University of Warwick

Department of Statistics

ON THE ADDITION OF FURTHER TREATMENTS TO LATIN SQUARE DESIGNS

By

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Submitted: May 1997

Revised: August 1997

pp. i - xiii, 1 - 152

This thesis is submitted to the University of Warwick as a partial fulfilment for the award of the degree of Doctor of Philosophy.

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To my Father & Mother

with love

Three DS HD 3.5" computer diskettes are attached to this thesis.

LIST OF TABLES

Acknowledgements

I would like express my deep appreciation and gratitude to my friends, for their faith that I would successfully complete my work and their constant support giving me additional strength over this research course. Very special thanks go to my dear friend Demettra Pitta, who has been one of the most special characters I came across during my stay at this University. I will treasure the memory of the talks and everything we shared for life. She is very much appreciated, respected, and loved by the author.

I am very pleased to thank the members of the Housekeeping and Residential staff at Claycroft Halls, who are very special people who greatly contributed to making my stay on the University Campus a memorable one.

I owe special thanks to G. H. Freeman, PhD, who is very much respected and appreciated by the writer: not only as a human being but also because he is extremely well-qualified in the subject dealt with here. His interest in the topic and constant enquiries offered me overwhelming encouragement. It is indeed a pleasure to thank him.

I wish to express my gratitude and indebtedness to my former teachers and mentors (Universidad Autónoma de Yucatán, México), Mtra. Lucy Torres and Mtro. Eric Avila, as well as my lecturers in computer sciences Ing. Victor Villanueva and Fis. Juan Herrera. They taught me modern algebra and encouraged the development of my skills in the analysis, design and structure of computer algorithms.

Finally, I owe thanks also to the *Consejo Nacional de Ciencia y Technología* (CONACyT) for (i) following up and keeping record of the chronological progress of the investigation through the *Departamento de seguimiento académico* and (ii) providing financial assistance under grant 86014/(94-97) to enable the author undertake this piece of work.

Proof of Lemma 6.2 has been given by Denes and Keedwell [08] in a slightly different way.

J.A. GONGORA-ALDAZ

"Better thine own work is,

though done with fault,

than doing other's work,

ev'n excellently"

From: the Bhagavad-Gita

The author does not intend this manuscript to be a 'how-to-do-it' set of instructions but a source of ideas and questions to the free, creative thinker interested in the subject. Hopefully it will arise questions whose answers the reader should seek within him/her-self, and Ideas which may well inspire further development.

"Select what is useful and develop from there" *Bruce Lee*

Declaration

I, Mr Jose Antonio GONGORA-ALDAZ, declare that the creative use of multidisciplinary tools (Modern Algebra, Computing Science and Statistics) in problem-solving in order to produce those results, known as well as new, comprised in this thesis is entirely my own work. Indeed, information reported here was obtained in complete isolation and independently from that which other people may be doing on the topic. Therefore, any mistakes contained in here are mine only.

The importance of the results gathered over the candidate's research period made imperative to have a first edition of this thesis published as [33]. This has been used as the corner stone for some joint publications by J.A. GONGORA-ALDAZ and G.H. FREEMAN on similar arrays to the ones studied here. A follow up of the progress of the investigation is reflected in [30, 31, 32, 33]. Information in chapters 6 and 7 was presented at the 16th British Combinatorial Conference 1997. Two papers reporting a small amount of the information presented here have been submitted for publication in internationally reputed journals. A second edition of [33] will be published in the near future.

The amount of data produced in this research project is so vast as to be impossible to print within the manuscript. As an alternative, three DS HD 3.5" microdisks containing valuable information on the addition of treatments to Latin Squares of size six are attached to this thesis.

This thesis has not been submitted for a degree in any other university.

Sincerely,

J.A. GONGORA-ALDAZ

Summary

Statisticians have made use of Latin Squares for randomized trials in the design of comparative experiments since the 1920s. Through *cross-disciplinary* use of Group theory, Statistics and Computing Science the author looks at the applications of the Latin Square as row-column design for scientific comparative experiments.

The writer presents his argument, based on likelihood theory, for an F-test on Latin Square designs. A distinction between the combinatorial object and the row-column design known as the Latin Square is explicitly presented for the *first time.*

Using statistical properties together with the tools of group actions on sets of block designs, the author brings *new evidence* to bear on well known issues such as (i) non-existence of two mutually orthogonal Latin Squares of size six and (ii) enumeration and classification of combinatorial layouts obtainable from superimposing two and three symbols on Latin Squares of size six.

The possibility for devising non-parametric computer-intensive permutation tests in statistical experiments designed under 2 or 3 blocking constraints seems to have been *explored* by the author over the candidate's research period — See Appendix V: Part 2 — *for the first time.*

The *discovery* that a projective plane does not determine all F/Z-inequivalent complete sets of Mutually Orthogonal Latin Squares is proved by fully enumerating the possibilities for those of size $p \leq 7$.

The *discovery* of thousands of representatives of a class of balanced superimpositions of four treatments on Latin Squares of size six through a systematic computer search is reported. These results were presented at the 16th British Combinatorial Conference 1997.

Indications of openings for further research are given at the end of the manuscript.

Preface

In practical experimentation research workers study the mineral, vegetable, and animal kingdoms. Their approach generally involves careful observations on small groups of members of a given species in a kingdom of interest. The author appeals to likelihood statistical theories in order to justify a classical F-test for practical scientific investigation.

The writer presents the statistical theories found in the literature as well as the most known criticisms and controversies associated with them, not to mention certain ingenious alternatives which statisticians have devised to improve their tools for the development of science.

The overall aim of this thesis is twofold:(i) to show the mathematical foundations behind scientific experiments designed so that treatments under comparison are arranged under two or three blocking constraints, and (ii) to appeal to likelihood statistical theory in order to provide research workers with a reasonable and practical quantitative tool for significance testing on treatments. The primary objective is to set up the mathematical foundations underlying some combinatorial arrangements suggested [13, 24, 25] for use in the comparison of a new set of treatments on experimental units still affected by an earlier set. Emphasis is given to the case when the original set of treatments is arranged as a Latin Square design. Hopefully, this thesis will be used as a source of questions and ideas: Questions whose answers the reader should seek within him/her-self and Ideas which may well inspire further development.

The material of the thesis is presented in eight chapters. The first section introduces the ideas of successive experimentation on a set of experimental units. Elementary concepts are presented which are helpful to distinguish amongst combinatorial arrays suggested as useful in the design of statistical comparative experiments. In addition, the writer has tried to give as exact and definite a description of the principles of experimental design as possible. Chapter 2 intends to justify the use of the statistical F-test for comparative experiments in Latin Square designs. Furthermore, the ideas underlying a suggested non-parametric permutation test [01, 36] are presented in considerable detail.

In Chapter 3 some general mathematical concepts of modern algebra are presented, as well as computer algorithms, which the author found helpful for enumerating and categorizing Latin Squares.

Combinatorial and Statistical properties are brought together in Chapter 4 in order to study the possibility and practicality of devising an exact non-parametric test for Latin Square Designs. Examples underlying a computer-intensive permutation test [01] for treatments on Latin Square designs are presented as didactically as possible. Finally, problems associated with the above-mentioned technique are pointed out, followed by certain ingenious alternatives to overcome such difficulties that have been proposed.

Chapter 5 reviews the literature with regard to the problem of successive experimentation on Latin Square Designs. Emphasis is given to describing previous methods for the construction of certain combinatorial arrangements which satisfy statistical properties.

Chapter 6 is devoted to providing the reader with combinatorial arrays recommended as useful layouts for sequential experimentation on Latin Square designs. The case of orthogonal superimpositions is studied in detail. The discovery that a projective plane does not determine all F/Z-inequivalent sets of Mutually Orthogonal Latin Squares is proved and presented in writing for the first time.

In Chapter 7 the discovery, enumeration results and examples are reported of very many new mathematical arrangements obtainable from the addition of four symbols to Latin Squares of size six in such a manner that the set of superimposed symbols is balanced with respect to each of the three blocking constraints.

The thesis concludes with some constructive remarks in Chapter 8 about the use of the Latin Square as row-column design in statistical applications. A summary of main results in the thesis and indications of openings for further research are given at the end of the chapter.

1 Introduction

The Oxford Dictionary defines **science** and **scientific** as "the branch of knowledge requiring systematic study and method, especially one of those dealing with substances, animals and vegetable life, and natural laws" and "Using careful and systematic study, observations, and tests of conclusions etc.", respectively.

Most people these days regard so called **scientific results** as something that can be relied upon. Indeed the agricultural industry, for instance, frequently introduces a new pesticide into the market by advertising that it has been shown scientifically to be more efficient than another.

The means through which conclusions in scientific research are achieved generally involve a strong basis of observation of events under controlled conditions. This is followed by some sort of quantitative analysis and interpretation of the collected data. The quantitative side of scientific research is generally relegated to the statistician or to statistical methods.

Given the high regard for science in everyday life, not to mention the belief in the validity of scientific methods, importance is given to the study of the *Design and Analysis of Experiments*: a chapter of statistics whose build up is largely attributed to Sir R. A. Fisher and F. Yates. Both were active English researchers at Rothamsted Experimental Station in England. This seems to be the birthplace of many applied statistical methods in the, otherwise considered empirical, scientific world.

A review of the statistical literature highlights some discrepancies of opinion amongst the experts in statistics with regard to the way data should be analyzed. There seem to be two main schools of statistics: the Bayesian and the Frequentist [40, §4.15]. Well-known acknowledged experts such as Fisher [19, p. 6] as well as Hinkelmann and Kempthorne [40, §5.3.2] present their arguments in favour of statistical analysis of data which complies with the basic *assumptions* made by the frequentist school. This author will present some statistical theories recommended by the frequentist school in considerable detail. Not only will both the classical statistical analysis of data and the currently termed 'Computer-intensive methods' in statistics [01], [36, p. 7] be presented, but also a suggested relationship [40, §6.6] between the two approaches will be examined in § 4.4.1. The author's justification for an F-test based on likelihood theory is presented in chapter 2.

In certain comparative experiments research workers frequently find that experimental material must be used for several successive trials [23]. Sequential experiments are those in which a new set of treatments has to be applied to experimental units which are still affected by an earlier set. Take for instance horticultural research: a plantation of mature trees is quite an expensive resource to be removed or replaced after a single experiment has been completed on it. Similar cases often occur in trials on long-lived animals – See Appendix IV. For descriptions of particular problems found in practice the reader may find examples in the works by Freeman [23], Hoblyn et al. [41], and Pearce and Taylor [54].

Preece [60], [61] has pointed out that terminology such as orthogonality and balance first intended to distinguish amongst combinatorial block designs in the statistical literature has become confusing or even chaotic to the newcomer. Consequently, the forthcoming § 1.1 introduces the very elementary statistical concepts upon which the present work will gradually be developed.

1.1 Elementary concepts

Let Ω denote a set of experimental units. In addition, consider a set of t symbols $T = \{ 1, 2, \ldots, t \}$ to represent t different treatments which are about to enter a comparative controlled experiment.

Fisher [17, 18, 19] argued that whenever researchers undertake the task of designing an experiment for purposes of comparing the effects of a set of treatments on a characteristic of interest, the principles of **replication**, **randomization** and **blocking** should be borne in mind.

The importance of *replication*, in other words the condition for having more than one experimental unit per treatment, is to increase precision in an experiment. This is achieved by estimating the variation due to causes out of the control of the experimenter as well as possible. In other words, the element of variation found in nature, frequently referred to as experimental error or residual variation, is best estimated from groups of experimental units under each treatment rather than from one single unit.

Randomization in the sense of **E**qually **L**ikely **R**andom **A**llocation of treatments to experimental units not only guarantees the elimination of bias in the results, but also ensures independence amongst experimental units. As a consequence, a valid estimate of residual variation is provided. This is required in order to analyze the observed variation in the characteristic of interest by using, for example, likelihood based statistical techniques.

Consider those potentially influential conditions under which the experimental units are grouped into more or less homogeneous blocks prior to a given experiment. These will be referred to as the blocking ***constraints*** in the design. The collection of elements within a blocking constraint and treatment set/factor will be termed as its ***levels***.

Blocking followed by an equally likely random allocation of treatments to experimental units within the blocks: (i) *reduces* and (ii) contributes greatly to *validating the estimate for* the residual variation.

The forthcoming definitions will be helpful to distinguish amongst the many most frequently used combinatorial arrangements in the area of Design and Analysis of Experiments.

In the following Definitions 1.1, 1.3 to 1.5, the sets

U and V are meant to be blocking constraints and/or treatment factors. The context will determine their nature.

Definition 1.1

For any two sets U, V, the **occurrence matrix** for U and V is defined as that matrix whose (i,j)-th entry represents the number of experimental units at the i-th level of U in the j-th level of V.

Definition 1.2

A matrix will be said to be **flat** if its entries are all equal to a constant.

In the following Definitions 1.3 - 1.5, let $\boldsymbol{n}_{UV}$ be the occurrence matrix for U and V. Furthermore, let $\boldsymbol{r}_U$, $\boldsymbol{r}_V$ denote column vectors of replications of each level of U and V respectively.

Notation: In the reminder of the thesis the term "iff" will be used to mean "if and only if."

Definition 1.3

U is said to be **Orthogonal** with respect to V iff

$$n_{UV} = \frac{1}{N} r_U r'_V$$

where N = Total number of experimental units.

Definition 1.4

U is said to be **Totally balanced** with respect to V iff for some positive integers λ and s,

$$n_{UV} n'_{UV} = (s - \lambda) I + \lambda J$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{J}$ is a flat matrix with unity as constant entry.

Definition 1.5

U is said to be in **Supplemented balance** with respect to V iff for some positive integers s_0, λ_0, s and λ the resulting matrix from the product n_{UV} n'_{UV} follows a pattern similar to that illustrated in Table 1.1.

s,	λ,	...,	λ,	λ,	λ_0
λ,	s,	...		λ,	λ_0
.			.	.	.
.			.	.	.
λ,		...,	s,	λ,	λ_0
λ,		...,	λ,	s,	λ_0
λ_0,	λ_0,	...,	λ_0,	λ_0,	s_0

Table 1.1 Pattern for n_{UV} n'_{UV} in Definition 1.5

In description of designs, the following notation, first introduced by Hoblyn et al. [41], is adopted. The letters O, T and S will be used to mean Orthogonality, Total balance and Supplemented balance, respectively. An order for the letters and colons will identify the relationship between the various pairs of blocking constraints and/or treatment factors. This is illustrated in the following example: suppose the original trial was a comparative experiment on experimental material whose blocking constraints were termed as rows and columns. The notation O:TO associated to the resulting combinatorial array will give the following description amongst blocking constraints and treatment sets:

- the letter O preceding the colon indicates that the second blocking constraint, columns, is orthogonal with respect to the first, rows;

- The TO symbol following the colon means that the set of treatment is orthogonal with respect to columns, and totally balanced with respect to rows.

Note that in the case of sequential experimentation on the above-mentioned design, the set of original treatments will *become* a blocking constraint for the new trial.

This thesis will mainly be concerned with designs of type O:OO, O:OO:OOO, as well as some of type O:OO:TTT.

1.2 Sequential Experimentation : Background material

The writer will now present a brief overview of the subject of successive experimentation. The mathematical and statistical tools will gradually be introduced in the subsequent chapters as they are needed.

Suppose a researcher wishes to study and compare the effects of certain different treatments on experimental material subject to a controlled experiment. Let us describe the simplest and most currently available arrays for using the same material for a new unspecified trial. The initial experiment is conducted under the following two conditions:

(i) Experimental units are divided into homogeneous equal-sized groups frequently referred to as blocks;

(ii) Treatments under comparison are randomly allocated to the experimental units in each block so that each treatment appears exactly once per block.

The aforementioned two conditions lead statisticians to refer to the arrangement so obtained as a Randomized

Complete Block Design (RCBD) [40].

Let us take for example a trial on fruit trees. A row of 16 rootstocks, ω_i for i = 1, 2, ..., 16, was divided into four homogeneous blocks containing four rootstocks each, as illustrated in Table 1.2(a). Four spraying treatments, represented by the symbols A, B, C and D, were applied in a randomized complete block design in the order shown in Table 1.2(b).

Three years later, and after examination of the fruit plantation, it was concluded that all the trees were still in good condition.

(a)	ω_1,	ω_2,	ω_3,	ω_4	ω_5,	ω_6,	ω_7,	ω_8	ω_9,	ω_{10},	ω_{11},	ω_{12}	ω_{13},	ω_{14},	ω_{15},	ω_{16}
(b)	C	B	D	A	B	A	D	C	C	D	A	B	B	A	C	D
(c)	α	γ	δ	β	δ	α	β	γ	δ	α	γ	β	α	δ	β	γ

Table 1.2 Example of a successive experiment on an RCBD

For economic reasons it was then decided to make use of the entire plantation of trees in order to compare a different set of four spraying treatments. As a consequence, the set of treatments in the first experiment now *becomes* a blocking constraint for the successive experiment. Indeed, an array for the treatments in the new experiment can be designed so that the variation due to the two blocking constraints is eliminated from the comparative analysis. Let the new treatments be represented by the greek letters α, β, γ and δ. Thus, one possible way to allocate the new treatments to the experimental units under the aforementioned conditions is the O:OO arrangement given in Table 1.2(c).

Sequential experimentation on the resulting randomized block design – from the original trial – has been discussed in considerable detail by Pearce and Taylor [54] and Hoblyn et al. [41].

Pearce and Taylor [54] seem to be the first authors to address the problems underlying sequential experimentation in long-term trials. Their paper is divided into four parts. The first part reviews problems involved in successive experimentation on perennial plants when the original trial was set up as a randomized block experiment. A description of the possible future for the original treatments is given there in general terms. Parts II and III account for the cases when all and some interactions between the two sets of treatments are retained, respectively. Finally, possibly the most important case in practical applications is presented: that of having two non-interactive sets of treatments. Emphasis on some balanced threefold classifications is made therein: those obtainable from allocating four treatments to a 6×6 layout under the two blocking constraints of a row-column design. Furthermore, the Classical statistical analysis is recommended for studying the significance of treatment differences in designs of this kind. For an enumeration of these arrays the reader is referred to [35]. The approach adopted by Pearce and Taylor [54] is very practical in essence, hence the applied statistician may find it interesting reading material.

A fuller account of the design of successive experiments for two non-interactive sets of treatments in fruit trees is due to Hoblyn et al. [41]. Their paper is divided into two main parts. The first part highlights several considerations, with regard to the initial and sequential trials, which should be borne in mind at the stage of planning the experiment. Furthermore, some layouts for successive experimentation are presented and their design well described therein. The second part is devoted to describing examples of some practical applications at a research station in England.

A thorough review, in general terms, of the problem of sequential experimentation on material which outlives more than one set of treatments was given by Freeman [23]. This author distinguishes the cases where (i) residual effects are unlikely, (ii) both residual effects and interactions are likely, and (iii) residual effects are likely but interactions unlikely. Applications of the concepts in the paper to problems found in practice are discussed.

In successive experimentation it is ideal to plan the original trial anticipating the use of the same material for new unspecified experiments. However, this study will focus on the case where both (i) an experiment *has taken* place and, (ii) a new trial will be carried out on the same material. Design of such experiments should take into account the possibility of the existence of residual effects from the original trial (i.e. effects which one set of treatments may leave on the experimental units and which may, of course, be carried over to the succeeding experiment). Residual effects could be detected in the experimental material during the following experiment or even later. Therefore, as much information as possible regarding the nature of the treatments is required. This is in order to increase the likelihood for deciding whether residual effects disappear or not after the objective of the original experiment has been achieved.

In practice, there is no guarantee that any wash-out[1] period of time will be enough for eliminating the residual effects from the experimental material. Therefore, the case of two successive experiments on the same material will be considered here. Emphasis is given to the case when each treatment in the original trial has left a residual effect.

There are problems associated with the addition of a further set of treatments to experimental units which have been initially arranged as a randomized block design [54], especially when the array devised for the new trial should satisfy certain statistical properties. The problems are inevitably more complex when the original trial is set up subject to two blocking constraints. A frequent situation is when a Latin Square (to be defined later) design has been used to compare the first set of treatments [24].

All designs for two sets of treatments to be described here assume the forthcoming two conditions:

(i) A Latin Square design was used for the original

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[1]A wash-out period of time is that intended to allow experimental units to recover from the treatment effects which outlive the original trial.

trial;

(ii) the new treatments – for the successive experiment – do not interact with the residual effects of the previous set.

1.3 Latin Squares : A brief survey

The study of the mathematical properties of so-called Latin Squares date back to the 18th century with the works of the Swiss mathematician Leonhard Euler (1707 - 1783) [12].

Definition 1.6

Let T be a set of p symbols $T = \{ 1, 2, \ldots, p \}$, and Ω be a $p{\times}p$ square array of cells. A Latin Square L has symbols $t \in T$ allocated to cells $\omega \in \Omega$ in such a way that each symbol appears exactly once in every row and column.

Note: The rows and columns represent the blocking constraints under which symbols are allocated to cells in the square array. To specify the underlying set T, L is called a Latin Square on T.

Examples of Latin Squares are given in the forthcoming Table 1.3(a, b).

The name Latin Square is attributed to Euler [12] because he made use of Latin letters as elements in the set of symbols T. He seems to have been the first author to consider the problem of superimposing a set of symbols on a Latin Square. He was studying methods to construct squares referred to as magic. These were defined in [12, p. 441] as "Quadratum magicum dici solet, cuius sellulis numeri naturales ita sunt inscripti, ut summae numerorum per omnes fascias tam horizontales quam verticales, tum vero etiam per binas diagonales prodeant inter se aequales." To this end, Euler introduced the concept of *formule directrice et de carré* [12]. In addition, he made use of pairs of Latin Squares which satisfied the property of orthogonality – to be defined later. Euler's *formule directrice* is well defined by Hedayat et al. [39] as a Transversal or Directrix of the square.

Definition 1.7

Let T be a set of p symbols { 1, 2, ..., p }, Ω a $p \times p$ square array of cells, and L denote a Latin Square on T. A collection of p cells such that these cells exhaust the set T, and every level in each of the two blocking constraints of L is represented in this collection, is said to be a Transversal or Directrix for L.

Definition 1.8

Let T be a set of p symbols { 1, 2, ..., p }, and Ω a $p \times p$ square array of cells. Then two Latin Squares L_1, L_2 on T are said to be orthogonal iff upon superimposition of one upon the other, each symbol of the superimposed square appears only once with each symbol of the square used as basis. The notation $L_1 \perp L_2$ will be used henceforth.

The arrangement obtained from such superimposition of one Latin Square upon another is referred to in the literature as a Graeco-Latin Square. This is because Euler [12, p. 445] made use of greek symbols for the set superimposed on that of Latin letters. In Table 1.3, $L_1 \perp L_2$ leads to that Graeco-Latin Square shown as (c).

Euler [12, pp. 444 - 456] made use of Graeco-Latin Squares of sizes 3, 4 and 5 in the construction of some magic squares of those sizes.

1	2	3	4	1	2	3	4	1_1	2_2	3_3	4_4
4	3	2	1	2	1	4	3	2_4	1_3	4_2	3_1
3	4	1	2	4	3	2	1	4_3	3_4	2_1	1_2
2	1	4	3	3	4	1	2	3_2	4_1	1_4	2_3
(a) Square L_1				(b) Square L_2				(c) Graeco-Latin Square			

Table 1.3 Some Latin (a,b) and Graeco-Latin (c) Squares

Making use of the Graeco-Latin Square in Table 1.3(c) the writer followed a parallel procedure to the one shown by Euler [12, p. 445] in order to produce the magic square presented in Table 1.4.

01	14	08	11
15	04	10	05
12	07	13	02
06	09	03	16

Table 1.4. A magic square of size four

The attention of the reader is drawn to the first 16 natural numbers corresponding to the number of cells in the squared array. These have been allocated to the 16 cells of the 4×4 layout so that the sum of entries per row, column or cross diagonal totals 34. This example clearly satisfies the definition for one such magic square [13, p. 441].

After extensive trials to construct Graeco-Latin Squares of size six, Euler [12, p. 383] concluded that it was impossible to have such an array. His quest for rigorous mathematical proof lead him to undertake the task of studying the more general problem of constructing Graeco-Latin Squares of size p for $p \geq 2$. Euler demonstrated that Graeco-Latin Squares can always be constructed for the cases where p is odd or if p is divisible by 4. He then raised his famous conjecture that no Graeco-Latin square of size $p = 4 \times k + 2$ for any positive integer k can be constructed.

G. Tarry [73] confirmed the non-existence of Graeco-Latin Squares of size six in 1900.

In the study of the validity of agricultural experiments R. A. Fisher [17, 18, 19] introduced the use of Latin Squares for randomized statistical trials. The concept of an equally likely random choice of arrays of this kind renewed interest in their enumeration.

From the combinatorial point of view, Fisher and Yates [20] enumerated and classified Latin Squares of size six into 22 *transformation sets* or 12 *main classes*. To this end, they introduced an operation termed *intramutation* and the concept of *intercalate* properties (to be defined later) in Latin Squares. Their work lead them to confirm the impossibility of constructing Graeco-Latin Squares of size six.

From the 1920s, further developments on the properties of Latin Squares were gradually made by the mathematical community. In 1960 rigorous mathematical demonstration of the existence of Graeco-Latin squares of all sizes except $p = 2, 6$ was given by Bose, Shrinkhande and Parker [04], a result which proved that Euler's conjecture about Graeco-Latin Squares did not hold in general.

In 1984 Bose and Manvel [03] reported applications of Latin Squares in the development of the subject of Coding in communications. A listing of references for following up leads is given in [03]. Furthermore, one year later Mandl [49] proposed orthogonal Latin Squares as a very useful resource for implementing a novel method for testing compilers in Computer Science. The increasing number of practical applications of the Latin Square has motivated statisticians, mathematicians and combinatorialists to pursue further research on these interesting and useful squares. A fuller historical account of the developments on Latin Squares can be found in the books by Dénes and Keedwell [08, 09]. Of particular interest here is the application of Latin Squares as row-column designs in randomized statistical trials [18, 19].

In chapter 2 the author presents an argument grounded in statistical likelihood properties for significance testing in comparative trials. This intends to justify clearly and reasonably the validity of treatment comparisons in Latin Square designs. The probabilistic foundations underlying validity of estimation of the residual variation – to be described later – for such tests are given there. Furthermore, the author describes how the discrete probabilistic argument gave origin to a non-parametric permutation test for treatment effects. In Chapter 3 some concepts from the algebraic theory of groups will be presented. What is more, their applications for enumerating and *categorizing* Latin Squares will be illustrated in detail. Then an exact permutation test for treatment effects on Latin Square designs is given in Chapter 4. Problems associated with such techniques followed by the most known proposed alternatives are presented as well. Having done that we will be in the position to review the combinatorial aspects involved in the problem of **sequential experimentation** on Latin Square designs in present day terminology.

2 Latin Squares as experimental designs

In this chapter a likelihood based test for Latin Square designs is presented. A constructive assessment of the assumptions – behind the mathematical equation – which lead to the analysis of variance tables is made. Furthermore, the probabilistic principle underlying a variance ratio test for treatment effects is described in considerable detail. Finally, the author describes the principle of these exact permutation tests.

2.1 Background

An investigator wishes to compare different dietary regimes on lactating dairy cows. Early empirical studies indicate that race and previous milk production of the cows are potential sources of variation. Therefore, there could for example be a functional relationship f between given diets $\tau_1, \ldots, \tau_p$, subject to the blocking constraints race and previous milk production $\mathbb{C}_1$, $\mathbb{C}_2$; Y is identified as the **milk yield** in order to have $\mathrm{Y} = f[\tau_1, \ldots, \tau_p \mid (\mathbb{C}_1, \mathbb{C}_2)]$. That is to say, variability of Y is dependent on that of treatments subject to two blocking constraints. The object of incorporating blocking constraints into the equation is to reduce variation on the dependent variable [17, 18, 19].

At its simplest, the function which models the characteristic of interest – eg. milk yield – is an additive linear relationship of the form

$$Y_{l,m(n)} = \mu_{l,m} + \tau_n \quad \text{...(2-1)}$$

where $\mu_{l,m}$ represents the average yield observed on that experimental unit located at levels *l* and *m* of the blocking constraints $\mathbb{C}_1$ and $\mathbb{C}_2$ respectively. τ_n represents the effect due to the *n*th treatment.

Not only to aid the interpretation of the $\mu_{l,m}$ in equation (2-1), but also to verify that appropriate blocking constraints have been used, the re-parameterized model in (2-2) is adopted

$$Y_{l,m(n)} = \mu + \rho_l + \kappa_m + \tau_n \quad \text{...(2-2)}$$

where $Y_{l,m(n)}$ represents the yield value observed in that experimental unit which was subjected to the influence of the *n*th treatment, and located at the *l*th and *m*th levels of the corresponding blocking constraints. In other words, μ is the overall average yield, ρ_l is the variability accounted for in the *l*th level in constraint $\mathbb{C}_1$, κ_m is the variability accounted for in the *m*th level in constraint $\mathbb{C}_2$, and τ_n represents the effect due to the *n*th treatment.

Of course, the proposed linear equation is intended to represent only the more likely influential factors that are substantial enough to model directly the characteristic of interest.

Many combinatorial arrays to fit linear additive models subject to two blocking constraints are reported in the literature. Choice amongst them depends, for instance, on (i) availability of experimental units, and (ii) the number of treatments under comparison. Experimental units subject to two mutually orthogonal blocking constraints are generally set up as $L \times M$ arrangements, where L and M denote the number of levels for each of the two blocking constraints, respectively. Some examples can be found in Fisher [18, 19], Kiefer [43, 44], Pearce [55], Potthoff [58], and Preece [62].

Let us describe one kind of design for material subjected to two mutually orthogonal blocking constraints. Amongst the designs most frequently used in practice are those which satisfy the following two conditions: (i) the number of levels in each of the two blocking constraints, and that of treatments under comparison, are all equal to a constant p, and (ii) treatments are allocated in a Latin Square array with equal probability. Thus, an emphasis on studying them as row-column designs will be given here.

To take an example, suppose sixteen experimental units ω_i for i = 1, 2, ..., 16, are arranged under two blocking constraints $\mathbf{C}_1$, $\mathbf{C}_2$ as in Table 2.1(i). Let the treatments under comparison be denoted by the symbols 1, 2, 3 and 4. Thus, one possible way to allocate the treatments to the experimental units in a Latin Square design is given in Table 2.1(ii).

The reader can see that Latin Square arrays satisfy the combinatorial property given by Definition 1.3. That is to say, Latin Squares are examples of orthogonal and equi-replicate arrangements. In the notation of the thesis, Latin Squares are of type O:OO.

		$\mathbf{C}_2$		
	ω_1,	ω_2,	ω_3,	ω_4
$\mathbf{C}_1$	ω_5,	ω_6,	ω_7,	ω_8
	ω_9,	ω_{10},	ω_{11},	ω_{12}
	ω_{13},	ω_{14},	ω_{15},	ω_{16}

(i) Layout for experimental units under $\mathbf{C}_1$, $\mathbf{C}_2$.

2	1	3	4
1	3	4	2
4	2	1	3
3	4	2	1

(ii) Latin Square array

Table 2.1 Allocation of treatments to experimental units in a Latin Square design.

The mathematical equation intended to assess variability on the characteristic of interest should take into account that variation which is inherent in nature. In other words, the equation representing the observed response should aim to isolate effects due to treatments from those due to natural variation. The latter is that which cannot be subjected to control on part of the investigator. Therefore, an element ν is incorporated in the mathematical equation as $Y = f[\tau_1, \ldots, \tau_p \mid (\mathbf{C}_1, \mathbf{C}_2), \nu]$. A further assumption that the element of natural variation ν is additive is then made. Thus, equation (2-2) takes the form

$$Y_{l,m(n)} = \mu + \rho_l + \kappa_m + \tau_n + \nu_{l,m(n)}$$

$$l = 1, 2, \ldots, p; \; m = 1, 2, \ldots, p; \; n = 1, 2, \ldots, p \qquad \ldots (2\text{-}3)$$

where $\nu_{l,m(n)}$ denotes the variability or disturbance which is found in nature. These disturbance or background-noise effects, which can take positive or negative values, are expected as an effect of natural variation.

For the sake of scientific progress the author combines his philosophical perspective to inference by introducing the following fundamental axiom: when the population under study is part of the mineral, vegetable, and animal kingdoms then inferences drawn from the observed data are justified by appealing to natural laws.

Let us move on to present a mathematical argument which shall complement the aforementioned axiom for significance testing in comparative experiments.

2.2 Assumptions of the linear model

The primary objective of this section is to obtain estimates for the parameters in equation (2-3) from the observed data. To this end, an appeal to the tool of Analysis of Variance (ANOVA) and likelihood theory is made.

2.2.1 Weak assumptions

The elementary properties underlying equation (2-3) are:

1. Additive linear functional form

   The relationship between the dependent variable Y and the independent variable τ is linear on its parameters as in equation (2-3);

2. Disturbance effects are independent and identically distributed with mean $E(\nu_{l,m(n)}) = 0$ and constant variance $Var(\nu_{l,m(n)}) = \sigma^2$, $l = 1, 2, \ldots, p$; $m = 1, 2, \ldots, p$; $n = 1, 2, \ldots, p$;

3. The parameters in the equation are assumed to satisfy: $\sum_l \rho_l = \sum_m \kappa_m = \sum_n \tau_n = 0$.

Freeman [23 to 29], in personal communication, pointed out that there are practical situations when there is a control in each blocking constraint as well as an untreated control. Hence, he argues that in such cases assumption 3 may be replaced by $\rho_1 = \kappa_1 = \tau_1 = 0$.

2.2.2 Least Squares

Estimation of the parameters in equation (2-3) is generally achieved through the fitting criterion of least squares, that is, by minimizing the sum of squares of the disturbances. The reader is referred to [46, chapter 2] for details.

Assumption 3, for instance, is required to solve the normal equations. Their solution is given by

$$\hat{\mu} = \frac{1}{p^2} \sum_{lm(n)} Y_{lm(n)} \quad ; \quad \hat{\rho}_l = \frac{1}{p} \sum_l Y_{lm(n)} - \hat{\mu}$$

$$\hat{\kappa}_m = \frac{1}{p} \sum_m Y_{lm(n)} - \hat{\mu} \quad ; \quad \hat{\tau}_n = \frac{1}{p} \sum_n Y_{lm(n)} - \hat{\mu}$$

... (2-4)

Only assumption 2 in §2.2.1 is required in order to appeal to the Gauss-Markov theorem for General Linear Models [46, p. 27]. This theorem guarantees that the least squares method has actually given us the best linear unbiased estimators for the parameters in equation (2-3).

2.2.3 Goodness of fit: An examination

The fitted response produced by equation (2-3) depends on values attributed to treatment effects. Therefore, it seems reasonable and natural to assess whether the observed variation on the characteristic of interest is due to treatments by studying their estimated effects.

Variation inherent to the dependent variable is defined in terms of deviations from the overall mean $(Y_{lm(n)} - \hat{\mu})$. Therefore, the total variation in the explained variable can well be studied from the sum of squared deviations $\sum_{lm(n)} (Y_{lm(n)} - \hat{\mu})^2$. It has been shown [19, 40] that this sum of squares can be decomposed as

$$p \sum_l (\bar{Y}_{l(\cdot)} - \hat{\mu})^2 + p \sum_m (\bar{Y}_{m(\cdot)} - \hat{\mu})^2 + p \sum_n (\bar{Y}_{..(n)} - \hat{\mu})^2 + SS(\varepsilon)$$

That is to say, $TSS = SS(\rho) + SS(\kappa) + SS(\tau) + SS(\nu)$.

Note that equation (2-3) would be good for modelling variation in the observed data if variation *TSS* in the observed yield is more largely explained from treatment variation $SS(\tau)$ rather than from residual variation $SS(\nu)$.

The attention of the reader is drawn to Table 2.2, where arithmetic calculations are summarized in the form of the ANOVA table. Note that assumption number 2 in §2.2.1 contributes greatly to calculate the expected residual variation σ^2. Furthermore, information in that table was obtained on the basis of the weak assumptions, underlying equation (2-3) only.

Source	d.f	Variation	Mean Square (MS)	Expected MS (EMS)
Between				
Constraint 1	p - 1	SS(ρ)	***	***
Constraint 2	p - 1	SS(κ)	***	***
Treatments	p - 1	SS(τ)	$\frac{SS(\tau)}{(p-1)}$	$\sigma^2 + \frac{p}{(p-1)}\sum_n \tau_n^2$
Residual	(p-1)(p-2)	SS(ν)	$\frac{SS(\nu)}{(p-1)(p-2)}$	σ^2
Total	p^2 - 1	TSS		

Table 2.2 ANOVA table for Latin Square designs

In order to test for the significance of the observed treatment differences, through the ANOVA technique, it is assumed that the disturbances follow a Normal distribution.

2.2.4 Strong assumptions

4. $v_{1,m(n)} \sim N[0, \sigma^2]$

Note that it is the **natural variation** which is assumed to follow a Normal distribution. In other words, this assumption **does not** aim at inferences about the population which comprises the experimental units. It simply cannot, because there was no random sampling of experimental units. As explained in §2.1, natural variation can take positive or negative values. Therefore, it is reasonable to make the assumption number 2 in §2.2.1, and then appeal to a Normal distribution.

The author emphasizes that the assumption introduced here affects the residual variation only. That is to say, at no time is the writer assuming random sampling of experimental units.

The immediate implication of this perspective is that the ratio F_o, referred to as the **variance ratio**, defined as the ratio of the mean square for treatments to that for natural variation, follows an F-distribution with $[(p-1),(p-1)(p-2)]$ degrees of freedom, under the null hypothesis. This ratio F_o may now be used to test the hypotheses H_o: all treatment means are equal versus H_A: not all treatment means are equal.

What the author is doing is analyzing variance in order to test equality of means. This test is grounded on likelihood theory [51, p. 437], hence equally likely allocation of treatments to experimental units is of the utmost importance.

All things considered, both a reasonable and theoretical justification for validity of estimation of residual variation has been given. Consequently, *equally likely random allocation* of treatments to plots turns out to be of the utmost importance. This is one device which has great influence in justifying validity of estimation of the residual variation.

Let us move on to review the probabilistic argument underlying validity of estimation of the residual variation, i.e. equally likely random allocation of treatments to experimental units in §2.3. In this section the writer shall also make as clear as possible the way in which the idea of a non-parametric permutation test of treatment effects came into existence.

2.3 Equally likely allocation of treatments and the permutation test principle

Suppose the study and comparison of the effects of a set of treatments indexed as 1, 2, . . ., p is of concern. In addition, suppose there are N experimental units at hand. Let the null hypothesis under test be that treatments will make an equal contribution on the characteristic of interest. For the moment, consider the case where p divides

N exactly. That is to say, the experimental units can be divided into p groups of equal size $r = N / p$.

Fisher [19] introduces the idea of a test for the null hypothesis of equality of treatment effects based on the theory of discrete probabilities. An element of randomness is incorporated in the experiment by a similar mechanism to that applied to games of chance.

Suppose treatments are allocated to experimental units at random: there will certainly be a finite number of outcomes. Furthermore, it can 'realistically' be *assumed* that the probability associated with such outcomes is the same for all of them. In other words, it can be assumed that the outcomes are **equally likely**.

In summary, random allocation of treatments to units has been suggested in order to model treatment effects on experimental units under the assumption that the outcomes are equally likely.

In order to justify that assumption – equally likely outcomes – in the experiment, it is necessary to know about all possible **different** ways of allocating the p treatments to the N experimental units. Of course, in order to distinguish between all possible allocations of treatments to units, information given by the carefully selected **statistic** should be incorporated. This should be selected so that it distinguishes the null hypothesis from the alternative. Fisher [17, §40] made use of the between-treatments sum of squares $SS(\tau)$ in the Analysis of Variance technique as test statistic. His viewpoint will be endorsed here.

The aforementioned argument was also considered in proposing a non-parametric test of equality of treatment effects. This is referred to as the permutation test [36]. Let us describe the principle underlying such a test in the forthcoming paragraphs.

Suppose an experiment is performed by randomly allocating treatments to units with equal probability, followed by careful recording of the characteristic of interest. Then, at the conclusion of the experiment, the collected data is used to calculate the observed value $SS(\tau)_o$ of the statistic $SS(\tau)$.

Let us assume that the null hypothesis of equality of treatment effects is true. Then the treatment labels, on the experimental units, contribute nothing to the observed outcome. Therefore, the corresponding discrete probability density function associated with the statistic used in the experiment can be calculated. This is achieved by considering other assignments of the treatment labels to the units. In other words, should the null hypothesis be true, then the calculated statistics for each reassignment of treatment labels are expected to have approximately the same values. Except, of course, for random fluctuation.

Let $SS(\tau)_r$ be the value of the statistic under a given reassignment of the treatment labels. Furthermore, let the event A be given by $A = \{ SS(\tau)_r \geq SS(\tau)_o \}$. The probability P(A) of the event A is given by the proportion of values the statistic $SS(\tau)_r$ takes which are larger than or equal to the observed value $SS(\tau)_o$ over all of the *different* assignments.

Clearly, the proposed exact permutation test reduces to a combinatorial problem to calculate discrete probabilities. That is to say, a problem of counting. The mathematical concepts of group theory and combinatorics which are helpful to overcome the counting problem will be presented in the following chapter.

The permutation test procedure is summarized as follows:

1.- Choose a test Statistic;

2.- Amongst all **different** ways of allocating treatments to experimental units select one at random;

3.- Calculate the value of the statistic from the observed data;

4.- Compute the probability of having a result at least as extreme as the observed one. Do so by *systematically* reallocating treatments to experimental units.

In chapter 4 several examples to illustrate the aforementioned procedure in Latin Square Designs will be presented in detail. Furthermore, problems associated with the technique as well as certain ingenious alternatives which have been proposed will be described therein.

3 Some group theoretical concepts and their applications to Latin Square arrays

This chapter is devoted to pointing out to statisticians the definitions, notation and terminology of algebraic concepts. These represent the grounding to overcome the enumeration problem raised in Chapter 2 which affects both (i) the computation of probabilities for a non-parametric exact permutation test for treatments, and (ii) equally likely random allocation of treatments to plots. Furthermore, the author gives special attention to computer algorithms found in the literature which are very useful in enumerating Latin Squares. The elementary concepts of group theory can be found in any book of Modern Algebra such as Durbin [10].

3.1 A few preliminary concepts

Definition 3.1.1

Let Δ be a non-empty set. A **Partition** of Δ is a collection $\wp$ of subsets of Δ having the following properties :

1) $\cup_{V \in \wp} V = \Delta$
2) for any U, V $\in \wp$, $U \cap V = \emptyset$.

Definition 3.1.2

Let Δ be a non-empty set and ~ be a relation between elements of Δ, then ~ is said to be an **equivalence**

relation on Δ iff

1) $D \sim D$, for any element $D \in \Delta$;

2) For any elements D_1, $D_2 \in \Delta$ such that $D_1 \sim D_2$ then $D_2 \sim D_1$;

3) For any D_1, D_2, $D_3 \in \Delta$ such that $D_1 \sim D_2$ and $D_2 \sim D_3$ then $D_1 \sim D_3$.

The three properties in Definition 3.1.2 are frequently referred to in the literature as reflexive, symmetric and transitive, respectively.

Definition 3.1.3

Let Δ be a non-empty set and $\sim$ be an equivalence relation on Δ. For any $D \in \Delta$, the **equivalence class** of D denoted by [D], is defined to be the set $[D] = \{ D' \in \Delta : D' \sim D \}$.

Lemma 3.1.1

Let Δ be a non-empty set and $\sim$ be an equivalence relation on Δ, then $\sim$ induces a natural partition of Δ.

Proof:

Let $\sim$ be an equivalence relation on Δ and let element $D \in \Delta$. Then Definition 3.1.3 implies that D belongs to at least one equivalence class [D], say.

Then $[D] \subseteq \Delta \Rightarrow \cup_{D \in \Delta} [D] \subseteq \Delta$...(i). In addition, $D \in [D] \subseteq \cup_{D \in \Delta} [D] \Rightarrow \Delta \subseteq \cup_{D \in \Delta} [D]$...(ii). From both (i) and (ii) it follows that $\Delta = \cup_{D \in \Delta} [D]$.

By Definition 3.1.1(2), it remains to show that any pair of different equivalence classes are mutually disjoint.

In other words, if any pair of equivalence classes are not mutually disjoint, then they are equal. Suppose $D \in [D] \cap [D']$. In order to show that D cannot be in more than one equivalence class we need to show that $[D] = [D']$. Let $D_1 \in [D]$. then $D_1 \sim D$. On the other hand, $D \in [D] \cap [D']$ implies that $D \sim D'$. Hence, by Definition 3.1.2(3) $D_1 \sim D'$. i.e. $D_1 \in [D']$. Therefore, $[D] \subseteq [D']$ . . . (i)

Now, let $D_1 \in [D']$. Then $D_1 \sim D'$. On the other hand, $D \in [D] \cap [D']$ implies that $D \sim D'$. Hence, by Definition 3.1.2(2) $D' \sim D$. Therefore, Definition 3.1.2(3) implies that $D_1 \sim D$. Hence, $D_1 \in [D]$. Consequently, $[D'] \subseteq [D]$ . . . (ii). Finally, both (i) and (ii) imply that $[D] = [D']$ ♯.

Lemma 3.1.2

Let Δ be a non-empty set and $\wp$ be a partition, then $\wp$ gives rise to an equivalence relation $\sim$ if $D_1 \sim D_2$ is defined to mean that there exists an element V in $\wp$ such that $D_1, D_2 \in V$.

Proof:

Let $D \in \Delta$.

1) Definition 3.1.1(1) implies that $D \in V$ for some V in $\wp$. It is trivial that $D \sim D$.

2) If $D_1 \sim D_2$ then there exist $V \in \wp$ such that $D_1, D_2 \in V$. It is immediate that $D_2, D_1 \in V \Rightarrow D_2 \sim D_1$.

3) If $D_1 \sim D_2$ and $D_2 \sim D_3$ then there exist $V_1, V_2 \in \wp$ such that $D_1, D_2 \in V_1$ and $D_2, D_3 \in V_2$. Now $D_2 \in V_1 \cap V_2$. But $\wp$

is a partition, therefore $V_1 = V_2$, which implies that D_1, D_2, $D_3 \in V_1$. Thus, $D_1 \sim D_3$.

Finally, Definition 3.1.2 completes the proof ♯.

3.2 Definitions and elementary properties

Definition 3.2.1

Let Δ be a non-empty set and $(G, *)$ be a group. It is said that G **acts on** Δ iff for each $g \in G$ and each $D \in \Delta$ there exists an element $g \bullet D \in \Delta$ such that

1) If $\mathbf{e}$ denotes the identity element in G then $\mathbf{e}$ fixes D, i.e. $\mathbf{e} \bullet D = D$;

2) for all $g_1, g_2 \in G$ and each $D \in \Delta$, it follows that $g_1 \bullet (g_2 \bullet D) = (g_1 * g_2) \bullet D$.

Lemma 3.2.1

Let Δ be a non-empty set and $(G, *)$ be a group which acts on Δ, then for every $g \in G$, and all $D_1, D_2 \in \Delta$, $g \bullet D_1 = D_2$ iff $D_1 = g^{-1} \bullet D_2$.

Proof:

Let $g \in G$ and D_1 , $D_2 \in \Delta$ such that $g \bullet D_1 = D_2$. As G is a group there exist $g^{-1} \in G$ such that $g^{-1} * g = \mathbf{e}$.

Then $g^{-1} \bullet D_2 =$

$$= g^{-1} \bullet (g \bullet D_1) = (g^{-1} * g) \bullet D_1$$

by Definition 3.2.1(2)

$$= \mathbf{e} \bullet D_1 = D_1$$

by Definition 3.2.1(1)

In an exactly parallel manner, the converse implication can be shown ♯.

Theorem 3.2.1

Let Δ be a non-empty set and (G, *) be a group which acts on Δ. The relation ~ on Δ, defined as $D_1 \sim D_2$ iff there exist $g \in G$ such that $g \bullet D_1 = D_2$, is an equivalence relation on Δ.

Proof:

From Definition 3.1.2 we need to show that the relation is reflexive, symmetric and transitive.

1) Let $D \in \Delta$ and **e** be the identity element in G. Definition 3.2.1(1) implies that $\mathbf{e} \bullet D = D$. Therefore, $D \sim D$.

2) Let $D_1, D_2 \in \Delta$ such that $D_1 \sim D_2$. Then, there exist $g \in G$ such that $g \bullet D_1 = D_2$. From lemma 3.2.1, there exist an element $g^{-1} \in G$ such that $g^{-1} \bullet D_2 = D_1$. Therefore, $D_2 \sim D_1$.

3) Let $D_1, D_2, D_3 \in \Delta$ such that $D_1 \sim D_2$, $D_2 \sim D_3$. Then, there exist $g_1, g_2 \in G$ such that $g_1 \bullet D_1 = D_2$, $g_2 \bullet D_2 = D_3$. Since G is a group, $g_2 * g_1 \in G$. Then Definition 3.2.1(2) implies that $(g_2 * g_1) \bullet D_1 = g_2 \bullet (g_1 \bullet D_1)$. Then it follows that $(g_2 * g_1) \bullet D_1 = D_3$. Hence, $D_1 \sim D_3$.

1), 2) and 3) show that the relation is reflexive, symmetric and transitive. Therefore, the theorem follows ♯.

Definition 3.2.2

Let Δ be a non-empty set and (G, *) be a group which acts on Δ. The equivalence classes defined by theorem 3.2.1 are named **orbits**. i.e. if $D \in \Delta$ then the orbit of

D, denoted by Orb(D) is given by :

$$\text{Orb}(D) = \{ g \bullet D \in \Delta : g \in G \}.$$

Lemma 3.2.2

Let Δ be a non-empty set and (G, *) be a group which acts on Δ. The orbits partition Δ into disjoint sets of equivalent elements.

Definition 3.2.3

Let Δ be a non-empty set and (G, *) be a group which acts on Δ. For any $D \in \Delta$, the **stabilizer** of D is defined to be the set $G_D = \{ g \in G : g \bullet D = D\}$.

Lemma 3.2.3

Let Δ be a non-empty set and (G, *) be a group which acts on Δ. Then for each $D \in \Delta$, G_D is a subgroup of G.

Proof:

Let $D \in \Delta$,

1) By Definition 3.2.1(1), $\mathbf{e} \bullet D = D$. Hence, $\mathbf{e} \in G_D$. Therefore, G_D is not empty.

2) Let $g_1, g_2 \in G_D$. Then, $g_1 \bullet D = D$ and $g_2 \bullet D = D$. From Definition 3.2.1(2), $(g_1 * g_2) \bullet D = g_1 \bullet (g_2 \bullet D) = g_1 \bullet D = D$. Thus, $g_1 * g_2 \in G_D$.

3) Now, let $g \in G_D$. Then, $g \bullet D = D$. Lemma 3.2.1 implies that $D = g^{-1} \bullet D$. Thus, $g^{-1} \in G_D$.

Consequently, from 1), 2) and 3) the proof is now complete[2] ♯.

In order to calculate the number of elements in every equivalence class, we have Theorem 3.2.2.

Theorem 3.2.2

Let Δ be a non-empty set and (G, *) be a group which acts on Δ. For any $D \in \Delta$, it follows that

$$|Orb(D)||G_D| = |G|.$$

Proof:

From Lemma 3.2.3, G_D is a subgroup of G. Then, Lagrange's theorem[3] implies that

$$[G: G_D]\ |G_D| = |G|.$$

The proof will be complete if we show that

$$|Orb(D)| = [G: G_D] \tag{1}$$

We know that:

$$Orb(D) = \{ g \cdot D \in \Delta : g \in G \}$$

Suppose the (left) cosets[4] of G_D are of the form gG_D for $g \in G$. If the mapping $\phi : [G: G_D] \rightarrow Orb(D)$, defined by $\phi(gG_D) = g \cdot D$ is a bijection then (1) follows immediately. Thus, let us see first that ϕ is well defined. To do so, let gG_D be a Coset of G_D in G and let g_1, g_2 be any two elements in G such that $g_1, g_2 \in gG_D$. Then, there exists $g'' \in G_D$ such that $g_2 = g_1 * g''$. This implies that $g_2 \cdot D = g_1 \cdot D$. In other words, the value

---

[2] [10, p. 44, Theorem 7.1]

[3] [10, p. 92]

[4] [10, pp. 89 - 91]

of $g_1 \bullet D$ is independent of the choice of coset representative. That is to say, ϕ is well defined.

Now, suppose that $g_1 \bullet D = g_2 \bullet D$

<=>	$g_2^{-1} \bullet (g_1 \bullet D) = D$	[From Lemma 3.2.1]
<=>	$(g_2^{-1} * g_1) \bullet D = D$	[From Definition 3.2.1(2)]
<=>	$g_2^{-1} * g_1 \in G_D$	[From Definition 3.2.3]
<=>	$g_1 G_D = g_2 G_D$	[10, Lemma 16.1 (For left cosets)]

Therefore, ϕ is injective.

Finally, let $\boldsymbol{y}$ be an element in Orb(D). Then, by Definition 3.2.2 there exist $g \in G$ such that $g \bullet D = \boldsymbol{y}$. Thus, ϕ is Onto Orb(D).

Therefore, ϕ is a bijection and the proof is now complete

#.

3.3 Permutation groups and Latin Squares

Present day combinatorial terminology and statistical properties are brought together in order to identify a *categorisation* for Latin Squares of size $p \leq 6$. This will, for instance, be useful for studying both (i) equally likely random allocation of treatments to plots, and (ii) an exact permutation test of treatment effects.

Let T be a set of p symbols $\{ 1, 2, . . ., p \}$. Amongst the groups that could be used for enumerating Latin Squares of size p from the combinatorial point of view are:

- the permutation group S_p on T [10, p. 38];
- the direct product $S_p \times S_p$ [10, p. 86] of the permutation groups on T and on row levels, respectively;

- the direct product $S_p \times S_p \times S_p$ of the permutation groups on *T*, column and row levels, respectively;
- the group of symmetries of the square [10, p. 170].

In the remaining parts of the thesis, permutation groups and their properties will be the main algebraic tools to be used for the enumeration and categorization of combinatorial arrays useful for statistical applications. To quote Durbin [10, p. 38] "any group whose elements are permutations, with composition as the operation, is called a permutation group."

3.3.1 Illustrative example on Latin Squares of size 3

Let *T* be a set comprising the symbols { 1, 2, 3 }, and Ω be a 3×3 array. Further, let Λ be the set of all Latin Squares on *T*. That is to say,

Λ = {

123	132	213	231	312	321	
312	213	321	123	231	132	
231,	321,	132,	312,	123,	213,	
L_1	L_2	L_3	L_4	L_5	L_6	
123	132	231	213	321	312	
231	321	312	132	213	123	
312,	213,	123,	321,	132,	231	
L_7	L_8	L_9	L_{10}	L_{11}	L_{12}	}.

Should S_3 denote a permutation group on *T*, then Lemma 3.2.2 guarantees that the action of S_3 on the set Λ induces a partition on Λ into orbits. This is illustrated in Table 3.1. Without loss of generality, let us take L_1 and L_7 as representatives of each orbit. The stabilizer of each orbit representative consists of the identity element in S_3 only. In symbols, $G_L = \{ \mathbf{e} \}$ for the Latin Square $L \in \{ L_1, L_7 \}$. In other words, Theorem 3.2.2 shows that each coset of G_L in S_3 corresponds to precisely one Latin Square equivalent to L. Thus, the number of Latin Squares in the orbit of which L is class representative is given by $|Orb(L)| = |S_3|/|G_L|$. Thus, $|Orb(L)| = 6$.

Element of Λ	Orbit	Stabilizer
L_1	$\{ L_1, L_2, L_3, L_4, L_5, L_6 \}$	$\{ \mathbf{e} \}$
L_2		$\{ \mathbf{e} \}$
L_3		$\{ \mathbf{e} \}$
L_4		$\{ \mathbf{e} \}$
L_5		$\{ \mathbf{e} \}$
L_6		$\{ \mathbf{e} \}$
L_7	$\{ L_7, L_8, L_9, L_{10}, L_{11}, L_{12} \}$	$\{ \mathbf{e} \}$
L_8		$\{ \mathbf{e} \}$
L_9		$\{ \mathbf{e} \}$
L_{10}		$\{ \mathbf{e} \}$
L_{11}		$\{ \mathbf{e} \}$
L_{12}		$\{ \mathbf{e} \}$

Table 3.1 Partition of Λ into two orbits

3.3.2 Computer algorithms

A number of computer algorithms to tackle the problem of enumeration of different combinatorial objects can be found in the literature. The ones this author found most useful are that by Lam and Thiel [47] as well as that by Butler and Lam [06]. The latter presents a general method for isomorphism testing which can be used to distinguish and enumerate combinatorial objects. In [47], the authors provide a backtrack algorithm for non-isomorphism testing under the action of symmetry groups. An additional feature to test for the consistency of the results, by counting the equivalent designs in two independent ways, is introduced.

Some algorithms to generate permutation groups S_p can be found in the paper by Ives [42]. Very useful permutation generators in dictionary or lexicographical order [52, p. 85], as well as pseudo-random generator functions, are available in the book by Flamig [22].

3.3.3 Enumeration results

Not only does Table 3.2 show the distribution for Latin Square structures under the action of (i) the trivial symmetry group S_p on symbols *T*, (ii) the direct product $S_p \times S_p$ of the trivial groups on symbols and rows, and (iii) the symmetry group $S_p \times S_p \times S_p$. The group in (iii) above represents the direct product of the permutation groups on treatment labels, and on the levels for the blocking constraints 2 and 1 – Columns and Rows –, respectively. In what follows the notations $S_p \times S_p \times S_p$ and S(P) will be used exchangeably.

p	S_p (on *T*)	$S_p \times S_p$ (on *T* & rows)	$S_p \times S_p \times S_p$
2	1	1	1
3	2	1	1
4	24	4	2
5	1,344	56	2
6	1,128,960	9,408	22

Table 3.2 Distribution of Orbit representatives for Latin Squares of size $p \le 6$ under group actions.

Particular orbit representatives under the action of the group $S_p \times S_p$ (on *T* & rows) are referred to in the literature as Reduced Latin Squares: namely, those for which the symbols 1, 2, ..., *p* appear in natural order in the first column and row. From Table 3.2 and Theorem 3.2.2 the total numbers of allocations of symbols *t* in *T* to cells in a Latin Square array of size 2 to 6 are 2, 12, 576, 161280 and 812851200, respectively. As they should be [19, p. 81], [08, p. 146].

The action of S(P) on the set of Latin Squares of size six induces a partition of this set into 22 orbits, a result which seems to have been given for the first time by Schönhardt [68] under the name of ***isotopy classes*** and later confirmed by Fisher and Yates [20] under the name of ***transformation sets***.

3.3.4 A backtrack algorithm with isomorphic rejection for enumerating combinatorial objects: a description

In 1989, Lam and Thiel [47] presented a general computer algorithm for the backtrack search of combinatorial objects under the action of symmetry groups. The paper is divided into five sections. Section 1 introduces the topic in general terms. The authors highlight the advantages of their proposed methodology over previously devised backtracking methods.

The second section is devoted not only to introducing the terminology of the paper but also to describing the ideas underlying the backtrack algorithm and the consistency check criterion. For each level k of the partial solutions the symbols G_k, G_k' and G_k'' denote subgroups of that which acts on the objects by permuting the indices so that partial solutions are fixed, candidate vectors are partitioned into orbits, orbit sizes under G_{k-1}' of its parent are computed respectively. Street and Street give some examples in [71, chapter 4] which are helpful to understand this section.

Section 3 describes how the backtrack search with isomorphic rejection works in considerable detail.

The applications of the theory are well illustrated in Section 4. In this section, Lam and Thiel [47, pp. 478 - 483] present detailed examples in which the symmetry group under consideration is defined as the direct product of two permutation groups. For instance, they show how the action of the group $S_5 \times S_5$ (on rows & columns) induces a partition on the set of all the 5×5 (0,1)-matrices, under the condition that two ones appear in each row and column, into two orbits containing 600 and 1440 arrays respectively.

Finally, a proof of the correctness of the program is given in Section 5.

The reader is strongly encouraged to read this paper because therein lies a well documented methodology for the enumeration of combinatorial arrays. Some of these are, particularly useful to statisticians in the design of controlled experiments.

3.3.5 The rationale behind the combinatorial methodology

Following Lam and Thiel [47], all different class representatives of Latin Squares under the action of S(P) can be generated in a row-by-row fashion. Only one representative from each equivalence class at each stage i (i.e. row i), under the action of the appropriate subgroup of S(P), is further extended. An illustration is given in Table 3.3 for enumerating Latin Squares of size four. This table additionally shows that increasing the symmetry of the group, for enumeration purposes, summarizes the information on Latin Squares greatly.

1 2 3 4 * * * * * * * * * * * * 1.1(24)	1 2 3 4 2 1 4 3 * * * * * * * * 2.1(3)	1 2 3 4 2 1 4 3 3 4 1 2 * * * * 3.1(2)	1 2 3 4 2 1 4 3 3 4 1 2 4 3 2 1 4.1(1)
		1 2 3 4 2 1 4 3 3 4 2 1 * * * * 3.2(2)	1 2 3 4 2 1 4 3 3 4 2 1 4 3 1 2 4.2(1)
	1 2 3 4 2 3 4 1 * * * * * * * * 2.2(6)	1 2 3 4 2 3 4 1 3 4 1 2 * * * * 3.3(2)--->3.2	

Table 3.3 Enumeration of Latin Squares of size four under the action of S(P).

Appropriate discrimination will enumerate different Latin Squares in a particular orbit. The total number of allocations is given by the product of those counts of isomorphic designs which are equivalent to each subdesign at every stage - rows - in the search tree leading to the completed orbit representative under the action of S(P). For instance, there are only two non-isomorphic Latin Squares of size 4 under the action of S(P). The numbers within brackets in Table 3.3 denote all those partial arrays isomorphic to that taken as their representative. Thus, 24×3×2×1 = 144 Latin Square arrangements can be generated from the orbit representative labelled as node 4.1. When subdesign at node 3.3 is generated, it is found to be equivalent to subdesign at node 3.2 (denoted as 3.3 --> 3.2). The repetition count of the former equals 24×6×2 = 288. Therefore, the repetition count of subdesign at node 3.2 is increased from 144 to 432. This implies that subdesign at node 3.2 contributes with 24×3×2 + 24×6×2 = 432 layouts to the counts at node 4.2. The record of repetition counts is given in Table 3.4

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
1.1	-	144	144	48	24	24	24
2.1	1.1	16	32	16	3	72	72
2.2	1.1	8	16	8	6	144	144
3.1	2.1	8	24	24	2	144	144
3.2	2.1	8	8	8	2	432	144
3.3	2.2	4	8	8	2	432	288
4.1	3.1	24	96	-	1	144	144
4.2	3.2	8	32	-	1	432	432

Table 3.4 Record of repetition counts for Latin Squares of size 4.

The search tree and record of repetition counts for Latin Squares of size 5 are given in Tables 3.5 and 3.6, respectively. Those for Latin Squares of size 6 are given in Appendix I.

12345	21453	34512	45231	53124
1.1(120)	2.1(20)	3.1(12)	4.1(2)	5.1(1)
	23451	31524	45132-->4.1	
	2.2(24)	3.2(5)	4.2(2)	
			54132	45213-->5.1
			4.3(2)	5.2(1)
		34512	45123	51234
		3.3(2)	4.4(2)	5.3(1)
		35124-->3.1		
		3.4(5)		
		45123-->3.3		
		3.5(1)		

Table 3.5 Search tree for Latin Squares of size 5

The representative at node 5.1 accounts for 144,000 layouts whilst that at node 5.3 stands for 17,280 arrays. The record of the counts is given in Table 3.6.

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
1.1	-	2880	2880	720	120	120	120
2.1	1.1	36	72	24	20	2400	2400
2.2	1.1	30	60	20	24	2880	2880
3.1	2.1	2	4	2	12	43200	28800
3.2	2.2	4	12	6	5	14400	14400
3.3	2.2	10	20	10	2	8640	5760
3.4	2.2	4	4	2	5	43200	14400
3.5	2.2	20	20	10	1	8640	2880
4.1	3.1	1	3	3	2	115200	86400
4.2	3.2	3	3	3	2	115200	28800
4.3	3.2	3	12	12	2	28800	28800
4.4	3.3	5	20	20	2	17280	17280
5.1	4.1	3	12	-	1	144000	115200
5.2	4.3	12	12	-	1	144000	28800
5.3	4.4	20	100	-	1	17280	17280

Table 3.6 Record of repetition counts for Latin Squares of size 5

The reader is referred to Lam and Thiel [47] for the general method.

3.3.6 Categorization of Latin Square arrays

A complete list of orbit representatives – under the action of S(P) – for Latin Squares of size $p \leq 6$ is presented in the following pages. Those for $p = 6$ have been indexed so that the first 17 orbit representatives are in a 1–1 correspondence with those transformation set representatives reported by Fisher and Yates [20].

Categorization list for orbit representatives – under the action of S(P) – for Latin Squares of size $p \leq 6$.

$p = 2$

1	2
2	1

[1, 1]

$p = 3$

1	2	3
2	3	1
3	1	2

[1,2]

$p = 4$

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

[1, 18]

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

[2, 6]

$p = 5$

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

[1, 1200]

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

[2, 144]

$p = 6$

1	2	3	4	5	6
2	1	4	5	6	3
3	4	5	6	1	2
4	3	6	1	2	5
5	6	2	3	4	1
6	5	1	2	3	4

[1, 129600]

1	2	3	4	5	6
2	1	4	5	6	3
3	4	2	6	1	5
4	5	6	2	3	1
5	6	1	3	4	2
6	3	5	1	2	4

[2, 129600]

1	2	3	4	5	6
2	1	4	5	6	3
3	4	2	6	1	5
4	6	5	2	3	1
5	3	6	1	2	4
6	5	1	3	4	2

[3, 129600]

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	4	2
4	6	5	1	2	3
5	3	6	2	1	4
6	4	2	5	3	1

[4, 129600]

1	2	3	4	5	6
2	1	4	5	6	3
3	4	5	6	2	1
4	6	2	1	3	5
5	3	6	2	1	4
6	5	1	3	4	2

[5, 64800]

1	2	3	4	5	6
2	1	4	5	6	3
3	4	1	6	2	5
4	5	6	1	3	2
5	6	2	3	4	1
6	3	5	2	1	4

[6, 64800]

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	2	4
4	6	2	5	1	3
5	3	6	2	4	1
6	4	5	1	3	2

[7, 64800]

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	2	4
4	6	2	5	3	1
5	4	6	2	1	3
6	3	5	1	4	2

[8, 43200]

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	4	2
4	6	5	2	1	3
5	3	6	1	2	4
6	4	2	5	3	1

[9, 43200]

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	3	6	5	2	1
5	6	1	2	4	3
6	5	2	1	3	4

[10, 21600]

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4
6	3	2	5	4	1

[11, 14400]

1	2	3	4	5	6
2	1	4	5	6	3
3	4	5	6	1	2
4	5	6	3	2	1
5	6	1	2	3	4
6	3	2	1	4	5

[12, 14400]

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	3	6	5	2	1
5	6	1	2	3	4
6	5	2	1	4	3

[13, 7200]

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	6	5	2	1	3
5	4	6	3	2	1
6	5	4	1	3	2

[14, 4800]

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	2	4
4	6	5	1	3	2
5	4	6	2	1	3
6	3	2	5	4	1

[15, 4320]

1	2	3	4	5	6
2	1	4	5	6	3
3	4	1	6	2	5
4	5	6	1	3	2
5	6	2	3	1	4
6	3	5	2	4	1

[16, 4320]

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	2	4
4	6	2	5	1	3
5	3	6	1	4	2
6	4	5	2	3	1

[17, 2400]

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	5	6	2	3	1
5	6	2	1	4	3
6	3	1	5	2	4

[18, 129600]

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	4	2
4	6	5	1	2	3
5	4	6	2	3	1
6	3	2	5	1	4

[19, 64800]

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	4	2
4	6	2	5	3	1
5	3	6	2	1	4
6	4	5	1	2	3

[20, 43200]

1	2	3	4	5	6
2	1	4	5	6	3
3	4	5	6	1	2
4	3	6	1	2	5
5	6	1	2	3	4
6	5	2	3	4	1

[21, 14400]

1	2	3	4	5	6
2	1	4	5	6	3
3	5	1	6	4	2
4	6	2	1	3	5
5	3	6	2	1	4
6	4	5	3	2	1

[22, 4320]

The symbol [***x***, ***y***] below each Latin Square layout is to be interpreted *as orbit representative number **x** under S(P) generates **y** orbit representatives of Latin Squares under* S_p (symmetry group on *T*).

The combinatorialist may increase the symmetry of the group $S_6 \times S_6 \times S_6$ to a group *G* which also includes transpositions along the left-to-right diagonal (one operation of the group of symmetries of the square [10, p. 170]). In the forthcoming §4.5 the reader will find that a different equivalence relation reduces the number of representatives of Latin Squares of size 6 from 22 to: (i) 17 families [08, p. 142], these being represented by the first 17 Latin Squares of size 6 in the above table (ii) 12 main classes (species or adjugate sets) of 6×6 Latin Squares [08, p. 142], [20].

For a listing for follow up leads on 7×7 Latin Squares see Dénes and Keedwell [08]. For further information on Latin Squares of size 8 see Kolesova et al. [45].

4 An exact permutation test on Latin Square designs

It was pointed out in Chapter 2 that probabilistic arguments such as the ones used for gambling play an important role as the foundations for (i) likelihood based and (ii) exact permutation tests. Allocation problems were then taken into consideration. Theoretical foundations from the tools of group theory were sought and the relevant results to aid in solving the problem of calculating probabilities were presented in Chapter 3.

In this chapter, Combinatorial and Statistical properties are brought together in order to describe the non-parametric exact permutation test on Latin Square designs. Illustrative examples are presented as didactically as possible. The chapter concludes by presenting the problems associated with this technique as well as certain ingenious alternatives that have been proposed. For further information on randomization and permutation tests the reader is referred to Manly [50] and Good [36].

4.1 Statistical tools

Fast development of computer power today allows us to reconsider an alternative to classical tests of hypotheses. The Computer-Intensive Permutation Test [01], [36 p. 7].

Fisher [17] proposed the Statistic :

$$Z = \frac{1}{2} Ln \left[\frac{SS(\tau)}{SS(\nu)} \right] \qquad \ldots(\text{ 4-1 })$$

where SS(τ), SS(ν) account for the variation due to treatments and natural variation (residual), respectively. That is to say, SS(τ) measures differences amongst treatment means whilst SS(ν) accounts for the sum of squares of the differences of observations within treatments. Fisher argues in [17] that (i) the variance of z depends only on the size of the sample, and (ii) z is then distributed very nearly in a Normal distribution. The argument is defended by Eden and Yates [11] through simulation of a randomization test [50] (taking random samples from all possible permutations of treatment labels). Fisher's [17] viewpoint will be endorsed in this chapter.

According to Good [36, § 1.3.2] research workers are free to choose a test statistic provided it discriminates between null and alternative hypotheses.

It is important to remind the reader that treatments under comparison are randomly allocated to experimental units subject to two blocking constraints. Consequently, sums of squares for the blocking constraints are given in Tables 4.2, 4.5, and 4.8 for checking purposes and not for comparison between the levels of the blocking constraints.

The statistic

$$W = \frac{SS(\tau)}{SS(\varepsilon) + SS(\tau)} \qquad \ldots(\text{ 4-2 })$$

which is a monotone increasing function of Z in equation (4.1), was introduced in the statistical literature [74] with the objective of comparing Fisher's Z distribution with that from Normal theory.

Yates [75] emphasized that, for experimental statistical trials, a random choice of Latin Squares based on all possible different values of the Z-statistic in the ANOVA table should be made. That is to say, an equally likely selection of Latin Squares was pointed out. In the notation of this thesis, the action of the group S_p (trivial group on T) on the set Λ of Latin Square arrays induces a partition on Λ such that all elements of a given orbit yield the same value for the Z-statistic. The concept of an orbit representative will hopefully be made clearer to the reader after his/her study of the forthcoming §§ 4.2.2, 4.2.3.

4.2 The permutation test

There are only two orbit representatives of Latin Squares of size 3 under the action of S_3 (symmetry group on T). Hence, there can be only two different values for the Z-statistic in the ANOVA table. Therefore, a permutation test on this kind of square is out of the question.

In order to evaluate the response of lactating dairy cows to four dietary regimes, sixteen lactating cows – four of a kind amongst Hereford, Holstein, Jersey and British Friesian – were blocked according to race and previous milk production. This in order for the cows to enter a comparative experiment for a period of three weeks. The four diets under comparison comprised 89% alfalfa silage supplemented with (i) 0 and, (ii) 9.3% soybean, (iii) 8.2% fish, and (iv) 4.7% soybean + 4.1% fish meal, respectively. These would be allocated at random to the selected cows in a 4x4 Latin Square design.

Note: There will be only 6 residual degrees of freedom in the ANOVA table, very few for a sensitive test. However, for the purposes of **elucidating** the exact permutation test on Latin Squares *in detail* the aforementioned example will be made use of though this remark is important in practice.

Clearly, the protocol is almost ready for a permutation test. In order to obtain the appropriate equally likely probability distribution we need to randomly select one layout for allocating the diets to the cows.

4.2.1 Random allocation of treatments in Latin Square form

As was mentioned at the end of §4.1.1, all different Latin Square structures for statistical applications are given by the set of orbit representatives under the action of the group S_p (trivial group on T) — See the forthcoming Table 4.3. In our particular case $p = 4$. A backtracking algorithm which makes use of that by Ives [42] can certainly be devised for such purposes. However, examination of one alternative way of obtaining the required information by using those orbit representatives given in §3.3.6 will now be made. The retrieval/generation procedure – and therefore the randomization scheme as well – is **implicit** in Fisher and Yates' [21, p. 24] random selection method. They grouped Latin Square layouts by intuitive use of the group $S_p \times S_p$ – direct product of the permutation groups on treatment labels and rows, respectively. Thus, the author uses those orbit representatives under S(P) to retrieve those under $S_p \times S_p$ (*T* & rows). The squares so generated are used to obtain those orbit representatives under S_p (on *T*).

In order to extend the orbit representatives under S(P) to those under $S_4 \times S_4$ (*T* & rows), the forthcoming systematic procedure may be followed:

1. Consider G = { $(2\ 3)^3$, $(2\ 4\ 3)^3$ } ⊂ S(P). Then, apply G to orbit representative number 1 – Under S(P) – from Latin Squares of size four – listed in §3.3.6 – in order to produce two additional Latin Squares. Thus , the resulting set of four Latin Squares may certainly be taken as the Orbit representatives of Latin Squares under the action of $S_4 \times S_4$ (*T* & rows).
2. To extend the orbit representatives under $S_4 \times S_4$ (*T* & rows) to those under S_4 (on *T*), let the permutation group on the levels of the – blocking constraint – rows except the first be S_3. Then apply S_3 to the row-levels of orbit representatives under $S_4 \times S_4$ except the first.

This unpacking method can be easily implemented in a computer program so that a listing of orbit representatives under S_4, as shown in column 1 (From the left) in Table 4.3, is produced readily for an exact permutation test.

The corresponding sets G_1, $G_2 \subset S(P)$ for Latin Squares of size five are given in Appendix I.

Let Λ be the set of Latin Squares of size 4 and S_4 denote the symmetry group on T acting on Λ. Let us assume an orbit representative D has been selected at random with equal probability. The random allocation procedure is completed by random choice of an element $g \in S_4$. The array given by $g \bullet D$ does determine the order in which cows will receive their diets.

The procedure outlined above (i) upholds the second rule for the permutation test procedure given in § 2.4 (equally likely random choice), and (ii) strengthens the assumption of independence in § 2.2.1.

In the example under consideration, the resulting randomly selected structure of a Latin Square layout is that given by L = 1234 2143 4312 3421 – in row order. The randomly chosen $g \in S_4$ was (1 4). Then $g \bullet L$ produced the array : 4231 2413 1342 3124.

In [02, §5] Bailey and Rowley write "Let Ω be a finite set of plots and $\mathfrak{I}$ a finite set of treatments...If $\phi: \Omega \to \mathfrak{I}$ is a plan and g is any permutation of $\mathfrak{I}$ then $g\phi$ is also a plan... If Φ is any set of plans and G any set of permutations of $\mathfrak{I}$ then we may enlarge Φ to $G\Phi$, where $G\Phi$ = $(g\phi$: $g \in G$, $\phi \in \Phi)$... Random choice of a plan from $G\Phi$ is equivalent to random choice of a plan from Φ followed by random permutation of the treatments using G...that is, random choice of plan from some comparatively small set, followed by randomization of treatments."

The reader can verify that, for the case of Latin Square designs, the following equivalence in the author's notation and that by Bailey and Rowley [02] holds.

Author	Bailey and Rowley [02]
(i) Set of representatives of Latin Squares induced by the action of the group S_p (on *T*).	(i) Set of plans Φ.
(ii) random choice of Latin square structure.	(ii) random choice of a plan from Φ.
(iii) The array $g \bullet D$, where g was randomly chosen from S_p, determines allocation scheme.	(iii) random permutation of treatments using *G*.

Likewise, the reader can check that what the present author refers to as *selecting* a Latin Square structure D *at random with equal probability* is, in general terms, referred to by Bailey and Rowley [02, §2] as "the randomization is valid that consists of *randomly choosing* a plan from Φ *with uniform probability*."

The randomization scheme (equally likely random allocation of treatments to plots) clearly described in this section will be extended to sequential experimentation on Latin Square designs in the forthcoming chapters of the thesis. For general theory on randomization (allocation of treatments to plots) the reader is referred to [02].

4.2.2 Calculate the value for Fisher's Z-statistic

At the conclusion of the experiment, the observed milk yields in kg/d were as shown in Table 4.1.

The analysis of variance associated to the data in Table 4.1 is presented in Table 4.2.

	BREED			
Previous Milk Production (kg/d)	Hereford	Holstein	Jersey	British Friesian
32.55	25.375(4)	41.025(2)	36.208(3)	27.691(1)
29.05	19.275(2)	37.925(4)	36.241(1)	22.858(3)
27.8	20.691(1)	29.608(3)	36.525(4)	24.975(2)
37.3	29.958(3)	43.741(1)	42.325(2)	33.275(4)

Table 4.1 Yields (kg/d) of milk from sixteen lactating cows in a Latin Square layout. Allocation of treatments was according to the symbols shown within brackets.

Source of variation	D.F.	S.S.	M.S.	Zc	Z(1%)
Prior Milk Prod.	3	213.422	71.141	1.323	
Breed	3	641.672	213.891	1.873	
Treatments	3	27.358	9.119	0.296	1.1401
Residual	6	30.294	5.049		
Total	15	912.745			

Table 4.2 Analysis of variance table for data in Table 4.1

In Table 4.2, the critical value for Fisher's Z is given for purposes of comparison only.

4.2.3 Compute the permutation distribution

The exact discrete probability distribution, conditional on the data at hand, is now obtained by calculating all possible different values for the statistic

Z through the set of orbit representatives under the action of S_4 (on *T*).

Orbit rep. under S_4	**	SS(τ)	**	SS(ε)	**	Z
1234214334124321	**	15.52	**	42.13	**	00.131
1234214343213412	**	23.61	**	34.04	**	00.237
1234341221434321	**	10.02	**	47.63	**	00.069
1234341243212143	**	24.90	**	32.75	**	00.257
1234432121433412	**	17.23	**	40.42	**	00.151
1234432134122143	**	24.02	**	33.63	**	00.243
1234214334214312	**	11.77	**	45.88	**	00.088
1234214343123421	**	27.36	**	30.29	**	00.296
1234342121434312	**	24.61	**	33.04	**	00.252
1234342143122143	**	26.62	**	31.04	**	00.284
1234431221433421	**	02.64	**	55.01	**	-0.003
1234431234212143	**	22.31	**	35.34	**	00.219
1234234134124123	**	04.01	**	53.64	**	00.010
1234234141233412	**	15.60	**	42.05	**	00.132
1234341223414123	**	27.91	**	29.74	**	00.305
1234341241232341	**	07.02	**	50.63	**	00.039
1234412323413412	**	25.23	**	32.42	**	00.262
1234412334122341	**	35.53	**	22.12	**	00.453
1234241331424321	**	11.21	**	46.44	**	00.082
1234241343213142	**	23.34	**	34.31	**	00.233
1234314224134321	**	14.33	**	43.32	**	00.117
1234314243212413	**	25.17	**	32.48	**	00.261
1234432124133142	**	22.51	**	35.14	**	00.221
1234432131422413	**	18.74	**	38.91	**	00.170

Table 4.3 Exact Permutation distribution for data in Table 4.1

The exact probability distribution function is calculated assuming the null hypothesis is true, i.e. treatment labels contribute nothing to the observed yields.

The discrete probability distribution associated to the data in Table 4.1 is summarized in the histogram given in Table 4.4.

The probability value obtained by this process is $P(Z \geq Zc \mid \text{null is true}) = 3/24 = 0.1250$.

Midpoint	Count	
0.00	2	**
0.05	2	**
0.10	3	***
0.15	4	****
0.20	2	**
0.25	7	*******
0.30	3	***
0.35	0	
0.40	0	
0.45	1	*

Table 4.4 Histogram of Z, 24 Orbit representatives

4.2.4 Decision making

This test indicates that the probability of rejecting the null when it is true is as high as 12.5%. Therefore, it may be argued that there is no evidence to reject the null hypothesis of equality of treatment effects.

4.3 Examples on Latin Squares of size $p = 5, 6$.

Empirical comparative studies using Latin Squares seem to date back to 1788 when de Palluel used a 4x4 Latin Square array to compare diets on sheep [72, 76].However, Latin Square arrangements were first introduced for use in randomized statistical trials on agricultural experiments by Fisher [17, 18, 19]. Gradients in fertility may run approximately parallel to two nearly perpendicular sides of a field, hence defining the two blocking constraints for the plots of land. Many examples are found in the literature where the effects of such potential sources of variation are controlled by using Latin Squares as row-column designs.

Example I.- Spacing on Millet plants in a 5x5 Latin Square arrangement [69, p. 313].

The ANOVA table associated to these data is given in Table 4.5.

Source of variation	D.F.	S.S.	M.S.	Zc	Z(5%)
Rows	4	13601	3400	0.585	
Cols	4	6146	1537	0.188	
Treatments	4	4157	1039	-0.008	0.5907
Residual	12	12667	1056		
Total	24	36571			

Table 4.5 Analysis of variance table for data in Example I.

Behaviour of the exact discrete probability distribution conditional on the data at hand in Example I is well illustrated by the histogram in Table 4.6.

The permutation process yields the value, $P(Z \geq Zc \mid \text{null is true}) = 293/1344 = 0.2180$. Therefore, it may be argued that there is no evidence to reject the null hypothesis of equality of treatment effects.

Midpoint	Count	
-2.4	2	*
-2.0	3	*
-1.6	11	**
-1.2	29	***
-0.8	111	************
-0.4	361	*************************************
-0.0	478	**
0.4	284	*****************************
0.8	58	******
1.2	7	*

Table 4.6 Histogram of Z, 1344 Orbit representatives. Each * represents 10 obs.

Example II.- Potato yields in a 6x6 Latin Square arrangement [19, Table 9].

633(5)	527(2)	652(6)	390(1)	504(3)	416(4)
489(2)	475(3)	415(4)	488(5)	571(6)	282(1)
384(1)	481(5)	483(3)	422(2)	334(4)	646(6)
620(6)	448(4)	505(5)	439(3)	323(1)	384(2)
452(4)	432(1)	411(2)	617(6)	594(5)	466(3)
500(3)	505(6)	259(1)	366(4)	326(2)	420(5)

Table 4.7 Data set from Fisher [19, Table 9].

In Table 4.7, the symbols in brackets represent the allocated treatments in the Latin Square design.

The ANOVA table associated to these data is given in Table 4.8.

Source of variation	D.F.	S.S.	M.S.	Zc	Z(5%)
Rows	5	54199	10840	0.980***	
Cols	5	24467	4893	0.582*	
Treatments	5	248180	49636	1.741***	0.4986
Residual	20	30541	1527		
Total	35	357387			

Table 4.8 Analysis of variance table for data in Example II.

The calculated probability is P(Z ≥ Zc | null is true) = 1/1128960. Therefore, it may be argued that there is very strong evidence against the null model. Consequently, statisticians would move on to carefully study relationships amongst treatment means, e.g. contrasts [40, §7.2].

All examples considered so far seem to indicate that the only object of the statistical procedure in a permutation test is to test significance of treatment differences. However, the reader is referred to Manly [50, § 1.4] who argues in favour of exact tests as the means to devise 'exact' confidence intervals in estimation, for instance, of the difference between two treatment means.

An illustration on how statisticians proceed after the F-test for treatments in the ANOVA table was found significant will be described in §8.1.1.

Even though the idea of an exact permutation test for treatment effects in Latin Squares is implicit in Fisher's [19] work, it was not until [30] appeared that the exact permutation test for the Latin Square Design was set down in writing for the first time. This piece of work was later included in [33] – an earlier version of this thesis.

4.4 Identified problems and their proposed solutions.

Using the computer facilities for intensive computations available at the University of Warwick, the permutation test in Example II took a few minutes through the server pansy (academic year 1996-97). This has two ultra SPARC-II processors clocked at 200 Mhz, with 256 Mb of RAM and a total of 2.1 Gb of virtual memory. However, not all research workers have access to computers of such capacity and speed as to execute the tests in reasonable time. Even when the number of experimental units is not very large, sometimes a change in the number of treatment groups may increase the number of permutations greatly. Let us take for example 36 experimental units. Should they be subject to the requirements for a Latin Square design of size 6, then the number of computer iterations for an exact permutation test would be of the order of 1,128,960. This should be clear to the reader from the examples given in § 4.2. In contrast, should the number of treatments be 4, say, then a permutation test on a balanced design such as the ones proposed by Pearce and Taylor [54,pp. 406-407] would involve as many as 538,789,708,800 computer iterations !! which is *impractical*.

4.4.1 Suggested alternatives.

In order to overcome the aforementioned drawback some statisticians, such as Good [36] and Manly [50] appeal to **tests based on re-sampling** (randomization tests, bootstrap, Monte-Carlo). These generally consist on taking samples of all possible permutations – as introduced by Eden and Yates [11]. Examples are, bootstrap [36, §2.3], [50, §2.3] or Monte-Carlo simulations [36, §13.2], [50, §2.1]. For information on limitations of randomization tests the reader is referred to [50, §1.3.4].

Let us assume we have a data set, and let f_P denote the probability distribution function associated to the exact permutation test. What is more, let f_R denote the distribution associated to re-sampling techniques (random samples, bootstrap etc.). Then in order to demonstrate a valid approximation to the permutation test by the re-sampling technique the following statement must hold: For every value ε greater than zero, there exists a natural number *N* such that when the number of computer iterations *n* is greater than or equal to N then the *distance* between f_P and f_R is less than or equal to ε. In symbols,

$$\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N} \ni n \geq N \Rightarrow \| f_P - f_R \| \leq \varepsilon.$$

The above-mentioned condition seems very difficult to prove because f_R is estimating the *unknown* function f_P. Furthermore, f_P depends on the observed data set.

On the other hand, there are statisticians who recommend their scheme for random allocation of treatments followed by the F-test based on Classical Normal theory.

For instance, in [64, p. 113] Preece et al. write "the randomization procedure then consists of random permutation of rows of the basic plan and random permutation of columns". It seems the only published justification for automatic use of the classical Normal F-test is that due to Hinkelmann and Kempthorne [40]. These authors present their argument in favour of an approximation to the randomization test by the Classical F-test – See also [50, §1.3.3]. This approximation consists of *taking samples of all possible permutations* and comparing the results with those of Normal theory. Eden and Yates [11] as well as Hinkelmann and Kempthorne [40] claim to have obtained very good approximations to the randomization test by the Classical F-test in the particular cases they investigated. The arguments for approximations are generally based on computer simulated studies on **particular** data sets.

As a theoretical argument for the approximation to the randomization test by the classical F-test it has been argued that a comparison between the Permutation and Normal theory distributions can be made through their corresponding *first two moments*. However, to quote Mood, Graybill and Boes [51] "In general, a sequence of moments μ_1, μ_2, ... does not determine a unique distribution function". In addition, the "approximations by the F-test" in [40, p. 167] are justified for large numbers of experimental units only. Furthermore, Manly in [50, §1.3.3] warns "If the data contains even just one or two anomalous values then the two tests may not agree."

All things considered, the attention of the reader is drawn to §2.2. S/he may now compare and contrast the theoretical foundations underlying the tests reported in the literature with those given in §2.2 for a variance ratio test based on likelihood theory. An illustrative example for the test presented in chapter 2 will be given in the forthcoming § 8.1.1.

4.5 The Latin Square: a distinction between the combinatorial object and the row-column design.

Combinatorialists have enumerated Latin Squares under the action of various different symmetry groups. In 1990, Kolesova et al. [45] reported their results from enumerating Latin Squares of size 8. They define 2 Latin Squares L_1, L_2 as equivalent iff one is obtainable from the other by:

i) permuting the rows; ii) permuting the columns; iii) relabelling the treatments; and the two forthcoming operations: iv) transposing the array along the left-to-right diagonal, and (v) row inverse.

According to Kolesova et al. [45] the row inverse operation is defined "by treating each row of the Latin Square as a permutation in image form and by replacing it with its inverse."

A slightly different combinatorial approach for Latin Squares of size $p \leq 6$ is that reported by Fisher and Yates [20]. In the terminology given in [20], an *intramutation* of a Latin Square is obtained by permuting the allocated symbols 2, 3,..., p and then permuting rows and columns so as to put the new square thus obtained into standard form. Their *conjugacy* operation interchanges the rows and columns of a given square. In a parallel manner, an *adjugacy* operation interchanges (i) the rows and allocated symbols or (ii) the columns and allocated symbols.

Fisher and Yates [20] enumerated all reduced squares of size 6 into 22 orbits or 12 adjugate sets. The tabulation of 6×6 Latin Squares in §3.3.6 is consistent with this, as orbit representatives numbers 18, 19, 20, 21 and 22 fall into the same adjugacy sets as do orbit representatives 1, 5, 8, 11 and 15, as representatives 2, 6, 9, 12 and 16 respectively.

Clearly in [20], Latin Squares L_1 and L_2 of a certain size are equivalent in the sense of Kolesova et al. iff they belong to the same adjugate set.

Let Λ be the set of Latin Squares. From Table 3.2 it is clear that the number of Latin Squares depends on the group acting on the set Λ.

In the case of Latin Squares of size 6, Dénes and Keedwell write 'of the 22 isotopy classes' (orbits) '10 can be arranged into pairs such that one member of each of these pairs is obtained from the other by interchange of rows and columns (that is, by transposition). If the squares of each such pair of isotopy classes' (orbits) 'are regarded as forming a single "family", the total number of families is 17'. Indeed, the additional operations (i) transposition of the array along the left-to-right diagonal,and (ii) row inverse reduce the number of *combinatorialy 'different'* Latin Squares further to what is referred to in the literature as *12 Main classes* or *species*. However, this manuscript concentrates on the use of the Latin Square for *statistical* comparative experiments. Therefore, a clear distinction between the combinatorial object and the row-column design known as the Latin Square should be made. Let S_6 denote the symmetry group on the set of 6 allocated treatments *T*, Furthermore, let us consider the data set in Example II and the following 3 Latin Squares:

5,2,6,1,3,4	2,5,6,1,3,4	5,2,1,6,4,3
2,3,4,5,6,1	5,3,4,2,6,1	2,3,5,4,1,6
1,5,3,2,4,6	1,2,3,5,4,6	6,4,3,5,2,1
6,4,5,3,1,2	6,4,2,3,1,5	1,5,2,3,6,4
4,1,2,6,5,3	4,1,5,6,2,3	3,6,4,1,5,2
3,6,1,4,2,5	3,6,1,4,5,2	4,1,6,2,3,5
(a) L_1	(b) L_2	(c) L_3

The reader can check that the action of (2 5) on L_1 yields Latin Square L_2. On the other hand, Latin Square L_3 is obtainable from L_1 by transposition along the left-to-right diagonal. The reader can verify that the values for Fisher's Z-statistic are (a) 1.741, (b) 1.741, and (c) 1.255, respectively. Similar results hold true should we choose the F-statistic in the ANOVA table instead of Fisher's Z-statistic. Consequently, in the reminder of the thesis the term z-equivalence or F-equivalence between two Latin Squares will be used. When there is no room for ambiguity the term equivalent will be used instead.

It has therefore been **proved** that F-equivalence and Z-equivalence properties are NOT INVARIANT under transpositions along the left-to-right diagonal. In a parallel manner it can be proved that the row inverse operation [45, p. 144] *does not* preserve F/Z-equivalence properties.

The reader can verify that F/Z-equivalence properties are invariant **only** when Latin Squares of size *p* are **classified under the action of** the permutation group on allocated symbols ***Sp*** (on *T*).

We are now in the position to review the combinatorial aspects around the problem of **sequential experimentation** on Latin Square designs in present day combinatorial terminology. This will be the main object of study in the following three chapters. In other words, from now on emphasis will be given to the combinatorics of the problem of enumerating arrays for **sequential experimentation** on Latin Square designs. The enumeration and categorization of arrays helps to handle equally likely random allocation of treatments to plots. This is a requirement for obtaining a valid estimate of the residual variation. The seed-idea of this concept was given in chapter 2. Readers interested in the general theory behind random allocation of treatments to plots presented here are referred to Bailey and Rowley [02]. Note the equivalence of notation pointed out in § 4.2.1 of this manuscript.

5 The addition of further treatments to Latin Square designs: A review

This chapter presents a review of the literature with regard to the problem of comparing a new set of treatments on experimental units still affected by an earlier set. Emphasis is given to describing the construction methods for superimposing a new set of treatments on a Latin Square design.

5.1 Background and the applied problem

The study of the mathematical properties of the so-called Latin Square has a long history. However, they were first used in randomized statistical trials by Fisher [17, 18] in the 1920s. His methods for statistical analysis were very practical in its approach. Therefore, they gained widespread popularity amongst research workers in diverse disciplines [11, p. 6], [36, §1.1.1].

In chapter 4 this author illustrated the application of the non-parametric exact Permutation Test of treatment effects in Latin Square designs. Furthermore, problems associated with the methodology as well as proposed alternatives reported in the literature were described in § 4.4. The reader is recommended to compare and contrast previously proposed tests with the variance-ratio test given in chapter 2.

In long-lived animal or agricultural trials it is frequently found in practice that economic considerations may not allow removing or replacing a set of experimental units after a single trial has been completed on it [23]. Therefore, suitable designs for sequential experimentation on the same experimental units are highly valued by the research worker.

There are problems associated with successive experimentation when the initial trial is first arranged as a Randomized Complete Block Design. These have been studied in considerable detail by Hoblyn, Pearce and Freeman [41] and will not be considered here. The problems are inevitably more complex when the original trial took place subject to two blocking constraints. The most frequent situation is when a Latin Square design has been used for the initial trial [24].

This author will illustrate how the principle of blocking naturally extends to sequential experimentation on Latin Square designs. Once the main idea behind the method is understood the reader is expected to devise the appropriate *variance ratio* tests when facing similar situations in his/her research work. For references on alternative methods of analysis of data proposed by statisticians in practical applications the reader is recommended to review § 4.4.1.

Let us make the idea of successive experimentation, to be studied here, as clear as possible. To this end one applied problem which motivated the quest for combinatorial arrays for *successive experimentation* on Latin Square designs is presented. Then, the combinatorics for superimposing a set of treatments on Latin Squares is considered. This plays an important role guaranteeing a valid estimate of residual variation in the *variance ratio* test.

5.1.1 The applied problem

Horticultural scientists wished to compare a new set of treatments on experimental material still affected by an earlier set. A Latin Square layout for six treatments *was used* for a phytotoxicity trial on pear trees at East Malling Research Station [24]. A year later, it was necessary to make use of the same plantation of pear trees for a successive phytotoxicity trial. The latter was intended to compare the effects of four dinitrophenolic compounds with an untreated control on that particular species of pear trees. The combinatorial arrangement which was devised for this experiment is given in the forthcoming Table 5.1.

Note that Latin letters in this table, representing treatments under comparison in the initial trial, *now form* a third blocking constraint for the experimental units, whilst those in subscript stand for the treatments to be compared in the successive experiment. The symbols – in subscripts – denoting the new set of treatments in Table 5.1 satisfy the following two properties :

E_4	F_0	A_2	C_0	B_3	D_1
C_3	B_2	D_3	A_1	E_0	F_4
B_2	E_4	F_0	D_2	C_1	A_3
A_0	D_3	C_4	E_1	F_2	B_0
D_0	C_1	B_1	F_3	A_4	E_2
F_1	A_0	E_3	B_4	D_4	C_2

Table 5.1 Layout for a successive experiment on a Latin Square of size six

(i) There is a control element $\lambda = 0$ say, which occurs twice in two levels of each of the three blocking constraints and exactly once in the remaining levels of each blocking constraint;

(i) Each of the remaining 4 symbols is duplicated in exactly one level of each of the three blocking constraints;

This is an example of a combinatorial arrangement intended to compare five treatments on thirty six experimental units subjected to three blocking constraints [24, p. 722]. In statistical jargon the array for the superimposed symbols in Table 5.1 is one of type O:OO:SSS.

5.2 Sequential experimentation on Latin Square designs: A Survey

In chapter 2 the author described the foundations for a variance ratio test for treatment effects in comparative experiments where experimental material was grouped under two blocking constraints. In Section § 4.4.1 the author drew the reader's attention to proposed alternative procedures which are most commonly used by statisticians in practice. In the context of the present study, on sequential experimentation on Latin Square designs, random allocation of treatments to plots as well as the variance ratio test for the new set of treatments are parallel to those given in § 4.2.1, and §§2.1 - 2.2.4 of this thesis.

Let us recall the procedure for equally likely random allocation of treatments to plots presented in Chapter 4 by extending it to our present situation, with a second set of treatments superimposed orthogonally on a Latin Square (e.g. addition of 2 or 4 treatments to Latin Squares of size 4): First a random choice of a layout *D* – representative amongst all F-inequivalent **superimposed** layouts – orthogonal to the Latin Square *L* is made. Secondly, choose at random one element *g* in S_k, where the subscript *k* denotes the number of superimposed symbols. Then the array given by $g \bullet D$ determines the allocation plan for the second set of treatments in the subsequent experiment. In order to illustrate what is meant in here let us consider the following 3 arrays for the orthogonal addition of 2 treatments to a Latin Squares of size 4:

A_1	B_1	C_2	D_2	A_2	B_2	C_1	D_1	A_1	B_1	C_2	D_2
B_2	A_2	D_1	C_1	B_1	A_1	D_2	C_2	B_2	A_2	D_1	C_1
C_1	D_1	B_2	A_2	C_2	D_2	B_1	A_1	C_2	D_2	B_1	A_1
D_2	C_2	A_1	B_1	D_1	C_1	A_2	B_2	D_1	C_1	A_2	B_2

(a) D_1 (b) D_2 (c) D_3

The value of k is 2, the superimposed layouts D_1, D_2, and D_3 above are clearly denoted by the treatment symbols in the set T = {1, 2}. The reader can verify that the element (1 2) in S_2 (on T) is such that the action of (1 2) on D_1 yields the superimposed layout D_2. In contrast, there is no element g in S_2 such that the action of g on D_1 yields D_3. Therefore, $D_1 \sim D_2$ and $D_1 \nsim D_3$. The equivalence terminology introduced in §4.5 extends naturally to superimposed layouts. That is, D_1 and D_2 are said to be F/Z-equivalent.

We recall from § 1.2 that the case of choosing a combinatorial layout **anticipating** the use of the same material for two successive experiments is out of the scope of the present study. This is due to the fact that the additional complexity of having the second allocation of treatments *conditional* to the first set is introduced in the statistical analysis.

Indeed, the main problems to overcome for significance testing of treatment effects are the combinatorics for the allocation of treatments to experimental units. This will be the main objective in the remainder of the thesis.

In [12] it is reported that Leonhard Euler seems to be the first mathematician to study the addition of further symbols to Latin Squares so that certain properties allow the construction of a kind of so-called magic squares. A number of examples may be obtained from [12, pp. 441-457]. In his study of the validity of agricultural experiments Fisher [17, 18, 19] renewed interest on the enumeration of those types of combinatorial arrays which seemed to be suitable for the design of two successive comparative experiments.

The importance of devising combinatorial arrangements for sequential experimentation on Latin Square designs was highlighted by Finney [13, 14, 16]. He expressed the problem of superimposing a further set of treatments on Latin Squares as that of "partitioning the n^2 cells of the $n \times n$ Latin Square into sets of nk_1, nk_2, nk_3, . . ., nk_r, where $k_1 + k_2 + k_3 + \ldots + k_r = n$, in such a way that the i-th set has k_i cells in each row, k_i cells in each column, and k_i cells in each letter." He describes a solution as a "(k_1, k_2, k_3, ... , k_r) orthogonal partition of an $n \times n$ Latin Square" and introduces some new terminology. He writes "A part of which k = 1 is generally known as a directrix of the square, and the terminology may conveniently be extended by introducing the names duplex (k = 2), triplex (k = 3), and so on." What is more, Finney [15] describes relationships amongst directrices as well as those between directrices and duplexes when presenting his results on the enumeration of partitions of Latin Squares of size six. This line of research was followed by Finney [16], Freeman [25, 26, 28] and Saidi [67].

The next author to write about the topic seems to be Freeman [24]. He presents his classical statistical analysis for the addition of treatments to Latin Square designs when the number of new and old treatments differ by one only. Examples of practical applications of the theory as well as methods of construction for these combinatorial arrangements are also given in [24]. Freeman develops the combinatorial aspects of the line of research opened by

Finney [13, 14, 16] and moves on to present different methods of constructing non-orthogonal partitions for Latin Squares of sizes 4, 5 and 6 in [25]. Some alternative superimpositions for Latin Squares of size less than or equal to six are described in detail by the same author in [26]. More general statistical theory for non-orthogonal layouts with any number of factors is due to Bradu [05]. This comprises the cases illustrated in [24].

In 1982, Finney [16] reported his enumeration results for all types of simple orthogonal partitions for Latin Squares of size six through a systematic computer search.

In 1983, Saidi [67] — under the guidance of Freeman — reports the first study of intersections of what she referred to as a Duplect (or duplex) of Latin Squares of sizes 4 and 5. For that Latin Square of size 4 representative of set II in [21] she lists all 12 duplexes [67, p. 30], studies the intersections of duplects (or duplexes) [67, pp. 30 - 34], and concludes that "Duplexes from different pairs may have 0 or 2 occurrences in common in both squares" of size 4 "but one occurrence in common in only one of the squares" [67, p. 51]. A parallel work is reported for one of the representatives of Latin Squares of size 5 as listed in [21].

In [27] Freeman describes the intersections of duplexes of Latin Squares of sizes 4 and 5 in some detail. Furthermore, he reports a full enumeration of parallel pairs of duplexes for those Latin Square representatives of size six listed in [21]. Furthermore, interrelations of directrices, duplexes and triplexes for those Latin Squares of size six listed in [21] are described by the same author in [29].

The present information on the addition of further treatments to Latin Squares of size $p = 4, 5, 6$ for the case when the new set of k treatments does not exceed the size of the Latin Square layout is summarized in Table 5.2. In contrast, readers interested in cases when $k > p$ are referred to Freeman [24, 25, 26].

p	k	Resulting type of design	Whether superimposed designs exist; refs.
4	2	0:00:000	Yes; Finney [13]
	3	0:00:SSS	Yes; Freeman [25]
	4	0:00:000	Yes; Euler [12]
5	4	0:00:SSS	Yes; Freeman [24, 25]
	5	0:00:000	Yes; Euler [12]
6	2	0:00:000	Yes; Finney [15]
	3	0:00:000	Yes; Finney [13]
	4	0:00:TTT	Yes; Freeman [24]
	5	0:00:SSS	Yes; Freeman [24, 25]
	6	0:00:000	No; Theorem 6.1 Alternatively, See Stinson [70]

Table 5.2 Present information on the addition of further treatments to Latin Squares of size $p \leq 6$.

From the literature survey this author found that all developments on the addition of treatments to Latin Square designs made use of a combinatorial classification of Latin Squares due to Fisher and Yates [20].

Since 1983, work on enumeration and categorization of partitions of Latin squares, both published and unpublished, has been disconnected from the practical statistical applications which motivated their study. Let us move on to Chapter 6, where the author takes the topic from its basics and completely rebuilds its foundations in a gradual manner. Then further developments unfold naturally.

6 Orthogonal superimposition of treatments to Latin Square designs

Combinatorial and statistical properties are brought together in order to review the problem of comparing a new set of treatments on experimental units still affected by an earlier set. Emphasis is given to the case when both (i) the original trial was arranged as a Latin Square design, and (ii) the new set of treatments is to be orthogonally added to the Latin Square design.

Full listing of F/Z-inequivalent sets of mutually orthogonal Latin Squares of sizes 4 and 5 for statistical applications is given. What is more, by making use of that categorisation of Latin Squares given in §3.3.6 and directrix properties, a simple counting argument shows the well known result that no Latin Square of size six possesses an orthogonal mate. Furthermore, with the aid of an electronic computer the author enumerated all complete sets of F/Z-inequivalent mutually orthogonal Latin Squares of size 7.

Concepts on Graeco-Latin Squares as well as alternative orthogonal superimpositions are reviewed in present day combinatorial terminology.

6.1 Orthogonal superimpositions

When facing the problem of comparing a new set of treatments on experimental units still affected by an earlier set we may, for example, find that the number of treatments in the new set equals the size of the Latin Square used in the original trial. In that situation, the simplest case of superimposition would be a further orthogonal (and equireplicate) addition of treatments. That is to say, when a design of type 0:00:000 can be completed.

An example of such arrangements was given in Table 1.3(c).

F/Z-inequivalence between two superimposed layouts can be deduced from Theorem 3.2.1 and Lemma 3.2.2. Namely, two orthogonal superimpositions D_1, D_2 are F/Z-equivalent, denoted $D_1 \sim D_2$,iff there exist $g \in S_k$ such that $g \bullet D_1 = D_2$ where k denotes the number of superimposed symbols. A thorough illustration of the terminology involved in the above-mentioned definition was given in §5.2.

Lemma 6.1

Let T be a set of p symbols { 1, 2, ..., p }. Let L_1 denote a Latin Square layout on T. Then, a second Latin Square L_2 on T can be superimposed orthogonally to L_1 iff L_1 has p disjoint transversals.

Proof:

Let $t \in T$ with respect to L_2.

$L_2 \perp L_1$ implies that t occurs with each symbol of L_1 only once by Definition 1.8. In addition, Definition 1.6 states that t appears only once with every level in each of the two blocking constraints. Therefore, the collection of cells containing symbol t forms a transversal by Definition 1.7. Note that t was arbitrarily taken from the set T which contains p different symbols. Therefore, we have p directrices. It remains to show that any pair of different directrices is mutually disjoint. Let D, D′ be any two different directrices. We will see that D ∩ D′ = ∅. Suppose D ∩ D′ ≠ ∅, then there exists a cell w such that w ∈ D ∩ D′.

Now, w ∈ D implies that there exist an element $t \in T$ with respect to L_2 such that $t \in$ w by Definition 1.7. On the other hand, there exists an element $t' \in T$ with respect to L_2 such that $t' \in$ w as w ∈ D′ as well. Therefore, we have two elements t, $t' \in T$ with respect to L_2 such that t, $t' \in$ w which contradicts Definition 1.8. Therefore, D ∩ D′ = ∅. The converse implication is trivial, namely, fill all cells of each transversal with each of the p different symbols in order to construct the superimposed orthogonal mate ♯.

Euler [12] seems to be the first person to construct Graeco-Latin Squares through directrix properties. He made use of arrays of this kind in the construction of magic squares [12, pp. 444 - 456].

In 1960, Bose, Shrikhande and Parker [04] demonstrated the existence of such designs for Latin Squares of all sizes except $p = 2, 6$.

Construction methods which show the existence of Graeco-Latin Squares found in the literature involve :

- Transversal properties [12]
- Theory of Galois Fields [65]
- Theory of quasigroups and Loops [07]
- Projective geometry [65]

Graeco-Latin Squares are of special interest as mathematical objects [12]. Furthermore, their applications in the areas of coding theory [03] and compiler testing [49] have been reported in the literature. Of our particular concern here is to clarify the role they play in statistical randomized trials and such usage will be discussed henceforth.

Definition 6.1

A collection of Latin Squares is said to be a complete set of **M**utually **O**rthogonal **L**atin **S**quares (MOLS) iff the forthcoming two conditions hold (i) there are p-1 elements in the collection and, (ii) any two elements in the collection form a Graeco-Latin Square array when one is superimposed upon the other.

The action of S(P) on the set of Latin Square arrays of a given size not only defines an equivalence relation, but also *preserves* the **relationship** amongst the transversals – if any such transversal exists – within the elements of each orbit. Consequently, in order to enumerate and categorize all Graeco-Latin Squares of a given size use of those orbit representatives – induced by the action of S(P) – which admit an orthogonal superimposition will be made. Useful subsets of S(P) to unpack those Latin Squares in a given orbit are listed in § 4.2.1 and Appendix I for Latin Squares of sizes 4 and 5, respectively.

A complete listing of transversals for those orbit representatives listed in §3.3.6 is given in Appendix II. By careful study around orbit representatives of Latin

Squares of sizes 4 and 5 which possess a complete set of disjoint transversals the author obtained the distribution of F/Z-inequivalent MOLS given in Table 6.1. With aid of an electronic computer the author enumerated those for p = 7.

p =	3	4	5	6	7
Sets of MOLS	1	2	36	None	21,211,200

Table 6.1 Distribution of F/Z-inequivalent sets of MOLS of size $p \leq 7$

Readers interested on the combinatorial relationship between projective planes and complete sets of MOLS of size 9 are referred to Owens and Preece [53]. Owens and Preece [53] define "two complete sets $\mathcal{L}$ and $\mathcal{L}'$ of order n equivalent if and only if there exist permutations θ and ϕ of the first n natural numbers such that the following transformation converts $\mathcal{L}$ into $\mathcal{L}'$: Permute the rows of every square so that row i becomes row iθ, 1 ≤ i ≤ n. Permute the columns of every square so that column j becomes column jϕ, 1 ≤ j ≤ n. Then permute the symbols, in each square separately, so that the new first rows are finally in natural order." With this definition of equivalence, Owens and Preece [53] found there to be 19 inequivalent sets, even though these come from just 4 projective planes. For each of p = 3, 4, 5 and 7 there is just a single projective plane.

Information in Table 6.1 shows that for the statistician a projective plane does not determine all F/Z-inequivalent sets of mutually orthogonal Latin Squares. From Table 6.4 and appendix III the reader can verify that no two Graeco-Latin squares of a given size from different sets yield the same value for the F/Z-statistic in the ANOVA table. Likewise, any two Graeco-Latin squares in a given set are F/Z-inequivalent.

Theorem 6.1

No Latin Square layout of size six has *six* disjoint transversals.

Proof.

First of all, in order to have six disjoint transversals the cells in the first row of each orbit representative, under the action of S(P), must be an element in each of them. Therefore, from Appendix II the only candidates to have a set of six disjoint transversals are the orbit representatives numbered as 10, 15, 16 and 22 in the categorization given in §3.3.6.

For the sake of the forthcoming combinatorial argument, an order for the transversals is introduced as follows: the transversals from left to right will be referred to as First, Second, ..., and Sixth transversal, respectively. Let N_1, N_2, ..., N_6 be the corresponding number of each so ordered transversal. The counting argument is best presented in algorithmic pseudo-code in Table 6.2. This will allow the reader to implement a small computer program in the language of the reader's preference or follow the procedure manually.

Variables:

d1, d2, d3 and d4 denote vectors of first four transversals, respectively;
i1max, i2max, i3max and i4max denote the maximum number of first four transversals, respectively;
i1, i2, i3 and i4 denote auxiliary variables, respectively.

Auxiliary Procedures:

Procedure direct(k, dk, ikmax): determines all k-transversals into dk, record counts into ikmax;
Boolean function disjoint(): returns true if transversals are disjoint.

Algorithm:

```
i1max = i2max = i3max = i4max = 0; { initialize counts for
                        first directrices }
direct(1,d1,i1Max);
if i1Max > 0 then {if at least one directrix then proceed}
  For i1 = 1 to i1max
    direct(2,d2,i2Max);
    if i2Max > 0 then {if at least one directrix then proceed}
    For i2:= 1 to i2max
      if disjoint(d1[i1],d2[i2]) then
        direct(3,d3,i3Max);
        if i3Max > 0 then {if at least one directrix then proceed}
        For i3:= 1 to i3max
          if disjoint(d3[i3],d2[i2],d1[i1]) then
           direct(4,d4,i4Max);
           if i4Max > 0 then { if at least one directrix then
                 proceed }
             For i4:= 1 to i4max
               if disjoint(d4[i4],d3[i3],d2[i2],d1[i1]) then
                           PRINT FIRST FOUR DISJOINT TRANSVERSALS
               end {if}
             end {for}
           end {if}
          end {if}
        end {for}
        end {if}
      end {if}
    end {for}
    end {if}
  end {for}
end {if}
```

Table 6.2 Algorithmic pseudo-code in Theorem 6.1

For instance, it can be seen that there are only two sets S_1, S_2 of first four disjoint transversals for orbit representative number 15 in §3.3.6. These are as follows, S_1 = {{1, 9, 17, 20, 30, 34}, {2, 10, 18, 21, 29, 31}, {3, 8, 16, 24, 25, 35}, {4, 7, 14, 23, 27, 36}} and S_2 = {{1, 10, 14, 24, 27, 35}, {2, 9, 16, 23, 25, 36}, {3, 7, 18, 20, 29, 34}, {4, 8, 17, 21, 30, 31}}. In Table 6.3, a simple visual examination will convince the reader that none admits a Fifth transversal disjoint to all of the previous four. Greek letters highlight the transversals in both sets S_1 and S_2. A parallel argument applies to the other three orbit representatives. Therefore, the theorem follows ♯.

1α	2β	3γ	4δ	5	6
2δ	1γ	4α	3β	6	5
3	5δ	1	6γ	2α	4β
4	6α	5β	1	3δ	2γ
5γ	4	6δ	2	1β	3α
6β	3	2	5α	4γ	1δ

(i) Set S_1

1α	2β	3γ	4δ	5	6
2γ	1δ	4β	3α	6	5
3	5α	1	6β	2δ	4γ
4	6γ	5δ	1	3β	2α
5β	4	6α	2	1γ	3δ
6δ	3	2	5γ	4α	1β

(ii) Set S_2

Table 6.3 Sets of first four ordered disjoint transversals for orbit representative number 15 in §3.3.6.

The complete sets of F/Z-inequivalent sets of MOLS for p = 4 are given in Table 6.4. Those for p = 5 are given in Appendix III.

1_1^1	2_2^2	3_3^3	4_4^4
2_3^4	1_4^3	4_1^2	3_2^1
3_4^2	4_3^1	1_2^4	2_1^3
4_2^3	3_1^4	2_4^1	1_3^2

[1]

1_1^1	2_2^2	3_3^3	4_4^4
2_3^4	1_4^3	4_1^2	3_2^1
4_2^3	3_1^4	2_4^1	1_3^2
3_4^2	4_3^1	1_2^4	2_1^3

[2]

Table 6.4 Full enumeration of F/Z-inequivalent sets of MOLS of size p = 4 (trivial group on added symbols)

In §4.1.1 the reader was reminded that, for randomized statistical trials, the array in the experimental design is required to be randomly chosen with equal probability on the basis of all F/Z-inequivalent layouts. In present-day notation, the action of S_p on the set of superimposed layouts (2nd set of treatments) induces a partition on the latter such that all elements in the same orbit are **F/Z-equivalent**. In addition, section §3.3.5 shows that we can always choose the non-isomorphic – under S_p – first row with ordered symbols 1, 2, ..., *p*. The reader is referred to review §4.2.1.

Lemma 6.2

Let *T* be a set of *p* symbols { 1, ..., *p* }. Let L denote a Latin Square on *T* such that *L* has *p* disjoint transversals. If Σ is the set of MOLS which contains *L*, then the number of elements in Σ is at most *p* - 1.

Proof

Let Σ be that collection of MOLS which contains *L*. In addition, let us suppose there are $q \geq p$ MOLS in Σ and arrive at a contradiction. Without loss of generality, assume that the first row in every element in Σ contains the elements in *T* ordered as 1, 2, ..., *p* – See for instance Table 6.4 or Appendix III. Consider the symbols appearing in the cell (2,1). Evidently, the symbol 1 cannot appear in this cell by Definition 1.6. Then, as we have only *p* - 1 different symbols left which could appear in this cell there is a symbol *k* say, which appears twice in cell (2,1). In that case we would have a Graeco-Latin square such that cells (2, 1) and (1, k) contain symbol k twice. This contradicts Definition 1.8. Therefore, there can be no more than p - 1 MOLS in Σ. ♯

Suppose a comparative trial *has taken place* on a Latin Square design of size p. Furthermore, suppose a new trial to compare a set of p treatments is desirable on the same experimental units. When the Latin Square in the original trial contains a complete set of p disjoint transversals, then Lemma 6.2 guarantees that there will be at most p - 2 **F/Z-inequivalent** superimpositions to choose from. To recapitulate from §4.2.1, Bailey and Rowley [02, §2] stated "the randomization is valid that consists of randomly choosing a plan from Φ with **uniform** probability ". Equally likely random choice will here imply that the probability of having a particular value for the test statistic is at most 1/(p - 2). Consequently, should the reader be willing to risk a 5% probability of making a Type I error – rejecting the null hypothesis of no difference in treatment effects when the null is true –, the value of p is required to be 22. In other words, orthogonal superimposition of p symbols for sequential experimentation on Latin Square block designs of size $p \leq 21$ will lead to tests of low power. This is stated in the following:

Lemma 6.3

Let Ω be a set of p^2 experimental units which are blocked in a Latin Square design L, and let L contain a set of p disjoint transversals. In addition, let the collection of symbols T = { 1, ..., p } denote a set of treatments for a ***sequential experiment*** on L. If $p \leq 21$ and α = P(Type Error I) = 5%, then no orthogonal superimposition of p symbols for statistical **sequential experimentation** is suitable for a *variance ratio test* with significance level α.

All things considered, alternative methods to add a further classification to Latin Square designs are needed.

6.2 Alternative orthogonal equireplicate superimpositions.

Finney [13 to 16], Saidi [67] and Freeman [25, 26, 28] have studied the possibility of adding a further set of treatments orthogonally to Latin Square designs but from a different perspective to the one presented in this manuscript, namely, partitions of Latin Squares and transversal properties, interrelationships of directrices, duplexes and triplexes.

There is no description of the methodology for enumeration of duplexes or triplexes in the literature. Therefore, previous enumeration techniques cannot be precisely compared with the one used in here. However, Saidi [67] seems to require a full enumeration of duplexes followed by the study of their intersections. In contrast, a simple row-by-row backtrack search eliminated any partial extension which would not lead to a complete array at run-time. Consequently, the method described here is likely to be far faster for generating the possibilities than the ones used in the past.

Some examples are given in Table 6.5 for the orthogonal superimposition of (1) two treatments on Latin Squares of sizes four and six, respectively, and (2) three symbols on Latin Squares of size six.

1_1	2_1	3_2	4_2
2_2	1_2	4_1	3_1
3_2	4_2	2_1	1_1
4_1	3_1	1_2	2_2

(i)

1_1	2_1	3_1	4_2	5_2	6_2
2_1	1_2	4_2	3_1	6_2	5_1
3_2	5_1	1_2	6_1	2_2	4_1
4_2	6_2	2_1	5_1	1_1	3_2
5_2	3_2	6_1	2_2	4_1	1_1
6_1	4_1	5_2	1_2	3_1	2_2

(ii)

1_1	2_1	3_2	4_2	5_3	6_3
2_1	1_1	4_2	3_2	6_3	5_3
3_3	4_3	5_1	6_1	1_2	2_2
4_3	3_3	6_1	5_1	2_2	1_2
5_2	6_2	1_3	2_3	3_1	4_1
6_2	5_2	2_3	1_3	4_1	3_1

(iii)

Table 6.5 Orthogonal addition of two treatments to Latin Squares of sizes four and six respectively, as well as that of three treatments on Latin Squares of size six.

Note that none of the Latin Squares used as basis in Table 6.5 has transversal properties.

6.3 Combinatorial methodology.

The attention of the reader is drawn to the applied problem described in § 5.1.1. Namely, first a Latin Square *was used* for the original trial and then a superimposed layout was needed for the **sequential experiment**. Therefore, the three blocking constraints for the final array remain **fixed** during the process of seeking for all possible superimposed layouts.

A simple computer algorithm for the enumeration process can be *deduced* from that in Lam and Thiel [47]. Complications of group symmetries are eliminated from the program but computer power is needed. In this case: (i) the constraints of the orthogonal superimposition remain fixed (picture a field orchard), and (ii) non-isomorphism testing is required only for the first row. The computer implementation includes:

- Use of the action of the group S_k, where k denotes the number of superimposed symbols;
- Definition of the boolean predicate function [47, p. 474] according to the desired properties for the overall arrangement under the three blocking constraints.

Different first rows are found, followed by an exhaustive in-depth backtrack search in order to produce the remaining rows. A check for extendable subdesigns through the boolean function is performed for each subsequent row.

6.4 Enumeration results and examples

Orbit Representative

[1]:	1234 2143 3421 4312	[2]:	1234 2143 3412 4321
1.-	1122 2211 1122 2211		1122 2211 1122 2211
2.-	1122 2211 2211 1122		1122 2211 2211 1122
3.-	1212 1212 2121 2121		1212 1212 2121 2121
4.-	1212 2121 2112 1221		1212 2121 2121 1212
5.-	1221 1221 2112 2112		1221 1221 2112 2112
6.-	1221 2112 1212 2121		1221 2112 1221 2112

Table 6.6 Full enumeration of orthogonal superimpositions of two treatments on Latin Squares of size four.

We have seen in § 3.3.6 that the orbit representative – under S(P) – for Latin Squares of size 4 indexed as number 1 generates 18 orbit representatives of Latin Squares under S_4. The attention of the reader is drawn to Table 6.6. This table lists all *F/Z-inequivalent* orthogonal superimpositions of two symbols for each of the representatives listed in § 3.3.6. Therefore, from orbit number 1 in § 3.3.6, the combinatorialist can generate 18×6 = 108 possible arrangements for the addition of two symbols to Latin Squares of size $p = 4$. On the other hand, 36 arrangements can be produced from the orbit representative indexed as number 2.

The problem we are studying here is that in which the original experiment **was designed** as a Latin Square and then a second experiment is considered for a new trial on the same experimental units. In such a situation the present study of the combinatorics of adding two treatments to Latin Squares of size four implies that there are only six possible different values for the statistic in the ANOVA table for the **sequential** experiment. Therefore, these layouts would lead to tests of low power. That is to say, they are of no use for statistical comparative tests, a conclusion that seems to have been given for the first time by Finney [13]. In contrast, the number of possibilities for the orthogonal addition of treatments to Latin Squares of size six is more promising for the statistician / research-worker.

Full enumeration of combinatorial layouts obtainable from the addition of (a) two, and (b) three treatments to Latin Squares of size six resulted in :

(a) 272, 272, 400, 352, 320, 320, 256, 320, 320, 896, 0, 0, 0, 0, 320, 320, 0, 272, 320, 320, 0 and 320;

(b) 155, 155, 187, 161, 319, 319, 173, 361, 361, 497, 1215, 1215, 5949, 1215, 295, 295, 945, 155, 319, 361, 1215 and 295 possibilities for each of the 22 orbit representatives for Latin Squares of size six as indexed in §3.3.6, respectively (For both (a) and (b), the counts for representatives 1, 2 and 18 are the same. These representatives come from the same species. Similarly for representatives 5, 6 and 19, for 8, 9 and 20, for 11, 12 and 21, and for 15, 16 and 22). This **confirms** the results obtained independently by Finney [16] and Freeman [27, 29] for those representatives listed in [21]. The requirement of equally likely random allocation of treatments over this type of arrays calls for a probability of making a type I error of at least 1/256 when two symbols can be superimposed whilst that when three symbols are superimposed is at least 1/155.

It was proved in §4.5 that transpositions along the left-to-right diagonal in the set of Latin Squares DO NOT preserve F/Z-equivalence properties.

Examples for each representative listed in §3.3.6 are given in Table 6.7. Superimposition of two symbols is given in subscripts whilst superscripts are used for that of three. A full listing of arrays similar to the ones illustrated in Table 6.7 can be obtained from the diskettes attached to this manuscript.

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_1^1 & 4_2^2 & 5_1^3 & 6_2^3 & 3_2^2 \\ 3_2^3 & 4_1^3 & 5_1^1 & 6_1^1 & 1_2^2 & 2_2^2 \\ 4_2^3 & 3_2^3 & 6_1^1 & 1_2^2 & 2_1^2 & 5_1^1 \\ 5_2^2 & 6_2^2 & 2_2^3 & 3_1^1 & 4_1^1 & 1_1^3 \\ 6_1^2 & 5_2^2 & 1_2^3 & 2_2^3 & 3_1^1 & 4_1^1 \end{array}$$

[1]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_2^1 & 4_1^2 & 5_2^2 & 6_1^3 & 3_2^3 \\ 3_2^2 & 4_1^3 & 2_2^3 & 6_1^1 & 1_2^2 & 5_1^1 \\ 4_2^3 & 5_1^2 & 6_2^1 & 2_2^3 & 3_1^1 & 1_1^2 \\ 5_2^3 & 6_2^2 & 1_2^3 & 3_1^1 & 4_1^1 & 2_1^2 \\ 6_1^2 & 3_2^3 & 5_1^1 & 1_1^3 & 2_2^2 & 4_2^1 \end{array}$$

[2]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_1^1 & 4_2^3 & 5_1^2 & 6_2^2 & 3_2^3 \\ 3_2^2 & 4_1^2 & 2_2^3 & 6_1^1 & 1_2^3 & 5_1^1 \\ 4_2^3 & 6_2^2 & 5_1^1 & 2_2^3 & 3_1^1 & 1_1^2 \\ 5_2^2 & 3_2^3 & 6_1^1 & 1_2^3 & 2_1^2 & 4_1^1 \\ 6_1^3 & 5_2^3 & 1_2^2 & 3_1^1 & 4_1^1 & 2_2^2 \end{array}$$

[3]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_1^1 & 4_2^2 & 3_2^2 & 6_1^3 & 5_2^3 \\ 3_2^3 & 5_1^2 & 1_2^3 & 6_1^1 & 4_1^1 & 2_2^2 \\ 4_1^3 & 6_2^2 & 5_1^1 & 1_2^3 & 2_2^2 & 3_1^1 \\ 5_2^2 & 3_2^3 & 6_1^1 & 2_1^3 & 1_2^2 & 4_1^1 \\ 6_2^2 & 4_2^3 & 2_2^3 & 5_1^1 & 3_1^1 & 1_1^2 \end{array}$$

[4]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_1^1 & 4_2^3 & 5_2^2 & 6_1^2 & 3_2^3 \\ 3_2^3 & 4_1^3 & 5_1^1 & 6_1^1 & 2_2^2 & 1_2^2 \\ 4_2^2 & 6_2^2 & 2_2^3 & 1_1^3 & 3_1^1 & 5_1^1 \\ 5_1^2 & 3_2^2 & 6_1^1 & 2_2^3 & 1_2^3 & 4_1^1 \\ 6_2^3 & 5_2^3 & 1_2^2 & 3_1^1 & 4_1^1 & 2_1^2 \end{array}$$

[5]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_2^1 & 4_1^2 & 5_1^2 & 6_2^3 & 3_2^3 \\ 3_1^2 & 4_2^3 & 1_2^3 & 6_1^1 & 2_2^2 & 5_1^1 \\ 4_2^3 & 5_1^2 & 6_1^1 & 1_2^3 & 3_1^1 & 2_2^2 \\ 5_2^3 & 6_1^2 & 2_2^3 & 3_2^1 & 4_1^1 & 1_1^2 \\ 6_2^2 & 3_2^3 & 5_2^1 & 2_1^3 & 1_1^2 & 4_1^1 \end{array}$$

[6]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_2^1 & 4_2^2 & 3_1^2 & 6_2^3 & 5_1^3 \\ 3_2^3 & 5_1^2 & 1_2^3 & 6_1^1 & 2_2^2 & 4_1^1 \\ 4_2^3 & 6_2^2 & 2_1^3 & 5_1^1 & 1_1^2 & 3_2^1 \\ 5_2^2 & 3_2^3 & 6_1^1 & 2_2^3 & 4_1^1 & 1_1^2 \\ 6_1^2 & 4_1^3 & 5_2^1 & 1_2^3 & 3_1^1 & 2_2^2 \end{array}$$

[7]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_2^1 & 4_1^2 & 3_2^2 & 6_1^3 & 5_2^3 \\ 3_2^3 & 5_1^2 & 1_2^3 & 6_1^1 & 2_2^2 & 4_1^1 \\ 4_2^3 & 6_1^2 & 2_2^3 & 5_2^1 & 3_1^1 & 1_1^2 \\ 5_1^2 & 4_2^3 & 6_2^1 & 2_1^3 & 1_2^2 & 3_1^1 \\ 6_2^2 & 3_2^3 & 5_1^1 & 1_1^3 & 4_1^1 & 2_2^2 \end{array}$$

[8]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_2^1 & 4_1^2 & 3_2^2 & 6_2^3 & 5_1^3 \\ 3_2^3 & 5_1^2 & 1_2^3 & 6_1^1 & 4_1^1 & 2_2^2 \\ 4_1^3 & 6_1^2 & 5_2^1 & 2_2^3 & 1_2^2 & 3_1^1 \\ 5_2^2 & 3_2^3 & 6_1^1 & 1_1^3 & 2_1^2 & 4_2^1 \\ 6_2^2 & 4_2^3 & 2_2^3 & 5_1^1 & 3_1^1 & 1_1^2 \end{array}$$

[9]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_2^1 & 4_1^2 & 3_1^2 & 6_2^3 & 5_2^3 \\ 3_2^3 & 4_2^3 & 5_1^1 & 6_1^1 & 1_1^2 & 2_2^2 \\ 4_2^3 & 3_2^3 & 6_2^1 & 5_1^1 & 2_1^2 & 1_1^2 \\ 5_2^2 & 6_1^2 & 1_2^3 & 2_2^3 & 4_1^1 & 3_1^1 \\ 6_1^2 & 5_1^2 & 2_2^3 & 1_2^3 & 3_2^1 & 4_1^1 \end{array}$$

[10]

$$\begin{array}{cccccc} 1^1 & 2^1 & 3^2 & 4^2 & 5^3 & 6^3 \\ 2^1 & 1^1 & 4^2 & 3^2 & 6^3 & 5^3 \\ 3^3 & 4^3 & 5^1 & 6^1 & 1^2 & 2^2 \\ 4^3 & 5^2 & 6^1 & 1^3 & 2^2 & 3^1 \\ 5^2 & 6^2 & 1^3 & 2^3 & 3^1 & 4^1 \\ 6^2 & 3^3 & 2^3 & 5^1 & 4^1 & 1^2 \end{array}$$

[11]

$$\begin{array}{cccccc} 1^1 & 2^1 & 3^2 & 4^2 & 5^3 & 6^3 \\ 2^1 & 1^1 & 4^2 & 5^2 & 6^3 & 3^3 \\ 3^3 & 4^3 & 5^1 & 6^1 & 1^2 & 2^2 \\ 4^3 & 5^3 & 6^1 & 3^1 & 2^2 & 1^2 \\ 5^2 & 6^2 & 1^3 & 2^3 & 3^1 & 4^1 \\ 6^2 & 3^2 & 2^3 & 1^3 & 4^1 & 5^1 \end{array}$$

[12]

$$\begin{array}{cccccc} 1^1 & 2^1 & 3^2 & 4^2 & 5^3 & 6^3 \\ 2^1 & 1^1 & 4^2 & 3^2 & 6^3 & 5^3 \\ 3^3 & 4^3 & 5^1 & 6^1 & 1^2 & 2^2 \\ 4^3 & 3^3 & 6^1 & 5^1 & 2^2 & 1^2 \\ 5^2 & 6^2 & 1^3 & 2^3 & 3^1 & 4^1 \\ 6^2 & 5^2 & 2^3 & 1^3 & 4^1 & 3^1 \end{array}$$

[13]

$$\begin{array}{cccccc} 1^1 & 2^1 & 3^2 & 4^2 & 5^3 & 6^3 \\ 2^1 & 3^2 & 1^3 & 5^1 & 6^2 & 4^3 \\ 3^3 & 1^2 & 2^3 & 6^1 & 4^2 & 5^1 \\ 4^3 & 6^3 & 5^2 & 2^2 & 1^1 & 3^1 \\ 5^2 & 4^1 & 6^1 & 3^3 & 2^3 & 1^2 \\ 6^2 & 5^3 & 4^1 & 1^3 & 3^1 & 2^2 \end{array}$$

[14]

$$\begin{array}{cccccc} 1_1^1 & 2_1^1 & 3_1^2 & 4_2^2 & 5_2^3 & 6_2^3 \\ 2_1^1 & 1_1^1 & 4_2^2 & 3_2^2 & 6_1^3 & 5_2^3 \\ 3_2^3 & 5_1^2 & 1_2^3 & 6_1^1 & 2_2^2 & 4_1^1 \\ 4_1^3 & 6_2^2 & 5_1^1 & 1_2^3 & 3_1^1 & 2_2^2 \\ 5_2^2 & 4_2^3 & 6_1^1 & 2_1^3 & 1_2^2 & 3_1^1 \\ 6_2^2 & 3_2^3 & 2_2^3 & 5_1^1 & 4_1^1 & 1_1^2 \end{array}$$

[15]

1_1^1	2_1^1	3_1^2	4_2^2	5_2^3	6_2^3
2_1^1	1_1^1	4_2^2	5_1^2	6_2^3	3_2^3
3_2^2	4_1^3	1_2^3	6_1^1	2_2^2	5_1^1
4_2^3	5_2^2	6_1^1	1_2^3	3_1^1	2_1^2
5_2^3	6_2^2	2_2^3	3_1^1	1_1^2	4_1^1
6_1^2	3_2^3	5_1^1	2_2^3	4_1^1	1_2^2

[16]

1^1	2^1	3^2	4^2	5^3	6^3
2^1	1^1	4^2	3^2	6^3	5^3
3^3	5^2	1^3	6^1	2^2	4^1
4^3	6^2	2^3	5^1	1^2	3^1
5^2	3^3	6^1	1^3	4^1	2^2
6^2	4^3	5^1	2^3	3^1	1^2

[17]

1_1^1	2_1^1	3_1^2	4_2^2	5_2^3	6_2^3
2_1^1	1_2^1	4_2^2	3_2^2	6_1^3	5_1^3
3_2^3	4_1^3	5_1^1	6_1^1	1_2^2	2_2^2
4_1^3	5_2^2	6_1^1	2_2^3	3_2^1	1_1^2
5_2^2	6_2^2	2_2^3	1_1^3	4_1^1	3_1^1
6_2^2	3_1^3	1_2^3	5_1^1	2_1^2	4_2^1

[18]

1_1^1	2_1^1	3_1^2	4_2^2	5_2^3	6_2^3
2_1^1	1_2^1	4_1^2	3_1^2	6_2^3	5_2^3
3_2^3	5_1^2	1_2^3	6_1^1	4_1^1	2_2^2
4_2^3	6_1^2	5_2^1	1_2^3	2_1^2	3_1^1
5_1^2	4_2^3	6_1^1	2_2^3	3_2^1	1_1^2
6_2^2	3_2^3	2_2^3	5_1^1	1_1^2	4_1^1

[19]

1_1^1	2_1^1	3_1^2	4_2^2	5_2^3	6_2^3
2_1^1	1_2^1	4_1^2	3_2^2	6_1^3	5_2^3
3_2^3	5_1^2	1_2^3	6_1^1	4_1^1	2_2^2
4_2^3	6_1^2	2_2^3	5_2^1	3_1^1	1_1^2
5_1^2	3_2^3	6_2^1	2_1^3	1_2^2	4_1^1
6_2^2	4_2^3	5_1^1	1_1^3	2_2^2	3_1^1

[20]

1^1	2^1	3^2	4^2	5^3	6^3
2^1	1^1	4^2	5^2	6^3	3^3
3^3	4^3	5^1	6^1	1^2	2^2
4^3	3^2	6^1	1^3	2^2	5^1
5^2	6^2	1^3	2^3	3^1	4^1
6^2	5^3	2^3	3^1	4^1	1^2

[21]

1_1^1	2_1^1	3_1^2	4_2^2	5_2^3	6_2^3
2_1^1	1_1^1	4_2^2	5_2^3	6_1^3	3_2^2
3_2^3	5_1^2	1_2^3	6_1^1	4_1^1	2_2^2
4_1^3	6_2^2	2_2^3	1_2^2	3_1^1	5_1^1
5_2^2	3_2^3	6_1^1	2_1^3	1_2^2	4_1^1
6_2^2	4_2^3	5_1^1	3_1^1	2_2^2	1_1^3

[22]

Table 6.7 Superimpositions of two (in subscript) and three (in superscript) symbols to Latin Squares of size six

It is sometimes found in practice that the number of treatments in the new trial or the size of the Latin Square in the original one do not allow a further orthogonal superimposition. As an alternative, statisticians resort to the property of balance [61].

In Chapter 7 the writer reports his results from a systematic computer search for balanced superimpositions of four symbols on Latin Squares of size six.

7 Enumerating balanced superimpositions of four treatments on Latin Squares of size six

This chapter briefly reviews recent literature on the addition of a further set of symbols to Latin Squares and their applications in problems found in diverse disciplines. The discovery of a large number of combinatorial arrangements referred to in the literature as balanced superimpositions of four treatments on Latin Squares of size six is reported in modern combinatorial terminology.

7.1 Background

A historical account of the developments on Latin Squares can be found in the books by Dénes and Keedwell [08, 09].

Renewed interest on the study of superimposing symbols on Latin Squares is attributed to Fisher in the 1920s [17, 18]. After he introduced the use of Latin Squares for randomized statistical trials in order to validate experiments in agriculture.

Developments in the areas of Compiler Testing in Computer Science [49] as well as Coding in Communications [03] have motivated statisticians, mathematicians and combinatorialists to pursue further research on these interesting squares.

In §6.1 a full listing of all F/Z-inequivalent sets of mutually orthogonal Latin Squares of sizes 4 and 5 for statistical applications, by making use of group actions, was given. Furthermore, in §5.2 a survey of the problem of superimposing a further set of symbols whether orthogonal or not to Latin Square designs was presented. In addition, §6.4 included a full enumeration of arrays obtainable from the orthogonal addition of 2 and 3 treatments to Latin squares of size six **for sequential** experimentation. These confirmed the results independently obtained by Finney [16] and Freeman [27, 29].

It is frequently found in long-lived animal, agricultural and pharmaceutical experimentation that a further set of treatments cannot be added orthogonally to a Latin Square design. As one alternative Statisticians resort to the property of balance [61].

Due to their usefulness in practical experimentation as well as their interesting combinatorial properties, special attention is given to the study of balanced superimpositions of four symbols to Latin Squares of size six. Freeman [24, p. 726] postulated that such arrays exist for all Latin Squares of size six. One example of this kind of combinatorial arrays which statisticians may use in sequential experimentation is given in Table 7.1.

1_1	2_1	3_2	4_2	5_3	6_4
2_1	1_1	4_2	5_3	6_3	3_4
3_3	4_4	5_1	6_1	1_4	2_2
4_4	3_3	6_3	1_2	2_2	5_1
5_2	6_2	2_4	3_4	4_1	1_3
6_2	5_4	1_4	2_3	3_1	4_3

Table 7.1 A balanced superimposition of four treatments on a Latin Square of size six.

Despite the fact that a variance ratio test calls for an equally likely selection from a complete aggregate of arrays, no attempt for enumerating them in full seems to have been made.

In [28] Freeman presented all the known orthogonal properties of a Latin Square representative of orbit number 10. He emphasized on the interrelations of directrices, duplexes and triplexes. That piece of work could have been used to study the possibility of enumerating all balanced superimpositions of 4 symbols to the above-mentioned orbit representative. However, we would have no information at all about the other squares. Specially those which possess no transversal. That is to say, those which lack orthogonal properties.

The study on this topic was therefore divided into two parts.

The first was intended to look for any pattern by fixing both (i) one different first row, and (ii) a lexicographical order for the repeated symbols in the rows.

The second part of the study aimed to relax the constraint on the first rows. That is to say, a full enumeration would be performed producing the repeated symbols in lexicographical order for the rows.

This chapter reports the results of a systematic computer search for arrays of this kind. Even though the discovery of examples has been made by the thousands, the search for them is still far from complete.

Before the writer moves on to describe the enumeration method the appropriate terminology for their classification must be introduced.

7.2 Preliminary Concepts

Definition 7.2.1

Let S be a set of 6 symbols $S = \{ A, B, C, D, E, F \}$, T be a set of 4 symbols $T = \{ 1, 2, 3, 4 \}$, Ω be an 6×6 array of cells, and L be a Latin Square on S then a superimposed layout D on L is said to complete a balanced superimposition of four symbols on L if and only if it is obtainable from the allocation of symbols $t \in T$ to cells $\omega \in \Omega$ such that the following four conditions hold (i) symbols $t \in T$ are equally replicated in the 6x6 layout, (ii) each symbol $t \in T$ appears either once or twice at every level of each of the three blocking constraints, (iii) Two and only two symbols appear twice at every level of each of the three blocking constraints, (iv) every distinct pair of symbols occurs twice in exactly one level of each of the three blocking constraints.

Notation: Rows, Columns and Symbols in the Latin Square array in Definition 7.2.1 will be referred to as the *constraints* of the balanced superimposition of four symbols on the Latin Square used as basis.

The reader can verify that the set *T* in Definition 7.2.1 satisfies the conditions of being totally balanced with respect to each of the three blocking constraints.

The appeal to a random choice of array with uniform probability aims to providing a valid estimate of the residual variation in the ANOVA table. Therefore, a discrimination criterion to distinguish amongst balanced superimpositions of four symbols on Latin Squares of size six from both the statistical and combinatorial points of view must incorporate the information given by the test statistic.

When Δ is the set of Latin Squares of size six, the action of S(P) on Δ induces a partition of the latter into 22 orbits – See § 3.3.6. These Orbit representatives can generate all F/Z-inequivalent Latin Squares of this size. We shall then make use of each *category* representative in order to enumerate the possible balanced superimpositions of four symbols on block designs of this size.

Let *S*, *T* and *L* be as in definition 7.2.1 and Δ be the set of superimposed layouts which complete a balanced superimposition of four symbols on *L*. Then the action of the symmetry group on *T*, S_4 say, on Δ induces a partition of Δ such that all elements in each orbit are **F/Z-equivalent**.

In other words, two superimposed (2nd Set of treatments) layouts B_1 and B_2 on L which satisfy Definition 7.2.1 are **F/Z-equivalent**, denoted $B_1 \sim B_2$, if and only if there exist an element g in S_4 such that $g \bullet B_1 = B_2$. For an illustrative example see §5.2.

7.3 Combinatorial method and enumeration results.

Note that the three constraints of a Latin Square remain fixed during the process of seeking for all possible superimpositions of an additional set of symbols. Picture a field orchard for instance, no gradients of fertility (rows, columns) move, nor do the residual effects from the Latin Square design in the original trial move.

As it was pointed out in § 6.3, in the computer program care must be given to :

- Use of the action of the group S_4 on the set of superimposed layouts;
- Definition of a boolean function in order for the superimposed symbols to satisfy Definition 7.2.1.

Different first rows are found, followed by an exhaustive enumeration through an in-depth backtrack search in order to produce the remaining rows. A check for extendable subdesigns, through the boolean function, is performed for each subsequent row.

The following constraints, in the computer search, have been implemented for the initial examination:

- Only one different first row was considered

- Subsequent rows were produced such that the pair of repeated symbols appears in lexicographical order.

The resulting counts were: 22312, 22523, 19888, 20432, 18966, 22606, 21781, 21131, 20981, 19602, 23451, 22635, 19820, 23673, 20962, 22554, 25374, 19221, 21384, 19906, 23003 and 20018 possibilities for the 22 category representatives for Latin Squares of size six listed in § 3.3.6. These layouts are fully listed and enclosed to this manuscript in the attached diskettes.

The significance of these results is that the number of possible superimpositions per square is nearly uniform. Yet, no two category representatives allow an equal number of superimpositions so far. Nevertheless, for representatives within the same species (e.g. 1, 2 and 18) the total number of superimpositions is expected to be the same. The probably most important piece of information to highlight here is that orthogonal properties in some squares seem to be unrelated to the construction of the superimposed arrays given by Definition 7.2.1.

Examples for each representative are given in Table 7.2. Let us move on to relax the constraint on the first row and see what happens.

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 5_2 & 6_1 & 3_1 \\ 3_4 & 4_4 & 5_2 & 6_1 & 1_1 & 2_3 \\ 4_1 & 3_2 & 6_3 & 1_3 & 2_2 & 5_4 \\ 5_3 & 6_2 & 2_1 & 3_4 & 4_4 & 1_2 \\ 6_2 & 5_1 & 1_4 & 2_4 & 3_3 & 4_3 \end{array}$$
[1]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 5_2 & 6_1 & 3_1 \\ 3_4 & 4_4 & 2_1 & 6_3 & 1_2 & 5_1 \\ 4_2 & 5_4 & 6_1 & 2_3 & 3_3 & 1_2 \\ 5_3 & 6_2 & 1_4 & 3_4 & 4_1 & 2_2 \\ 6_3 & 3_2 & 5_4 & 1_1 & 2_4 & 4_3 \end{array}$$
[2]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 5_2 & 6_1 & 3_1 \\ 3_4 & 4_4 & 2_3 & 6_1 & 1_2 & 5_1 \\ 4_1 & 6_2 & 5_4 & 2_3 & 3_3 & 1_2 \\ 5_2 & 3_2 & 6_4 & 1_4 & 2_1 & 4_3 \\ 6_3 & 5_4 & 1_1 & 3_3 & 4_4 & 2_2 \end{array}$$
[3]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 5_4 & 1_3 & 6_1 & 4_1 & 2_2 \\ 4_3 & 6_2 & 5_1 & 1_4 & 2_2 & 3_3 \\ 5_2 & 3_1 & 6_4 & 2_3 & 1_2 & 4_4 \\ 6_3 & 4_4 & 2_1 & 5_3 & 3_4 & 1_2 \end{array}$$
[4]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 5_2 & 6_1 & 3_1 \\ 3_4 & 4_4 & 5_1 & 6_3 & 2_2 & 1_1 \\ 4_1 & 6_2 & 2_4 & 1_3 & 3_2 & 5_3 \\ 5_2 & 3_3 & 6_4 & 2_1 & 1_4 & 4_2 \\ 6_3 & 5_4 & 1_2 & 3_4 & 4_1 & 2_3 \end{array}$$
[5]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 5_2 & 6_1 & 3_1 \\ 3_4 & 4_4 & 1_1 & 6_3 & 2_2 & 5_1 \\ 4_2 & 5_4 & 6_1 & 1_3 & 3_2 & 2_3 \\ 5_3 & 6_2 & 2_4 & 3_4 & 4_1 & 1_2 \\ 6_2 & 3_3 & 5_4 & 2_1 & 1_4 & 4_3 \end{array}$$
[6]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 5_4 & 1_3 & 6_1 & 2_2 & 4_1 \\ 4_3 & 6_2 & 2_1 & 5_4 & 1_2 & 3_3 \\ 5_2 & 3_1 & 6_4 & 2_3 & 4_4 & 1_2 \\ 6_3 & 4_4 & 5_2 & 1_4 & 3_1 & 2_3 \end{array}$$
[7]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 5_4 & 1_3 & 6_1 & 2_2 & 4_1 \\ 4_3 & 6_2 & 2_1 & 5_3 & 3_4 & 1_2 \\ 5_2 & 4_4 & 6_2 & 2_4 & 1_1 & 3_3 \\ 6_3 & 3_1 & 5_4 & 1_4 & 4_2 & 2_3 \end{array}$$
[8]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 5_4 & 1_2 & 6_1 & 4_1 & 2_3 \\ 4_2 & 6_2 & 5_3 & 2_3 & 1_4 & 3_1 \\ 5_2 & 3_3 & 6_4 & 1_1 & 2_2 & 4_4 \\ 6_3 & 4_4 & 2_1 & 5_4 & 3_3 & 1_2 \end{array}$$
[9]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 4_4 & 5_2 & 6_1 & 1_1 & 2_3 \\ 4_1 & 3_3 & 6_3 & 5_4 & 2_2 & 1_2 \\ 5_3 & 6_2 & 1_4 & 2_4 & 4_2 & 3_1 \\ 6_2 & 5_4 & 2_1 & 1_3 & 3_4 & 4_3 \end{array}$$
[10]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 4_4 & 5_1 & 6_1 & 1_3 & 2_2 \\ 4_1 & 5_3 & 6_2 & 1_4 & 2_2 & 3_3 \\ 5_2 & 6_2 & 1_4 & 2_3 & 3_4 & 4_1 \\ 6_3 & 3_1 & 2_3 & 5_4 & 4_4 & 1_2 \end{array}$$
[11]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 5_2 & 6_1 & 3_1 \\ 3_4 & 4_4 & 5_2 & 6_1 & 1_1 & 2_3 \\ 4_3 & 5_1 & 6_3 & 3_4 & 2_2 & 1_2 \\ 5_3 & 6_2 & 1_4 & 2_4 & 3_2 & 4_1 \\ 6_2 & 3_3 & 2_1 & 1_3 & 4_4 & 5_4 \end{array}$$
[12]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 4_4 & 5_3 & 6_1 & 1_1 & 2_2 \\ 4_3 & 3_1 & 6_2 & 5_4 & 2_3 & 1_2 \\ 5_2 & 6_2 & 1_4 & 2_3 & 3_4 & 4_1 \\ 6_3 & 5_4 & 2_1 & 1_4 & 4_2 & 3_3 \end{array}$$
[13]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 3_3 & 1_3 & 5_2 & 6_1 & 4_1 \\ 3_4 & 1_4 & 2_1 & 6_3 & 4_1 & 5_2 \\ 4_2 & 6_2 & 5_4 & 2_3 & 1_3 & 3_1 \\ 5_3 & 4_4 & 6_4 & 3_1 & 2_2 & 1_2 \\ 6_2 & 5_1 & 4_3 & 1_4 & 3_4 & 2_3 \end{array}$$
[14]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 5_4 & 1_3 & 6_1 & 2_2 & 4_1 \\ 4_2 & 6_2 & 5_1 & 1_4 & 3_3 & 2_3 \\ 5_2 & 4_4 & 6_2 & 2_1 & 1_4 & 3_3 \\ 6_3 & 3_1 & 2_4 & 5_3 & 4_4 & 1_2 \end{array}$$
[15]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 5_2 & 6_1 & 3_1 \\ 3_4 & 4_4 & 1_2 & 6_1 & 2_3 & 5_1 \\ 4_3 & 5_4 & 6_3 & 1_1 & 3_2 & 2_2 \\ 5_2 & 6_2 & 2_1 & 3_3 & 1_4 & 4_4 \\ 6_2 & 3_3 & 5_4 & 2_4 & 4_1 & 1_3 \end{array}$$
[16]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 5_4 & 1_3 & 6_1 & 2_2 & 4_1 \\ 4_2 & 6_2 & 2_1 & 5_3 & 1_4 & 3_3 \\ 5_2 & 3_4 & 6_2 & 1_1 & 4_4 & 2_3 \\ 6_3 & 4_3 & 5_4 & 2_4 & 3_1 & 1_2 \end{array}$$
[17]

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_2 & 4_2 & 5_3 & 6_4 \\ 2_4 & 1_3 & 4_3 & 3_2 & 6_1 & 5_1 \\ 3_4 & 4_4 & 5_3 & 6_1 & 1_1 & 2_2 \\ 4_1 & 5_2 & 6_4 & 2_3 & 3_3 & 1_2 \\ 5_2 & 6_2 & 2_1 & 1_3 & 4_4 & 3_4 \\ 6_3 & 3_1 & 1_4 & 5_4 & 2_2 & 4_3 \end{array}$$
[18]

1_1	2_1	3_2	4_2	5_3	6_4
2_4	1_3	4_3	3_2	6_1	5_1
3_4	5_4	1_2	6_1	4_1	2_3
4_3	6_2	5_1	1_4	2_2	3_3
5_2	4_4	6_3	2_1	3_4	1_2
6_2	3_1	2_4	5_3	1_3	4_4

[19]

1_1	2_1	3_2	4_2	5_3	6_4
2_4	1_3	4_3	3_2	6_1	5_1
3_4	5_4	1_3	6_1	4_1	2_2
4_3	6_2	2_1	5_3	3_4	1_2
5_2	3_1	6_4	2_3	1_2	4_4
6_3	4_4	5_1	1_4	2_2	3_3

[20]

1_1	2_1	3_2	4_2	5_3	6_4
2_4	1_3	4_3	5_2	6_1	3_1
3_4	4_4	5_1	6_3	1_1	2_2
4_1	3_2	6_4	1_3	2_2	5_3
5_2	6_2	1_4	2_4	3_3	4_1
6_3	5_4	2_3	3_1	4_4	1_2

[21]

1_1	2_1	3_2	4_2	5_3	6_4
2_4	1_3	4_3	5_2	6_1	3_1
3_4	5_4	1_1	6_3	4_1	2_2
4_3	6_2	2_3	1_4	3_2	5_1
5_2	3_3	6_4	2_1	1_2	4_4
6_3	4_2	5_1	3_4	2_4	1_3

[22]

Table 7.2. Examples of balanced superimpositions of four treatments to Latin Squares of size six as listed in §3.3.6.

Full enumeration of non-isomorphic (Under S_4) superimpositions subject to the constraints of having the pairs of repeated symbols in lexicographical order for the rows has been made for the first two category representatives listed in §3.3.6. The resulting counts were 908,671 and 889,108 F/Z-inequivalent superimpositions respectively.

Study of the combinatorics of balanced superimpositions for four symbols on Latin Squares of size six for **sequential experimentation** strongly suggests that the total number of this type of arrays is very large. There are 120 ways of permuting the repeated pairs of symbols in rows 2 to 6. Therefore, should the number of possible superimpositions remain around 900,000 for each square, as in the previous paragraph, then this study suggests that the total number of arrangements could well be of the order of 100 million possibilities per Latin Square.

Even though no complete enumeration was made for one specific category representative we have learnt much more about the problem, for we now know that (i) their construction is unrelated to orthogonal properties, and (ii) the number of possible superimpositions is approximately the same amongst category representatives.

The examples produced by the present systematic computer search may well be used to study the existence of a pattern which may lead us to obtain their enumeration in full. For instance, should we take the Latin Square from set X in [20] as representative of orbit number 10 in §3.3.6 then the symmetries of that square may be additional valuable information to obtain a pattern for categorising the possibilities in full for that equivalence class.

Evidence from combinatorial concepts, and information produced from a systematic computer search in enumerating combinatorial layouts obtainable from allocating four symbols in a 6×6 layout subject to two blocking constraints were combined. The author calculated that the number of orbit representatives under the action of S_4 is of the order of 538,789,708,800 F/Z-inequivalent Layouts. The complexity of the problem of superimposing four symbols to Latin squares of size six can now be best appreciated. Further information about arrays of this kind may be found in [35]. The ANOVA table for the combinatorial layouts obtained by superimposing four treatments to Latin Squares of size six, can be obtained from that general theory given by Potthoff [59].

8 Discussion and conclusions

In this chapter the author summarizes his justification for the classical statistical analysis in Latin Square row-column designs. Secondly, the writer highlights the main results comprised in this manuscript and finally, the chapter concludes with openings for further research.

8.1 Classical F-test on Latin Square designs: a justification.

The concepts underlying the classical statistical analysis in Latin Square designs have been the source of some confusion in the past. Thus, this study on the combinatorial and statistical properties of the Latin Square will finish with a discussion of those issues over which there has been much disagreement.

People have attempted to justify their inferential procedures by assuming their data come from a known distribution. That is to say, appealing to sampling techniques followed by model based inferences.

A brief survey of 77 papers published through internationally reputed journals – Appendix IV – between 1989 and 1994 on applications of Latin Squares as row-column designs was made. It was generally found that, in practical experimentation, research workers do not choose their experimental material through sampling techniques. Furthermore, Latin Squares are most useful as row-column designs for trials where the number of experimental units is small.

All things considered, the author aims to give here a useful and practical solution to significance testing of treatment effects in scientific comparative trials. He summarizes his justification for a test of significance of treatment effects in comparative experimentation, presented in chapter 2, in the forthcoming paragraphs. An application of the test is illustrated in §8.1.1.

The linear equation proposed in chapter 2 to explain the variability on the observed response is reproduced here:

$$Y_{l,m(n)} = \mu + \rho_l + \kappa_m + \tau_n + \nu_{l,m(n)} \qquad \ldots \; (\text{2-3})$$

$l = 1, 2, \ldots, p;\ m = 1, 2, \ldots, p;\ n = 1, 2, \ldots, p$

The Latin Square is used as a device to study the observed variability under two blocking constraints. In other words, variation explained from the blocking device is represented by the parameters ρ, κ in (2-3). An attempt to isolate the effects due to treatments from those due to natural variation is made through the additive parameters τ and ν, respectively.

The elementary assumptions to studying variation on the observed response are:

1. The equation is linear and additive on its parameters;
2. Natural variation effects are independent and identically distributed with mean $E(\nu_{l,m(n)}) = 0$ and constant variance $Var(\nu_{l,m(n)}) = \sigma^2$, $l = 1, 2, \ldots, p$; $m = 1, 2, \ldots, p$; $n = 1, 2, \ldots, p$;
3. The parameters in (2-3) satisfy: $\sum_l \rho_l = \sum_m \kappa_m = \sum_n \tau_n = 0$.

The mechanism of random allocation of treatments to experimental units with uniform probability transmutes the assumption of independence in (2) above into a valid **axiom**. It was shown in §2.2.3 that this axiom of independence plays a key-role in (i) partitioning the observed variability on the characteristic of interest as presented in Table 2.2, and (ii) justifying the variance-ratio test.

Natural variation v can take positive or negative values. This leads us to reasonably assume both that values of v (i) satisfy the distributional assumptions in (2) above, and (ii) follow a Normal distribution.

Based on the aforementioned argument it seems reasonably justified to appeal to a variance-ratio test [51, p. 437]. Not only for Latin Squares but also for any other designs with or without blocking.

Of course, the author recommends the reader to research all other previously published explanations on alternative tests of significance.

8.1.1 Illustrative example

Let us consider the data in Example II. The ANOVA table associated to this data set is given here as:

Source of variation	D.F.	S.S.	M.S.	v.r.
Rows	5	54199	10840	
Cols	5	24467	4893	
Treatments	5	248180	49636	32.50***
Residual	20	30541	1527	
Total	35	357387		

In this example we have very strong evidence to reject the null hypothesis of equality of treatment means. From the literature review, statisticians would attempt to compare the treatments amongst themselves. To this end, it has been argued [40, §7.5.2] that a two tailed t-test for two means may be used to obtain an *indication* on how two treatments compare with each other.

Let us calculate the critical difference for our particular example; it is

$$t_{(20)}\sqrt{\frac{2\times 1527}{6}} = 22.561\times t_{(20)}$$

The critical values of $t_{(20)}$ are 2.086 (5%), 2.845 (1%), and 3.85 (0.1%). Therefore, the *Least Significant Differences* (Fisher's LSDs) are 47.0622 (5%), 64.186 (1%), and 86.8599 (0.1%). The treatment means in increasing order for their size are 1, 345; 4, 405.2; 2, 426.5; 3, 477.8; 5, 520.2; 6, 601.8.

As an indication for testing in future trials the reader may note that treatment mean 6 is greater than that of 1,2,3, and 4 at the 0.1% level, and greater than that of 5 at the 1% level. Treatment mean 5 is greater than that of 1,2 and 4 at the 0.1%. Treatment mean 3 is greater than that of (i) 1 at the 0.1% level, (ii) 2 at the 5% level, and (iii) 4 at the 1% level. Treatment mean 2 is greater than that of 1 at the 1% level. Treatment mean 4 is greater than that of 1 at the 5% level.

Using computer simulated studies on particular data sets Chew [07] compares and contrasts a the most known multiple comparison techniques after a test has been found significant. He argues that "If the objective of to find as many significantly different pairs as possible, Fishers LSD is best". The reader is referred to his work for details.

8.2 Main results

In summary, the main results presented in this thesis are stated in the forthcoming paragraphs.

• A *distinction* between the combinatorial object and the row-column design known as the Latin Square *is* clearly *presented* in writing *for the first time*;

• Statisticians argue [36, 50] that under a more relaxed set of *assumptions* and present day computer power, statistical comparative permutation tests are practicable. This author pioneered researching the possibility of implementing exact permutation tests for Latin Square designs. It was found that, with the computer equipment at the University of Warwick, exact permutation tests on Latin Squares of size $p \leq 6$ are possible. Furthermore, the exact permutation test is also possible for sequential experimentation on Latin Squares of size six when a new set of two or three treatments is orthogonally added to the original set. However, as it was illustrated in § 4.4, there are situations in practice where possibility does not imply practicality. As an alternative, an F-test based on statistical likelihood theory is proposed in chapter 2 and well summarized in § 8.1. This intends to justify use of the classical F-test for treatment effects in the theoretical framework discussed in here.

• A counting argument brings *new evidence* to show the well known result, that no Latin Square of size six possesses an orthogonal mate.

• A systematic backtrack computer search produced a full enumeration of orthogonal additions of 2 and 3 treatments to Latin Squares of size 6. The counts so obtained *confirmed* the results independently reported by Finney [16] and Freeman [27, 29].

• When studying Graeco-Latin squares for sequential experimentation the author listed all F/Z-inequivalent sets of mutually orthogonal Latin Squares of sizes 4 and 5. Furthermore, with aid of an electronic computer the writer enumerated all *F/Z-inequivalent* sets of MOLS of size 7.

All information produced on complete sets of MOLS comprises a *discovery* to the statistician. It proves to the statistical community that projective planes may be used to show existence results on F/Z-inequivalent sets of MOLS but by no means provide their enumeration in full (For statistical applications that is.)

• Before the present study was undertaken, no attempt at enumerating those combinatorial arrangements – suggested as useful layouts for two successive trials [24] – obtainable from superimposing four symbols on Latin Squares of size six seems to have been reported in the literature. Even though the *discovery* of many examples has been made, the search for this kind of combinatorial layout is still far from complete. The results from this study lead the author to conjecture that the number of possibilities for such balanced superimpositions may be of the order of 100 million possibilities per Latin Square.

This study not only confirms that the addition of three and four symbols to all Latin Squares of size six is possible, but also clearly indicates that superimpositions are unrelated to properties of the transversals (if any). The latter also holds true for the addition of two treatments to most Latin Squares of this size.

8.3 Further work

Combinatorics

We now know that the enumeration of balanced superimpositions of 4 symbols to 6×6 Latin Squares is unrelated to the orthogonal properties of some Latin Squares. This valuable piece of information would not have been obtained if the author had concentrated on one single category representative.

The examples of balanced additions of symbols to Latin Squares of size 6 produced in the systematic computer search may well be used to find a pattern which may lead to their full enumeration and categorization. Use of the symmetries of some particular Latin Squares such as that representative of set X in [21] may additionally be made for such purposes.

The systematic search here illustrated for enumerating designs of type O:OO:OOO and O:OO:TTT has shown to be fruitful, yet further investigation is required, to enumerate layouts of type O:OO:SSS such as those illustrated in Table 8.1. Freeman [25] has given examples of arrays of this type and suggested that such arrays may be useful to the statistician.

This writer found many examples of combinatorial arrays similar to those illustrated in [25] and Table 8.1. Yet, the total number of F/Z-inequivalent ones remains unknown.

1_1	2_1	3_2	4_3		1_1	2_1	3_2	4_3
2_2	1_3	4_1	3_1		2_2	1_3	4_1	3_2
3_3	4_2	2_3	1_1		3_1	4_2	2_3	1_1
4_1	3_3	1_2	2_2		4_3	3_3	1_2	2_1
(i)					(ii)			

Table 8.1 Examples of layouts of type O:OO:SSS on a Latin Square of size *4*.

Statistics

The complexity of the problem of enumerating balanced superimpositions of 4 symbols to Latin Square designs of size 6 points out that random allocation of treatments to plots with uniform probability is not always possible. It would be interesting to extend the variance-ratio test for the case of unconstrained random allocation of treatments to experimental units.

All things considered, clear indications of openings for further combinatorial and statistical entertainment have just been given.

Appendix I

Search tree for Latin Squares of size six.

The following sets G_1, $G_2 \subset S(P)$ when applied to the orbit representatives numbers 1 and 2 (§3.3.6) of Latin Squares of size five retrieve the set of representatives under the action of $S_5 \times S_5$.

G_1 = { ((4 5))((1 2)(3 5))^2, ((1 2)(4 5))((3 5))((1 2)(4 5)), ((1 2)^2)((3 4 5)), (1 2)(3 4 5)(1 2), ((1 2)(4 5))^2 (3 5), ((e))((1 2)(3 4 5))^2, (4 5)^3, (2 5 3)(1 4 2 3 5)(1 3 5 4), (2 4 3)(1 4 5 2)^2, (2 5 4 3)(1 5 2)^2, ((2 4 5 3))((1 5)(2 3 4))((1 3 4)), ((2 5 4 3))((1 2 3 4))((1 3)(2 4)), (2 4 5 3)^3, ((1 2))((1 3 5)(2 4))((1 5 4 2 3)), ((2 3))((1 5 4 3))((1 2 4 3 5)),((4 5))((1 4 2 3))((1 3 4 2 5)), (2 5 3)^3, ((2 4 3))((1 2 3 5 4))((1 3)(2 5 4)), ((1 2)(4 5))((1 3 4 2 5))((1 4 2 3)), ((2 3)(4 5))((1 4 3))((1 2 5)(3 4)), ((e))((1 5 4 2 3))((1 3 5)(2 4)), (2 5 4 3)^3, ((2 4 5 3))((1 3))((1 2 5)), ((4 5))((1 5)(2 4 3))((1 4)(2 3)), ((2 3))((1 2 4)(3 5))((1 4 2 5 3)), ((1 2))((1 4 3)(2 5))((1 3 5 2 4)), ((2 5 3))((1 5 2)(3 4))^2, ((2 3)(4 5))((1 5)(2 4))((1 4)), ((2 3))((1 3 4 5 2))^2, ((2 5 4 3))((1 3 5)(2 4))((1 4)(3 5)), ((2 4 3))((1 5 3))((1 2 3 5)), ((e))((1 3 5 2 4))((1 4 3)(2 5)), ((1 2)(4 5))((1 4)(2 3))((1 5)(2 4 3)), ((2 3)(4 5))^3, ((2 5 3))((1 2 4))((1 4 2 3)), ((2 5 3))((1 3)(4 5))((1 2 4 5)), (2 4 3)^3, ((1 2)(4 5))((1 5 2 4 3))((1 3 4)(2 5)), ((2 3)(4 5))((1 2 5 3 4)

)((1 5 3)(2 4)), ((e))((1 4 3 2 5))((1 5 4)(2 3)), ((2 5 4 3))((1 4 5 3))((1 2 3 4 5)), ((1 2))((1 5 4)(2 3))((1 4 3 2 5)), ((4 5))((1 3 4)(2 5))((1 5 2 4 3)), (2 3)^{3}, ((2 4 5 3))((1 2 5 4))((1 5 4 2 3)), ((2 4 5 3))((1 4 3 5 2))^{2}, ((2 3))((1 4 2 5))((1 5 4)), ((2 3)(4 5))((1 3 5 2))((1 3 5 2)), ((2 4 3))((1 3 4 2 5))((1 5 3 4))}.

G_2 = { (4 5)^{3}, (3 4 5)^{3}, (3 4)^{3}, (3 5)^{3}, (3 5 4)^{3} }.

Note to the reader: According to §3.3.3 in this manuscript, elements in $S_p \times S_p \times S_p$ are of the form (g_1, g_2, g_3). Where, g_1, g_2, g_3 are elements of the corresponding S_p and to be applied to treatment labels and levels of constraints 2 and 1, respectively. Hence, the action of ((4 5))((1 2)(3 5))^{2} , for example, on the orbit representative number 1 for Latin Squares of size five yields the array: 12345 21453 34521 45132 53214.

The search tree and record of repetition counts for Latin Squares of size six are given in Fig. 1 and Fig. 2, respectively.

Lev. 1	Lev. 2	Lev. 3	Lev. 4	Lev. 5	Lev. 6
123456	214365	345612	436521	561234	652143
				561243	652134
			456123	561234	632541
				632541	561234 --> 6.3
			456231	562143	631524
				631524	562143 --> 6.5
			561234	436521 --> 5.1	
				456123 --> 5.3	
			561243	436521 --> 5.2	
			562143	436521 --> 5.2	
				456231 --> 5.5	
			652143	436521 --> 5.1	

Lev. 1	Lev. 2	Lev. 3	Lev. 4	Lev. 5	Lev. 6
		351624	436512 --> 4.2		
			462513	536142	645231
				536241	645132
			462531	536142 --> 5.14	
				546213	635142
				635142	546213 --> 6.9
				645213	536142 --> 6.8
			465132	536241 --> 5.16	
				546213	632541
			465231	536142 --> 5.14	
				642513 --> 5.18	
			645231	436512 --> 5.4	
				462513 --> 5.13	
		351642	436521 --> 4.3		
			462513 --> 4.10		
			462531	536124 --> 5.18	
				536214	645123
			465123	536214	642531
				546231	632514
			465213	536124	642531
			465231	536124 --> 5.29	
				546123 --> 5.27	
			536124	462513 --> 5.18	
				465213 --> 5.29	
			536214	462531 --> 5.26	
				465123 --> 5.27	
			546213 --> 4.12		
			546231	462513 --> 5.17	
				465123 --> 5.28	
			635124 --> 4.10		
			635214	462531 --> 5.26	
			645213	436521 --> 5.6	
				462531 --> 5.18	
			645231 --> 4.12		
	214563	341625	456132	562314	635241
				562341	635214
			456231	562314 --> 5.42	
				635142	562314 --> 6.18
			465132 --> 4.19		
			465231	536142	652314 --> 6.14
				652314 --> 5.27	
		342615	456132 --> 4.29		
			456231	561324 --> 5.44	
				561342	635124
			456321	561234	635142 --> 6.21
				635142 --> 5.48	
			465132 --> 4.18		
			465231	536124	651342
				536142	651324 --> 6.14
				651324 --> 5.27	
				651342	536124 --> 6.23
			465321	536142 --> 5.51	
				651234 --> 5.54	
			561234	456321 --> 5.49	
				635142 --> 5.48	
			561324	456132 --> 5.44	

Lev. 1	Lev. 2	Lev. 3	Lev. 4	Lev. 5	Lev. 6
			651234	465321 --> 5.54	
			651324 --> 4.19		
		345612	436125	561234	652341
				562341	651234
			456231 --> 4.34		
			456321	561234	632145
				562134 --> 5.49	
			461235	536124	652341 --> 6.27
				652341 --> 5.62	
			461325	536241 --> 5.28	
				652134	536241 -->6.15
			462135 --> 4.31		
			536124	461235 --> 5.65	
				652341 --> 5.62	
			536241 --> 4.17		
			561234	436125 --> 5.61	
				456321 --> 5.63	
			561324 --> 4.38		
		345621	436215	561342 --> 5.65	
			456132 --> 4.29		
			456312	561234 --> 5.48	
			461235	536142 --> 5.54	
			462135	536214	651342
				651342	536214 --> 6.31
			462315 --> 4.3		
			536142	461235 --> 5.54	
				462315 --> 5.6	
				651234	462315 --> 6.5
			536214 --> 4.56		
			561234 --> 4.38		
			561342 --> 4.48		
			562134	436215 --> 5.65	
				456312 --> 5.48	
			562314 --> 4.29		
			631245 --> 4.33		
			632145 --> 4.34		
			651234	462315 --> 5.5	
				536142 --> 5.80	
			651342	436215 --> 5.65	
				462135 --> 5.77	
			652134 --> 4.45		
			652314 --> 4.40		
		351624 --> 3.3			
		351642	436125 --> 4.31		
			436215 --> 4.56		
			462135	536214	645321
				546321 --> 5.76	
			462315 --> 4.21		
		352614	436125 --> 4.46		
			461235 --> 4.36		
			461325	536142	645231 --> 6.9
				536241 --> 5.17	
				546132 --> 5.52	
				546231 --> 5.45	
			465132 --> 4.26		
			465231	536142 --> 5.89	

Lev. 1	Lev. 2	Lev. 3	Lev. 4	Lev. 5	Lev. 6
				641325 --> 5.16	
			536241 --> 4.10		
			546231 --> 4.31		
			546321 --> 4.58		
			645231 --> 4.23		
		352641	436125 --> 4.45		
			436215 --> 4.55		
			461235 --> 4.67		
			461325 --> 4.36		
			465132 --> 4.25		
			465312 --> 4.56		
			536124	461235 --> 5.77	
			536214 --> 4.21		
			546132 --> 4.37		
			546312 --> 4.48		
	231564	312645	456123 --> 4.4		
			456132 --> 4.50		
			465132 --> 4.13		
			465213	546132	654321 --> 6.3
				546321	654132
				654321 --> 5.98	
		314625	456132 --> 4.38		
			456213 --> 4.45		
			456231 --> 4.66		
			456312 --> 4.5		
			465132 --> 4.12		
			465213	546132	652341 --> 6.5
				652341 --> 5.6	
			562143 --> 4.6		
			562341 --> 4.62		
			652341 --> 4.26		
		345612	416235	562143	654321 --> 6.12
				562341	654123 --> 6.15
				564321 --> 5.68	
				654321 --> 5.20	
			416325	562143 --> 5.102	
				564231 --> 5.5	
				652143 --> 5.99	
				654231 --> 5.28	
			456123 --> 4.6		
			456321 --> 4.46		
			462135 --> 4.78		
			654321 --> 4.11		
		345621	416235 --> 4.107		
			456132 --> 4.48		
			456213 --> 4.34		
			456312 --> 4.3		
			462135 --> 4.36		
			462315	516243 --> 5.51	
			612345 --> 4.62		
			614235 --> 4.102		
			654132 --> 4.37		
			654312 --> 4.17		
		346215 --> 3.10			
		456123 --> 3.1			
		456132 --> 3.6			

```
Lev. 1 Lev. 2 Lev. 3     Lev. 4    Lev. 5    Lev. 6
              456231     312645 --> 4.4
                         314625 --> 4.66
                         345612 --> 4.107
                         365142 --> 4.23
                         564123    312645 --> 5.3
                                   315642 --> 5.80
                                   345612 --> 5.99
                                   645312 --> 5.96
                         564312 --> 4.7
              465132 --> 3.2
              465213     312645 --> 4.96
                         314625 --> 4.102
                         346125 --> 4.37
                         354621 --> 4.58
                         546132 --> 4.124
                         546321    312645 --> 5.97
                         654321 --> 4.2
       234561 312645 --> 3.13
              315624 --> 3.11
              341625 --> 3.5
              345612 --> 3.19
              346125 --> 3.7
              351624 --> 3.3
              356214 --> 3.15
              365214 --> 3.21
              456123 --> 3.1
              512634 --> 3.26
```

Fig. 1 Search tree for Latin Squares of size 6

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
1.1	-	86400	86400	17280	720	720	720
2.1	1.1	1152	2304	576	15	10800	10800
2.2	1.1	192	384	96	90	64800	64800
2.3	1.1	432	864	216	40	28800	28800
2.4	1.1	144	288	72	120	86400	86400
3.1	2.1	36	36	12	16	518400	172800
3.2	2.1	36	72	24	16	259200	172800
3.3	2.1	12	12	4	48	1555200	518400
3.4	2.2	12	36	12	8	518400	518400
3.5	2.2	12	12	4	8	1555200	518400
3.6	2.2	6	12	4	16	1555200	1036800
3.7	2.2	6	6	2	16	3110400	1036800
3.8	2.2	12	12	4	8	1555200	518400
3.9	2.2	12	36	12	8	518400	518400
3.10	2.2	12	24	8	8	777600	518400
3.11	2.2	12	12	4	8	1555200	518400
3.12	2.3	216	648	216	1	28800	28800
3.13	2.3	24	24	8	9	777600	259200
3.14	2.3	24	72	24	9	259200	259200
3.15	2.3	12	12	4	18	1555200	518400
3.16	2.3	24	24	8	9	777600	259200
3.17	2.3	36	36	12	6	518400	172800
3.18	2.3	12	12	4	18	1555200	518400
3.19	2.3	72	72	24	3	259200	86400
3.20	2.3	72	72	24	3	259200	86400
3.21	2.3	36	36	12	6	518400	172800
3.22	2.4	12	24	8	6	777600	518400
3.23	2.4	6	12	4	12	1555200	1036800
3.24	2.4	6	12	4	12	1555200	1036800
3.25	2.4	36	72	24	2	259200	172800
3.26	2.4	6	6	2	12	3110400	1036800
3.27	2.4	12	12	4	6	1555200	518400
3.28	2.4	6	12	4	12	1555200	1036800
3.29	2.4	18	36	12	4	518400	345600
3.30	2.4	36	36	12	2	518400	172800
3.31	2.4	6	6	2	12	3110400	1036800
4.1	3.1	12	48	24	1	518400	518400
4.2	3.1	6	12	6	2	2073600	1036800
4.3	3.1	2	2	1	6	12441600	3110400
4.4	3.1	12	24	12	1	1036800	518400
4.5	3.1	4	8	4	3	3110400	1555200
4.6	3.1	4	8	4	3	3110400	1555200
4.7	3.1	12	24	12	1	1036800	518400
4.8	3.2	12	12	6	2	2073600	518400
4.9	3.2	12	48	24	2	518400	518400
4.10	3.2	4	4	2	6	6220800	1555200
4.11	3.2	8	24	12	3	1036800	777600
4.12	3.2	4	4	2	6	6220800	1555200
4.13	3.2	24	72	36	1	345600	259200

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
4.14	3.3	2	2	1	2	12441600	3110400
4.15	3.3	4	4	2	1	6220800	1555200
4.16	3.3	4	16	8	1	1555200	1555200
4.17	3.3	2	4	2	2	6220800	3110400
4.18	3.3	2	4	2	2	6220800	3110400
4.19	3.3	2	4	2	2	6220800	3110400
4.20	3.3	4	16	8	1	1555200	1555200
4.21	3.3	2	4	2	2	6220800	3110400
4.22	3.3	4	4	2	1	6220800	1555200
4.23	3.3	4	8	4	1	3110400	1555200
4.24	3.3	4	4	2	1	6220800	1555200
4.25	3.3	2	4	2	2	6220800	3110400
4.26	3.3	4	8	4	1	3110400	1555200
4.27	3.3	4	4	2	1	6220800	1555200
4.28	3.4	4	16	8	3	1555200	1555200
4.29	3.4	2	2	1	6	12441600	3110400
4.30	3.4	4	4	2	3	6220800	1555200
4.31	3.4	2	2	1	6	12441600	3110400
4.32	3.5	2	2	1	2	12441600	3110400
4.33	3.5	2	4	2	2	6220800	3110400
4.34	3.5	2	2	1	2	12441600	3110400
4.35	3.5	2	4	2	2	6220800	3110400
4.36	3.5	2	2	1	2	12441600	3110400
4.37	3.5	2	2	1	2	12441600	3110400
4.38	3.5	2	2	1	2	12441600	3110400
4.39	3.5	4	16	8	1	1555200	1555200
4.40	3.5	2	4	2	2	6220800	3110400
4.41	3.5	4	4	2	1	6220800	1555200
4.42	3.6	2	8	4	2	3110400	3110400
4.43	3.6	2	2	1	2	12441600	3110400
4.44	3.6	2	8	4	2	3110400	3110400
4.45	3.6	2	2	1	2	12441600	3110400
4.46	3.6	2	4	2	2	6220800	3110400
4.47	3.6	2	2	1	2	12441600	3110400
4.48	3.6	2	2	1	2	12441600	3110400
4.49	3.6	4	4	2	1	6220800	1555200
4.50	3.6	4	12	6	1	2073600	1555200
4.51	3.6	2	2	1	2	12441600	3110400
4.52	3.7	2	8	4	1	3110400	3110400
4.53	3.7	2	2	1	1	12441600	3110400
4.54	3.7	2	8	4	1	3110400	3110400
4.55	3.7	2	4	2	1	6220800	3110400
4.56	3.7	2	2	1	1	12441600	3110400
4.57	3.7	2	2	1	1	12441600	3110400
4.58	3.7	2	4	2	1	6220800	3110400
4.59	3.7	2	2	1	1	12441600	3110400
4.60	3.7	2	2	1	1	12441600	3110400
4.61	3.7	2	2	1	1	12441600	3110400
4.62	3.7	2	4	2	1	6220800	3110400
4.63	3.7	2	2	1	1	12441600	3110400
4.64	3.7	2	4	2	1	6220800	3110400
4.65	3.7	2	2	1	1	12441600	3110400
4.66	3.7	2	4	2	1	6220800	3110400
4.67	3.7	2	4	2	1	6220800	3110400

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
4.68	3.7	2	2	1	1	12441600	3110400
4.69	3.7	2	4	2	1	6220800	3110400
4.70	3.9	2	2	1	6	12441600	3110400
4.71	3.9	2	2	1	6	12441600	3110400
4.72	3.9	4	16	8	3	1555200	1555200
4.73	3.9	4	4	2	3	6220800	1555200
4.74	3.10	4	4	2	2	6220800	1555200
4.75	3.10	2	2	1	4	12441600	3110400
4.76	3.10	4	16	8	2	1555200	1555200
4.77	3.10	8	8	4	1	3110400	777600
4.78	3.10	4	12	6	2	2073600	1555200
4.79	3.10	4	4	2	2	6220800	1555200
4.80	3.10	2	2	1	4	12441600	3110400
4.81	3.10	4	4	2	2	6220800	1555200
4.82	3.10	8	8	4	1	3110400	777600
4.83	3.11	2	2	1	2	12441600	3110400
4.84	3.11	2	4	2	2	6220800	3110400
4.85	3.11	2	4	2	2	6220800	3110400
4.86	3.11	2	2	1	2	12441600	3110400
4.87	3.11	2	4	2	2	6220800	3110400
4.88	3.11	2	2	1	2	12441600	3110400
4.89	3.11	4	16	8	1	1555200	1555200
4.90	3.11	4	4	2	1	6220800	1555200
4.91	3.11	2	2	1	2	12441600	3110400
4.92	3.11	2	2	1	2	12441600	3110400
4.93	3.12	24	24	12	9	1036800	259200
4.94	3.12	12	12	6	18	2073600	518400
4.95	3.12	72	72	36	3	345600	86400
4.96	3.12	36	36	18	6	691200	172800
4.97	3.13	2	2	1	4	12441600	3110400
4.98	3.13	2	2	1	4	12441600	3110400
4.99	3.13	4	4	2	2	6220800	1555200
4.100	3.13	4	8	4	2	3110400	1555200
4.101	3.13	4	4	2	2	6220800	1555200
4.102	3.13	4	4	2	2	6220800	1555200
4.103	3.13	8	8	4	1	3110400	777600
4.104	3.13	4	4	2	2	6220800	1555200
4.105	3.13	8	8	4	1	3110400	777600
4.106	3.14	12	48	24	2	518400	518400
4.107	3.14	4	4	2	6	6220800	1555200
4.108	3.14	8	8	4	3	3110400	777600
4.109	3.14	4	4	2	6	6220800	1555200
4.110	3.14	12	12	6	2	2073600	518400
4.111	3.14	24	24	12	1	1036800	259200
4.112	3.15	2	4	2	2	6220800	3110400
4.113	3.15	2	2	1	2	12441600	3110400
4.114	3.15	2	2	1	2	12441600	3110400
4.115	3.15	2	2	1	2	12441600	3110400
4.116	3.15	2	2	1	2	6220800	3110400
4.117	3.15	2	8	4	2	3110400	3110400
4.118	3.15	4	4	2	1	6220800	1555200
4.119	3.15	2	4	2	2	6220800	3110400
4.120	3.15	2	2	1	2	12441600	3110400
4.121	3.15	4	4	2	1	6220800	1555200

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
4.122	3.19	24	24	12	1	1036800	259200
4.123	3.19	4	4	2	6	6220800	1555200
4.124	3.19	4	4	2	6	6220800	1555200
4.125	3.19	8	8	4	3	3110400	777600
4.126	3.19	12	24	12	2	1036800	518400
4.127	3.19	12	24	12	2	1036800	518400
4.128	3.21	12	36	18	1	691200	518400
4.129	3.21	4	4	2	3	6220800	1555200
4.130	3.21	2	2	1	6	12441600	3110400
4.131	3.21	4	4	2	3	6220800	1555200
4.132	3.21	12	24	12	1	1036800	518400
4.133	3.21	6	24	12	2	1036800	1036800
4.134	3.21	12	12	6	1	2073600	518400
5.1	4.1	12	12	12	2	5184000	1036800
5.2	4.1	4	4	4	6	15552000	3110400
5.3	4.2	6	12	12	1	5184000	2073600
5.4	4.2	6	18	18	1	3456000	2073600
5.5	4.3	1	2	2	1	31104000	12441600
5.6	4.3	1	2	2	1	31104000	12441600
5.7	4.4	6	12	12	2	5184000	2073600
5.8	4.4	6	12	12	2	51844000	2073600
5.9	4.5	2	4	4	2	15552000	6220800
5.10	4.6	2	4	4	2	15552000	6220800
5.11	4.6	2	2	2	2	31104000	6220800
5.12	4.7	6	12	12	2	5184000	2073600
5.13	4.9	12	36	36	2	1728000	1036800
5.14	4.9	4	4	4	6	15552000	3110400
5.15	4.10	2	4	4	1	15552000	6220800
5.16	4.10	2	6	6	1	10368000	6220800
5.17	4.10	2	4	4	1	15552000	6220800
5.18	4.10	2	2	2	1	31104000	6220800
5.19	4.11	6	6	6	2	10368000	2073600
5.20	4.11	6	24	24	2	2592000	2073600
5.21	4.12	2	4	4	1	15552000	6220800
5.22	4.12	2	2	2	1	31104000	6220800
5.23	4.13	18	18	18	2	3456000	691200
5.24	4.13	18	36	36	2	1728000	691200
5.25	4.16	2	2	2	4	31104000	6220800
5.26	4.16	2	2	2	4	31104000	6220800
5.27	4.17	1	1	1	2	62208000	12441600
5.28	4.17	1	2	2	2	31104000	12441600
5.29	4.18	1	2	2	2	31104000	12441600
5.30	4.19	1	2	2	2	31104000	12441600
5.31	4.19	1	1	1	2	62208000	12441600
5.32	4.20	2	2	2	4	31104000	6220800
5.33	4.20	2	2	2	4	3110400	6220800
5.34	4.21	1	2	2	2	31104000	12441600
5.35	4.21	1	1	1	2	62208000	12441600
5.36	4.23	2	4	4	2	15552000	6220800
5.37	4.23	2	2	2	2	31104000	6220800
5.38	4.25	1	2	2	2	31104000	12441600
5.39	4.26	2	2	2	2	31104000	6220800
5.40	4.26	2	2	2	2	31104000	6220800

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
5.41	4.28	4	20	20	2	3110400	3110400
5.42	4.28	4	4	4	2	15552000	3110400
5.43	4.29	1	4	4	1	15552000	12441600
5.44	4.29	1	2	2	1	31104000	12441600
5.45	4.31	1	4	4	1	15552000	12441600
5.46	4.31	1	1	1	1	62208000	12441600
5.47	4.33	1	2	2	2	31104000	12441600
5.48	4.33	1	1	1	2	62208000	12441600
5.49	4.34	1	2	2	1	31104000	12441600
5.50	4.34	1	1	1	1	62208000	12441600
5.51	4.36	1	2	2	1	31104000	12441600
5.52	4.36	1	4	4	1	15552000	12441600
5.53	4.36	1	1	1	1	62208000	12441600
5.54	4.36	1	1	1	1	62208000	12441600
5.55	4.37	1	2	2	1	31104000	12441600
5.56	4.37	1	1	1	1	62208000	12441600
5.57	4.38	1	2	2	1	31104000	12441600
5.58	4.38	1	1	1	1	62208000	12441600
5.59	4.39	2	2	2	4	31104000	6220800
5.60	4.40	1	1	1	2	62208000	12441600
5.61	4.42	2	6	6	2	10368000	6220800
5.62	4.42	2	2	2	2	31104000	6220800
5.63	4.44	2	6	6	2	10368000	6220800
5.64	4.44	2	2	2	2	31104000	6220800
5.65	4.45	1	1	1	1	62208000	12441600
5.66	4.45	1	2	2	1	31104000	12441600
5.67	4.46	2	2	2	1	31104000	6220800
5.68	4.46	2	8	8	1	7776000	6220800
5.69	4.48	1	1	1	1	62208000	12441600
5.70	4.48	1	2	2	1	31104000	12441600
5.71	4.50	3	6	6	2	10368000	4147200
5.72	4.50	3	6	6	2	10368000	4147200
5.73	4.52	1	1	1	4	62208000	12441600
5.74	4.54	1	1	1	4	62208000	12441600
5.75	4.55	1	1	1	2	62208000	12441600
5.76	4.56	1	4	4	1	15552000	12441600
5.77	4.56	1	2	2	1	31104000	12441600
5.78	4.58	1	1	1	2	62208000	12441600
5.79	4.68	2	2	2	1	31104000	6220800
5.80	4.58	2	4	4	1	15552000	6220800
5.81	4.62	1	1	1	2	62208000	12441600
5.82	4.62	1	1	1	2	62208000	12441600
5.83	4.66	2	2	2	1	31104000	6220800
5.84	4.66	2	4	4	1	15552000	6220800
5.85	4.67	1	1	1	2	62208000	12441600
5.86	4.67	1	2	2	2	31104000	12441600
5.87	4.72	4	20	20	2	3110400	3110400
5.88	4.72	4	4	4	2	15552000	3110400
5.89	4.76	4	12	12	2	5184000	3110400
5.90	4.76	4	4	4	2	15552000	3110400
5.91	4.76	4	4	4	2	15552000	3110400
5.92	4.76	4	4	4	2	15552000	3110400
5.93	4.78	6	12	12	1	5184000	2073600
5.94	4.78	6	6	6	1	10368000	2073600

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
5.95	4.89	2	2	2	4	31104000	6220800
5.96	4.96	18	36	36	1	1728000	691200
5.97	4.96	9	18	18	2	3456000	1382400
5.98	4.96	18	18	18	1	3456000	691200
5.99	4.102	2	4	4	1	15552000	6220800
5.100	4.102	2	2	2	1	31104000	6220800
5.101	4.106	24	120	120	1	518400	518400
5.102	4.106	8	8	8	3	7776000	1555200
5.103	4.106	8	8	8	3	7776000	1555200
5.104	4.106	24	24	24	1	2592000	518400
5.105	4.107	2	8	8	1	7776000	6220800
5.106	4.107	2	2	2	1	31104000	6220800
5.107	4.107	2	4	4	1	15552000	6220800
5.108	4.107	2	2	2	1	31104000	6220800
5.109	4.117	2	2	2	2	31104000	6220800
5.110	4.126	12	12	12	1	5184000	1036800
5.111	4.126	4	4	4	3	15552000	3110400
5.112	4.126	4	4	4	3	15552000	3110400
5.113	4.126	12	36	36	1	1728000	1036800
5.114	4.133	6	18	18	2	3456000	2073600
6.1	5.1	12	72	-	1	5184000	5184000
6.2	5.2	4	24	-	1	15552000	15552000
6.3	5.3	12	36	-	1	10368000	5184000
6.4	5.4	18	36	-	1	10368000	3456000
6.5	5.5	2	4	-	1	93312000	31104000
6.6	5.6	2	4	-	1	93312000	31104000
6.7	5.13	36	216	-	1	1728000	1728000
6.8	5.14	4	8	-	1	46656000	15552000
6.9	5.16	6	12	-	1	31104000	10368000
6.10	5.17	4	12	-	1	31104000	15552000
6.11	5.18	2	8	-	1	46656000	31104000
6.12	5.20	24	120	-	1	3110400	2592000
6.13	5.26	2	12	-	1	31104000	31104000
6.14	5.27	1	4	-	1	93312000	62208000
6.15	5.28	2	8	-	1	46656000	31104000
6.16	5.29	2	12	-	1	31104000	31104000
6.17	5.41	20	120	-	1	3110400	3110400
6.18	5.42	4	8	-	1	46656000	15552000
6.19	5.44	2	8	-	1	46656000	31104000
6.20	5.45	4	4	-	1	93312000	15552000
6.21	5.48	1	4	-	1	93312000	62208000
6.22	5.49	2	4	-	1	93312000	31104000
6.23	5.51	2	4	-	1	93312000	31104000
6.24	5.52	4	4	-	1	93312000	15552000
6.25	5.54	1	4	-	1	93312000	62208000
6.26	5.61	6	36	-	1	10368000	10368000
6.27	5.62	2	4	-	1	93312000	31104000
6.28	5.63	6	36	-	1	10368000	10368000
6.29	5.65	1	4	-	1	93312000	62208000
6.30	5.68	8	8	-	1	46656000	7776000
6.31	5.76	4	8	-	1	46656000	15552000
6.32	5.77	2	8	-	1	46656000	31104000
6.33	5.80	4	4	-	1	93312000	15552000

Node	Parent	Auto G_k''	Group G_k	Size G_k'	Num Twin	Expected Ocurr.	Repetition Counts
6.34	5.87	20	120	-	1	3110400	3110400
6.35	5.89	12	12	-	1	31104000	5184000
6.36	5.96	36	36	-	1	10368000	1728000
6.37	5.97	18	108	-	1	3456000	3456000
6.38	5.99	4	4	-	1	93312000	15552000
6.39	5.101	120	120	-	1	3110400	518400
6.40	5.102	8	8	-	1	46656000	7776000

Fig. 2 Record of repetition counts for Latin Squares of size 6

Appendix II

Complete listing of directrices for orbit representatives of Latin Squares of size $p \leq 6$

Note: The numbering of plots/cells in the directrices is to be interpreted from left-to-right in the order of rows from top-to-bottom, in the squared array. See Table 2.1(i) for an example.

$p = 3$

Latin Square : 123 231 312

Directrices:

directrix 1: {1, 5, 9}

directrix 2: {2, 6, 7}

directrix 3: {3, 4, 8}

$p = 4$

Latin Square 1: 1234 2143 3421 4312

Directrices: None

Latin Square 2: 1234 2143 3412 4321

Directrices:

directrix 1: { {1, 7,12,14}, {1, 8,10,15} }

directrix 2: { {2, 7, 9,16}, {2, 8,11,13} }

directrix 3: { {3, 5,10,16}, {3, 6,12,13} }

directrix 4: { {4, 5,11,14}, {4, 6, 9,15} }

$p = 5$

Latin Square 1 12345 21453 34512 45231 53124

Directrices:

directrix 3: 3, 6,14,17,25,

directrix 4: 4, 6,13,20,22,

directrix 5: 5, 6,12,19,23,

Latin Square 2 12345 23451 34512 45123 51234

Directrices:

directrix 1: { {1, 7,13,19,25}, {1, 8,15,17,24}, {1, 9,12,20,23} }

directrix 2: { {2, 8,14,20,21}, {2, 9,11,18,25}, {2,10,13,16,24} }

directrix 3: { {3, 6,14,17,25}, {3, 9,15,16,22},

```
                  {3,10,12,19,21} }
directrix 4:  { {4, 6,13,20,22}, {4, 7,15,18,21},
                  {4,10,11,17,23} }
directrix 5:  { {5, 6,12,18,24}, {5, 7,14,16,23},
                  {5, 8,11,19,22} }
```

p = 6

```
 1 123456 214563 345612 436125 562341 651234
Directrices:
directrix 2:  { {2, 9,17,24,28,31}, {2,12,15,22,29,31} }
directrix 3:  { {3, 8,16,23,25,36}, {3,10,14,23,30,31} }
directrix 4:  { {4, 8,18,21,25,35}, {4,11,18,20,25,33} }
directrix 5:  {5,12,14,22,27,31}
directrix 6:  {6, 9,17,20,25,34}
 2 123456 214563 342615 456231 561342 635124
Directrices:
directrix 1:  { {1,10,15,23,26,36}, {1,11,15,20,28,36} }
directrix 3:  {3, 8,18,22,29,31}
directrix 4:  {4,11,15,24,25,32}
directrix 5:  { {5,12,14,22,27,31}, {5,12,15,19,26,34} }
directrix 6:  { {6, 8,13,22,29,33}, {6, 9,17,22,25,32} }
 3 123456 214563 342615 465231 536124 651342
Directrices:
directrix 1:  { {1, 9,18,20,29,34}, {1,10,14,23,27,36} }
directrix 2:  { {2, 9,18,23,28,31}, {2,10,13,24,27,35} }
directrix 3:  { {3, 7,18,20,28,35}, {3,10,14,24,29,31} }
directrix 5:  { {5, 7,14,24,27,34}, {5, 9,13,20,28,36} }
 4 123456 214365 351642 465123 536214 642531
Directrices:
directrix 3:  { {3,11,14,19,28,36}, {3,11,18,22,25,32} }
directrix 4:  { {4,12,13,20,29,33}, {4,12,15,23,26,31} }
directrix 5:  { {5, 9,13,20,28,36}, {5, 9,18,22,26,31} }
directrix 6:  { {6,10,14,19,29,33}, {6,10,15,23,25,32} }
 5 123456 214563 345621 462135 536214 651342
Directrices:
directrix 1:  { {1,10,14,23,27,36}, {1,12,15,20,28,35} }
directrix 3:  { {3,11,14,22,25,36}, {3,11,18,19,28,32} }
directrix 4:  { {4, 7,18,23,27,32}, {4,12,17,20,25,33} }
```

```
directrix 6:  { {6, 7,15,22,26,35}, {6,10,17,19,26,33} }
 6 123456 214563 341625 456132 562341 635214
Directrices:
directrix 1:  { {1, 9,18,23,26,34}, {1,11,14,24,28,33} }
directrix 4:  { {4, 8,18,23,27,31}, {4,11,15,24,25,32} }
directrix 5:  { {5,12,14,22,27,31}, {5,12,15,19,26,34} }
directrix 6:  { {6, 8,17,19,28,33}, {6, 9,17,22,25,32} }
 7 123456 214365 351624 462513 536241 645132
Directrices: None
 8 123456 214365 351624 462531 546213 635142
Directrices: None
 9 123456 214365 351642 465213 536124 642531
Directrices: None
10 123456 214365 345612 436521 561243 652134
Directrices:
directrix 1:  { {1, 9,16,23,30,32}, {1, 9,18,22,26,35},
                {1,10,15,23,26,36},  {1,10,18,21,29,32},
                {1,11,14,22,30,33}, {1,11,15,20,28,36},
                {1,12,14,21,28,35}, {1,12,16,20,29,33} }
directrix 2:  { {2, 9,16,24,25,35}, {2, 9,17,22,30,31},
                {2,10,15,24,29,31}, {2,10,17,21,25,36},
                {2,11,13,22,27,36}, {2,11,15,19,30,34},
                {2,12,13,21,29,34}, {2,12,16,19,27,35} }
directrix 3:  { {3, 7,16,24,29,32}, {3, 7,17,22,26,36},
                 {3, 8,16,23,25,36}, {3, 8,18,22,29,31} }
Directrix 4:  { {4, 7,15,24,26,35}, {4, 7,17,21,30,32},
                {4, 8,15,23,30,31}, {4, 8,18,21,25,35} }
directrix 5:  { {5, 7,14,21,30,34}, {5, 7,16,20,27,36},
                {5, 8,13,21,28,36}, {5, 8,16,19,30,33} }
directrix 6:  { {6, 7,14,22,27,35}, {6, 7,15,20,29,34},
                {6, 8,13,22,29,33}, {6, 8,15,19,28,35} }
11 123456 214365 345612 456123 561234 632541
Directrices: None
12 123456 214563 345612 456321 561234 632145
Directrices: None
13 123456 214365 345612 436521 561234 652143
Directrices: None
```

```
14 123456 231564 312645 465213 546321 654132
Directrices: None
15 123456 214365 351624 465132 546213 632541
Directrices:
directrix 1:  { {1, 9,17,20,30,34}, {1,10,14,24,27,35},
                {1,11,18,21,28,32}, {1,12,16,23,26,33} }
directrix 2:  { {2, 9,16,23,25,36}, {2,10,18,21,29,31},
                {2,11,15,19,30,34}, {2,12,13,22,27,35} }
directrix 3:  { {3, 7,18,20,29,34}, {3, 8,16,24,25,35},
                {3,11,14,19,28,36}, {3,12,17,22,26,31} }
directrix 4:  { {4, 7,14,23,27,36}, {4, 8,17,21,30,31},
                {4,11,15,24,25,32}, {4,12,13,20,29,33} }
directrix 5:  { {5, 7,18,22,27,32}, {5, 8,16,19,30,33},
                {5, 9,13,20,28,36}, {5,10,15,24,26,31} }
directrix 6:  { {6, 7,15,23,26,34}, {6, 8,13,21,28,35},
                {6, 9,17,22,25,32}, {6,10,14,19,29,33}
16 123456 214563 341625 456132 562314 635241
Directrices:
directrix 1:  { {1, 9,18,23,26,34}, {1,10,17,21,30,32},
                {1,11,14,24,28,33}, {1,12,16,20,27,35} }
directrix 2:  { {2, 9,16,23,25,36}, {2,10,15,23,30,31},
                {2,11,13,22,30,33}, {2,12,16,19,29,33} }
directrix 3:  { {3, 7,18,22,26,35}, {3, 8,16,24,25,35},
                {3,10,14,24,29,31}, {3,10,17,19,26,36} }
directrix 4:  { {4, 7,18,21,29,32}, {4, 8,18,23,27,31},
                {4,11,13,20,27,36}, {4,11,15,24,25,32} }
directrix 5:  { {5, 7,14,21,28,36}, {5, 8,13,21,30,34},
                {5,12,14,22,27,31}, {5,12,15,19,26,34} }
directrix 6:  { {6, 7,15,20,28,35}, {6, 8,17,19,28,33},
                {6, 9,13,20,29,34}, {6, 9,17,22,25,32} }
17 123456 214365 351624 462513 536142 645231
Directrices: None
18 123456 214365 345612 456231 562143 631524
Directrices:
directrix 1:  { {1, 9,16,20,30,35}, {1, 9,18,23,26,34} }
directrix 2:  { {2,10,15,24,29,31}, {2,10,17,21,25,36} }
```

```
directrix 5:  { {5,10,14,24,27,31}, {5,10,18,19,26,33} }
directrix 6:  { {6, 9,13,20,28,35}, {6, 9,17,22,25,32} }
19 123456 214365 351642 465123 546231 632514
Directrices:
directrix 1:  { {1, 9,18,20,29,34}, {1,10,14,23,27,36} }
directrix 2:  { {2, 9,16,24,25,35}, {2,10,17,21,30,31} }
directrix 3:  { {3, 7,17,20,30,34}, {3, 8,16,23,25,36} }
directrix 4:  { {4, 7,14,24,27,35}, {4, 8,18,21,29,31} }
20 123456 214365 351642 462531 536214 645123
Directrices: None
21 123456 214563 345612 436125 561234 652341
Directrices: None
22 123456 214563 351642 462135 536214 645321
Directrices:
directrix 1:  { {1, 9,16,24,26,35}, {1,10,18,23,27,32},
                {1,11,14,21,30,34}, {1,12,17,20,28,33} }
directrix 2:  { {2, 9,16,23,25,36}, {2,10,15,23,30,31},
                {2,11,13,22,30,33}, {2,12,16,19,29,33} }
directrix 3:  { {3, 7,16,24,29,32}, {3, 8,17,24,28,31},
                {3,11,14,19,28,36}, {3,11,18,22,25,32} }
directrix 4:  { {4, 7,14,23,27,36}, {4, 8,13,24,27,35},
                {4,12,14,21,29,31}, {4,12,15,20,25,35} }
directrix 5:  { {5, 7,15,20,30,34}, {5, 8,18,19,27,34},
                {5, 9,13,20,28,36}, {5, 9,18,22,26,31} }
directrix 6:  { {6, 7,17,22,26,33}, {6, 8,17,21,25,34},
                {6,10,13,21,29,32}, {6,10,15,19,26,35} }.
```

Appendix III

F/Z-inequivalent sets of MOLS of size $p = 5$ (trivial group on added symbols)

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_4{}^5{}_1 \quad 4_5{}^1{}_2 \quad 5_1{}^2{}_3 \quad 1_2{}^3{}_4$
$3_5{}^2{}_4 \quad 4_1{}^3{}_5 \quad 5_2{}^4{}_1 \quad 1_3{}^5{}_2 \quad 2_4{}^1{}_3$
$4_2{}^5{}_3 \quad 5_3{}^1{}_4 \quad 1_4{}^2{}_5 \quad 2_5{}^3{}_1 \quad 3_1{}^4{}_2$
$5_4{}^3{}_2 \quad 1_5{}^4{}_3 \quad 2_1{}^5{}_4 \quad 3_2{}^1{}_5 \quad 4_3{}^2{}_1$

[1]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_4{}^5{}_1 \quad 4_5{}^1{}_2 \quad 5_1{}^2{}_3 \quad 1_2{}^3{}_4$
$3_5{}^2{}_4 \quad 4_1{}^3{}_5 \quad 5_2{}^4{}_1 \quad 1_3{}^5{}_2 \quad 2_4{}^1{}_3$
$5_4{}^3{}_2 \quad 1_5{}^4{}_3 \quad 2_1{}^5{}_4 \quad 3_2{}^1{}_5 \quad 4_3{}^2{}_1$
$4_2{}^5{}_3 \quad 5_3{}^1{}_4 \quad 1_4{}^2{}_5 \quad 2_5{}^3{}_1 \quad 3_1{}^4{}_2$

[2]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_4{}^5{}_1 \quad 4_5{}^1{}_2 \quad 5_1{}^2{}_3 \quad 1_2{}^3{}_4$
$4_2{}^5{}_3 \quad 5_3{}^1{}_4 \quad 1_4{}^2{}_5 \quad 2_5{}^3{}_1 \quad 3_1{}^4{}_2$
$3_5{}^2{}_4 \quad 4_1{}^3{}_5 \quad 5_2{}^4{}_1 \quad 1_3{}^5{}_2 \quad 2_4{}^1{}_3$
$5_4{}^3{}_2 \quad 1_5{}^4{}_3 \quad 2_1{}^5{}_4 \quad 3_2{}^1{}_5 \quad 4_3{}^2{}_1$

[3]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_4{}^5{}_1 \quad 4_5{}^1{}_2 \quad 5_1{}^2{}_3 \quad 1_2{}^3{}_4$
$4_2{}^5{}_3 \quad 5_3{}^1{}_4 \quad 1_4{}^2{}_5 \quad 2_5{}^3{}_1 \quad 3_1{}^4{}_2$
$5_4{}^3{}_2 \quad 1_5{}^4{}_3 \quad 2_1{}^5{}_4 \quad 3_2{}^1{}_5 \quad 4_3{}^2{}_1$
$3_5{}^2{}_4 \quad 4_1{}^3{}_5 \quad 5_2{}^4{}_1 \quad 1_3{}^5{}_2 \quad 2_4{}^1{}_3$

[4]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_4{}^5{}_1 \quad 4_5{}^1{}_2 \quad 5_1{}^2{}_3 \quad 1_2{}^3{}_4$
$5_4{}^3{}_2 \quad 1_5{}^4{}_3 \quad 2_1{}^5{}_4 \quad 3_2{}^1{}_5 \quad 4_3{}^2{}_1$
$3_5{}^2{}_4 \quad 4_1{}^3{}_5 \quad 5_2{}^4{}_1 \quad 1_3{}^5{}_2 \quad 2_4{}^1{}_3$
$4_2{}^5{}_3 \quad 5_3{}^1{}_4 \quad 1_4{}^2{}_5 \quad 2_5{}^3{}_1 \quad 3_1{}^4{}_2$

[5]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_4{}^5{}_1 \quad 4_5{}^1{}_2 \quad 5_1{}^2{}_3 \quad 1_2{}^3{}_4$
$5_4{}^3{}_2 \quad 1_5{}^4{}_3 \quad 2_1{}^5{}_4 \quad 3_2{}^1{}_5 \quad 4_3{}^2{}_1$
$4_2{}^5{}_3 \quad 5_3{}^1{}_4 \quad 1_4{}^2{}_5 \quad 2_5{}^3{}_1 \quad 3_1{}^4{}_2$
$3_5{}^2{}_4 \quad 4_1{}^3{}_5 \quad 5_2{}^4{}_1 \quad 1_3{}^5{}_2 \quad 2_4{}^1{}_3$

[6]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_5{}^1{}_4 \quad 5_4{}^2{}_1 \quad 1_2{}^5{}_3 \quad 4_1{}^3{}_2$
$3_4{}^5{}_2 \quad 5_1{}^4{}_3 \quad 4_2{}^1{}_5 \quad 2_5{}^3{}_1 \quad 1_3{}^2{}_4$
$4_5{}^2{}_3 \quad 1_4{}^3{}_5 \quad 2_1{}^5{}_4 \quad 5_3{}^1{}_2 \quad 3_2{}^4{}_1$
$5_2{}^3{}_4 \quad 4_3{}^5{}_1 \quad 1_5{}^4{}_2 \quad 3_1{}^2{}_5 \quad 2_4{}^1{}_3$

[7]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_5{}^1{}_4 \quad 5_4{}^2{}_1 \quad 1_2{}^5{}_3 \quad 4_1{}^3{}_2$
$3_4{}^5{}_2 \quad 5_1{}^4{}_3 \quad 4_2{}^1{}_5 \quad 2_5{}^3{}_1 \quad 1_3{}^2{}_4$
$5_2{}^3{}_4 \quad 4_3{}^5{}_1 \quad 1_5{}^4{}_2 \quad 3_1{}^2{}_5 \quad 2_4{}^1{}_3$
$4_5{}^2{}_3 \quad 1_4{}^3{}_5 \quad 2_1{}^5{}_4 \quad 5_3{}^1{}_2 \quad 3_2{}^4{}_1$

[8]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_5{}^1{}_4 \quad 5_4{}^2{}_1 \quad 1_2{}^5{}_3 \quad 4_1{}^3{}_2$
$4_5{}^2{}_3 \quad 1_4{}^3{}_5 \quad 2_1{}^5{}_4 \quad 5_3{}^1{}_2 \quad 3_2{}^4{}_1$
$3_4{}^5{}_2 \quad 5_1{}^4{}_3 \quad 4_2{}^1{}_5 \quad 2_5{}^3{}_1 \quad 1_3{}^2{}_4$
$5_2{}^3{}_4 \quad 4_3{}^5{}_1 \quad 1_5{}^4{}_2 \quad 3_1{}^2{}_5 \quad 2_4{}^1{}_3$

[9]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_5{}^1{}_4 \quad 5_4{}^2{}_1 \quad 1_2{}^5{}_3 \quad 4_1{}^3{}_2$
$4_5{}^2{}_3 \quad 1_4{}^3{}_5 \quad 2_1{}^5{}_4 \quad 5_3{}^1{}_2 \quad 3_2{}^4{}_1$
$5_2{}^3{}_4 \quad 4_3{}^5{}_1 \quad 1_5{}^4{}_2 \quad 3_1{}^2{}_5 \quad 2_4{}^1{}_3$
$3_4{}^5{}_2 \quad 5_1{}^4{}_3 \quad 4_2{}^1{}_5 \quad 2_5{}^3{}_1 \quad 1_3{}^2{}_4$

[10]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_5{}^1{}_4 \quad 5_4{}^2{}_1 \quad 1_2{}^5{}_3 \quad 4_1{}^3{}_2$
$5_2{}^3{}_4 \quad 4_3{}^5{}_1 \quad 1_5{}^4{}_2 \quad 3_1{}^2{}_5 \quad 2_4{}^1{}_3$
$3_4{}^5{}_2 \quad 5_1{}^4{}_3 \quad 4_2{}^1{}_5 \quad 2_5{}^3{}_1 \quad 1_3{}^2{}_4$
$4_5{}^2{}_3 \quad 1_4{}^3{}_5 \quad 2_1{}^5{}_4 \quad 5_3{}^1{}_2 \quad 3_2{}^4{}_1$

[11]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 3_5{}^1{}_4 \quad 5_4{}^2{}_1 \quad 1_2{}^5{}_3 \quad 4_1{}^3{}_2$
$5_2{}^3{}_4 \quad 4_3{}^5{}_1 \quad 1_5{}^4{}_2 \quad 3_1{}^2{}_5 \quad 2_4{}^1{}_3$
$4_5{}^2{}_3 \quad 1_4{}^3{}_5 \quad 2_1{}^5{}_4 \quad 5_3{}^1{}_2 \quad 3_2{}^4{}_1$
$3_4{}^5{}_2 \quad 5_1{}^4{}_3 \quad 4_2{}^1{}_5 \quad 2_5{}^3{}_1 \quad 1_3{}^2{}_4$

[12]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 4_1{}^5{}_3 \quad 1_5{}^2{}_4 \quad 5_2{}^3{}_1 \quad 3_4{}^1{}_2$
$3_2{}^5{}_4 \quad 1_4{}^3{}_5 \quad 5_1{}^4{}_2 \quad 2_5{}^1{}_3 \quad 4_3{}^2{}_1$
$4_5{}^3{}_2 \quad 5_3{}^1{}_4 \quad 2_4{}^5{}_1 \quad 3_1{}^2{}_5 \quad 1_2{}^4{}_3$
$5_4{}^2{}_3 \quad 3_5{}^4{}_1 \quad 4_2{}^1{}_5 \quad 1_3{}^5{}_2 \quad 2_1{}^3{}_4$

[13]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 4_1{}^5{}_3 \quad 1_5{}^2{}_4 \quad 5_2{}^3{}_1 \quad 3_4{}^1{}_2$
$3_2{}^5{}_4 \quad 1_4{}^3{}_5 \quad 5_1{}^4{}_2 \quad 2_5{}^1{}_3 \quad 4_3{}^2{}_1$
$5_4{}^2{}_3 \quad 3_5{}^4{}_1 \quad 4_2{}^1{}_5 \quad 1_3{}^5{}_2 \quad 2_1{}^3{}_4$
$4_5{}^3{}_2 \quad 5_3{}^1{}_4 \quad 2_4{}^5{}_1 \quad 3_1{}^2{}_5 \quad 1_2{}^4{}_3$

[14]

$1_1{}^1{}_1 \quad 2_2{}^2{}_2 \quad 3_3{}^3{}_3 \quad 4_4{}^4{}_4 \quad 5_5{}^5{}_5$
$2_3{}^4{}_5 \quad 4_1{}^5{}_3 \quad 1_5{}^2{}_4 \quad 5_2{}^3{}_1 \quad 3_4{}^1{}_2$
$4_5{}^3{}_2 \quad 5_3{}^1{}_4 \quad 2_4{}^5{}_1 \quad 3_1{}^2{}_5 \quad 1_2{}^4{}_3$
$3_2{}^5{}_4 \quad 1_4{}^3{}_5 \quad 5_1{}^4{}_2 \quad 2_5{}^1{}_3 \quad 4_3{}^2{}_1$
$5_4{}^2{}_3 \quad 3_5{}^4{}_1 \quad 4_2{}^1{}_5 \quad 1_3{}^5{}_2 \quad 2_1{}^3{}_4$

[15]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_1{}^5{}_3 & 1_5{}^2{}_4 & 5_2{}^3{}_1 & 3_4{}^1{}_2 \\ 4_5{}^3{}_2 & 5_3{}^1{}_4 & 2_4{}^5{}_1 & 3_1{}^2{}_5 & 1_2{}^4{}_3 \\ 5_4{}^2{}_3 & 3_5{}^4{}_1 & 4_2{}^1{}_5 & 1_3{}^5{}_2 & 2_1{}^3{}_4 \\ 3_2{}^5{}_4 & 1_4{}^3{}_5 & 5_1{}^4{}_2 & 2_5{}^1{}_3 & 4_3{}^2{}_1 \end{array}$$

[16]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_1{}^5{}_3 & 1_5{}^2{}_4 & 5_2{}^3{}_1 & 3_4{}^1{}_2 \\ 5_4{}^2{}_3 & 3_5{}^4{}_1 & 4_2{}^1{}_5 & 1_3{}^5{}_2 & 2_1{}^3{}_4 \\ 3_2{}^5{}_4 & 1_4{}^3{}_5 & 5_1{}^4{}_2 & 2_5{}^1{}_3 & 4_3{}^2{}_1 \\ 4_5{}^3{}_2 & 5_3{}^1{}_4 & 2_4{}^5{}_1 & 3_1{}^2{}_5 & 1_2{}^4{}_3 \end{array}$$

[17]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_1{}^5{}_3 & 1_5{}^2{}_4 & 5_2{}^3{}_1 & 3_4{}^1{}_2 \\ 5_4{}^2{}_3 & 3_5{}^4{}_1 & 4_2{}^1{}_5 & 1_3{}^5{}_2 & 2_1{}^3{}_4 \\ 4_5{}^3{}_2 & 5_3{}^1{}_4 & 2_4{}^5{}_1 & 3_1{}^2{}_5 & 1_2{}^4{}_3 \\ 3_2{}^5{}_4 & 1_4{}^3{}_5 & 5_1{}^4{}_2 & 2_5{}^1{}_3 & 4_3{}^2{}_1 \end{array}$$

[18]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_5{}^3{}_1 & 5_2{}^1{}_4 & 3_1{}^5{}_2 & 1_4{}^2{}_3 \\ 3_5{}^2{}_4 & 5_1{}^4{}_3 & 2_4{}^5{}_1 & 1_2{}^3{}_5 & 4_3{}^1{}_2 \\ 4_2{}^5{}_3 & 3_4{}^1{}_5 & 1_5{}^4{}_2 & 5_3{}^2{}_1 & 2_1{}^3{}_4 \\ 5_4{}^3{}_2 & 1_3{}^5{}_4 & 4_1{}^2{}_5 & 2_5{}^1{}_3 & 3_2{}^4{}_1 \end{array}$$

[19]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_5{}^3{}_1 & 5_2{}^1{}_4 & 3_1{}^5{}_2 & 1_4{}^2{}_3 \\ 3_5{}^2{}_4 & 5_1{}^4{}_3 & 2_4{}^5{}_1 & 1_2{}^3{}_5 & 4_3{}^1{}_2 \\ 5_4{}^3{}_2 & 1_3{}^5{}_4 & 4_1{}^2{}_5 & 2_5{}^1{}_3 & 3_2{}^4{}_1 \\ 4_2{}^5{}_3 & 3_4{}^1{}_5 & 1_5{}^4{}_2 & 5_3{}^2{}_1 & 2_1{}^3{}_4 \end{array}$$

[20]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_5{}^3{}_1 & 5_2{}^1{}_4 & 3_1{}^5{}_2 & 1_4{}^2{}_3 \\ 4_2{}^5{}_3 & 3_4{}^1{}_5 & 1_5{}^4{}_2 & 5_3{}^2{}_1 & 2_1{}^3{}_4 \\ 3_5{}^2{}_4 & 5_1{}^4{}_3 & 2_4{}^5{}_1 & 1_2{}^3{}_5 & 4_3{}^1{}_2 \\ 5_4{}^3{}_2 & 1_3{}^5{}_4 & 4_1{}^2{}_5 & 2_5{}^1{}_3 & 3_2{}^4{}_1 \end{array}$$

[21]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_5{}^3{}_1 & 5_2{}^1{}_4 & 3_1{}^5{}_2 & 1_4{}^2{}_3 \\ 4_2{}^5{}_3 & 3_4{}^1{}_5 & 1_5{}^4{}_2 & 5_3{}^2{}_1 & 2_1{}^3{}_4 \\ 5_4{}^3{}_2 & 1_3{}^5{}_4 & 4_1{}^2{}_5 & 2_5{}^1{}_3 & 3_2{}^4{}_1 \\ 3_5{}^2{}_4 & 5_1{}^4{}_3 & 2_4{}^5{}_1 & 1_2{}^3{}_5 & 4_3{}^1{}_2 \end{array}$$

[22]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_5{}^3{}_1 & 5_2{}^1{}_4 & 3_1{}^5{}_2 & 1_4{}^2{}_3 \\ 5_4{}^3{}_2 & 1_3{}^5{}_4 & 4_1{}^2{}_5 & 2_5{}^1{}_3 & 3_2{}^4{}_1 \\ 3_5{}^2{}_4 & 5_1{}^4{}_3 & 2_4{}^5{}_1 & 1_2{}^3{}_5 & 4_3{}^1{}_2 \\ 4_2{}^5{}_3 & 3_4{}^1{}_5 & 1_5{}^4{}_2 & 5_3{}^2{}_1 & 2_1{}^3{}_4 \end{array}$$

[23]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 4_5{}^3{}_1 & 5_2{}^1{}_4 & 3_1{}^5{}_2 & 1_4{}^2{}_3 \\ 5_4{}^3{}_2 & 1_3{}^5{}_4 & 4_1{}^2{}_5 & 2_5{}^1{}_3 & 3_2{}^4{}_1 \\ 4_2{}^5{}_3 & 3_4{}^1{}_5 & 1_5{}^4{}_2 & 5_3{}^2{}_1 & 2_1{}^3{}_4 \\ 3_5{}^2{}_4 & 5_1{}^4{}_3 & 2_4{}^5{}_1 & 1_2{}^3{}_5 & 4_3{}^1{}_2 \end{array}$$

[24]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 5_1{}^3{}_4 & 1_4{}^5{}_2 & 3_5{}^2{}_1 & 4_2{}^1{}_3 \\ 3_2{}^5{}_4 & 1_5{}^4{}_3 & 4_1{}^2{}_5 & 5_3{}^1{}_2 & 2_4{}^3{}_1 \\ 4_5{}^3{}_2 & 3_4{}^1{}_5 & 5_2{}^4{}_1 & 2_1{}^5{}_3 & 1_3{}^2{}_4 \\ 5_4{}^2{}_3 & 4_3{}^5{}_1 & 2_5{}^1{}_4 & 1_2{}^3{}_5 & 3_1{}^4{}_2 \end{array}$$

[25]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 5_1{}^3{}_4 & 1_4{}^5{}_2 & 3_5{}^2{}_1 & 4_2{}^1{}_3 \\ 3_2{}^5{}_4 & 1_5{}^4{}_3 & 4_1{}^2{}_5 & 5_3{}^1{}_2 & 2_4{}^3{}_1 \\ 5_4{}^2{}_3 & 4_3{}^5{}_1 & 2_5{}^1{}_4 & 1_2{}^3{}_5 & 3_1{}^4{}_2 \\ 4_5{}^3{}_2 & 3_4{}^1{}_5 & 5_2{}^4{}_1 & 2_1{}^5{}_3 & 1_3{}^2{}_4 \end{array}$$

[26]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 5_1{}^3{}_4 & 1_4{}^5{}_2 & 3_5{}^2{}_1 & 4_2{}^1{}_3 \\ 4_5{}^3{}_2 & 3_4{}^1{}_5 & 5_2{}^4{}_1 & 2_1{}^5{}_3 & 1_3{}^2{}_4 \\ 3_2{}^5{}_4 & 1_5{}^4{}_3 & 4_1{}^2{}_5 & 5_3{}^1{}_2 & 2_4{}^3{}_1 \\ 5_4{}^2{}_3 & 4_3{}^5{}_1 & 2_5{}^1{}_4 & 1_2{}^3{}_5 & 3_1{}^4{}_2 \end{array}$$

[27]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 5_1{}^3{}_4 & 1_4{}^5{}_2 & 3_5{}^2{}_1 & 4_2{}^1{}_3 \\ 4_5{}^3{}_2 & 3_4{}^1{}_5 & 5_2{}^4{}_1 & 2_1{}^5{}_3 & 1_3{}^2{}_4 \\ 5_4{}^2{}_3 & 4_3{}^5{}_1 & 2_5{}^1{}_4 & 1_2{}^3{}_5 & 3_1{}^4{}_2 \\ 3_2{}^5{}_4 & 1_5{}^4{}_3 & 4_1{}^2{}_5 & 5_3{}^1{}_2 & 2_4{}^3{}_1 \end{array}$$

[28]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 5_1{}^3{}_4 & 1_4{}^5{}_2 & 3_5{}^2{}_1 & 4_2{}^1{}_3 \\ 5_4{}^2{}_3 & 4_3{}^5{}_1 & 2_5{}^1{}_4 & 1_2{}^3{}_5 & 3_1{}^4{}_2 \\ 3_2{}^5{}_4 & 1_5{}^4{}_3 & 4_1{}^2{}_5 & 5_3{}^1{}_2 & 2_4{}^3{}_1 \\ 4_5{}^3{}_2 & 3_4{}^1{}_5 & 5_2{}^4{}_1 & 2_1{}^5{}_3 & 1_3{}^2{}_4 \end{array}$$

[29]

$$\begin{array}{ccccc} 1_1{}^1{}_1 & 2_2{}^2{}_2 & 3_3{}^3{}_3 & 4_4{}^4{}_4 & 5_5{}^5{}_5 \\ 2_3{}^4{}_5 & 5_1{}^3{}_4 & 1_4{}^5{}_2 & 3_5{}^2{}_1 & 4_2{}^1{}_3 \\ 5_4{}^2{}_3 & 4_3{}^5{}_1 & 2_5{}^1{}_4 & 1_2{}^3{}_5 & 3_1{}^4{}_2 \\ 4_5{}^3{}_2 & 3_4{}^1{}_5 & 5_2{}^4{}_1 & 2_1{}^5{}_3 & 1_3{}^2{}_4 \\ 3_2{}^5{}_4 & 1_5{}^4{}_3 & 4_1{}^2{}_5 & 5_3{}^1{}_2 & 2_4{}^3{}_1 \end{array}$$

[30]

$1_1{}^1{}_1$ $2_2{}^2{}_2$ $3_3{}^3{}_3$ $4_4{}^4{}_4$ $5_5{}^5{}_5$

$2_3{}^4{}_5$ $5_4{}^1{}_3$ $4_2{}^5{}_1$ $1_5{}^3{}_2$ $3_1{}^2{}_4$

$3_4{}^5{}_2$ $4_1{}^3{}_5$ $2_5{}^1{}_4$ $5_3{}^2{}_1$ $1_2{}^4{}_3$

$4_5{}^2{}_3$ $1_3{}^5{}_4$ $5_1{}^4{}_2$ $3_2{}^1{}_5$ $2_4{}^3{}_1$

$5_2{}^3{}_4$ $3_5{}^4{}_1$ $1_4{}^2{}_5$ $2_1{}^5{}_3$ $4_3{}^1{}_2$

[31]

$1_1{}^1{}_1$ $2_2{}^2{}_2$ $3_3{}^3{}_3$ $4_4{}^4{}_4$ $5_5{}^5{}_5$

$2_3{}^4{}_5$ $5_4{}^1{}_3$ $4_2{}^5{}_1$ $1_5{}^3{}_2$ $3_1{}^2{}_4$

$3_4{}^5{}_2$ $4_1{}^3{}_5$ $2_5{}^1{}_4$ $5_3{}^2{}_1$ $1_2{}^4{}_3$

$5_2{}^3{}_4$ $3_5{}^4{}_1$ $1_4{}^2{}_5$ $2_1{}^5{}_3$ $4_3{}^1{}_2$

$4_5{}^2{}_3$ $1_3{}^5{}_4$ $5_1{}^4{}_2$ $3_2{}^1{}_5$ $2_4{}^3{}_1$

[32]

$1_1{}^1{}_1$ $2_2{}^2{}_2$ $3_3{}^3{}_3$ $4_4{}^4{}_4$ $5_5{}^5{}_5$

$2_3{}^4{}_5$ $5_4{}^1{}_3$ $4_2{}^5{}_1$ $1_5{}^3{}_2$ $3_1{}^2{}_4$

$4_5{}^2{}_3$ $1_3{}^5{}_4$ $5_1{}^4{}_2$ $3_2{}^1{}_5$ $2_4{}^3{}_1$

$3_4{}^5{}_2$ $4_1{}^3{}_5$ $2_5{}^1{}_4$ $5_3{}^2{}_1$ $1_2{}^4{}_3$

$5_2{}^3{}_4$ $3_5{}^4{}_1$ $1_4{}^2{}_5$ $2_1{}^5{}_3$ $4_3{}^1{}_2$

[33]

$1_1{}^1{}_1$ $2_2{}^2{}_2$ $3_3{}^3{}_3$ $4_4{}^4{}_4$ $5_5{}^5{}_5$

$2_3{}^4{}_5$ $5_4{}^1{}_3$ $4_2{}^5{}_1$ $1_5{}^3{}_2$ $3_1{}^2{}_4$

$4_5{}^2{}_3$ $1_3{}^5{}_4$ $5_1{}^4{}_2$ $3_2{}^1{}_5$ $2_4{}^3{}_1$

$5_2{}^3{}_4$ $3_5{}^4{}_1$ $1_4{}^2{}_5$ $2_1{}^5{}_3$ $4_3{}^1{}_2$

$3_4{}^5{}_2$ $4_1{}^3{}_5$ $2_5{}^1{}_4$ $5_3{}^2{}_1$ $1_2{}^4{}_3$

[34]

$1_1{}^1{}_1$ $2_2{}^2{}_2$ $3_3{}^3{}_3$ $4_4{}^4{}_4$ $5_5{}^5{}_5$

$2_3{}^4{}_5$ $5_4{}^1{}_3$ $4_2{}^5{}_1$ $1_5{}^3{}_2$ $3_1{}^2{}_4$

$5_2{}^3{}_4$ $3_5{}^4{}_1$ $1_4{}^2{}_5$ $2_1{}^5{}_3$ $4_3{}^1{}_2$

$3_4{}^5{}_2$ $4_1{}^3{}_5$ $2_5{}^1{}_4$ $5_3{}^2{}_1$ $1_2{}^4{}_3$

$4_5{}^2{}_3$ $1_3{}^5{}_4$ $5_1{}^4{}_2$ $3_2{}^1{}_5$ $2_4{}^3{}_1$

[35]

$1_1{}^1{}_1$ $2_2{}^2{}_2$ $3_3{}^3{}_3$ $4_4{}^4{}_4$ $5_5{}^5{}_5$

$2_3{}^4{}_5$ $5_4{}^1{}_3$ $4_2{}^5{}_1$ $1_5{}^3{}_2$ $3_1{}^2{}_4$

$5_2{}^3{}_4$ $3_5{}^4{}_1$ $1_4{}^2{}_5$ $2_1{}^5{}_3$ $4_3{}^1{}_2$

$4_5{}^2{}_3$ $1_3{}^5{}_4$ $5_1{}^4{}_2$ $3_2{}^1{}_5$ $2_4{}^3{}_1$

$3_4{}^5{}_2$ $4_1{}^3{}_5$ $2_5{}^1{}_4$ $5_3{}^2{}_1$ $1_2{}^4{}_3$

[36]

Appendix IV

Some Journals which have published practical applications of Latin Squares

A number of examples of applications of Latin Squares to problems faced by research workers in everyday practice may well be found in the following internationally reputed journals: Journal of animal science, Animal feed Science and technology, Journal of dairy science, PAIN, Antibiotiki I khimioterapiya, Canadian Journal of animal science, Journal of animal physiology and animal nutrition, Korean journal of animal nutrition and feedstuffs, Theriogenology, Acta pharmacologica sinica, Journal of rheumatology, Animal production, Japanese journal of zootechnical science, Mikrobiologicheskii zhurnal (Kiev), Biometrics, Medical and veterinary entomology, Small ruminant research, Environmental technology, Agricultural science in Finland, Swedish journal of agricultural research, Bulletin of entomological research, Biometrical journal, Journal of pharmaceutical sciences, Asian-australasian journal of animal sciences, Grass and forage science, American journal of veterinary research, Ochrana rostlin, Australian journal of experimental agriculture, Current science (Bangalore), Statistics in medicine, Journal of animal and feed sciences, Applied animal behaviour science, Scandinavian journal of forest research, Wirstschaftseigene futter, Animal science (Pencaitland), Revista da sociedade brasileira de zootecnia.

Program execution times in Unix-computer units (u)

Number of added symbols:	2	3	4*
User time per square:	311.30u	288.95u	1 month-machine-time

* Enumeration producing repeated symbols in lexicographical order for the rows.

Appendix V

This appendix comprises copies of two documents issued by the Department of statistics. These are:

I. Advice on the presentation of PhD theses (Department of statistics, University of Warwick, April 1996), pp. 139-141.

1 Timetable

2 Theses length

3 Before writing

4 Writing

5 Copying (Plagiarism)

6 Style

7 After writing

8 Regulations

II. Regulations and advice for PhD students (Department of statistics, University of Warwick, October 1994), pp. 142-144.

Background

1. Year one

2. Year two

3. Year three

UNIVERSITY OF WARWICK

DEPARTMENT OF STATISTICS

ADVICE ON THE PREPARATION OF Ph.D THESES

A Ph.D thesis should be presented within three years after the start of full-time research. The following notes are intended to help students to do that.

1. TIMETABLE

The time required to write a thesis is longer than is often supposed. A respectable timetable, allowing for the possibility arising of the need for further analysis or calculations and for checking the literature, for a Ph.D thesis might be:

Introductory chapter(s) and co-ordination of review material	four weeks
Initial final versions of the five(?) chapters of new material	ten weeks
Consultations and revisions of these chapters	three weeks
Preparations of tables and diagrams	two weeks
Proof reading, copying	two weeks
Binding	one week
Allowance for hiccups, false starts etc.	two weeks
	twenty-four weeks

This means that writing should normally be begun 2 years and 6 months after beginning the course, that is, Easter of the third year for an October start. Note that this timetable assumes that hand-written or preferably word processed draft versions of the material in the chapters have been worked out and all major analyses and results have been produced.

2. THESES LENGTH

The thesis should be no longer than necessary to **provide a succinct introduction** to the field of study for the non-specialist, to **present** all the new results, to **discuss** their implications in the context of current knowledge of the filed by **providing adequate references** to the literature. Omit unessential information - it has been said that the art of writing consists largely of knowing what to leave in the inkwell. Examiners are just as critical of theses that are too long as ones that are too short. Rewriting is a very painful business.

The Statistics Departmental guidelines for the presentation of theses for the degrees of Ph.D, MPhil and MSc are as follows:

a. All theses must conform in style content and presentation to the "Higher Degree Regulations" published in the University Calendar, and the "Requirements for the presentation of research theses" provided to research students by Registry. Candidates attention is drawn to section E "Presentation and typing of theses".

b. The guidelines for the appropriate lengths for Statistics theses are as follows :

PhD	90 to 150 pages of text plus essential diagrams, tables etc.
MPhil	70 to 100 pages of text plus essential diagrams, tables etc.
MSc by Research	50 to 70 pages of text plus essential diagrams, tables etc.

Students are advised not to include unessential detailed data or computer analyses, even as appendices, in the bound thesis.

3. BEFORE WRITING

Adequate preparation before beginning to write can help greatly to obtain a logically arranged, readable thesis and to shorten both the thesis and the writing time. First analyse the problem by answering the following questions. What information do I wish to present? What background can I assume? What is the most sensible sequence in which to present the information.

Make a detailed outline. Identify as many subdivisions as possible. It is easier to combine subheadings or eliminate them than to insert new ones later. Plan tables and figures. Avoid duplication of results in tables and figures unless there is specific justification. Consign material that would disturb the smooth flow of an argument to Appendices. Bulky material such as computer programmes should normally be omitted; if appropriate, copies should be left with the supervisor.

Some excellent tips are contained in a short article "Writing your thesis" by J M Pratt (Chemistry in Britain, 20 (December 1984) 1114-5) which you would do well to read and in "Communicating in Science: Writing and Speaking" by V Booth, CUP 1985.

4. WRITING

Scientific writing is not exempt from the rules of good grammar, spelling usage and punctuation! Keep a dictionary handy. Avoid long, meandering and contorted sentences, but do not achieve brevity by becoming telegraphic - do not omit a's and the's. Remember that it is an invariable rule that every sentence begins with a capital letter, contains at least one verb, and ends with a full stop. Good punctuation is an aid to clarity; if someone familiar with the subject has to re-read a sentence to understand it, the sentence probably needs more punctuation, or reconstruction. Go through paragraphs when you have written them, trying to put yourself in the place of a reader rather than the writer.

Avoid vague and inexact terms: remember you should be able to use mathematics to say precisely what you mean. Define all non-standard terms, symbols and abbreviations where first used, and stick to them. In particular try to keep the same notation from one chapter to the next. Try to develop your arguments in a logical manner. This may be quite different from the chronological order in which you performed the research!

5. COPYING (Plagiarism)

Any material copied word for word MUST be placed in quotation marks and the original source fully referenced. This principle applies to diagrams as well as text. Students are reminded that plagiarism - reproducing another person's work as your own - is considered a very serious offence. Your attention is drawn to the following paragraph in the University booklet 'Guidance to Candidates, Supervisors and Examiners Concerning Higher Degrees by Research'.

'The Thesis must be entirely the candidate's own work, and all sources used should be fully referenced and acknowledged in the thesis. There is no distinction to be made between plagiarism of reviews or summaries of existing knowledge on a subject and original research work. The University's regulations on plagiarism appear in the University Calendar (Regulation 12).'

6. STYLE

The general style of presentation should conform to that required for scientific papers in reputable journals, for example the Journals of the Royal Statistical Society whose layout is summarised each year within the text of the journal. The thesis will be longer than typical research papers. It will therefore require a list of contents. Number all pages including diagrams, illustrations and tables. Collect all references and put them at the end of the thesis rather than with individual chapters.

7. AFTER WRITING

When you have completed the first draft (of a chapter, say) lay it aside for a day or two. Then, coming to it afresh, read it carefully for final revision, making sure notation and symbols are uniform throughout and consistent with what you have used in other chapters. Look out for obscurities, duplication or omissions. Adequate marginal annotation of your manuscript will help the typist and minimise the number of corrections to the typescript, if you choose not the type the thesis yourself.

Proof read the typescript for typographical errors and accidental omissions. This requires the utmost care if the thesis is not to be spoiled by residual minor errors. Allow yourself enough time for this essential final stage; it cannot be hurried. You can expect your supervisor to read and comment on your first or second drafts in general terms, but not to rewrite it for you.

8. REGULATIONS

The University provides three relevant documents which should be read earlier rather than later

i. Requirements for the presentation of research theses (sent out by Registry two and a half years after starting research).

ii. Higher Degree Regulations (to be found in the University Calendar)

iii. Guidance to students, supervisors and examiners concerning higher degrees by research (sent out by Registry on first registration).

Spare copies of these documents can always be obtained from the Registry. You should note that, amongst other requirements, the University insists that the thesis has an abstract and a declaration regarding any joint work.

UNIVERSITY OF WARWICK

DEPARTMENT OF STATISTICS

REGULATIONS AND ADVICE FOR PHD STUDENTS

BACKGROUND

The course regulations are framed with the following objectives:-

a. To ensure that each postgraduate student continues to receive a training in statistics that will deepen his or her understanding of the general subject area underlying the research project and broaden knowledge of related areas so that the results of the project can be interpreted fully and set in context.

b. To assess the progress of the student at all stages of the project; judge whether it is compatible with completion of the project for PhD submission within a three year period and initiate any changes necessary.

c. To provide practice in writing and oral communication in order that the student can present his thesis and defend it adequately in the subsequent viva.

Most, but not all, of the training/assessment will take place in the first nine months so that the student's ongoing registration can be confirmed well in advance of the start of the second year of study. The progress of all postgraduate students is monitored by the Graduate Tutor, The Chairman of the Department and another informed member of the academic staff who act in collaboration with the project supervisor.

The regulations are presented in terms of chronological course requirements and assume an October 1st start; the schedules for students beginning at other dates will be adjusted accordingly.

1. YEAR ONE

Upon arrival you will discuss with your supervisor your programme of work and the theme of the research.

You will be expected to attend those courses your supervisor deems necessary supplements to your research programme.

It will be necessary for you to become familiar with as much current literature in the area of your program as early as possible. This search will be directed by your supervisor.

In the case of the more practically based theses, the research will be relatively well defined - for example the application of a new statistical technique to a novel area. In the case of theoretical theses the direction of the project will be quickly developed through the student's acquisition of knowledge in this preparatory stage of research. You can expect to have supervisions on this material about once a week. You should aim to prepare written material for your supervisor the day before most supervisions - this will aid continuity as well as provide a record of your work through the year. In any case your progress will be continuously monitored by your supervisor.

In June of your first year of study you will produce a 25 minute oral presentation of your work so far to a panel of academic staff, normally including the Chairman, Postgraduate Tutor, your supervisor and an appointed examining member of staff who is not your supervisor. Following your presentation there will be a 15 minute discussion lead by the examining member of staff. You will need to convince the panel:

i. that you have a good understanding of the literature underpinning your area of research.

ii. that your research area is developing in a fruitful way.

and

iii. you have the creative ability to produce new work in your area of research which is essentially your own.

On the evidence of your presentation and any other written material you would like to be taken into account you will :

a. be allowed to progress to a PhD programme

or

b. be instructed to produce further evidence to support your case for progress on to a PhD programme - this work would normally be required by the end of September.

or

c. be allowed to proceed on an MPhil programme

or

d. be recommended to withdraw.

The results of your appraisal will be communicated to you by the Head of Department as soon as possible after your presentation. Of course, all decisions made are subject to the subsequent formal agreement of Senate.

2. YEAR TWO

By July of year two of a Ph.D programme you will be expected to have produced at least one research report (possibly jointly with your supervisor) communicating your progress to that time. This piece of work will be examined by your supervisor and another examining member of staff and Head of Department together with any other information you believe is relevant. This material will be assessed with regard to the yardstick that your thesis will need to contain material on which it would be possible to publish at least 2 substantial papers in internationally recognised journals before it can be awarded a Ph.D On the basis of this work you will :

a. be permitted to proceed on the Ph.D programme

or

b. be requested to convert to an MPhil degree

or

c. recommend to withdraw

In exceptional circumstances at the discretion of the department you may be allowed to give an oral presentation as in Year 1 instead of the research report mentioned above and this presentation will then be examined .

3. YEAR THREE

Although there are no formal assessments in the last year of a Ph.D student's study, you should expect between July of the second year and April of the third to be actively developing the framework built on the initial 22 months study. You should expect at this stage to be more expert in some of the fields of your research than your supervisor and that you now work as colleagues within the project. You should be able to provide a thesis plan by the end of April of your last year and plan to start writing up your thesis at this time.

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