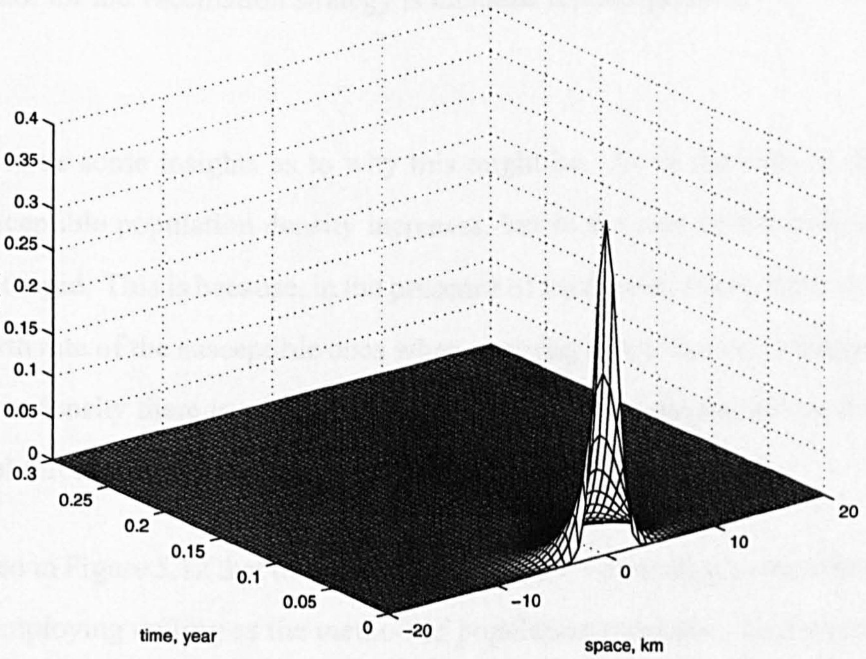
Anderson et al. (1981) argue that the control of rabies by culling would be difficult to achieve in all but poor habitats. Murray et al. (1986) also argue in favour of vaccination over culling concluding that culling would free territory meaning a more rapid colonisation by young foxes. More recently, Harris and Smith (1990) have challenged this view by suggesting that in Britain, with the poor uptake of vaccinated bait by foxes in urban areas, culling might be the only effective strategy.

To consider this question the previous example is considered with a strategy involving only culling. All model parameters remain the same, as does the target density y_d. For this case the initial control is a cull of 60% of the susceptible foxes uniformly across the whole of the spatial domain. The control u_c returned by the iterative process is compared with u_v in Figure 5.10. The evolution of the densities of the susceptible and rabid classes of fox when this control is applied is plotted in Figure 5.11.

A similar profile for the controls is observed, but a smaller level of population reduction is required in the case of culling. The evolution of the susceptible population density (see Fig-



(a) Susceptible population



(b) Rabid population

Figure 5.11: The effect of the control u_c during a time-span of 0.3 year.

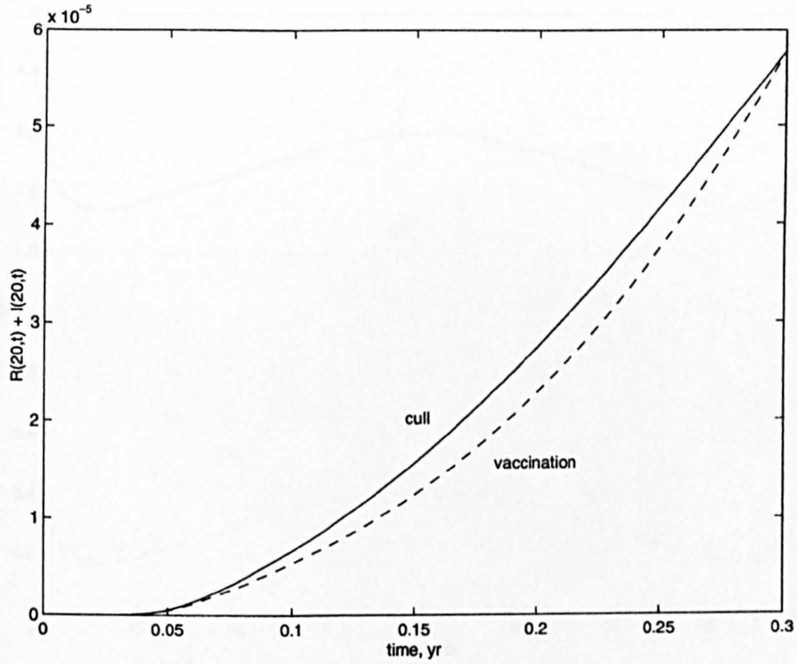


Figure 5.12: The total infected fox population density at the boundary of the spatial domain. The corresponding plot for the vaccination strategy is included for comparison.

ure 5.11(a)) gives us some insights as to why this might be. As in the case of the vaccination strategy, the susceptible population density increases, but in the case of the culling strategy this increase is not as rapid. This is because, in the presence of vaccinated foxes, there is a contribution to the overall birth rate of the susceptible ones when applying a vaccination strategy. With a much slower increase in density there is a longer period for the rabies to die out before the density rises above the critical threshold.

It is confirmed in Figure 5.12 that the disease is contained—according to the criterion discussed earlier—when employing culling as the method of population reduction. In comparison with the profile produced in the vaccination program it is seen that the rate of change of the population density of infected foxes at the boundary, that is the gradient of the plots in Figure 5.12, is shallower for the culling program. Hence the flux of infected foxes across the boundary of the spatial domain is less for the culling strategy than for the vaccination one.

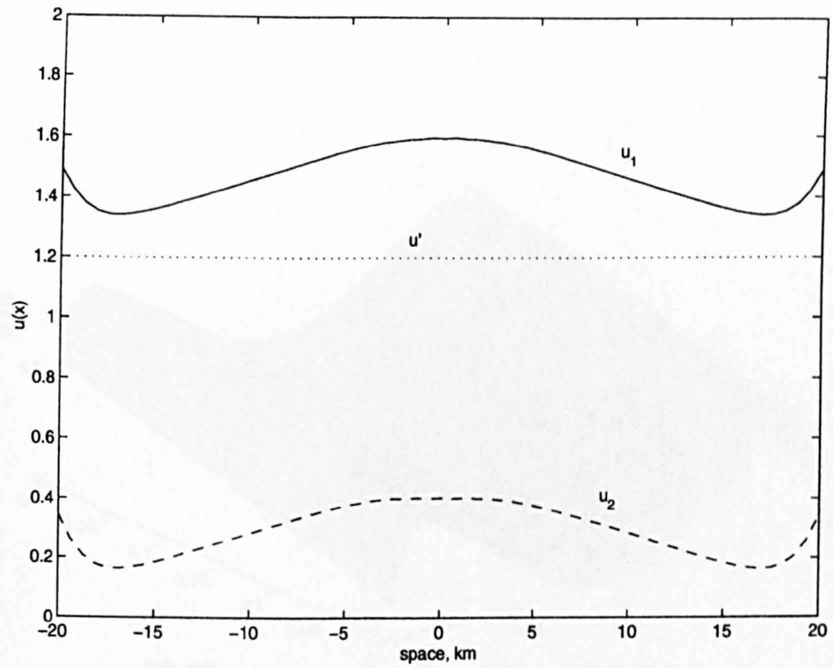


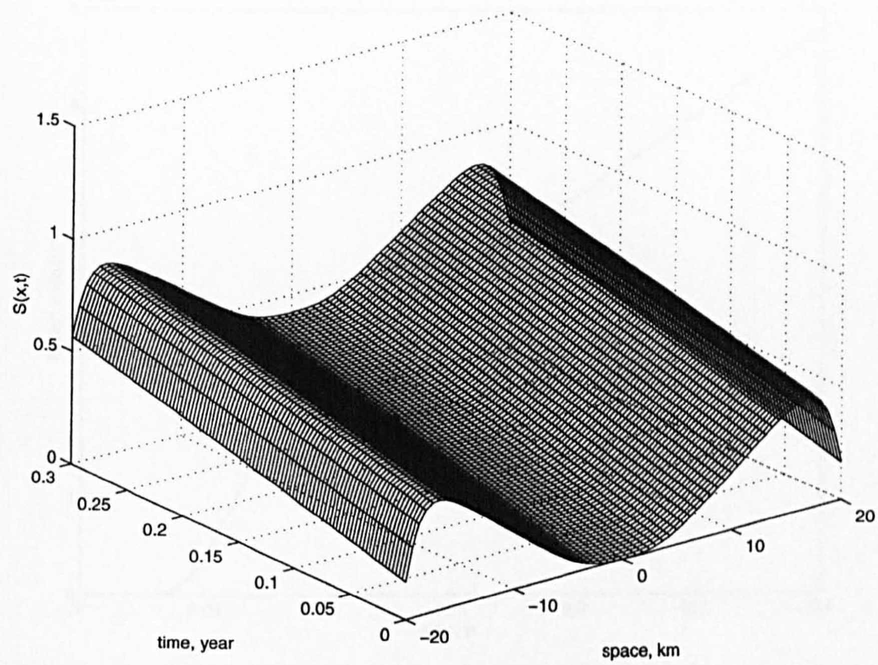
Figure 5.13: The control produced by the iterative scheme when employing a combined strategy of vaccination and culling. The levels of vaccination, u_1, and cull, u_2, are shown.

5.3.3 Combined vaccination and culling strategy

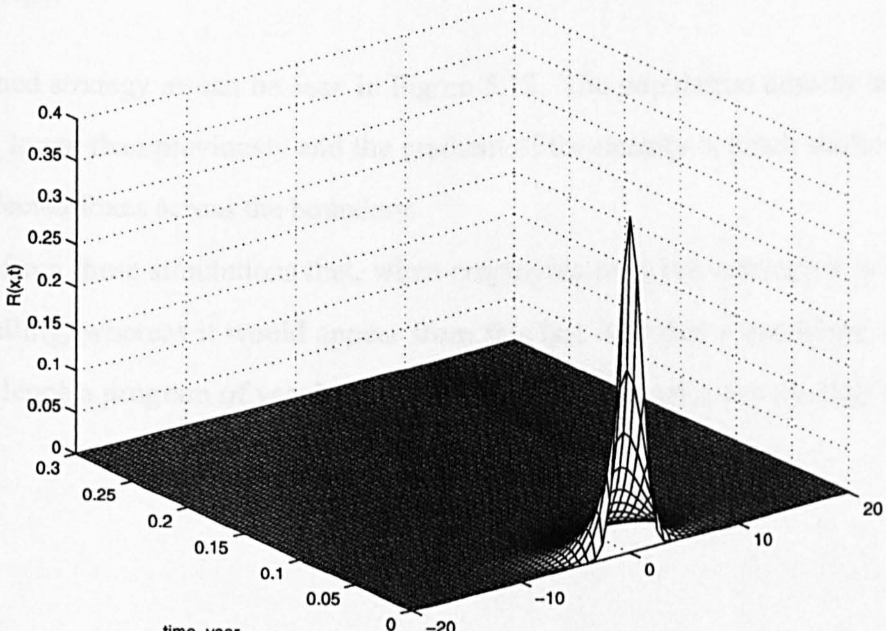
It has been suggested (Anderson, 1986) that in high fox population densities a program of vaccination could be supplemented by culling. This will be the case considered here for the same example.

The initial control applied is a 60% reduction of the susceptible foxes performed uniformly across the spatial domain using vaccination. The target density y_d is again of the form (5.25) but with $\mu_1 = 5.3 \times 10^{-4}$ and $\mu_2 = 9 \times 10^{-3}$. Applying the iterative scheme developed in this thesis leads to the control $u_{vc} = (u_1\ u_2)^{\top}$ shown in Figure 5.13, where u_1, u_2 are the levels of vaccination and cull respectively. Interestingly the level of vaccination is much higher than the level of cull, though both profiles are shallower than for the respective single strategies.

When applying the combined control the resulting evolution of the susceptible and rabid population densities is given in Figure 5.14. Notice that the susceptible population density does not rise above the critical threshold (Figure 5.14(a)). The containment of the outbreak is also improved



(a) Susceptible population



(b) Rabid population

Figure 5.14: The effect of the control u_{vc} for a time-span of 0.3 year.

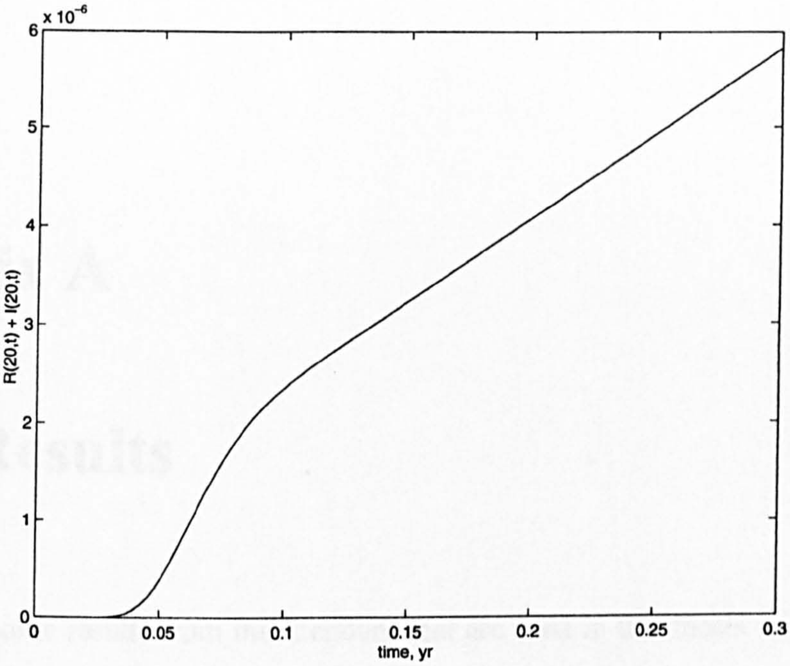


Figure 5.15: The total infected fox population density at the boundary of the spatial domain for the combined strategy.

for the combined strategy as can be seen in Figure 5.15. The population density at the boundary remains much lower than previously and the gradient of the density is much shallower indicating less flux of infected foxes across the boundary.

It appears from these simulations that, when employing only one strategy it is better to use a program of culling, whereas it would appear from this last case that a combined strategy works much better. Hence a program of vaccination can be made more effective through the addition of a small cull.

Appendix A

Useful Results

In this appendix some results from the literature that are used in this thesis are listed for convenience. In the proof of the main result of this thesis (Theorem 2.2.10) the following version of the Contraction Mapping Theorem from Collatz (1966) is used to show the existence of a fixed point.

Theorem A.1. *Suppose $\varphi : W \longrightarrow W$, where W is a Banach space, satisfies*

$$\|\varphi x - \varphi y\| \leq k\|x - y\|, \qquad 0 \leq k < 1$$

where k is a constant, for each $x, y \in D$, a subset of W. If both the ball

$$S = \left\{ w \in w : \|w - w_1\| \leq \frac{k}{1-k}\|w_1 - w_0\| \right\}$$

and w_0 lie in D then the iterative process

$$w_{n+1} = \varphi w_n$$

converges to a unique solution in D.

The following version of Gronwall's Lemma from Curtain and Zwart (1995) is used in Chapter 3.

Lemma A.2. *Let $\alpha(\cdot) \in L^1(0, T; \mathbb{R})$ with $\alpha(T) \geq 0$. If $z(\cdot) \in L^\infty(0, T; \mathbb{R})$ satisfies, for some $\beta \geq 0$,*

$$z(t) \leq \beta + \int_0^t \alpha(s) z(s)\, ds$$

then

$$z(t) \leq \beta \exp\left(\int_0^t \alpha(s)\, ds\right).$$

Appendix B

Source Code for Numerical Simulations

The programs listed in this appendix have been written by the author for simulating the effect and control of an outbreak of rabies in a one-dimensional example. The functions realft(), four1(), cosft() for performing (Fast) Fourier Transform calculations are taken from Press et al. (1992).

The programs are based on example source code for the M.Sc. course "Numerical methods for PDEs" given by Dr. Dwight Barkley at the University of Warwick, 1997. The programs are based on a pseudospectral procedure using a combined Crank-Nicolson (Ames, 1977) and (second order) Adams-Bashford (Press et al., 1992) method.

On the spatial domain $[0, L]$ consider the following differential equation

$$\dot{z} = \mathcal{L}z + \mathcal{N}(z), \qquad z(0) = z_0$$

where $\mathcal{L}$ is a linear operator and $\mathcal{N}$ a nonlinear one. Writing z in terms of cosine functions gives

$$z(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

it is clear that z can be approximated by the $N+1$ dimensional vector given by

$$\underline{a} = \begin{pmatrix} a_0 \\ \vdots \\ a_N \end{pmatrix}.$$

If, for each $n \in \mathbb{N}$, $\cos\left(\frac{n\pi\cdot}{L}\right)$ is an eigenvector of $\mathcal{L}$ then the linear problem can be easily treated as

$$\mathcal{L}^N \underline{a} = \begin{pmatrix} \lambda_0 a_0 \\ \vdots \\ \lambda_N a_N \end{pmatrix}$$

where $\{\lambda_n\}$ are the eigenvalues. However the nonlinear operator maybe difficult to express for this spectral approximation of z. Hence the nonlinearity will be dealt with for the discrete approximation of z given by

$$\underline{z} = \begin{pmatrix} z_0 \\ \vdots \\ z_j \\ \vdots \\ z_N \end{pmatrix} = \begin{pmatrix} z(0) \\ \vdots \\ z(jh_x) \\ \vdots \\ z(L) \end{pmatrix}$$

where $h_x = L/N$.

Example B.3. Suppose that $\mathcal{L}z = \partial^2 z/\partial x^2$ and $\mathcal{N}(z) = z^2$. Then

$$\mathcal{L}^N \underline{a} = \begin{pmatrix} 0 \\ \frac{\pi^2}{L^2} a_1 \\ \vdots \\ \frac{N^2\pi^2}{L^2} a_N \end{pmatrix}$$

and

$$\mathcal{N}^N(\underline{z}) = \begin{pmatrix} z_0^2 \\ \vdots \\ z_N^2 \end{pmatrix}.$$

The linear and nonlinear parts of the partial differential equation can be combined to form a procedure for time-stepping the solution. Suppose that the time step has length Δt and that the solution z is approximated at $t = n\Delta t$ in discrete space by $\underline{z}^n$ and spectral space by $\underline{a}^n$. The method for generating the approximations at $t = (n+1)\Delta t$ is as follows: The nonlinearity is treated in discrete space using the second order Adams-Bashford scheme given by

$$\underline{z}' = \frac{\Delta t}{2}\left(3\mathcal{N}^N(\underline{z}^n) - \mathcal{N}^N(\underline{z}^{n-1})\right).$$

A discrete cosine transform is then applied to $\underline{z}'$ to give the spectral representation $\underline{z}^*$. Then the spectral approximation $\underline{a}^{n+1}$ is calculated by the scheme

$$\underline{a}^{n+1} = \left(I - \frac{\Delta t}{2}\mathcal{L}^N\right)^{-1}\left[\left(I + \frac{\Delta t}{2}\mathcal{L}^N\right)\underline{a}^n + \underline{z}^*\right].$$

The discrete (inverse) cosine transform is then applied to give the discrete approximation $\underline{z}^{n+1}$.

All of the programs are written in C and have a common header file rabid.h, and source file rabid.c. The common source file contains those functions required by all of the programs to perform a time-step. Each time-step is divided into two parts: the first corresponds to the second order Adams-Bashford approximation of the nonlinear terms; and the second to the actual time-stepping using a Crank-Nicolson scheme. The file also contains functions for determining the reaction terms for each of the classes of fox.

The file also contains a function for determining the value of a particular distribution of the form (5.25) at a specified point in space with specified arguments μ_1 and μ_2. This function is used for setting the initial density of rabid foxes and the target density y_d.

The first program phi.c calculates an approximation of the linear operator $\phi = CU(T,0)B$ for a specified initial control. The approximation of ϕ is output in a form that can be easily read into MATLAB to calculate the pseudoinverse. This programs works by solving the differential equation given by

$$\dot{h}(t) = (A + P(t))h(t), \qquad h(0) = Bu \tag{B.1}$$

with output

$$y = Ch(T),$$

where the operators are as given in Chapter 5. Since the operator $P(t)$ depends on the trajectory $z'(\cdot)$ corresponding to the initial control u' the program calculates the former after the latter has been input. Let $\mathcal{L} = CU(T,0)B$, where $U(t,s)$ is the mild evolution operator with generator $A + P(\cdot)$.

If the number of grid-points (or Fourier coefficients) is $N+1$ then letting $u = e_n$, the n^{th} vector of the standard basis of $\mathbb{R}^{N+1}$, the approximation of the output of (B.1) is

$$y^N = \mathcal{L}^N e_n.$$

Hence the output y^N is the n^{th} column of $\mathcal{L}^N$. Therefore by using each of the vectors e_n in the initial condition the $(N+1) \times (N+1)$ matrix approximation, in discrete space, of $\mathcal{L}$ is determined.

The output of the program phi.c is then entered into MATLAB to calculate the generalised inverse $(\mathcal{L}^N)^\dagger$. This is then read into the second program spect.c which uses the adaptive scheme of Section 2.3 to calculate the desired control based on a specified target density. The target is specified by (5.25) with the parameters μ_1 and μ_2.

Only uniform distributions are considered for the initial control in this thesis and so if $u'(x) = \lambda$, the initial condition for the susceptible class of fox is updated by

$$s((n+1)\Delta t) = (1-\lambda)K - u_n - (\mathcal{L}^N)^\dagger (y_d - y_n) = (1-\lambda)K - \sum_{k=1}^{n} (\mathcal{L}^N)^\dagger (y_d - y_k)$$

where K is the environmental carrying capacity, y_n is the output after the n^{th} iteration, and $u_0 = 0$.

The control output by spect.c is then applied in the final program rabies.c to produce the data files used to generate the plots in Chapter 5. The $N+1$ dimensional vector that approximates $s(t)$ (or $r(t)$) is output by the program at intervals of specified length δt.

Program listings

```
/*-------------------------------------------------------------------*
 *                                                                   *
 *                Header file for rabies programs                    *
 *                                                                   *
 *-------------------------------------------------------------------*/

/*-------------------------- Definitions ----------------------------*/

#define X(j)       ((j)*h_x)        /*  Convert to spatial coord */
#define NJ         64               /*  Number of grid points (MUST be an
                                        integer power of 2 for FFT) */
#define     PI    3.141592653589793
#define     SWAP(a,b) tempr=(a);(a)=(b);(b)=tempr

#define T      0.3
#define DT     1.e-4

/* Model parameters [Taken from Anderson et al. (1981)] */

#define AA       1.              /* avg birth rate  */
#define BB       0.5             /* avg death rate  */
#define DD       200.            /* diffusion coefficient */
#define KK       2.              /* carrying capacity */
#define ALPHA    365.25/5.       /* [avg duration for clinical disease]^-1 */
#define BETA     80.             /* disease transmission coefficient  */
#define SIGMA    365.25/28.      /* [avg incubation period]^-1 */

/* Test features */

#define TEST      0

/*----------------------- Shared Function Prototypes ------------------*/

/* Normal distribution for i.c. and updator */

double norm    (double z, double c, double b) ;  /* Need this for i.c. */

/*  Functions needed for timestepping original dynamics z=f(z)  */

void Step     (double *s, double *q, double *r, double *v, double *ar,
               double h_x, int k) ;      /* Function for timestep */

void Step_ab   (double *s, double *q, double *r, double *v,
                int k) ;          /* Adams-Bashford part of step */

void Step_sp   (double *r, double *ar, double h_x, int k) ;
                                       /* Implicit part of step */

/*  Reaction terms for original dynamics  */

double f_react    (double  s, double q, double r, double v) ; /*  For s  */
double g_react    (double  s, double q, double r, double v) ; /*  For q  */
double h_react    (double  s, double q, double r, double v) ; /*  For r  */
double p_react    (double  s, double q, double r, double v) ; /*  For v  */

/* Numerical Recipes routines with double precision data */

void cosft (double *y, int n) ;  /*  Cosine Transform */

void realft  (double data[], unsigned long n, int isign) ; /* Real FFT */

void four1 (double data[], unsigned long nn, int isign) ;
```

```
/*-----------------------------------------------------------------*
 * The shared functions between the rabies programs                *
 *-----------------------------------------------------------------*/

/*-------------------- Include Headers ---------------------------*/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "rabid.h"

/*---------------------- Step ---------------------------------*/

void Step (double *s, double *q, double *r, double *v,
           double *ar, double h_x, int k)
    /* Function for taking one timestep using Crank-Nicolson
       and Adams-Bashford schemes                                */
{
  Step_ab (s, q, r, v, k) ;
  Step_sp (r, ar, h_x, k) ;
}
/*---------------------- Step_ab -------------------------------*/

 void Step_ab (double *s, double *q, double *r, double *v,
               int k)
     /*  2nd order Adams-Bashforth terms  */
{
  double f_new, g_new, h_new, p_new ;
  double dt_o_2 = DT / (2. - DT) ;
  static double ns[NJ+1], nq[NJ+1], nr[NJ+1], nv[NJ+1];
  int j ;

  if ( k )
    {
      /* If first_time then cannot take a 2nd order step. By setting old *
       * values to current values, the reaction terms are stepped by      *
       * first-order (Explicit Euler) method the first step.  All other  *
       * steps are second order                                           */

      for ( j=0; j<NJ+1; j++)
        {
          ns[j] = f_react(s[j], q[j], r[j], v[j]) ;
          nq[j] = g_react(s[j], q[j], r[j], v[j]) ;
          nr[j] = h_react(s[j], q[j], r[j], v[j]) ;
          nv[j] = p_react(s[j], q[j], r[j], v[j]) ;
        }
    }

  for ( j=0; j<NJ+1; j++)
    {
      f_new = f_react(s[j], q[j], r[j], v[j]) ;
      g_new = g_react(s[j], q[j], r[j], v[j]) ;
      h_new = h_react(s[j], q[j], r[j], v[j]) ;
      p_new = p_react(s[j], q[j], r[j], v[j]) ;

      s[j] += dt_o_2 * (3.*f_new - ns[j]) ;
      q[j] += dt_o_2 * (3.*g_new - nq[j]) ;
      r[j]  = (DT / 2.) * (3.*h_new - nr[j]) ;
      v[j] += dt_o_2 * (3.*p_new - nv[j]) ;

      ns[j] = f_new ;
      nq[j] = g_new ;
      nr[j] = h_new ;
      nv[j] = p_new ;
    }
}

/*---------------------- Step_sp ---------------------------*/

void Step_sp (double *r, double *ar, double h_x, int k)

{
  static double forward[NJ+1], back[NJ+1] ;
  static int first_time = 1 ;
  int j ;

  if ( first_time )
    {
      double kn, dt_o2_kn2;  /*  only needed first time */

      /* Set evolution coefficients
       *   forward = ( 1. + (dt/2 * k_n^2 * DD)  )
       *   back    = 1. / ( 1. - (dt/2 * k_n^2 * DD)  )  */

      forward[0] = 1. ;
      back[0]    = 1. ;
      for (j=1; j<NJ+1; j++)
        {
          kn = (j * M_PI) / (h_x * NJ)  ;
          dt_o2_kn2 = -(DT * kn * kn) / 2. ;
          forward[j] = 1. + (dt_o2_kn2 * DD);
          back[j]   = 1. / ( 1. - (dt_o2_kn2*DD) ) ;
        }
      forward[NJ] = 0.5*forward[NJ];
      back[NJ] = 0.5*forward[NJ];

      first_time = 0;
    }

   /*  Transform to spectral space for initial amplitudes  */

#if TEST

        printf("Amplitudes\n");
        for (j=0;j<NJ+1;j++)
        {
                printf("ar[%d] = %g\n",j,ar[j]);
        }

#endif
  if (k)
  {
        cosft(ar-1,NJ);
  }
```

```
#if TEST
   for (j=0; j<NJ+1;j++)
     {
       ar[j] = (2. * ar[j]) / NJ ;
     }
   cosft(ar-1,NJ);
   for (j=0; j<NJ+1; j++)
     {
       printf("ar[%d] = %g\n",j,ar[j]);
     }
   exit(1);
#endif

   /* transform nonlinear terms to spectral space  */

   cosft(r-1,NJ) ;
   /* for (j=0; j<NJ+1;j++)
   {
       r[j] = (2. * r[j]) / NJ ;
   } */

   /* Time step amplitude  */

   for (j=0; j<NJ+1; j++)
     {
       ar[j] = back[j] * ( forward[j]*ar[j] + r[j] ) ;
     }

   for (j=0; j<NJ+1;j++)
   {
       r[j] = (2. * ar[j]) / NJ;
   }
   /* Transform back to physical space */
   cosft(r-1,NJ) ;
}

/*------------------------ f_react ------------------------*/

double f_react( double s, double q, double r, double v )

  /* Reaction terms for s  */

{
  double f ;
  double NN = s + q + r + v ;

  f = (AA - BB)*(1. - (NN/KK))*s ;

  f += AA*v - BETA*r*s ;

  return( f );
}

/*------------------------ g_react ------------------------*/

double g_react( double s, double q, double r, double v )

  /* Reaction terms for q */

{
  double g ;
  double NN = s + q + r + v ;

  g = -q*(BB + ((AA - BB)*(NN/KK))) ;

  g += BETA*r*s - SIGMA*q ;

  return( g ) ;
}

/*------------------------ h_react ------------------------*/

double h_react( double s, double q, double r, double v )

  /* Reaction terms for r */

{
  double h ;
  double NN = s + q + r + v ;

  h = -r*(BB + ((AA - BB)*(NN/KK))) ;

  h += SIGMA*q - ALPHA*r ;

  return( h ) ;
}

/*------------------------ p_react ------------------------*/

double p_react( double s, double q, double r, double v )

  /* Reaction terms for v */

{
  double p ;
  double NN = s + q + r + v ;

  p = -v*(BB + ((AA - BB)*( NN / KK ))) ;

  return( p ) ;
}

/*------------------------ norm ---------------------------*/

double norm(double z, double c, double b)

{
  double a, t_1, t_2, c_1 ;

  a = 20. ;

  t_1 = -b*(z - a)*(z - a) ;

  c_1 = c / (sqrt(2. * M_PI)) ;

  t_2 = c_1*exp(t_1) ;

  return(t_2) ;
}
```

```
/*-----------------------------------------------------------------*
 *    This program constructs a approximation of phi =             *
 *    CU(T,0)B using NJ+1 coefficients in the series expansions *
 *    (cosines only).  This is output in a form that can be        *
 *    readily imported into MATLAB to calculated the               *
 *    generalised inverse of phi.                                  *
 *-----------------------------------------------------------------*/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "rabid.h"  /*  The common header file  */

/*-------------  Specific Function prototypes  -----------------*/

/*  In the following functions UPPER case letters correspond to *
 *  the dynamics of dot(h)(t) = A(t) h(t) used to calculate phi *
 *  and lower case letters to the solution z' corresponding to  *
 *  the initial guess u'                                        */

/*  Modified functions needed to calculate the operator phi  */

void StepH    (double *s, double *q, double *r, double *v,
               double *S, double *Q, double *R, double *V, double *aR,
               double h_x, int k);

void StepH_ab (double *s, double *q, double *r, double *v,
               double *S, double *Q, double *R, double *V,
               int k) ;

/*  Reactions terms for calculating operator phi */

double F_react( double s, double q, double r, double v,
                double S, double Q, double R, double V ) ;
double G_react( double s, double q, double r, double v,
                double S, double Q, double R, double V ) ;
double H_react( double s, double q, double r, double v,
                double S, double Q, double R, double V ) ;
double P_react( double s, double q, double r, double v,
                double S, double Q, double R, double V ) ;

/*------------  MAIN  BODY  -----------------------------------*/

int main (void)

{
  double S[NJ+1], V[NJ+1], Q[NJ+1], R[NJ+1] ;  /*  components of h  */
  double s[NJ+1], v[NJ+1], q[NJ+1], r[NJ+1] ;  /*  components of z  */
  double aR[NJ+1], ar[NJ+1] ;  /*  amplitudes  */
  double  h_x, vv;  /* length between grid pts and the level
                       of the (uniform) initial control */
  double phi[NJ+1][NJ+1] ;  /* The operator (matrix) phi  */

  int nn, nsteps, i, j, k;

  FILE *fp ;

  /*  Set the length between grid pts and number of time steps  */
  h_x = 40. / NJ ;
  nsteps  = T / DT ;

  /*  Check we have an integer number of time steps  */

  if ((double) nsteps != T / DT)
  {
    printf("Error... Not an integer number of time steps \n");
    printf("%g %g\n", (float) T, (float) DT);
    exit(1);
  }

  /* Prompt user for the level of the initial control  */
  printf("Input initial control = \n");
  scanf("%lg", &vv);

  /* Write out parameters */
  printf(" Parameters : \n");
  printf(" NJ = %d, Length_x = 40, h_x = %f \n", NJ,(float) h_x ) ;
  printf(" nsteps = %d, t_f = %f, dt = %f \n", nsteps, (float)T,
          (float)DT ) ;

  /*  Each column of phi is calculated separately                 *
   *  the n+1 column of phi is obtained by choosing the initial  *
   *  state of dot(h)(t) = A(t) h(t) to be B e_n where {e_n} are *
   *  the standard basis of R^(NJ+1).                             */

  /*  Column k+1  */
  for (k=0; k<NJ+1; k++)
   {
    printf("%d \n", k);  /* Visual indicator of program progress */

    /*  Set up initial conditions for dot(h)(t) = A(t) h(t)  *
    for (j=0; j<NJ+1; j++)
      {
        S[j] = V[j] = aR[j] = R[j] = Q[j] = 0.;
      }
    /*  Input operator is B_v */
    S[k] = -1. ; V[k] = 1. ;

    /*  Input operator is B_c
    S[k] = -1. ; */

    /*  Set up initial conditions for z'(t) (initial guess) */
    for (j = 0; j < NJ+1; j++)
      {
        s[j] = KK - vv*KK ;  /*  initial control is a (spatially) */
        v[j] = vv*KK ;       /*  uniform vaccination or cull      */
        q[j] = 0. ;
        ar[j] = r[j] = norm(X(j),1.0,0.5) ;
      }

    /*  Numerical simulation of dynamics  */

    /*  The dynamics of h depend on the components of the initial *
     *  guess z' at each timestep and so the former are stepped   *
     *  before the latter                                         */

    /*  Initial step  */
    StepH( s, q, r, v, S, Q, R, V, aR, h_x, 1) ;
    Step( s, q, r, v, ar, h_x, 1) ;
```

```
    /*  The main loop for the timesteps */
    for (nn=1; nn<nsteps; nn++)
      {
       StepH(s, q, r, v, S, Q, R, V, aR, h_x, 0) ;
       Step( s, q, r, v, ar, h_x, 0) ;
      }

    /*  The output C h(T) is the k+1 column of phi */
    for (j=0; j<NJ+1; j++)
      {
       phi[j][k] = Q[j] + R[j] ;
      }
   } /*  Finished calculating k+1 column of phi  */

  /*  Output results  */

  /*  We have now calculated phi and must output it */
  fp = fopen("phi.m","w");
  fprintf(fp,"T = [");
  for (i=0; i<NJ+1; i++)
    {
     for (j=0; j<NJ+1; j++)
       {
        fprintf(fp,"%g ", phi[i][j]) ;
       }
     fprintf(fp,"\n");
    }
  fprintf(fp,"] ; \nP = pinv(T) ; \nsave phi.dat P -ascii");
  fclose(fp);

  /*  We also have Cz'(T) so output this as well */
  fp = fopen("guess.dat","w");
  fprintf(fp,"%g \n", vv) ;
  for (j=0; j<NJ+1; j++)
    {
     fprintf(fp,"%g \n", r[j] + q[j]) ;
    }
  fclose(fp);

  return(1);
}

/*--------------------- StepH --------------------------------*/

void StepH   (double *s, double *q, double *r, double *v,
              double *S, double *Q, double *R, double *V, double *aR,
              double h_x, int k)

{
  StepH_ab (s, q, r, v, S, Q, R, V, k) ;
  Step_sp (R, aR, h_x, k) ;
}

/*--------------------- StepH_ab -----------------------------*/

void StepH_ab (double *s, double *q, double *r, double *v,
               double *S, double *Q, double *R, double *V,
               int k)

     /*  This function calls those required to perform one *
      *  time step of the dynamics                         */
{
  double F_new, G_new, H_new, P_new ;
  double dt_o_2 = DT / (2. - DT) ;

  /* The following arrays store the nonlinear contributions *
   * from the previous time-step                            */
  static double nS[NJ+1], nQ[NJ+1], nR[NJ+1], nV[NJ+1];
  int j ;

  if ( k )
    {
     /* If first_time then cannot take a 2nd order step. By setting old *
      * values to current values, the reaction terms are stepped by     *
      * first-order (Explicit Euler) method the first step.  All other  *
      * steps are second order                                          */

     for ( j=0; j<NJ+1; j++)
       {
        nS[j] = F_react(s[j], q[j], r[j], v[j],
                        S[j], Q[j], R[j], V[j]) ;
        nQ[j] = G_react(s[j], q[j], r[j], v[j],
                        S[j], Q[j], R[j], V[j]) ;
        nR[j] = H_react(s[j], q[j], r[j], v[j],
                        S[j], Q[j], R[j], V[j]) ;
        nV[j] = P_react(s[j], q[j], r[j], v[j],
                        S[j], Q[j], R[j], V[j]) ;
       }
    }

  for ( j=0; j<NJ+1; j++)
    {
     F_new = F_react(s[j], q[j], r[j], v[j],
                     S[j], Q[j], R[j], V[j]) ;
     G_new = G_react(s[j], q[j], r[j], v[j],
                     S[j], Q[j], R[j], V[j]) ;
     H_new = H_react(s[j], q[j], r[j], v[j],
                     S[j], Q[j], R[j], V[j]) ;
     P_new = P_react(s[j], q[j], r[j], v[j],
                     S[j], Q[j], R[j], V[j]) ;

     S[j] += dt_o_2 * (3.*F_new - nS[j]) ;
     Q[j] += dt_o_2 * (3.*G_new - nQ[j]) ;
     R[j]  = (DT / 2.) * (3.*H_new - nR[j]) ;
     V[j] += dt_o_2 * (3.*P_new - nV[j]) ;

     /*  Update nonlinear terms from previous time-step */
     nS[j] = F_new ;
     nQ[j] = G_new ;
     nR[j] = H_new ;
     nV[j] = P_new ;
    }
}

/*--------------------- F_react ------------------------------*/

double F_react( double s, double q, double r, double v,
                double S, double Q, double R, double V )
```

```
    /* Reaction terms for S  */

  {
    double F ;
    double nn = s + q + r + v ;
    double NN = S + Q + R + V ;

    F = (AA - BB)*((1. - (nn/KK))*S - (s/KK)*NN)   ;

    F += - BETA*r*S - BETA*s*R + AA*V ;

    return( F );
  }

  /*--------------------- G_react ------------------------------*/

  double G_react( double s, double q, double r, double v,
                  double S, double Q, double R, double V )

    /* Reaction terms for Q */

  {
    double G ;
    double nn = s + q + r + v ;
    double NN = S + Q + R + V ;

    G = -1.*(BB + (AA - BB)*(nn/KK))*Q - (AA - BB)*(q/KK)*NN ;

    G += BETA*S*r + BETA*R*s - SIGMA*Q ;

    return( G ) ;
  }

  /*--------------------- H_react ------------------------------*/

  double H_react( double s, double q, double r, double v,
                  double S, double Q, double R, double V )

    /* Reaction terms for R */

  {
    double H ;
    double nn = s + q + r + v ;
    double NN = S + Q + R + V ;

    H = -1.*(BB + (AA - BB)*(nn/KK))*R - (AA - BB)*(r/KK)*NN ;

    H += SIGMA*Q - ALPHA*R ;

    return( H ) ;
  }

  /*--------------------- P_react ------------------------------*/

  double P_react( double s, double q, double r, double v,
                  double S, double Q, double R, double V )

    /* Reaction terms for V */

  {
    double P ;
    double nn = s + q + r + v ;
    double NN = S + Q + R + V ;

    P = -1.*(BB + (AA - BB)*(nn/KK))*V ;

    P += -1.*(AA - BB)*(v/KK)*NN ;

    return( P ) ;
  }

  /*--------------------- The End ------------------------------*/
```

```c
/*-------------------------------------------------------------------*
 *                                                                   *
 *     Simulates the dynamics of rabies in a fox population          *
 *        using Crank-Nicolson / Adams-Bashforth Method.             *
 *                       Spectral Method.                            *
 *                                                                   *
 *-------------------------------------------------------------------*/

/*--------------------- Include Headers ---------------------------*/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "rabid.h"

/*------------------ Specific Function Prototypes -----------------*/

double dot (double *x, double *y);
                                /*  Calculates the inner product x.y */

void writem (double *x) ;

void set_ic ( double *s, double *q, double *r, double *v,
              double *ar, double phi[2*NJ+2][NJ+1], double *y,
              double h_x, double vv, int k)  ; /* Sets ic */

/*----------------------- Main -----------------------------------*/

int main ()

{
  double  s[NJ+1], q[NJ+1], r[NJ+1], v[NJ+1] ; /* The components of z */
  double  ar[NJ+1], y[NJ+1];  /* The amplitudes of r and the output */
  double  Cz[NJ+1];  /*  the value of the initial guess  */
  double  h_x, xlen, vv, yd1, yd2 ;

  double phi[NJ+1][NJ+1]  ;  /* The operator (matrix) phi   */

  int nn, nsteps, i, j, kk, stop ;

  FILE *fp ;

  xlen      = 40. ;             /*  Length of spatial domain */
  h_x       = xlen / NJ ;       /*  Length between grid points  */

  nsteps  = T / DT ;

  /*  Check we have an integer number of time steps   */

  if ((double) nsteps != T / DT)
  {
    printf("Error... Not an integer number of time steps \n");
    printf("%g %g\n", (float)T, (float)DT);
    exit(1);
  }

  /* Input target y_d */
  printf("Input y_d.  Height =  \n");
  scanf("%lg", &yd1);
  printf("Width = \n");
  scanf("%lg", &yd2);
  printf("Input no. of iterations = \n");
  scanf("%d", &stop);

  /*  Write out parameters  */

  printf(" Parameters : \n");
  printf("    NJ = %d,  Length_x = %g,  h_x = %f  \n", NJ, (float)xlen,
                (float) h_x ) ;
  printf(" nsteps = %d, t_f = %f, dt = %f \n", nsteps, (float)T,
             (float)DT ) ;

  /*  Get all the relevant stuff from phi.c  */
  fp = fopen("guess.dat","r");
  fscanf(fp,"%lg \n", &vv);
  for (j=0; j<NJ+1; j++)
    {
      fscanf(fp,"%lg \n", &Cz[j]) ;
    }
  fclose(fp);

  fp = fopen("phi.dat","r");
  for (i=0; i<NJ+1; i++)
    {
      for (j=0; j<NJ; j++)
        {
          fscanf(fp,"%lg ", &phi[i][j]) ;
        }
      fscanf(fp,"%lg \n", &phi[i][NJ]);
    }
  fclose(fp);

  /* The iterative loop where the control will be updated via *
   * the adaptive scheme                                      */
  for (kk=0; kk<stop; kk++)
    {
      /*  Set initial conditions  */
      if (kk == stop-1)
        {
          set_ic(s, q, r, v, ar, phi, y, h_x, vv, 2) ;

          /* In the last loop of the iterative scheme the control *
           * is output in a form that can be used in rabies.c     */
          fp = fopen("ic.m","w");
          for (j=0; j<NJ+1; j++)
            {
              fprintf(fp,"%g \n", v[j]); /* Vaccination */
              /* fprintf(fp,"%g \n", s[j]) ; Cull */
            }
          fclose(fp);
        }
      else if (kk == 0)
        {
          set_ic(s, q, r, v, ar, 0, 0, h_x, vv, 0) ;
        }
      else
        {
          set_ic(s, q, r, v, ar, phi, y, h_x, vv, 1) ;
        }
```

```
      /*  Main Loop for timesteps */
      Step( s, q, r, v, ar, h_x, 1);

      for (nn=1; nn<nsteps; nn++)
        {
          Step( s, q, r, v, ar, h_x, 0) ;
        }

      if (kk == stop-1)
        {
          for (j=0; j<NJ+1; j++)
            {
              y[j] = r[j] + q[j] ;
            }
        }
      else
        {
          /* Set y_d - y_(k+1)  */
          for (j=0; j<NJ+1; j++)
            {
              y[j] = norm(X(j),yd1,yd2) - r[j] - q[j] ;
            }
        }

      printf("Output after %d = %g \n", kk+1, y[NJ/2]);

    }

  /* Write out the final results */

  fp = fopen("spect.out","w");
  for( j=0; j<NJ+1; j++)
  {
    fprintf(fp,"%g  %g  %g \n", (float)X(j), (float) y[j],
                               (float) Cz[j] ) ;
  }
  fclose(fp);

  return(1);
}

/*---------------------- Set Initial Conditions ----------------*/

void set_ic ( double *s, double *q, double *r, double *v,
              double *ar, double phi[][NJ+1], double *y,
              double h_x, double vv, int k)

      /*  Updates and sets the initial condition z(0)  */

{
  static double ic[NJ+1] ;
  int i,j ;

  for (j=0; j<NJ+1; j++)
    {
      /* In every loop other than the first we add    *
       * phi^(dagger) (y_d - y_k) to existing control */
      if ( k )
        {
          ic[j] += dot(phi[j],y) ;
        }
      else /* otherwise u_0 = 0 */
        {
          ic[j] = 0. ;
        }
      s[j] = KK - vv*KK - ic[j] ;
      q[j] = 0. ;
      ar[j] = r[j] = norm(X(j),1.0,0.5) ;
      v[j] = vv*KK + ic[j] ; /*  Vaccination */
      /* v[j] = 0. ;  Cull  */
    }

  /*  Combined control strategy
  for (j=0; j<NJ+1; j++)
    {
      i = NJ + 1 + j ;
      if (k)
        {
          ic[i] += dot(phi[i],y);
        }
      else
        {
          ic[i] = 0. ;
        }
      s[j] += - ic[i];
    }  */

  if (k == 2) writem(ic);
}

/*------------------------ Inner Product ---------------------------*/

double dot (double *x, double *y)

     /*  This function calculates the inner product of x and y  */

{
  double product = 0. ;
  int j;

  for (j=0; j<NJ+1; j++)
    {
      product += x[j] * y[j] ;
    }

  return( product  );
}

/*------------------------ Writem ---------------------------------*/

void writem ( double *x )

{
  int j ;

  for (j=0; j<NJ+1; j++)
    {
      printf("u[%d] = %g ", j, x[j]);
    }
  printf("\n");
}
```

```
/*---------------------------------------------------------------*
 *                                                               *
 *     Simulates 1d Diffusion-reaction Equation governing the    *
 *     temporal and spatial evolution of rabies among foxes      *
 *     using the Crank-Nicolson / Adams-Bashford Method.         *
 *                                                               *
 *---------------------------------------------------------------*/

#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include "rabid.h"

#define TSPT     0   /* spatial pt for output time series */

/*-------------  Specific Function Prototypes  ------------------*/

/*  These functions produce a plot on the screen while the *
 *  program is in progress                                  */
void plot        (double *u, double *v, int nj);
void plot_ini    (int nj, double max, double min);
int keyboard_chk();
/*---------------------------------------------------------------*/

int main ()

{
  double  h_x, xlen, tmp;
  double  s[NJ+1], q[NJ+1], r[NJ+1], v[NJ+1], r_10[NJ+1], ar[NJ+1] ;
  int     nn, dt, nsteps, j, final ;
  FILE *fp;
  /* Read, set and write out problem parameters */

  /* Set the length between outputs of the program */
  printf("dt = ?");
  scanf("%d", &dt);

  /* If we are checking the boundary condition this is output *
   * otherwise after each period of dt the level of s or r is */
  printf("Check boundary? (0 No, 1 Yes) \n");
  scanf("%d", &final);

  /* Set simulation parameters */
  xlen     = 40. ;
  h_x      = xlen / NJ ;
  nsteps   = T / DT ;

  /* Check we have an integer number of time steps */
  if ((double) nsteps != T / DT)
  {
    printf("Error... Not an integer number of time steps \n");
    printf("%d %g\n", nsteps, T);
    exit(1);
  }
  /* Output simulation parameters */
  printf(" Parameters : \n");
  printf("   NJ = %d,  Length_x = %g,  h_x = %f  \n", NJ, xlen, h_x ) ;
  printf("   nsteps = %d,  t_f = %f,  DT = %f,  dt = %d \n",
          nsteps, T, DT, dt ) ;

/*  Set initial conditions   */

/* The initial (given) distribution z_0 */
for (j=0; j<NJ+1; j++)
  {
    s[j] = KK;
    v[j] = 0. ;
    q[j] = 0. ;
    ar[j] = r[j] = norm(X(j),1.0,0.5);
  }

/* Now for the part affected by the control, namely Bu *
 * This is read in from a file produced by spect.c the *
 * program for calculating the control                 */
fp = fopen("ic.m","r");
for (j=0; j<NJ+1; j++)
  {
    tmp=0. ;
    fscanf(fp,"%lg \n", &tmp);
    /* vaccination */
    s[j] += -tmp ;
    v[j] = tmp ;
    /* cull
    s[j] = tmp ; */
  }
fclose(fp) ;

plot_ini(NJ+1, KK, -0.01);

/*  Main Loop  */

/*  Open appropriate file to output results into */
if ( final )
  {
    fp = fopen("endpt.out","w");
  }
else
  {
    fp = fopen("res.out","w");
    fprintf(fp,"y = [");
  }

/*  The initial step  */
if ( final )
      {
        fprintf(fp,"%g  \n", r[TSPT] + q[TSPT]);
      }
else
      {
        for (j=0; j<NJ+1;j++)
              {
                        fprintf(fp,"%g ",s[j]);
              }
        fprintf(fp,"\n");
      }
Step( s, q, r, v, ar, h_x, 1);
```

```
/*  The main loop  */
for (nn=1; nn<nsteps; nn++)
  {
    /* Output results after each period of dt */
    if (nn / dt == (double) nn/dt)
      {
        if ( final )
          {
            fprintf(fp,"%g  \n", r[TSPT] + q[TSPT]);
          }
        else
          {
            for (j=0; j<NJ+1;j++)
              {
                    fprintf(fp,"%g ",s[j]);
              }
            fprintf(fp,"\n");
          }
      }

    /* Step dynamics */
    Step( s, q, r, v, ar, h_x, 0) ;
    for (j=0;j<NJ+1;j++)
      {
        r_10[j] = 10 * r[j] ;
      }
    plot(r_10, s, NJ+1);
    if(keyboard_chk())
      {
            break ;
      }

  } /* End of main loop */

/* Close appropriate files */
if ( final )
  {
    fprintf(fp,"%g  \n", r[TSPT] + q[TSPT]);
    fclose(fp);
  }
else
  {
     for (j=0; j<NJ+1;j++)
      {
            fprintf(fp,"%g ",s[j]);
      }
      fprintf(fp,"] ; \n");
      fclose(fp);
  }

/*  Now for the final output  */

fp = fopen("outpt.res","w");
fprintf(fp,"y = [");
for (j=0; j<NJ+1; j++)
  {
     fprintf(fp,"%g\n",r[j]+q[j]);
  }
fprintf(fp,"] ;");
fclose(fp);

return(1);

}
/*-----------------The End---------------------------------------*/
```

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